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Global infinite energy solutions of the critical semilinear wave equation

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Abstract

We consider the critical semilinear wave equation

$$(NLW)_{2^*-1} \quad \begin{cases} \Box u + |u|^{2^*-2}u = 0 \\ u_{|t=0} = u_0 \\ \partial_t u_{|t=0} = u_1 \ , \end{cases}$$

set in $\mathbb{R}^d$, $d \geq 3$, with $2^* = \frac{2d}{d-2} \cdot$ Shatah and Struwe [22] proved that, for finite energy initial data (ie if $(u_0, u_1) \in \dot{H}^1 \times L^2$), there exists a global solution such that $(u, \partial_t u) \in \mathcal{C}(\mathbb{R}, \dot{H}^1 \times L^2)$. Planchon [17] showed that there also exists a global solution for certain infinite energy initial data, namely, if the norm of (u_0, u_1) in $\dot{B}^1_{2,\infty} \times \dot{B}^0_{2,\infty}$ is small enough. In this article, we build up global solutions of $(NLW)_{2^*-1}$ for arbitrarily big initial data of infinite energy, by using two methods which enable to interpolate between finite and infinite energy initial data: the method of Calderón, and the method of Bourgain. These two methods give complementary results.

1. Introduction

1.1. Wave equations with a power non-linearity

Consider the following Cauchy problem, set in $\mathbb{R}^d$, where $u(t,x)$ is a real-valued function, and we denote $\Box = \partial_t^2 - \Delta$.

$$(NLW)_p \quad \begin{cases} \Box u + |u|^{p-1}u = 0 \\ u_{|t=0} = u_0 \\ \partial_t u_{|t=0} = u_1 \ . \end{cases}$$

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This Cauchy problem has two interesting features. First, it enjoys a scaling invariance: if u is a solution of $(NLW)_p$ for the initial data (u_0, u_1), then

$$\lambda^{\frac{2}{p-1}} u(\lambda x, \lambda t) \tag{1.1}$$

will be a solution for the initial data

$$\big(\lambda^{\frac{2}{p-1}} u_0(\lambda x), \lambda^{\frac{2}{p-1}+1} u_1(\lambda x)\big).$$

Furthermore, denoting by $\|\cdot\|_r$ the norm of the Lebesgue space L^r, the energy

$$E(u,t) \stackrel{\text{def}}{=} \frac{1}{2}\|\nabla u(t,\cdot)\|_2^2 + \frac{1}{2}\|\partial_t u(t,\cdot)\|_2^2 + \frac{1}{p+1}\|u(t,\cdot)\|_{p+1}^{p+1}$$

is (at least formally) conserved by the flow of $(NLW)_p$. The index $p_c = 2^* - 1$, with $2^* \stackrel{\text{def}}{=} \frac{2d}{d-2}$ appears to be critical because for this value of p the scaling transformation (1.1) leaves E invariant. The equation is said to be subcritical for $p < p_c$, critical for $p = p_c$, and supercritical for $p > p_c$.

1.2. Finite energy solutions

Using the formal conservation of energy, Segal [19] (see also Shatah and Struwe [22]) proved for any p the existence of a weak finite energy solution such that $(u, \partial_t u) \in L^\infty(\mathbb{R}, \dot{H}^1 \times L^2)$. Recall that the homogeneous Sobolev spaces $\dot{H}^s$ are given by the norm

$$\|f\|_{\dot{H}^s}^2 \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} |\xi|^{2s} |\widehat{f}(\xi)|^2 \, d\xi \ .$$

The uniqueness of this weak solution is known in the subcritical case (Ginibre and Velo [9]). In the critical case one must add a supplementary condition: we have the following theorem, which goes back to Grillakis [10], but whose proof has been simplified and generalized by Shatah and Struwe [20] [21] [22].

Theorem 1.1 *Let $d \geq 3$, and consider initial data $(u_0, u_1) \in \dot{H}^1 \times L^2$. Then the Cauchy problem $(NLW)_{2^*-1}$ has a unique solution u such that*

$$(u, \partial_t u) \in \mathcal{C}(\mathbb{R}, \dot{H}^1 \times L^2) \cap L^\mu_{\text{loc}}(\mathbb{R}, \dot{B}^{1/2}_{\mu,2} \times \dot{B}^{-1/2}_{\mu,2}) \ ,$$

with $\mu = \frac{2(d+1)}{d-1}$.

1.3. Infinite energy solutions

We refer for this topic to the paper of Lindblad and Sogge [15]. These authors show in particular that in the range $p \geq \widehat{p} \stackrel{\text{def}}{=} \frac{d+3}{d-1}$, the optimal space for the local resolution of $(NLW)_p$ is given by the scaling of the equation. Set

$$s(p) \stackrel{\text{def}}{=} \frac{d}{2} - \frac{2}{p-1} \ ;$$

it is the Sobolev exponent s such that the norm of $\dot{H}^s$ is invariant by the scaling of $(NLW)_p$.

Theorem 1.2 *Let $p \geq \widehat{p}$. Then*

1. *$(NLW)_p$ is not locally well-posed for initial data $(u_0, u_1) \in \dot{H}^s \times \dot{H}^{s-1}$ if $s < s(p)$.*
2. *$(NLW)_p$ is locally well-posed for initial data $(u_0, u_1) \in \dot{H}^s \times \dot{H}^{s-1}$ if $s \geq s(p)$.*

In the critical case $p = 2^* - 1$, $s(p) = 1$; so we cannot hope to obtain infinite energy solutions by taking initial data in Sobolev spaces. We must therefore use Besov spaces, see the appendix for a definition. Planchon [17] built up solutions of $(NLW)_p$ for initial data in Besov spaces; we state his result only in the critical case.

Theorem 1.3

- *If the initial data (u_0, u_1) belong to the closure of the Schwartz class $\mathcal{S}$ in $\dot{B}^1_{2,\infty} \times \dot{B}^0_{2,\infty}$, there exists a local solution of $(NLW)_{2^*-1}$.*
- *If the initial data (u_0, u_1) have a small enough norm in $\dot{B}^1_{2,\infty} \times \dot{B}^0_{2,\infty}$, there exists a global solution of $(NLW)_{2^*-1}$.*

1.4. Interpolation between finite and infinite energy solutions: the subcritical case

Given the existence results stated above, one would like to interpolate between finite and infinite energy solutions in order to obtain the existence of global solutions of $(NLW)_{2^*-1}$ for large infinite energy initial data.

Let us focus here on the case $d = 3$, $p = 3$, which is quite typical (see the article of Kenig, Ponce and Vega [13] for a larger range for p). The equation reads

$$(NLW)_3 \quad \begin{cases} \Box u + u^3 = 0 \\ u_{|t=0} = u_0 \\ \partial_t u_{|t=0} = u_1 \ . \end{cases}$$

The results stated above give for $(u_0,u_1)\in \dot H^1\times L^2$ a finite energy global solution, and for (u_0,u_1) small in $\dot H^{1/2}\times \dot H^{-1/2}$ an infinite energy global solution.

By interpolating between finite and infinite energy solutions, the following theorem can be proved.

Theorem 1.4 *If the initial data* (u_0,u_1) *belongs to* $\dot H^s\times\dot H^{s-1}$, *with* $s>3/4$, *there exists a global solution of* $(NLW)_3$.

This theorem was first proved by Kenig, Ponce and Vega [13] using a method developed by Bourgain [3] to study non-linear Schrödinger equations. To explain very briefly this idea, consider (u_0,u_1) in $\dot H^s\times\dot H^{s-1}$; we can also write this initial data as

$$(u_0,u_1)=(v_0,v_1)+(w_0,w_1) \tag{1.2}$$

with $(v_0,v_1)\in\dot H^1\times L^2$, and (w_0,w_1) small in $\dot H^s\times\dot H^{s-1}$. Let v be the finite energy solution associated to the initial data (v_0,v_1). In order to get a global solution of our problem, we just have to solve the perturbed equation

$$\begin{cases} \Box w+(v+w)^3-v^3=0\\ w_{|t=0}=w_0\\ \partial_t w_{|t=0}=w_1\ , \end{cases} \tag{1.3}$$

and this can be done by using repetitively a fixed point argument.

Theorem 1.4 has been proved by a different method by Gallagher and Planchon [7]. They adapted to this setting an idea which had been introduced by Calderón [4], and applied by the same authors [6], in order to study the Navier-Stokes equation. This method is somewhat dual of Bourgain's one. As a first step, one splits the data as in (1.2). One considers then a solution of $(NLW)_3$ for the initial data (w_0,w_1), so one obtains the perturbed equation

$$\begin{cases} \Box v+(v+w)^3-w^3=0\\ v_{|t=0}=v_0\\ \partial_t v_{|t=0}=v_1\ . \end{cases} \tag{1.4}$$

The idea is then to show that v remains of finite energy and to use energy methods.

1.5. Statement of the theorems

Our aim in this article is to show how the methods of Bourgain and Calderón can be applied to the critical equation $(NLW)_{2^*-1}$, in order to show global existence results for arbitrarily large infinite energy initial data.

To adapt the methods explained above to the critical case, it suffices to replace $\dot{H}^s \times \dot{H}^{s-1}$ by $\dot{B}^1_{2,\infty} \times \dot{B}^0_{2,\infty}$. Interestingly, both methods give complementary results in the critical case.

First, one obtains with the method of Bourgain the following theorem.

Theorem A *Let $d = 3, 4$ or 6. There exist constants $C > 0$ and $\kappa > 0$ such that, for initial data (u_0, u_1) of the form*

$$(u_0, u_1) = (v_0 + w_0, v_1 + w_1)$$

with

$$E = \|v_0\|_{\dot{H}^1} + \|v_1\|_{L^2} < \infty$$

and

$$\|w_0\|_{\dot{B}^1_{2,\infty}} + \|w_1\|_{\dot{B}^0_{2,\infty}} \leq C \exp(-\exp(E^\kappa)) \ ,$$

there exists a global solution u of $(NLW)_{2^*-1}$.

Furthermore, if one denotes

$$(1.5) \quad \mu = \frac{2(d+1)}{d-1}\,, \quad \nu = \frac{2(d+1)}{d+3}\,, \quad \alpha = \frac{\mu}{\nu(2^*-1)} \quad \text{and} \quad \frac{1}{\rho} = \frac{\alpha}{\mu} + \frac{1-\alpha}{2},$$

then

$$u \in X \stackrel{\text{def}}{=} \widetilde{L}^{\mu/\alpha}(\mathbb{R}, \dot{B}^{1-\alpha/2}_{\rho,\infty})$$

(see the appendix for the definition of the spaces $\widetilde{L}^q(\mathbb{R}, \dot{B}^s_{r,k})$). Finally, u is unique in the set

$$(1.6) \qquad \mathcal{E} \stackrel{\text{def}}{=} \{u\,,\, d_X(u, \mathcal{S}) < \epsilon_1\}$$

where $\epsilon_1 > 0$ is a universal constant and

$$d_X(u, \mathcal{S}) = \inf\left\{\|f - u\|_X\,,\ f \in \mathcal{S}\right\}.$$

This theorem is proved in Part 2.

Remark 1.1 The restriction to $d = 3, 4$ or 6 is simply due to the fact that 2^* is an integer for these values of d. We could not prove the product lemma 2.4 for other values of d, but we believe that the theorem is true for any $d \geq 3$.

The method of Calderón gives the following theorem.

Theorem B *Let $d = 6$. There exists $\epsilon > 0$ such that the Cauchy problem $(NLW)_{2^*-1}$ has a global solution u provided the initial data (u_0, u_1) can be written*

$$u_0(x) = v_0(x) + \frac{c_0}{|x|^2} \qquad \text{and} \qquad u_1(x) = v_1(x) + \frac{c_1}{|x|^3} \ ,$$

with $(v_0, v_1) \in \dot{H}^1 \times L^2$, $c_1 < \epsilon$ and $c_2 < \epsilon$.

