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Boundedness of Cauchy singular integral operator under Hölder norm

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Abstract. In this article, we will investigate the boundedness of the Cauchy singular integral operator under Hölder norm. This boundedness can be obtained by improving the method used in the early proof of the famous 2P theorems (Plemelj–Sokhotski formula and Privalov–Muskhelishvili theorem) by Jinyuan Du, which deepens the 2P theorems. Then, various Cauchy singular integral operators on the system of curves or open arcs are also proved to be bounded.

1. Introduction and notations

Assume that f is defined on a set Ω of the complex plane $\mathbb{C}$. If

$$|f(z_1) - f(z_2)| \leq A|z_1 - z_2|^{\mu} \quad (0 < \mu \leq 1),$$

for arbitrary points z_1, z_2 on Ω, where A and μ are definite constants, or

$$\mathcal{M}_{\Omega}[f, \mu] = \sup\left\{\frac{|f(z) - f(w)|}{|z - w|^{\mu}}, z, w \in \Omega, z \neq w\right\} \tag{1.1}$$

is a definite constant, then f is said to satisfy Hölder condition of order μ on Ω, denoted by $f \in H^{\mu}(\Omega)$.

Let Γ be a simple arc-wise (positive) smooth curve (closed or open), oriented counterclockwise. If Γ is closed, then it divides the complex plane $\mathbb{C}$ into two domains, a bounded region and a unbounded region, denoted, respectively, by Ω^+ and Ω^-. Clearly, $\Omega^- = \mathbb{C} \backslash \overline{\Omega^+}$. Sometimes, we write them in detail as $\Omega^+(\Gamma)$ and $\Omega^-(\Gamma)$, respectively.

For $\varphi \in H^{\mu}(\Gamma)$, let

$$(C_{\Gamma}[\varphi])(z) = \frac{1}{2\pi i}\int_{\Gamma}\frac{\varphi(\tau)}{\tau - z}\mathrm{d}\tau, \quad z \in \mathbb{C}\backslash\Gamma, \tag{1.2}$$

and

$$(S_{\Gamma}[\varphi])(t) = \frac{1}{\pi i}\int_{\Gamma}\frac{\varphi(\tau)}{\tau - t}\,\mathrm{d}\tau, \quad t \in \Gamma \quad (t \text{ is not an endpoint if } \Gamma \text{ is open}). \tag{1.3}$$

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It is obvious that $(C_\Gamma[\varphi])(z)$ and $(S_\Gamma[\varphi])(t)$ are, respectively, the Cauchy type integral and the Cauchy principal value integral or Cauchy singular integral. As is well known, the Cauchy type integral and the Cauchy singular integral are the most important tools in the theory of boundary value problems for analytic functions. The two important properties of $C_\Gamma[\varphi]$ and $S_\Gamma[\varphi]$ are, respectively, the famous Plemelj–Sokhotski formula and Privalov–Muskhelishvili theorem, for short, 2P theorems [1,8,11,13,14], which are the cornerstones of the boundary value theory of analytic functions. In [5,6], the 2P theorems are unified under the framework of Hölder property of the projections of the Cauchy type integral stated as follows.

Theorem 1.1 (Hölder continuity of projections of the Cauchy type integral). *If Γ is a closed arc-wise smooth curve and $\varphi \in H^\mu(\Gamma)$ with $0 < \mu < 1$, then*

$$C_\Gamma^+[\varphi] \in H^\mu(\overline{\Omega^+}), \quad C_\Gamma^-[\varphi] \in H^\mu(\overline{\Omega^-}), \tag{1.4}$$

where the projections C_Γ^+ and C_Γ^- are defined by

$$(C_\Gamma^+[\varphi])(z) = \begin{cases} (C_\Gamma[\varphi])(z) & \text{if } z \in \Omega^+, \\ \left(1 - \dfrac{\theta_t}{2\pi}\right)\varphi(t) + \dfrac{1}{2}(S_\Gamma[\varphi])(t) & \text{if } z = t \in \Gamma, \end{cases} \tag{1.5}$$

and

$$(C_\Gamma^-[\varphi])(z) = \begin{cases} (C_\Gamma[\varphi])(z) & \text{if } z \in \Omega^-, \\ -\dfrac{\theta_t}{2\pi}\varphi(t) + \dfrac{1}{2}(S_\Gamma[\varphi])(t) & \text{if } z = t \in \Gamma, \end{cases} \tag{1.6}$$

while θ_t is the angle spanned by two one-sided tangents of Γ at t towards the positive side of Γ ($0 \le \theta_t \le 2\pi$), called the opening angle at t. In particular, if t is a smooth point on Γ, denoted by $t \in \mathrm{smooth}(\Gamma)$, $\theta_t = \pi$, then

$$(C_\Gamma^\pm[\varphi])(t) = \pm\frac{1}{2}\varphi(t) + \frac{1}{2}(S_\Gamma[\varphi])(t) \quad \text{if } t \in \mathrm{smooth}(\Gamma). \tag{1.7}$$

And a simple proof of this theorem is given in [5,6]. Monograph [11] reproduced this proof. This theorem was later extended to the field of hypercomplex analysis [3,4].

In addition, the Cauchy singular integral operator is also the cornerstone of the theory of singular integral equations [9,10,12].

For $f \in H^\mu(\Omega)$, the classical Hölder norm is defined by

$$\|f\|_{H^\mu(\Omega)} = \|f\|_\Omega + \mathcal{M}_\Omega[f,\mu], \tag{1.8}$$

where

$$\|f\|_\Omega = \sup\left\{|f(t)|, t \in \Omega\right\} \tag{1.9}$$

is the Chebyshev norm of f on Ω and $\mathcal{M}_\Omega[\varphi,\mu]$ is the Hölder semi-norm of f on Ω given in (1.1).

Later, the notation $H^\mu(\Omega)$ is also used to denote the function space of all $f \in H^\mu(\Omega)$ with Hölder norm $\|f\|_{H^\mu(\Omega)}$ given by (1.8). $H^\mu(\Omega)$ is also called Hölder space, which is a Banach space [2, 10]. In particular, $H^{\mu}(\Gamma)$ is a Banach space. When $\Gamma = L$ is a closed smooth curve, $S_L[\varphi] \in H^{\mu}(L)$ $(0 < \mu < 1)$ [11, 14]. So, one wants to explore the boundedness of the Cauchy singular integral operator

$$S_L : H^{\mu}(L) \to H^{\mu}(L), \quad \varphi \mapsto S_L[\varphi],$$

where $0 < \mu < 1$.

In [9, 12], the researchers got the following classical result, see [9, Theorem 6.1] or [12, Theorem 4.7].

Theorem 1.2 (Boundedness of the Cauchy singular integral operator S_L). *If L is a closed smooth curve then the Cauchy singular integral operator S_L from the Banach space $H^{\mu}(L)$ into the Banach space $H^{\mu}(L)$ is bounded for $\mu \in (0, 1)$.*

In [9], the proof of Theorem 1.1 is not explicitly presented but the authors pointed out that it could be easily obtained from Muskhelishvili's estimates in the excellent monograph [14]. In fact, the proof of Theorem 1.1 is fairly tedious although it can be derived from Muskhelishvili's results. The classical proof of Theorem 1.1 is so complicated that its proof is not included in a large number of references after Muskhelishvili. In 1980, a simpler proof of Plemelj–Sokhotski formula and Privalov–Muskhelishvili theorem was presented by J. Du in [5, 6], so it is possible to simplify the classical proof of Theorem 1.2 in [14] by use of Du's method. In [12], Theorem 1.1 is verified by the closed graph theorem because there is the additional requirement that Γ is a Lyapunov curve. And hence, the proof in [12] is not appropriate if Γ is a piecewise smooth curve. In fact, if L is piecewise smooth with corner points including cusps, Theorem 1.2 is still valid but the proof cannot be found in the pertinent literature.

From the view of application [7], we need a stronger result than Theorem 1.1. To do so, by (1.5), we introduce the projection operators

$$C_{\Gamma}^{\pm} : H^{\mu}(\Gamma) \to H^{\mu}(\overline{\Omega^{\pm}}), \quad \varphi \mapsto C_{\Gamma}^{\pm}[\varphi],$$

where $0 < \mu < 1$. Then, we may obtain a nice result.

Theorem 1.3 (Boundedness of the projection operators). *Let Γ be a closed arc-wise smooth curve. The projection operators $C_{\Gamma}^{\pm}$ from the Banach space $H^{\mu}(\Gamma)$ into the Banach space $H^{\mu}(\overline{\Omega^{\pm}})$ are bounded for $\mu \in (0, 1)$.*

Obviously, Theorem 1.2 is a direct corollary of Theorem 1.3. Under the assumption that Γ is of class C^2, an analog of Theorem 1.3 with respect to the Cauchy operator C_{Γ} is verified in [10]. But its method of proof could not be applied to prove Theorem 1.3. More importantly, Theorem 1.3 has important applications in the asymptotic analysis of orthogonal polynomials, which will be presented in a forthcoming paper.

In the present article, we will prove this theorem and give some of its corollaries. The context is organized as follows. In [5, 6, 11], we proved

$$|(C_{\Gamma}^{\pm}[\varphi])(z) - (C_{\Gamma}^{\pm}[\varphi])(w)| \leq B|z - w|^{\mu} \quad \forall z, w \in \overline{\Omega^{\pm}} \quad (0 < \mu < 1) \tag{1.10}$$

for some constant B. In the next section, using Du's method [5, 6, 11], we will prove the inequality

$$\|C_\Gamma^{\pm}[\varphi]\|_{H^\mu(\overline{\Omega^+})} \leq D\mathcal{M}_\Gamma[\varphi,\mu] \quad \text{when } \varphi \in H^\mu(\Gamma) \quad (0<\mu<1), \tag{1.11}$$

where D is some positive constant and Γ is piecewise smooth. In Section 3, we ulteriorly transfer the results of the previous section to the case of a system of curves consisting of a number of simple closed arc-wise smooth curves. In Section 4, we generalize the results of closed curves in Section 2 to the case of open curve.

2. Boundedness of the projection operators

In this section, we will make use of Du's method [5, 6, 11] to verify the inequality (1.11). To do so, we only need to give some details of Du's proof for Theorem 1.1 in [5, 6, 11]. In particular, the relation between the Hölder coefficient B in (1.10) involved in the proof of [5, 6, 11] and $\mathcal{M}_\Gamma[\varphi,\mu]$ must be specified.

A symbol is introduced beforehand. For an arbitrary point $z \in \mathbb{C}$, we denote the point on Γ nearest to z by z_Γ (if such points are more than one in number, z_Γ may be any one of them). In particular, if $z = t \in \Gamma$, then $t_\Gamma = t$. Obviously,

$$|t - z_\Gamma| \leq 2|t-z| \quad \forall t \in \Gamma,\ \forall z \in \mathbb{C}. \tag{2.1}$$

In addition, if $\varphi \in H^\mu(\Gamma)$, one immediately has

$$|\varphi(t) - \varphi(z_\Gamma)| \leq 2^\mu \mathcal{M}_\Gamma[\varphi,\mu]|t-z|^\mu \quad \forall t \in \Gamma,\ \forall z \in \mathbb{C}. \tag{2.2}$$

Denote the natural equation of Γ by the arc-length parameter as

$$\Gamma : z = \phi(s), \quad 0 \leq s \leq \Gamma, \tag{2.3}$$

where the length of Γ is also denoted by Γ. For the sake of reference, let us give Γ a little technical treatment.

