Ann. Inst. H. Poincaré C
Anal. Non Linéaire (Online first)
DOI 10.4171/AIHPC/112

Published by EMS Press

Asymptotic stability of the fourth-order ϕ^4 kink for general perturbations in the energy space

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Abstract. The fourth-order ϕ^4 model extends the classical ϕ^4 model of quantum field theory to the fourth-order case, but sharing the same kink solution. It is also the dispersive counterpart of the well-known parabolic Cahn–Hilliard equation. Mathematically speaking, the kink is characterized by a fourth-order nonnegative linear operator with a simple kernel at the origin but no spectral gap. In this paper, we consider the kink of this theory, and prove orbital and asymptotic stability for any perturbation in the energy space.

Contents

1. Introduction

1.1. Setting

In this work we consider the *fourth-order ϕ^4 model*, or wave-Cahn–Hilliard equation. In one dimension this model is written as

$$\partial_t^2\phi + \partial_x^2(\partial_x^2\phi + \phi - \phi^3) = 0, \quad (t,x) \in \mathbb{R}^2, \tag{1.1}$$

Mathematics Subject Classification 2020: 35B35 (primary); 35B40,37K40 (secondary).
Keywords: wave Cahn—Hilliard, fourth-order ϕ^4, kink, asymptotic stability, orbital stability.

where $\phi = \phi(t, x)$ is assumed real valued. Equation (1.1) with quadratic nonlinearity and an additional linear term of order zero was considered by Bretherton [11] as a model for studying weakly nonlinear wave dispersion. It also appears with an additional parabolic term in [24].

There are several interesting and profound motivations to study (1.1). The first and most obvious one is given by its similarity to a recognized model of degenerate phase transitions. Indeed, the "parabolic" version of (1.1) is the well-known Cahn–Hilliard model

$$\partial_t \phi + \partial_x^2(\partial_x^2 \phi + \phi - \phi^3) = 0, \quad x \in \mathbb{R},\ t \geq 0, \tag{1.2}$$

introduced by Cahn and Hilliard in the study of phase separation in cooling binary solutions such as alloys, glasses and polymer mixtures [14]; see additionally [76, 77]. Major mathematical advances have been obtained for this model during past years, not only in one dimension but also in several dimensions. In this setting, a key question is the long-time behavior of kinks, usually referred to in the community as fronts. Among other essential works, one finds the foundational result by Bricmont, Kupiainen and Taskinen [12], who showed the stability and asymptotic stability of fronts in one dimension for (1.2). Many subsequent works have extended and improved this achievement. These results will be described in detail in Section 1.3.

Additionally, (1.1) can be recast as the natural fourth-order extension of the well-known ϕ^4 model of quantum field theory,

$$\partial_t^2 \phi - (\partial_x^2 \phi + \phi - \phi^3) = 0, \quad (t, x) \in \mathbb{R}^2, \tag{1.3}$$

that has been extensively studied during past years. The ϕ^4 kink

$$H(x) = \tanh\Big(\frac{x}{\sqrt{2}}\Big), \tag{1.4}$$

is a notable example of a simple solution of (1.3) with challenging dynamical behavior, and its general behavior is still unknown to us. In [42] (see also [41]), the authors showed asymptotic stability for odd perturbations in the energy space. A proof for the general case is still missing.

Another interesting motivation for the study of (1.1) is given by its defocusing character, opposite to the focusing one present in the good-Boussinesq model. See [68] for a detailed introduction to the soliton problem in good-Boussinesq models. The latter model and the dynamics of their (unstable) solitons has been studied by many authors during past years. But, contrary to good-Boussinesq, the model studied in this work will have more physical interest.

Precisely, the fourth and final motivation to study (1.1) comes from the fact that it possesses, as well as Cahn–Hilliard and ϕ^4, stable kink solutions. In the fourth-order ϕ^4 case, the kink coincides with the one for ϕ^4, and as far as we understand, for (1.1) there is not even a single kink stability result. Consequently, in this paper our main objective is to initiate the study of the dynamics of kinks for (1.1) dealing with the one-dimensional case for general data in the energy space.

Following Linares [52], equation (1.1) can be rewritten formally taking

$$\boldsymbol{\phi} = (\phi_1, \phi_2) = (\phi, \partial_x^{-1}\partial_t \phi),$$

and obtaining the following representation as a 2×2 system:

$$\begin{cases} \partial_t \phi_1 = \partial_x \phi_2, \\ \partial_t \phi_2 = -\partial_x(\partial_x^2 \phi_1 + \phi_1 - \phi_1^3). \end{cases} \tag{1.5}$$

This system admits kink solutions given by

$$\boldsymbol{H}_c = (H_c, -cH_c)(x - ct - x_0), \quad c \in \mathbb{R},\ x_0 \in \mathbb{R},$$

where $\pm H_c$ solves the ODE

$$H_c'' + (c^2+1)H_c - H_c^3 = 0, \quad H_c(\pm\infty) = \pm\sigma = \pm\sqrt{1+c^2}. \tag{1.6}$$

One has that H_c is a rescaled version of the ϕ^4 kink:

$$H_c(x) = \sigma H(\sigma x), \quad H(x) \text{ given in (1.4)},$$

which differs from the general kink from (1.3) in the sense that the former only has the Lorentz boost as its scaling symmetry. This subtle difference will remain very important for the main results of this paper.

System (1.5) will be the exact model worked in this paper. It has the following associated conserved quantities:

$$\begin{aligned} E[\boldsymbol{\phi}] &= E[\phi_1, \phi_2] = \frac{1}{2}\int \Big[(\partial_x \phi_1)^2 + \phi_2^2 + \frac{1}{2}(\phi_1^2 - 1)^2\Big] && \text{(energy)}, \\ P[\boldsymbol{\phi}] &= P[\phi_1, \phi_2] = \int \phi_1 \phi_2 && \text{(momentum)}. \end{aligned} \tag{1.7}$$

(Here $\int$ means $\int_{\mathbb{R}} dx$.) The quantity P has no good meaning around the kink solution if one works barely in the energy space, and will be loosely used in this general form. However, its second variation around the kink is perfectly well defined and will be the key element to study the long-time behavior of kinks. Following [45], the set of functions $\boldsymbol{\phi} \in L^1_{\mathrm{loc}}(\mathbb{R}) \times L^1_{\mathrm{loc}}(\mathbb{R})$ for which the energy is finite is

$$\boldsymbol{E} = \big\{(\phi_1, \phi_2) \in L^1_{\mathrm{loc}}(\mathbb{R}) \times L^1_{\mathrm{loc}}(\mathbb{R}) \ \big|\ \partial_x \phi_1, \phi_2 \in L^2(\mathbb{R}),\ (\phi_1^2 - 1) \in L^2\big\}. \tag{1.8}$$

To study the stability of the static kink

$$\boldsymbol{H}(x) = \boldsymbol{H}_0 = (H, 0) = \Big(\tanh\Big(\frac{x}{\sqrt{2}}\Big), 0\Big), \tag{1.9}$$

we introduce the subset of $\boldsymbol{E}$ in (1.8) given by

$$\boldsymbol{E}_{\boldsymbol{H}} = \big\{\boldsymbol{\phi} \in \boldsymbol{E} \ \big|\ \boldsymbol{\phi} - \boldsymbol{H} \in H^1(\mathbb{R}) \times L^2(\mathbb{R})\big\}. \tag{1.10}$$

Let us consider a shift modulation $\rho(t)$ induced by a perturbation $\boldsymbol{\phi} \in \boldsymbol{E_H}$ in (1.5) of $\boldsymbol{H}$ of the form

$$\phi_1(t,x) = H(x-\rho(t)) + u_1(t,x), \quad \phi_2(t,x) = u_2(t,x),$$

such that

$$\langle \boldsymbol{u}(t), \boldsymbol{H}'(\cdot - \rho(t))\rangle = \langle u_1(t), H'(x-\rho(t))\rangle = 0, \quad \langle a,b\rangle := \int ab\,dx. \tag{1.11}$$

Then from (1.5) one can see that the perturbation satisfies the following space-time, variable coefficients system:

$$\begin{cases} \partial_t u_1 = \partial_x u_2 + \rho' H'(\cdot - \rho), \\ \partial_t u_2 = \partial_x \mathcal{L} u_1 + \partial_x (3H(\cdot - \rho)u_1^2 + u_1^3), \end{cases} \tag{1.12}$$

where

$$\mathcal{L} = -\partial_x^2 + V_0(\cdot - \rho), \quad \text{with } V_0(x) = -1 + 3H^2(x) = 2 - 3\,\mathrm{sech}^2\Big(\frac{x}{\sqrt{2}}\Big). \tag{1.13}$$

Note that $\mathcal{L}$ coincides with the unbounded linear operator appearing around the ϕ^4 kink. Recalling [42,45,62] (see also Lemma 2.1 for further details), one has that $\mathcal{L}$ is nonnegative and it has absolutely continuous spectrum $[2,\infty)$. Additionally, the discrete spectrum consists of the simple eigenvalues $\lambda_0 = 0$ and $\lambda_1 = \frac{\sqrt{3}}{2}$, with eigenfunctions

$$Y_0(x) = H' \quad \text{and} \quad Y_1(x) = \mathrm{sech}\Big(\frac{x}{\sqrt{2}}\Big)\tanh\Big(\frac{x}{\sqrt{2}}\Big), \tag{1.14}$$

respectively. Finally, $\lambda_2 = 2$ is a threshold resonance, in the sense that $\mathcal{L}\phi = 2\phi$ possesses a smooth $L^\infty \backslash L^2$ solution with spatial derivative in L^2.

In the case of (1.12) the situation has its own particularities, similar to the ones already present in [68], in the sense that now the corresponding linear operator is $-\partial_x^2\mathcal{L}$, which is of fourth order and the composition of two second-order operators. The properties of this operator differ from $\mathcal{L}$ itself in the sense that (Lemma 2.2) $-\partial_x^2\mathcal{L}$ is nonnegative and it has absolutely continuous spectrum $[0,\infty)$, namely, as in KdV, there is no spectral gap. There are no embedded eigenvalues in the continuous spectrum nor resonances, but $\lambda_0 = 0$ is an eigenvalue with kernel Y_0 in (1.14), generated by the invariance under shifts of the model. There is an additional linearly growing odd mode at the origin, which in one dimension is usually harmless. Notice that any suitable scaling modulation of the kink costs infinite energy and for this reason will not be present in this work.

1.2. Main results

Our first result is the orbital stability of the fourth-order ϕ^4 kink:

Theorem 1.1. *Let $\boldsymbol{H}$ be the kink introduced in (1.9). There exist $\delta_* > 0$ and $C^* > 0$ such that for any $\boldsymbol{\phi}^{\mathbf{in}} \in \boldsymbol{E_H}$, with*

$$\|\boldsymbol{\phi}^{\mathbf{in}} - \boldsymbol{H}\|_{H^1\times L^2} \leq \delta_*, \tag{1.15}$$

there exists a unique global solution $\boldsymbol{\phi} \in C(\mathbb{R}, \boldsymbol{E_H})$ of (1.5) with $\boldsymbol{\phi}(0) = \boldsymbol{\phi}^{\mathbf{in}}$. Moreover, for some smooth $\rho(t) \in \mathbb{R}$, one has

$$\sup_{t\in\mathbb{R}} \|\boldsymbol{\phi}(t, \cdot + \rho(t)) - \boldsymbol{H}\|_{H^1\times L^2} \leq C^*\|\boldsymbol{\phi}^{\mathbf{in}} - \boldsymbol{H}\|_{H^1\times L^2}. \tag{1.16}$$

Some previous results are needed for the proof of this theorem, among them a well-posedness theory for perturbations of the kink in the energy space whose proof relies on standard energy arguments; see e.g. [45]. The proof of this fact is given in Section 2 and follows the ideas of Linares [52].

In the case of scalar field models, orbital stability was proved by Henry, Perez and Wreszinski [30] in the case of static kinks. The general case is contained in [45].

Although the proof of Theorem 1.1 will follow standard ideas, it is worth mentioning that no result of this type has appeared in the literature for the case of the fourth-order ϕ^4 model. An interesting open question is to prove (1.16) in the case of infinite energy perturbations that allow for scaling variations. In that case, the energy will no longer be useful, at least in the classical sense. Our main results, orbital and asymptotic stability, stated for data in the energy space only, provide a satisfactory answer as in the parabolic setting [12]. Indeed, in the "hyperbolic" case, the final state asymptotics will also be found in the case of general data perturbations.

Theorem 1.2 (Asymptotic stability). *Under assumption (1.15) in Theorem 1.1, and by making $\delta_* > 0$ smaller if necessary, one has that for any compact interval I of $\mathbb{R}$ and $\gamma > 0$ small enough,*

$$\lim_{t\to+\infty} \left(\|\phi(t, \cdot + \rho(t)) - H\|_{L^\infty(I)} + \|(1 - \gamma\partial_x^2)^{-1}\partial_t\phi(t, \cdot + \rho(t))\|_{H^1(I)}\right) = 0. \tag{1.17}$$

Recall that convergence on the whole line will imply that the data is the kink itself. In that sense, Theorem 1.2 is optimal if one considers the energy space topology only. Compared with the results in [12] for the parabolic Cahn–Hilliard model and subsequent improvements, the Hamiltonian character of the dynamics forbids a better understanding of the exterior regions without the use of well-chosen weighted norms. Additionally, the parabolic dynamics possesses several key elements that are not present here, the most important being the presence of good decaying functionals to measure the long-time dynamics.

The parameter $\gamma > 0$ in Theorem 1.2 depends on δ, in particular, any fixed $0 < \gamma \sim \delta^{2/5}$ suffices to prove (1.17). Additionally, general data with no restriction naturally induce shifts in the dynamics. This shift parameter in (1.17) satisfies a particular equation that suggests that, unless one asks for additional space decay at time zero, there will not be convergence of $\rho(t)$ as time tends to infinity. This is in strong contrast with scalar field models as in [37, 38, 45], where convergence was ensured by quadratic estimates on the shift parameters.

It is also interesting to compare Theorem 1.2 with the foundational asymptotic stability result for the defocusing mKdV kink obtained by Merle and Vega [70]. The fourth-order ϕ^4 kink requires more care because of its character as a nonlinear system of equations. They showed weak convergence of any (suitably shifted) perturbation of the kink towards the kink itself, once again thanks to the impossibility of performing scalings without spending an infinite amount of energy. This structural rigidity is present in other models; see e.g. the case of the Peregrine breather [72]. We believe that our techniques provide a locally stronger version of the results in [70], and probably improvements of [71] as well.

We believe that our results open the door to the understanding of the long-time solitary wave dynamics in more general fourth-order models, as well as several other Boussinesq models. We mention for instance the asymptotic stability, without parity condition, of the good-Boussinesq standing wave (see [68]), of some particular $abcd$ solitary waves, at least in the zero-speed even-data case [7,8]. Some of these models have important physical meaning and are of the utmost interest as well.

1.3. Previous results

We briefly comment now on the main previous contributions to the kink asymptotic stability problem in our setting. We classify them into three different lines: the Boussinesq models, the Cahn–Hilliard model and the ϕ^4 and similar models.

The fourth-order ϕ^4 model (1.1) is part of the family of Boussinesq [10] models highly studied in the literature. One important aspect that is not completely understood is the behavior of solitary waves in the long-time dynamics. For a complete review of this model in the "focusing case", the reader may consult the introduction in [68]. The literature is extensive and we will concentrate ourselves on the soliton problem. Bona and Sachs [9] showed the stability of fast solitary waves for the so-called good Boussinesq model. Slow solitary waves are unstable and may develop blow-up [61]. Pego and Weinstein [79, 80] addressed the asymptotic stability problem for the first time in the case of the so-called improved Boussinesq model (see also [69]) revealing its difficulty compared with other fluid models. Precisely, compared with (1.1), the former has strongly unstable directions, specially in the case of the standing wave which was proved asymptotically stable in [68], provided one works orthogonal to the unstable manifold. Our proofs follow the spirit of the results in [68] (see also a previous work in the case of the improved Boussinesq system [69]); however, unlike in previous works where parity was needed, here we are able to consider the problem in full generality. Previous decay results are available for the Bona, Chen and Saut $abcd$ model [7, 8]; see the references [47, 48].

The long-time behavior of kinks (or fronts) in the parabolic Cahn–Hilliard model [14] has been addressed by many authors during the past decades. Novick-Cohen and Segel [77] and Novick-Cohen [76] provided foundational energy methods to describe the dynamics in the case of the originally motivated strongly degenerate Cahn–Hilliard model. In [23], local and global well-posedness and long-time behavior for Cahn–Hilliard on an interval were first obtained. Pego [78] used matched asymptotics to describe the evolution in higher dimensions of the phase separation in the singular perturbative regime. Later, Caffarelli and Muler [13] provided rigorous L^∞ bounds for this regime. Alikakos, Bates and Chen [4] showed that level surfaces of solutions to the Cahn–Hilliard equation tend to solutions of the Hele-Shaw problem under the assumption that classical solutions of the latter exist. As previously mentioned, Bricmont, Kupiainen and Taskinen [12] provided a foundational result proving that the kink in one dimension is asymptotically stable: if the perturbation is continuous and for some $p > 2$, $\|\langle x\rangle^p(\phi - H)\|_{L^\infty}$ is small enough, then for some $x_0 \in \mathbb{R}$,

$$\lim_{t\to+\infty} \|\phi(t) - H(\cdot - x_0)\|_{L^\infty} = 0.$$

A more precise asymptotics of the remainder term is also given. The method of proof involves but it is not limited to the renormalization group technique. It is interesting to mention that in the energy space, the convergence of the shift parameter in Theorem 1.2 towards a final state is probably not possible unless one adds additional information on the initial data of the problem. The reader may also consult a simplified proof [15] of the kink asymptotic stability by using free energy techniques. Later, Korvola, Kupiainen and Taskinen [40] (see also [39]) extended the one-dimensional result to dimension $d \geq 3$ and any $p > d + 1$. There is an interesting anomalous decay for the problem in higher dimensions, and the shift must take into account the transversal perturbations; however, it converges to zero as time tends to infinity, unlike in the one-dimensional case.

In the case of a generalized version of the one-dimensional Cahn–Hilliard equation, Howard [32] established that linear stability of fronts implies nonlinear stability. Nonlinear orbital stability is established for waves with initial perturbations of algebraic decay, under the spectral stability assumption, described in terms of the Evans function. Later, Howard [31] established that the planar wave solutions are asymptotically stable, in the d-dimensional Cahn–Hilliard equation, with $d \geq 2$. In this result, it is required that the initial perturbations decay at an appropriate algebraic rate in an L^1 norm of the transverse variables; and in [33], the same author considers the multidimensional Cahn–Hilliard system, showing for the planar transition front solutions that spectral stability implies nonlinear stability.

Finally, Theorem 1.2 can be recast as an extension into the fourth-order ϕ^4 model of the stable [30] kink asymptotic stability proved in [42] using essentially virial identities (see [41] for a simplified proof and [21] for an extension of the previous result). As previously explained, the data considered in [42] is odd and the general case is still open. Earlier results in this direction were obtained by Cuccagna [20], who considered the ϕ^4 kink in three dimensions, and using vector field methods showed the asymptotic stability of the ϕ^4 kink. Under higher-order weighted norms, Kopylova and Komech [37, 38] showed asymptotic stability of kinks of highly degenerate scalar field theories. Delort and Masmoudi [22] applied Fourier analysis techniques and proved a detailed asymptotics for odd perturbations of the kink up to times $O(\varepsilon^{-4})$, where ε is the size of the perturbation. In [45], a sufficient condition is given to describe the long-time dynamics of kink perturbations for any data in the energy space, including many models of interest in quantum field theory [62] (except sine-Gordon and ϕ^4). In this case, kinks must be modulated in terms of scaling (the Lorentz boost) and shifts, which makes computations harder than usual. Cuccagna and Maeda provided a new sufficient condition to have asymptotic stability [21] in the odd data case. Snelson [81] considered the case of the ϕ^4 kink in the presence of variable coefficients. The case of the integrable sine-Gordon kink has attracted attention during past years due to its complexity and lack of kink asymptotic stability in the energy space; see [3, 16, 63, 73] and references therein. The case of collision of kink structures was treated in [35].

Another point of view, equivalent to the treatment of kinks under symmetry assumptions (essentially no shifts or Lorentz boosts), is given by the study of one-dimensional nonlinear Klein-Gordon models under variable coefficients. Foundational works in three dimensions were obtained by Soffer and Weinstein [82–84]. In this direction we mention previous scattering results by Lindblad and Soffer [57–59], Hayashi and Naumkin [27–29], Sternbenz [85], Bambusi and Cuccagna [5], Lindblad and Tao [60] and Lindblad et. al. [54–56]. These results have recently been improved by considering quadratic nonlinearities; see the scattering results by Germain and Pusateri [25]; see also [26]. The dynamics of solitons in nonlinear Klein–Gordon models has concentrated much effort during past years. We mention three-dimensional and one-dimensional works on the description of the manifold by Krieger, Nakanishi and Schlag [46] and Nakanishi and Schlag [75], and earlier results by Ibrahim, Masmoudi and Nakanishi [34]. Recently, one- and three-dimensional subcritical dynamics around the soliton have been addressed in great detail in [6, 17, 18, 36, 43, 44, 49–51, 64] in the presence of at least one unstable mode.

1.4. Idea of proofs

We will use localized virial estimates to show the asymptotic stability of the fourth-order ϕ^4 kink. Virials have been previously used in many complex dynamics; see e.g. [1, 2, 42, 44, 66, 67]. In this paper, we follow ideas from [68], which are based on previous ones from Kowalczyk, Martel and the second author in [44] and Kowalczyk, Martel, the second author and Van Den Bosch [45] to study the stability properties of kinks for (1+1)-dimensional nonlinear scalar field theories.

The first step is to decompose the solution close to the kink as follows: we choose $\rho(t)$ such that

$$\begin{cases} \phi_1(t,x) = H(y) + u_1(t,x), \quad y = x - \rho(t), \\ \phi_2(t,x) = u_2(t,x), \end{cases}$$

and $\langle u_1, H'(y)\rangle = 0$ and $\|(u_1(t), u_2(t))\|_{H^1\times L^2} \leq \delta$. Notice that we have chosen not to follow the standard centering performed in [45]. There are several reasons to follow a different approach. The most relevant is that multi-kink structures do not center well, specially in the case of several kinks. Another reason is that centering and multiple derivatives as in (1.1) do not cooperate well with each other.

Then we will focus on $(u_1, u_2) \in H^1 \times L^2$, which satisfy the linearized equation (1.12). Following [68, 74], for an adequate weight function φ_A placed at scale A large, we obtain the virial estimate (see (4.19))

$$\begin{aligned}\frac{d}{dt}\int \varphi_A(y)u_1u_2 \leq -\frac{1}{2}\int [w_2^2 + 2(\partial_x w_1)^2 + (2 - C_1\delta)w_1^2] \\ + C_1\int \operatorname{sech}(y)u_1^2 + C_1\rho'^2,\end{aligned} \tag{1.18}$$

where (w_1, w_2) is a localized version of (u_1, u_2) at A scale, and C_1 denotes a fixed constant. This virial estimate is standard now but it is not enough to conclude because of the terms $C_1 \int \operatorname{sech}(y)u_1^2$ and ρ'^2 (which is only of quadratic nature). Then we require to transform the system to a new one which has better virial estimates, in the spirit of Martel [65]. For any $\gamma > 0$ small enough, we define new variables $(v_1, v_2) \in H^1 \times H^2$ by

$$\begin{cases} v_1 = (1 - \gamma\partial_x^2)^{-1}\mathcal{L}u_1, \\ v_2 = (1 - \gamma\partial_x^2)^{-1}u_2 \end{cases}$$

(see (5.1)). In [68], this change of variable was enough to describe the stable manifold related to the unstable static soliton. Here we have additional complications since shifts are nontrivial perturbations and $(v_1, v_2) \in H^1 \times H^2$ follow modified dynamics. In particular, $|\rho'|$ is only linear in (u_1, u_2), unlike in scalar field models. However, a surprising miracle happens and the new system for (v_1, v_2) (see (5.2)) satisfies, for an adequate weight function $\psi_{A,B}$, $B \ll A$, the new virial estimate (see (5.7))

$$\frac{d}{dt}\int \psi_{A,B}v_1v_2 \leq -\frac{1}{2}\int [z_1^2 + (V_0 - C_2\delta^{1/10})z_2^2 + 2(\partial_x z_2)^2] + \text{l.o.t.}, \tag{1.19}$$

where (z_1, z_2) is a localized version of (v_1, v_2), at the smaller scale B, V_0 is given by (1.13) and C_2 denotes a fixed constant. An important point here is that the operator $-2\partial_y^2 + V_0$ is now positive, and ρ' has a critical almost zero contribution in (1.19), which is not the case in (1.18). The cancelation of the contribution by ρ' is part of a new idea that reveals that shift and scaling modulations should be treated as if they were additional internal modes (recall that they are not), just as is done in scalar field models. This is the main new outcome of this paper, which allows us to treat shifts in a simple fashion in the case of fourth-order ϕ^4 kink perturbations.

Following [44], in order to combine estimates (1.18) and (1.19) we need an estimate for the term before last in (1.18). A coercive estimate is proved in terms of the variables (w_1, w_2) and (z_1, z_2) (see (7.2)):

$$\int \operatorname{sech}(y)u_1^2 \lesssim \delta^{1/20}(\|w_1\|_{L^2}^2 + \|\partial_x w_1\|_{L^2}^2) + \delta^{-1/20}\|z_1\|_{L^2}^2 + \delta^{2/5}\|\partial_x z_1\|_{L^2}^2. \tag{1.20}$$

We can directly observe that the term $\partial_x z_1$ does not appear in (1.19), leading to the main obstruction present in this paper. This problem is deeply related to the fact that $(v_1, v_2) \in H^1 \times H^2$, i.e. the new variables are in opposite order of regularity.

In order to overcome this problem, we introduce a series of modifications that will allow us to close estimates (1.18) and (1.19) properly. First, we must gain derivatives. In a new virial estimate for the system of $(\partial_x v_1, \partial_x v_2)$ (see (6.2)), we obtain the third virial estimate

$$\begin{aligned} &\frac{d}{dt}\int \psi_{A,B}\partial_x v_1 \partial_x v_2 \\ &\quad \le -\frac{1}{2}\int ((\partial_x z_1)^2 + (V_0 - C_3\delta^{1/10})(\partial_x z_2)^2 + 2(\partial_x^2 z_2)^2) + \text{l.o.t.}, \end{aligned} \tag{1.21}$$

with $C_3 > 0$ fixed. This new estimate give us local L^2 control on $\partial_x z_1$ and $\partial_x^2 z_2$, which was not present before. Together with a similar estimate as (1.20), and in order to control ρ'^2 and combine estimates (1.18) and (1.19), we need an estimate for the last terms in (1.18). A coercive estimate is proved in terms of the variables (w_1, w_2) and (z_1, z_2) (see (7.11)):

$$\rho'^2 \lesssim \delta^{1/20}\|w_2\|_{L^2}^2 + \delta^{-1/20}\|z_2\|_{L^2}^2 + \delta^{2/5}\|\partial_x^2 z_2\|_{L^2}^2 + \delta^{3/5}\|\partial_x z_2\|_{L^2}^2. \tag{1.22}$$

Finally, we consider a functional $\mathcal{H}$ being a well-chosen linear combination of (1.18), (1.19), (1.21), (1.20) and (1.22). We get

$$\frac{d}{dt}\mathcal{H}(t) \le -\delta^{1/10}(\|w_1\|_{L^2}^2 + \|\partial_x w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2) \quad \text{for all } t \ge 0.$$

This final estimate allows us to close estimates, and prove local decay for u_1 after some standard change of variables from w_j to u_j.

