Topological models of abstract commensurators

Edgar A. Bering IV and Daniel Studenmund

Abstract. The full solenoid over a topological space X is the inverse limit of all finite covers. When X is a compact Hausdorff space admitting a locally path-connected universal cover, we relate the pointed homotopy equivalences of the full solenoid to the abstract commensurator of the fundamental group $\pi_1(X)$. The relationship is an isomorphism when X is an aspherical CW complex. If X is additionally a geodesic metric space and $\pi_1(X)$ is residually finite, we show that this topological model is compatible with the realization of the abstract commensurator as a subgroup of the quasi-isometry group of $\pi_1(X)$. This is a general topological analog of work of Biswas, Nag, Odden, Sullivan, and others on the universal hyperbolic solenoid, the full solenoid over a closed surface of genus at least two.

1. Introduction

The abstract commensurator of a group G, denoted by $\mathrm{Comm}(G)$, is the collection of isomorphisms $\phi\colon H \to K$ between finite-index subgroups $H, K \leq G$, modulo the equivalence relation which identifies isomorphisms that agree on a finite-index domain. When G is infinite, $\mathrm{Comm}(G)$ is a natural relaxation of the automorphism group $\mathrm{Aut}(G)$, but is often more difficult to study. For example, it is not known whether $\mathrm{Comm}(F_2)$ is simple, where F_2 is a two-generated free group. Our goal is to provide topological and metric perspectives on $\mathrm{Comm}(G)$ parallel to those used to study $\mathrm{Aut}(G)$.

Given a group G with Eilenberg–MacLane space $X = K(G, 1)$, the automorphism group $\mathrm{Aut}(G)$ is topologically modeled by $\mathcal{E}(X, *)$, the group of pointed homotopy classes of pointed homotopy equivalences of X. When G is finitely generated, $\mathrm{Aut}(G)$ is metrically modeled by a natural map to $\mathrm{QI}(G)$, the quasi-isometry group of G. If X is a compact geodesic metric space, these two models of $\mathrm{Aut}(G)$ are compatible: each pointed homotopy equivalence of X lifts to a proper homotopy equivalence of the universal cover $\widetilde{X}$, preserving the G-orbit of the basepoint, which determines an element of $\mathrm{QI}(G)$ by the Milnor–Schwarz lemma.

The metric model of $\mathrm{Aut}(G)$ extends to a natural map $\mathrm{Comm}(G) \to \mathrm{QI}(G)$, a metric model of $\mathrm{Comm}(G)$. This article develops a topological model of $\mathrm{Comm}(G)$ compatible with this metric model and analogous to the homotopy model of $\mathrm{Aut}(G)$.

1.1. Statement of results

Our results are applied to fundamental groups of compact Hausdorff spaces admitting a locally path-connected universal cover; spaces satisfying the latter condition are called *unloopable*. Following classic constructions of McCord [19], for any such space $(X, *)$ we construct the *full solenoid* over X as the inverse limit $(\widehat{X}, *)$ of all finite-sheeted covers of $(X, *)$.

Theorem 5.6. *Suppose* $(X, *)$ *is an unloopable pointed compact Hausdorff space. Then there is a homomorphism*

$$C \colon \mathcal{E}(\widehat{X}, *) \to \mathrm{Comm}(\pi_1(X, *)).$$

If X is homotopy equivalent to an aspherical CW complex, this map is an isomorphism.

For example, let $X = T^n$ be an n-dimensional torus with $G = \pi_1(X, *) \cong \mathbb{Z}^n$. There is an isomorphism that $\mathrm{Comm}(\mathbb{Z}^n) \cong \mathrm{GL}_n(\mathbb{Q})$, and so there is an isomorphism $\mathcal{E}(\widehat{T^n}, *) \cong \mathrm{GL}_n(\mathbb{Q})$.

After discussing abstract commensurators in Section 3 and reviewing the structure of full solenoids in Section 4, we prove Theorem 5.6 in Section 5. The proof combines the structure of the full solenoid as a fiber bundle over X with totally disconnected fibers with standard facts in shape theory applied to the inverse system of covering spaces used to define $\widehat{X}$. We present the arguments with care, starting with a review of inverse systems in Section 2, because the techniques may be unfamiliar to some readers.

Continuing the analogy with the automorphism group, for a residually finite group G we connect the topological model of $\mathrm{Comm}(G)$ to its realization in $\mathrm{QI}(G)$ in Section 7. To do so, we assume the base space X is a geodesic metric space and then use the metric structure on $\widehat{X}$ as a space foliated by leaves quasi-isometric to G, which we develop in Section 6.

Theorem 7.7. *Suppose* $(X, *)$ *is an unloopable compact geodesic metric space with residually finite fundamental group $G = \pi_1(X, *)$. There is a homomorphism $Q \colon \mathcal{E}(\widehat{X}, *) \to \mathrm{QI}(G)$ which factors through the homomorphism $C \colon \mathcal{E}(\widehat{X}, *) \to \mathrm{Comm}(G)$ of Theorem 5.6.*

We were motivated to study full solenoids by the work of Sullivan, with Biswas and Nag, who initiated the study of the universal hyperbolic solenoid $\widehat{\Sigma}$ [2, 25, 26], the full solenoid over a closed surface Σ of genus $g \geq 2$. Odden then proved that the pointed mapping class group

$$\mathrm{Homeo}(\widehat{\Sigma}, *)/\mathrm{Homeo}_0(\widehat{\Sigma}, *) \cong \mathrm{Comm}(\pi_1(\Sigma))$$

provides a topological model of the abstract commensurator of the fundamental group of a closed surface [20]. Odden connected the homeomorphisms of the solenoid to the action of $\mathrm{Comm}(\pi_1(\Sigma))$ on the boundary of the hyperbolic plane. Section 8 records a generalization in the setting of hyperbolic groups, relating the action of $\mathrm{Comm}(G)$ on ∂G to the solenoid model.