Use all corners (including cusps) and $z_\ell = \phi(\ell\Gamma/2)$ ($\ell = 0,1,2$), which are denoted as $c_0, c_1, \ldots, c_{\mathfrak{m}}$ in increasing order of parameter, to divide Γ into $\mathfrak{m}$ (finite number) smooth arcs $\Gamma_j = \widehat{c_{j-1}c_j}$ ($j = 1,\ldots,\mathfrak{m}$). Such $\{\Gamma_j = \widehat{c_{j-1}c_j}, j = 1,\ldots,\mathfrak{m}\}$ is referred to as the standard smooth segments of the curve Γ and $\mathfrak{m}$ is called its number of segments later. It must be pointed out that some z_ℓ's may not be corner points, called artificial corner points of Γ. Now, by the chord-arc inequality (see, e.g., [11, Lemma 2.2.2]), we know that, if both $\phi(s_1)$ and $\phi(s_2)$ are on Γ_j ($j = 1,\ldots,\mathfrak{m}$), there exists some constant C ($0 < C \leq 1$) such that

$$C|s_1-s_2| \leq |\phi(s_1)-\phi(s_2)| \leq |s_1-s_2| \quad \text{for } \phi(s_1),\phi(s_2) \in \Gamma_j \quad (j=1,\ldots,\mathfrak{m}), \tag{2.4}$$

where C is called the standard segmented chord-arc coefficient of Γ.

Lemma 2.1. *Let* Γ *be a simple arc-wise smooth curve (closed or open) and* $\varphi \in H^{\mu}(\Gamma)$ *with* $0 < \mu < 1$. *If* γ *is a subarc of* Γ *(the length of* γ *is also denoted by* γ *for brevity), then*

$$\int_{\gamma} \left| \frac{\varphi(t) - \varphi(z_\Gamma)}{t - z} \right| |\mathrm{d}t| \leq M_\mu \mathcal{M}_\Gamma[\varphi, \mu] \gamma^\mu \quad \forall z \in \mathbb{C}, \tag{2.5}$$

where $\mathcal{M}_\Gamma[\varphi, \mu]$ *is the Hölder semi-norm of* φ *on* Γ *and* M_μ *is a constant independent of* γ *and* z.

Proof. Let $\gamma_j = \gamma \cap \Gamma_j$ $(j = 1, \ldots, \mathbb{m})$, where $\{\Gamma_j, j = 1, \ldots, \mathbb{m}\}$ is the standard smooth segments of Γ. Some γ_j is allowed to be an empty set, which does not affect the subsequent proof.

Denote the arc-length parameter of any point $t \in \Gamma$ by s_t. Then, we have

$$\begin{aligned}
&\int_{\gamma_j} \left| \frac{\varphi(t) - \varphi(z_\Gamma)}{t - z} \right| |\mathrm{d}t| \qquad (j = 1, 2, \ldots, \mathbb{m}) \\
&\leq 2A \int_{\gamma_j} \frac{|\mathrm{d}t|}{|t - z_{\gamma_j}|^{1-\mu}} \quad \text{(by (2.1) and (2.2) with } A = \mathcal{M}_\Gamma[\varphi, \mu]) \\
&\leq \frac{2A}{C^{1-\mu}} \int_{\gamma_j} \frac{\mathrm{d}s}{|s - s_{z_{\gamma_j}}|^{1-\mu}} \quad \text{(by (2.4))} \\
&= \frac{2A}{\mu C^{1-\mu}} \left[(s_{z_{\gamma_j}} - s_{a_j})^\mu + (s_{b_j} - s_{z_{\gamma_j}})^\mu \right] \quad (\gamma_j = \widehat{a_j b_j} \text{ with } s_{a_j} \leq s_{b_j}) \\
&\leq \frac{4}{\mu C^{1-\mu}} \mathcal{M}_\Gamma[\varphi, \mu][\gamma_j]^\mu \quad \text{(the length of } \gamma_j \text{ is also denoted by } \gamma_j),
\end{aligned}$$

which results in the desired estimate (2.5) with

$$M_\mu = \frac{4\mathbb{m}}{\mu C^{1-\mu}}, \tag{2.6}$$

where $\mathbb{m}$ is the number of segments of the standard smooth division of the curve Γ given in (2.4). ∎

Example 2.1. Let Γ be a simple arc-wise smooth curve (closed or open) and $\varphi \in H^{\mu}(\Gamma)$ with $0 < \mu < 1$. Then, we have

$$\left| \int_{\Gamma} \frac{\varphi(t) - \varphi(z_\Gamma)}{t - z} \mathrm{d}t \right| \leq M_\mu \Gamma^\mu \mathcal{M}_\Gamma[\varphi, \mu] \quad \forall z \in \mathbb{C}, \tag{2.7}$$

where M_μ is given in (2.6).

Lemma 2.2. *Under the same conditions as in Lemma 2.1, one has*

$$\int_{\gamma} \left| \frac{\varphi(t) - \varphi(z_\Gamma)}{(t - z)^2} \right| |\mathrm{d}t| \leq N_\mu \mathcal{M}_\Gamma[\varphi, \mu] |z - z_\gamma|^{\mu - 1} \quad \forall z \in \mathbb{C} \backslash \gamma, \tag{2.8}$$

where $\mathcal{M}_\Gamma[\varphi, \mu]$ *is the Hölder semi-norm of* φ *on* Γ *and* N_μ *is a constant independent of* γ, φ *and* z.

Proof. All the notations used here are the same as in Lemma 2.1. We have

$$\begin{aligned}
&\int_{\gamma_j}\left|\frac{\varphi(t)-\varphi(z_\Gamma)}{(t-z)^2}\right||\mathrm{d}t| \quad (j=1,\ldots,\mathbb{m})\\
&\le 2^\mu 3^{2-\mu}A\int_{\gamma_j}\frac{|\mathrm{d}t|}{(3|t-z|)^{2-\mu}} \quad (\text{by (2.2) with } A=\mathcal{M}_\Gamma[\varphi,\mu])\\
&\le 2^\mu 3^{2-\mu}A\int_{\gamma_j}\frac{|\mathrm{d}t|}{(|t-z_{\gamma_j}|+|z-z_{\gamma_j}|)^{2-\mu}} \quad (\text{by (2.1)})\\
&\le 2^\mu 3^{2-\mu}A\left[\int_{s_{a_j}}^{s_{z_{\gamma_j}}}+\int_{s_{z_{\gamma_j}}}^{s_{b_j}}\right]\frac{\mathrm{d}s}{(C(s-s_{z_{\gamma_j}})+|z-z_{\gamma_j}|)^{2-\mu}} \quad (\text{by (2.4)})\\
&\le \frac{2^{\mu+1}3^{2-\mu}}{(1-\mu)C}\mathcal{M}_\Gamma[\varphi,\mu]|z-z_\gamma|^{\mu-1},
\end{aligned}$$

which results in the desired estimate (2.8) with

$$N_\mu=\frac{18\mathbb{m}}{(1-\mu)C} \quad (0<\mu<1), \tag{2.9}$$

where $\mathbb{m}$ is the number of segments of the standard smooth segmentation of the curve Γ given in (2.4). ■

Example 2.2. Under the same conditions as in Lemma 2.1 but Γ is a closed arc-wise smooth curve, then

$$|(C_\Gamma^+[\varphi])'(z)|=\left|\frac{1}{2\pi i}\oint_\Gamma\frac{\varphi(t)-\varphi(z_\Gamma)}{(t-z)^2}\mathrm{d}t\right|\le\frac{N_\mu}{2\pi}\mathcal{M}_\Gamma[\varphi,\mu]|z-z_\Gamma|^{\mu-1}, \quad z\in\mathbb{C}\backslash\Gamma, \tag{2.10}$$

where N_μ is given by (2.9).

Remark 2.1. In Lemmas 2.1 and 2.2, we have actually proved

$$\int_\gamma\frac{1}{|t-z|^\alpha}|\mathrm{d}t|\le\begin{cases}E_\mu\gamma^{1-\alpha} & \text{if } 0<\alpha<1,\ z\in\mathbb{C},\\ E_\mu|z-z_\gamma|^{1-\alpha} & \text{if } \alpha>1,\ z\in\mathbb{C}\backslash\gamma,\end{cases} \tag{2.11}$$

where E_μ is a constant independent of γ and z.

Remark 2.2. We have seen that the nearest point z_Γ plays a very important role in the above proofs. If we select a fixed z_Γ (for example, the closest point with the smallest arc parameter), (2.2) indicates that the function $\varphi_\Gamma(z)=\varphi(z_\Gamma)$ $(z\in\mathbb{C})$ has the $(\Gamma,\mathbb{C})$-Hölder continuity, i.e.,

$$|\varphi_\Gamma(t)-\varphi_\Gamma(z)|\le 2^\mu\mathcal{M}_\Gamma[\varphi,\mu]|t-z|^\mu \quad \forall t\in\Gamma,\ \forall z\in\mathbb{C}. \tag{2.12}$$

And in general, let Σ and $\mathbb{u}$ be two sets of the complex plane $\mathbb{C}$. If the function f defined on $\Sigma\cup\mathbb{u}$ satisfies the condition

$$|f(t)-f(z)|\le M|t-z|^\mu \quad \forall t\in\mathbb{u},\ \forall z\in\Sigma, \tag{2.13}$$

where M is a constant, then we say that it satisfies the $(\Sigma, \mathbb{m})$-Hölder condition, denoted by $f \in H^\mu(\Sigma, \mathbb{m})$. Obviously,

$$H^\mu(\Omega, \Omega) = H^\mu(\Omega).$$

For $f \in H^\mu(\Sigma, \mathbb{m})$, we introduce the $(\Sigma, \mathbb{m})$-Hölder norm

$$\|f\|_{H^\mu(\Sigma,\mathbb{m})} = \|f\|_{\Sigma \cup \mathbb{m}} + \mathcal{M}_{\Sigma,\mathbb{m}}[f, \mu], \tag{2.14}$$

where

$$\mathcal{M}_{\Sigma,\mathbb{m}}[f, \mu] = \sup \left\{ \frac{|f(z) - f(w)|}{|z - w|^\mu}, z \in \Sigma, w \in \mathbb{m}, z \neq w \right\} \tag{2.15}$$

is referred to as the $(\Sigma, \mathbb{m})$-Hölder semi-norm.

It is not difficult to prove that $\mathcal{M}_{\Sigma,\mathbb{m}}[\cdot, \mu]$ and $\|\cdot\|_{H^\mu(\Sigma,\mathbb{m})}$ are, respectively, a semi-norm and a norm of $H^\mu(\Sigma, \mathbb{m})$. Installed the norm (2.14) it is still referred to as $H^\mu(\Sigma, \mathbb{m})$, which is a Banach space (the proof is exactly the same as for the case $H^\mu(\Omega)$ [2, 10]).

Example 2.3. The extended operator

$$\mathbb{E} : H^\mu(\Gamma) \to H^\mu(\mathbb{C}, \Gamma) \quad (0 < \mu < 1), \quad \varphi \mapsto \varphi_\Gamma$$

from the Banach space $H^\mu(\Gamma)$ into Banach space $H^\mu(\Gamma, \mathbb{C})$ is bounded, where $\varphi_\Gamma(z) = \varphi(z_\Gamma)$ $(z \in \mathbb{C})$. More precisely,

$$\begin{cases} \|\mathbb{E}[\varphi]\|_{\mathbb{C}} \leq \|\varphi\|_\Gamma, \\ \mathcal{M}_{\mathbb{C},\Gamma}[\varphi_\Gamma, \mu] \leq 2^\mu \mathcal{M}_\Gamma[\varphi, \mu]. \end{cases}$$

Now, let us introduce an auxiliary operator given earlier in [6] by Du,

$$\mathfrak{D}_\Gamma : \varphi \mapsto \mathfrak{D}_\Gamma[\varphi],$$

where $\varphi \in H^\mu(\Gamma)$ $(0 < \mu < 1)$ and

$$(\mathfrak{D}_\Gamma[\varphi])(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\varphi(\tau) - \varphi(z_\Gamma)}{\tau - z} \mathrm{d}\tau, \quad z \in \mathbb{C}, \tag{2.16}$$

in which the integral on the right-hand side is the Cauchy type integral of the function Δ when $z \notin \Gamma$ and an ordinary (improper) integral of the function Δ when $z \in \Gamma$ where $\Delta = \varphi - \varphi_\Gamma$ with φ_Γ given in Remark 2.2.