Organization of this paper

This paper is organized as follows. In Section 2 we provide several basic but not less important elements for the study of the fourth-order ϕ^4 model. Section 3 is devoted to the proof of Theorem 1.1. Finally, Sections 4, 5, 6, 7 and 8 deal with the proof of the asymptotic stability of the kink, Theorem 1.2.

2. Preliminaries

2.1. Linear operators

Recall $\mathcal{L}$ introduced in (1.13). The following results are standard; see [62].

Lemma 2.1 (Properties of $\mathcal{L}$). *The linear unbounded operator $\mathcal{L}$, defined in L^2 with domain H^2, satisfies the following properties:*

(1) *The absolutely continuous spectrum of* $\mathcal{L}$ *is* $[2,\infty)$.

(2) $\mathcal{L}$ *is self-adjoint and nonnegative.*

(3) *The discrete spectrum consists of simple eigenvalues* $\lambda_0 = 0$ *and* $\lambda_1 = \frac{\sqrt{3}}{2}$, *with eigenfunctions*

$$Y_0 = H' \quad \textit{and} \quad Y_1(x) = \operatorname{sech}\Big(\frac{x}{\sqrt{2}}\Big)\tanh\Big(\frac{x}{\sqrt{2}}\Big),$$

respectively.

In a similar way, we have the following properties for the more involved operator $-\partial_x^2\mathcal{L}$:

Lemma 2.2 (Properties of $-\partial_x^2\mathcal{L}$). *The linear operator* $-\partial_x^2\mathcal{L}$ *defined in* (1.13), *posed in* L^2 *with domain* H^4, *satisfies the following properties:*

(1) *The absolutely continuous spectrum of* $-\partial_x^2\mathcal{L}$ *is* $[0,\infty)$.

(2) $-\partial_x^2\mathcal{L}$ *is nonnegative.*

(3) $\ker(-\partial_x^2\mathcal{L}) = \operatorname{span}\{H'\}$.

The proofs of the first two statements are direct. For the proof of the last result, see Appendix A.

Remark 2.1. Unlike good Boussinesq, the operator $\mathcal{L}$ appearing in the case of kinks has even kernel H', implying that the equation $\mathcal{L}A = 1$ cannot have bounded solutions. Resonances are therefore excluded in this case. See [68] for the case where resonances but no shifts are allowed.

2.2. Coercivity

The linearization of (1.5) around $\boldsymbol{H}$ involves the operator (1.13) and $-\partial_x^2\mathcal{L}$. Here, we recall a few properties of the operator $\mathcal{L}$.

Let the bilinear form

$$H(u,v) = \langle \mathcal{L}(u), v\rangle = \int (\partial_x u \partial_x v + V_0 uv),$$

where V_0 is given in (1.13).

Lemma 2.3 (Coercivity [42]). *If* $u \in H^1(\mathbb{R})$ *satisfies* $\langle u, Y_0\rangle = 0$, *where* Y_0 *is the even eigenfunction associated to eigenvalue* $\lambda_0 = 0$ *of operator* $\mathcal{L}$, *then*

$$H(u,u) \geq \frac{3}{7}\|u\|_{H^1}^2. \tag{2.1}$$

We will need a weighted version of the previous result, which uses (2.1). See [68] for a similar proof.

Lemma 2.4 (Coercivity with weight function). *Consider the bilinear form*

$$H_{\phi_L}(u,v) = \langle \sqrt{\phi_L}\mathcal{L}(u), \sqrt{\phi_L}v\rangle = \int \phi_L(\partial_x u \partial_x v + V_0 uv),$$

for ϕ_L such that $|\phi_L'| \leq CL\phi_L$, with C not depending on L. Then there exists $\lambda > 0$ independent of L small such that

$$H_{\phi_L}(u,u) \geq \lambda \int \phi_L((\partial_x u)^2 + u^2)$$

for all $u \in H^1(\mathbb{R})$ satisfying $\langle u, Y_0\rangle = 0$, and provided L is taken small enough, independent of the size of u.

2.3. Technical identities related to the operator $\mathcal{L}$

The next lemma will be useful for the first computations related to virial identities, which allow us to prove the asymptotic stability of the static kink.

Lemma 2.5. *Let η be a smooth bounded function, and $(f,g) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$. The following identities hold:*

$$\langle \eta\mathcal{L}(f), g\rangle = \int \eta[\partial_x f \partial_x g + V_0 fg] + \langle \eta' \partial_x f, g\rangle, \tag{2.2}$$

$$\langle \eta\partial_x\mathcal{L}(f), f\rangle = -\frac{1}{2}\int \eta'[3(\partial_x f)^2 + V_0 f^2] + \frac{1}{2}\int \eta V_0' f^2 + \frac{1}{2}\int \eta''' f^2 \tag{2.3}$$

and

$$\langle \eta\mathcal{L}(\partial_x f), f\rangle = -\frac{1}{2}\int \eta'[3(\partial_x f)^2 + V_0 f^2] - \frac{1}{2}\int \eta V_0' f^2 + \frac{1}{2}\int \eta''' f^2. \tag{2.4}$$

For the proof see Appendix B.

2.4. Ill-posedness

Before considering perturbations of the kink solution, we will make some remarks about the case of small data around the zero solution. It is expected, as in the case of ϕ^4 or Allen–Cahn, that this state is unstable. However, some interesting remarks about local existence can be made in the fourth-order ϕ^4 case.

System (1.1) can be written as

$$\begin{cases} \partial_t\phi + \partial_x^2\varphi = 0, \\ \partial_t\varphi - \partial_x^2\phi - \phi = -\phi^3. \end{cases}$$

Let $\psi = \phi + i\varphi$; then

$$i\partial_t\psi + \partial_x^2\psi + \operatorname{Re}\psi = (\operatorname{Re}\psi)^3.$$

From this we can see that w has interesting similarities with the equations appearing in the study of the Peregrine breather [72]. Indeed, the linear fourth-order ϕ^4 equation has the same instability issues as NLS around a nonzero background. From a standard frequency analysis, we get for a formal standing wave $\phi = e^{i(kx-wt)}$ solution to the linear (1.1), one has

$$w(k) = \pm|k|\sqrt{k^2-1},$$

which reveals that for small wave numbers ($|k| < 1$) the linear equation behaves in an "elliptic" setting, and exponentially in time growing modes are present from small perturbation of the vacuum solution. Unlike [72], where one still has well-posedness in H^s, $s > \frac{1}{2}$ in the nonlinear case, that approach fails here because of the two derivatives in the nonlinearity.

2.5. Local well-posedness in a neighborhood of the kink

Contrary to Section 2.4, perturbations of heteroclinic kinks do not suffer from the lack of well-posedness. To prove well-posedness around the kink solution we will adapt the well-posedness proof by Linares [52] (see also [53]). Most of the concern is given by how the linear flow can recover the two derivatives present in the nonlinearity. Let $\delta > 0$ small enough be chosen later. We consider an initial data $\boldsymbol{u}^{\text{in}} \in \boldsymbol{E_H}$ (see (1.10)) such that (1.15) is satisfied.

In this section, we are looking for a solution $\boldsymbol{\phi}(t)$ of (1.5) in $\boldsymbol{E_H}$ for all time with initial data $\boldsymbol{\phi}^{\text{in}}$. Now we will focus on the local well-posedness of the perturbed system around the static kink for equation (1.5). Here, it is remarkable that around the static kink the nature of the solution changes drastically: here, the exponential growth of the section above is not present.

We consider an initial data which is perturbation around the static kink, i.e. an initial data of system (1.5) which has the form

$$\boldsymbol{\phi}^{\mathbf{in}} = (H(x) + u_1^0(x), u_2^0(x)), \tag{2.5}$$

and setting $\boldsymbol{u}(t,x) = \boldsymbol{\phi}(t) - \boldsymbol{H}$, we reduce our problem to solving the system

$$\begin{aligned} \partial_t u_1 &= \partial_x u_2, \\ \partial_t u_2 &= \partial_x(-\partial_x^2 u_1 + 2u_1 + F(t,x,u_1)), \end{aligned} \tag{2.6}$$

in $H^1 \times L^2$, where

$$F(t,x,u_1) = 3u_1(H^2-1) + u_1^2(u_1 + 3H). \tag{2.7}$$

Notice that F is locally Lipschitz in the third variable, i.e. there exists $C > 0$, such that for any v, w, if $\|u\|_{L^\infty} \le 1$, $\|w\|_{L^\infty} \le 1$, then

$$\begin{aligned} |F(t,x,u) - F(t,x,w)| &= |u-w|\,|3(H^2-1) + u^2 + (u+w)(w+3H)| \\ &\le C|u-w|. \end{aligned} \tag{2.8}$$

The fourth-order ϕ^4 linearization around the static kink (see (2.6)) hides a slightly modified good-Boussinesq equation structure. That means that the linear part of (2.6) is essentially the same as the linear good-Boussinesq equation. We can see that by Duhamel's principle, the associated solution is represented as

$$u_1(t,x) = \mathcal{G}(t)u_1^0(x) + \mathcal{K}(t)u_2^0(x) + \int_0^t \mathcal{K}(t-s)\partial_x^2 F(s,x,u_1)\,ds,$$

where

$$\mathcal{G}(t) = \mathcal{F}^{-1}G(t,\xi)\mathcal{F}, \quad \mathcal{K}(t) = \mathcal{F}^{-1}K(t,\xi)\mathcal{F},$$

and $\mathcal{F}$ and $\mathcal{F}^{-1}$ represent the Fourier transform and its inverse, respectively. The Fourier multipliers are given by

$$G(t,\xi) = \cos(\omega(\xi)t), \quad K(t,\xi) = \frac{\sin(\omega(\xi)t)}{\omega(\xi)},$$

where $\omega(\xi) = |\xi|\sqrt{\xi^2+2}$. Then, following Linares' work [52,53] and noticing that F in (2.7) is locally Lipschitz and satisfies

$$|F| \lesssim |u_1|e^{-\sqrt{2}|x-\rho|} + u_1^2 + |u_1|^3, \tag{2.9}$$

we conclude that the equation (2.6) is locally well posed in $H^1 \times L^2(\mathbb{R})$. Using the conserved quantities (1.7), we obtain the global well-posedness of (2.6). To see the complete proof, the reader may consult Appendix C.

In this paper, we will only need the above notion of solution $\boldsymbol{\phi} = (\phi_1, \phi_2)$ of (1.5). Now we are ready to face the orbital stability of the static kink.

3. Orbital stability

First, we will establish the energy and momentum of the static kink, as well as their perturbations.

Lemma 3.1. *Let E and P be defined as in (1.7). The following are satisfied:*

(1) *Conservation laws for the static kink:*

$$E[\boldsymbol{H}] = \|H'\|_{L^2}^2, \quad P[\boldsymbol{H}] = 0. \tag{3.1}$$

(2) *Additionally, for any $\boldsymbol{u} = (u_1, u_2)$ such that $\|u_1\|_{L^\infty} \lesssim 1$,*

$$E[\boldsymbol{H} + \boldsymbol{u}] = E[\boldsymbol{H}] + \frac{1}{2}[\|u_2\|_{L^2}^2 + \langle \mathcal{L}u_1, u_1\rangle] + R_{u_1}, \tag{3.2}$$

where $|R_{u_1}| \le C\|u_1\|_{L^\infty}\|u_1\|_{L^2}^2$.

Proof of Lemma 3.1. *Proof of* (3.1). Recalling that $\boldsymbol{H} = (H, 0)$, one immediately obtains that $P[\boldsymbol{H}] = 0$. Furthermore, from (1.6) for $c = 0$, we have $(H')^2 = (1 - H^2)^2/2$, concluding that $E[\boldsymbol{H}] = \|H'\|_{L^2}^2$.

Proof of (3.2). Expanding the energy, we get

$$
\begin{aligned}
E[\boldsymbol{H} + \boldsymbol{u}] &= \frac{1}{2}\int ((\partial_x u_1)^2 + u_2^2 + (\partial_x H)^2 + 2H'\partial_x u_1) \\
&\quad + \frac{1}{4}\int [u_1^2(u_1 + 2H)^2 + 2u_1(u_1 + 2H)(H^2 - 1) + (H^2 - 1)^2] \\
&= \frac{1}{2}\int (\partial_x u_1)^2 + u_2^2 + (\partial_x H)^2 - \int H'' u_1 \\
&\quad + \frac{1}{4}\int [u_1^2(u_1 + 2H)^2 + 2u_1(u_1 + 2H)(H^2 - 1) + (H^2 - 1)^2].
\end{aligned}
$$

Using (1.6), we obtain

$$
\begin{aligned}
E[\boldsymbol{H} + \boldsymbol{u}] &= \frac{1}{2}\int (\partial_x u_1)^2 + u_2^2 + (-1 + 3H^2)u_1^2 + (\partial_x H)^2 + \frac{1}{2}(H^2 - 1)^2 \\
&\quad + \frac{1}{4}\int [u_1^2(u_1 + 2H)^2 + 2u_1(u_1 + 2H)(H^2 - 1)] \\
&\qquad\qquad + 4u_1 H(1 - H^2) + 2(1 - 3H^2)u_1^2 \\
&= E[\boldsymbol{H}] + \frac{1}{2}(\langle \mathcal{L}u_1, u_1\rangle + \|u_2\|_{L^2}^2) + \frac{1}{4}\int u_1^3[u_1 + 4H].
\end{aligned}
$$

Letting $R_{u_1} = \frac{1}{4}\int u_1^3[u_1 + 4H]$, we quickly obtain that $|R_{u_1}| \le C\|u_1\|_{L^\infty}\|u_1\|_{L^2}^2$. This concludes the proof of lemma. ■

Proof of Theorem 1.1. Let $\boldsymbol{\phi}$ be the local-in-time solution of (1.5) with initial data $\boldsymbol{\phi}^{\mathbf{in}}$ given in (2.5), satisfying (1.15). For $C > 1$ to be chosen later, define

$$
\begin{aligned}
T^* = \sup\Big\{t \ge 0 \;\Big|\; &\boldsymbol{\phi} \text{ is well defined on } [0, t] \text{ and} \\
&\sup_{s\in[0,t]} \inf_{\rho\in\mathbb{R}} \|\boldsymbol{\phi}(s) - \boldsymbol{H}(\cdot - \rho)\|_{H^1\times L^2} \le C^*\delta.\Big\}
\end{aligned}
$$

By continuity and (1.15), we get T^* is well defined and $T^* > 0$. Then, if $T^* < \infty$, using a continuity argument and the smallness of the initial data (see (1.15)), $\boldsymbol{\phi}$ is well defined on $[0, T^1]$ for some $T^1 > T^*$ and it will hold that

$$
\inf_{\rho\in\mathbb{R}} \|\boldsymbol{\phi}(T^*) - \boldsymbol{H}(\cdot - \rho)\|_{H^1\times L^2} = C^*\delta. \tag{3.3}
$$

Assuming that T^* is finite, we work on the interval $[0, T^*]$. Considering that $\boldsymbol{\phi}$ has the form $\boldsymbol{H} + \boldsymbol{u}$, we get from (3.2),

$$
E[\boldsymbol{H} + \boldsymbol{u}](t) - E[\boldsymbol{H}] = \frac{1}{2}[\|u_2\|_{L^2}^2 + \langle \mathcal{L}u_1, u_1\rangle] + R_{u_1},
$$

so that using the conservation of energy (Lemma 3.1) and (3.3),

$$\begin{aligned}\frac{1}{2}[\|u_2\|_{L^2}^2 + \langle \mathcal{L}u_1, u_1\rangle] &= \frac{1}{2}[\|u_2(0)\|_{L^2}^2 + \langle \mathcal{L}u_1(0), u_1(0)\rangle] + R_{u_1(0)} - R_{u_1}\\ &\leq C\delta^2 + C(C^*)^3\delta^3.\end{aligned}$$

The coercivity of the operator $\mathcal{L}$ (Lemma 2.3 and (1.11)) gives that for some fixed $\mu, C > 0$,

$$\mu\|\boldsymbol{u}\|_{H^1\times L^2}^2 \leq C\delta^2 + C(C^*)^3\delta^3.$$

Thus, for all $t \in [0, T^*]$, if δ is small enough,

$$\|\boldsymbol{u}\|_{H^1\times L^2}^2 \leq \frac{2C}{\mu}\delta^2; \tag{3.4}$$

here C and μ are independent of C^*. Fixing $(C^*)^2 > C/\mu$ we get a contradiction with (3.3). Therefore, $\boldsymbol{\phi}$ is a global solution for $t \geq 0$, T^* and (3.4) hold for any $t \geq 0$. This concludes the proof of Theorem 1.1. ■

4. Asymptotic stability: First estimates

Consider a small perturbation of the kink solution $\boldsymbol{H} = (H, 0)$. In what follows we will describe this decomposition, introduce some notation, and develop a virial estimate for the fourth-order ϕ^4 system (1.5).

4.1. Decomposition of the solution in the vicinity of a kink

Let $\boldsymbol{\phi} = (\phi, \partial_t\partial_x^{-1}\phi) = (\phi_1, \phi_2)$ be a solution of (1.5) satisfying

$$\|\boldsymbol{\phi}(t, \cdot + \rho(t)) - \boldsymbol{H}\|_{H^1\times L^2} = \|\boldsymbol{u}(t)\|_{H^1\times L^2} \leq C_0\delta \tag{4.1}$$

for all $t \geq 0$, where δ is defined from the initial data $\boldsymbol{\phi}^{\text{in}}$ (2.5), to be taken small enough. Recall the decomposition for $\boldsymbol{\phi}$:

$$\boldsymbol{\phi}(t,x) = \boldsymbol{H}(x - \rho(t)) + \boldsymbol{u}(t,x), \quad \begin{aligned}&\text{where } \boldsymbol{u} = (u_1(t,x), u_2(t,x))\\ &\text{satisfies } \langle \boldsymbol{u}(t), \boldsymbol{H}'(\cdot - \rho(t))\rangle =\\ &\langle u_1(t), H'(\cdot - \rho(t))\rangle = 0.\end{aligned} \tag{4.2}$$

Note that (4.1) and Theorem 1.1 now read

$$\|u_1(t)\|_{H^1} + \|u_2(t)\|_{L^2} \leq C_0\delta. \tag{4.3}$$

Additionally, from (1.12) $\boldsymbol{u}$ satisfies the equation

$$\partial_t\boldsymbol{u} = \partial_x \boldsymbol{J}\boldsymbol{L}\boldsymbol{u} + \partial_x\boldsymbol{F}(x, \boldsymbol{u}) + \rho'\boldsymbol{H}', \tag{4.4}$$

where

$$\boldsymbol{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{L} = \begin{pmatrix} \mathcal{L} & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{F}(x, \boldsymbol{u}) = \begin{pmatrix} 0 \\ u_1^3 + 3H(\cdot - \rho)u_1^2 \end{pmatrix},$$

with $\mathcal{L}$ given by (1.13). Equation (4.4) is equivalent to the system

$$\begin{cases} \dot{u}_1 = \partial_x u_2 + \rho' H', \\ \dot{u}_2 = \partial_x \mathcal{L} u_1 + \partial_x (u_1^3 + 3H u_1^2). \end{cases} \tag{4.5}$$

Notice that from (4.2) one has

$$\langle u_1, H' \rangle = 0 \implies \langle \dot{u}_1, H' \rangle = \rho' \langle u_1, H'' \rangle,$$

and from (4.5),

$$\begin{aligned} &\langle \dot{u}_1, H' \rangle = -\langle u_2, H'' \rangle + \rho' \|H'\|^2 \\ &\implies |\rho'| \lesssim |\langle u_2, H'' \rangle| \lesssim \Big(\int e^{-\sqrt{2}|x-\rho|} u_2^2 \Big)^{1/2}. \end{aligned} \tag{4.6}$$

4.2. Weighted functions

In this section we recall all the necessary auxiliary results that will be needed in forthcoming sections. The notation is taken essentially from [45], with some particular choices from [68]. We start by describing the weighted functions used to define our local norms.

We consider a smooth even function $\chi\colon \mathbb{R} \to \mathbb{R}$ satisfying

$$\chi = 1 \text{ on } [-1, 1], \quad \chi = 0 \text{ on } (-\infty, 2] \cup [2, \infty), \quad \chi' \leq 0 \text{ on } [0, \infty). \tag{4.7}$$

For any scale $K > 0$, we define the functions ζ_K and φ_K as

$$\begin{aligned} \zeta_K(x) &= \exp\Big(-\frac{1}{K}(1 - \chi(x))|x| \Big), \\ \varphi_K(x) &= \int_0^x \zeta_K^2(y)\, dy, \ x \in \mathbb{R}. \end{aligned} \tag{4.8}$$

We consider the function $\psi_{A,B}$ defined as

$$\psi_{A,B}(x) = \chi_A^2(x)\varphi_B(x), \quad \text{where } \chi_A(x) = \chi\Big(\frac{x}{A}\Big), x \in \mathbb{R}. \tag{4.9}$$

We recall that $\psi_{A,B}$ and φ_K are *odd functions*.

These functions will be used several times in arguments of virial type, and with different scales, A and B, under the following constraint:

$$1 \ll B \ll B^8 \ll A. \tag{4.10}$$

Basic set of technical estimates related to weighted functions. The following technical estimates on functions ζ_K and χ_A will be useful throughout this work. These estimates have been used in a similar context (for the proofs, see [42, 68]).

Lemma 4.1. *Let ζ_K and χ be defined by (4.8) and (4.7), respectively. Then one has*

$$\frac{\zeta_K'}{\zeta_K} = -\frac{1}{K}[-\chi'(x)|x| + (1-\chi(x))\operatorname{sgn}(x)],$$
$$\frac{\zeta_K''}{\zeta_K} = \left(\frac{\zeta_K'}{\zeta_K}\right)^2 + \frac{1}{K}[\chi''(x)|x| + 2\chi'(x)\operatorname{sgn}(x)],$$

and

$$\frac{\zeta_K'''}{\zeta_K} = 3\frac{\zeta_K''}{\zeta_K}\frac{\zeta_K'}{\zeta_K} - 2\left(\frac{\zeta_K'}{\zeta_K}\right)^3 + \frac{1}{K}[\chi'''(x)|x| + 3\chi''(x)\operatorname{sgn}(x)],$$
$$\frac{\zeta_K^{(4)}}{\zeta_K} = 4\frac{\zeta_K'''}{\zeta_K}\frac{\zeta_K'}{\zeta_K} + 3\left(\frac{\zeta_K''}{\zeta_K}\right)^2 - 12\frac{\zeta_K''}{\zeta_K}\left(\frac{\zeta_K'}{\zeta_K}\right)^2 + 6\left(\frac{\zeta_K'}{\zeta_K}\right)^4 + \frac{1}{K}[\chi^{(4)}(x)|x| + 4\chi'''(x)\operatorname{sgn}(x)].$$

The previous identities are key through the paper. From the previous lemma we observe that for any $K > 0$ sufficiently large,

$$\left|\frac{\zeta_K''}{\zeta_K} - 2\left(\frac{\zeta_K'}{\zeta_K}\right)^2\right| \lesssim \frac{1}{K}. \tag{4.11}$$

Furthermore,

$$\left|\frac{\zeta_K'}{\zeta_K}\right| \lesssim K^{-1}\mathbf{1}_{\{|x|>1\}}(x), \qquad \left|\frac{\zeta_K''}{\zeta_K}\right| \lesssim K^{-2} + K^{-1}\operatorname{sech}(x), \tag{4.12}$$

and

$$\left|\frac{\zeta_K'''}{\zeta_K}\right| \lesssim K^{-3} + K^{-1}\operatorname{sech}(x), \qquad \left|\frac{\zeta_K^{(4)}}{\zeta_K}\right| \lesssim K^{-4} + K^{-1}\operatorname{sech}(x).$$

Then,

$$\left|\frac{\zeta_K''}{\zeta_K}\right| + \left|\frac{\zeta_K'''}{\zeta_K}\right| + \left|\frac{\zeta_K^{(4)}}{\zeta_K}\right| \lesssim K^{-1}. \tag{4.13}$$

In particular, for A large enough, the following estimate holds:

$$\left|\mathbf{1}_{\{A<|x|<2A\}}\frac{\zeta_K^{(n)}}{\zeta_K}\right| \lesssim \frac{1}{K^n} \quad \text{for } n \in \mathbb{N}. \tag{4.14}$$

Now we will focus on χ_A. Considering that χ_A is a cut-off function at A scale, we notice the following relation between χ_A and ζ_A: for each function v,

$$\int \chi_A^2 v^2 \le \int_{|x|\le 2A} v^2 \le C\int_{|x|\le 2A} e^{-4|x|/A}v^2 \lesssim \int v^2\zeta_A^4. \tag{4.15}$$

This estimate will be essential for the well-boundedness of some nonlinear terms in what follows (see Sections 5.6.1 and 5.6.2). Notice that this estimate will be used with the translated variable $y = x - \rho(t)$.

4.3. A first virial estimate

Following the ideas in [68], we set

$$\mathcal{I}(t) = \int_{\mathbb{R}} \varphi_A(y) u_1(t,x) u_2(t,x)\, dx, \quad y = x - \rho(t), \tag{4.16}$$

and

$$w_i(t,x) = \zeta_A(y) u_i(t,x), \quad i = 1, 2. \tag{4.17}$$

Here, $\boldsymbol{w} = (w_1, w_2)$ represents a localized version of $\boldsymbol{u} = (u_1, u_2)$ at scale A. The following virial argument has been used in [42,45,68] in a similar context.

Proposition 4.2. *There exist $C_1 > 0$ and $\delta_1 > 0$ such that for any $0 < \delta \le \delta_1$, the following holds. Fix*

$$A = \delta^{-1}. \tag{4.18}$$

Assume that for all $t \ge 0$, (4.1) holds. Then

$$\begin{aligned}\frac{d}{dt}\mathcal{I} \le -\frac{1}{2}\int (w_2^2 + 3(\partial_x w_1)^2 + (V_0 - 4C_0\delta) w_1^2) + \int \varphi_A (3H + u_1) H' u_1^2 \\ + C_0 \delta |\rho'|^2 + \rho' \int \varphi_A H' u_2.\end{aligned} \tag{4.19}$$

The proof of this result requires several computations. We start with a first identity.