Furthermore, u is unique in the set

$$\mathcal{E} \stackrel{\text{def}}{=} \left\{ u \, , \, d_X(u, \mathcal{S}) < \epsilon_1 \right\} \tag{1.7}$$

where $\epsilon_1 > 0$, and the notations are those of Theorem A.

This theorem is proved in Part 3.

Remark 1.2

- First, since $d = 6$,

$$x \mapsto \frac{1}{|x|^2} \in \dot{B}^1_{2,\infty} \setminus \dot{H}^1 \qquad \text{and} \qquad x \mapsto \frac{1}{|x|^3} \in \dot{B}^0_{2,\infty} \setminus L^2 \ ;$$

  the theorem would otherwise have no interest.

- As opposed to Theorem A, the infinite energy perturbations must here have a precise form; but the bound on the size of these perturbations does not depend any more on v_0 or v_1.

- We believe that the restriction $d = 6$ could be replaced by a bound $d \geq 4$, but we could not do it because of technical problems.

2. Proof of Theorem A

2.1. Strichartz estimates

Strichartz estimates will be an essential tool in the proof of Theorem A.

Let us denote simply by $L^q L^r$ the space $L^q(\mathbb{R}, L^r(\mathbb{R}^d))$, i.e.

$$\|u(t,x)\|_{L^q L^r} = \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^d} |u(t,x)|^r \, dx \right)^{q/r} dt \right)^{1/q} .$$

Strichartz estimates give information on the space time-norm of the solution U of a linear wave equation. This kind of estimates goes back to Strichartz [23], and has been proved in its greatest generality by Ginibre and Velo [9], and Keel and Tao [12].

Theorem 2.1 *Suppose $d \geq 3$, and take $j \in \mathbb{Z}$. If the Fourier transforms of $U_0(x)$ and $U_1(x)$ are supported in the annulus $\{2^j \leq |\xi| \leq 2^{j+2}\}$, and if $F(t,x)$ is a real function of $(t,x) \in \mathbb{R} \times \mathbb{R}^d$, whose x-Fourier transform is supported in $\{2^j \leq |\xi| \leq 2^{j+2}\}$, then the solution U of*

$$\begin{cases} \Box U = F \\ U_{|t=0} = U_0 \\ \partial_t U_{|t=0} = U_1 \end{cases} \tag{2.1}$$

verifies

$$\begin{aligned} &\|U\|_{L^q L^r} + 2^{-j} \|\partial_t U\|_{L^q L^r} \\ &\quad \leq C 2^{j\left(-\frac{1}{q}-\frac{d}{r}+\frac{d}{2}\right)} \big(\|U_0\|_2 + 2^{-j}\|U_1\|_2\big) + C 2^{j\left(-\frac{d}{r}-\frac{d}{\widetilde{r}}-\frac{1}{q}-\frac{1}{\widetilde{q}}+d-1\right)} \|F\|_{L^{\widetilde{q}'} L^{\widetilde{r}'}}, \end{aligned}$$

provided (q,r) and $(\widetilde{q},\widetilde{r})$ verify

$$q \geq 2 \ , \ (q,r,d) \neq (2,\infty,3) \quad \text{and} \quad \frac{1}{q} + \frac{d-1}{2r} \leq \frac{d-1}{4} \cdot \tag{2.2}$$

The above theorem gives estimates for solutions of wave equations which are localized in frequency in dyadic annuli. We will make use in the following of estimates in Besov and Sobolev spaces; they are easily deduced from this theorem.

The space-time norms at the scaling of Strichartz estimates control the critical wave equation $(NLW)_{2^*-1}$.

We will need in the following to control, for a finite energy solution, these space-time norms by the energy. Bahouri and Gérard [1] showed that such a control exists, but without giving an explicit bound. Nakanishi [16] proved the following theorem.

Theorem 2.2 (Nakanishi [16]) *Let $d \geq 3$, consider initial data $(u_0, u_1) \in \dot{H}^1 \times L^2$, and let us denote*

$$E = \|u_0\|_{\dot{H}^1} + \|u_1\|_{L^2} \ .$$

Let u be the solution of $(NLW)_{2^*-1}$ given by Theorem 1.1. There exist constants $C > 0$ and $\kappa > 0$ such that

$$\|u\|_{L^{q_0}(\mathbb{R}, \dot{B}^{s_0}_{r_0,2})} \leq C \exp E^\kappa \ ,$$

with

$$q_0 = \frac{2(d^2+2)}{(d+1)(d-2)} \qquad r_0 = 2^* \qquad s_0 = \frac{1}{q_0} \cdot$$

2.2. Core of the proof of Theorem A

1. Let us take (v_0, v_1) as in the theorem, and let v be the finite energy solution of $(NLW)_{2^*-1}$ associated to this initial data by Theorem 1.1. In order to prove the theorem, it suffices to show that there exists a global solution of

$$\begin{cases} \Box w + (v+w)|v+w|^{2^*-2} - v|v|^{2^*-2} = 0 \\ w_{|t=0} = w_0 \\ \partial_t w_{|t=0} = w_1 \end{cases} \tag{2.3}$$

for then $v + w$ is a solution of $(NLW)_{2^*-1}$ associated to the initial data $(v_0 + w_0, v_1 + w_1)$.

2. The idea of the proof is now the following: we split $[0, +\infty[$ into N intervals $[T_n, T_{n+1}]$, with $T_0 = 0$ and $T_N = +\infty$.

Then we solve the perturbed equation (2.3) successively on the intervals $[T_n, T_{n+1}]$, with the help of a fixed point theorem, in

$$X_{[T_n,T_{n+1}]} \overset{\text{def}}{=} \widetilde{L}^{\mu/\alpha}([T_n, T_{n+1}], \dot{B}^{1-\alpha/2}_{\rho,\infty}) \ ,$$

where μ, α and ρ have been defined in (1.5) and spaces $\widetilde{L}^r([0,T], \dot{B}^s_{p,q})$ are defined in the appendix.

The use of this fixed point theorem will be made possible by the smallness of v in $X_{[T_n,T_{n+1}]}$, for T_n close enough to T_{n+1}. We will also have to control the norm of $w(T_n)$ for each $n \leq N$, and check that it remains small enough. Finally, the crucial point is that N is indeed finite. This is a consequence of Theorem 2.2, which will give us a bound on the norm of v in $X_{[0,\infty[}$.

The proof sketched above will give us a solution of (2.3) for $t \in [0, +\infty[$; since the case $t \in]-\infty, 0]$ is identical, we will actually get a solution defined on $\mathbb{R}$, and the theorem will be proved.

3. Let us now implement the program described above. By Theorem 2.2, we have

$$\|v\|_{L^{q_0}\dot{B}^{s_0}_{r_0,2}} \leq C \exp E^\kappa \ . \tag{2.4}$$

Moreover, Theorem 2.1 implies that the solution v of $(NLW)_{2^*-1}$ associated to the initial data (v_0, v_1) verifies the estimate (we denote as in the statement of the theorem $X_{]-\infty,\infty[}$ simply by X)

$$\begin{aligned} \|v\|_X &\leq \|v\|_{\widetilde{L}^{\mu/\alpha}(\mathbb{R}, \dot{B}^{1-\alpha/2}_{\rho,2})} \\ &\leq C \left(\|v_0\|_{\dot{H}^1} + \|v_1\|_{L^2} + \|v|v|^{2^*-2}\|_{L^{q_1}\dot{B}^{s_1}_{r_1,2}} \right) \end{aligned} \tag{2.5}$$

with
$$q_1 = \frac{2(d^2+2)}{(d+1)(d+2)} \qquad r_1 = \frac{2(d^2+2)}{d^2+2d-2} \qquad s_1 = \frac{1}{q_0} \; .$$
We conclude as in Nakanishi's paper [16, page 34], that

$$\|v\|_X \le C\left(\|v_0\|_{\dot H^1} + \|v_1\|_{L^2} + \|v\|^{2^*-1}_{L^{q_0}\dot B^{s_0}_{r_0,2}}\right) \; . \tag{2.6}$$

Combining (2.4) and (2.6), we get, for new positive constants κ and C,

$$\|v\|_X \le C\exp(E^\kappa) \; . \tag{2.7}$$

Let us take $\epsilon > 0$ (we shall set the value of ϵ later), and let us build up

$$0 = T_0 < T_1 < \cdots < T_n < T_{n+1} < \cdots < T_N = \infty$$

with the help of the following lemma, whose proof we postpone for the moment.

Lemma 2.1 *Let v be as above, and $\epsilon > 0$. There exists an integer N and $N+1$ numbers $T_0, \ldots, T_N$ belonging to $[0,\infty]$ such that*

$$0 = T_0 < T_1 < \cdots < T_n < T_{n+1} < \cdots < T_N = \infty$$

and

$$\|v\|_{X_{[T_n,T_{n+1}]}} \le \epsilon \; .$$

Furthermore, we have the estimate

$$N^{\alpha/\mu}\epsilon \le C\exp(E^\kappa) \; . \tag{2.8}$$

4. The following lemma gives conditions such that (2.3) has a solution on $[0,T_1]$; it also gives a bound on the norm of this solution in T_1.

Lemma 2.2 *Let $T > 0$. There exist constants c_0 and C_0 such that if*

$$\begin{cases} \|w_0\|_{\dot B^1_{2,\infty}} + \|w_1\|_{\dot B^0_{2,\infty}} \le c_0 \\ \|v\|_{X_{[0,T]}} \le c_0 \end{cases}$$

then the Cauchy problem (2.3) *has a solution $w \in X_{[0,T]}$ which verifies*

$$\|w\|_{X_{[0,T]}} \le C_0\left(\|w_0\|_{\dot B^1_{2,\infty}} + \|w_1\|_{\dot B^0_{2,\infty}}\right)$$

and

$$\|w(T)\|_{\dot B^1_{2,\infty}} + \|\partial_t w(T)\|_{\dot B^0_{2,\infty}} \le C_0\left(\|w_0\|_{\dot B^1_{2,\infty}} + \|w_1\|_{\dot B^0_{2,\infty}}\right) \; .$$

We will not prove this lemma at once, and will first show how it enables us to conclude the proof of the theorem. Let us first set $\epsilon = c_0$. If $\|w_0\|_{\dot{B}^1_{2,\infty}} + \|w_1\|_{\dot{B}^0_{2,\infty}} \leq c_0$, we can use Lemma 2.2, which gives us a solution $w \in X_{[0,T_1]}$ such that

$$\|w(T_1)\|_{\dot{B}^1_{2,\infty}} + \|\partial_t w(T_1)\|_{\dot{B}^0_{2,\infty}} \leq C_0 \left(\|w_0\|_{\dot{B}^1_{2,\infty}} + \|w_1\|_{\dot{B}^0_{2,\infty}} \right) \ .$$

Let us apply once again Lemma 2.2 to the Cauchy problem

$$\text{(2.9)} \qquad \begin{cases} \Box w + (v+w)|v+w|^{2^*-2} - v|v|^{2^*-2} = 0 \\ w_{|t=T_1} = w(T_1) \\ \partial_t w_{|t=T_1} = w(T_1) \ , \end{cases}$$

so as to get a solution on $[T_1, T_2]$.