Theorem 2.1 (Boundedness of $\mathfrak{D}_\Gamma$). *Let Γ be a simple arc-wise smooth closed curve. If $\varphi \in H^\mu(\Gamma)$ with $0 < \mu < 1$, then $\mathfrak{D}_\Gamma[\varphi] \in H^\mu(\Gamma, \mathbb{C})$, more precisely,*

$$|(\mathfrak{D}_\Gamma[\varphi])(z) - (\mathfrak{D}_\Gamma[\varphi])(t)| \leq B_\mu \mathcal{M}_\Gamma[\varphi, \mu]|z - t|^\mu, \quad t \in \Gamma,\ z \in \mathbb{C}, \tag{2.17}$$

where B_μ is some constant. And $\mathfrak{D}_\Gamma$ from the Banach space $H^\mu(\Gamma)$ into the Banach space $H^\mu(\Gamma, \mathbb{C})$ is bounded.

Proof. From Example 2.1, we know that

$$\|\mathfrak{D}_\Gamma[\varphi]\|_{\mathbb{C}} \leq \frac{M_\mu \Gamma^\mu}{2\pi} \mathcal{M}_\Gamma[\varphi,\mu] \leq \frac{M_\mu \Gamma^\mu}{2\pi} \|\varphi\|_{H^\mu(\Gamma)}. \tag{2.18}$$

So, to prove the boundedness of $\mathfrak{D}_\Gamma : H^\mu(\Gamma) \to H^\mu(\Gamma,\mathbb{C})$, we only need to verify (2.17). To do so, we just have to prove (2.17) for sufficiently small $|z-t|$, i.e.,

$$|(\mathfrak{D}_\Gamma[\varphi])(z) - (\mathfrak{D}_\Gamma[\varphi])(t)| \leq B_\mu \mathcal{M}_\Gamma[\varphi,\mu]|z-t|^\mu, \quad |z-t| \leq \delta, \tag{2.19}$$

where δ is some positive constant because by (2.18) we already have

$$|(\mathfrak{D}_\Gamma[\varphi])(z) - (\mathfrak{D}_\Gamma[\varphi])(t)| \leq \frac{M_\mu \Gamma^\mu}{\pi \delta^\mu} \mathcal{M}_\Gamma[\varphi,\mu]|z-t|^\mu, \quad |z-t| \geq \delta > 0.$$

To do this, take

$$\delta = \frac{1}{2} \min\{|c_j - c_k|, 0 \leq j < k \leq \mathbb{m}\}, \tag{2.20}$$

where $\{\Gamma_j = \widehat{c_{j-1}c_j}, j = 1,2,\ldots,\mathbb{m}\}$ is the standard smooth segmentation of Γ. We draw a circle $D_\eta = \{w, |w-t| = \eta\}$ with center at t and radius $\eta = 2|z-t| < \delta$. Obviously, t is inside the circle D_η, and there is at most one corner point, say, $c_{\mathbb{k}}$, inside the circle D_η. Consider the subarc on Γ between the points first departing from and last entering into D_η, denoted by α and β'. In detail, α and β' on the circle D_η are, respectively, two definite points with the longest and shortest arc-lengths if the curve Γ is parameterized counterclockwise from t. Some subarcs of this curve may be located outside of D_η. Cancel those subarcs (finite in number) which contain corner points outside D_η and whose endpoints are on the circle, and denote the remaining subarcs by Γ_η (see Figure 1). Then, obviously, $\Gamma_\eta \leq 2\mathbb{m}\eta/C$, where C is the constant given in (2.4).

Therefore, now we may see that

$$\begin{aligned} &|(\mathfrak{D}_\Gamma)[\varphi](z) - (\mathfrak{D}_\Gamma)[\varphi](t)| \\ &\quad \leq |(\mathfrak{D}_{\Gamma_\eta})[\varphi](z)| + |(\mathfrak{D}_{\Gamma_\eta})[\varphi](t)| \\ &\qquad + \left|\frac{z-t}{2\pi} \int_{\Gamma\setminus\Gamma_\eta} \frac{\varphi(\tau)-\varphi(z_\Gamma)}{(\tau-z)(\tau-t)} \mathrm{d}\tau\right| + |\varphi(t)-\varphi(z_\Gamma)| \left|\frac{1}{2\pi i} \int_{\Gamma\setminus\Gamma_\eta} \frac{1}{\tau-t} \mathrm{d}\tau\right| \\ &\quad \triangleq \frac{1}{2\pi}(\delta_1 + \delta_2 + |z-t|\delta_3 + |\varphi(t)-\varphi(z_\Gamma)|\delta_4). \end{aligned} \tag{2.21}$$

Let us estimate δ_j $(j = 1,2,3,4)$ in turn. By Lemma 2.1,

$$\delta_1, \delta_2 \leq M'_\mu \mathcal{M}_\Gamma[\varphi,\mu](\Gamma_\eta)^\mu \leq M''_\mu \mathcal{M}_\Gamma[\varphi,\mu]|z-t|^\mu. \tag{2.22}$$

where M'_μ and M''_μ are some constants. By the inequality

$$|\tau - z| \leq 2|\tau - t| \leq 4|z-\tau| \quad \text{when } \tau \in \Gamma\setminus\Gamma_\eta \tag{2.23}$$

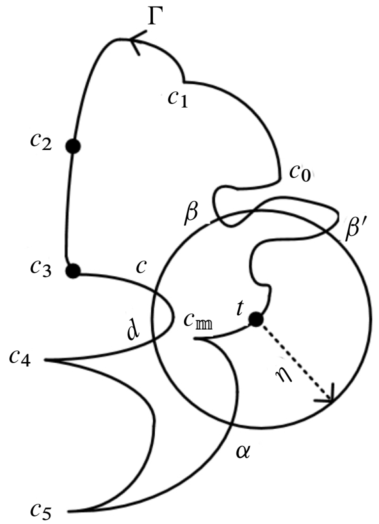

Figure 1. Diagram of $\Gamma_\eta = \widehat{\alpha\beta} + \widehat{cd}$.

and Lemma 2.2, we get

$$\begin{aligned}\delta_3 &\le 2\int_{\Gamma\backslash\Gamma_\eta}\left|\frac{\varphi(\tau)-\varphi(z_\Gamma)}{(\tau-z)^2}\right||\mathrm{d}\tau| \quad (\text{by } (2.23))\\ &\le 2^{1+\mu}\mathscr{M}_\Gamma[\varphi,\mu]\int_{\Gamma\backslash\Gamma_\eta}\frac{|\mathrm{d}\tau|}{|\tau-z|^{2-\mu}} \quad (\text{by } (2.2))\\ &\le N'_\mu\,\mathscr{M}_\Gamma[\varphi,\mu]\,|z-z_{\Gamma\backslash\Gamma_\eta}|^{\mu-1} \quad (\text{by } (2.11))\\ &\le N''_\mu\,\mathscr{M}_\Gamma[\varphi,\mu]|z-t|^{\mu-1} \quad (\text{by } (2.23)),\end{aligned} \tag{2.24}$$

where N'_μ and N''_μ are some constants.

Clearly, $\Gamma\backslash\Gamma_\eta$ is made up of a finite number (less than $\mathbb{m}$) of subarcs whose endpoints are on D_η. So, we have

$$\delta_4 = \left|\int_{\Gamma\backslash\Gamma_\eta}\frac{\mathrm{d}\tau}{\tau-t}\right| \le (\mathbb{m}+1)(2\pi). \tag{2.25}$$

Inserting (2.22), (2.24), and (2.25) into (2.21) and noting (2.2), one gets (2.19). ■

By installing Theorem 2.1 in a variety of concrete situations, we can recover some famous theorems. First, noting

$$(C_\Gamma^{\pm}[\varphi])(z) = \begin{cases}(\mathfrak{D}_\Gamma[\varphi])(z)+\varphi(z_\Gamma), & z\in\overline{\Omega^+},\\ (\mathfrak{D}_\Gamma[\varphi])(z), & z\in\overline{\Omega^-},\end{cases} \tag{2.26}$$

and

$$(S_\Gamma[1])(t) = \frac{\theta_t}{\pi}, \quad t \in \Gamma, \tag{2.27}$$

where θ_t is the opening angle at t, we have the following boundary value formulae.

Corollary 2.1 (Plemelj–Sokhotski formulae). *Suppose that*

$$(C_\Gamma[\varphi])^{\pm}(t) = \lim_{z \to t, z \in \Omega^{\pm}} (C_\Gamma[\varphi])(z), \quad t \in \Gamma, \tag{2.28}$$

which are, respectively, known as the positive and negative boundary valves of Cauchy type integral (1.2) [11, 14]. *Then, they actually exist, and*

$$\begin{cases} (C_\Gamma[\varphi])^{-}(t) = (\mathfrak{D}_\Gamma[\varphi])(t) = -\dfrac{\theta_t}{2\pi}\varphi(t) + \dfrac{1}{2}(S_\Gamma[\varphi])(t) = (C_\Gamma^{-}[\varphi])(t), \\ (C_\Gamma[\varphi])^{+}(t) = (\mathfrak{D}_\Gamma[\varphi])(t) + \varphi(t) = (C_\Gamma^{-}[\varphi])(t) + \varphi(t) = (C_\Gamma^{+}[\varphi])(t), \end{cases} \quad t \in \Gamma, \tag{2.29}$$

where $(C_\Gamma^{\pm}[\varphi])(t)$ *are, respectively, given in* (1.5) *and* (1.6)*, which are just the famous Plemelj–Sokhotski formulae.*

Remark 2.3. The projections given in (1.5) and (1.6) can be rewritten as

$$(C_\Gamma^{\pm}[\varphi])(z) = \begin{cases} (C_\Gamma[\varphi])(z), & \text{if } z \in \Omega^{\pm}, \\ (C_\Gamma[\varphi])^{\pm}(z), & \text{if } z = t \in \Gamma. \end{cases}$$

This expression can be used with the equivalent definition of projections (1.5) and (1.6), which will particularly come in handy on system of curves in the next section.

Using (2.29) again and noting

$$\frac{1}{2}S_L[\varphi] = (\mathfrak{D}_L[\varphi])|_L + \frac{1}{2}\varphi, \tag{2.30}$$

where L is smooth, we have no trouble getting $S_L[\varphi] \in H^{\mu}(L)$ and the Cauchy singular integral operator S_L from the Banach space $H^{\mu}(L)$ into the Banach space $H^{\mu}(L)$ is bounded by using Theorem 2.1.

Corollary 2.2 (Boundedness of S_L). *Theorem* 1.2 *stated in Section* 1 *holds.*

Similarly,

$$C_\Gamma^{\pm}[\varphi] = (\mathfrak{D}_\Gamma[\varphi])\big|_{\overline{\Omega^{\pm}}} \pm \frac{1}{2}\varphi_\Gamma\, C_\Gamma^{\pm}[1].$$

From Theorem 2.1 and (2.12), noting Remark 2.3 and Example 2.3, by (2.29) we get a generalization of the classical Privalov theorem.