Lemma 4.3. *Let $(u_1, u_2) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ be a solution of (4.5). Consider $\varphi_A = \varphi_A(y)$ a smooth bounded function to be chosen later. Then*

$$\begin{aligned}\frac{d}{dt}\mathcal{I} &= -\frac{1}{2}\int \varphi_A' (u_2^2 + 3(\partial_x u_1)^2 + V_0 u_1^2) + \frac{1}{2}\int \varphi_A''' u_1^2 \\ &\quad - \frac{1}{2}\int \varphi_A' u_1^3 (\frac{3}{2} u_1 + 4H) + \int \varphi_A u_1^2 H' (3H + u_1) \\ &\quad - \rho' \int \varphi_A' u_1 u_2 + \rho' \int \varphi_A H' u_2.\end{aligned} \tag{4.20}$$

Proof. Taking the derivative in (4.16) and using (4.5),

$$\begin{aligned}\frac{d}{dt}\mathcal{I} &= \int \varphi_A (\dot{u}_1 u_2 + u_1 \dot{u}_2) - \rho' \int \varphi_A' u_1 u_2 \\ &= \int \varphi_A \big(\partial_x u_2 u_2 + \rho' H' u_2 + u_1 (\partial_x \mathcal{L}(u_1) + \partial_x (u_1^3 + 3H u_1^2))\big) \\ &= -\frac{1}{2}\int \varphi_A' u_2^2 + \int \varphi_A u_1 \partial_x \mathcal{L}(u_1) - \int \varphi_A u_1 \partial_x (u_1^3 + 3H u_1^2) \\ &\quad - \rho' \int \varphi_A' u_1 u_2 + \rho' \int \varphi_A H' u_2.\end{aligned} \tag{4.21}$$

For the second integral on the right-hand side of the above equation, by (2.3) in Lemma 2.5, we have

$$\int \varphi_A u_1 \partial_x \mathcal{L}(u_1) = -\frac{1}{2}\int \varphi_A'[3(\partial_x u_1)^2 + V_0 u_1^2] + \frac{1}{2}\int \varphi_A V_0' u_1^2 + \frac{1}{2}\int \varphi_A''' u_1^2.$$

For the third integral on the right-hand side of (4.21), separating terms and integrating by parts we obtain

$$\begin{aligned}\int \varphi_A u_1 \partial_x (u_1^3 + 3Hu_1^2) &= -\int \varphi_A \partial_x u_1 (u_1^3 + 3Hu_1^2) - \int \varphi_A'(u_1^4 + 3Hu_1^3)\\ &= \frac{1}{4}\int \varphi_A' u_1^4 + \int (\varphi_A' H + \varphi_A H')u_1^3 - \int \varphi_A'(u_1^4 + 3Hu_1^3)\\ &= -\frac{3}{4}\int \varphi_A' u_1^4 - 2\int \varphi_A' H u_1^3 + \int \varphi_A H' u_1^3.\end{aligned}$$

Finally, by collecting, noticing that $V_0' = 6HH'$ and regrouping terms, we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{I} = &-\frac{1}{2}\int \varphi_A'(u_2^2 + 3(\partial_x u_1)^2 + V_0 u_1^2) + \frac{1}{2}\int \varphi_A''' u_1^2\\ &-\frac{1}{2}\int \varphi_A' u_1^2\Big(\frac{3}{2}u_1^2 + 4Hu_1\Big) + \int \varphi_A u_1^2 H'(3H + u_1)\\ &-\rho'\int \varphi_A' u_1 u_2 + \rho'\int \varphi_A H' u_2.\end{aligned}$$

This concludes the proof of (4.20). ■

Proof of Proposition 4.2. Recalling that $\varphi_A' = \zeta_A^2$, we have in (4.20),

$$\int \varphi_A'(u_2^2 + V_0 u_1^2) = \int (w_2^2 + V_0 w_1^2). \tag{4.22}$$

Also,

$$\int \varphi_A'(\partial_x u_1)^2 = \int (\partial_x w_1)^2 + \int w_1^2 \frac{\zeta_A''}{\zeta_A}. \tag{4.23}$$

Additionally,

$$\int \varphi_A''' u_1^2 = \int \frac{(\zeta_A^2)''}{\zeta_A^2} w_1^2 = 2\int \Big(\frac{\zeta_A''}{\zeta_A} + \frac{(\zeta_A')^2}{\zeta_A^2}\Big) w_1^2. \tag{4.24}$$

Next, we deal with the nonlinear terms. Using that $w_1 = \zeta_A u_1$, we obtain

$$\int \varphi_A'\Big(\frac{3}{2}u_1^4 + 4Hu_1^3\Big) = \int w_1^2\Big(\frac{3}{2}u_1^2 + 4Hu_1\Big).$$

Finally, gathering the previous computation and (4.22), (4.23) and (4.24), we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{I} = &-\frac{1}{2}\int (w_2^2 + 3(\partial_x w_1)^2 + V w_1^2) + \int \varphi_A u_1^2 H'(3H + u_1)\\ &-\rho'\int \varphi_A' u_1 u_2 + \rho'\int \varphi_A H' u_2,\end{aligned}$$

where (with variable x, y and t)

$$V = V_0 + \frac{\zeta_A''}{\zeta_A} - 2\frac{(\zeta_A')^2}{\zeta_A^2} + u_1\Big(\frac{3}{2}u_1^2 + 4H\Big). \tag{4.25}$$

Now we consider the range of parameter $A \gg 1$. Then from (4.11) in (4.25) one has

$$V \geq V_0 - C(A^{-1} + \delta).$$

This last estimate follows from (4.11) and $\|u_1\|_{L^\infty} \leq C_0\delta$.

Finally, consider the penultimate term in (4.20). Using (4.3) and (4.18) one gets $\delta = A^{-1}$, and $\|u_2\|_{L^2}, \|u_1\|_{L^\infty} \leq C_0\delta$. Therefore,

$$\left|\rho' \int \varphi_A' u_1 u_2\right| \lesssim |\rho'| \int |w_1 \zeta_A u_2| \lesssim |\rho'| \|w_1\|_{L^2} \|\zeta_A u_2\|_{L^2} \leq C_0\delta|\rho'| \|w_1\|_{L^2}.$$

Replacing this bound in (4.20), and using the Cauchy inequality, we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{I} &= -\frac{1}{2}\int (w_2^2 + 3(\partial_x w_1)^2 + V w_1^2) + \frac{1}{2}\int \varphi_A u_1^2 (V_0' + 2H' u_1) \\ &\quad - \rho' \int \varphi_A' u_1 u_2 + \rho' \int \varphi_A H' u_2 \\ &\leq -\frac{1}{2}\int (w_2^2 + 3(\partial_x w_1)^2 + (V_0 - 4CA^{-1}) w_1^2) + \int \varphi_A (3H + u_1) H' u_1^2 \\ &\quad + C_0\delta|\rho'|^2 + \rho' \int \varphi_A H' u_2.\end{aligned}$$

This concludes the proof of (4.19). ∎

5. Duality and second virial estimates

Following Martel [65], we consider the function $v_1 = \mathcal{L}u_1$ instead of u_1 to obtain a transformed problem with better virial properties. Unlike generalized KdV, in our case we have a system of unknowns and some care is needed with the dual transformation. In the case of the good-Boussinesq equation this was done in [68] with notable success. Since our original variables (u_1, u_2) belong to $H^1(\mathbb{R}) \times L^2(\mathbb{R})$, by using $\mathcal{L}$, the new variables are not well defined. Therefore, we need a regularization procedure, as in many other works [42, 45, 68].

5.1. The transformed problem

Let $\gamma > 0$ small, to be determined later. Set

$$\boldsymbol{v} = (1 - \gamma\partial_x^2)^{-1}\boldsymbol{L}\boldsymbol{u},$$

or equivalently,

$$\begin{cases} v_1 = (1-\gamma\partial_x^2)^{-1}\mathcal{L}u_1, \\ v_2 = (1-\gamma\partial_x^2)^{-1}u_2. \end{cases} \tag{5.1}$$

From system (4.5) we have $\boldsymbol{v} = (v_1, v_2) \in H^1(\mathbb{R}) \times H^2(\mathbb{R})$. Furthermore, from system (4.4), it follows that

$$\partial_t \boldsymbol{v} = \boldsymbol{L}\boldsymbol{J}\partial_x \boldsymbol{v} + (1-\gamma\partial_x^2)^{-1}\widetilde{\boldsymbol{F}},$$

where

$$\widetilde{\boldsymbol{F}} = \begin{pmatrix} \gamma[V_0''\partial_x v_2 + 2V_0'\partial_x^2 v_2] - \rho' V_0' u_1 \\ \partial_x[u_1^3 + 3Hu_1^2] \end{pmatrix}.$$

The above system is equivalent to

$$\begin{cases} \dot{v}_1 = \mathcal{L}(\partial_x v_2) + G, \\ \dot{v}_2 = \partial_x v_1 + F, \end{cases} \tag{5.2}$$

where

$$\begin{aligned} F &= (1-\gamma\partial_x^2)^{-1}\partial_x[u_1^3 + 3Hu_1^2], \\ G &= \gamma(1-\gamma\partial_x^2)^{-1}[V_0''\partial_x v_2 + 2V_0'\partial_x^2 v_2] - \rho'(1-\gamma\partial_x^2)^{-1}(V_0' u_1). \end{aligned} \tag{5.3}$$

Now we compute a second virial estimate, this time on (v_1, v_2).

5.2. Virial functional for the transformed problem

Recall that $y = x - \rho(t)$. Now set

$$\mathcal{J}(t) = \int \psi_{A,B}(y) v_1(t,x) v_2(t,x)\, dx, \tag{5.4}$$

with

$$\begin{aligned} \psi_{A,B} &= \chi_A^2 \varphi_B, \\ z_i(t,x) &= \chi_A(y)\zeta_B(y) v_i(t,x), \quad i = 1, 2. \end{aligned} \tag{5.5}$$

Here, $\boldsymbol{z} = (z_1, z_2)$ represents a localized version of the variables $\boldsymbol{v} = (v_1, v_2)$ at the scale B. This scale is intermediate, and $\mathcal{J}$ involves a cut-off at scale A, which is needed to bound some bad error and nonlinear terms; see [45, 68] for a similar procedure.

Proposition 5.1. *Under* (4.1), (4.10) *and* (4.18), *the following is satisfied. There exist* $C_2 > 0$ *and* $\delta_2 > 0$ *such that for any* $0 < \delta \leq \delta_2$, *the following holds. Fix*

$$B = A^{1/10} = \delta^{-1/10}, \quad \gamma = B^{-4} = \delta^{2/5}. \tag{5.6}$$

Then for all $t \geq 0$,

$$\begin{aligned} \frac{d}{dt}\mathcal{J} \leq &-\frac{1}{2}\int (z_1^2 + (V_0 - C_2\delta^{1/10})z_2^2 + 2(\partial_x z_2)^2) + C_2\delta\|z_1\|_{L^2}^2 \\ &+ C_2\delta^{1/10}(\|w_1\|_{L^2}^2 + \|\partial_x w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2) + C_2\delta^{9/10}|\rho'|^2, \end{aligned} \tag{5.7}$$

where V_0 *is given by* (1.13).

The rest of this section is devoted to the proof of this proposition, which has been divided into several subsections.

5.3. Change of variable

The following identities were proved in [68]; they are useful in the next subsection and in the proof of Claim 6.3.

Claim 5.2. *Let $P \in W^{1,\infty}(\mathbb{R})$, $P = P(y)$, v_i be as in (5.1) and z_i be as in (5.5). Then*

$$\int P\chi_A^2\zeta_B^2(\partial_x v_i)^2 = \int P(\partial_x z_i)^2 + \int \left[P'\frac{\zeta_B'}{\zeta_B} + P\frac{\zeta_B''}{\zeta_B}\right]z_i^2 + \int \mathcal{E}_1\zeta_B^2 v_i^2, \tag{5.8}$$

where

$$\mathcal{E}_1(P) = P\left[\chi_A''\chi_A + (\chi_A^2)'\frac{\zeta_B'}{\zeta_B}\right] + \frac{1}{2}P'(\chi_A^2)', \tag{5.9}$$

and

$$|\mathcal{E}_1(P)| \lesssim A^{-1}\|P'\|_{L^\infty(A\le|y|\le 2A)} + (AB)^{-1}\|P\|_{L^\infty(A\le|y|\le 2A)}. \tag{5.10}$$

Remark 5.1. For further purposes, we need the following easy consequences: for $P \equiv 1$, we get

$$\int \chi_A^2\zeta_B^2(\partial_x v_i)^2 = \int (\partial_x z_i)^2 + \int \frac{\zeta_B''}{\zeta_B}z_i^2 + \int \mathcal{E}_1\zeta_B^2 v_i^2, \tag{5.11}$$

where

$$\mathcal{E}_1 = \mathcal{E}_1(1) = \chi_A''\chi_A + (\chi_A^2)'\frac{\zeta_B'}{\zeta_B}. \tag{5.12}$$

Additionally, from (5.11), (4.12) and (5.10), one has the following estimate:

$$\|\chi_A\zeta_B\partial_x v_i\|^2 \lesssim \|\partial_x z_i\|_{L^2}^2 + B^{-1}\|z_i\|_{L^2}^2 + (AB)^{-1}\|\zeta_B v_i\|_{L^2}^2. \tag{5.13}$$

This estimate will be particularly useful later in the paper; see e.g. (6.41).

5.4. Proof of Proposition 5.1: First computations

Recall the definition of $\mathcal{J}$ in (5.4). We have from (5.4) and (5.2),

$$\begin{aligned}\frac{d}{dt}\mathcal{J} &= \int \psi_{A,B}\left[(\mathcal{L}\partial_x v_2)v_2 + \frac{1}{2}\partial_x(v_1^2) + Fv_1 + Gv_2\right] - \rho'\int \psi_{A,B}'v_1v_2\\ &= \int \psi_{A,B}(\mathcal{L}\partial_x v_2)v_2 - \frac{1}{2}\int \psi_{A,B}'v_1^2 + \int \psi_{A,B}[Gv_2 + Fv_1] - \rho'\int \psi_{A,B}'v_1v_2.\end{aligned}$$

Applying (2.4) in Lemma 2.5, we obtain

$$\begin{aligned}\int \psi_{A,B}(\mathcal{L}\partial_x v_2)v_2 = &-\frac{1}{2}\int \psi_{A,B}'(3(\partial_x v_2)^2 + V_0 v_2^2)\\ &+ \frac{1}{2}\int \psi_{A,B}'''v_2^2 - \frac{1}{2}\int \psi_{A,B}V_0'v_2^2.\end{aligned}$$

Now, we consider the following decomposition

$$
\begin{aligned}
\frac{d}{dt}\mathcal{J} = &-\frac{1}{2}\int \psi_{A,B}'(v_1^2+V_0v_2^2+3(\partial_x v_2)^2)+\frac{1}{2}\int \psi_{A,B}'''v_2^2-\frac{1}{2}\int \psi_{A,B}V_0'v_2^2\\
&+\int \psi_{A,B}Gv_2+\int \psi_{A,B}Fv_1-\rho'\int \psi_{A,B}'v_1v_2\\
=&: (J_1+J_2+J_3)+(J_4+J_5+J_6).
\end{aligned}
\tag{5.14}
$$

By the definition of $\psi_{A,B}$ (see (5.5)), it follows that

$$
\begin{aligned}
\psi_{A,B}' &= \chi_A^2\zeta_B^2+(\chi_A^2)'\varphi_B,\\
\psi_{A,B}''' &= \chi_A^2(\zeta_B^2)''+3(\chi_A^2)'(\zeta_B^2)'+3(\chi_A^2)''\zeta_B^2+(\chi_A^2)'''\varphi_B.
\end{aligned}
\tag{5.15}
$$

Also, by the definition of $z=(z_1,z_2)$ in (5.5), we have

$$
\begin{aligned}
-2J_1 &= \int \psi_{A,B}'(v_1^2+V_0v_2^2+3(\partial_x v_2)^2)\\
&= \int (z_1^2+V_0z_2^2)+\int(\chi_A^2)'\varphi_B(v_1^2+V_0v_2^2+3(\partial_x v_2)^2)+3\int \chi_A^2\zeta_B^2(\partial_x v_2)^2.
\end{aligned}
$$

For the last term of the above equation, applying Remark 5.1, we obtain

$$
\int \chi_A^2\zeta_B^2(\partial_x v_2)^2 = \int (\partial_x z_2)^2+\int \frac{\zeta_B''}{\zeta_B}z_2^2+\int\left[\chi_A''+2\chi_A'\frac{\zeta_B'}{\zeta_B}\right]\chi_A\zeta_B^2v_2^2.
$$

Then, for J_1 in (5.14), we obtain

$$
\begin{aligned}
J_1 =& -\frac{1}{2}\int (z_1^2+3(\partial_x z_2)^2+V_0z_2^2)-\frac{3}{2}\int \frac{\zeta_B''}{\zeta_B}z_2^2\\
&-\frac{3}{2}\int\left[\chi_A''+2\chi_A'\frac{\zeta_B'}{\zeta_B}\right]\chi_A\zeta_B^2v_2^2-\frac{1}{2}\int(\chi_A^2)'\varphi_B(v_1^2+V_0v_2^2+3(\partial_x v_2)^2).
\end{aligned}
\tag{5.16}
$$

Now we turn to J_2. By (5.15), J_2 in (5.14) satisfies the decomposition

$$
\begin{aligned}
J_2 &= \frac{1}{2}\int \big(\chi_A^2(\zeta_B^2)''+3(\chi_A^2)'(\zeta_B^2)'+3(\chi_A^2)''\zeta_B^2+(\chi_A^2)'''\varphi_B\big)v_2^2\\
&= \int\left[\left(\frac{\zeta_B'}{\zeta_B}\right)^2+\frac{\zeta_B''}{\zeta_B}\right]z_2^2+\frac{1}{2}\int\big(3(\chi_A^2)'(\zeta_B^2)'+3(\chi_A^2)''\zeta_B^2+(\chi_A^2)'''\varphi_B\big)v_2^2.
\end{aligned}
\tag{5.17}
$$

As for J_3 in (5.14), using the definition of z_2 in (5.5) we obtain

$$
J_3 = -\frac{1}{2}\int \psi_{A,B}V_0'v_2^2 = -\frac{1}{2}\int \frac{\varphi_B}{\zeta_B^2}V_0'z_2^2.
\tag{5.18}
$$

Finally, gathering (5.16), (5.17) and (5.18), we obtain that the first part in (5.14) can be written as

$$
J_1+J_2+J_3 = -\frac{1}{2}\int [z_1^2+Vz_2^2+3(\partial_x z_2)^2]+\tilde{J}_1,
$$

where

$$V = V_0 + \frac{\zeta_B''}{\zeta_B} - 2\frac{(\zeta_B')^2}{\zeta_B^2} + V_0'\frac{\varphi_B}{\zeta_B^2}, \tag{5.19}$$

and the error term is given by

$$\begin{aligned}\tilde{J}_1 = &-\frac{1}{2}\int\Big[3(\chi_A''\chi_A - (\chi_A^2)'')\zeta_B^2 - \frac{3}{2}(\chi_A^2)'(\zeta_B^2)' + ((\chi_A^2)'V_0 - (\chi_A^2)''')\varphi_B\Big]v_2^2\\ &-\frac{1}{2}\int(\chi_A^2)'\varphi_B v_1^2 - \frac{3}{2}\int(\chi_A^2)'\varphi_B(\partial_x v_2)^2. \end{aligned}\tag{5.20}$$

In order to control the main part of the virial term, a lower bound for the potential V is necessary. We have the following result:

Lemma 5.3. *There are $C > 0$ and $B_0 > 0$ such that for all $B \geq B_0$, one has*

$$V \geq V_0 - CB^{-1}, \quad \textit{where } V_0 = -1 + 3H^2.$$

Proof. First, noticing that $V_0' = 6HH' > 0$ for $y > 0$ and using that for $y \in [0,\infty) \mapsto \zeta_B(y)$ is a nonincreasing function, we have for $y > 0$,

$$\frac{\varphi_B}{\zeta_B^2} = \frac{\int_0^y \zeta_B^2}{\zeta_B^2} \geq y > 0.$$

Then, from this estimate, (5.19) and (4.11) with $K = B$,

$$V(y) \geq V_0(y) - CB^{-1} + |V_0'(y)y| \geq V_0(y) - CB^{-1}.$$

The case $y \leq 0$ is similar. These estimates hold for any $y \in \mathbb{R}$. This concludes the proof. ∎

First conclusion. Using this lemma, and $\tilde{J}_1$ in (5.20), we conclude that

$$\frac{d}{dt}\mathcal{J} \leq -\frac{1}{2}\int[z_1^2 + (V_0 - CB^{-1})z_2^2 + 3(\partial_x z_2)^2] + \tilde{J}_1 + J_4 + J_5 + J_6, \tag{5.21}$$

where J_4 and J_5 are related to the nonlinear term in (5.14) and J_6 is related to the shift considered on the kink. To control the terms $\tilde{J}_1$, J_4, J_5 and J_6, and the terms that will appear in the sections below, some technical estimates will be needed.

5.5. First technical estimates

The following estimates have been used to establish asymptotic stability for the good-Boussinesq equation, as well as the $1+1$ scalar fields equation (see [45, 68] for proof). For $\gamma > 0$, let $(1-\gamma\partial_x^2)^{-1}$ be the bounded operator from L^2 to H^2 defined by its Fourier transform as

$$\mathcal{F}((1-\gamma\partial_x^2)^{-1}g)(\xi) = \frac{\hat{g}(\xi)}{1+\gamma\xi^2}, \quad \text{for any } g \in L^2.$$

We start with a basic but essential result, in the spirit of [45].

Lemma 5.4. *Let $f \in L^2(\mathbb{R})$ and $0 < \gamma < 1$ fixed. We have the following estimates:*

(i) $\|(1-\gamma\partial_x^2)^{-1} f\|_{L^2(\mathbb{R})} \leq \|f\|_{L^2(\mathbb{R})}$,

(ii) $\|(1-\gamma\partial_x^2)^{-1}\partial_x f\|_{L^2(\mathbb{R})} \leq \gamma^{-1/2}\|f\|_{L^2(\mathbb{R})}$,

(iii) $\|(1-\gamma\partial_x^2)^{-1} f\|_{H^2(\mathbb{R})} \leq \gamma^{-1}\|f\|_{L^2(\mathbb{R})}$.

We also enunciate the following results that appear in [42,45,68]. Notice that now the variable in the weights is $y = x - \rho(t)$, but the shift does not affect the final outcome.

Lemma 5.5. *There exist $\gamma_1 > 0$ and $C > 0$ such that for any $\gamma \in (0, \gamma_1)$, $0 < K \leq 1$ and $g \in L^2$, the following estimates hold:*

$$\|\operatorname{sech}(Ky)(1-\gamma\partial_x^2)^{-1} g\|_{L^2} \leq C\|(1-\gamma\partial_x^2)^{-1}[\operatorname{sech}(Ky)g]\|_{L^2}, \tag{5.22}$$

$$\|\cosh(Ky)(1-\gamma\partial_x^2)^{-1}[\operatorname{sech}(Ky)g]\|_{L^2} \leq C\|(1-\gamma\partial_x^2)^{-1} g\|_{L^2}, \tag{5.23}$$

$$\|\operatorname{sech}(Ky)(1-\gamma\partial_x^2)^{-1}\partial_x g\|_{L^2} \leq C\gamma^{-1/2}\|\operatorname{sech}(Ky)g\|_{L^2}, \tag{5.24}$$

and

$$\|\operatorname{sech}(Ky)(1-\gamma\partial_x^2)^{-1}(1-\partial_x^2) g\|_{L^2} \leq C\gamma^{-1}\|\operatorname{sech}(Ky)g\|_{L^2}, \tag{5.25}$$

where the explicit constant C is independent of γ and K.

The following results are also contained in [68], and are essentially obtained from Lemma 5.4.

Lemma 5.6. *Recall v_i, w_i and z_i, $i = 1, 2$ defined in (5.1), (4.17) and (5.5), respectively. Then one has*

(a) *estimates on v_1 and u_1:*

$$\begin{aligned} \|v_1\|_{L^2} &\lesssim \gamma^{-1}\|u_1\|_{L^2}, \\ \|\partial_x v_1\|_{L^2} &\lesssim \gamma^{-1/2}\|u_1\|_{L^2} + \gamma^{-1}\|\partial_x u_1\|_{L^2}. \end{aligned} \tag{5.26}$$

(b) *estimates on v_2 and u_2:*

$$\begin{gathered} \|v_2\|_{L^2} \lesssim \|u_2\|_{L^2}, \quad \|\partial_x v_2\|_{L^2} \lesssim \gamma^{-1/2}\|u_2\|_{L^2}, \\ \|\partial_x^2 v_2\|_{L^2} \lesssim \gamma^{-1}\|u_2\|_{L^2}. \end{gathered} \tag{5.27}$$

Lemma 5.7. *Let $1 \leq K \leq A$ fixed. Then*

(a) *estimates on v_1 and w_1:*

$$\begin{aligned} \|\zeta_K v_1\| &\lesssim \gamma^{-1}\|w_1\|_{L^2}, \\ \|\zeta_K \partial_x v_1\| &\lesssim \gamma^{-1}(\|w_1\|_{L^2} + \|\partial_x w_1\|_{L^2}). \end{aligned} \tag{5.28}$$

(b) *estimates on v_2 and w_2:*

$$\begin{gathered} \|\zeta_K v_2\|_{L^2} \lesssim \|w_2\|_{L^2}, \quad \|\zeta_K \partial_x v_2\|_{L^2} \lesssim \gamma^{-1/2}\|w_2\|_{L^2}, \\ \|\zeta_K \partial_x^2 v_2\|_{L^2} \lesssim \gamma^{-1}\|w_2\|_{L^2}. \end{gathered} \tag{5.29}$$

(c) *estimates on u_1 and w_1:*

$$\begin{aligned}\Big\|\operatorname{sech}\Big(\frac{y}{K}\Big)u_1\Big\|_{L^2} &\lesssim \|w_1\|_{L^2},\\ \Big\|\operatorname{sech}\Big(\frac{y}{K}\Big)\partial_x u_1\Big\|_{L^2} &\lesssim \|\partial_x w_1\|_{L^2}+\|w_1\|_{L^2}.\end{aligned} \tag{5.30}$$

(d) *estimates on u_2 and w_2:*

$$\Big\|\operatorname{sech}\Big(\frac{y}{K}\Big)u_2\Big\|_{L^2} \lesssim \|w_2\|_{L^2}. \tag{5.31}$$

For the purposes of this work, we also include a refined version of Lemma 5.7 (a).

Lemma 5.8. *Let $1 \le K \le A$ fixed. Then*

$$\begin{aligned}\|(1-\gamma\partial_x^2)^{-1}\mathcal{L}f\|_{L^2} &\lesssim \|f\|_{L^2}+\gamma^{-1/2}\|\partial_x f\|_{L^2},\\ \|\zeta_K v_1\|_{L^2} &\lesssim \Big(1+\frac{1}{K\gamma^{1/2}}\Big)\|\zeta_K u_1\|_{L^2}+\gamma^{-1/2}\|\zeta_K\partial_x u_1\|_{L^2}.\end{aligned} \tag{5.32}$$

Proof. We have

$$\begin{aligned}\|(1-\gamma\partial_x^2)^{-1}\mathcal{L}f\|_{L^2} &= \Big\|(1-\gamma\partial_x^2)^{-1}\Big[(2-\partial_x^2)f-3\operatorname{sech}^2\Big(\frac{y}{\sqrt{2}}\Big)f\Big]\Big\|_{L^2}\\ &\lesssim \|(1-\gamma\partial_x^2)^{-1}(2-\partial_x^2)f\|_{L^2}+\Big\|\operatorname{sech}^2\Big(\frac{\cdot-\rho}{\sqrt{2}}\Big)f\Big\|_{L^2}\\ &\lesssim \|(1-\gamma\partial_x^2)^{-1}(2-\partial_x^2)f\|_{L^2}+\|f\|_{L^2}.\end{aligned}$$

Focus on the first term on the right-hand side. Using Plancherel's theorem, we get

$$\begin{aligned}\|(1-\gamma\partial_x^2)^{-1}(2-\partial_x^2)f\|_{L^2} &= \Big\|\frac{(2+\xi^2)}{(1+\gamma\xi^2)}\hat{f}\Big\|_{L^2}\\ &= \Big\|\Big(\frac{(2+\xi^2)}{(1+\gamma\xi^2)}1_{[-1,1]}+\frac{(2+\xi^2)}{(1+\gamma\xi^2)}1_{[-1,1]^c}\Big)\hat{f}\Big\|_{L^2}\\ &\le \Big\|\frac{3}{(1+\gamma\xi^2)}1_{[-1,1]}\hat{f}\Big\|_{L^2}+\Big\|\frac{3\operatorname{sgn}(\xi)|\xi|}{(1+\gamma\xi^2)}1_{[-1,1]^c}\xi\hat{f}\Big\|_{L^2}\\ &\le \Big\|\frac{3}{(1+\gamma\xi^2)}1_{[-1,1]}\hat{f}\Big\|_{L^2}\\ &\quad+\Big\|3\gamma^{-1/2}\frac{\operatorname{sgn}(\xi)\sqrt{1+\gamma|\xi|^2}}{(1+\gamma\xi^2)}1_{[-1,1]^c}\xi\hat{f}\Big\|_{L^2}\\ &\lesssim \|f\|_{L^2}+\gamma^{-1/2}\|\partial_x f\|_{L^2}.\end{aligned}$$

This concludes the proof of the first estimate in (5.32).