Provided $C_0\big(\|w_0\|_{\dot{B}^1_{2,\infty}} + \|w_1\|_{\dot{B}^0_{2,\infty}}\big) \leq c_0$, the lemma gives a solution $w \in X_{[T_1,T_2]}$ such that

$$\|w(T_2)\|_{\dot{B}^1_{2,\infty}} + \|\partial_t w(T_2)\|_{\dot{B}^0_{2,\infty}} \leq C_0^2 \left(\|w_0\|_{\dot{B}^1_{2,\infty}} + \|w_1\|_{\dot{B}^0_{2,\infty}} \right) \ .$$

By iterating this procedure, we obtain a solution w on $[0,\infty[$ under the condition that

$$C_0^N \left(\|w_0\|_{\dot{B}^1_{2,\infty}} + \|w_1\|_{\dot{B}^0_{2,\infty}} \right) \leq c_0 \ .$$

This inequality and (2.8) imply that a sufficient condition in order to obtain a global solution w is that

$$\|w_0\|_{\dot{B}^1_{2,\infty}} + \|w_1\|_{\dot{B}^0_{2,\infty}} \leq C \exp\left(-\exp\left(E^\kappa\right)\right) \ ,$$

for universal constants C and κ; this proves the "existence" part of the theorem.

5. The "uniqueness" part is left. Let us observe first that the solution $u = v + w$ which has been built up in the previous paragraph belongs to the set $\mathcal{E}$ defined in (1.6). Indeed, we know by (2.6) that $v \in \widetilde{L}^{\mu/\alpha}(\mathbb{R}, \dot{B}^{1-\alpha/2}_{\rho,2})$. This implies that

$$\text{(2.10)} \qquad d_X(v, \mathcal{S}) = 0 \ .$$

Moreover, we will majorize $\|w\|_{X_{[0,\infty[}}$ (a bound for $\|w\|_X$ can be obtained identically); we shall use the times $T_0, T_1 \dots T_N$ defined above. Let

$$a_{n,j} \stackrel{\text{def}}{=} 2^{j(1-\alpha/2)} \|\Delta_j w\|_{L^{\mu/\alpha}([T_{n-1},T_n],L^\rho)} \ .$$

By Minkowski's inequality, we have

(2.11)

$$\|w\|_{X_{[0,\infty[}} = \sup_{j\in\mathbb{Z}} \left(\sum_{n=1}^{N} a_{n,j}^{\mu/\alpha} \right)^{\alpha/\mu} \leq \sup_j \sum_{n=1}^{N} a_{n,j} \leq \sum_{n=1}^{N} \sup_j a_{n,j} = \sum_{n=1}^{N} \|w\|_{X_{[T_{n-1},T_n]}} .$$

Furthermore, we deduce from Lemma 2.2 and from the paragraph **4**. that

(2.12)

$$\sum_{n=1}^{N} \|w\|_{X_{[T_{n-1},T_n]}} \leq \left(C_0 + C_0^2 + \cdots + C_0^N\right) \left(\|w_0\|_{\dot{B}^1_{2,\infty}} + \|w_1\|_{\dot{B}^0_{2,\infty}} \right) \leq C c_0$$

and this last quantity is smaller than ϵ_1 (whose value will be set later on) provided that c_0 be chosen small enough, which we can assume.

Gathering (2.10), (2.11) and (2.12), we get

$$d_X(u,\mathcal{S}) \leq d_X(v,\mathcal{S}) + d_X(w,\mathcal{S}) \leq \|w\|_X \leq \epsilon_1 ,$$

and therefore $u \in \mathcal{E}$.

6. Let us now show that uniqueness indeed holds in $\mathcal{E}$. Suppose that u and $\widetilde{u}$ are two solutions of $(NLW)_{2^*-1}$, belonging to $\mathcal{E}$, and associated to the same initial data (u_0, u_1), which of course satisfies the assumptions of Theorem A. We finally assume that these two solutions are not equal for all time; we can take 0 to be the maximal time until which they are equal.

The following lemma, proved in Section 2.3, will enable us to conclude.

Lemma 2.3 *There exist $\epsilon > 0$ and $\zeta > 0$ such that, if (u_0, u_1) are initial data verifying*

$$d_{\dot{B}^1_{2,\infty}\times\dot{B}^0_{2,\infty}}((u_0,u_1),\mathcal{S}) \stackrel{\text{def}}{=} \inf\left\{ \|(u_0,u_1) - f\|_{\dot{B}^1_{2,\infty}\times\dot{B}^0_{2,\infty}} ,\ f \in \mathcal{S} \right\} \leq \epsilon ,$$

then there exist $T > 0$ and u such that

- *u is a solution of $(NLW)_{2^*-1}$ on $[0,T]$ for the initial data (u_0, u_1)*
- *for any $t \leq T$, u is unique on $[0,t]$ if one adds the condition*

$$\|u\|_{\widetilde{L}^{\mu/\alpha}([0,t],\dot{B}^{1-\alpha/2}_{\rho,\infty})} < \zeta .$$

We observe then that, if $u \in \mathcal{E}$,

$$\limsup_{t\to 0} \|u\|_{\widetilde{L}^{\mu/\alpha}([0,t],\dot{B}^{1-\alpha/2}_{\rho,\infty})} \leq \epsilon_1 .$$

In particular, if $\epsilon_1 < \zeta$ (we choose ϵ_1 such that this inequality holds), for t small enough, one can apply the uniqueness criterion of the above lemma to u and $\widetilde{u}$. We obtain that $u = \widetilde{u}$ on $[0,t]$, which is absurd. This concludes the proof of the theorem. ■

2.3. Proof of the auxiliary lemmas

This section is dedicated to the proof of Lemmas 2.1, 2.2 and 2.3, which have been used in the proof of Theorem A.

Proof of Lemma 2.1. First pick T_1 such that

$$\|v\|_{\widetilde{L}^{\mu/\alpha}([0,T_1],\dot{B}^{1-\alpha/2}_{\rho,2})} = \epsilon$$

(if $\|v\|_{\widetilde{L}^{\mu/\alpha}([0,\infty[,\dot{B}^{1-\alpha/2}_{\rho,2})} < \epsilon$, we are done). Pick then T_2 such that

$$\|v\|_{\widetilde{L}^{\mu/\alpha}([T_1,T_2],\dot{B}^{1-\alpha/2}_{\rho,2})} = \epsilon \ ,$$

then T_3, and so on. This procedure stops for some N, and the estimate (2.8) of Lemma 2.1 holds, because of the inequality which we are about to prove. Suppose $T_1 < T_2 < \dots < T_K$ have been built up.

Let us define as above the sequence $(a_{n,j})$, $n \in \{1 \dots K\}$ and $j \in \mathbb{Z}$, by

$$a_{n,j} \overset{\text{def}}{=} 2^{j(1-\alpha/2)} \|\Delta_j v\|_{L^{\mu/\alpha}([T_{n-1},T_n],L^\rho)} \ .$$

We have then

$$\sum_{n=1}^{K} \Big(\sum_{j\in\mathbb{Z}} a_{n,j}^2 \Big)^{1/2} = \sum_{n=1}^{K} \|v\|_{\widetilde{L}^{\mu/\alpha}([T_{n-1},T_n],\dot{B}^{1-\alpha/2}_{\rho,2})} \geq K\epsilon \ . \tag{2.13}$$

We will estimate the left-hand side of the above equation with the help of the concavity inequality

$$\sum_{n=1}^{M} b_n^\beta \leq M^{1-\beta} \Big(\sum_{n=1}^{M} b_n \Big)^\beta$$

which holds if $M \in \mathbb{N}$, $b_1 \dots b_M$ are positive, and $\beta \in [0,1]$. We get

$$\begin{aligned}
\sum_{n=1}^{K} \Big(\sum_{j\in\mathbb{Z}} a_{n,j}^2 \Big)^{1/2} &\leq \sqrt{K} \Big(\sum_{n=1}^{K} \sum_{j\in\mathbb{Z}} a_{n,j}^2 \Big)^{1/2} \\
&\leq \sqrt{K} \Big(\sum_{j\in\mathbb{Z}} K^{1-2\alpha/\mu} \Big[\sum_{n=1}^{K} a_{n,j}^{\mu/\alpha} \Big]^{2\alpha/\mu} \Big)^{1/2} \\
&\leq K^{1-\alpha/\mu} \Big(\sum_{j\in\mathbb{Z}} \Big[\sum_{n=1}^{K} a_{n,j}^{\mu/\alpha} \Big]^{2\alpha/\mu} \Big)^{1/2} ,
\end{aligned} \tag{2.14}$$

where we used in the second inequality that $0 < 2\alpha/\mu < 1$. Now we notice that

$$\left(\sum_{j\in\mathbb{Z}}\left[\sum_{n=1}^{K} a_{n,j}^{\mu/\alpha}\right]^{2\alpha/\mu}\right)^{1/2} = \|v\|_{\widetilde{L}^{\mu/\alpha}([T_0,T_K],\dot{B}^{1-\alpha/2}_{\rho,2})} \,. \tag{2.15}$$

It follows from (2.13), (2.14) and (2.15) that

$$K\epsilon \leq K^{1-\alpha/\mu}\|v\|_{\widetilde{L}^{\mu/\alpha}([T_0,T_K],\dot{B}^{1-\alpha/2}_{\rho,2})} \,,$$

and, combined with (2.7), this concludes the proof of the lemma. ■

Proof of Lemma 2.2. Solving (2.3) on $[0,T]$ is equivalent to solving on the same interval the integral equation

$$\begin{aligned} w(t) &= \dot{W}(t)w_0 + W(t)w_1 \\ &\quad + \int_0^t W(t-s)\left[(v+w)|v+w|^{2^*-2}(s) - v|v|^{2^*-2}(s)\right] ds \\ &\stackrel{\text{def}}{=} G(w)(t) \,, \end{aligned} \tag{2.16}$$

where $W(t) \stackrel{\text{def}}{=} \dfrac{\sin(t|D|)}{|D|}$ is the wave operator. For $\eta > 0$, let us denote

$$Y = \left\{w \in X_{[0,T]} \,,\, \|w\|_{X_{[0,T]}} < \eta\right\} \,;$$

and let us determine for which $\eta > 0$ does G stabilize Y. We shall need Lemma 2.4, which is proved below and states that the mapping

$$\left\{\begin{array}{ccc} (u_1,\ldots,u_{2^*-1}) & \mapsto & u_1|u_2|\ldots|u_{2^*-1}| \\ \left(X_{[0,T]}\right)^{2^*-1} & \to & \widetilde{L}^{\nu}\left([0,T],\dot{B}^{1/2}_{\nu,\infty}\right) \end{array}\right. \tag{2.17}$$

is continuous (ν was defined in (1.5)).