Corollary 2.3 (Generalized Privalov theorem of $C_\Gamma^{\pm}$). *Let* Γ *be an arc-wise smooth closed curve and* $\varphi \in H^{\mu}(\Gamma)$ *with* $0 < \mu < 1$. *Then*

$$|(C_\Gamma^{\pm}[\varphi])(z) - (C_\Gamma^{\pm}[\varphi])(t)| \le B_\mu \mathcal{M}_\Gamma[\varphi, \mu]|z - t|^{\mu}, \quad t \in \Gamma,\ z \in \overline{\Omega^{\pm}},$$

where B_μ is some constant. And the projection operators $C_\Gamma^\pm$ from the Banach spaces $H^\mu(\Gamma)$ into the Banach space $H^\mu(\Gamma, \overline{\Omega^\pm})$ are bounded.

In particular, noting

$$\left(1 - \frac{\theta_t}{\pi}\right)\varphi(t) + (S_\Gamma[\varphi])(t) = (C_\Gamma^+[\varphi])(t) + (C_\Gamma^-[\varphi])(t), \quad t \in \Gamma,$$

where Γ could be arc-wise smooth and θ_t is the opening angle at t.

Suppose that

$$S_\Gamma^\theta : H^\mu(\Gamma) \to H^\mu(\Gamma), \quad \varphi \mapsto S_\Gamma^\theta[\varphi], \tag{2.31}$$

where $0 < \mu < 1$ and

$$(S_\Gamma^\theta[\varphi])(t) = \left(1 - \frac{\theta_t}{\pi}\right)\varphi(t) + (S_\Gamma[\varphi])(t), \quad t \in \Gamma.$$

We also have the following conclusion, which is the generalization of Theorem 1.2.

Corollary 2.4. *If Γ is a closed arc-wise smooth curve, then the operator S_Γ^θ from the Banach space $H^\mu(\Gamma)$ into the Banach space $H^\mu(\Gamma)$ is bounded for $\mu \in (0, 1)$.*

Next, introduce two so-called restricted operators of C_Γ,

$$C_{\Gamma,\Omega^+} : \varphi \mapsto (C_\Gamma[\varphi])|_{\Omega^+}, \quad C_{\Gamma,\Omega^-} : \varphi \mapsto (C_\Gamma[\varphi])|_{\Omega^-}. \tag{2.32}$$

We also have the following corollary.

Corollary 2.5 (Generalized Muskhelishvili theorem of C_Γ). *Let Γ be an arc-wise smooth closed curve and $\varphi \in H^\mu(\Gamma)$ $(0 < \mu < 1)$. Then, $C_\Gamma \in H^\mu(\Omega^+)$ and $C_\Gamma \in H^\mu(\Omega^-)$, more precisely,*

$$|(C_\Gamma[\varphi])(z) - (C_\Gamma[\varphi])(w)| \leq B_\mu \mathcal{M}_\Gamma[\varphi, \mu]|z - w|^\mu, \quad z, w \in \Omega^\pm, \tag{2.33}$$

where B_μ is some constant. And C_{Γ,Ω^+} and C_{Γ,Ω^-} are bounded from the Banach space $H^\mu(\Gamma)$, respectively, into Banach spaces $H^\mu(\Omega^+)$ and $H^\mu(\Omega^-)$.

Proof. We are only going to prove the case $z, w \in \Omega^+$. The proof for the case $z, w \in \Omega^-$ is similar. Denote the distance between the segment $\overline{zw}$ and Γ by ρ. Then, there exists a point $\mathcal{K}$ on $\overline{zw}$ such that $|\mathcal{K} - \mathcal{K}_\Gamma| = \rho$.

Case 1. If $|z - w| \geq |\mathcal{K} - \mathcal{K}_\Gamma|$, one has

$$\begin{cases} |z - \mathcal{K}_\Gamma| \leq |z - \mathcal{K}| + |\mathcal{K} - \mathcal{K}_\Gamma| \leq 2|z - w|, \\ |w - \mathcal{K}_\Gamma| \leq |w - \mathcal{K}| + |\mathcal{K} - \mathcal{K}_\Gamma| \leq 2|z - w|. \end{cases}$$

Therefore, by Corollary 2.3,

$$\begin{aligned} &|C_\Gamma[\varphi](z) - C_\Gamma[\varphi](w)| \\ &\quad \leq |C_\Gamma[\varphi](z) - C_\Gamma[\varphi](\mathcal{K}_\Gamma)| + |C_\Gamma[\varphi](w) - C_\Gamma[\varphi](\mathcal{K}_\Gamma)| \\ &\quad \leq B'_\mu \mathcal{M}_\Gamma[\varphi, \mu](|z - \mathcal{K}_\Gamma|^\mu + |w - \mathcal{K}_\Gamma|^\mu) \\ &\quad \leq B_\mu \mathcal{M}_\Gamma[\varphi, \mu]|z - w|^\mu. \end{aligned} \tag{2.34}$$

Case 2. If $|z-w| \leq |\mathcal{K}-\mathcal{K}_\Gamma|$, one easily knows that $\overline{zw} \subset \Omega^+$. And hence, by Example 2.2,

$$\begin{aligned}
&|C_\Gamma[\varphi](z) - C_\Gamma[\varphi](w)| \\
&= |\int_z^w (C_\Gamma[\varphi])'(\tau)\mathrm{d}\tau| \\
&\leq B_\mu \mathcal{M}_\Gamma[\varphi,\mu] \int_{\overline{zw}} |\tau-\tau_\Gamma|^{\mu-1}|\mathrm{d}\tau| \\
&\leq B'_\mu \mathcal{M}_\Gamma[\varphi,\mu]|z-w|^\mu.
\end{aligned} \tag{2.35}$$

Equations (2.34) and (2.35) imply (2.33) for both z and w on Ω^+.

Then, by use of the inequality

$$\|C_{\Gamma,\Omega^\pm}[\varphi]\|_{\Omega^\pm} \leq \|\mathfrak{D}_\Gamma[\varphi]\|_\Gamma + \|\varphi\|_\Gamma,$$

the proof is complete. ∎

The proof of Theorem 1.3. Generalized Privalov theorem of $C_\Gamma^\pm$ and generalized Muskhelishvili theorem of C_Γ, in Corollary 2.3 and Corollary 2.5, result in the validity of Theorem 1.2, which goes deeper than the classical 2P theorems [8, 11, 14].

Remark 2.4. Let Γ be a simply closed arc-wise smooth curve, oriented clockwise, denoted as Γ^- in detail. It also divides the complex plane $\mathbb{C}$ into two domains, a bounded region and a unbounded region, denoted, respectively, as Ω^- and Ω^+. Theorem 1.2 still holds for such Γ. And just to make the difference, we are going to write Ω^+ and Ω^- divided by the curve Γ with the direction, respectively, as $\Omega^+(\Gamma)$ and $\Omega^-(\Gamma)$. Thus, $\Omega^+(\Gamma^-) = \Omega^-(\Gamma)$ and $\Omega^-(\Gamma^-) = \Omega^+(\Gamma)$.

Let f be defined on $\Omega^- \cup \Omega^+$. Inspired by (2.33), we introduce the sectional Hölder semi-norm

$$\mathcal{M}_{(\Omega^-|\Omega^+)}[f,\mu] = \sup\left\{\frac{|f(z)-f(w)|}{|z-w|^\mu}, z \text{ and } w \text{ both in } \Omega^- \text{ or } \Omega^+\right\}, \tag{2.36}$$

and the sectional Hölder norm is introduced accordingly:

$$\|f\|_{H^\mu(\Omega^-|\Omega^+)} = \|f\|_{\mathbb{C}\setminus\Gamma} + \mathcal{M}_{(\Omega^-|\Omega^+)}[f,\mu].$$

If $\|f\|_{H^\mu(\Omega^-|\Omega^+)}$ is finite, let us say $f \in H^\mu(\Omega^- \mid \Omega^+)$. Such a space, in which the sectional Hölder norm is fitted, is still denoted as $H^\mu(\Omega^- \mid \Omega^+)$. Obviously, it is a Banach space.

Theorem 2.2 (Boundedness of Cauchy operator). *If Γ is a simple closed arc-wise smooth curve (oriented clockwise or counterclockwise) and $\varphi \in H^\mu(\Gamma)$ with $0 < \mu < 1$, then*

$$C_\Gamma[\varphi] \in H^\mu(\Omega^- \mid \Omega^+).$$

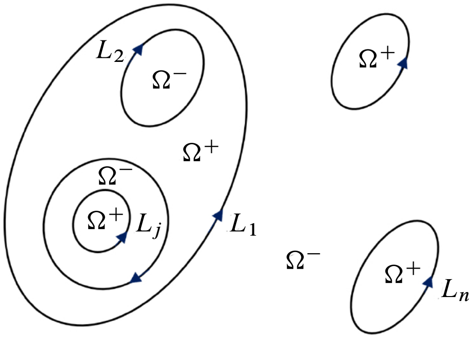

Figure 2. The system $L = \sum_{j=1}^{n} L_j$.

And the Cauchy type integral operator (briefly, Cauchy operator)

$$C_\Gamma : H^\mu(\Gamma) \to H^\mu(\Omega^- \mid \Omega^+), \quad \varphi \mapsto C_\Gamma[\varphi], \tag{2.37}$$

from the Banach space $H^\mu(\Gamma)$ *into the Banach space* $H^\mu(\Omega^- \mid \Omega^+)$ *is bounded.*

Proof. Obviously, we have

$$\|C_\Gamma[\varphi]\|_{H^\mu(\Omega^-|\Omega^+)} \le \|C_\Gamma[\varphi]\|_{H^\mu(\Omega^-)} + \|C_\Gamma[\varphi]\|_{H^\mu(\Omega^+)}. \tag{2.38}$$

Then, by Corollary 2.5, we get

$$\|C_\Gamma\| < +\infty. \tag{2.39}$$

The proof is now complete. ■

3. Cauchy singular integral on a system of curves

As in [11], assume L is a system of n simply closed arc-wise smooth curves $L_1, \ldots, L_n$ on the complex plane $\mathbb{C}$, non-intersecting each other, denoted by $L = \sum_{j=1}^{n} L_j$. L divides the extended complex plane into a finite number of regions. The region containing the point at infinity is Ω^-, the regions neighboring to it are collected as Ω^+, and so on. Then, the entire plane is divided into two parts Ω^+ and Ω^- (not necessarily connected). Sometimes, we write them in detail as $\Omega^+(L)$ and $\Omega^-(L)$. Orient each L_j and hence L positively such that each connected component of Ω^+ lies on the positive (left) side of L while that of Ω^- lies on the negative (right) side (see Figure 2).

A bounded domain enclosed by the subcurve L_k is called the inner domain, denoted by $\mathrm{id}(L_k)$. If $\mathrm{id}(L_k)$ contains no other subcurve L_ℓ ($\ell \neq k$), the L_k is said to be an internal subcurve, such as L_2, L_j, and L_n in Figure 2. Obviously, there must be an internal subcurve in the system of curves; otherwise, there exists an L_{j_2} inside of L_1. Similarly, there exists an L_{j_3} inside of L_{j_2}, and so on. Thus, L cannot be made up of finitely many subcurves. Ω^+ and Ω^- are, respectively, divided into a number of small regions $\Omega_j^{\pm}(L)$ by L, say,

$$\begin{cases} \Omega^+(L) = \Omega_1^+(L) \cup \cdots \cup \Omega_{k^+}^+(L), \\ \Omega^-(L) = \Omega_1^-(L) \cup \cdots \cup \Omega_{k^-}^-(L). \end{cases} \tag{3.1}$$

Clearly, $k^+ + k^- = n + 1$.

In this section, we try to transfer the results of the previous section to systems of curves. For this reason, we need to explore some characteristics of the curve system L.