For the second inequality, by (5.22) and using that $\zeta_K\mathcal{L}(u_1)=\mathcal{L}(\zeta_K u_1)+2\zeta_K'\partial_x u_1+\zeta_K''u_1$, we obtain

$$\begin{aligned}\|\zeta_K v_1\|_{L^2} &\lesssim \|(1-\gamma\partial_x^2)^{-1}(\zeta_K\mathcal{L}u_1)\|_{L^2}\\ &\lesssim \|(1-\gamma\partial_x^2)^{-1}(\mathcal{L}(\zeta_K u_1)+2\zeta_K'\partial_x u_1+\zeta_K''u_1)\|_{L^2}\\ &\lesssim \|(1-\gamma\partial_x^2)^{-1}\mathcal{L}(\zeta_K u_1)\|_{L^2}+\|2\zeta_K'\partial_x u_1\|_{L^2}+\|\zeta_K''u_1\|_{L^2}.\end{aligned}$$

Therefore, from the first identity in (5.32), (4.12) and (4.13),

$$
\begin{aligned}
\|\zeta_K v_1\|_{L^2} &\lesssim \|\zeta_K u_1\|_{L^2} + \gamma^{-1/2}\|\partial_x(\zeta_K u_1)\|_{L^2} + \|\zeta_K' \partial_x u_1\|_{L^2} + \|\zeta_K'' u_1\|_{L^2} \\
&\lesssim \left(1+\frac{1}{K\gamma^{1/2}}\right)\|\zeta_K u_1\|_{L^2} + \left(\gamma^{-1/2}+\frac{1}{K}\right)\|\zeta_K\partial_x u_1\|_{L^2} \\
&\lesssim \left(1+\frac{1}{K\gamma^{1/2}}\right)\|\zeta_K u_1\|_{L^2} + \gamma^{-1/2}\|\zeta_K\partial_x u_1\|_{L^2}.
\end{aligned}
$$

This ends the proof. ∎

Remark 5.2. Using (4.23), for $K \in [1, A]$, and $\zeta_A \partial_x u_1 = \partial_x w_1 - \frac{\zeta_A'}{\zeta_A} w_1$, we obtain

$$
\int \zeta_K^2 (\partial_x u_1)^2 = \int \frac{\zeta_K^2}{\zeta_A^2}(\partial_x w_1)^2 + \int w_1^2\left(\frac{\zeta_K^2}{\zeta_A^2}\frac{\zeta_A''}{\zeta_A} + 2\frac{\zeta_K}{\zeta_A}\frac{\zeta_A'}{\zeta_A}\left(\frac{\zeta_K'}{\zeta_A} - \frac{\zeta_K}{\zeta_A}\frac{\zeta_A'}{\zeta_A}\right)\right).
$$

Hence, a crude estimate gives

$$
\|\zeta_K \partial_x u_1\|_{L^2}^2 \lesssim \|\partial_x w_1\|_{L^2}^2 + A^{-1}\|w_1\|_{L^2}^2.
$$

Taking $K = A$ we obtain

$$
\|\zeta_A \partial_x u_1\|_{L^2} \lesssim \|\partial_x w_1\|_{L^2} + A^{-1/2}\|w_1\|_{L^2}, \tag{5.33}
$$

and from (5.32) and $K = A$,

$$
\|\zeta_A v_1\|_{L^2} \lesssim \left(1+\frac{1}{(A\gamma)^{1/2}}\right)\|w_1\|_{L^2} + \gamma^{-1/2}\|\partial_x w_1\|_{L^2}.
$$

Finally, using (5.6), $A\gamma = \delta^{-1+2/5} \gg 1$, and

$$
\|\zeta_A v_1\|_{L^2} \lesssim \|w_1\|_{L^2} + \gamma^{-1/2}\|\partial_x w_1\|_{L^2}. \tag{5.34}
$$

5.6. Controlling error and nonlinear terms

By the definition of ζ_B and χ_A in (4.8) and (4.9), it holds that

$$
\begin{gathered}
\zeta_B(y) \lesssim e^{-\frac{|y|}{B}}, \quad |\zeta_B'(y)| \lesssim \frac{1}{B}e^{-\frac{|y|}{B}}, \quad |\varphi_B| \lesssim B, \\
|\chi_A'| \lesssim A^{-1}, \quad |(\chi_A^2)'| \lesssim A^{-1}, \quad |(\chi_A^2)''| \lesssim A^{-2}, \quad |(\chi_A^2)'''| \lesssim A^{-3}.
\end{gathered} \tag{5.35}
$$

5.6.1. Control of $\tilde{J}_1$. Considering the following decomposition for $\tilde{J}_1$:

$$
\begin{aligned}
\tilde{J}_1 = &-\frac{1}{2}\int (\chi_A^2)'\varphi_B(v_1^2 + 3(\partial_x v_2)^2) - \frac{1}{2}\int ((\chi_A^2)'V_0 - (\chi_A^2)''')\varphi_B v_2^2 \\
&-\frac{1}{2}\int \left[3(\chi_A''\chi_A - (\chi_A^2)'') - 3(\chi_A^2)'\frac{\zeta_B'}{\zeta_B}\right]\zeta_B^2 v_2^2 \\
=:&\ H_1 + H_2 + H_3.
\end{aligned}
$$

In the case of H_1 and H_2, using $|(\chi_A^2)'\varphi_B| \lesssim A^{-1}B$ and (4.15), we obtain

$$\begin{aligned}|H_1| &\lesssim A^{-1}B(\|v_1\|^2_{L^2(|y|\le 2A)} + \|\partial_x v_2\|^2_{L^2(|y|\le 2A)})\\ &\lesssim A^{-1}B(\|\zeta_A^2 v_1\|^2_{L^2} + \|\zeta_A^2\partial_x v_2\|^2_{L^2}),\end{aligned}$$

and

$$|H_2| \lesssim A^{-1}B\|v_2\|^2_{L^2(A\le|y|\le 2A)} \lesssim A^{-1}B\|v_2\|^2_{L^2(|y|\le 2A)} \lesssim A^{-1}B\|\zeta_A^2 v_2\|^2_{L^2}.$$

Finally, in the case of H_3, using (5.35), we have

$$|H_3| \lesssim (AB)^{-1}\|\zeta_B v_2\|^2_{L^2(|y|\le 2A)} \lesssim (AB)^{-1}\|\zeta_B v_2\|^2_{L^2}.$$

We obtain, using that $\zeta_B \lesssim \zeta_A$,

$$|\tilde{J}_1| \lesssim A^{-1}B(\|\zeta_A v_1\|^2_{L^2} + \|\zeta_A v_2\|^2_{L^2} + \|\zeta_A\partial_x v_2\|^2_{L^2}).$$

Applying (5.34) and (5.29) with $K = A$, and finally using (5.6), we get

$$\begin{aligned}|\tilde{J}_1| &\lesssim A^{-1}B(\|w_1\|^2_{L^2} + \gamma^{-1}\|\partial_x w_1\|^2_{L^2} + \gamma^{-1}\|w_2\|^2_{L^2})\\ &\lesssim A^{-1}B\gamma^{-1}(\|w_1\|^2_{L^2} + \|\partial_x w_1\|^2_{L^2} + \|w_2\|^2_{L^2})\\ &\lesssim \delta^{1/2}(\|w_1\|^2_{L^2} + \|\partial_x w_1\|^2_{L^2} + \|w_2\|^2_{L^2}).\end{aligned} \tag{5.36}$$

5.6.2. Control of J_4. Using the value of G in (5.3), J_4 is bounded as follows:

$$\begin{aligned}J_4 &= \gamma\int \psi_{A,B}v_2(1-\gamma\partial_x^2)^{-1}[2\partial_x(V_0'\partial_x v_2) - V_0''\partial_x v_2]\\ &\quad - \rho'\int \psi_{A,B}v_2(1-\gamma\partial_x^2)^{-1}(V_0'u_1)\\ &=: J_{41} + J_{42}.\end{aligned} \tag{5.37}$$

Bound on J_{41}. Using the Hölder inequality,

$$|J_{41}| \lesssim \gamma\|\psi_{A,B}v_2\|_{L^2}\|(1-\gamma\partial_x^2)^{-1}[2\partial_x(V_0'\partial_x v_2) - V_0''\partial_x v_2]\|_{L^2}.$$

Using $\psi_{A,B} = \chi_A^2\varphi_B$ in (5.5), and (4.15), one can see that

$$\|\psi_{A,B}v_2\|_{L^2} \lesssim B\|\chi_A v_2\|_{L^2} \lesssim B\|v_2\|_{L^2(|y|<2A)} \lesssim B\|\zeta_A^2 v_2\|_{L^2}. \tag{5.38}$$

On the other hand, using Lemma 5.4,

$$\|(1-\gamma\partial_x^2)^{-1}[2\partial_x(V_0'\partial_x v_2) - V_0''\partial_x v_2]\|_{L^2} \lesssim \gamma^{-1/2}\|V_0'\partial_x v_2\|_{L^2} + \|V_0''\partial_x v_2\|_{L^2}.$$

Recall that $|V_0'|, |V_0''| \sim \operatorname{sech}^2(y/\sqrt{2}) \lesssim e^{-\sqrt{2}|y|}$. Therefore, we are led to the estimate

$$\|e^{-\sqrt{2}|y|}\partial_x v_2\|_{L^2}.$$

First of all, write

$$
\begin{aligned}
e^{-\sqrt{2}|y|}\partial_x v_2 &= (e^{-\sqrt{2}|y|/2}\zeta_B^{-1})e^{-\sqrt{2}|y|/2}\zeta_B(1-\chi_A)\partial_x v_2\\
&\quad + (e^{-\sqrt{2}|y|/2}\zeta_B^{-1})e^{-\sqrt{2}|y|/2}\chi_A\zeta_B\partial_x v_2.
\end{aligned}
$$

Noticing that since $e^{-\sqrt{2}|y|/2}\zeta_B^{-1} \lesssim 1$,

$$
\begin{aligned}
\|e^{-\sqrt{2}|y|}\partial_x v_2\|_{L^2} &\lesssim \|e^{-\sqrt{2}|y|/2}\zeta_B(1-\chi_A)\partial_x v_2\|_{L^2} + \|e^{-\sqrt{2}|y|/2}\zeta_B\chi_A\partial_x v_2\|_{L^2}\\
&\lesssim e^{-\sqrt{2}A/2}\|\zeta_B\partial_x v_2\|_{L^2} + \|e^{-\sqrt{2}|y|/2}\chi_A\zeta_B\partial_x v_2\|_{L^2}.
\end{aligned} \tag{5.39}
$$

Using (5.8) and (5.10) with $P = e^{-\sqrt{2}|y|/2}$ we get

$$
\|e^{-\sqrt{2}|y|/2}\chi_A\zeta_B\partial_x v_2\|_{L^2} \lesssim \|\partial_x z_2\|_{L^2} + B^{-1/2}\|z_2\|_{L^2} + \frac{1}{A^{1/2}}e^{-\sqrt{2}A/4}\|\zeta_B v_2\|_{L^2}.
$$

Coming back to the bound on $\|e^{-\sqrt{2}|y|}\partial_x v_2\|_{L^2}$, we get

$$
\begin{aligned}
\|e^{-\sqrt{2}|y|}\partial_x v_2\|_{L^2} \leq e^{-\sqrt{2}A/2}\|\zeta_B\partial_x v_2\|_{L^2} + \|\partial_x z_2\|_{L^2} + B^{-1/2}\|z_2\|_{L^2}\\
+ (Ae^{\sqrt{2}A/2})^{-1/2}\|\zeta_B v_2\|_{L^2}.
\end{aligned}
$$

The previous inequality allows us to conclude that

$$
\begin{aligned}
&\|(1-\gamma\partial_x^2)^{-1}[2\partial_x(V_0'\partial_x v_2) - V_0''\partial_x v_2]\|_{L^2}\\
&\quad\lesssim \gamma^{-1/2}(\|\partial_x z_2\|_{L^2} + B^{-1/2}\|z_2\|_{L^2})\\
&\qquad + e^{-A/4}\gamma^{-1/2}(\|\zeta_B v_2\|_{L^2} + \|\zeta_B\partial_x v_2\|_{L^2}).
\end{aligned} \tag{5.40}
$$

Gathering (5.38) and this last estimate, we conclude first that

$$
|J_{41}| \lesssim \gamma^{1/2}B\|\zeta_A^2 v_2\|_{L^2}\big(\|\partial_x z_2\|_{L^2} + \|z_2\|_{L^2} + e^{-A/4}(\|\zeta_B v_2\|_{L^2} + \|\zeta_B\partial_x v_2\|_{L^2})\big).
$$

Using (5.29) with $K = A, B$, the Cauchy inequality and (5.6), we obtain $e^{-A/4}\gamma^{-1/2} \ll 1$ and

$$
\begin{aligned}
|J_{41}| &\lesssim \gamma^{1/2}B\|w_2\|_{L^2}(\|\partial_x z_2\|_{L^2} + \|z_2\|_{L^2} + e^{-A/2}\gamma^{-1/2}\|w_2\|_{L^2})\\
&\lesssim \gamma^{1/2}B(\|w_2\|_{L^2}^2 + \|\partial_x z_2\|_{L^2}^2 + \|z_2\|_{L^2}^2).
\end{aligned}
$$

We conclude that $\gamma^{1/2}B = B^{-1} = \delta^{1/10}$ and

$$
|J_{41}| \lesssim \delta^{1/10}[\|w_2\|_{L^2}^2 + \|\partial_x z_2\|_{L^2}^2 + \|z_2\|_{L^2}^2]. \tag{5.41}
$$

Bound on J_{42}. Now we focus on J_{42} in (5.37). By the Hölder inequality, we get

$$
\begin{aligned}
|J_{42}| &= |\rho'|\left|\int \psi_{A,B}v_2(1-\gamma\partial_x^2)^{-1}(V_0'u_1)\right|\\
&\lesssim |\rho'|\|\chi_A\varphi_B v_2\|_{L^2}\|\chi_A(1-\gamma\partial_x^2)^{-1}(V_0'u_1)\|_{L^2}.
\end{aligned}
$$

Now, using (5.27) and $|\chi_A\varphi_B| \lesssim B$, and (4.3),

$$\|\chi_A\varphi_B v_2\|_{L^2} \lesssim B\|u_2\|_{L^2} \lesssim B\delta.$$

Additionally, from Lemma 5.4, the fact that

$$|V_0'\zeta_A^{-1}| = 6|\zeta_A^{-1}H'(y)H(y)| \lesssim 1$$

and (4.17), we obtain

$$\|\chi_A(1-\gamma\partial_x^2)^{-1}(V_0'u_1)\|_{L^2} \lesssim \|\zeta_A u_1\|_{L^2} = \|w_1\|_{L^2}.$$

Since $\delta B = \delta^{9/10}$, we conclude

$$|J_{42}| \lesssim \delta B|\rho'|\|w_1\|_{L^2} \lesssim \delta^{9/10}(|\rho'|^2 + \|w_1\|_{L^2}^2). \tag{5.42}$$

Gathering (5.41) and (5.42), we finally arrive at the estimate

$$|J_4| \lesssim \delta^{1/10}[\|w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2 + \|z_2\|_{L^2}^2 + \|\partial_x z_2\|_{L^2}^2] + \delta^{9/10}\rho'^2. \tag{5.43}$$

5.6.3. Control of J_5. Recalling that $\psi_{A,B} = \chi_A^2\varphi_B$, using the Hölder inequality and Remark 4.15, we get

$$\begin{aligned}|J_5| = \left|\int \psi_{A,B}Fv_1\right| &\lesssim \|\chi_A\varphi_B v_1\|_{L^2}\|\chi_A F\|_{L^2}\\ &\lesssim \|\chi_A\varphi_B v_1\|_{L^2}\|\zeta_A^2(1-\gamma\partial_x^2)^{-1}\partial_x[u_1^3 + 3Hu_1^2]\|_{L^2}.\end{aligned}$$

First of all, since $|\varphi_B| \lesssim B$, we have

$$\|\chi_A\varphi_B v_1\|_{L^2} \lesssim B\|\chi_A v_1\|_{L^2} \lesssim B\|v_1\|_{L^2(|y|<2A)} \lesssim B\|\zeta_A^2 v_1\|_{L^2}. \tag{5.44}$$

Furthermore, using (5.22) and Lemma 5.4, we have

$$\|\zeta_A^2(1-\gamma\partial_x^2)^{-1}\partial_x[u_1^3 + 3Hu_1^2]\|_{L^2} \lesssim \|\zeta_A^2\partial_x[u_1^3 + 3Hu_1^2]\|_{L^2},$$

and expanding the above term, one gets

$$\begin{aligned}\|\zeta_A^2\partial_x[u_1^3 + 3Hu_1^2]\|_{L^2} &= \|\zeta_A^2[3u_1^2\partial_x u_1 + 3H'u_1^2 + 6Hu_1\partial_x u_1]\|_{L^2}\\ &\le \|u_1\|_{L^\infty}\|\zeta_A^2[3u_1\partial_x u_1 + 3H'u_1 + 6H\partial_x u_1]\|_{L^2}\\ &\lesssim \|u_1\|_{L^\infty}[\|w_1\|_{L^2}\|\zeta_A\partial_x u_1\|_{L^2} + \|w_1\|_{L^2} + \|\zeta_A\partial_x u_1\|_{L^2}]\\ &\lesssim \|u_1\|_{L^\infty}[\|w_1\|_{L^2} + \|\zeta_A\partial_x u_1\|_{L^2}].\end{aligned} \tag{5.45}$$

Finally, by (5.45), (5.44) and (5.34), we conclude that

$$\begin{aligned}|J_5| &\lesssim B\|u_1\|_{L^\infty}\|\zeta_A^2 v_1\|_{L^2}[\|w_1\|_{L^2} + \|\zeta_A\partial_x u_1\|_{L^2}]\\ &\lesssim B\delta[\|w_1\|_{L^2} + \gamma^{-1/2}\|\partial_x w_1\|_{L^2}][\|w_1\|_{L^2} + \|\zeta_A\partial_x u_1\|_{L^2}]\\ &\lesssim \delta^{7/10}[\|w_1\|_{L^2}^2 + \|\partial_x w_1\|_{L^2}^2].\end{aligned} \tag{5.46}$$

In the last inequality we have used (5.30) combined with (5.35), and also $B\delta\gamma^{-1/2} = \delta^{1-1/10-2/10} = \delta^{7/10}$.

5.6.4. Control of J_6 in (5.14). Replacing $\psi'_{A,B}$ from (5.15), we have

$$\begin{aligned}|J_6| &= \left|\rho' \int \psi'_{A,B} v_1 v_2\right| \\ &\le |\rho'|\left|\int (\chi_A^2\zeta_B^2 + 2\chi_A'\chi_A\varphi_B)v_1v_2\right| \\ &= \rho' \int z_1\chi_A\zeta_B v_2 + 2\rho' \int \chi_A'\chi_A\varphi_B v_1v_2 =: J_{6,1} + J_{6,2}.\end{aligned}$$

Bound on $J_{6,1}$. Applying the Hölder inequality, estimate (5.27) and (4.3), we get

$$|J_{6,1}| \le |\rho'|\|z_1\|_{L^2}\|u_2\|_{L^2} \le C_0\delta|\rho'|\|z_1\|_{L^2}.$$

Bound on $J_{6,2}$. Here, after invoking (5.35) and the Hölder inequality, we get

$$|J_{6,2}| \lesssim A^{-1}B|\rho'|\|\chi_A v_1\|_{L^2}\|v_2\|_{L^2}.$$

Using (5.1), Lemma 5.4 and (4.3), we easily have $\|v_2\|_{L^2} \lesssim \|u_2\|_{L^2} \lesssim \delta$. From (4.15), we also get

$$\|\chi_A v_1\|_{L^2} \lesssim \|\zeta_A^2 v_1\|_{L^2} \lesssim \|\zeta_A v_1\|_{L^2}.$$

Hence, by (5.34),

$$|J_{6,2}| \lesssim \delta A^{-1}B|\rho'|(\|w_1\|_{L^2} + \gamma^{-1/2}\|\partial_x w_1\|_{L^2}).$$

Since $A^{-1}B\gamma^{-1/2} = \delta^{7/10}$, after applying the Cauchy–Schwarz inequality we conclude that

$$\begin{aligned}|J_6| &\lesssim \delta A^{-1}B|\rho'|(\|w_1\|_{L^2} + \gamma^{-1/2}\|\partial_x w_1\|_{L^2}) + C\delta|\rho'|\|z_1\|_{L^2} \\ &\lesssim \delta^{17/10}|\rho'|(\|w_1\|_{L^2} + \|\partial_x w_1\|_{L^2}) + C\delta|\rho'|\|z_1\|_{L^2} \\ &\lesssim \delta^{17/10}(\|w_1\|_{L^2}^2 + \|\partial_x w_1\|_{L^2}^2) + C\delta\|z_1\|_{L^2}^2 + C\delta|\rho'|^2. \end{aligned} \tag{5.47}$$

5.7. End of proof of Proposition 5.1

From (5.36), (5.43), (5.46) and (5.47), we obtain

$$\begin{aligned}&|\tilde{J}_1 + J_4 + J_5 + J_6| \\ &\quad\lesssim \delta^{1/2}(\|w_1\|_{L^2}^2 + \|\partial_x w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2) + \delta^{7/10}(\|w_1\|_{L^2}^2 + \|\partial_x w_1\|_{L^2}^2) \\ &\qquad + \delta^{17/10}(\|w_1\|_{L^2}^2 + \|\partial_x w_1\|_{L^2}^2) + \delta^{1/10}(\|w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2) \\ &\qquad + \delta^{1/10}(\|\partial_x z_2\|_{L^2}^2 + \|z_2\|_{L^2}^2) + \delta\|z_1\|_{L^2}^2 + \delta^{9/10}\rho'^2 + \delta\rho'^2.\end{aligned}$$

Finally, simplifying, we obtain that (5.48) is bounded as

$$\begin{aligned}|\tilde{J}_1 + J_4 + J_5 + J_6| \lesssim{}& \delta^{1/10}(\|w_1\|_{L^2}^2 + \|\partial_x w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2 + \|z_2\|_{L^2}^2 + \|\partial_x z_2\|_{L^2}^2) \\ &+ \delta\|z_1\|_{L^2}^2 + \delta^{9/10}|\rho'|^2. \end{aligned} \tag{5.48}$$

Finally, the virial estimate is concluded as follows: for some $C_2 > 0$ independent of δ small, (5.21) becomes

$$\frac{d}{dt}\mathcal{J} \leq -\frac{1}{2}\int [z_1^2 + (V_0 - C_2\delta^{1/10})z_2^2 + 2(\partial_x z_2)^2] + C_2\delta\|z_1\|_{L^2}^2 \\ + C_2\delta^{1/10}(\|w_1\|_{L^2}^2 + \|\partial_x w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2) + C_2\delta^{9/10}|\rho'|^2.$$

This ends the proof of (5.7) and Proposition 5.1.

6. Gain of regularity

We will focus now on a new virial estimate obtained from the new system of equations involving the variables

$$\tilde{v}_i = \partial_x v_i, \quad i = 1, 2. \tag{6.1}$$

Formally taking derivatives in (5.2), we have

$$\begin{cases} \dot{\tilde{v}}_1 = \partial_x \mathcal{L}\tilde{v}_2 + \tilde{G}, & \tilde{G} = \partial_x G, \\ \dot{\tilde{v}}_2 = \partial_x \tilde{v}_1 + \tilde{F}, & \tilde{F} = \partial_x F, \end{cases} \tag{6.2}$$

where G and F are given in (5.3) and $\tilde{v}_i$ given in (6.1). For this new system, we consider the virial functional

$$\mathcal{M}(t) = \int \phi_{A,B}(y)\tilde{v}_1(t,x)\tilde{v}_2(t,x)\,dx = \int \phi_{A,B}(y)\partial_x v_1(t,x)\partial_x v_2(t,x)\,dx. \tag{6.3}$$

Later we will choose $\phi_{A,B} = \psi_{A,B} = \chi_A^2\varphi_B$ (see (5.5)).

6.1. A virial estimate related to $\mathcal{M}$

Lemma 6.1. *Let $(v_1, v_2) \in H^1(\mathbb{R}) \times H^2(\mathbb{R})$ be a solution of (5.2). Consider $\phi_{A,B}$ an odd smooth bounded function to be chosen later. Then*

$$\frac{d}{dt}\mathcal{M} = -\frac{1}{2}\int \phi_{A,B}'[(\partial_x v_1)^2 + V_0(\partial_x v_2)^2 + 3(\partial_x^2 v_2)^2] + \frac{1}{2}\int \phi_{A,B}'''(\partial_x v_2)^2 \\ + \frac{1}{2}\int \phi_{A,B} V_0'(\partial_x v_2)^2 + \int \phi_{A,B}[\partial_x G \partial_x v_2 + \tilde{F}\partial_x v_1] \\ - \rho'\int \phi_{A,B}'\partial_x v_1 \partial_x v_2. \tag{6.4}$$

Proof of Lemma 6.1. We compute from (6.3),

$$\frac{d}{dt}\mathcal{M} = \int \phi_{A,B}\dot{\tilde{v}}_1\tilde{v}_2 + \int \phi_{A,B}\tilde{v}_1\dot{\tilde{v}}_2 - \rho'\int \phi_{A,B}'\tilde{v}_1\tilde{v}_2.$$

From (6.2) and (2.3), we have

$$
\begin{aligned}
\frac{d}{dt}\mathcal{M} &= \int \phi_{A,B}(\partial_x \mathcal{L}\tilde{v}_2 + \widetilde{G})\tilde{v}_2 + \int \phi_{A,B}\tilde{v}_1(\partial_x \tilde{v}_1 + \widetilde{F}) - \rho' \int \phi_{A,B}' \partial_x v_1 \partial_x v_2 \\
&= -\frac{1}{2}\int \phi_{A,B}'[\tilde{v}_1^2 + V_0\tilde{v}_2^2 + 3(\partial_x \tilde{v}_2)^2] + \frac{1}{2}\int \phi_{A,B}''' \tilde{v}_2^2 \\
&\quad + \frac{1}{2}\int \phi_{A,B} V_0' \tilde{v}_2^2 + \int \phi_{A,B}[\widetilde{G}\tilde{v}_2 + \widetilde{F}\tilde{v}_1] - \rho' \int \phi_{A,B}' \partial_x v_1 \partial_x v_2.
\end{aligned}
$$

Rewriting the above identity in terms of the variables (v_1, v_2) and using the definition of $\widetilde{G}$, we have

$$
\begin{aligned}
\frac{d}{dt}\mathcal{M} = &-\frac{1}{2}\int \phi_{A,B}'[(\partial_x v_1)^2 + V_0(\partial_x v_2)^2 + 3(\partial_x^2 v_2)^2] + \frac{1}{2}\int \phi_{A,B}'''(\partial_x v_2)^2 \\
&+ \frac{1}{2}\int \phi_{A,B} V_0'(\partial_x v_2)^2 + \int \phi_{A,B}[\partial_x G \partial_x v_2 + \widetilde{F}\partial_x v_1] - \rho' \int \phi_{A,B}' \partial_x v_1 \partial_x v_2.
\end{aligned}
$$

This ends the proof of Lemma 6.1. ■

The following proposition connects two virial identities in the variables (z_1, z_2). Finally, let z_i, $i = 1, 2$ be as in (5.5).