We will also need a Strichartz estimate: Theorem 2.1 implies that the solution U of (2.1) verifies

$$\|U\|_{X_{[0,T[}} \leq C\left(\|U_0\|_{\dot{B}^1_{2,\infty}} + \|U_1\|_{\dot{B}^0_{2,\infty}} + \|F\|_{\widetilde{L}^{\nu}([0,T],\dot{B}^{1/2}_{\nu,\infty})}\right) \,. \tag{2.18}$$

Denoting

$$\left\{\begin{array}{l} \delta = \|w_0\|_{\dot{B}^1_{2,\infty}} + \|w_1\|_{\dot{B}^0_{2,\infty}} \\ \epsilon = \|v\|_{X_{[0,T]}} \,, \end{array}\right.$$

the inequality (2.18) combined to (2.17) gives, if $w \in Y$,

$$
\begin{aligned}
\|G(w)\|_{X_{[0,T]}} &\leq C\left(\delta + \|(v+w)|v+w|^{2^*-2} - v|v|^{2^*-2}\|_{\widetilde{L}^\nu([0,T],\dot{B}^{1/2}_{\nu,\infty})}\right) \\
&\leq C\left(\delta + \left\|(2^*-1)w\int_0^1 |v+\tau w|^{2^*-2} d\tau\right\|_{\widetilde{L}^\nu([0,T],\dot{B}^{1/2}_{\nu,\infty})}\right) \\
&\leq C\left(\delta + (2^*-1)\|w\|_{X_{[0,T]}}\int_0^1 \|v+\tau w\|^{2^*-2}_{X_{[0,T]}} d\tau\right) \\
&\leq C(\delta + \eta(\eta+\epsilon)^{2^*-2}) \; .
\end{aligned} \tag{2.19}
$$

As a consequence, G stabilizes Y if

$$
\delta \leq c_1\eta\,, \qquad \eta \leq c_2 \qquad \text{and} \qquad \epsilon \leq c_2 \tag{2.20}
$$

(c_1 and c_2 are positive constants). Let us now find conditions such that G be contracting on Y. If w and w' belong to Y, we have, due to (2.18) and Lemma 2.4,

$$
\begin{aligned}
&\|G(w)-G(w')\|_{X_{[0,T]}} \\
&= \left\|\int_0^t W(t-s)\left[(v+w)|v+w|^{2^*-2}(s) - (v+w')|v+w'|^{2^*-2}(s)\right] ds\right\|_{X_{[0,T]}} \\
&\leq C\|(v+w)|v+w|^{2^*-2} - (v+w')|v+w'|^{2^*-2}\|_{\widetilde{L}^\nu([0,T],\dot{B}^{1/2}_{\nu,\infty})} \\
&\leq C\left\|(2^*-1)(w-w')\int_0^1 |v+w+\tau(w'-w)|^{2^*-2} d\tau\right\|_{\widetilde{L}^\nu([0,T],\dot{B}^{1/2}_{\nu,\infty})} \\
&\leq C(\eta+\epsilon)^{2^*-2}\|w-w'\|_{X_{[0,T]}} \; .
\end{aligned}
$$

So G is contracting if

$$
\eta \leq c_3 \qquad \text{and} \qquad \epsilon \leq c_3 \; . \tag{2.21}
$$

Let us set $\eta = \dfrac{\delta}{c_1}$. Then (2.20) and (2.21) are equivalent to

$$
\delta \leq c_0 \qquad \text{and} \qquad \epsilon \leq c_0 \; .
$$

If this last condition is verified, G is contracting on Y, and Picard's fixed point theorem implies the existence of a solution of (2.3) belonging to Y, hence such that

$$
\|w\|_{X_{(0,T]}} \leq \eta = \frac{\delta}{c_1} \; .
$$

We are left with estimating its norm in T. Applying once again Theorem 2.1 gives the inequality

$$\begin{aligned}
&\|w\|_{L^\infty([0,T],\dot B^1_{2,\infty})} + \|\partial_t w\|_{L^\infty([0,T],\dot B^0_{2,\infty})} \\
&\leq C\left(\|w_0\|_{\dot B^1_{2,\infty}} + \|w_1\|_{\dot B^0_{2,\infty}} + \|(v+w)|v+w|^{2^*-2} - v|v|^{2^*-2}\|_{\widetilde L^\nu([0,T],\dot B^{1/2}_{\nu,\infty})}\right) \\
&\leq C_1\delta\ .
\end{aligned}$$

The last majorization is justified by (2.19) and by the choice of η which we made. ■

We will now prove a product lemma which has already been useful in the proof of Lemma 2.2, and which we will use again in Part 3.

Recall we set

$$(2.22)\quad \mu = \frac{2(d+1)}{d-1}\ ,\ \nu = \frac{2(d+1)}{d+3}\ ,\ \alpha = \frac{\mu}{\nu(2^*-1)}\ \text{ and }\ \frac{1}{\rho} = \frac{\alpha}{\mu} + \frac{1-\alpha}{2}\ .$$

Lemma 2.4 *Take d equal to 3, 4 or 6, and take $T \leq T'$ two real numbers.*

1. *The mapping*

$$\begin{cases} (u_1, u_2, \dots, u_{2^*-1}) & \mapsto\ u_1|u_2|\dots|u_{2^*-1}| \\ \left(\widetilde L^{\mu/\alpha}([T,T'], \dot B^{1-\alpha/2}_{\rho,\infty})\right)^{2^*-1} & \to\ \widetilde L^\nu([T,T'], \dot B^{1/2}_{\nu,\infty}) \end{cases}$$

is continuous.

2. *The mapping*

$$\begin{cases} (u_1, u_2, \dots, u_{2^*-1}) & \mapsto u_1|u_2|\dots|u_{2^*-1}| \\ \widetilde L^{\mu/\alpha}([T,T'], \dot B^{1-\alpha/2}_{\rho,2}) \times \left(\widetilde L^{\mu/\alpha}([T,T'], \dot B^{1-\alpha/2}_{\rho,\infty})\right)^{2^*-2} & \to \widetilde L^\nu([T,T'], \dot B^{1/2}_{\nu,2}) \end{cases}$$

is continuous.

Proof of Lemma 2.4. We shall only prove point 1.; the proof of point 2. is very similar.

To simplify notations, we will take $z_1 \dots z_{2^*-1}$ in $\dot B^{1-\alpha/2}_{\rho,\infty}$ and prove that their product belongs to $\dot B^{1/2}_{\nu,\infty}$. This corresponds to the statement of the lemma, except for the Lebesgue spaces in time, but one simply needs to use Hölder's inequality to add them; and except for the absolute values, but it is possible to add them due to the boundedness of the mapping $f \mapsto |f|$ on $\dot B^s_{p,q}$ if $s < 1$.

So as to use Bony's paraproduct algorithm [2], we write the product $z_1 \dots z_{2^*-1}$ as a telescopic sum

$$z_1 \dots z_{2^*-1} = \sum_{j\in\mathbb{Z}} \left[\prod_{i=1}^{2^*-1} S_{j+1} z_i - \prod_{i=1}^{2^*-1} S_j z_i \right] \stackrel{\text{def}}{=} \sum_j A_j$$

(see the appendix for the definition of S_j, and of Δ_j, which we are about to use). It is clear that for any j, A_j is localized in frequency in a ball whose radius is proportional to 2^j. We would like to show that $\sum_j A_j$ belongs to $\dot{B}^{1/2}_{\nu,\infty}$, and since this space has a positive regularity index, it suffices, by a classical result (see [17]), to show that

$$\sup_{j\in\mathbb{Z}} 2^{\frac{j}{2}} \|A_j\|_\nu < \infty \ .$$

If one looks closer at the A_j, it appears that each one of them is a linear combination of products $a_1 ... a_{2^*-1}$, where each a_i is either equal to $\Delta_j z_i$ or $S_j z_i$, but with at least one of the a_i equal to $\Delta_j z_i$. Let us now estimate the size of the a_i.

- First, by definition of $\dot{B}^{1-\alpha/2}_{\rho,\infty}$, $\|\Delta_j z_i\|_\rho \leq C 2^{(1-\alpha/2)j} \|z_i\|_{\dot{B}^{1-\alpha/2}_{\rho,\infty}}$.

- Secondly, the classical Sobolev embedding

$$\dot{B}^{1-\alpha/2}_{\rho,\infty} \hookrightarrow \dot{B}^{1-\frac{\alpha}{2}+d\left(\frac{1}{R}-\frac{1}{\rho}\right)}_{R,\infty} \qquad \text{for } R > \rho$$

  gives

$$\|\Delta_j z_i\|_R \leq C 2^{-j\left[1-\frac{\alpha}{2}+d\left(\frac{1}{R}-\frac{1}{\rho}\right)\right]} \|z_i\|_{\dot{B}^{1-\alpha/2}_{\rho,\infty}}$$

  and

$$\|S_j z_i\|_R \leq C 2^{-j\left[1-\frac{\alpha}{2}+d\left(\frac{1}{R}-\frac{1}{\rho}\right)\right]} \|z_i\|_{\dot{B}^{1-\alpha/2}_{\rho,\infty}}$$

  provided that $R > \rho$ and $1 - \frac{\alpha}{2} + d\left(\frac{1}{R} - \frac{1}{\rho}\right) < 0$.

Consider one of the terms $a_1 ... a_{2^*-1}$; by symmetry, we may assume that $a_1 = \Delta_j z_1$. Hölder's inequality added to the two previous estimates gives

$$\begin{aligned} \|A_j\|_\nu &\leq \|a_1\|_\rho \|a_2\|_R \dots \|a_{2^*-1}\|_R \\ &\leq C 2^{-j(1-\alpha/2)} \left(2^{-j\left[1-\frac{\alpha}{2}+d\left(\frac{1}{R}-\frac{1}{\rho}\right)\right]} \right)^{2^*-2} = C 2^{-\frac{j}{2}} \ , \end{aligned}$$

and the last inequality holds provided

$$
(2.23) \qquad \begin{cases} \dfrac{1}{\rho} + \dfrac{2^*-2}{R} = \dfrac{1}{\nu} \\ 1 - \dfrac{\alpha}{2} + (2^*-2)\left[1 - \dfrac{\alpha}{2} + d\left(\dfrac{1}{R} - \dfrac{1}{\rho}\right)\right] = \dfrac{1}{2} \\ R > \rho \\ 1 - \dfrac{\alpha}{2} + d\left(\dfrac{1}{R} - \dfrac{1}{\rho}\right) < 0 \ . \end{cases}
$$

It turns out that the two first lines of (2.23) are equivalent and lead to a value of R which verifies the two last inequalities. This proves the lemma. ■

Proof of Lemma 2.3. The proof of this lemma is quite similar to the one of Lemma 2.2, so we will only give the outline of the proof. As above, we work on the integral equation

$$
u(t) = \dot{W}(t)u_0 + W(t)u_1 + \int_0^t W(t-s)u|u|^{2^*-2}(s)\, ds \ ,
$$

which we solve in

$$
X_T \stackrel{\text{def}}{=} \widetilde{L}^{\mu/\alpha}([0,T], \dot{B}^{1-\alpha/2}_{\rho,\infty}) \ .
$$

We use the two following facts:

- The mapping
$$
\begin{array}{rcl} X_T^{2^*-1} & \longrightarrow & \widetilde{L}^{\nu}([0,T], \dot{B}^{1/2}_{\nu,\infty}) \\ (z_1 \ldots, z_\nu) & \longmapsto & z_1|z_2| \ldots |z_{2^*-1}| \end{array}
$$
is bounded - that is Lemma 2.4.

- Theorem 2.1 gives the following Strichartz estimate, for U solution of (2.1)
$$
\|U\|_{X_T} \leq C\|\dot{W}(t)U_0\|_{X_T} + \|W(t)U_1\|_{X_T} + \|F\|_{\widetilde{L}^{\nu}([0,T],\dot{B}^{1/2}_{\nu,\infty})} \ .
$$

We deduce from the two previous points the following a priori estimate, for u solution of $(NLW)_{2^*-1}$,

$$
\|u\|_{X_T} \leq C\left[\|\dot{W}(t)u_0\|_{X_T} + \|W(t)u_1\|_{X_T} + \|u\|_{X_T}^{2^*-1}\right] \ .
$$

But

$$
\lim_{T\to 0}\left(\|\dot{W}(t)u_0\|_{X_T} + \|W(t)u_1\|_{X_T}\right) \leq C d_{\dot{B}^1_{2,\infty}\times\dot{B}^0_{2,\infty}}((u_0,u_1),\mathcal{S}) \leq C\epsilon
$$

by hypothesis. Hence if ϵ is chosen small enough, the fixed point problem has a solution. This solution is unique in a ball of radius ζ small enough. ■

2.4. Complementary result

The proof which we have given above actually shows a result which is slightly better than Theorem A. We state it as a corollary

Corollary 2.1 *Let* $d = 3, 4$ *or* 6*, and let* v *be a solution of* $(NLW)_{2^*-1}$ *associated to initial data* $(v_0, v_1) \in \dot{B}^1_{2,\infty} \times \dot{B}^0_{2,\infty}$*. We assume that*

$$\|v\|_X < \infty \ .$$

Then if (w_0, w_1) *verifies*

$$\|w_0\|_{\dot{B}^1_{2,\infty}} + \|w_1\|_{\dot{B}^1_{2,\infty}} \leq C \exp(-C\|v\|_X) \ ,$$

there exists a global solution of $(NLW)_{2^*-1}$ *associated to the initial data* $(v_0 + w_0, v_1 + w_1)$*.*

3. Proof of Theorem B

In order to keep notations as light as possible, we shall show the theorem in the case where $c_1 = 0$.