Let $\mathbb{X}$ and $\mathbb{Y}$ be two sets on the complex plane $\mathbb{C}$, and call them to be separated from each other, if

$$d(\mathbb{X}, \mathbb{Y}) = \inf\{|x - y|, x \in \mathbb{X},\ y \in \mathbb{Y}\} > 0. \tag{3.2}$$

Lemma 3.1 (Separability of L_j, $\Omega_j^{\pm}(L)$). *L_j's are separated from each other, i.e., there is a positive* $\mathbb{g}$ *such that*

$$0 < \mathbb{g} < \min\{d(L_j, L_\ell), j, \ell = 1, 2, \ldots, n, j \neq \ell\} \quad (n > 1).$$

When $k^+ > 1$, $\Omega_1^+(L), \ldots, \Omega_{k^+}^+(L)$ are separated from each other, i.e., there is a positive $\mathbb{g}^+$ such that

$$0 < \mathbb{g}^+ < \min\{d(\Omega_j^+(L), \Omega_\ell^+(L)), j, \ell = 1, \ldots, k^+, j \neq \ell\}.$$

When $k^- > 1$, $\Omega_1^-(L), \ldots, \Omega_{k^-}^-(L)$ are also separated form each other, i.e., there is a positive $\mathbb{g}^-$ such that

$$0 < \mathbb{g}^- < \min\{d(\Omega_j^-(L), \Omega_\ell^-(L)), j, \ell = 1, \ldots, k^-, j \neq \ell\}.$$

Proof. In fact, we may take

$$\mathbb{g}^+ = \mathbb{g}^- = \mathbb{g} = \min\{d(L_j, L_\ell), j, \ell = 1, 2, \ldots, n, j \neq \ell\} \triangleq \text{gap}(L) \quad (n > 1), \tag{3.3}$$

where $\mathbb{g} > 0$ since L_j's are non-intersecting each other and for all $j = 1, 2, \ldots, k^{\pm}$ the boundary $\partial(\Omega_j^{\pm}) \subset L \cup \{\infty\}$. Such $\mathbb{g}$ is called the gap distance of the curve system L, denoted as gap(L). ■

For the sake of clarity, we must also establish some lemmas for Hölder norm.

Lemma 3.2. *Let $\mathbb{X}$ and $\mathbb{Y}$ be two sets on the complex plane $\mathbb{C}$. If $\mathbb{X} \subseteq \mathbb{Y}$ and $f \in H^\mu(\mathbb{Y})$ then*

$$\|f\|_{\mathbb{X}} \leq \|f\|_{\mathbb{Y}}, \quad \mathcal{M}_{\mathbb{X}}[f, \mu] \leq \mathcal{M}_{\mathbb{Y}}[f, \mu], \quad \|f\|_{H^\mu(\mathbb{X})} \leq \|f\|_{H^\mu(\mathbb{Y})}. \tag{3.4}$$

Proof. This lemma is almost self-evident because $\|f\|_\Omega$ and $\mathcal{M}_\Omega[f, \mu]$ is increasing with respect to Ω in the sense of inclusion of sets, so is $\|f\|_{H^\mu(\Omega)}$. But it often plays an obvious role in the subsequent proofs. ■

Lemma 3.3. *Let* $\mathbb{X}$ *and* $\mathbb{Y}$ *be two sets separated from each other on the complex plane* $\mathbb{C}$. *If* $f \in H^{\mu}(\mathbb{X})$ *and* $f \in H^{\mu}(\mathbb{Y})$, *then* $f \in H^{\mu}(\mathbb{X} \cup \mathbb{Y})$,

$$\mathcal{M}_{\mathbb{X}\cup\mathbb{Y}}[f,\mu] \leq \max\left\{\mathcal{M}_{\mathbb{X}}[f,\mu], \mathcal{M}_{\mathbb{Y}}[f,\mu], \frac{\|f\|_{\mathbb{X}} + \|f\|_{\mathbb{Y}}}{[d(\mathbb{X},\mathbb{Y})]^{\mu}}\right\} \tag{3.5}$$

and

$$\|f\|_{\mathbb{X}\cup\mathbb{Y}} \leq \max\left\{\|f\|_{\mathbb{X}}, \|f\|_{\mathbb{Y}}\right\}. \tag{3.6}$$

So,

$$\|f\|_{H^{\mu}(\mathbb{X}\cup\mathbb{Y})} \leq \max\left\{1, \frac{1}{[d(\mathbb{X},\mathbb{Y})]^{\mu}}\right\}\left[\|f\|_{H^{\mu}(\mathbb{X})} + \|f\|_{H^{\mu}(\mathbb{Y})}\right]. \tag{3.7}$$

Proof. Equation (3.6) is obvious. And to verify (3.5), one just has to pay attention to

$$\mathcal{M}_{\mathbb{X},\mathbb{Y}}[f,\mu] \leq \frac{\|f\|_{\mathbb{X}} + \|f\|_{\mathbb{Y}}}{[d(\mathbb{X},\mathbb{Y})]^{\mu}},$$

where $\mathcal{M}_{\mathbb{X},\mathbb{Y}}[\cdot,\mu]$ is the $(\mathbb{X},\mathbb{Y})$-Hölder semi-norm defined by (2.15). Then, (3.5) and (3.6) result in (3.7). ■

Sometimes, we need to define a restricted Hölder semi-norm

$$\mathcal{M}_{\Omega}^{\rho}[f,\mu] = \sup\left\{\frac{|f(z) - f(w)|}{|z - w|^{\mu}}, z, w \in \Omega, 0 < |z - w| \leq \rho\right\}, \tag{3.8}$$

where $\rho > 0$.

Lemma 3.4. *Let* $\mathbb{X}$ *and* $\mathbb{Y}$ *be two sets separated from each other on the complex plane* $\mathbb{C}$. *If* $f \in H^{\mu}(\mathbb{X})$ *and* $f \in H^{\mu}(\mathbb{Y})$, *then* $f \in H^{\mu}(\mathbb{X} \cup \mathbb{Y})$,

$$\mathcal{M}_{\mathbb{X}\cup\mathbb{Y}}^{\rho}[f,\mu] \leq \max\left\{\mathcal{M}_{\mathbb{X}}^{\rho}[f,\mu], \mathcal{M}_{\mathbb{Y}}^{\rho}[f,\mu]\right\} \quad \textit{when } \rho < d(\mathbb{X},\mathbb{Y}), \tag{3.9}$$

and

$$\mathcal{M}_{\mathbb{X}\cup\mathbb{Y}}[f,\mu] \leq \max\left\{1, \frac{1}{\rho^{\mu}}\right\}\max\left\{\mathcal{M}_{\mathbb{X}}^{\rho}[f,\mu], \mathcal{M}_{\mathbb{Y}}^{\rho}[f,\mu], \|f\|_{\mathbb{X}} + \|f\|_{\mathbb{Y}}\right\}. \tag{3.10}$$

Proof. Notice the fact that z and w in (3.8) are both in $\mathbb{X}$ or both in $\mathbb{Y}$ when $\rho < d(\mathbb{X},\mathbb{Y})$, and we get (3.9). Then, (3.9) and (3.7) result in (3.10). ■

Let

$$(C_L[\varphi])(z) = \sum_{j=1}^{n}(C_{L_j}[\varphi])(z), \quad z \in \mathbb{C}\backslash L,$$

where L_j is oriented (see Remark 2.4). Its boundary values are defined by

$$(C_L[\varphi])^{\pm}(t) = \lim_{z\to t, z\in\Omega^{\pm}(L)}(C_L[\varphi])(z), \quad t \in L. \tag{3.11}$$

Lemma 3.5 (Plemelj–Sokhotski formulae). *Let $L = \sum_{j=1}^{n} L_j$ be a system of simply closed arc-wise smooth curves defined as above. Then, the boundary values (3.11) exist and*

$$(C_L[\varphi])^{\pm}(t) = \sum_{j=1}^{n} (C_{L_j}^{\pm}[\varphi])(t), \quad t \in L, \tag{3.12}$$

where the orientation of L_j is described earlier in this section. More precisely, there is an ℓ $(1 \le \ell \le n)$ such that

$$(C_L[\varphi])^{\pm}(t) = \sum_{j=1, j \neq \ell}^{n} (C_{L_j}[\varphi])(t) + (C_{L_\ell}^{\pm}[\varphi])(t), \quad t \in L_\ell, \tag{3.13}$$

where $(C_{L_j}[\varphi])(t)$ $(j \neq \ell)$ is the Cauchy type integral while $(C_{L_\ell}^{\pm}[\varphi])(t)$ are boundary values of the Cauchy type integral.

Proof. This conclusion is directly obtained from Lemma 2.1 and Remark 2.3. Noting the separability of L_j's, then $t \in L$ if and only if there is a unique ℓ $(1 \le \ell \le n)$ such that $t \in L_\ell$. So, (3.12) and (3.13) are equivalent. ∎

The projections on the system of curves are now defined by

$$(C_L^{\pm}[\varphi])(z) = \begin{cases} (C_L[\varphi])(z) & \text{if } z \in \Omega^{\pm}, \\ (C_L[\varphi])^{\pm}(t) & \text{if } z = t \in L, \end{cases}$$

which are equivalent to (1.5) and (1.6) when $n = 1$ by Remark 2.3.

Lemma 3.6. *Let*

$$L = \sum_{j=1}^{n} L_j$$

be a system of closed arc-wise smooth curves defined as above. Then, for $\varphi \in H^{\mu}(L)$ with $\mu \in (0,1)$, we have

$$\|C_L^{\pm}[\varphi]\|_{\overline{\Omega^{\pm}}} \le \mathcal{S}_n \|\varphi\|_{H^{\mu}(L)}, \tag{3.14}$$

where

$$\mathcal{S}_n = \sum_{j=1}^{n} \|C_{L_j}\|,$$

and $C_{L_j} : H^{\mu}(L_j) \to H^{\mu}(\Omega^-(L_j) \mid \Omega^+(L_j))$ is given in (2.37) with $\Gamma = L_j$.

Proof. In fact, by Theorem 2.2 and Lemma 3.2, we have

$$\|C_L[\varphi]\|_{\mathbb{C}\setminus L} \le \sum_{j=1}^{n} \|C_{L_j}[\varphi]\|_{\mathbb{C}\setminus L_j} \le \sum_{j=1}^{n} \|C_{L_j}\| \|\varphi\|_{H^{\mu}(L_j)} \le \mathcal{S}_n \|\varphi\|_{H^{\mu}(L)}, \tag{3.15}$$

which is just (3.14). ∎

Similarly to (3.8), we introduce

$$\mathcal{M}^{\rho}_{(\Omega^-|\Omega^+)}[f,\mu] = \sup\left\{\frac{|f(z)-f(w)|}{|z-w|^{\mu}}, z, w \in \Omega^- \text{ or } z, w \in \Omega^+, 0 < |z-w| \le \rho\right\}.$$

Lemma 3.7. *Let $L = \sum_{j=1}^{n} L_j$ be a system of closed arc-wise smooth curves defined as above. Then, for $\varphi \in H^{\mu}(L)$ with $\mu \in (0,1)$,*

$$\mathcal{M}^{\rho}_{(\Omega^-|\Omega^+)}[C_L[\varphi],\mu] \le \mathcal{M}_n \|\varphi\|_{H^{\mu}(L)} \quad (0 < \rho < \text{gap}(L)) \tag{3.16}$$

and

$$\mathcal{M}_{(\Omega^-|\Omega^+)}[C_L[\varphi],\mu] \le \max\left\{1, \frac{2}{[\text{gap}(L)]^{\mu}}\right\}\mathcal{S}_n \|\varphi\|_{H^{\mu}(L)}, \tag{3.17}$$

where

$$\mathcal{M}_n = \max\left\{\|C_{L_j}\|, j = 1, \ldots, n\right\}, \tag{3.18}$$

$\|C_{L_j}\|$ and $\mathcal{S}_n$ are the same as those in Lemma 3.6 above and

$$\mathbb{g} = \text{gap}(L)$$

is the gap distance of the curve system L given in (3.3) (when $n = 1$, we agree that gap(L) *is any positive number).*

Proof. Let us prove (3.16) by using the induction in n. When $n = 1$, it is just the Theorem 2.2 proved in the last section. Without loss of generality, we suppose that L_n is an internal subcurve in the system L of curves. Then, its inner domain id(L_n) is going to have two possibilities.