Proposition 6.2. *There exist $C_3 > 0$ and $\delta_3 > 0$ such that for any $0 < \delta \le \delta_3$, the following holds. Assume that for all $t \ge 0$, (4.1) and (5.6) hold. Then for all $t \ge 0$,*

$$
\begin{aligned}
\frac{d}{dt}\mathcal{M} \le &-\frac{1}{2}\int [(\partial_x z_1)^2 + 2(\partial_x^2 z_2)^2] + C_3(\|z_1\|_{L^2}^2 + \|z_2\|_{L^2}^2 + \|\partial_x z_2\|_{L^2}^2) \\
&+ C_3\delta^{1/10}(\|w_1\|_{L^2}^2 + \|\partial_x w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2) + C_3\delta^{7/10}|\rho'|^2.
\end{aligned} \tag{6.5}
$$

The proof of the above result requires some technical estimates. We first state them, and then prove Proposition 6.2 (Section 6.3).

6.2. Second set of technical estimates

Recall the following technical estimates on the variables ζ_B and other related error terms. These estimates have been proved and used in [68]; therefore we only enunciate the main results.

Claim 6.3. *Let R be a $W^{2,\infty}(\mathbb{R})$ function, $R = R(y)$, v_i be as in (5.1) and z_i be as in (5.5). Then*

$$
\begin{aligned}
\int R\chi_A^2\zeta_B^2(\partial_x^2 v_i)^2 = &\int R(\partial_x^2 z_i)^2 + \int \tilde{R} z_i^2 + \int P_R(\partial_x z_i)^2 \\
&+ \int \left[P_R'\frac{\zeta_B'}{\zeta_B} + P_R\frac{\zeta_B''}{\zeta_B}\right] z_i^2 + \int \mathcal{E}_2\zeta_B^2 v_i^2 \\
&+ \int \mathcal{E}_1(P_R)\zeta_B^2 v_i^2 + \int \mathcal{E}_3\zeta_B^2(\partial_x v_i)^2,
\end{aligned}
$$

where

$$\tilde{R} = \tilde{R}_R = -2R\left[\frac{\zeta_B^{(4)}}{\zeta_B} + \frac{\zeta_B'''}{\zeta_B}\frac{\zeta_B'}{\zeta_B}\right] - 2R'\frac{\zeta_B'''}{\zeta_B} - R''\frac{\zeta_B''}{\zeta_B}, \tag{6.6}$$

$$P_R = R\left[4\frac{\zeta_B''}{\zeta_B} - 2\left(\frac{\zeta_B'}{\zeta_B}\right)^2\right] + 2R'\frac{\zeta_B'}{\zeta_B}, \tag{6.7}$$

$\mathcal{E}_1$ *is defined in* (5.9),

$$\begin{aligned}\mathcal{E}_2 = \mathcal{E}_2(R) = &-R\left(\chi_A^{(4)}\chi_A + 4\chi_A'''\chi_A\frac{\zeta_B'}{\zeta_B^2} + 6\chi_A''\chi_A\frac{\zeta_B''}{\zeta_B} + 2(\chi_A^2)'\frac{\zeta_B'''}{\zeta_B}\right)\\ &- R'\left(2\chi_A'''\chi_A + 6\chi_A''\chi_A\frac{\zeta_B'}{\zeta_B} + 6\chi_A'\chi_A\frac{\zeta_B''}{\zeta_B}\right)\\ &- R''\left(\chi_A''\chi_A + \frac{1}{2}(\chi_A^2)'\frac{\zeta_B'}{\zeta_B}\right)\end{aligned} \tag{6.8}$$

and

$$\mathcal{E}_3 = \mathcal{E}_3(R) = R\left[4\chi_A''\chi_A - 2(\chi_A')^2 + 2\frac{\zeta_B'}{\zeta_B}(\chi_A^2)'\right] + R'(\chi_A^2)'. \tag{6.9}$$

Finally, P_R*,* $\mathcal{E}_2$ *and* $\mathcal{E}_3$ *satisfy the bounds*

$$\begin{aligned}|P_R| &\lesssim B^{-1}\|R'\|_{L^\infty} + B^{-1}\|R\|_{L^\infty},\\ |P_R'| &\lesssim B^{-1}\|R''\|_{L^\infty} + B^{-1}\|R'\|_{L^\infty} + B^{-1}\|R\|_{L^\infty},\\ |\mathcal{E}_2| &\lesssim (AB)^{-1}\|R''\|_{L^\infty(A\le|y|\le 2A)} + (AB^2)^{-1}\|R'\|_{L^\infty(A\le|y|\le 2A)}\\ &\quad + (AB^3)^{-1}\|R\|_{L^\infty(A\le|y|\le 2A)},\\ |\mathcal{E}_3| &\lesssim A^{-1}\|R'\|_{L^\infty(A\le|y|\le 2A)} + (AB)^{-1}\|R\|_{L^\infty(A\le|y|\le 2A)}.\end{aligned} \tag{6.10}$$

Remark 6.1. Estimates (6.10) are of a technical type, needed at some particular stage to control the error terms on the gain regularity virial (see (6.20)). A proof of these estimates is present in [68], but without the localization terms. Here we slightly improve these estimates by considering the region of space where these functions are supported.

Remark 6.2. For further purposes, we will need the previous identities in the simplest case $R = 1$. We obtain

$$\begin{aligned}\int \chi_A^2\zeta_B^2(\partial_x^2 v_i)^2 = &\int (\partial_x^2 z_i)^2 + \int \tilde{R}_1 z_i^2 + \int P_1(\partial_x z_i)^2 + \int \left[P_1'\frac{\zeta_B'}{\zeta_B} + P_1\frac{\zeta_B''}{\zeta_B}\right]z_i^2\\ &+ \int \mathcal{E}_2\zeta_B^2 v_i^2 + \int \mathcal{E}_1(P_1)\zeta_B^2 v_i^2 + \int \mathcal{E}_3\zeta_B^2(\partial_x v_i)^2,\end{aligned} \tag{6.11}$$

where (see (6.6) and (6.7))

$$\tilde{R}_1 = -2\left[\frac{\zeta_B^{(4)}}{\zeta_B} + \frac{\zeta_B'''}{\zeta_B}\frac{\zeta_B'}{\zeta_B}\right], \quad P_1 = 4\frac{\zeta_B''}{\zeta_B} - 2\left(\frac{\zeta_B'}{\zeta_B}\right)^2, \tag{6.12}$$

$\mathcal{E}_1$ is defined in (5.9), $\mathcal{E}_2$ in (6.8) becomes

$$\mathcal{E}_2(1) = -\left[\chi_A^{(4)}\chi_A + 4\chi_A'''\chi_A\frac{\zeta_B'}{\zeta_B^2} + 6\chi_A''\chi_A\frac{\zeta_B''}{\zeta_B} + 2(\chi_A^2)'\frac{\zeta_B'''}{\zeta_B}\right], \tag{6.13}$$

and (6.9) now reads

$$\mathcal{E}_3(1) = 4\chi_A''\chi_A - 2(\chi_A')^2 + 2\frac{\zeta_B'}{\zeta_B}(\chi_A^2)'. \tag{6.14}$$

Finally, by (4.13) and (4.14), we obtain the simplified estimate

$$\begin{aligned}\|\chi_A\zeta_B\partial_x^2 v_i\| \lesssim{}& \|\partial_x^2 z_i\|_{L^2}^2 + B^{-1}\|\partial_x z_i\|_{L^2}^2 + B^{-1}\|z_i\|_{L^2}^2\\ &+ (AB)^{-1}(\|\zeta_B v_i\|_{L^2}^2 + \|\zeta_B\partial_x v_i\|_{L^2}^2).\end{aligned}$$

6.3. Start of proof of Proposition 6.2

The proof of this result needs the following computation:

Lemma 6.4. *Let $(v_1, v_2) \in H^1(\mathbb{R}) \times H^2(\mathbb{R})$ be a solution of (5.2). Consider $\phi_{A,B} = \psi_{A,B} = \chi_A^2\varphi_B$. Then*

$$\begin{aligned}\frac{d}{dt}\mathcal{M} ={}& -\frac{1}{2}\int[(\partial_x z_1)^2 + V_0(\partial_x z_2)^2 + 3(\partial_x^2 z_2)^2] + \mathcal{R}_z + \mathcal{R}_v + \mathcal{D}\mathcal{R}_v\\ &+ \frac{1}{2}\int \phi_{A,B}V_0'(\partial_x v_2)^2 + \int \phi_{A,B}[\partial_x G\partial_x v_2 + \widetilde{F}\partial_x v_1]\\ &- \rho'\int \phi_{A,B}'\partial_x v_1\partial_x v_2,\end{aligned} \tag{6.15}$$

where $\mathcal{R}_z(t)$, $\mathcal{R}_v(t)$ and $\mathcal{D}\mathcal{R}_v(t)$ are error terms that, under the constraint (5.6), satisfy the bound

$$\begin{aligned}|\mathcal{R}_z + \mathcal{R}_v + \mathcal{D}\mathcal{R}_v| \lesssim{}& \delta^{1/10}(\|w_1\|_{L^2}^2 + \|\partial_x w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2\\ &+ \|z_1\|_{L^2}^2 + \|z_2\|_{L^2}^2 + \|\partial_x z_2\|_{L^2}^2).\end{aligned} \tag{6.16}$$

Proof. First, we recall that $z_i = \chi_A\zeta_B v_i$, and by (5.15) and Remark 5.1, one gets

$$\int \phi_{A,B}'(\partial_x v_1)^2 = \int(\partial_x z_1)^2 + \int\frac{\zeta_B''}{\zeta_B}z_1^2 + \int \mathcal{E}_1\zeta_B^2 v_1^2 + \int(\chi_A^2)'\varphi_B(\partial_x v_1)^2, \tag{6.17}$$

where $\mathcal{E}_1$ is given by (5.12). For the second term of (6.4), applying (5.8), we obtain

$$\begin{aligned}\int \phi_{A,B}'V_0(\partial_x v_2)^2 ={}& \int V_0(\partial_x z_2)^2 + \int\left[V_0'\frac{\zeta_B'}{\zeta_B} + V_0\frac{\zeta_B''}{\zeta_B}\right]z_2^2\\ &+ \int \mathcal{E}_1(V_0)\zeta_B^2 v_2^2 + \int(\chi_A^2)'\varphi_B V_0(\partial_x v_2)^2,\end{aligned}$$

where following (5.9),

$$\mathcal{E}_1(V_0) = V_0\left[\chi_A''\chi_A + (\chi_A^2)'\frac{\zeta_B'}{\zeta_B}\right] + \frac{1}{2}V_0'(\chi_A^2)'.$$

Now, using (6.11) in Remark 6.2, we get

$$\begin{aligned}\int \phi_{A,B}'(\partial_x^2 v_2)^2 &= \int (\partial_x^2 z_2)^2 + \int P_1(\partial_x z_2)^2 + \int \left[\tilde{R}_1 + P_1'\frac{\zeta_B'}{\zeta_B} + P_1\frac{\zeta_B''}{\zeta_B}\right]z_2^2\\ &\quad + \int \mathcal{E}_2(1)\zeta_B^2 v_2^2 + \int \mathcal{E}_1(P_1)\zeta_B^2 v_2^2 + \int \mathcal{E}_3(1)\zeta_B^2(\partial_x v_2)^2\\ &\quad + \int (\chi_A^2)'\varphi_B(\partial_x^2 v_2)^2, \end{aligned} \tag{6.18}$$

where $\tilde{R}_1, P_1, \mathcal{E}_2(1), \mathcal{E}_3(1)$ are given by (6.12), (6.13), (6.14), and $\mathcal{E}_1$ is given by (5.9).

Now, continuing with the second integral on the right-hand side of (6.4), we have

$$\begin{aligned}\int \phi_{A,B}'''(\partial_x v_2)^2 &= \int \frac{(\zeta_B^2)''}{\zeta_B^2}\chi_A^2\zeta_B^2(\partial_x v_2)^2\\ &\quad + \int \left[6(\chi_A^2)'\frac{\zeta_B'}{\zeta_B} + 3(\chi_A^2)'' + (\chi_A^2)'''\frac{\varphi_B}{\zeta_B^2}\right]\zeta_B^2(\partial_x v_2)^2,\end{aligned}$$

and using Claim 5.2,

$$\begin{aligned}\int \phi_{A,B}'''(\partial_x v_2)^2 &= \int \frac{(\zeta_B^2)''}{\zeta_B^2}(\partial_x z_2)^2 + \int \frac{(\zeta_B^2)''}{\zeta_B^2}\frac{\zeta_B''}{\zeta_B}z_2^2 + \int \left(\frac{(\zeta_B^2)''}{\zeta_B^2}\right)'\frac{\zeta_B'}{\zeta_B}z_2^2\\ &\quad + \int \frac{(\zeta_B^2)''}{\zeta_B^2}\left[\chi_A''\chi_A + (\chi_A^2)'\frac{\zeta_B'}{\zeta_B}\right]\zeta_B^2 v_2^2 + \frac{1}{2}\int \left(\frac{(\zeta_B^2)''}{\zeta_B^2}\right)'(\chi_A^2)'\zeta_B^2 v_2^2\\ &\quad + \int \left[6(\chi_A^2)'\frac{\zeta_B'}{\zeta_B} + 3(\chi_A^2)'' + (\chi_A^2)'''\frac{\varphi_B}{\zeta_B^2}\right]\zeta_B^2(\partial_x v_2)^2. \end{aligned} \tag{6.19}$$

Collecting (6.17), (6.18) and (6.19), we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{M} &= -\frac{1}{2}\int [(\partial_x z_1)^2 + V_0(\partial_x z_2)^2 + 3(\partial_x^2 z_2)^2] + \mathcal{R}_z + \mathcal{R}_v + \mathcal{D}\mathcal{R}_v\\ &\quad + \frac{1}{2}\int \phi_{A,B}V_0'(\partial_x v_2)^2 + \int \phi_{A,B}[\partial_x G\partial_x v_2 + \widetilde{F}\partial_x v_1] - \rho'\int \phi_{A,B}'\partial_x v_1\partial_x v_2,\end{aligned}$$

where the error terms are the following: associated to (z_1, z_2) is

$$\begin{aligned}\mathcal{R}_z &= -\frac{1}{2}\int \frac{\zeta_B''}{\zeta_B}z_1^2 - \frac{1}{2}\int \left[V_0'\frac{\zeta_B'}{\zeta_B} + V_0\frac{\zeta_B''}{\zeta_B}\right]z_2^2 - \frac{3}{2}\int \left[\tilde{R}_1 + P_1'\frac{\zeta_B'}{\zeta_B} + P_1\frac{\zeta_B''}{\zeta_B}\right]z_2^2\\ &\quad + \frac{1}{2}\int \frac{(\zeta_B^2)''}{\zeta_B^2}\frac{\zeta_B''}{\zeta_B}z_2^2 + \frac{1}{2}\int \left[\frac{(\zeta_B^2)''}{\zeta_B^2}\right]'\frac{\zeta_B'}{\zeta_B}z_2^2\\ &\quad + \frac{1}{2}\int \left[\frac{(\zeta_B^2)''}{\zeta_B^2} - 3P_1\right](\partial_x z_2)^2, \end{aligned} \tag{6.20}$$

associated to (v_1, v_2) is

$$\mathcal{R}_v = -\frac{1}{2}\int \mathcal{E}_1(1)\zeta_B^2 v_1^2 - \frac{1}{2}\int \mathcal{E}_1(V_0)\zeta_B^2 v_2^2 + \frac{1}{4}\int \frac{(\zeta_B^2)''}{\zeta_B^2}\left[\chi_A''\chi_A + 2(\chi_A^2)'\frac{\zeta_B'}{\zeta_B}\right]\zeta_B^2 v_2^2$$
$$+ \frac{1}{4}\int \left[\frac{(\zeta_B^2)''}{\zeta_B^2}\right]'(\chi_A^2)'\zeta_B^2 v_2^2 - \frac{3}{2}\int \mathcal{E}_2(1)\zeta_B^2 v_2^2 - \frac{3}{2}\int \mathcal{E}_1(P_1)\zeta_B^2 v_2^2 \tag{6.21}$$

and associated to $(\partial_x v_1, \partial_x v_2)$ is

$$\mathcal{D}\mathcal{R}_v = -\frac{3}{2}\int \mathcal{E}_3(1)\zeta_B^2(\partial_x v_2)^2 - \frac{1}{2}\int (\chi_A^2)'\varphi_B(\partial_x v_1)^2 - \frac{1}{2}\int (\chi_A^2)'\varphi_B V_0(\partial_x v_2)^2$$
$$+ \frac{1}{2}\int \left[6(\chi_A^2)'\frac{\zeta_B'}{\zeta_B} + 3(\chi_A^2)'' + (\chi_A^2)'''\frac{\varphi_B}{\zeta_B^2}\right]\zeta_B^2(\partial_x v_2)^2$$
$$- \frac{3}{2}\int (\chi_A^2)'\varphi_B(\partial_x^2 v_2)^2. \tag{6.22}$$

We have obtained the identity (6.15). To conclude the proof of Lemma 6.4, we must estimate the error terms.

6.4. Controlling error terms

Recall $\mathcal{R}_z(t)$ from (6.20). Decompose

$$\mathcal{R}_z(t) = \mathcal{R}_z^1(t) + \mathcal{R}_z^2(t) + \mathcal{R}_z^3(t),$$

where

$$\mathcal{R}_z^1 = -\frac{1}{2}\int \frac{\zeta_B''}{\zeta_B}[z_1^2 + V_0 z_2^2] - \frac{1}{2}\int V_0'\frac{\zeta_B'}{\zeta_B}z_2^2,$$
$$\mathcal{R}_z^2 = -\frac{3}{2}\int \left[\tilde{R}_1 + P_1'\frac{\zeta_B'}{\zeta_B} + P_1\frac{\zeta_B''}{\zeta_B}\right]z_2^2 + \frac{1}{2}\int \left[\frac{(\zeta_B^2)''}{\zeta_B^2} - 3P_1\right](\partial_x z_2)^2,$$
$$\mathcal{R}_z^3 = \frac{1}{2}\int \left[\frac{(\zeta_B^2)''}{\zeta_B^2}\frac{\zeta_B''}{\zeta_B} + \left(\frac{(\zeta_B^2)''}{\zeta_B^2}\right)'\frac{\zeta_B'}{\zeta_B}\right]z_2^2.$$

In the case of $\mathcal{R}_z^1$, recalling the estimate (4.13), we obtain

$$|\mathcal{R}_z^1| \lesssim B^{-1}(\|z_1\|_{L^2}^2 + \|z_2\|_{L^2}^2). \tag{6.23}$$

As for $\mathcal{R}_z^2$, we recall the form of P_1 (see (6.12)), $\tilde{R}_1$ (see (6.12)), and by (4.13), we conclude that

$$|\mathcal{R}_z^2| \lesssim B^{-1}(\|z_2\|_{L^2}^2 + \|\partial_x z_2\|_{L^2}^2). \tag{6.24}$$

Now we consider the term $\mathcal{R}_z^3$. From (4.14) we obtain

$$|\mathcal{R}_z^3| \lesssim B^{-1}\|z_2\|_{L^2}^2. \tag{6.25}$$

Collecting (6.23), (6.24) and (6.25), and considering (5.6), we finally get

$$|\mathcal{R}_z| \lesssim \delta^{1/10}(\|z_1\|_{L^2}^2 + \|z_2\|_{L^2}^2 + \|\partial_x z_2\|_{L^2}^2). \tag{6.26}$$

Controlling $\mathcal{R}_v$. For the term $\mathcal{R}_v$ given by (6.21), we consider the following decomposition

$$\mathcal{R}_v^1 = -\frac{1}{2}\int \mathcal{E}_1(1)\zeta_B^2 v_1^2 - \frac{1}{2}\int (\mathcal{E}_1(V_0) + 3\mathcal{E}_2(1) + 3\mathcal{E}_1(P_1))\zeta_B^2 v_2^2,$$
$$\mathcal{R}_v^2 = \frac{1}{4}\int \frac{(\zeta_B^2)''}{\zeta_B^2}\left[\chi_A''\chi_A + 2(\chi_A^2)'\frac{\zeta_B'}{\zeta_B}\right]\zeta_B^2 v_2^2 + \frac{1}{4}\int \left(\frac{(\zeta_B^2)''}{\zeta_B^2}\right)'(\chi_A^2)'\zeta_B^2 v_2^2.$$

We note that the terms $\mathcal{E}_1(P_1)$ and $\mathcal{E}_2(1)$ (see (5.9), (6.12) and (6.13)), by (4.14) and (6.10), are bounded and satisfy the estimates

$$|\mathcal{E}_2(1)| \lesssim (AB^3)^{-1} \quad \text{and} \quad |\mathcal{E}_1(P_1)| \lesssim (AB^3)^{-1},$$

and for $\mathcal{E}_1(1)$ in (5.12) and $\mathcal{E}_1(V_0)$ in (5.10) (replacing P by V_0),

$$|\mathcal{E}_1(1)| \lesssim (AB)^{-1}, \quad |\mathcal{E}_1(V_0)| \lesssim (AB)^{-1}.$$

Consequently,

$$|\mathcal{R}_v^1| \lesssim \frac{1}{AB}(\|\zeta_B v_1\|_{L^2}^2 + \|\zeta_B v_2\|_{L^2}^2).$$

Also,

$$|\mathcal{R}_v^2| \lesssim \frac{1}{AB}(\|\zeta_B v_1\|_{L^2}^2 + \|\zeta_B v_2\|_{L^2}^2).$$

Then, applying (5.34) and (5.29), we have

$$\begin{aligned}|\mathcal{R}_v| &\lesssim (AB)^{-1}(\|\zeta_B v_1\|_{L^2}^2 + \|\zeta_B v_2\|_{L^2}^2)\\ &\lesssim (AB)^{-1}\gamma^{-1}(\|w_1\|_{L^2}^2 + \|\partial_x w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2),\end{aligned}$$

By (5.6), we have $(AB)^{-1}\gamma^{-1} = \delta^{7/10}$. Then we get

$$|\mathcal{R}_v| \lesssim \delta^{7/10}(\|w_1\|_{L^2}^2 + \|\partial_x w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2). \tag{6.27}$$

Controlling $\mathcal{D}\mathcal{R}_v$. In the case of the term $\mathcal{D}\mathcal{R}_v$ given by (6.22), first of all we have from (6.14) and (5.35),

$$|\mathcal{E}_3(1)| \lesssim \frac{1}{AB}.$$

Using (5.35) and (4.15) again, we have

$$\begin{aligned}|\mathcal{D}\mathcal{R}_v| \lesssim{}& \int |\mathcal{E}_3(1)|\zeta_B^2(\partial_x v_2)^2 + \int (\chi_A^2)'|\varphi_B|(\partial_x v_1)^2 + \int (\chi_A^2)'|\varphi_B V_0|(\partial_x v_2)^2\\ &+ \int \left|6(\chi_A^2)'\frac{\zeta_B'}{\zeta_B} + 3(\chi_A^2)'' + (\chi_A^2)'''\frac{\varphi_B}{\zeta_B^2}\right|\zeta_B^2(\partial_x v_2)^2 + \int |(\chi_A^2)'\varphi_B|(\partial_x^2 v_2)^2\\ \lesssim{}& BA^{-1}(\|\zeta_A^2\partial_x v_1\|_{L^2}^2 + \|\zeta_A^2\partial_x v_2\|_{L^2}^2 + \|\zeta_A^2\partial_x^2 v_2\|_{L^2}^2).\end{aligned}$$

Now, applying (5.28) and (5.29) with $K = A$, and by (5.6), we get

$$|\mathcal{D}\mathcal{R}_v| \lesssim \delta^{1/10}(\|w_1\|_{L^2}^2 + \|\partial_x w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2). \tag{6.28}$$

And, by (6.26), (6.27) and (6.28), we obtain

$$
\begin{aligned}
&|\mathcal{R}_z + \mathcal{R}_v + \mathcal{D}\mathcal{R}_v| \\
&\quad \lesssim \delta^{1/10}(\|w_1\|_{L^2}^2 + \|\partial_x w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2 + \|z_1\|_{L^2}^2 + \|z_2\|_{L^2}^2 + \|\partial_x z_2\|_{L^2}^2), \quad (6.29)
\end{aligned}
$$

which proves (6.16). This ends the proof of Lemma 6.4. ■

6.5. Controlling nonlinear terms

Recall the second line of error terms in (6.15):

$$
\frac{1}{2}\int \phi_{A,B} V_0'(\partial_x v_2)^2 + \int \phi_{A,B}[\partial_x G \partial_x v_2 + \widetilde{F}\partial_x v_1] - \rho' \int \phi_{A,B}' \partial_x v_1 \partial_x v_2.
$$

In order to control them in a well-ordered fashion, we set

$$
\begin{aligned}
\mathcal{M}_0 &= \frac{1}{2}\int \phi_{A,B} V_0'(\partial_x v_2)^2, & \mathcal{M}_2 &= \int \phi_{A,B}\partial_x G \partial_x v_2, \\
\mathcal{M}_1 &= \int \phi_{A,B}\partial_x F \partial_x v_1, & \mathcal{M}_3 &= -\rho' \int \phi_{A,B}' \partial_x v_1 \partial_x v_2.
\end{aligned} \tag{6.30}
$$

6.5.1. Control of $\mathcal{M}_0$. Noticing that $V_0' = 6HH'$, $|V_0'| \lesssim H'$, $\phi_{A,B} = \chi_A^2 \varphi_B$ and that

$$
|H'(y)\varphi_B(y)| \lesssim \left| y \operatorname{sech}^2\left(\frac{y}{\sqrt{2}}\right)\right| \lesssim e^{-|y|},
$$

we get

$$
\frac{1}{2}\int \phi_{A,B} V_0'(\partial_x v_2)^2 \lesssim \|e^{-|y|/2}\partial_x v_2\|_{L^2}^2.
$$

Following a decomposition similar to that in (5.39), using (5.29), and recalling the values for B and γ in (5.6), one gets

$$
\begin{aligned}
|\mathcal{M}_0| &= \left|\frac{1}{2}\int \phi_{A,B} V_0'(\partial_x v_2)^2\right| \\
&\lesssim e^{-A/2}\|\zeta_B \partial_x v_2\|_{L^2}^2 + \|\partial_x z_2\|_{L^2}^2 + B^{-1}\|z_2\|_{L^2}^2 + (Ae^{A/4})^{-1}\|\zeta_B v_2\|_{L^2}^2 \\
&\lesssim \|\partial_x z_2\|_{L^2}^2 + \delta^{1/10}\|z_2\|_{L^2}^2 + \delta\|w_2\|_{L^2}^2.
\end{aligned} \tag{6.31}
$$

6.5.2. Control of $\mathcal{M}_1$. Recalling $\mathcal{M}_1$ in (6.30) and that $\widetilde{F} = \partial_x F$, with F given by (5.3), we can say the following:

$$
|\mathcal{M}_1| \lesssim \|\chi_A \varphi_B \partial_x v_1\|_{L^2} \|\chi_A (1-\gamma\partial_x^2)^{-1}\partial_x^2[u_1^3 + 3Hu_1^2]\|_{L^2}.
$$