3.1. Global solutions for infinite energy and small initial data

The following proposition is very close to a result proved by Planchon [17]. This proposition can be seen as a particular case of Lemma 2.2.

Proposition 3.1 (Planchon [17]) *For* $d = 3, 4, 6$ *there exists* $\epsilon, \epsilon_1 > 0$ *such that, for any initial data verifying*

$$\|w_0\|_{\dot{B}^1_{2,\infty}} + \|w_1\|_{\dot{B}^0_{2,\infty}} < \epsilon \ ,$$

there exists w *a global solution of* $(NLW)_{2^*-1}$*. Moreover,* w *is unique in the set of functions such that*

$$\|w\|_{\widetilde{L}^{\mu/\alpha}(\mathbb{R},\dot{B}^{1-\alpha/2}_{\rho,\infty})} < \epsilon_1$$

(*the notations are the same as in theorem* A)*, and verifies the estimate*

$$\|w\|_{L^\infty(\mathbb{R},\dot{B}^1_{2,\infty})\cap\widetilde{L}^\mu(\mathbb{R},\dot{B}^{1/2}_{\mu,\infty})} \leq C \left(\|w_0\|_{\dot{B}^1_{2,\infty}} + \|w_1\|_{\dot{B}^0_{2,\infty}} \right) . \tag{3.1}$$

Proof. We do not give the details of the proof, since it relies on a fixed point argument identical to the one appearing in Lemma 2.3; one should simply substitute

$$\widetilde{L}^{\mu/\alpha}(\mathbb{R}, \dot{B}^{1-\alpha/2}_{\rho,\infty}) \qquad \text{for } X_T.$$

■

Let us consider w a solution of $(NLW)_{2^*-1}$ for small initial data given by Proposition 3.1; we examine from now on on the following Cauchy problem

$$\text{(3.2)} \qquad \begin{cases} \Box v + (v+w)|v+w|^{2^*-2} - w|w|^{2^*-2} = 0 \\ v_{|t=0} = v_0 \\ \partial_t v_{|t=0} = v_1 \ , \end{cases}$$

where $(v_0, v_1) \in \dot{H}^1 \times L^2$. Our aim is to prove the existence of a global solution of $(NLW)_{2^*-1}$ for the initial data (u_0, u_1). Such a solution is given by $u = v + w$, if v is a global solution of the above Cauchy problem. So we just have to prove the existence of (3.2).

The first step is to build up a local solution; this is done in the next section.

3.2. Local solution of the perturbed equation and blow up criterion

Proposition 3.2 *Let $d = 3, 4, 6$.*

(i) There exists $\epsilon > 0$ such that, for initial data (w_0, w_1) verifying

$$\|w_0\|_{\dot{B}^1_{2,\infty}} + \|w_1\|_{\dot{B}^0_{2,\infty}} < \epsilon \ ,$$

the Cauchy problem $(NLW)_{2^*-1}$ has a global solution w which is unique in a ball centered in 0, of positive radius, of $\widetilde{L}^{\mu/\alpha}(\mathbb{R}, \dot{B}^{1-\alpha/2}_{\rho,\infty})$. For this w and initial data $(v_0, v_1) \in \dot{H}^1 \times L^2$, there exists $T > 0$ such that the Cauchy problem (3.2) has a solution v such that

$$(v, \partial_t v) \in \mathcal{C}([0,T], \dot{H}^1 \times L^2) \ .$$

Let T^* be the maximal time with that property.

(ii) There exists $\eta > 0$ such that if one denotes

$$\beta(t) \stackrel{\text{def}}{=} \sup \Big\{ r > 0 \, , \, \|v(t,\cdot)\|_{\dot{H}^1(B(x,r))} + \|\partial_t v(t,\cdot)\|_{L^2(B(x,r))} + \|v(t,\cdot)\|_{L^{2^*}(B(x,r))} < \eta \ \forall x \in \mathbb{R}^d \Big\}$$

for $t < T^*$, and if moreover $T^* < \infty$, then

$$\beta(t) \stackrel{t \to T^*}{\longrightarrow} 0 \ .$$

We have the more precise estimate

$$\beta(t) \leq T^* - t \ .$$

Proof. (i) The existence and uniqueness of w for ϵ small enough correspond to Proposition 3.1.

In order to prove the local existence of a solution v of (3.2), let us recall first that (3.2) is equivalent to the integral equation

(3.3)
$$v(t) = \dot{W}(t)v_0 + W(t)v_1 + \int_0^t W(t-s)\left[(v+w)|v+w|^{2^*-2} - w|w|^{2^*-2}\right] ds \; .$$

We will now apply Picard's fixed point theorem in the space

$$X \stackrel{\text{def}}{=} \widetilde{L}^{\mu/\alpha}([0,T], \dot{B}^{1-\alpha/2}_{\rho,2}) \; ,$$

where $T > 0$ will be set up in the following, and where μ, α and ρ are defined as in (1.5). We need two estimates

- On the one hand, using Lemma 2.4 and its notations, we have

$$\begin{aligned} &\|(v+w)|v+w|^{2^*-2} - w|w|^{2^*-2}\|_{\widetilde{L}^{\nu}([0,T],\dot{B}^{1/2}_{\nu,2})} \\ &\qquad = \left\| (2^*-1)v \int_0^1 |w+\tau v|^{2^*-2} d\tau \right\|_{\widetilde{L}^{\nu}([0,T],\dot{B}^{1/2}_{\nu,\infty})} \\ &\qquad \leq C\|v\|_X \left(\|v\|_X + \|w\|_X\right)^{2^*-2} \\ &\qquad \leq C\|v\|_X (\epsilon + \|v\|_X)^{2^*-2} \; , \end{aligned} \tag{3.4}$$

  where the bound (3.1) was used in the last inequality.

- On the other hand, Theorem 2.1 gives, for U solution of (2.1),

$$\begin{aligned} &\|U\|_{L^\infty([0,T],\dot{H}^1)} + \|\partial_t U\|_{L^\infty([0,T],L^2)} + \|U\|_X \\ &\qquad \leq C\left(\|U_0\|_{\dot{H}^1} + \|U_1\|_{L^2} + \|F\|_{\widetilde{L}^{\nu}([0,T],\dot{B}^{1/2}_{\nu,2})}\right) \; . \end{aligned} \tag{3.5}$$

Taking the norm of (3.3) in X, and using (3.4) and (3.5), we get the following a priori estimate for v solution of (3.2)

$$\|v\|_X \leq C\left(\|\dot{W}(t)v_0\|_X + \|W(t)v_1\|_X + \|v\|_X(\epsilon + \|v\|_X)^{2^*-2}\right) \; .$$

This estimate enables us to apply Picard's fixed point theorem, provided ϵ is small enough, which we assume from now on, and provided that

$$\|\dot{W}(t)v_0\|_X + \|W(t)v_1\|_X \leq c \; , \tag{3.6}$$

for a universal constant c. Pick T so that this last condition is fulfilled; this is possible since $(v_0, v_1) \in \dot{H}^1 \times L^2$.

We get a solution v of (3.2) on $[0,T]$.

We are left with showing that

$$(v, \partial_t v) \in \mathcal{C}([0,T], \dot{H}^1 \times L^2) .$$

But this is a consequence of the following a priori estimate, which is implied by (3.4) and (3.5),

$$\|v\|_{L^\infty([0,T],\dot{H}^1)} + \|\partial_t v\|_{L^\infty([0,T],L^2)} \leq C\big(\|v_0\|_{\dot{H}^1} + \|v_1\|_{L^2} + \|v\|_X(\epsilon + \|v\|_X)^{2^*-2}\big) .$$

Indeed, v is built up using an iterative scheme. At each step, we compute the solution of a linear equation, and this solution therefore has the desired continuity property. Due to the above estimate, this iterative scheme converges in $L^\infty \dot{H}^1 \times L^\infty L^2$, therefore the limit function v is also continuous.

(ii) The following lemma will be useful.

Lemma 3.1 *For any $r > 0$ and $x \in \mathbb{R}^d$, there exists an extension operator*

$$E(x,r) \ : [\dot{H}^1 \cap L^{2^*}](B(x,r)) \to [\dot{H}^1 \cap L^{2^*}](\mathbb{R}^d)$$

which is bounded and verifies $(E(x,r)v)_{|B(x,r)} = v$. Furthermore, its bound does not depend on x or r.

Proof of Lemma 3.1. Take two functions ϕ and ψ such that

- $\phi : \mathbb{R} \to \mathbb{R}$, $\phi(x) = x$ if $0 \leq x \leq 1$ and $\phi(x) = 2 - x$ if $1 \leq x \leq 2$.
- $\psi : \mathbb{R}^d \to \mathbb{R}$, $\psi \in \mathcal{C}^\infty$, $\mathrm{Supp}(\psi) \subset B(0,3/2)$ and $\psi = 1$ on $B(0,1)$.

One can check that the following operator has all the desired properties

$$E(x,r)(v)(y) = \psi\left(\frac{y-x}{r}\right) v\left(x + \frac{r(y-x)}{|y-x|}\phi\left(\frac{|y-x|}{r}\right)\right) .$$

■

Back to the proof of Proposition 3.2: Let us take $x \in \mathbb{R}^d$ and $t < T^*$. By definition of β, we have if $r \leq \beta(t)$

$$\|v(t,\cdot)\|_{\dot{H}^1(B(x,r))} + \|v(t,\cdot)\|_{L^{2^*}(B(x,r))} + \|\partial_t v(t,\cdot)\|_{L^2(B(x,r))} \leq \eta ,$$

(the value of η has not been set yet) and therefore

$$\|E(x,r)(v(t,\cdot))\|_{\dot{H}^1(\mathbb{R}^d)} + \|\chi_{B(x,r)}\partial_t v(t,\cdot)\|_{L^2(\mathbb{R}^d)} \leq C_0\eta ,$$

where C_0 is the norm of $E(x,r)$ and $\chi_{B(x,r)}$ the characteristic function of $B(x,r)$.

Moreover, Strichartz estimates (Theorem 2.1) give

$$\begin{aligned}\|\dot W(t')E(x,r)(v(t,\cdot))\|_{\tilde L^{\mu/\alpha}([0,\infty[,\dot B^{1-\alpha/2}_{\rho,2})} + \|W(t')\chi_{B(x,r)}\partial_t v(t,\cdot)\|_{\tilde L^{\mu/\alpha}([0,\infty[,\dot B^{1-\alpha/2}_{\rho,2})} \\ \leq C_1\Big(\|E(x,r)(v(t,\cdot))\|_{\dot H^1(\mathbb{R}^d)} + \|\chi_{B(x,r)}\partial_t v(t,\cdot)\|_{L^2(\mathbb{R}^d)}\Big) \\ \leq C_1 C_0 \eta\ .\end{aligned}$$

We now set η such that

$$C_1 C_0 \eta \leq c\ .$$

Then the inequality (3.6) holds, with

$$v_0 = E(x,r)(v(t,\cdot)) \qquad \text{and} \qquad v_1 = \chi_{B(x,r)}\partial_t v(t,\cdot)\ ,$$

and one can apply the construction of (i). One gets a global solution of (3.2) for the initial data $(E(x,r)(v(t,\cdot)), \chi_{B(x,r)}\partial_t v(t,\cdot))$. By finite propagation speed, this is, in the space-time truncated cone with base $\{t\}\times B(x,r)$ and vertex $(t+r,x)$, a solution for the initial data $(v(t,\cdot),\partial_t v(t,\cdot))$.