Case 1. If id$(L_n) \subset \Omega^+$, then L_n is oriented counterclockwise.

Case 2. If id$(L_n) \subset \Omega^-$, then L_n is oriented clockwise.

For Case 1, letting

$$L' = L_1 + \cdots + L_{n-1},$$

we know that

$$\begin{cases} \Omega^-(L) = \Omega^-(L') \backslash \Omega^+(L_n) \subset \Omega^-(L'), \\ \Omega^-(L) \subset \Omega^-(L_n), \\ \Omega^+(L) = \Omega^+(L') \cup \Omega^+(L_n) \quad (\mathbb{g} \le d(\Omega^+(L'), \Omega^+(L_n))), \end{cases} \tag{3.19}$$

For Case 2, we know that

$$\begin{cases} \Omega^+(L) = \Omega^+(L') \backslash \Omega^-(L_n) \subset \Omega^+(L'), \\ \Omega^+(L) \subset \Omega^+(L_n), \\ \Omega^-(L) = \Omega^-(L') \cup \Omega^-(L_n) \quad (\mathbb{g} \le d(\Omega^-(L'), \Omega^-(L_n))). \end{cases}$$

The proof of these two cases is exactly the same. We will only prove (3.16) for Case 1. By Lemma 3.2 and the inductive assumption, we immediately have

$$\begin{aligned}
&\mathcal{M}^{\rho}_{\Omega^-(L)}[C_L[\varphi],\mu] \quad (0<\rho<\mathrm{gap}(L))\\
&\quad\le \mathcal{M}^{\rho}_{\Omega^-(L)}[C_{L'}[\varphi],\mu]+\mathcal{M}^{\rho}_{\Omega^-(L)}[C_{L_n}[\varphi],\mu] \quad (\text{by } L=L'+L_n)\\
&\quad\le \mathcal{M}^{\rho}_{\Omega^-(L')}[C_{L'}[\varphi],\mu]+\mathcal{M}^{\rho}_{\Omega^-(L_n)}[C_{L_n}[\varphi],\mu] \quad (\text{by (3.19) and Lemma 3.2})\\
&\quad\le \mathcal{M}_{n-1}\|\varphi\|_{H^\mu(L')}+\|C_{L_n}\|\|\varphi\|_{H^\mu(L_n)} \quad (\text{by the inductive assumption})\\
&\quad\le \mathcal{M}_n\|\varphi\|_{H^\mu(L)} \quad (\text{by } L=L'+L_n),
\end{aligned} \tag{3.20}$$

where $\mathcal{M}_n$ is defined by (3.18). We also have

$$\begin{aligned}
&\mathcal{M}^{\rho}_{\Omega^+(L)}[C_L[\varphi],\mu] \quad (0<\rho<\mathrm{gap}(L))\\
&\quad\le \max\left\{\mathcal{M}^{\rho}_{\Omega^+(L')}[C_L[\varphi],\mu],\mathcal{M}^{\rho}_{\Omega^+(L_n)}[C_L[\varphi],\mu]\right\} \quad (\text{by (3.19) and (3.9)})\\
&\quad= \max\left\{\mathcal{M}^{\rho}_{\Omega^+(L')}[C_{L'}[\varphi]+C_{L_n}[\varphi],\mu],\mathcal{M}^{\rho}_{\Omega^+(L_n)}[C_{L'}[\varphi]+C_{L_n}[\varphi],\mu]\right\}\\
&\quad\le \max\left\{\mathcal{M}^{\rho}_{(\Omega^+(L')|\Omega^-(L'))}[C_{L'}[\varphi],\mu],\mathcal{M}^{\rho}_{(\Omega^-(L_n)|\Omega^+(L_n))}[C_{L_n}[\varphi],\mu]\right\}\\
&\quad\le \mathcal{M}_{n-1}\|\varphi\|_{H^\mu(L')}+\|C_{L_n}\|\|\varphi\|_{H^\mu(L_n)} \quad (\text{by the inductive assumption})\\
&\quad\le \mathcal{M}_n\|\varphi\|_{H^\mu(L)}.
\end{aligned} \tag{3.21}$$

Equations (3.20) and (3.21) result in (3.16). Then, we easily see that

$$\begin{aligned}
&\mathcal{M}_{(\Omega^-|\Omega^+)}[C_L[\varphi],\mu]\\
&\quad\le \max\left\{\mathcal{M}^{\rho}_{(\Omega^-|\Omega^+)}[C_L[\varphi],\mu],\frac{2}{\rho^\mu}\|C_L[\varphi]\|_{\mathbb{C}\setminus L}\right\} \quad (\text{by Lemma 3.3})\\
&\quad\le \max\left\{1,\frac{2}{\rho^\mu}\right\}\mathcal{S}_n\|\varphi\|_{H^\mu(L)} \quad (\text{by (3.16) and (3.14)})\\
&\quad\le \max\left\{1,\frac{2}{[\mathrm{gap}(L)]^\mu}\right\}\mathcal{S}_n\|\varphi\|_{H^\mu(L)} \quad (\text{by } \rho\to\mathrm{gap}(L)),
\end{aligned}$$

which is just (3.17). ■

Theorem 3.1 (Boundedness of Cauchy operator C_L). *Let $L=\sum_{j=1}^n L_j$ be the system of closed arc-wise smooth curves defined as above. Then, for $\varphi\in H^\mu(L)$ with $\mu\in(0,1)$,*

$$\|C_L[\varphi]\|_{H^\mu(\Omega^-|\Omega^+)}\le 2\max\left\{1,\frac{2}{[\mathrm{gap}(L)]^\mu}\right\}\sum_{j=1}^n\|C_{L_j}\|\|\varphi\|_{H^\mu(L)}, \tag{3.22}$$

i.e., the operator C_L is a bounded one from the Banach space $H^\mu(L)$ into the Banach space $H^\mu(\Omega^-\mid\Omega^+)$.

Proof. Clearly, (3.22) follows directly from (3.14) and (3.17). ■

The projections are now defined by

$$(C_L^{\pm}[\varphi])(z) = \begin{cases} (C_L[\varphi])(z) & \text{if } z \in \Omega^{\pm}, \\ (C_L[\varphi])^{\pm}(t) & \text{if } z = t \in L. \end{cases}$$

which are equivalent to (1.5) and (1.6) when $n = 1$ by Remark 2.3.

Lemma 3.8. *Let $L = \sum_{j=1}^{n} L_j$ be the system of closed arc-wise smooth curves defined as above. Then, for $\varphi \in H^{\mu}(L)$ with $\mu \in (0, 1)$,*

$$\|C_L[\varphi]\|_{\mathbb{C}\setminus L} = \max\left\{\|C_L^-[\varphi]\|_{\overline{\Omega^-}}, \|C_L^+[\varphi]\|_{\overline{\Omega^+}}\right\}, \tag{3.23}$$

and

$$\mathcal{M}_{(\Omega^-|\Omega^+)}[C_L[\varphi], \mu] = \max\left\{\mathcal{M}_{\overline{\Omega^-}}[C_L^-[\varphi], \mu], \mathcal{M}_{\overline{\Omega^+}}[C_L^+[\varphi], \mu]\right\}. \tag{3.24}$$

Proof. Using continuous extensions of $C_L[\varphi]$ from Ω^- and Ω^+ to L, respectively, we obtain (3.23) and (3.24). ∎

Auxiliary Theorem 3.1. *Let $L = \sum_{j=1}^{n} L_j$ be a system of closed arc-wise smooth curves defined as above. Then, for $\varphi \in H^{\mu}(L)$ with $\mu \in (0, 1)$, the following* (1) *and* (2) *are equivalent.*

(1) *The operator C_L is a bounded one from the Banach space $H^{\mu}(L)$ into the Banach space $H^{\mu}(\Omega^- \mid \Omega^+)$.*

(2) *Both C_L^- and C_L^+ are bounded from the Banach space $H^{\mu}(L)$ into Banach spaces $H^{\mu}(\overline{\Omega^-})$ and $H^{\mu}(\overline{\Omega^+})$, respectively.*

Proof. By Lemma 3.8, we have

$$\begin{cases} \max\left\{\|C_L^-[\varphi]\|_{H^{\mu}(\overline{\Omega^-})}, \|C_L^+[\varphi]\|_{H^{\mu}(\overline{\Omega^+})}\right\} \le \|C_L[\varphi]\|_{H^{\mu}(\Omega^-|\Omega^+)}, \\ \|C_L[\varphi]\|_{H^{\mu}(\Omega^-|\Omega^+)} \le \|C_L^-[\varphi]\|_{H^{\mu}(\overline{\Omega^-})} + \|C_L^+[\varphi]\|_{H^{\mu}(\overline{\Omega^+})}. \end{cases}$$

Thus, (1) and (2) are equivalent. ∎

So, we have the following theorem.

Theorem 3.2 (Boundedness of the projection operators $C_{\Gamma}^{\pm}$). *Let $L = \sum_{j=1}^{n} L_j$ be a system of closed arc-wise smooth curves defined as above. Then, for $\varphi \in H^{\mu}(L)$ with $\mu \in (0, 1)$, C_L^- and C_L^+ are both bounded operators from the Banach space $H^{\mu}(L)$ into Banach spaces $H^{\mu}(\overline{\Omega^-})$ and $H^{\mu}(\overline{\Omega^+})$, respectively.*

4. Cauchy type integral on open arcs

In this section, let $\Gamma = \widehat{ab}$ be a simple piecewise smooth open arc, oriented positively from a to b. In this case, Theorem 2.1 is still valid for $\mathfrak{D}_{\Gamma}$ given in (2.16), but its proof must be appropriately modified.

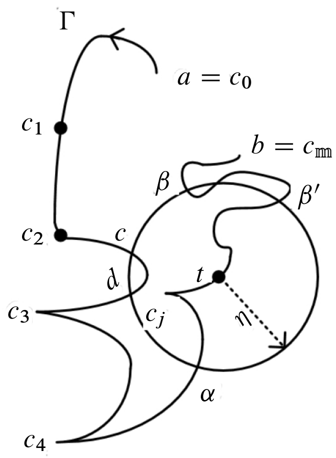

Figure 3. $\Gamma_\eta = \widehat{\alpha\beta} + \widehat{cd}$.