Using (4.15),

$$
|\mathcal{M}_1| \lesssim \|\chi_A \varphi_B \partial_x v_1\|_{L^2} \|\zeta_A^2 (1-\gamma\partial_x^2)^{-1}\partial_x^2[u_1^3 + 3Hu_1^2]\|_{L^2}.
$$

Now, using (5.24) and by (5.45), we obtain

$$\begin{aligned}\|\zeta_A^2(1-\gamma\partial_x^2)^{-1}\partial_x^2[u_1^3+3Hu_1^2]\|_{L^2} &\le \gamma^{-1/2}\|\zeta_A^2\partial_x[u_1^3+3Hu_1^2]\|_{L^2}\\ &\le \gamma^{-1/2}\|u_1\|_{L^\infty}[\|w_1\|_{L^2}+\|\zeta_A\partial_x u_1\|_{L^2}]. \end{aligned}\tag{6.32}$$

Then, by (5.44), (5.28) and the above estimates, we get

$$\begin{aligned}|\mathcal{M}_1| &\lesssim \gamma^{-1/2}B\delta\|\zeta_A^2\partial_x v_1\|_{L^2}[\|w_1\|_{L^2}+\|\zeta_A\partial_x u_1\|_{L^2}]\\ &\lesssim \gamma^{-3/2}B\delta[\|w_1\|_{L^2}^2+\|\partial_x w_1\|_{L^2}^2].\end{aligned}$$

Considering that $\delta\gamma^{-3/2}B=\delta^{3/10}$, by (5.6), we conclude that

$$|\mathcal{M}_1| \lesssim \delta^{3/10}[\|w_1\|_{L^2}^2+\|\partial_x w_1\|_{L^2}^2]. \tag{6.33}$$

6.5.3. Control of $\mathcal{M}_2$. Recall that $\widetilde{G}=\partial_x G$, G given by (5.3). First of all, we have

$$\begin{aligned}\mathcal{M}_2 &= \int \phi_{A,B}\partial_x G\partial_x v_2\\ &= -\gamma\int((\chi_A^2)'\zeta_B^2+\chi_A^2(\zeta_B^2)')\partial_x v_2(1-\gamma\partial_x^2)^{-1}[V_0''\partial_x v_2+2V_0'\partial_x^2 v_2]\\ &\quad -\gamma\int \chi_A^2\varphi_B\partial_x^2 v_2(1-\gamma\partial_x^2)^{-1}[V_0''\partial_x v_2+2V_0'\partial_x^2 v_2]\\ &\quad -\rho'\int \chi_A^2\varphi_B\partial_x v_2(1-\gamma\partial_x^2)^{-1}\partial_x(V_0'u_1)\\ &=: M_{21}+M_{22}+M_{23}.\end{aligned}$$

First, we focus on M_{21}. Using (5.35), (5.13) in Remark 5.1 and (5.29), we have

$$\begin{aligned}&\|((\chi_A^2)'\zeta_B^2+\chi_A^2(\zeta_B^2)')\partial_x v_2\|_{L^2}\\ &\quad = 2\|(\chi_A'\zeta_B+\chi_A\zeta_B')\chi_A\zeta_B\partial_x v_2\|_{L^2}\\ &\quad \lesssim (A^{-1}+B^{-1})\|\chi_A\zeta_B\partial_x v_2\|_{L^2}\\ &\quad \lesssim B^{-1}(\|\partial_x z_2\|_{L^2}+B^{-1/2}\|z_2\|_{L^2}+(AB)^{-1/2}\|\zeta_B v_2\|_{L^2})\\ &\quad \lesssim B^{-1}(\|\partial_x z_2\|_{L^2}+B^{-1/2}\|z_2\|_{L^2}+(AB)^{-1/2}\|w_2\|_{L^2})\\ &\quad \lesssim B^{-1}(\|\partial_x z_2\|_{L^2}+\|z_2\|_{L^2}+\|w_2\|_{L^2}).\end{aligned}$$

Additionally, by (5.40) and (5.29), and $e^{-A/4}\gamma^{-1}\ll 1$,

$$\begin{aligned}\|(1-\gamma\partial_x^2)^{-1}[V_0''\partial_x v_2+2V_0'\partial_x^2 v_2]\|_{L^2} &\lesssim \gamma^{-1/2}(\|\partial_x z_2\|_{L^2}+B^{-1/2}\|z_2\|_{L^2})\\ &\quad + e^{-A/4}\gamma^{-1/2}(\|\zeta_B v_2\|_{L^2}+\|\zeta_B\partial_x v_2\|_{L^2})\\ &\lesssim \gamma^{-1/2}(\|\partial_x z_2\|_{L^2}+\|z_2\|_{L^2})+\|w_2\|_{L^2}.\end{aligned}$$

After applying the Cauchy–Schwarz inequality on M_{21}, we conclude that

$$|M_{21}| \lesssim B^{-1}\gamma^{1/2}[\|\partial_x z_2\|_{L^2}^2+\|z_2\|_{L^2}^2+\|w_2\|_{L^2}^2]. \tag{6.34}$$

Second, for M_{22} we set $\varrho(y) = \operatorname{sech}(y/10)$ and we perform the following separation:

$$
\begin{aligned}
|M_{22}| &\lesssim \left|\gamma \int \chi_A^2 \varphi_B \partial_x^2 v_2 (1-\gamma\partial_x^2)^{-1}[V_0''\partial_x v_2 + 2V_0'\partial_x^2 v_2]\right| \\
&\lesssim \gamma \|\varrho \chi_A^2 \varphi_B \partial_x^2 v_2\|_{L^2} \|\varrho^{-1}(1-\gamma\partial_x^2)^{-1}[V_0''\partial_x v_2 + 2V_0'\partial_x^2 v_2]\|_{L^2}.
\end{aligned}
$$

Since $|\varrho\varphi_B\zeta_B^{-1}| \lesssim B$, one gets

$$
|M_{22}| \lesssim \gamma B \|\chi_A \zeta_B \partial_x^2 v_2\|_{L^2} \|\varrho^{-1}(1-\gamma\partial_x^2)^{-1}[V_0''\partial_x v_2 + 2V_0'\partial_x^2 v_2]\|_{L^2}.
$$

Now we focus on the second term on the right-hand side above. Using (5.23) from Lemma 5.5, we obtain

$$
\begin{aligned}
&\|\varrho^{-1}(1-\gamma\partial_x^2)^{-1}[V_0''\partial_x v_2 + 2V_0'\partial_x^2 v_2]\|_{L^2} \\
&\quad \lesssim \|(1-\gamma\partial_x^2)^{-1}[V_0''\varrho^{-1}\partial_x v_2 + 2V_0'\varrho^{-1}\partial_x^2 v_2]\|_{L^2} \\
&\quad \lesssim \|V_0''\varrho^{-1}\partial_x v_2\|_{L^2} + \|V_0'\varrho^{-1}\partial_x^2 v_2\|_{L^2}.
\end{aligned} \tag{6.35}
$$

Since $|V_0''\varrho^{-1}| \lesssim e^{-|y|}$, following a similar decomposition to (5.39), we obtain

$$
\begin{aligned}
\|e^{-|y|}\partial_x v_2\|_{L^2} &\lesssim \|\partial_x z_2\|_{L^2} + B^{-1/2}\|z_2\|_{L^2} + A^{-1}e^{-A/4}\|\zeta_B v_2\|_{L^2} + e^{-A}\|\zeta_A \partial_x v_2\|_{L^2} \\
&\lesssim \|\partial_x z_2\|_{L^2} + B^{-1/2}\|z_2\|_{L^2} + A^{-1}e^{-A/4}\|w_2\|_{L^2}.
\end{aligned}
$$

The case of the second term on the right-hand side of (6.35) requires more care. First of all, we note again that $V_0'\rho^{-1} \sim e^{-|y|}$ and repeating the same decomposition, we have

$$
\|e^{-|y|}\partial_x^2 v_2\|_{L^2} \le e^{-A/2}\|\zeta_B \partial_x^2 v_2\|_{L^2} + \|e^{-|y|/2}\chi_A \zeta_B \partial_x^2 v_2\|_{L^2}.
$$

Applying Remark 6.2 and Lemma 5.7 (5.29), we have

$$
\begin{aligned}
\|e^{-|y|}\partial_x^2 v_2\|_{L^2} \le{}& \|\partial_x^2 z_2\|_{L^2} + B^{-1/2}\|\partial_x z_2\|_{L^2} + B^{-1/2}\|z_2\|_{L^2} \\
&+ (AB)^{-1/2}e^{-A/4}\gamma^{-1/2}\|w_2\|_{L^2} + e^{-A/2}\gamma^{-1}\|w_2\|_{L^2}.
\end{aligned}
$$

Finally, gathering the previous estimates, for M_{22} we have

$$
\begin{aligned}
|M_{22}| \lesssim{}& \gamma B (\|\partial_x^2 z_2\|_{L^2}^2 + B^{-1}\|\partial_x z_2\|_{L^2}^2 + B^{-1}\|z_2\|_{L^2}^2 \\
&+ (AB)^{-1}\gamma^{-1/2}\|w_2\|_{L^2}^2).
\end{aligned} \tag{6.36}
$$

Third, we treat the term M_{23}. By the Hölder inequality and Lemma 5.4 (i), we get

$$
\begin{aligned}
|M_{23}| &= |\rho'| \left| \int \chi_A \varphi_B \partial_x v_2 (1-\gamma\partial_x^2)^{-1}\partial_x(V_0' u_1) \right| \\
&\lesssim |\rho'| \|\chi_A \varphi_B \partial_x v_2\|_{L^2} \|(1-\gamma\partial_x^2)^{-1}\partial_x(V_0' u_1)\|_{L^2} \\
&\lesssim |\rho'| \|\chi_A \varphi_B \partial_x v_2\|_{L^2} \|\partial_x(V_0' u_1)\|_{L^2}.
\end{aligned}
$$

Expanding the derivatives, by (5.35) one gets

$$\begin{aligned}|M_{23}| &\lesssim |\rho'|\|\chi_A\varphi_B\partial_x v_2\|_{L^2}\|V_0''u_1 + V_0'\partial_x u_1\|_{L^2}\\ &\lesssim B|\rho'|\|\partial_x v_2\|_{L^2}\|V_0''u_1 + V_0'\partial_x u_1\|_{L^2},\end{aligned}$$

and by (5.27), we obtain

$$|M_{23}| \lesssim B\gamma^{-1/2}|\rho'|\|u_2\|_{L^2}\|V_0''u_1 + V_0'\partial_x u_1\|_{L^2}.$$

Then having in mind (4.17) and (4.3), using (5.33) (since $|V_0'|, |V_0''| \sim e^{-\sqrt{2}|x|}$), we obtain

$$|M_{23}| \lesssim B\gamma^{-1/2}\delta|\rho'|(\|w_1\|_{L^2} + \|\partial_x w_1\|_{L^2}). \tag{6.37}$$

Collecting (6.34), (6.36) and (6.37), one obtains

$$\begin{aligned}|\mathcal{M}_2| &\lesssim B^{-1}\gamma^{1/2}[\|\partial_x z_2\|_{L^2}^2 + \|z_2\|_{L^2}^2 + \|w_2\|_{L^2}^2]\\ &\quad + \gamma B[\|\partial_x^2 z_2\|_{L^2}^2 + B^{-1}\|\partial_x z_2\|_{L^2}^2 + B^{-1}\|z_2\|_{L^2}^2]\\ &\quad + B\gamma^{-1/2}\delta|\rho'|[\|w_1\|_{L^2} + \|\partial_x w_1\|_{L^2}].\end{aligned}$$

Using (5.6) and the Cauchy–Schwarz inequality, we conclude that

$$\begin{aligned}|\mathcal{M}_2| &\lesssim \delta^{3/10}[\|\partial_x^2 z_2\|_{L^2}^2 + \|\partial_x z_2\|_{L^2}^2 + \|z_2\|_{L^2}^2]\\ &\quad + \delta^{7/10}[\|w_1\|_{L^2}^2 + \|\partial_x w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2 + |\rho'|^2].\end{aligned} \tag{6.38}$$

6.5.4. Control of $\mathcal{M}_3$. Replacing (5.15), we get

$$\begin{aligned}\mathcal{M}_3 &= -\rho'\int(\chi_A^2\zeta_B^2 + (\chi_A^2)'\varphi_B)\partial_x v_1\partial_x v_2\\ &= -\rho'\int\chi_A^2\zeta_B^2\partial_x v_1\partial_x v_2 - 2\rho'\int\chi_A'\varphi_B\chi_A\partial_x v_1\partial_x v_2 =: M_{31} + M_{32}.\end{aligned}$$

Control of M_{31}. By the Hölder inequality, recalling that $v_1 = (1-\gamma\partial_x^2)^{-1}\mathcal{L}u_1$ (see (5.1)), and using (5.26), we get

$$\begin{aligned}|M_{31}| &\le |\rho'|\|\chi_A\zeta_B\partial_x v_1\|_{L^2}\|\chi_A\zeta_B\partial_x v_2\|_{L^2}\\ &\lesssim \gamma^{-1}|\rho'|\|u_1\|_{H^1}\|\chi_A\zeta_B\partial_x v_2\|_{L^2}.\end{aligned}$$

From (4.3),

$$|M_{31}| \lesssim \gamma^{-1}\delta|\rho'|\|\chi_A\zeta_B\partial_x v_2\|_{L^2}. \tag{6.39}$$

Control of M_{32}. Applying the Hölder inequality, we get

$$|M_{32}| \lesssim BA^{-1}|\rho'|\|\partial_x v_2\|_{L^2(A<|y|<2A)}\|\partial_x v_1\|_{L^2}.$$

Using (5.26) and (4.15), we obtain

$$|M_{32}| \lesssim \gamma^{-1}BA^{-1}|\rho'|\|u_1\|_{H^1}\|\zeta_A^2\partial_x v_2\|_{L^2},$$

and

$$|M_{32}| \lesssim \gamma^{-1}\delta BA^{-1}|\rho'|\|\zeta_A^2\partial_x v_2\|_{L^2}. \tag{6.40}$$

Collecting (6.39) and (6.40), we get

$$|\mathcal{M}_3| \lesssim \gamma^{-1}\delta|\rho'|[\|\chi_A\zeta_B\partial_x v_2\|_{L^2} + BA^{-1}\|\zeta_A^2\partial_x v_2\|_{L^2}].$$

Then, by (5.13), (5.29) and (5.6), we obtain

$$\begin{aligned}|\mathcal{M}_3| &\lesssim \gamma^{-1}\delta|\rho'|[\|\partial_x z_2\|_{L^2} + B^{-1/2}\|z_2\|_{L^2} + (AB)^{-1/2}\|w_2\|_{L^2} \\ &\quad + BA^{-1}\gamma^{-1/2}\|w_2\|_{L^2}] \\ &\lesssim \delta^{3/5}|\rho'|[\|\partial_x z_2\|_{L^2} + \delta^{1/20}\|z_2\|_{L^2} + \delta^{11/20}\|w_2\|_{L^2}].\end{aligned}$$

Hence, using the Cauchy–Schwarz inequality, we conclude that

$$|\mathcal{M}_3| \lesssim \delta^{3/5}[|\rho'|^2 + \|\partial_x z_2\|_{L^2}^2 + \delta^{1/10}\|z_2\|_{L^2}^2 + \delta^{11/10}\|w_2\|_{L^2}^2]. \tag{6.41}$$

6.6. End of proof of Proposition 6.2

Gathering (6.29), (6.31), (6.33), (6.38) and (6.41), we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{M} &\leq -\frac{1}{2}\int[(\partial_x z_1)^2 + V_0(\partial_x z_2)^2 + 3(\partial_x^2 z_2)^2] + C_3\delta^{3/10}\|\partial_x^2 z_2\|_{L^2}^2 \\ &\quad + C_3\delta^{1/10}(\|z_1\|_{L^2}^2 + \|z_2\|_{L^2}^2) + C_3\|\partial_x z_2\|_{L^2}^2 \\ &\quad + C_3\delta^{1/10}(\|w_1\|_{L^2}^2 + \|\partial_x w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2) + \delta^{7/10}|\rho'|^2.\end{aligned}$$

Finally, for δ small enough, we conclude that for some $C_3 > 0$ fixed,

$$\begin{aligned}\frac{d}{dt}\mathcal{M} &\leq -\frac{1}{2}\int[(\partial_x z_1)^2 + 2(\partial_x^2 z_2)^2] + C_3(\|z_1\|_{L^2}^2 + \|z_2\|_{L^2}^2 + \|\partial_x z_2\|_{L^2}^2) \\ &\quad + C_3\delta^{1/10}(\|w_1\|_{L^2}^2 + \|\partial_x w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2) + C_3\delta^{7/10}|\rho'|^2.\end{aligned}$$

This proves (6.5).

7. Coercivity estimates

Before starting the proof of Theorem 1.2, we need coercivity results to deal with the terms

$$\int \varphi_A(3H + u_1)H'u_1^2, \quad \delta|\rho'|^2, \quad \rho'\int\varphi_A H'u_2, \tag{7.1}$$

which appear in the virial estimates of $\mathcal{I}(t)$ (see (4.19)). We will decompose this term in terms of the variables (w_1, w_2) and (z_1, z_2). The last ones involve the variables (v_1, v_2); then we should be able to reconstruct the operator $\mathcal{L}$ from our computations.

7.1. First coercivity estimate

The key element of the proof of Theorem 1.2 is the following transfer estimate.

Lemma 7.1. *Let u_1 be in H^1, (w_1, w_2) be as in (4.17) and (z_1, z_2) as in (5.5). Then*

$$\begin{aligned}\left|\int \varphi_A(3H + u_1)H' u_1^2\right| &\lesssim \delta^{1/20}(\|w_1\|_{L^2}^2 + \|\partial_x w_1\|_{L^2}^2) \\ &\quad + \frac{1}{\delta^{1/20}}\|z_1\|_{L^2}^2 + \delta^{2/5}\|\partial_x z_1\|_{L^2}^2. \end{aligned} \tag{7.2}$$

Proof. First, we observe that $\varphi_A(y) \lesssim |y|$, and

$$|\varphi_A(3H + u_1)H'| \lesssim |y| \operatorname{sech}^2\Big(\frac{y}{\sqrt{2}}\Big) \lesssim \operatorname{sech}^2\Big(\frac{y}{2}\Big). \tag{7.3}$$

Set $\frac{2}{B} < \ell < \min\{\frac{1}{2}, \frac{1}{4}\sqrt{\lambda}\} \le \frac{1}{2}$, where λ is such that the coercivity on Lemma 2.4 holds. We note that

$$\int \operatorname{sech}^2\Big(\frac{y}{2}\Big)u_1^2 \lesssim \int \operatorname{sech}^2(\ell y)u_1^2.$$

Now, we focus on the term on the right-hand side of the last inequality. Applying Lemma 2.4 with $\phi_L = \operatorname{sech}^2(\ell y)$, and using that $|\phi'| \le C\ell\phi$ and that $\langle u_1, H'\rangle = 0$, we obtain for some $\lambda > 0$,

$$\begin{aligned}\int \operatorname{sech}^2(\ell y)u_1^2 &\le \int \operatorname{sech}^2(\ell y)[u_1^2 + (\partial_x u_1)^2] \\ &\le \frac{1}{\lambda}\int \operatorname{sech}^2(\ell y)[(\partial_x u_1)^2 + V_0 u_1^2].\end{aligned}$$

Now, integrating by parts, one gets

$$\int \operatorname{sech}(\ell y)(\partial_x u_1)^2 = -\int \operatorname{sech}^2(\ell y)u_1\partial_x^2 u_1 + \frac{1}{2}\int (\operatorname{sech}^2(\ell y))'' u_1^2.$$

Using that

$$|(\operatorname{sech}^2(\ell y))''| \le \ell^2 \operatorname{sech}^2(\ell y),$$

and choosing ℓ small enough ($0 < \ell \le \frac{\sqrt{\lambda}}{4}$), one gets

$$\int \operatorname{sech}^2(\ell y)u_1^2 \lesssim \int \operatorname{sech}^2(\ell y)\mathcal{L}(u_1)u_1.$$

Now, using the definition of v_1, we obtain

$$\int \operatorname{sech}^2(\ell y)\mathcal{L}(u_1)u_1 = \int \operatorname{sech}^2(\ell y)u_1 v_1 - \gamma\int \operatorname{sech}^2(\ell y)u_1\partial_x^2 v_1. \tag{7.4}$$

For the first integral on the right-hand side of (7.4), using the definitions of z_1 and w_1, one can see that

$$\begin{aligned}&\int \operatorname{sech}^2(\ell y)u_1 v_1 \\ &\quad = \int \chi_A^3 \operatorname{sech}^2(\ell y)u_1 v_1 + \int (1 - \chi_A^3)\operatorname{sech}^2(\ell y)u_1 v_1\end{aligned}$$

$$
\begin{aligned}
&= \int \chi_A^2 \operatorname{sech}^2(\ell y)(\zeta_A\zeta_B)^{-1} w_1 z_1 + \int (1-\chi_A^3)\operatorname{sech}^2(\ell y)\zeta_A^{-2} w_1(\zeta_A v_1)\\
&\lesssim \max_{|y|<2A}\{\operatorname{sech}^2(\ell y)(\zeta_A\zeta_B)^{-1}\}\|w_1\|_{L^2}\|z_1\|_{L^2}\\
&\quad + \max_{|y|>A}\{\operatorname{sech}^2(\ell y)\zeta_A^{-2}\}\|w_1\|_{L^2}\|\zeta_A v_1\|_{L^2}\\
&\lesssim \max_{|y|<2A}\{\operatorname{sech}^2(\ell y)(\zeta_A\zeta_B)^{-1}\}\|w_1\|_{L^2}\|z_1\|_{L^2} + \gamma^{-1}\max_{|y|>A}\{\operatorname{sech}^2(\ell y)\zeta_A^{-2}\}\|w_1\|_{L^2}^2\\
&\lesssim \varepsilon\|w_1\|_{L^2}^2 + \varepsilon^{-1}\|z_1\|_{L^2}^2 + \gamma^{-1}e^{-\frac{A}{4B}}\|w_1\|_{L^2}^2, \qquad (7.5)
\end{aligned}
$$

where ε is a positive number to be chosen later. Note that the last inequality holds if $2B^{-1} < \ell$. Now, for the second integral on the right-hand side of (7.4), integrating by parts we obtain the expression

$$
\begin{aligned}
&\int \partial_x[\operatorname{sech}^2(\ell y)u_1]\partial_x v_1\\
&\quad= \int [(\operatorname{sech}^2(\ell y))' u_1 + \operatorname{sech}^2(\ell y)\partial_x u_1]\partial_x v_1\\
&\quad= \int (\operatorname{sech}^2(\ell y))'\chi_A^2 u_1\partial_x v_1 + \int (1-\chi_A^2)(\operatorname{sech}^2(\ell y))' u_1\partial_x v_1\\
&\qquad + \int \operatorname{sech}^2(\ell y)\chi_A^2\partial_x u_1\partial_x v_1 + \int (1-\chi_A^2)\operatorname{sech}^2(\ell y)\partial_x u_1\partial_x v_1\\
&\quad=: \ell_1+\ell_2+\ell_3+\ell_4. \qquad (7.6)
\end{aligned}
$$

We treat each term ℓ_i in (7.6), starting with ℓ_1. Using the following decomposition and by the Hölder inequality, we get

$$
\begin{aligned}
&\left|\int (\operatorname{sech}^2(\ell y))'\chi_A^2 u_1\partial_x v_1\right|\\
&\quad\lesssim \left|\int (\operatorname{sech}^2(\ell y))'\chi_A^4 u_1\partial_x v_1\right| + \left|\int (\operatorname{sech}^2(\ell y))'(1-\chi_A^2)\chi_A^2 u_1\partial_x v_1\right|\\
&\quad\lesssim \left|\int (\operatorname{sech}^2(\ell y))'\chi_A^4 u_1\partial_x v_1\right| + \left|\int (\operatorname{sech}^2(\ell y))'\zeta_A^{-2}(1-\chi_A^2)w_1(\zeta_A\partial_x v_1)\right|\\
&\quad\lesssim \ell\|\chi_A u_1\|_{L^2}\|\zeta_B\chi_A^2\partial_x v_1\|_{L^2} + \ell\max_{|x|>A}\{\operatorname{sech}^2(\ell y)\zeta_A^{-2}\}\|w_1\|_{L^2}\|\zeta_A\partial_x v_1\|_{L^2}.
\end{aligned}
$$

Furthermore, by the definition of z_1, we can check

$$
\chi_A^2\zeta_B\partial_x v_1 = \chi_A\partial_x z_1 - \chi_A\frac{\zeta_B'}{\zeta_B}z_1 - \chi_A' z_1, \qquad (7.7)
$$

and by Lemma 5.7 (5.28) and Remark 4.15, we obtain

$$
\begin{aligned}
|\ell_1| &= \left|\int (\operatorname{sech}^2(\ell y))'\chi_A^2 u_1\partial_x v_1\right|\\
&\lesssim \ell\|w_1\|_{L^2}(\|\partial_x z_1\|_{L^2} + B^{-1}\|z_1\|_{L^2})\\
&\quad + \ell\gamma^{-1}\max_{|y|>A}\{\operatorname{sech}^2(\ell y)\zeta_A^{-2}\}(\|\partial_x w_1\|_{L^2}^2 + \|w_1\|_{L^2}^2). \qquad (7.8)
\end{aligned}
$$

In similar way, we obtain for ℓ_3,

$$\begin{aligned}&\left|\int \operatorname{sech}^2(\ell y)\chi_A^2\partial_x u_1\partial_x v_1\right|\\&\quad\lesssim \left|\int \operatorname{sech}^2(\ell y)\chi_A^4\partial_x u_1\partial_x v_1\right| + \left|\int \operatorname{sech}^2(\ell y)(1-\chi_A^2)\chi_A^2\partial_x u_1\partial_x v_1\right|\\&\quad\lesssim \|\chi_A\partial_x u_1\|_{L^2}\|\zeta_B\chi_A^2\partial_x v_1\|_{L^2} + \max_{|y|>A}\{\operatorname{sech}^2(\ell y)\zeta_A^{-2}\}\|\zeta_A\partial_x u_1\|_{L^2}\|\zeta_A\partial_x v_1\|_{L^2}.\end{aligned}$$

By (7.7), Lemma 5.7 and Remark 4.15, we get

$$\begin{aligned}\left|\int \operatorname{sech}^2(\ell y)\partial_x u_1\partial_x v_1\right| &\lesssim \|\zeta_A\partial_x u_1\|_{L^2}(\|\partial_x z_1\|_{L^2}+\|z_1\|_{L^2})\\&\quad+\gamma^{-1}\max_{|y|>A}\{\operatorname{sech}^2(\ell y)\zeta_A^{-2}\}\|\zeta_A\partial_x u_1\|_{L^2}^2.\end{aligned}$$

We conclude using (5.5) with $K = A$. We obtain

$$\begin{aligned}\left|\int \operatorname{sech}^2(\ell y)\partial_x u_1\partial_x v_1\right| &\lesssim (\|w_1\|_{L^2}+\|\partial_x w_1\|_{L^2})(\|\partial_x z_1\|_{L^2}+\|z_1\|_{L^2})\\&\quad+\gamma^{-1}\max_{|y|>A}\{\operatorname{sech}^2(\ell y)\zeta_A^{-2}\}(\|w_1\|_{L^2}^2+\|\partial_x w_1\|_{L^2}^2).\end{aligned}\tag{7.9}$$

Collecting (7.5), (7.8), (7.9) and by the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}\int \operatorname{sech}(y)u_1^2 &\lesssim \varepsilon\|w_1\|_{L^2} + \frac{1}{\varepsilon}\|z_1\|_{L^2}\\&\quad+\gamma(\|w_1\|_{L^2}^2+\|\partial_x w_1\|_{L^2}^2)+\gamma(\|\partial_x z_1\|_{L^2}^2+B^{-2}\|z_1\|_{L^2}^2)\\&\lesssim \max\{\varepsilon, A^{-1},\gamma\}\|w_1\|_{L^2}+\gamma\|\partial_x w_1\|_{L^2}^2\\&\quad+\max\{\varepsilon^{-1},B^{-2}\}\|z_1\|_{L^2}^2+\gamma\|\partial_x z_1\|_{L^2}^2.\end{aligned}$$

Finally, by (5.6) and choosing $\varepsilon = \delta^{1/20}$, we conclude that

$$\begin{aligned}\int \operatorname{sech}(y)u_1^2 &\lesssim B^{-1/2}(\|w_1\|_{L^2}^2+\|\partial_x w_1\|_{L^2}^2)+B^{1/2}\|z_1\|_{L^2}^2+\gamma\|\partial_x z_1\|_{L^2}^2\\&\lesssim \delta^{1/20}(\|w_1\|_{L^2}^2+\|\partial_x w_1\|_{L^2}^2)+\frac{1}{\delta^{1/20}}\|z_1\|_{L^2}^2+\delta^{2/5}\|\partial_x z_1\|_{L^2}^2.\end{aligned}\tag{7.10}$$

This ends the proof of Lemma 7.1. ∎

7.2. Second coercivity estimate

Now we consider the second term in (7.1). Recall the estimate for the shift ρ obtained in (4.6), where

$$|\rho'|^2 \lesssim \int e^{-\sqrt{2}|y|}u_2^2.$$

In order to manage this term, we will decompose it in localized terms for w_2 and z_2.