By repeating this construction for all couples (x,r), with $r<\beta(t)$, we get a solution of (3.2) on $[t,t+\beta(t)[\times\mathbb{R}^d$.

If $t+\beta(t)>T^*$, this contradicts the definition of T^*; so there must hold

$$\beta(t)\leq T^*-t\ .$$

This proves (ii). ■

By the above blow-up criterion, in order to show that v can be prolonged past T, we just need to prove that the energy of v does not concentrate.

We focus from now on on the case $d=6$

We will see in Section 3.4 that the energy of v does not concentrate if

$$w\in L^\infty_{\text{loc}}(]0,\infty[, L^{6,\infty}_{\text{loc}})\ .$$

We will first prove, in the next section, that there exists infinite energy initial data (w_0,w_1) in $\dot B^1_{2,\infty}\times\dot B^0_{2,\infty}$ such that the associated solution belongs to $L^\infty_{\text{loc}}(]0,\infty[, L^{6,\infty}_{\text{loc}})$. We will prove this result by studying a self-similar solution.

3.3. Study of a self-similar solution

Recall the space dimension has been set equal to 6; we denote the solution of the Cauchy problem by w.

$$\begin{cases}\Box w + |w|w = 0 \\ \quad w_{|t=0}(x) = w_0(x) \overset{\text{def}}{=} \dfrac{c}{|x|^2} \\ \quad \partial_t w_{|t=0} = 0\ .\end{cases}$$

Since $d = 6$, $2^* = 3$, and we are therefore considering the critical equation $(NLW)_{2^*-1}$. Notice that the initial data $w_0(x) = \frac{c}{|x|^2}$ belongs to $\dot{B}^1_{2,\infty}$. Choosing c small enough, we get, thanks to Proposition 3.1, a global solution of the above Cauchy problem, which is unique in a ball of $L^\infty(\mathbb{R}, \dot{B}^1_{2,\infty}) \cap \widetilde{L}^\mu(\mathbb{R}, \dot{B}^{1/2}_{\mu,\infty})$.

Moreover, w_0 is self-similar, hence there exists a profile ψ, radially symmetric, such that

$$w(t,x) = \frac{1}{t^2}\psi\left(\frac{x}{t}\right) .$$

Proposition 3.3 *The profile ψ belongs locally to $L^{6,\infty}$.*

Proof. We will first employ Strichartz estimates, in order to get as much regularity as possible on ψ. But this will not be enough, and we will have to use the ordinary differential equation satisfied by ψ.

1. Let us first examine ψ away from 0. Since $w \in L^\infty \dot{B}^1_{2,\infty}$, we also have $\psi \in \dot{B}^1_{2,\infty}$. Since ψ is radial, this implies that it is continuous away from 0.

So we just have to ensure that ψ does not have a too singular behavior in 0.

2. We use now Strichartz estimates: Theorem 2.1 gives

$$\|w\|_{\widetilde{L}^q \dot{B}^s_{r,\infty}} \le C\left(\|w_0\|_{\dot{B}^1_{2,\infty}} + \|w_1\|_{\dot{B}^0_{2,\infty}} + \|w|w|\|_{\widetilde{L}^\nu \dot{B}^{1/2}_{\nu,\infty}}\right)$$

if s is given by $s = \frac{1}{q} + \frac{6}{r} - 2$, and if q and r verify conditions (2.2). It follows by Lemma 2.4 that

$$\|w\|_{\widetilde{L}^q \dot{B}^s_{r,\infty}} \le C\left(\|w_0\|_{\dot{B}^1_{2,\infty}} + \|w_1\|_{\dot{B}^0_{2,\infty}} + \|w\|^2_{\widetilde{L}^{\mu/\alpha} \dot{B}^{1-\alpha/2}_{\rho,\infty}}\right) < \infty$$

since $w \in L^\infty(\mathbb{R}, \dot{B}^1_{2,\infty}) \cap \widetilde{L}^\mu(\mathbb{R}, \dot{B}^{1/2}_{\mu,2}) \hookrightarrow \widetilde{L}^{\mu/\alpha}(\mathbb{R}, \dot{B}^{1-\alpha/2}_{\rho,\infty})$. So we get

$$w \in \widetilde{L}^q \dot{B}^s_{r,\infty} \qquad \text{with } s = \frac{1}{q} + \frac{6}{r} - 2 , \tag{3.7}$$

if q and r satisfy conditions (2.2).

3. We will now deduce, from the belonging of w to the functional spaces above, that ψ belongs to certain Sobolev spaces, with the help of an argument used by Planchon [17]. If q, r and s verify conditions (2.2) and (3.7), we know that

$$\sup_j 2^{js} \|\Delta_j w\|_{L^q L^r} < \infty ;$$

till now j was an integer; from now on we consider it as a continuous parameter. Obviously,

$$\Delta_j w(x,t) = t^{-2}\left(\Delta_{j+\ln_2 t}\psi\right)\left(\frac{x}{t}\right) .$$

This implies that

$$\|\Delta_j w(\cdot,t)\|_r = t^{-2+6/r}\|\Delta_{j+\ln_2 t}\psi\|_r \ .$$

But s, r and q are linked by the relation $-2+\frac{6}{r} = s - \frac{1}{q}$. Taking the L^q norm in t of the above equality, for $j=0$, we obtain

$$\|\Delta_0 w\|_{L^q L^r} = \left[\int_{\mathbb{R}^+} \left(t^s \|\Delta_{\ln_2(t)}\psi\|_r\right)^q \frac{dt}{t}\right]^{1/q} \ .$$

This last expression is equivalent to the norm of ψ in $\dot{B}^s_{r,q}$. As a conclusion, if $s = \frac{1}{q} + \frac{6}{r} - 2$, and if q and r verify (2.2), ψ belongs to $\dot{B}^s_{r,q}$.

4. Does this last information suffice to affirm that $w \in L^{6,\infty}$ locally? Heuristically, this would correspond to $s=0$, $r=6$; due to the scaling, this implies $q=1$, but this is a forbidden value for q because of (2.2).

But we can choose $s=0$, $r=4$ and $q=2$; this gives $\psi \in \dot{B}^0_{4,2} \hookrightarrow L^4$.

Let us now denote by $\widetilde{\psi}$ the function of a real variable such that

$$\widetilde{\psi}(|x|) \overset{\text{def}}{=} \psi(x) \ .$$

It is easy to show that $\widetilde{\psi}$ verifies the following equation, already used by Kavian and Weissler in [11]

$$\text{if } r>0 \ , \qquad (r^2-1)\widetilde{\psi}''(r) + (6r - \frac{5}{r})\widetilde{\psi}'(r) + 6\widetilde{\psi}(r) + \widetilde{\psi}(r)|\widetilde{\psi}(r)| = 0 \ .$$

We consider this equation on $(0,\frac{1}{2})$.

We have seen that $r^{5/4}\widetilde{\psi}(r)$ belongs to L^4. This implies that, if $r \in (0,\frac{1}{2})$,

$$(r^3-r)\widetilde{\psi}''(r) + (6r^2-5)\widetilde{\psi}'(r) = \frac{g(r)}{r^{3/2}} \ ,$$

with

$$g(r) = -6r^{5/2}\widetilde{\psi}(r) - r^{5/2}\widetilde{\psi}(r)|\widetilde{\psi}(r)| \quad \in L^2([0,1/2]) \ .$$

We now change variables by setting $z(r) = r^5\widetilde{\psi}'(r)$. Then,

$$(r^3-r)\widetilde{\psi}''(r) + (6r^2-5)\widetilde{\psi}'(r) = -\frac{\sqrt{1-r^2}}{r^4}\left(\sqrt{1-r^2}z(r)\right)' \ .$$

We can also write this as

$$-\left(\sqrt{1-r^2}z(r)\right)' = r^{5/2}f(r) \ ,$$

with

$$f(r) = \frac{g(r)}{\sqrt{1-r^2}} \quad \in L^2([0,1/2]) \ .$$

But then $\sqrt{1-r^2}z(r)$ is a continuous function on $(0,\frac{1}{2})$, which necessarily has a zero limit in 0 (if this limit was $\theta > 0$, we would have $\widetilde{\psi}'(r) \overset{r\to 0}{\sim} \frac{\theta}{r^5}$, and $r^{5/4}\widetilde{\psi}(r)$ would not belong to L^4). By integrating the above expression, we get therefore

$$\left|\sqrt{1-r^2}z(r)\right| \leq \int_0^r |t^{5/2}f(t)|dt \leq \|f\|_2 \left(\int_0^r t^5 dt\right)^{1/2} \leq Cr^3 \ ,$$

if $r \in (0,\frac{1}{2})$. This last estimate enables us to come back to $\widetilde{\psi}'$, which can be bounded if $r \in (0,\frac{1}{2})$ by

$$|\widetilde{\psi}'(r)| \leq Cr^{-2} \ ,$$

and it suffices to integrate this inequality to see that

$$|\widetilde{\psi}(r)| \leq Cr^{-1} \ ,$$

so in other words ψ belongs to $L^{6,\infty}$ in a neighbourhood of 0. This concludes the proof of the proposition. ■

3.4. Control of the energy and proof of Theorem B

We shall, in this section, conclude the proof of Theorem B, by showing that the local solution v given by Theorem 3.2 can be prolonged to a global solution provided w is chosen as in section 3.3.

Proposition 3.4 *Let us suppose $d=6$, and let us take initial data (w_0,w_1) and (v_0,v_1) such that*

1. $w_0 = \dfrac{c}{|x|^2}$, *with c small enough*
2. $w_1 = 0$
3. $(v_0,v_1) \in \dot{H}^1 \times L^2$.

Then there exists a global solution v of (3.2) *such that*

$$(v,\partial_t v) \in \mathcal{C}(\mathbb{R},\dot{H}^1 \times L^2) \ .$$

Proof. Shatah and Struwe's theorem (Theorem 1.1) corresponds to the case $w=0$. We will follow the scheme of the proof of Shatah and Struwe, and show that this proof is "stable by a well-chosen perturbation".

Let us first summarize already known results

- By Proposition 3.3, we know that $w(t,x) = \frac{1}{t^2}\psi\left(\frac{x}{t}\right)$, with ψ locally in $L^{6,\infty}$.

- By Proposition 3.2, we know there exists a local solution v of (3.2), such that $(v, \partial_t v) \in \mathcal{C}([0,T[, \dot{H}^1 \times L^2)$, for some $T > 0$. Let us reason now by contradiction, and assume that T is finite and maximal. We shall prove that v can actually be prolonged past the time T, and this will prove the proposition.

Restriction to a bounded domain

The theorem of Shatah and Stuwe enables us to treat the case of finite energy initial data. The initial data we consider here is locally of finite energy on $\mathbb{R}^6 \setminus \{0\}$. By finite speed of propagation, the solution u can explode only if it lies on the influence cone of 0. In particular, we can prolong u on $[T, T+1] \times (\mathbb{R}^6 \setminus B(0, T+2))$. So we just have to prolong u on $[T, T+\epsilon] \times B(0, T+2)$ with $\epsilon > 0$ to conclude.

As a consequence, due to the finite propagation speed, we can **assume from now on that the initial data** (v_0, v_1) **and** $w_0, w_1)$ **are compactly supported**.