Theorem 4.1 (Boundedness of $\mathfrak{D}_\Gamma$ on open arc Γ – first version). *Let $\Gamma = \widehat{ab}$ be a simple piecewise smooth open arc. If $\varphi \in H^\mu(\Gamma)$ with $0 < \mu < 1$, then $\mathfrak{D}_\Gamma[\varphi] \in H^\nu(\Gamma, \mathbb{C})$, more precisely,*

$$\left|(\mathfrak{D}_\Gamma[\varphi])(z) - (\mathfrak{D}_\Gamma[\varphi])(t)\right| \le B_{\mu,\nu}\mathcal{M}_\Gamma[\varphi,\mu]|z-t|^\nu, \quad t \in \Gamma,\ z \in \mathbb{C} \quad (0 < \nu < \mu), \tag{4.1}$$

where $\mathfrak{D}_\Gamma$ is defined by (2.16) *and $B_{\mu,\nu}$ is a positive constant independent of φ, t and z. And $\mathfrak{D}_\Gamma$ from the Banach space $H^\mu(\Gamma)$ into the Banach space $H^\nu(\Gamma, \mathbb{C})$ is bounded.*

Proof. Let $\Gamma = \widehat{ab} = \sum_{j=0}^{\mathbb{m}} \Gamma_j$, where $\Gamma_j = \widehat{c_{j-1}c_j}$ $(j = 1, 2, \ldots, \mathbb{m})$ with $c_0 = a$ and $c_{\mathbb{m}} = b$ are the smooth subarcs in the increasing order of parameter. $\{\Gamma_j = \widehat{c_{j-1}c_j}, j = 1, 2, \ldots, \mathbb{m}\}$ is still called the standard smooth segmentation of the curve Γ. All the notations used here are the same as those in the proof of Theorem 2.1. And the idea of proof is also the same as that of Theorem 2.1, except that we need to reestimate the upper bound of δ_4. Under this case (see Figure 3), we easily see that

$$\begin{aligned}
\delta_4 &= \left| \int_{\Gamma\setminus\Gamma_\eta} \frac{\mathrm{d}\tau}{\tau - t} \right| \\
&\le 2(\mathbb{m}-2)\pi + \left| \left[\int_{\widehat{ac}} + \int_{\widehat{\beta b}} \right] \frac{\mathrm{d}\tau}{\tau - t} \right| \quad \left(\int_{\widehat{\beta b}} = 0 \text{ if } |b - t| < \eta \right) \\
&\le 2\mathbb{m}\pi + \ln\frac{|a-t|}{\eta} + \ln\frac{|b-t|}{\eta} \quad (\text{say } 0 < \eta < 1) \\
&\le 2\mathbb{m}\pi + 2|\ln \mathrm{dia}(\Gamma)| + |\ln 2|t - z|| \quad (\mathrm{dia}(\Gamma) \text{ is the diameter of } \Gamma) \\
&\le M + |\ln|t - z|| \quad (M \text{ is some positive constant}).
\end{aligned} \tag{4.2}$$

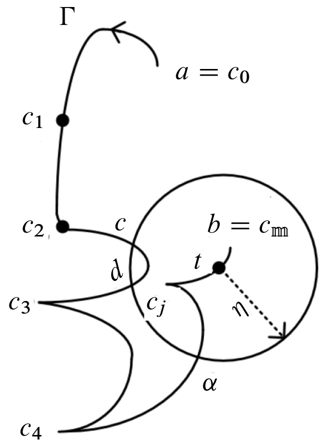

Figure 4. $\Gamma_\eta = \widehat{\alpha b} + \widehat{cd}$.

Then, we get

$$\begin{aligned}
&|(\mathfrak{D}_\Gamma[\varphi])(z) - (\mathfrak{D}_\Gamma[\varphi])(t)| \\
&\quad \le [B_\mu + |\ln|t-z||]\mathcal{M}_\Gamma[\varphi,\mu]|z-t|^\mu \quad \text{(by (2.21), (2.22), (2.24), and (4.2))} \\
&\quad \le [B_\mu[\mathrm{dia}(\Gamma)]^{\mu-\nu} + \mathbb{B}_{\mu-\nu}]\mathcal{M}_\Gamma[\varphi,\mu]|z-t|^\nu \quad (\mathbb{B}_\varsigma = \sup\{|x^\varsigma \ln x|, 0 < x \le 1\}) \\
&\quad = B_{\mu,\nu}|z-t|^\nu \quad (0 < \nu < \mu < 1).
\end{aligned}$$

The proof is now complete. ■

Remark 4.1. It must be noted that there are two possible scenarios for $\Gamma \backslash \Gamma_\eta$.

Case 1. Both a and b are not in D_η (see Figure 3).

Case 2. Only one of them is in D_η and the other is not in D_η (see Figure 4). The estimate (4.2) is true for both cases. If we can guarantee that both a and b are not in D_η, then the estimate (4.2) can be improved as follows:

$$\begin{aligned}
\delta_4 &= \left| \int_{\Gamma\backslash\Gamma_\eta} \frac{\mathrm{d}\tau}{\tau - t} \right| \quad (\gamma_1 = |t-a| > \eta, \gamma_2 = |t-b| > \eta) \\
&\le 2(\mathbb{m}-2)\pi + \left| \left[\int_{\widehat{ac}} + \int_{\widehat{\beta b}} \right] \frac{\mathrm{d}\tau}{\tau - t} \right| \\
&\le 2\mathbb{m}\pi + \left| \ln \frac{|b-t|}{|a-t|} \right| \quad (0 < \eta < 1) \\
&\le 2\mathbb{m}\pi + \ln \frac{\Gamma}{\gamma} \quad (\gamma = \min\{\gamma_1, \gamma_2\}).
\end{aligned} \tag{4.3}$$

If $\Gamma^{\text{inner}} = \widehat{a^+b^-}$ is a subarc of Γ with $a \neq a^+$ and $b \neq b^-$, which is called an inner subarc of Γ. Clearly,

$$\rho = \rho(\Gamma, \Gamma^{\text{inner}}) = d(\{a, b\}, \Gamma^{\text{inner}}) > 0, \tag{4.4}$$

where $d(A, B)$ is the distance between A and B, which is called the deviation of Γ^{inner} from Γ.

Theorem 4.2 (Boundedness of $\mathfrak{D}_\Gamma$ on open arc—second version). *Let $\Gamma = \widehat{ab}$ be a simple piecewise smooth open arc and $\Gamma^{\text{inner}} = \widehat{a^+b^-}$ an inner subarc of Γ. If $\varphi \in H^\mu(\Gamma)$ with $0 < \mu < 1$, then $\mathfrak{D}_\Gamma[\varphi] \in H^\mu(\Gamma^{\text{inner}}, \mathbb{C})$, more precisely,*

$$|(\mathfrak{D}_\Gamma[\varphi])(z) - (\mathfrak{D}_\Gamma[\varphi])(t)| \leq B_\mu \mathcal{M}_\Gamma[\varphi, \mu]|z - t|^\mu, \quad t \in \Gamma^{\text{inner}},\ z \in \mathbb{C}, \tag{4.5}$$

where $\mathfrak{D}_\Gamma$ is defined by (2.16) and B_μ is a positive constant independent of φ, t, and z. And $\mathfrak{D}_\Gamma$ from the Banach space $H^\mu(\Gamma)$ to the Banach space $H^\mu(\Gamma^{\text{inner}}, \mathbb{C})$ (equipped with the $H^\mu(\Gamma^{\text{inner}}, \mathbb{C})$-norm) is bounded.

Proof. By (2.21), (2.22), (2.24), and (4.3), when η is sufficiently small, we get

$$|(\mathfrak{D}_\Gamma[\varphi])(z) - (\mathfrak{D}_\Gamma[\varphi])(t)| \leq \left[B_\mu + \ln\frac{\Gamma}{\rho}\right]\mathcal{M}_\Gamma[\varphi, \mu]|z - t|^\mu \quad (t \in \Gamma^{\text{inner}}, z \in \mathbb{C}),$$

which results in (4.5), where B_μ is some constant and ρ is given in (4.4). ∎

For the open arc Γ^{inner}, we introduce $\Omega^+(\Gamma^{\text{inner}})$ and $\Omega^-(\Gamma^{\text{inner}})$ as follows. If $t \in \Gamma^{\text{inner}}$, we draw a circle $\mathbb{T}(t, \rho) = \{z, |z - t| = \rho\}$ with centre at t and radius ρ given in (4.4). Obviously, a and b are outside the circle $\mathbb{T}(t, \eta)$ $(0 < \eta < \rho)$, which is oriented counterclockwise. We consider the subarc $\widehat{\alpha\beta}$ on Γ where α and β are, respectively, the points first departing from and last entering into $\mathbb{T}(t, \eta)$. In detail, α and β are, respectively, two definite points with the shortest arc-lengths if two subarcs $\widehat{ta}$ and $\widehat{tb}$ of Γ are, respectively, parameterized from t. α and β divide $\mathbb{T}(t, \eta)$ into two subarcs $\widehat{\beta c\alpha}$ and $\widehat{\alpha d\beta}$. We assume that $\widehat{\beta c\alpha}$ is oriented from α to β and its orientation is consistent with the circle $\mathbb{T}(t, \eta)$, and $\widehat{\alpha d\beta}$ is similarly understood. Let γ^+ be the closed curve consisting of arc $\widehat{\beta c\alpha}$ on $\mathbb{T}(t, \eta)$ and the subcurve $\widehat{\alpha\beta}$ on Γ, γ^- the closed curve consisting of arc $\widehat{\alpha d\beta}$ on $\mathbb{T}(t, \eta)$ and the subcurve $\widehat{\alpha\beta}$ on Γ. We call, respectively, $\text{id}(\gamma^+)$ and $\text{id}(\gamma^-)$ the positive and negative half-neighborhoods of t, denoted as $O^+(t, \eta)$ and $O^-(t, \eta)$. Let

$$\begin{cases} \mathcal{N}^+(\Gamma^{\text{inner}}) = D(a, \rho) \cup \left[\bigcup_{t \in \Gamma^{\text{inner}}} O^+(t, \rho)\right] \cup D(b, \rho), \\ \mathcal{N}^-(\Gamma^{\text{inner}}) = D(a, \rho) \cup \left[\bigcup_{t \in \Gamma^{\text{inner}}} O^-(t, \rho)\right] \cup D(b, \rho), \end{cases} \tag{4.6}$$

which are, respectively, referred to as the positive and negative half-neighborhoods of Γ^{inner}. Then, we define

$$\Omega^+(\Gamma^{\text{inner}}) = \mathbb{C}\backslash\mathcal{N}^-(\Gamma^{\text{inner}}), \quad \Omega^-(\Gamma^{\text{inner}}) = \mathbb{C}\backslash\mathcal{N}^+(\Gamma^{\text{inner}}).$$

From (1.2), we have

$$(C_\Gamma[1])(z) = \frac{1}{2\pi i}\int_{\widehat{ab}} \frac{d\tau}{\tau - z} = \frac{1}{2\pi i}\log\frac{b-z}{a-z}, \quad z \notin \widehat{ab},$$

where

$$\mathcal{L}(z) = \log\frac{b-z}{a-z} \in A(\mathbb{C}\backslash\Gamma),$$
$$\mathcal{L}(\infty) = \lim_{z\to\infty}\log\frac{b-z}{a-z} = 0,$$

i.e., $\mathcal{L}$ is to be understood as a definite single-valued branch in the complex plane cut along $\Gamma = \widehat{ab}$ and it vanishes at infinity [11, 14].

In addition, let

$$\mathcal{L}^+(z) = \begin{cases} \log\dfrac{b-z}{a-z}, & z \in \mathbb{C}\backslash\Gamma, \\ \log\dfrac{b-t}{a-t}, & z = t \in \Gamma\backslash\{a,b\}, \end{cases} \tag{4.7}$$

and

$$\mathcal{L}^-(z) = \begin{cases} \log\dfrac{b-z}{a-z}, & z \in \mathbb{C}\backslash\Gamma, \\ \log\dfrac{b-t}{a-t} - 2\pi i, & z = t \in \Gamma\backslash\{a,b\}, \end{cases} \tag{4.8}$$

where $\mathcal{L}^+(t)$ and $\mathcal{L}^-(t)$ are, respectively, the positive and negative boundary values when $t \in \Gamma\backslash\{a,b\}$ [11, 14].