Lemma 7.2. *Let u_2 be in L^2, (w_1, w_2) be as in (4.17), and (z_1, z_2) as in (5.5). Then*

$$\int e^{-\sqrt{2}|y|}u_2^2 \lesssim \delta^{1/20}\|w_2\|_{L^2}^2 + \frac{1}{\delta^{1/20}}\|z_2\|_{L^2}^2 + \delta^{2/5}\|\partial_x^2 z_2\|_{L^2}^2 + \delta^{3/5}\|\partial_x z_2\|_{L^2}^2. \quad (7.11)$$

Proof. First, we consider the decomposition

$$\int e^{-\sqrt{2}|y|}u_2^2 = \int \chi_A^3 e^{-\sqrt{2}|y|}u_2^2 + \int (1-\chi_A^3)e^{-\sqrt{2}|y|}u_2^2. \quad (7.12)$$

For the second integral on the right-hand side it holds that

$$\int (1-\chi_A^3)e^{-\sqrt{2}|y|}u_2^2 \lesssim \sup_{|y|>A}\{e^{-\sqrt{2}|y|}\zeta_A^{-2}\}\int w_2^2 \ll \gamma\|w_2\|_{L^2}^2. \quad (7.13)$$

Now, we focus on the first integral on the right-hand side of (7.12). We observe that

$$\chi_A^2\zeta_B\partial_x v_2 = \chi_A\partial_x z_2 - (\chi_A\zeta_B)'\zeta_B^{-1}z_2,$$

and

$$\begin{aligned}\chi_A^3\zeta_B\partial_x^2 v_2 &= \chi_A^2\partial_x^2 z_2 - (\chi_A\zeta_B)''\zeta_B^{-1}\chi_A z_2 - 2(\chi_A\zeta_B)'\zeta_B^{-1}\chi_A^2\zeta_B\partial_x v_2\\ &= \chi_A^2\partial_x^2 z_2 - (\chi_A\zeta_B)''\zeta_B^{-1}\chi_A z_2\\ &\quad - 2(\chi_A\zeta_B)'\zeta_B^{-1}(\chi_A\partial_x z_2 - (\chi_A\zeta_B)'\zeta_B^{-1}z_2). \end{aligned} \quad (7.14)$$

Recalling that $v_2 = (1-\gamma\partial_x^2)^{-1}u_2$ (see (5.1)), (5.5) and using (7.14), we obtain

$$\begin{aligned}\left|\int \chi_A^3 e^{-\sqrt{2}|y|}u_2^2\right| &= \left|\int e^{-\sqrt{2}|y|}(\zeta_A\zeta_B)^{-1}w_2\chi_A^3\zeta_B(1-\gamma\partial_x^2)v_2\right|\\ &\lesssim \int |w_2|\chi_A^3\zeta_B|(1-\gamma\partial_x^2)v_2|\\ &\lesssim \|w_2\|_{L^2}\|z_2\|_{L^2} + \gamma\|w_2\|_{L^2}\|\chi_A^3\zeta_B\partial_x^2 v_2\|_{L^2}\\ &\lesssim \|w_2\|_{L^2}\|z_2\|_{L^2} + \gamma\|w_2\|_{L^2}(\|\partial_x^2 z_2\|_{L^2}\\ &\quad + B^{-1}\|\partial_x z_2\|_{L^2} + B^{-1}\|z_2\|_{L^2}).\end{aligned}$$

Using the Cauchy inequality and (7.13), we get for $\varepsilon > 0$,

$$\int e^{-\sqrt{2}|y|}u_2^2 \lesssim (\varepsilon+\gamma)\|w_2\|_{L^2}^2 + \frac{1}{\varepsilon}\|z_2\|_{L^2}^2 + \gamma\|\partial_x^2 z_2\|_{L^2}^2 + \gamma B^{-1}\|\partial_x z_2\|_{L^2}^2,$$

and, by (5.6), choosing $\varepsilon = \delta^{1/20}$, we get

$$\int e^{-\sqrt{2}|y|}u_2^2 \lesssim \delta^{1/20}\|w_2\|_{L^2}^2 + \frac{1}{\delta^{1/20}}\|z_2\|_{L^2}^2 + \delta^{2/5}\|\partial_x^2 z_2\|_{L^2}^2 + \delta^{1/2}\|\partial_x z_2\|_{L^2}^2,$$

which proves (7.11). ∎

7.3. Third coercivity estimate

Notice that on the variation of the functional $\mathcal{I}$ (see (4.19)), the integral term

$$\Big(\int \varphi_A H' u_2\Big)^2$$

can be treated in a similar way to (7.3) in the above lemma, since

$$|\varphi_A H'| \lesssim \operatorname{sech}^2\Big(\frac{y}{2}\Big),$$

and, by the Hölder inequality, we get

$$\Big(\int \operatorname{sech}^2\Big(\frac{y}{2}\Big)u_2\Big)^2 \lesssim \int \operatorname{sech}^2\Big(\frac{y}{2}\Big)u_2^2.$$

By (7.11),

$$\Big(\int \varphi_A H' u_2\Big)^2 \lesssim \delta^{1/20}\|w_2\|_{L^2}^2 + \frac{1}{\delta^{1/20}}\|z_2\|_{L^2}^2 + \delta^{2/5}\|\partial_x^2 z_2\|_{L^2}^2 + \delta^{1/2}\|\partial_x z_2\|_{L^2}^2.$$

We conclude the following corollary.

Corollary 7.3. *Under the assumptions in Proposition 4.2, (4.18) and (5.6), one has for some $C_1 \geq C_0 > 0$,*

$$\begin{aligned}\frac{d}{dt}\mathcal{I} \leq{}& -\frac{1}{2}\int (w_2^2 + 3(\partial_x w_1)^2 + (2-4C_1\delta)w_1^2)\\ &+ C_1\delta^{1/20}(\|w_1\|_{L^2}^2 + \|\partial_x w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2) + \frac{C_1}{\delta^{1/20}}(\|z_1\|_{L^2}^2 + \|z_2\|_{L^2}^2)\\ &+ C_1\delta^{2/5}(\|\partial_x z_1\|_{L^2}^2 + \|\partial_x z_2\|_{L^2}^2 + \|\partial_x^2 z_2\|_{L^2}^2).\end{aligned} \tag{7.15}$$

Proof. Recalling that $V_0 = 2 - 3\operatorname{sech}^2(y/\sqrt{2})$ (see (1.13)) and calling to mind the definition of w_1 (see (4.17)), we get

$$-\frac{1}{2}\int (V_0 - 4C_0\delta)w_1^2 = -\frac{1}{2}\int (2-4C_0\delta)w_1^2 + \frac{3}{2}\int \operatorname{sech}^2\Big(\frac{y}{\sqrt{2}}\Big)\zeta_A^2 u_1^2.$$

The proof concludes by applying (7.10) to the last term on the right-hand side of the above equality. ∎

7.4. Final coercivity estimate

Lastly, we need the following coercivity property for z_2.

Lemma 7.4. *There exists $m_0 > 0$ such that for any $z \in H^1$ it holds that*

$$\int (2(\partial_x z)^2 + V_0 z^2) \geq m_0\|z\|_{H^1}^2. \tag{7.16}$$

Remark 7.1. Notice that no orthogonality condition is needed in Lemma 7.4. Indeed, the term $2(\partial_x z)^2$ is sufficiently robust to elude the requirement for a very complicated condition to ensure orthogonality on z_2 in Proposition 5.1.

Proof of Lemma 7.4. Define

$$\mathcal{L}_\# := -2\partial_x^2 + V_0 = -2\partial_x^2 + 2 - 3\operatorname{sech}^2\Big(\frac{x}{\sqrt{2}}\Big).$$

Since $\mathcal{L}_\# \geq \mathcal{L}$, we easily have $\mathcal{L}_\# \geq 0$. Let us write

$$\mathcal{L}_\# = \frac{1}{5}(-\partial_x^2 + 1) + \mathcal{L}_{\#\#}, \quad \mathcal{L}_{\#\#} := -\frac{9}{5}(-\partial_x^2 + 1) - 3\operatorname{sech}^2\Big(\frac{x}{\sqrt{2}}\Big).$$

It is not difficult to check that $\mathcal{L}_{\#\#}$ is a classical self-adjoint operator, with first eigenfunction given by

$$\operatorname{sech}^m\Big(\frac{x}{\sqrt{2}}\Big), \quad m = \frac{1}{6}(\sqrt{129} - 3) \sim 1.39,$$

and eigenvalue $\ell = \frac{3}{20}(\sqrt{129} - 11) \sim 0.054 > 0$. Therefore, since $\mathcal{L}_{\#\#} \geq 0$, we conclude (7.16) with $m_0 = \frac{1}{5}$. ∎

From the previous result, we obtain the following corollary.

Corollary 7.5. *Under the hypotheses in Proposition 5.1, the following is satisfied: there exist $C_2 > 0$ and $m_0 > 0$ such that, for all $t \geq 0$,*

$$\begin{aligned}\frac{d}{dt}\mathcal{J} \leq &-\frac{m_0}{4}(\|z_1\|_{L^2}^2 + \|z_2\|_{L^2}^2 + \|\partial_x z_2\|_{L^2}^2)\\ &+ C_2\delta^{1/10}(\|w_1\|_{L^2}^2 + \|\partial_x w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2) + C_2\delta^{6/5}\|\partial_x^2 z_2\|_{L^2}^2.\end{aligned} \tag{7.17}$$

Proof. Direct from Proposition 5.1 and Corollary 7.4. ∎

8. Proof of Theorem 1.2

Now we are ready to conclude the proof of Theorem 1.2. Recall the constants $\delta_i > 0$ for $i = 1, 2, 3$, defined in Propositions 4.2, 5.1 and 6.2.

Proposition 8.1. *There exist $C_4 > 0$ and $0 < \delta_4 \leq \min\{\delta_1, \delta_2, \delta_3\}$ such that for any $0 < \delta \leq \delta_4$, the following is satisfied. Assume that for all $t \geq 0$, (4.1) holds. Let*

$$\mathcal{H} := \mathcal{J} + 16C_2\delta^{1/10}\mathcal{I} + 80C_1C_2C_4\delta^{1/5}\mathcal{M}.$$

Then, under (4.18) and (5.6), one has for all $t \geq 0$,

$$\frac{d}{dt}\mathcal{H} \lesssim -\delta^{1/10}(\|w_1\|_{L^2}^2 + \|\partial_x w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2). \tag{8.1}$$

Proof. First, from (7.15), we obtain for $\delta > 0$ small and some C_1 fixed,

$$\frac{d}{dt}\mathcal{I} \leq -\frac{1}{4}\int (w_2^2 + 3(\partial_x w_1)^2 + w_1^2) + \frac{C_1}{\delta^{1/20}}(\|z_1\|_{L^2}^2 + \|z_2\|_{L^2}^2) \\ + C_1\delta^{2/5}(\|\partial_x z_1\|_{L^2}^2 + \|\partial_x z_2\|_{L^2}^2 + \|\partial_x^2 z_2\|_{L^2}^2). \tag{8.2}$$

From (7.17) we also have

$$\frac{d}{dt}\mathcal{J} \leq -\frac{m_0}{4}(\|z_1\|_{L^2}^2 + \|z_2\|_{L^2}^2 + \|\partial_x z_2\|_{L^2}^2) \\ + C_2\delta^{1/10}(\|w_1\|_{L^2}^2 + \|\partial_x w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2) + C_2\delta^{6/5}\|\partial_x^2 z_2\|_{L^2}^2. \tag{8.3}$$

Gathering (8.2) and (8.3), we conclude that

$$\begin{aligned} &\frac{d}{dt}(\mathcal{J} + 16C_2\delta^{1/10}\mathcal{I}) \\ &\quad\leq -\frac{m_0}{8}(\|z_1\|_{L^2}^2 + \|z_2\|_{L^2}^2 + \|\partial_x z_2\|_{L^2}^2) \\ &\qquad - 3C_2\delta^{1/10}(\|w_1\|_{L^2}^2 + \|\partial_x w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2) \\ &\qquad + 16C_1C_2\delta^{1/2}(\|\partial_x z_1\|_{L^2}^2 + \|\partial_x^2 z_2\|_{L^2}^2) + C_2\delta^{6/5}\|\partial_x^2 z_2\|_{L^2}^2 \\ &\quad\leq -\frac{m_0}{8}(\|z_1\|_{L^2}^2 + \|z_2\|_{L^2}^2 + \|\partial_x z_2\|_{L^2}^2) \\ &\qquad - 3C_2\delta^{1/10}(\|w_1\|_{L^2}^2 + \|\partial_x w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2) \\ &\qquad + 16C_1C_2\delta^{1/2}(\|\partial_x z_1\|_{L^2}^2 + \|\partial_x^2 z_2\|_{L^2}^2) + C_2\delta^{6/5-1/2}\delta^{1/2}\|\partial_x^2 z_2\|_{L^2}^2. \end{aligned} \tag{8.4}$$

On the other hand, from (6.5) and (7.11), we get

$$\frac{d}{dt}\mathcal{M} \leq -\frac{1}{2}\int ((\partial_x z_1)^2 + (\partial_x^2 z_2)^2) + 2C_3(\|z_1\|_{L^2}^2 + \|z_2\|_{L^2}^2 + \|\partial_x z_2\|_{L^2}^2) \\ + 2C_3\delta^{1/10}(\|w_1\|_{L^2}^2 + \|\partial_x w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2).$$

Finally, define

$$\mathcal{H} := \mathcal{J} + 16C_2\delta^{1/10}\mathcal{I} + 40C_1C_2\delta^{1/2}\mathcal{M}. \tag{8.5}$$

We conclude from the last estimate and (8.4),

$$\begin{aligned} \frac{d}{dt}\mathcal{H} \leq &-\frac{m_0}{8}(\|z_1\|_{L^2}^2 + \|z_2\|_{L^2}^2 + \|\partial_x z_2\|_{L^2}^2) \\ &- 3C_2\delta^{1/10}(\|w_1\|_{L^2}^2 + \|\partial_x w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2) \\ &+ 16C_1C_2\delta^{1/2}(\|\partial_x z_1\|_{L^2}^2 + \|\partial_x^2 z_2\|_{L^2}^2) + C_2\delta^{6/5}\|\partial_x^2 z_2\|_{L^2}^2 \\ &- 20C_1C_2\delta^{1/2}(\|\partial_x z_1\|_{L^2}^2 + \|\partial_x^2 z_2\|_{L^2}^2) \\ &+ 80C_1C_2C_3\delta^{1/2}(\|z_1\|_{L^2}^2 + \|z_2\|_{L^2}^2 + \|\partial_x z_2\|_{L^2}^2) \\ &+ 80C_1C_2C_3\delta^{1/10+1/2}(\|w_1\|_{L^2}^2 + \|\partial_x w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2) \end{aligned}$$

$$
\begin{aligned}
&\leq -\frac{m_0}{32}(\|z_1\|_{L^2}^2 + \|z_2\|_{L^2}^2 + \|\partial_x z_2\|_{L^2}^2) \\
&\quad - C_2\delta^{1/10}(\|w_1\|_{L^2}^2 + \|\partial_x w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2) \\
&\quad - C_2\delta^{1/2}(4C_1 - \delta^{7/10})(\|\partial_x z_1\|_{L^2}^2 + \|\partial_x^2 z_2\|_{L^2}^2).
\end{aligned} \tag{8.6}
$$

From (8.6) we obtain (8.1). This ends the proof of the proposition. ∎

We will use now Proposition 8.1 as follows.

Lemma 8.2. *One has*

$$
\int_0^\infty [\|w_2\|_{L^2}^2 + \|\partial_x w_1\|_{L^2}^2 + \|w_1\|_{L^2}^2]\,dt \lesssim \delta^{11/10}.
$$

Proof. Integrating estimate (8.1) on $[0,t]$, we get

$$
\int_0^t [\|w_2\|_{L^2}^2 + \|\partial_x w_1\|_{L^2}^2 + \|w_1\|_{L^2}^2]\,dt \lesssim 2 \sup_{0\leq s\leq t} |\mathcal{H}(s)| \leq 2 \sup_{s\geq 0} |\mathcal{H}(s)|.
$$

Now we estimate the functional $\mathcal{H}(t)$. Indeed, from (8.5),

$$
\begin{aligned}
|\mathcal{H}| &= |\mathcal{J} + 16C_2\delta^{1/10}\mathcal{I} + 80C_1C_2C_4\delta^{1/5}\mathcal{M}| \\
&\lesssim |\mathcal{J}| + \delta^{1/10}|\mathcal{I}| + \delta^{1/5}|\mathcal{M}| \\
&\lesssim \left|\int \psi_{A,B} v_1 v_2\right| + \delta^{1/10}\left|\int_{\mathbb{R}} \varphi_A u_1 u_2\right| + \delta^{1/5}\left|\int \psi_{A,B}\partial_x v_1 \partial_x v_2\right| \\
&\lesssim B\|v_1v_2\|_{L^1} + A\delta^{1/10}\|u_1u_2\|_{L^1} + B\delta^{1/5}\|\partial_x v_1\partial_x v_2\|_{L^1}.
\end{aligned}
$$

Now, from estimates (5.26), (5.27) and (5.32), we conclude that

$$
|\mathcal{H}| \lesssim B\delta^2\gamma^{-1/2} + A\delta^{2+1/10} + B\delta^{1/5}\gamma^{-1}\delta\gamma^{-1/2}\delta.
$$

Finally, using (4.18), (5.6),

$$
|\mathcal{H}| \lesssim \delta^{1+1/10}(\delta^{6/10} + 1 + \delta^{10/10} + \delta^{4/10}) \lesssim \delta^{11/10}.
$$

Passing to the limit as $t \to \infty$, we conclude. ∎

By Lemma 5.7, estimates (5.30)–(5.31), one obtains

$$
\int_0^\infty \left(\int (u_1^2 + (\partial_x u_1)^2 + u_2^2)\,\mathrm{sech}(y)\right) dt \leq \delta. \tag{8.7}
$$

Using the above estimate, we will conclude the proof of Theorem 1.2. Let

$$
\mathcal{K}(t) = \int \mathrm{sech}(y)u_1^2 + \int \mathrm{sech}(y)((1-\gamma\partial_x^2)^{-1}\partial_x u_2)^2 =: \mathcal{K}_1(t) + \mathcal{K}_2(t).
$$

For $\mathcal{K}_1$, using (4.5) and integrating by parts, we have

$$\begin{aligned}\frac{d\mathcal{K}_1}{dt} &= 2\int \operatorname{sech}(y)(u_1\dot{u_1}) - \rho'\int \operatorname{sech}'(y)u_1^2\\ &= 2\int \operatorname{sech}(y)u_1(\partial_x u_2 + \rho' H') - \rho'\int \operatorname{sech}'(y)u_1^2\\ &= -2\int (\operatorname{sech}'(y)u_1 + \operatorname{sech}(y)\partial_x u_1)u_2 + 2\rho'\int \operatorname{sech}(y)H'(y)u_1\\ &\quad - \rho'\int \operatorname{sech}'(y)u_1^2.\end{aligned}$$

Then, applying the Hölder inequality, the Cauchy–Schwarz inequality and (4.6), we get

$$\begin{aligned}\left|\frac{d}{dt}\mathcal{K}_1(t)\right| &\lesssim \int \operatorname{sech}(y)(u_1^2 + (\partial_x u_1)^2 + u_2^2) + \left(\int e^{-\sqrt{2}|y|}u_2^2\right)^{1/2}\left(\int e^{-\sqrt{2}|y|}u_1^2\right)^{1/2}\\ &\lesssim \int \operatorname{sech}(y)(u_1^2 + (\partial_x u_1)^2 + u_2^2).\end{aligned}$$

For $\mathcal{K}_2$, passing to the variables (v_1, v_2) (see (5.1)),

$$\mathcal{K}_2 = \int \operatorname{sech}(y)(\partial_x v_2)^2,$$

and using (5.2), we get

$$\begin{aligned}\frac{d}{dt}\mathcal{K}_2 &= 2\int \operatorname{sech}(y)\partial_x v_2\partial_x^2 v_1 + 2\int \operatorname{sech}(y)\partial_x v_2\partial_x F - \rho'\int \operatorname{sech}'(y)(\partial_x v_2)^2\\ &=: K_{21} + K_{22} + K_{23}.\end{aligned}$$

Integrating by parts in K_{21}, we have

$$K_{21} = -2\int (\operatorname{sech}'(y)\partial_x v_2 + \operatorname{sech}(y)\partial_x^2 v_2)\partial_x v_1.$$

Using (5.1) we obtain

$$\begin{aligned}|K_{21}| &\lesssim \int \operatorname{sech}(y)\big((\partial_x(1-\gamma\partial_x^2)^{-1}u_2)^2 + (\partial_x^2(1-\gamma\partial_x^2)^{-1}u_2)^2\\ &\qquad + (\partial_x(1-\gamma\partial_x^2)^{-1}\mathcal{L}u_1)^2\big)\\ &\lesssim \|\operatorname{sech}^{1/2}(y)(1-\gamma\partial_x^2)^{-1}\partial_x u_2\|_{L^2}^2 + \|\operatorname{sech}^{1/2}(y)(1-\gamma\partial_x^2)^{-1}(\partial_x^2 - 1 + 1)u_2\|_{L^2}^2\\ &\quad + \|\operatorname{sech}^{1/2}(y)(1-\gamma\partial_x^2)^{-1}\partial_x(\mathcal{L}u_1)\|_{L^2}^2.\end{aligned}$$

Using Lemma 5.22, more precisely (5.24) and (5.25), (1.13) and (5.26), we obtain

$$\begin{aligned}|K_{21}| &\lesssim \frac{1}{\gamma^2}\|\operatorname{sech}^{1/2}(y)u_2\|_{L^2}^2 + \|\operatorname{sech}^{1/2}(y)(1-\gamma\partial_x^2)^{-1}(\mathcal{L}\partial_x u_1 + \partial_x V_0 u_1)\|_{L^2}^2\\ &\lesssim \gamma^{-2}\int \operatorname{sech}(y)(u_2^2 + (\partial_x u_1)^2 + u_1^2).\end{aligned}$$

For K_{22}, we use the Cauchy–Schwarz inequality, (5.24) and a similar computation of (6.32); then

$$
\begin{aligned}
|K_{22}| &\lesssim \int \operatorname{sech}(y)[((1-\gamma\partial_x^2)^{-1}\partial_x u_2)^2 + ((1-\gamma\partial_x^2)^{-1}\partial_x((u_1^3+3Hu_1^2)))^2]\\
&\lesssim_\gamma \int \operatorname{sech}(y)[u_2^2+u_1^2+(\partial_x u_1)^2].
\end{aligned}
$$

The term K_{23} is bounded as follows: using (5.1)

$$
\begin{aligned}
|K_{23}| &\lesssim \left(\int e^{-\sqrt{2}|y|}u_2^2\right)^{1/2}\left(\int \operatorname{sech}(y)(\partial_x v_2)^2\right)\\
&\lesssim \gamma^{-2}\int \operatorname{sech}(y)u_2^2 + \gamma^2\left(\int \operatorname{sech}(y)(\partial_x(1-\gamma\partial_x^2)^{-1}u_2)^2\right)^2\\
&\lesssim \gamma^{-2}\int \operatorname{sech}(y)u_2^2 + \left(\int \operatorname{sech}(y)u_2^2\right)^2.
\end{aligned}
$$

Then we conclude that

$$
\left|\frac{d}{dt}\mathcal{K}_2(t)\right| \lesssim_\gamma \int \operatorname{sech}(y)(u_1^2+(\partial_x u_1)^2+u_2^2).
$$

By (8.7), there exists an increasing sequence $t_n \to \infty$ such that

$$
\lim_{n\to\infty}[\mathcal{K}_1(t_n)+\mathcal{K}_2(t_n)] = 0.
$$

For $t \geq 0$, integrating on $[t, t_n]$, and passing to the limit as $n \to \infty$, we obtain

$$
\mathcal{K}(t) \lesssim \int_t^\infty \left[\int \operatorname{sech}(y)(u_1^2+(\partial_x u_1)^2+u_2^2)\right]dt.
$$

By (8.7), we deduce

$$
\lim_{t\to\infty}\mathcal{K}(t) = 0.
$$

By the decomposition of solution (4.2) and the boundedness in H^1 of u_1, this implies (1.17). This ends the proof of Theorem 1.2.