Control of the total energy

Our aim in this paragraph is to show that the energy of v remains bounded until T. To do so, let us multiply the equation verified by v by $\partial_t v$

$$\partial_t v\, \partial_t^2 v - \partial_t v\, \Delta v + \partial_t v\, v\, |v| + \partial_t v\, ((v+w)\,|v+w| - w\,|w| - v\,|v|) = 0$$

and then let us integrate the above equality on $[\frac{T}{2}, \frac{T}{2}+\tau] \times \mathbb{R}^6$. To justify this manipulation, we should first mollify the equation, and then use a limiting argument. In order to avoid supplementary technicalities, we do not perform this procedure, and refer rather to Shatah and Struwe [22]. We get

$$\begin{aligned} &E(v, T/2+\tau, \mathbb{R}^6) + \text{flux} \\ &\quad = E(v, T/2, \mathbb{R}^6) - \int_{T/2}^{T/2+\tau} \int_{\mathbb{R}^6} \partial_t v\, ((v+w)|v+w| - w|w| - v|v|)\, dx\, dt\ , \end{aligned}$$

where flux is a positive quantity, and we denote, if Ω is a space domain,

$$E(v, t, \Omega) \stackrel{\text{def}}{=} \frac{1}{2}\|\nabla v(t, \cdot)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\partial_t v(t, \cdot)\|^2_{L^2(\Omega)} + \frac{1}{3}\|v(t, \cdot)\|^3_{L^3(\Omega)}\ .$$

We have then

$$\begin{aligned} &E(v, T/2+\tau, \mathbb{R}^6) \\ &\leq E(v, T/2, \mathbb{R}^6) - 2\int_{T/2}^{T/2+\tau} \int_{\mathbb{R}^6} \int_0^1 \partial_t v\, v\, (|w+sv| - |sv|)\, ds\, dx\, dt\ . \end{aligned} \tag{3.8}$$

We notice that for any real numbers a and b, $\big|\,|a+b|-|b|\,\big| \leq |a|$, which implies

$$\big\| |w+sv|(t,\cdot) - |sv|(t,\cdot) \big\|_{L^{6,\infty}} \leq \|w(t,\cdot)\|_{L^{6,\infty}}$$

Let us now come back to (3.8): the above inequality and the product law

$$\|fg\|_{L^2} \leq C\|f\|_{\dot{H}^1}\|g\|_{L^{6,\infty}}$$

(which is a consequence of the sharp Sobolev embedding $\dot{H}^1 \hookrightarrow L^{3,2}$, see for instance Lemarié [14], Theorem 2.4.) imply that

$$\begin{aligned} &E(v, T/2+\tau, \mathbb{R}^6) \\ &\leq E(v, T/2, \mathbb{R}^6) + C\tau\|w\|_{L^\infty([\frac{T}{2},\frac{T}{2}+\tau], L^{6,\infty})}\|v\|_{L^\infty([\frac{T}{2},\frac{T}{2}+\tau],\dot{H}^1)}\|\partial_t v\|_{L^\infty([\frac{T}{2},\frac{T}{2}+\tau],L^2)} \,. \end{aligned}$$

(since we retricted to a bounded domain and are away from 0 in time, we can consider that $w \in L^\infty([\frac{T}{2}, \frac{T}{2}+\tau], L^{6,\infty})$). Denoting

$$\widetilde{E}(\tau) = \sup_{s\in\left[\frac{T}{2},\frac{T}{2}+\tau\right]} E(v, s, \mathbb{R}^6) \ ,$$

it follows from the last inequality that

$$\Big(1 - C\tau\|w\|_{L^\infty([\frac{T}{2},\frac{T}{2}+\tau],L^{6,\infty})}\Big)\, \widetilde{E}(v,\tau) \leq \widetilde{E}(v,0) \ .$$

Now if we pick τ such that $C\tau\|w\|_{L^\infty L^{6,\infty}(\mathcal{B}')} < \frac{1}{2}$, the above inequality enables to control $\widetilde{E}(v,\tau)$, and hence $E(v,s,\mathbb{R}^6)$ on $[\frac{T}{2}, \frac{T}{2}+\tau]$. We can iterate this argument to control $E(v,s,\mathbb{R}^6)$ on $[\frac{T}{2}, \frac{T}{2}+2\tau]$, and after a finite number of steps on $[\frac{T}{2}, T]$.

As a conclusion, $E(v,s,\mathbb{R}^6)$ is bounded by a constant, which we denote $\mathcal{E}$, on $[0,T]$.

Restriction to a cone

To make notations lighter, we perform a translation in time of $-T$. So from now on, we consider v a local solution of

$$\text{(3.9)} \qquad \begin{cases} \Box v + |v+w|\,(v+w) - |w|\,w = 0 \\ v_{|t=-T} = v_0 \\ \partial_t v_{|t=-T} = v_1 \ . \end{cases}$$

We know that $(v, \partial_t v) \in \mathcal{C}([-T, 0[, \dot{H}^1 \times L^2)$, and we would like to prolong v past $t=0$; since we restricted the problem to a compact domain, we can assume that the perturbation w belongs to $L^\infty_{\text{loc}}(]-T, +\infty[, L^{6,\infty})$.

We showed in the previous paragraph that the energy of v remains bounded until $t = 0$; we will now prove that it does not concentrate. We will work in a backward light cone with vertex $z_0 = (0, x_0)$; up to a space translation, we can always assume that $x_0 = 0$.

We will use the following notations

- If $s < s' < 0$, $K_s^{s'}$ is the slice of the backward light cone given by

$$K_s^{s'} \stackrel{\text{def}}{=} \{(t,x) \in [s,s'] \times \mathbb{R}^d \,,\, |x| \le |t|\} \ .$$

- If $s < s' < 0$, $M_s^{s'}$ is the mantel of the backward light cone given by

$$M_s^{s'} \stackrel{\text{def}}{=} \{(t,x) \in [s,s'] \times \mathbb{R}^d \,,\, |x| = |t|\} \ .$$

- If $s < 0$, the disk D_s is the section of the cone by the hyperplane $t = s$

$$D_s \stackrel{\text{def}}{=} \{(s,x) \,,\, x \in \mathbb{R}^d \,,\, |x| \le |s|\} \ .$$

The flux on the mantel of the cone goes to 0

If $-T < s < s' < 0$, the local energy identity is obtained by multiplying (3.2) by $\partial_t v$, and then by integrating on $K_s^{s'}$. It reads (see Shatah and Struwe [22])

$$\begin{aligned} E(v,s',D_{s'}) &+ \operatorname{flux}(v, M_s^{s'}) \\ &= E(v,s,D_s) + \int_{K_s^{s'}} \partial_t v\, v \int_0^1 (|w+\tau v| - |\tau v|)\, d\tau\, dx\, dt \ , \end{aligned}$$

with

$$\operatorname{flux}(v, M_s^{s'}) = \frac{1}{\sqrt{2}} \int_{M_s^{s'}} \left(\frac{|\nabla v - \frac{x}{|x|}\partial_t v|^2}{2} + \frac{1}{p+1}|v|^{p+1} \right) d\sigma \ ,$$

where σ is the surface measure on $M_s^{s'}$.

Here again, we should mollify the equation to obtain this identity, but we skip that step.

Using as above the identity $\big|\,|a+b| - |b|\,\big| \le |a|$ true for any real numbers a and b, we obtain

$$E(v,s',D_{s'}) + \operatorname{flux}(v, M_s^{s'}) \le E(v,s,D_s) + \mathcal{E} \int_s^{s'} \|w(\tau)\|_{L^{6,\infty}} d\tau$$

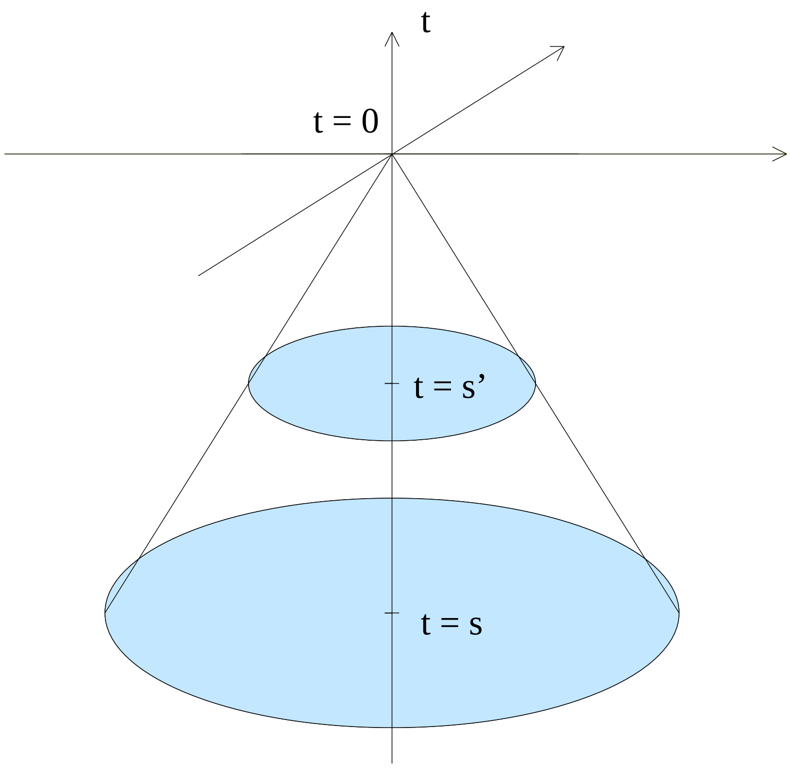

Figure 1: The backward light cone with vertex $(0,0)$. The disks D_s and $D_{s'}$ have been colored up.

and then

$$\mathrm{flux}(v, M_s^{s'}) \leq \mathcal{E} + \mathcal{E}\int_s^{s'} \|w(\tau)\|_{L^{6,\infty}} d\tau \ .$$

Let us suppose by contradiction that $\mathrm{flux}(v, M_s^{s'}) \overset{s,s'\to 0}{\not\longrightarrow} 0$. Then there exists $\delta > 0$ and an increasing sequence (s_n) of $[-\frac{T}{2}, 0[$, going to 0, such that

$$\mathrm{flux}(v, M_{s^n}^{s^{n+1}}) \geq \delta \ .$$

But this implies that, if $N \in \mathbb{N}$ is big enough,

$$\mathrm{flux}(v, M_{s^0}^{s^N}) \geq N\delta \geq \mathcal{E} + \mathcal{E}\int_{-T/2}^{0} \|w(\tau)\|_{L^{6,\infty}} d\tau \ ,$$

which is absurd. So if $-T < s < 0$, $\mathrm{flux}(v, M_s^0)$ is well defined and

$$\mathrm{flux}(v, M_s^0) \overset{s\to 0}{\longrightarrow} 0 \ . \tag{3.10}$$

The $L^\infty L^3$ norm in the cone goes to 0

If $-T < s < 0$, we get, using Morawetz' identity (see Shatah and Struwe [22]),

$$\begin{aligned}\int_{D_s} |v|^3\,dx &\le C\left(\mathrm{flux}(v, M_s^0) + |s|^a\,\mathrm{flux}(v, M_s^0)^b\right)\right) \\ &\qquad + \frac{C}{|s|}\left|\int_{K_s^0} v\int_0^1 (|w+\tau v| - |\tau v|)\,d\tau\left(t\partial_t v + x\cdot\nabla v + \frac{5}{2}v\right)dxdt\right| \\ &\overset{\mathrm{def}}{=} I(s) + II(s)\ ,\end{aligned}$$

where a and b are positive numbers. Since the flux goes to 0 (identity (3.10)), it is obvious that $I(s) \overset{s\to 0}{\longrightarrow} 0$. As for $II(s)$, we observe that, if $-T < t < 0$,

$$\begin{aligned}\|t\,\partial_t v\|_{L^2(D_t)} &\le |t|\mathcal{E}^{1/2} \\ \|x\cdot\nabla v\|_{L^2(D_t)} &\le |t|\mathcal{E}^{1/2} \\ \|v\|_{L^2(D_t)} &\le \|1\|_{L^6(D_t)}\|u\|_{L^3(D_t)} \le |t|\mathcal{E}^{1/3}\ ,\end{aligned}$$

where we used Hölder's inequality in the last line. The bounds above give

$$\begin{aligned}|II(s)| &\le \frac{C}{|s|}\int_s^0 \|v(t)\|_{\dot H^1(D_t)}\|w(t)\|_{L^{6,\infty}}\left\|t\partial_t v + x\cdot\nabla v + \frac{5}{2}v\right\|_{L^2(D_t)} dt \\ &\le \frac{C}{|s|}\int_s^0 \|w(t)\|_{L^{6,\infty}}(\mathcal{E}+\mathcal{E}^{5/6})|t|\,dt \overset{s\to 0}{\longrightarrow} 0\ ,\end{aligned}$$

since $\|w(t)\|_{L^{6,\infty}}$ is bounded if $t \ge C > -T$. So we reach the desired conclusion

$$\int_{D_s} |v|^3\,dx \overset{s\to 0}{\longrightarrow} 0\ . \tag{3.11}$$

The $L^\mu \dot B^{1/2}_{\mu,2}$ norm in the cone is finite

Let us first notice that, if B is a ball of radius r, and f a function defined on B,

$$\|f\|_{L^{7/2}(B)} \le C(r)\|f\|_{L^{6,\infty}(B)}$$

with $C(r) \overset{r\to 0}{\longrightarrow} 0$. In particular, since w is bounded in $L^\infty L^{6,\infty}$, we have

$$\|w\|_{(L^\infty L^{7/2})(K_s^0)} \overset{s\to 0}{\longrightarrow} 0\ . \tag{3.12}$$