Lemma 4.1 (Local Hölder continuity of $\mathcal{L}^\pm$ on $\Gamma^{+\sigma}$). *Let $\mathcal{L}^\pm$ be given in* (4.7) *and* (4.8)*, and ρ given in* (4.4)*. Then*

$$\mathcal{L}^\pm \in H^\mu(D^\pm(\zeta, \rho/2))$$

with $0 < \mu \le 1$ and $\zeta \in \Omega^\pm(\Gamma^{\text{inner}})$, where

$$D^\pm\left(\zeta, \frac{1}{2}\rho\right) = D\left(\zeta, \frac{1}{2}\rho\right) \cap \Omega^\pm(\Gamma^{\text{inner}}).$$

Proof. We only prove the inequality

$$|\mathcal{L}^+(w_1) - \mathcal{L}^+(w_2)| \le A|w_1 - w_2|^\mu, \quad w_j \in D^+\left(\zeta, \frac{1}{2}\rho\right), \tag{4.9}$$

where A is a positive constant which only depends on Γ and Γ^{inner}. To do so, we analytically extend $\mathcal{L}^+|_{D^+(\zeta,\rho/2)}$ to an analytic function on the disc $D(\zeta, \rho)$ if necessary ($D^+(\zeta, \rho/2)$ intersects Γ). For example, let

$$\log_L(\upsilon) = \log\frac{a-\upsilon}{b-\upsilon}, \quad \upsilon \in \mathbb{C}\backslash L,$$

where L is a cut line from a to b and does not intersect Γ except a and b, and $\log_L(\upsilon) = \mathcal{L}(\upsilon)$ when $\upsilon \in D^+(\zeta, \rho/2)$. Thus,

$$\begin{aligned}
&\left|\mathcal{L}^+(w_1) - \mathcal{L}^+(w_2)\right| \quad \left(w_1, w_2 \in D^+\left(\zeta, \frac{1}{2}\rho\right)\right) \\
&= \left|\log\frac{a-w_1}{b-w_1} - \log\frac{a-w_2}{b-w_2}\right| \quad \left(\mathcal{L}^+(w_j) = \log_L(w_j)\right) \\
&= \left|\int_{\overline{w_1w_2}} \left[\log\frac{a-\upsilon}{b-\upsilon}\right]' \mathrm{d}\upsilon\right| \quad (\overline{w_1w_2} \text{ is the segment form } w_1 \text{ to } w_2) \\
&\le \|\log_L'\|_{\overline{D(\zeta,\rho/2)}} |w_1 - w_2| \quad \left(\log_L'(\upsilon) = \frac{1}{\upsilon - b} - \frac{1}{\upsilon - a}\right) \\
&\le \frac{4}{\rho^\mu}|w_1 - w_2|^\mu \quad \left(|\upsilon - a| \ge \frac{1}{2}\rho, |\upsilon - b| \ge \frac{1}{2}\rho\right),
\end{aligned}$$

which is just (4.9). ∎

Example 4.1. Let Γ be a simple piecewise smooth open arc and Γ^{inner} an inner subarc of Γ. Then

$$\mathcal{L}^\pm \in H^\mu(\Omega^+(\Gamma^{\text{inner}})) \quad (0 < \mu \le 1).$$

In fact, if both w_1 and w_2 are in $\Omega^\pm(\Gamma^{\text{inner}})$, by Lemma 4.1 and the boundedness of $\mathcal{L}^\pm$ on $\Omega^\pm(\Gamma^{\text{inner}})$, then

$$|\mathcal{L}^\pm(w_1) - \mathcal{L}^\pm(w_2)| \le \begin{cases} \dfrac{4}{\rho^\mu}|w_1 - w_2|^\mu & \text{when } |w_1 - w_2| < \dfrac{1}{2}\rho, \\ \dfrac{4}{\rho^\mu}\|\mathcal{L}^\pm\|_{\Omega^\pm(\Gamma^{\text{inner}})}|w_1 - w_2|^\mu & \text{when } |w_1 - w_2| \ge \dfrac{1}{2}\rho. \end{cases}$$

Example 4.2. Let Γ be a simple piecewise smooth open arc, Γ^{inner} an inner subarc of Γ and $\varphi_\Gamma(z) = \varphi(z_\Gamma)$ as before. Then, when $\varphi \in H^\mu(\Gamma)$,

$$\varphi_\Gamma \mathcal{L}^+ \in H^\mu(\Gamma^{\text{inner}}, \Omega^+(\Gamma^{\text{inner}})) \quad (0 < \mu \le 1),$$

which is directly derived from Lemma 4.1 and Example 2.3.

We still denote the angle spanned by two one-sided tangents of Γ at t towards the positive side of Γ $(0 \le \theta_t \le 2\pi)$ by θ_t, i.e., the opening angle at t for $O^+(t, \eta)$ given in (4.6) with $0 < \eta < \min\{|a - t|, |b - t|\}$ (θ_t is not dependent of η), which is also called the opening angle at t for Γ.

Now, note that

$$C_\Gamma[\varphi](z) = (\mathfrak{D}_\Gamma[\varphi])(z) + \varphi_\Gamma(z)\mathcal{L}(z), \quad z \in \mathbb{C}\backslash\Gamma, \tag{4.10}$$

and

$$(S_\Gamma[1])(t) = \frac{1}{\pi i}\log\frac{b-t}{a-t} - \frac{2\pi - \theta_t}{\pi} = \frac{1}{\pi i}\log\frac{b-t}{a-t} - \frac{\theta_t^-}{\pi}, \quad t \in \Gamma, \tag{4.11}$$

where θ_t is the opening angle at t [11, 14] while θ_t^- is the opening angle at t for $O^-(t,\eta)$ given in (4.6).

Using (4.10) and (4.11), we get the Plemelj–Sokhotski formulae for the open curve.

Corollary 4.1 (Plemelj–Sokhotski formulae). *The boundary values of* $C_\Gamma[\varphi]$ *exist and*

$$(C_\Gamma[\varphi])^\pm(t) = (C_\Gamma^\pm[\varphi])(t), \quad t \in \Gamma\backslash\{a,b\}, \tag{4.12}$$

where

$$\begin{cases} (C_\Gamma^-[\varphi])(t) = -\dfrac{\theta_t}{2\pi}\varphi(t) + \dfrac{1}{2}(S_\Gamma[\varphi])(t), \\ (C_\Gamma^+[\varphi])(t) = \left(1 - \dfrac{\theta_t}{2\pi}\right)\varphi(t) + \dfrac{1}{2}(S_\Gamma[\varphi])(t), \end{cases} \quad t \in \Gamma\backslash\{a,b\},$$

which are just the famous Plemelj–Sokhotski formulae.

Remark 4.2. It must be pointed out that the proof of Plemelj–Sokhotski formulae for the open curve here is different from that in the present monographs; see, for example, [11, 14]. The proof here is more rigorous and concise.

Using Example 4.2, Theorem 4.2, and (4.10), one gets the desired conclusion as follows.

Theorem 4.3 (Generalized Privalov–Muskhelishvili theorem on open arc). *Let* Γ *be a simple piecewise smooth open arc and* $\Gamma^{\text{inner}} = \widehat{a^+b^-}$ *an inner subarc of* Γ*. If* $\varphi \in H^\mu(\Gamma)$ *with* $0 < \mu < 1$*, then* $C_\Gamma^\pm[\varphi] \in H^\mu(\Gamma^{\text{inner}}, \Omega^\pm(\Gamma^{\text{inner}}))$*, i.e.,*

$$\left|(C_\Gamma^\pm[\varphi])(z) - (C_\Gamma^\pm[\varphi])(t)\right| \le B_\mu \|\varphi\|_{H^\mu(\Gamma)} |z-t|^\mu, \quad t \in \Gamma^{\text{inner}},\ z \in \Omega^\pm(\Gamma^{\text{inner}}),$$

where B_μ *is a positive constant independent of* φ*,* t*, and* z*. And* $C_\Gamma^\pm$ *is bounded from the Banach space* $H^\mu(\Gamma)$ *into the Banach space* $H^\mu(\Gamma^{\text{inner}}, \Omega^\pm(\Gamma^{\text{inner}}))$*.*

Proof. We easily see that

$$\begin{aligned} &\left|(C_\Gamma^\pm[\varphi])(z) - (C_\Gamma^\pm[\varphi])(t)\right| \quad \left(t \in \Gamma^{\text{inner}}, z \in \Omega^\pm(\Gamma^{\text{inner}})\right) \\ &\quad \le \left|(C_\Gamma^\pm[\varphi])(z) - (C_\Gamma[\varphi])^\pm(t)\right| \quad (\text{by } (4.12)) \\ &\quad \le \left|(\mathfrak{D}_\Gamma[\varphi])(t) - (\mathfrak{D}_\Gamma[\varphi])(z)\right| + \left|\varphi(t)\mathcal{L}^\pm(t) - \varphi_\Gamma(z)\mathcal{L}^\pm(z)\right| \quad (\text{by } (4.10)) \\ &\quad \le B_\mu \|\varphi\|_{H^\mu(\Gamma)} |z-t|^\mu \quad (\text{by Theorem 4.2 and Example 4.2}) \end{aligned}$$

And the proof is complete. ∎

Still like (2.31), let us consider

$$S_\Gamma^\theta : H^\mu(\Gamma) \to H^\mu(\Gamma), \quad \varphi \mapsto S_\Gamma^\theta[\varphi],$$

where $0 < \mu < 1$ and

$$(S_\Gamma^\theta[\varphi])(t) = \left(1 - \frac{\theta_t}{\pi}\right)\varphi(t) + (S_\Gamma[\varphi])(t), \quad t \in \Gamma\backslash\{a,b\}. \tag{4.13}$$

with $S_\Gamma[\varphi]$ given in (1.3).

We also have the following conclusion.

Theorem 4.4 (Boundedness of singular integral operator S_Γ^θ on open arc Γ). *Let Γ be a simple piecewise smooth open arc and $\Gamma^{\text{inner}} = \widehat{a^+b^-}$ an inner subarc of Γ. If $\varphi \in H^\mu(\Gamma)$ $(0 < \mu < 1)$, then $S_\Gamma^\theta[\varphi] \in H^\mu(\Gamma^{\text{inner}})$, i.e.,*

$$\left|(S_\Gamma^\theta[\varphi])(t_1) - (S_\Gamma^\theta[\varphi])(t_2)\right| \leq A_\mu \|\varphi\|_{H^\mu(\Gamma)} |t_1 - t_2|^\mu, \quad t_1, t_2 \in \Gamma^{\text{inner}},$$

where A_μ is a positive constant independent of φ and t. And S_Γ^θ is bounded from the Banach space $H^\mu(\Gamma)$ into the Banach space $H^\mu(\Gamma^{\text{inner}})$.

In particular, we have the following common conclusion if L is a smooth open arc.

Corollary 4.2 (Boundedness of singular integral operator S_Γ on open arc Γ). *Let Γ be the simple smooth open arc and $\Gamma^{\text{inner}} = \widehat{a^+b^-}$ the inner subarc of Γ. If $\varphi \in H^\mu(\Gamma)$ $(0 < \mu < 1)$, then $S_\Gamma[\varphi] \in H^\mu(\Gamma^{\text{inner}})$, i.e.,*

$$\left|(S_\Gamma[\varphi])(t_1) - (S_\Gamma[\varphi])(t_2)\right| \leq C_\mu \|\varphi\|_{H^\mu(\Gamma)} |t_1 - t_2|^\mu, \quad t_1, t_2 \in \Gamma^{\text{inner}}, \tag{4.14}$$

where $S_\Gamma[\varphi]$ is given in (1.3) and C_μ is a positive constant independent of φ and t. And S_Γ is bounded from the Banach space $H^\mu(\Gamma)$ into the Banach space $H^\mu(\Gamma^{\text{inner}})$.

Proof. In this case, (4.13) becomes

$$(S_\Gamma^\pi[\varphi])(t) = \frac{1}{2}\varphi(t) + (S_\Gamma[\varphi])(t), \quad t \in \Gamma \backslash \{a, b\},$$

or

$$(S_\Gamma[\varphi])(t) = (S_\Gamma^\pi[\varphi])(t) - \frac{1}{2}\varphi(t), \quad t \in \Gamma \backslash \{a, b\}.$$

Thus, by Theorem 4.4 and $\varphi \in H^\mu(\Gamma)$, we get (4.14). ∎

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