A. Proof of Lemma 2.2

The first step is to understand the generalized null space of this operator. By Coppel [19], we know that the asymptotic behavior of the solutions of fourth-order differential equation

$$
-\partial_x^2\mathcal{L}\phi = 0 \tag{A.1}
$$

are determined by the asymptotic behavior of the solutions

$$
-\partial_x^2\mathcal{L}_0 u = 0, \quad \text{where } \mathcal{L}_0 = -\partial_x^2 + 2.
$$

One can see, analyzing the coefficient matrix of the first-order system associated to the above equation, that the eigenvalues are simple and given by $\pm\sqrt{2}$ and 0 is an eigenvalue of multiplicity two. Then, by [19, Theorem 4 of Chapter 4], the asymptotic behavior of the solutions of (A.1) are, modulo scaling, as $x \to +\infty$,

$$e^{-\sqrt{2}x}, \quad e^{\sqrt{2}x}, \quad 1, \quad x.$$

Using classical techniques of ODEs and the principle of superposition, one obtains that the solution of $-\partial_x^2 \mathcal{L}\phi = 0$ is a linear combination of the linearly independent functions

$$\begin{aligned}
u_0(x) &= H'(x), \quad u_1(x) = H'(x) \int_0^x (H'(y))^{-2}\, dy, \\
u_2(x) &= -2H'(x) \int_0^x (H'(y))^{-2} \int_{-\infty}^y H'(s)\, ds\, dy, \\
u_3(x) &= H'(x) \int_0^x (H'(y))^{-2} \int_{-\infty}^y sH'(s)\, ds\, dy.
\end{aligned}$$

We also have $\lim_{x\to-\infty} u_0(x) = 0$, but the others do not belong to L^2:

$$\begin{aligned}
\lim_{x\to-\infty} u_1(x) = \lim_{x\to-\infty} \frac{\int_0^x (H'(y))^{-2}\, dy}{(H'(x))^{-1}} &= \lim_{x\to-\infty} \frac{(H'(x))^{-2}}{-(H'(x))^{-2} H''(x)} \\
&= \lim_{x\to-\infty} \frac{1}{-H(H^2-1)} = +\infty.
\end{aligned}$$

Also,

$$\begin{aligned}
\lim_{x\to-\infty} u_2(x) &= \lim_{x\to-\infty} -2\frac{\int_0^x (H'(y))^{-2}(H(y)+1)\, dy}{(H'(x))^{-1}} \\
&= \lim_{x\to-\infty} -2\frac{(H'(x))^{-2}(H(x)+1)}{-(H'(x))^{-2}H''(x)} \\
&= \lim_{x\to-\infty} -2\frac{H(x)+1}{-H(H-1)(H+1)} = 1.
\end{aligned}$$

Finally,

$$\begin{aligned}
\lim_{x\to-\infty} u_3(x) &= \lim_{x\to-\infty} \frac{\int_0^x (H'(y))^{-2} \int_{-\infty}^y sH'(s)\, ds\, dy}{(H'(x))^{-1}} \\
&= \lim_{x\to-\infty} \frac{(H'(x))^{-2} \int_{-\infty}^x sH'(s)\, ds}{-(H'(x))^{-2}H''(x)} = \lim_{x\to-\infty} \frac{\int_{-\infty}^x sH'(s)\, ds}{-H''(x)} \\
&= \lim_{x\to-\infty} \frac{xH'(x)}{-H'''(x)} = \lim_{x\to-\infty} \frac{xH'(x)}{-H'(x)(3H^2-1)} \\
&= \lim_{x\to-\infty} \frac{x}{1-3H^2} = -\infty.
\end{aligned}$$

This proves Lemma 2.2.

B. Proof of Lemma 2.5

Replacing $\mathcal{L}$ (see (1.13)) and integrating by parts, we get

$$\langle \eta\mathcal{L}(f), g\rangle = \int \eta[\partial_x f\partial_x g + V_0 fg] + \langle \eta'\partial_x f, g\rangle.$$

In particular, we get

$$\begin{aligned}
\langle \eta\partial_x\mathcal{L}(f), f\rangle &= \int \eta[-\partial_x^3 ff + \partial_x(V_0 f)f] \\
&= \int \eta[\partial_x^2 f\partial_x f + V_0' f^2 + V_0\partial_x ff] + \int \eta'\partial_x^2 ff \\
&= \int \eta\Big(\frac{1}{2}\partial_x[(\partial_x f)^2] + V_0' f^2 + \frac{1}{2}V_0\partial_x[f^2]\Big) - \int \partial_x(\eta' f)\partial_x f \\
&= -\frac{1}{2}\int \eta'[(\partial_x f)^2 + V_0 f^2] + \int \eta V_0' f^2 - \frac{1}{2}\int \eta V_0' f^2 \\
&\quad - \int (\eta'' f + \eta'\partial_x f)\partial_x f \\
&= -\frac{1}{2}\int \eta'[3(\partial_x f)^2 + V_0 f^2] + \frac{1}{2}\int \eta V_0' f^2 + \frac{1}{2}\int \eta''' f^2.
\end{aligned}$$

Lastly, integrating by parts, we have

$$\begin{aligned}
\langle \eta\mathcal{L}(\partial_x f), f\rangle &= \int \eta[-\partial_x^3 ff + V_0\partial_x ff] \\
&= -\frac{1}{2}\int \eta'[3(\partial_x f)^2 + V_0 f^2] - \frac{1}{2}\int \eta V_0' f^2 + \frac{1}{2}\int \eta''' f^2.
\end{aligned}$$

The proof is complete.

C. Local well-posedness

In this section, we will prove that the small perturbations around the static kink are in fact locally well posed on the energy spaces associated, i.e. on $H^1 \times L^2$. It has been proved that the linear part of (2.6) is well posed on $H^1 \times L^2$ (see [53]). Now we will focus on whether the nonlinear equation associated to (2.6) is locally well posed. Notice that (2.6) is equivalent to

$$\partial_t^2 u_1 - 2\partial_x^2 u_1 + \partial_x^4 u_1 - \partial_x^2 F(t,x,u_1) = 0,$$

where $F(t,x,u_1) = 3u_1(H^2-1) + u_1^2(u_1+3H)$, satisfies (2.9). Let the operator Φ be given by

$$\Phi[u](t) = \mathcal{G}(t)u_1^0(x) + \mathcal{K}(t)u_2^0(x) + \int_0^t \mathcal{K}(t-s)\partial_x^2 F(s,x,u)\,ds, \tag{C.1}$$

where

$$\mathcal{G}(t) = \mathcal{F}^{-1} G(t,\xi)\mathcal{F}, \quad \mathcal{K}(t) = \mathcal{F}^{-1} K(t,\xi)\mathcal{F};$$

$\mathcal{F}$ and $\mathcal{F}^{-1}$ represent the Fourier transform and its inverse, respectively. The Fourier multipliers are given by

$$G(t,\xi) = \cos(\omega(\xi)t), \quad K(t,\xi) = \frac{\sin(\omega(\xi)t)}{\omega(\xi)}, \quad \text{and} \quad \omega(\xi) = |\xi|\sqrt{\xi^2+2}.$$

Following the ideas in the proof of [53], we will use a contraction mapping argument and analogous estimates in the linear case.

Remark C.1. One will notice that $\frac{1}{2}\omega(\xi) = \phi(\tilde{\xi}) = |\tilde{\xi}|\sqrt{1+\tilde{\xi}^2}$, where the last one is the same symbol studied by Linares for the good-Boussinesq equation. In fact, letting $v \in L^2$ and considering $\xi = \tau/\sqrt{2}$, we get

$$\begin{aligned} V_1(2t)v(\sqrt{2}x) &= \int e^{i(2t\phi(\xi)+\sqrt{2}x\xi)}\widehat{v(\sqrt{2}x)}(\xi)\,d\xi \\ &= \frac{1}{\sqrt{2}}\int e^{i(2t\phi(\xi)+\sqrt{2}x\xi)}\widehat{v(x)}\Big(\frac{\xi}{\sqrt{2}}\Big)\,d\xi \\ &= \frac{1}{2}\int e^{i(tw(\tau)+x\tau)}\hat{v}\Big(\frac{\tau}{2}\Big)\,d\tau. \end{aligned}$$

Let u be a function such that $\widehat{(u(x))}(\tau) = \widehat{(v(x))}(\tau/2)$; then

$$V_1(2t)v(x) = \frac{1}{2}\int e^{i(tw(\tau)+x\tau)}\widehat{u(y)}(\tau)\,d\tau = \widetilde{V}_1(t)u(y),$$

where $\widetilde{V}_1 = \mathcal{F}^{-1}[e^{iw(\tau)t}\mathcal{F}]$ is related to the Fourier multiplier $\mathcal{G}$. It is analogous for V_2 and the Fourier multiplier $\mathcal{K}$.

The following results have been proved in [52, 53].

Lemma C.1 ([53, Lemma 2.7]). *Let $f \in L^2$ and*

$$V_1(t)f(x) = \int e^{i(t\phi(\xi)+x\xi)}\hat{f}(\xi)\,d\xi,$$

where $\phi(\xi) = |\xi|(1+\xi^2)^{1/2}$. Then

$$\|V_1(t)f\|_{L^2} \le \|f\|_{L^2}$$

and

$$\Big(\int_0^T \|V_1(t)f\|_{L^\infty}^4\,dt\Big)^{1/4} \le c(1+T^{1/4})\|f\|_{L^2}.$$

Lemma C.2 ([53, Lemma 2.8]). *Let $g = h' \in L^2$ and*

$$V_2(t)h'(x) = \int e^{i(t\phi(\xi)+x\xi)}\frac{\operatorname{sgn}(\xi)\hat{h}(\xi)}{(1+\xi^2)^{1/2}}\,d\xi.$$

Then

$$\|V_2(t)h'\|_{L^2} \le c\|h\|_{H^{-1}}$$

and

$$\left(\int_0^T \|V_2(t)h'\|_{L^\infty}^4 \, dt\right)^{1/4} \le c(1+T^{1/4})\|h\|_{H^{-1}}.$$

Lemma C.3 ([53, Lemma 2.8]). *Let $g = p'' \in L^2$ and*

$$V_2(t)p''(x) = \int e^{i(t\phi(\xi)+x\xi)} \frac{\operatorname{sgn}(\xi)\xi\hat{h}(\xi)}{(1+\xi^2)^{1/2}} \, d\xi.$$

Then

$$\|V_2(t)p''\|_{L^2} \le c\|p\|_{L^2}$$

and

$$\left(\int_0^T \|V_2(t)p''\|_{L^\infty}^4 \, dt\right)^{1/4} \le c\|p\|_{L^2}.$$

Finally, we have the following lemma.

Lemma C.4 ([53, Proposition 2.12]). *Let*

$$(-\partial_x^2)^{-1/2}\partial_t V_1(t)f(x) = \int i|\xi|^{-1}\phi(\xi)e^{i(t\phi(\xi)+x\xi)}\hat{f}(\xi)\, d\xi,$$

$$(-\partial_x^2)^{-1/2}\partial_t V_2(t)h'(x) = \int i|\xi|^{-1}e^{i(t\phi(\xi)+x\xi)} \frac{\operatorname{sgn}(\xi)\hat{h}(\xi)}{(1+\xi^2)^{1/2}} \, d\xi$$

and

$$(-\partial_x^2)^{-1/2}\partial_t V_2(t)p''(x) = \int i|\xi|^{-1}e^{i(t\phi(\xi)+x\xi)} \frac{\xi\hat{p}(\xi)}{(1+\xi^2)^{1/2}} \, d\xi.$$

Then

$$\|(-\partial_x^2)^{-1/2}\partial_t V_1(t)f\|_{L^2} \le C\|f\|_{H^1},$$
$$\|(-\partial_x^2)^{-1/2}\partial_t V_2(t)h'\|_{L^2} \le C\|h\|_{L^2}$$

and

$$\|(-\partial_x^2)^{-1/2}\partial_t V_2(t)p''\|_{L^2} \le C\|p\|_{H^1}.$$

Let the operator Φ be given by (C.1) and, similarly to [53],

$$\begin{aligned} Y_T^a = \Big\{u \in C([0,T]\colon H^1(\mathbb{R})) \cap L^4([0,T];L_1^\infty(\mathbb{R})) \;\Big|\; &(-\partial_x^2)^{-1/2}\partial_t u \in C([0,T],L^2(\mathbb{R})),\\ &\sup_{[0,T]}\|(1-\partial_x^2)^{1/2}u(t)\|_{L^2} \le a,\\ &\sup_{[0,T]}\|(-\partial_x^2)^{-1/2}\partial_t u(t)\|_{L^2} \le a\\ &\text{and } |||u|||_1 \le a\Big\}, \end{aligned}$$

where

$$|||u|||_1 = \left(\int_0^T \|u(t)\|_{L^\infty}^4 \, dt\right)^{1/4} + \left(\int_0^T \|\partial_x u(t)\|_{L^\infty}^4 \, dt\right)^{1/4}.$$

Remark C.2. The multipliers $\mathcal{G}$ and $\mathcal{K}$ are directly related to V_1 and V_2, respectively. This means that, considering Remark C.1, Lemmas C.1, C.2, C.3 and C.4 are valid for $\mathcal{G}$ and $\mathcal{K}$, respectively.

First, we will show the following proposition.

Proposition C.5. *Let $u_1^0 \in H^1(\mathbb{R})$ and $u_2^0 = \partial_x \tilde{u}_0^2 \in L^2$, and define $\Phi[u]$ as in (C.1). Then*

$$\Phi\colon Y_T^a \to Y_T^a$$

for some $T = T(\delta)$, where δ such that $\|u_1^0\|_{H^1}, \|u_2^0\|_{L^2} \leq \delta$.

We will need the following technical lemma to manage the nonlinearity of F.

Lemma C.6. *Let F be as in (2.7), then the following inequalities hold:*

$$\begin{aligned} \|F(t,x,u)\|_{L^2} &\lesssim \|u(H^2-1)\|_{L^2} + (\|u\|_{L^\infty}^2 + \|u\|_{L^\infty})\|u\|_{L^2}, \\ \|\partial_x F\|_{L^2} &\lesssim (1+\|u\|_{L^\infty})\|uH'\|_{L^2} + \|\partial_x u(H^2-1)\|_{L^2} \\ &\quad + (\|u\|_{L^\infty}^2 + \|u\|_{L^\infty})\|\partial_x u\|_{L^2} \end{aligned} \tag{C.2}$$

and

$$\begin{aligned} &\|F(t,x,u) - F(t,x,v)\|_{L^2} \\ &\quad \lesssim \|u-v\|_{L^2}(\|u\|_{L^\infty}^2 + \|w\|_{L^\infty}^2 + \|u\|_{L^\infty} + \|v\|_{L^\infty}) \\ &\qquad + \|(u-v)(H^2-1)\|_{L^2}, \\ &\|\partial_x F(s,x,u) - \partial_x F(s,x,v)\|_{L^2} \\ &\quad \lesssim \|\partial_x u - \partial_x v\|(\|u\|_{L^\infty} + \|v\|_{L^\infty} + \|u\|_{L^\infty}^2 + \|v\|_{L^\infty}^2) \\ &\qquad + \|u-v\|_{L^2}(\|u\|_{L^\infty} + \|v\|_{L^\infty} + \|u\|_{L^\infty}^2 + \|v\|_{L^\infty}^2 + \|\partial_x u\|_{L^\infty}^2 \\ &\qquad\qquad + \|\partial_x v\|_{L^\infty}^2) \\ &\qquad + \|(H^2-1)(\partial_x u - \partial_x v)\|_{L^2} + \|6HH'(u-v)\|_{L^2}. \end{aligned} \tag{C.3}$$

Proof. The first inequality follows directly from the definition of F. For the second inequality, we notice that

$$\partial_x F(t,x,u) = 3uH'(2H+u) + \partial_x u(H^2-1) + (3u^2 + 6Hu)\partial_x u.$$

Then we get

$$\begin{aligned} \|\partial_x F(t,x,u)\|_{L_x^2} &\lesssim (1+\|u\|_{L^\infty})\|uH'\|_{L^2} + \|\partial_x u(H^2-1)\|_{L^2} \\ &\quad + (\|u\|_{L^\infty}^2 + \|u\|_{L^\infty})\|\partial_x u\|_{L^2}. \end{aligned}$$

Second, using (2.8), we get

$$\begin{aligned} \|F(t,x,u) - F(t,x,v)\|_{L^2} &\leq 3\|(u-v)(H^2-1)\|_{L^2} \\ &\quad + \|u-v\|_{L^2}(\|u\|_{L^\infty}^2 + \|v\|_{L^\infty}^2 + \|u\|_{L^\infty} + \|v\|_{L^\infty}). \end{aligned}$$

For the last inequality, we notice that

$$
\begin{aligned}
&|\partial_x(F(t,x,u)-F(t,x,v))| \\
&\quad \lesssim |\partial_x u-\partial_x v|\,|(H^2-1)+u^2+v^2+u+v| \\
&\qquad + |u-v|\,|6HH'+u+v+u^2+v^2+(\partial_x u)^2+(\partial_x v)^2|.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\|\partial_x F(t,x,u)-\partial_x F(t,x,v)\|_{L^2} \\
&\quad \le \|\partial_x u-\partial_x v\|(\|u\|_{L^\infty}+\|v\|_{L^\infty}+\|u\|_{L^\infty}^2+\|v\|_{L^\infty}^2) \\
&\qquad + \|u-v\|_{L^2}(\|u\|_{L^\infty}+\|v\|_{L^\infty}+\|u\|_{L^\infty}^2+\|v\|_{L^\infty}^2+\|\partial_x u\|_{L^\infty}^2+\|\partial_x v\|_{L^\infty}^2) \\
&\qquad + \|(H^2-1)(\partial_x u-\partial_x v)\|_{L^2}+\|6HH'(u-v)\|_{L^2}.
\end{aligned}
$$

This concludes the proof. ■

Now we prove Proposition C.5.

Proof of C.5. First, we will focus on the norms $\|\cdot\|_{L_t^\infty L_x^2}$. By Remark C.1, Lemmas C.1 and C.2 and the Hölder inequality, we get

$$
\begin{aligned}
\sup_{[0,T]}\|\Phi[u]\|_{L^2} &\le \sup_{[0,T]}\|\mathcal{G}(t)u_1^0(x)\|_{L^2}+\sup_{[0,T]}\|\mathcal{K}(t)u_2^0(x)\|_{L^2} \\
&\quad + \sup_{[0,T]}\left\|\int_0^t \mathcal{K}(t-s)\partial_x^2 F(s,x,u)\,ds\right\|_{L^2} \\
&\le c\|u_1^0(x)\|_{L^2}+\|\tilde{u}_2^0(x)\|_{H^{-1}}+\int_0^T \|F(s,x,u)\|_{L^2}\,ds.
\end{aligned}
$$

Using (C.2) and the Sobolev embedding, one has

$$
\sup_{[0,T]}\|\Phi[u]\|_{L^2} \le c\|u_1^0(x)\|_{H^1}+\|\tilde{u}_2^0(x)\|_{L^2}+cT\sup_{[0,T]}(\|u\|_{H^1}+\|u\|_{H^1}^3+\|u\|_{H^1}^2).
$$

Similarly, for the term $\partial_x\Phi$, using Lemmas C.1 and C.2 it is obtained that

$$
\sup_{[0,T]}\|\partial_x\Phi[u]\|_{L^2} \le \|\partial_x u_1^0(x)\|_{L^2}+\|u_2^0(x)\|_{L^2}+\int_0^T \|\partial_x F(s,x,u)\,ds\|_{L^2}.
$$

From (C.2) and the Sobolev embedding, we get

$$
\begin{aligned}
\sup_{[0,T]}\|\partial_x\Phi[u]\|_{L^2} &\le c(\|u_1^0(x)\|_{H^1}+\|u_2^0(x)\|_{L^2}) \\
&\quad + cT\sup_{[0,T]}(\|u\|_{H^1}+\|u\|_{H^1}^2+\|u\|_{H^1}^3).
\end{aligned}
$$

Then we arrive at

$$\begin{aligned}\sup_{[0,T]} \|\Phi[u]\|_{H^1} &\leq \sup_{[0,T]} \|\Phi[u]\|_{L^2} + \sup_{[0,T]} \|\partial_x \Phi[u]\|_{L^2} \\ &\leq 2c(\|u_1^0(x)\|_{H^1} + \|u_2^0(x)\|_{L^2}) + 2cT \sup_{[0,T]} (\|u\|_{H^1} + \|u\|_{H^1}^2 + \|u\|_{H^1}^3).\end{aligned}$$

Hence, if

$$a = 8c\delta \quad \text{and} \quad T = \delta^4/4c,$$

we obtain

$$4c\delta + 2cTa(1 + a + a^2) = 4c\delta(1 + 4cT(1 + a + a^2)) < 8c\delta.$$

Now, to estimate $(-\partial_x^2)^{-1/2}\partial_t\Phi[u]$, we will use Lemma C.4 and (C.2); we obtain

$$\begin{aligned}\sup_{[0,T]} \|(-\partial_x^2)^{-1/2}\partial_t\Phi[u]\|_{L^2} &\leq c(\|u_1^0(x)\|_{H^1} + \|u_2^0(x)\|_{L^2}) \\ &\quad + cT \sup_{[0,T]} (\|u\|_{H^1} + \|u\|_{H^1}^2 + \|u\|_{H^1}^3).\end{aligned}$$

As before,

$$2c\delta + acT(1 + a + a^2) = 2c\delta(1 + 2cT(1 + a + a^2)) < 8c\delta.$$

Finally, we will estimate the L^4L^∞ norm. Applying Lemmas C.3, C.2 and C.1 and using (C.2), we get

$$\begin{aligned}\|\Phi[u]\|_{L_T^4L_x^\infty} &\leq \|\mathcal{G}(t)u_1^0(x)\|_{L_T^4L_x^\infty} + \|\mathcal{K}(t)u_2^0(x)\|_{L_T^4L_x^\infty} \\ &\quad + \left\| \int_0^t \mathcal{K}(t-s)\partial_x^2 F(s,x,u)\,ds \right\|_{L_T^4L_x^\infty} \\ &\leq (1 + T^{1/4})\|u_1^0\|_{L^2} + \|\tilde{u}_2^0\|_{L^2} + cT \sup_{[0,T]} (\|u\|_{H^1} + \|u\|_{H^1}^2 + \|u\|_{H^1}^3).\end{aligned}$$

Similarly to the above estimates we obtain

$$\|\partial_x\Phi[u]\|_{L_T^4L_x^\infty} \leq (1 + T^{1/4})(\|u_1^0\|_{L^2} + \|\tilde{u}_2^0\|_{L^2}) + cT \sup_{[0,T]} (\|u\|_{H^1} + \|u\|_{H^1}^2 + \|u\|_{H^1}^3).$$

Finally,

$$\begin{aligned}|||\Phi[u]|||_1 &= \|\Phi[u]\|_{L_T^4L_x^\infty} + \|\partial_x\Phi[u]\|_{L_T^4L_x^\infty} \\ &\leq 2c\delta(1 + T^{1/4}) + 2cTa(1 + a + a^2),\end{aligned}$$

and we get

$$2c\delta[(1 + T^{1/4}) + 2cTa(1 + a + a^2)] = 2c\delta[(1 + T^{1/4}) + 2cT(1 + a + a^2)] < 8c\delta.$$

This ends the proof of the proposition. ∎

Second, we will prove that Φ is a contraction mapping.

Theorem C.7. *Let $u_1^0 \in H^1(\mathbb{R})$ and $u_2^0 = \partial_x \tilde{u}_0^2 \in L^2$. Then there exist $T = T(\delta)$ and a unique solution of the integral equation (C.1) in $[0,T]$ with*

$$u \in C([0,T]; H^1(\mathbb{R})) \cap L^4([0,T]; L^\infty)$$

and

$$(-\partial_x^2)^{-1/2}\partial_t u \in C([0,T]; L^2(\mathbb{R})).$$

Proof. Proposition C.5 ensures that $\Phi[u]\colon Y_T^a \to Y_T^a$, so we only need to show that Φ is a contraction. First, we notice that

$$(\Phi[u] - \Phi[v])(t) = \int_0^t \mathcal{K}(t-s)\partial_x^2(F(s,x,u) - F(x,s,v))\,ds.$$

Now, we will focus on the $L_t^\infty L_x^2$ norm. By the Hölder inequality and Lemma C.3, we obtain

$$\begin{aligned}
&\sup_{[0,T]} \|\Phi[u] - \Phi[v]\|_{L^2}(t) \\
&\quad \leq C \int_0^T \|u^2 + (u+v)(v+3H)\|_{L^\infty} \|u-v\|_{L^2}\,ds \\
&\qquad + C \int_0^T \| |u-v|\,|3(H^2-1)| \|_{L^2}\,ds \\
&\quad \leq C \sup_{[0,T]} \|u-v\|_{L^2} \bigg(\int_0^T (\|u\|_{L^\infty}^2 + \|v\|_{L^\infty}^2 + \|u\|_{L^\infty} + \|v\|_{L^\infty})\,ds + \int_0^T 1\,ds \bigg).
\end{aligned}$$

Applying the Hölder inequality,

$$\sup_{[0,T]} \|\Phi[u] - \Phi[v]\|_{L^2}(t) \leq C T^{1/2}[2a^2 + 2T^{1/4}a + T^{1/2}] \sup_{[0,T]} \|u-v\|_{L^2}.$$

Then, by (C.3),

$$\begin{aligned}
&\sup_{[0,T]} \|\partial_x \Phi[u] - \partial_x \Phi[v]\|_{L^2}(t) \\
&\quad \leq C \int_0^T \|\partial_x u - \partial_x w\|_{L^2} \|u^2 + v^2 + u + v\|_{L^\infty} \\
&\qquad + C \int_0^T \|u - w\|_{L^2} \|u + v + u^2 + v^2 + (\partial_x u)^2 + (\partial_x v)^2\|_{L^\infty}\,ds \\
&\qquad + C \int_0^T \|(\partial_x u - \partial_x w)(H^2-1)\|_{L^2} + C \int_0^T \|(u-w)6HH'\|_{L^2}\,ds. \qquad \text{(C.4)}
\end{aligned}$$

Hence, by Lemma C.3, we get

$$\begin{aligned}\sup_{[0,T]} &\|\partial_x \Phi[u] - \partial_x \Phi[v]\|_{L^2}(t) \\ &\leq C \sup_{[0,T]} \|\partial_x u - \partial_x w\|_{L^2}[2a^2 T^{1/2} + 2T^{3/4}a + T] \\ &\quad + C \sup_{[0,T]} \|u - w\|_{L^2}[2T^{3/4}a + 4T^{1/2}a^2 + T] \\ &\leq C\Big(\sup_{[0,T]} \|\partial_x u - \partial_x w\|_{L^2} + \sup_{[0,T]} \|u - w\|_{L^2}\Big)[4a^2 T^{1/2} + 2T^{3/4}a + T].\end{aligned}$$

Now, as in (C.4),

$$\begin{aligned}\sup_{[0,T]} &\|(-\partial_x^2)^{-1/2}\partial_t \Phi[u] - (-\partial_x^2)^{-1/2}\partial_t \Phi[v]\|_{L_x^2} \\ &\leq C\Big(\sup_{[0,T]} \|\partial_x u - \partial_x w\|_{L^2} + \sup_{[0,T]} \|u - w\|_{L^2}\Big)[4a^2 T^{1/2} + 2T^{3/4}a + T].\end{aligned}$$

Finally, we will estimate the $L_T^4 L_x^\infty$ norm. Using the Hölder inequality, we get

$$\|\Phi[u] - \Phi[v]\|_{L_t^4 L_x^\infty} \leq C T^{1/2}[2a^2 + 2T^{1/4}a + T^{1/2}] \sup_{[0,T]} \|u - v\|_{L^2}.$$

Furthermore,

$$\begin{aligned}&\|\partial_x \Phi[u] - \partial_x \Phi[v]\|_{L_T^4 L_x^\infty} \\ &\leq C\Big(\sup_{[0,T]} \|\partial_x u - \partial_x w\|_{L^2} + \sup_{[0,T]} \|u - w\|_{L^2}\Big)[4a^2 T^{1/2} + 2T^{3/4}a + T].\end{aligned}$$

Then, for $T = \delta^4/4c < 1$ and $a = 8c\delta$, we get

$$\begin{aligned}4a^2 T^{1/2} + 2T^{3/4}a + T &\leq \delta^4(3(8c)^2 + (8c+1)^2), \\ 2a^2 T^{1/2} + 2T^{3/4}a + T &\leq \delta^4((8c)^2 + (8c+1)^2),\end{aligned}$$

and for δ small enough we obtain that Φ is a contraction. ∎

Acknowledgments. We are grateful to the anonymous referees whose comments have helped us to improve the article. Ch. Ma. thanks Sebastian Herr and Lars Eric Hientzsch for support and interesting comments and suggestions. Both authors thank the Banff Center (Canada), where part of this work was done.

Funding. Ch.Ma. was partially funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 317210226 – SFB 1283, and Chilean research grants FONDECYT 1191412. Cl.Mu. was partially funded by Chilean research grants FONDECYT 1191412, 1231250, and Basal CMM FB210005.

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Received 12 June 2023; revised 1 November 2023; accepted 2 November 2023.

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