Moreover, it is possible to localize the Strichartz estimates given by Theorem 2.1 to a cone: see [20] [21]. So we get for $-T < s < s' < 0$

$$\begin{aligned}
\|v\|_{(L^\mu \dot B^{1/2}_{\mu,2})(K_s^{s'})} &\le C\Big(\|v(s,\cdot)\|_{\dot H^1(D_s)} + \|\partial_t v(s,\cdot)\|_{L^2(D_s)} \\
&\qquad + \Big\| |v+w|(v+w) - |w|w \Big\|_{(L^\nu \dot B^{1/2}_{\nu,2})(K_s^{s'})}\Big) \\
&\le C\Big(\mathcal{E}^{1/2} + \Big\| v\int_0^1 |w+\tau v| d\tau \Big\|_{(L^\nu \dot B^{1/2}_{\nu,2})(K_s^{s'})}\Big) \\
&\le C\Big(\mathcal{E}^{1/2} + \|v\|_{(L^\mu \dot B^{1/2}_{\mu,2})(K_s^{s'})}\Big[\|v\|_{L^{7/2}(K_s^{s'})} + \|w\|_{L^{7/2}(K_s^{s'})}\Big]\Big);
\end{aligned} \tag{3.13}$$

we used in the last inequality the product law

$$\|f\,g\|_{(L^\nu \dot B^{1/2}_{\nu,2})(K_s^{s'})} \le C\|f\|_{L^{7/2}(K_s^{s'})}\|g\|_{(L^\mu \dot B^{1/2}_{\mu,2})(K_s^{s'})} ,$$

for any functions f and g, see Shatah and Struwe [20]. It is also proved in that reference that, for any function f,

$$\|f\|_{L^{7/2}(K_s^{s'})} \le \|f\|^\theta_{(L^\infty L^3)(K_s^{s'})}\|f\|^{1-\theta}_{(L^\mu \dot B^{1/2}_{\mu,2})(K_s^{s'})} ,$$

with $\theta \in]0,1[$. Coming back to (3.13), we have

$$\begin{aligned}
&\|v\|_{(L^\mu \dot B^{1/2}_{\mu,2})(K_s^{s'})} \\
&\le C\Big(\mathcal{E}^{1/2} + \|v\|_{(L^\mu \dot B^{1/2}_{\mu,2})(K_s^{s'})}\|w\|_{L^{7/2}(K_s^{s'})} + \|v\|^{2-\theta}_{(L^\mu \dot B^{1/2}_{\mu,2})(K_s^{s'})}\|v\|^\theta_{(L^\infty L^3)(K_s^{s'})}\Big).
\end{aligned} \tag{3.14}$$

Thanks to (3.12), for s close enough to 0, we have $C\|w\|_{L^{7/2}(K_s^0)} < \frac{1}{2}$. If we denote by $G(s')$ the quantity $\|v\|_{(L^\mu \dot B^{1/2}_{\mu,2})(K_s^{s'})}$, the above bound gives

$$G(s') \le C\left(\mathcal{E}^{1/2} + G(s')^{2-\theta}\|v\|^\theta_{(L^\infty L^3)(K_s^0)}\right) .$$

The following lemma will enable us to conclude (for the proof, see for instance Bahouri and Gérard [1]).

Lemma 3.2 *Let $M(t)$ be a continuous function on $[0, A]$, with $A > 0$, such that*

$$M(t) \le a + bM(t)^k ,$$

where a and b are positive numbers, $k > 1$,

$$a < \left(1 - \frac{1}{k}\right)\frac{1}{(kb)^{1/(k-1)}} \qquad \text{and} \qquad M(0) \le \frac{1}{(kb)^{1/(k-1)}} . \tag{3.15}$$

Then, for any $t \in [0, A]$,

$$M(t) \le \frac{k}{k-1}a .$$

We know that

$$\|v\|_{(L^\infty L^3)(K_s^0)} \stackrel{s\to 0}{\longrightarrow} 0 \ ,$$

so we can choose s such that, setting

$$M(s'-s) = G(s') \quad , \quad a = C\mathcal{E}^{1/2} \quad , \quad b = C\|v\|^\theta_{(L^\infty L^3)(K_s^0)} \quad \text{and} \quad k = 2-\theta \ ,$$

the conditions (3.15) are verified. Let us then apply the above lemma; we get, for $s' > s$,

$$G(s') \le C \ ,$$

that is

$$\|v\|_{(L^\mu \dot{B}^{1/2}_{\mu,2})(K_s^0)} < \infty \ . \tag{3.16}$$

The energy in the cone goes to 0

Using Duhamel's formula, we can write v in integral form

$$\begin{aligned} v(t) = &\dot{W}(t-s)v(s) + W(t-s)\partial_t v(s) \\ &+ \int_0^{t-s} W(t-t')v(s+t') \int_0^1 |w(s+t') + \tau v(s+t')| \, d\tau \, dt' \ , \end{aligned}$$

if $-T < s < t < 0$. Denoting by $f(t)$ the last term of the right-hand side of the above expression, Theorem 2.1 localized on the cone K_s^0 gives

$$\begin{aligned} \|f\|_{(L^\infty \dot{H}^1)(K_s^0)} + \|\partial_t f\|_{(L^\infty L^2)(K_s^0)} &\le C \left\| v \int_0^1 |w+\tau v| \, d\tau \right\|_{(L^\nu \dot{B}^{1/2}_{\nu,2})(K_s^0)} \\ &\le C \left(\|v\|_{(L^\mu \dot{B}^{1/2}_{\mu,2})(K_s^0)} \|w\|_{L^{7/2}(K_s^0)} + \|v\|^{2-\theta}_{(L^\mu \dot{B}^{1/2}_{\mu,2})(K_s^0)} \|v\|^\theta_{(L^\infty L^3)(K_s^0)} \right) , \end{aligned}$$

where the last inequality is obtained using the same bounds as for (3.14). But we know by (3.16) that $\|v\|_{(L^\mu \dot{B}^{1/2}_{\mu,2})(K_s^0)}$ is bounded for $-T<s<s'<0$, whereas $\|w\|_{L^{7/2}(K_s^0)}$ and $\|v\|_{(L^\infty L^3)(K_s^0)}$ go to 0 with s (relations (3.12) and (3.11)). Consequently,

$$\|f\|_{(L^\infty \dot{H}^1)(K_s^0)} + \|\partial_t f\|_{(L^\infty L^2)(K_s^0)} \stackrel{s\to 0}{\longrightarrow} 0 \ .$$

Moreover, denoting $g(t) = \dot{W}(t-s)v(s) + W(t-s)\partial_t v(s)$, it is clear that, for fixed $s < 0$,

$$\|g\|_{(L^\infty \dot{H}^1)(K_t^0)} + \|\partial_t g\|_{(L^\infty L^2)(K_t^0)} \stackrel{t\to 0}{\longrightarrow} 0 \ .$$

The two above limits imply that

$$\|v\|_{(L^\infty \dot{H}^1)(K_s^0)} + \|\partial_t v\|_{(L^\infty L^2)(K_s^0)} \stackrel{s\to 0}{\longrightarrow} 0 \ , \tag{3.17}$$

that is to say that **the energy of v does not concentrate**.

Conclusion of the proof of Proposition 3.4

Thanks to the limits (3.11) and (3.17), there exists $\delta > 0$ such that

$$\|v(-\delta,\cdot)\|_{\dot H^1(B(0,\delta))} + \|v(-\delta,\cdot)\|_{L^{2^*}(B(0,\delta))} + \|\partial_t v(-\delta,\cdot)\|_{L^2(B(0,\delta))} < \frac{\eta}{2} .$$

We can therefore pick $\epsilon > 0$ such that

$$\|v(-\delta,\cdot)\|_{\dot H^1(B(0,\delta+\epsilon))} + \|v(-\delta,\cdot)\|_{L^{2^*}(B(0,\delta+\epsilon))} + \|\partial_t v(-\delta,\cdot)\|_{L^2(B(0,\delta+\epsilon))} < \frac{3\eta}{4} .$$

The proof of Proposition 3.2 shows then that one can build up a solution of (3.2) on the truncated cone with base $\{-\delta\}\times B(0,\delta+\epsilon)$ and vertex $(\epsilon, 0)$.

This result of course holds for any $x \in \mathbb{R}^d$: for any $x \in \mathbb{R}^d$, there exists $\epsilon_x > 0$ such that one can prolong v on the cone with base $\{0\}\times B(x,\epsilon_x)$ and vertex $(\epsilon_x, 0)$.

Since we restricted to a compact domain, there exists a finite number of balls $B(x,\epsilon_x)$ which cover this domain. We can then deduce the existence of $\epsilon > 0$ such that one can prolong v on $[0,\epsilon]$. This contradicts the definition of T and proves the "existence" part of the theorem. The proof of the "uniqueness" part is identical to the proof of uniqueness in Theorem A, so we do not give it here. Theorem B is proved. ■

4. Appendix: some functional spaces

4.1. Besov spaces

Besov spaces play a key role in this article. We define them briefly below. For more information on these spaces, see Runst and Sickel [18]. Let us first introduce a homogeneous Littlewood-Paley decomposition. Consider ψ such that

$$\psi \in \mathcal{S} \ , \quad \mathrm{Supp}(\psi) \subset \mathcal{C}(0, 3/4, 8/3) \ , \text{ and } \quad \sum_{j\in\mathbb{Z}} \psi(2^{-j}\xi) = 1 \quad \text{for } \xi \neq 0 .$$

Then we define the Fourier multipliers

$$\Delta_j \stackrel{\text{def}}{=} \psi(2^{-j}D) \qquad \text{and} \qquad S_j \stackrel{\text{def}}{=} 1 - \sum_{k\geq j-1} \Delta_k \ ,$$

and for $s < d/p$ the Besov spaces $\dot B^s_{p,q}$ by the norm

$$\|f\|_{\dot B^s_{p,q}} = \left(\sum_{j\in\mathbb{Z}} \left[2^{js}\|\Delta_j f\|_p\right]^q \right)^{1/q} .$$

4.2. Chemin-Lerner spaces

These spaces have been introduced first by Chemin and Lerner [5]. They are defined by the following norm

$$\|u\|_{\widetilde{L}^r([0,T],\dot{B}^s_{p,q})} = \left[\sum_{j\in\mathbb{Z}} \left(2^{js}\|\Delta_j u\|_{L^r([0,T],L^p)}\right)^q \right]^{1/q} .$$

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