Corollary 8.4. *Suppose $G \cong \pi_1(X, *)$, where $(X, *)$ is an unloopable compact geodesic metric space homotopy equivalent to an aspherical CW complex. Suppose G is a residually finite, torsion-free, non-elementary hyperbolic group. Then the homomorphism Q: $\mathcal{E}(\widehat{X}, *) \to QI(G)$ of Theorem 7.7 induces isomorphisms*

$$\mathcal{E}(\widehat{X}, *) \cong \mathrm{Comm}(G) \cong \mathrm{Comm}_{\mathrm{Homeo}(\partial G)}(G).$$

1.2. Relationship with known commensurator computations

Corollary 8.4 fits into a familiar scheme of understanding $\mathrm{Comm}(G)$ by finding a map $\mathrm{Comm}(G) \to \mathrm{Aut}(X)$ for some object X on which G acts faithfully, then proving the map is an isomorphism onto the relative commensurator $\mathrm{Comm}_{\mathrm{Aut}(X)}(G)$. When G is a sufficiently nice branch group acting on a tree T, such as Grigorchuk's group of intermediate growth, Röver [22] gave an isomorphism $\mathrm{Comm}(G) \cong \mathrm{Comm}_{\mathrm{Homeo}(\partial T)}(G)$. When $G = F$ is Thompson's group, Burillo, Cleary, and Röver [5] identified an isomorphism $\mathrm{Comm}(F) \cong \mathrm{Comm}_{\mathrm{PL}_2^{\pm}(\mathbb{R})}(F)$, where $\mathrm{PL}_2^{\pm}(\mathbb{R})$ is a certain group of piecewise linear homeomorphisms of $\mathbb{R}$. When $G = \mathrm{Mod}(\Sigma)$ is the extended mapping class group of a closed surface of genus $g \geq 3$ and $C(\Sigma)$ is the curve complex of Σ, Ivanov [16] gave an isomorphism $\mathrm{Comm}(\mathrm{Mod}(\Sigma)) \cong \mathrm{Comm}_{\mathrm{Aut}(C(\Sigma))}(\mathrm{Mod}(\Sigma))$. This has been generalized to other subgroups of mapping class groups, notably in recent work of Brendle and Margalit [3].

In each of these cases, the structure of $\mathrm{Aut}(X)$ has allowed for more information about $\mathrm{Comm}(G)$ to be determined. Obtaining the strongest possible conclusion, Ivanov proved a rigidity statement $\mathrm{Aut}(C(\Sigma)) \cong \mathrm{Mod}(\Sigma)$, which has the consequence that the natural map $\mathrm{Mod}(\Sigma) \to \mathrm{Comm}(\mathrm{Mod}(\Sigma))$ is an isomorphism.

1.3. Bibliographic remarks

Solenoids are well-studied objects in classical topology, as a rich source of examples and counterexamples, as well as in dynamics, where they arise naturally in the study of foliations and actions on Cantor sets. Their appearance in dynamics goes back to the works of Smale [23] and Williams [29], who studied solenoids arising as the inverse limit of iterated self-immersions of branched manifolds.

McCord's work, summarized in Theorem 4.5 in Section 4, identifies $\widehat{X}$ with the suspension of the action of $\pi_1(X, *)$ on its profinite completion by right translation. This is an example of a minimal, equicontinuous group action on a Cantor set. From this point of view, the full solenoid is an example of a foliated space called a matchbox manifold. Such dynamical systems have been studied systematically by Clark, Dyer, Hurder, and Lukina [7, 8, 10].

The structure of $\widehat{X}$ as a principal bundle over X, which is used in the proof of Theorem 5.6, has been further studied by Clark and Fokkink [6] in the case that X is a closed manifold. Weak solenoids over closed manifolds were shown by Fokkink and Oversteegen to be fiber bundles with profinite group fiber [12, §7].

2. Inverse systems and pro-categories

Solenoids are by definition inverse limits of certain topological spaces. In order to relate their topology to the abstract commensurator, we start by recalling the relevant categorical framework. We follow the language and conventions of Mardešić and Segal [18].

Given a category $\mathcal{C}$, an *inverse system* in $\mathcal{C}$ consists of a directed set Λ, a collection of objects $\{X_\lambda\}_{\lambda \in \Lambda}$ and bonding morphisms $p_{\lambda\lambda'} \colon X_{\lambda'} \to X_\lambda$ for each $\lambda \leq \lambda'$ satisfying $p_{\lambda\lambda} = \mathrm{id}_{X_\lambda}$ and $p_{\lambda\lambda'} \circ p_{\lambda'\lambda''} = p_{\lambda\lambda''}$ for all $\lambda \leq \lambda' \leq \lambda''$. To ease the burden of notation, when referring to the entire system, we will use boldface without indexing information when such information is not needed.

Given two inverse systems $(\{X_\lambda\}, p_{\lambda\lambda'}, \Lambda)$ and $(\{Y_\mu\}, q_{\mu\mu'}, M)$ a *morphism* $\mathbf{f} = (f_\mu, \phi) \colon \mathbf{X} \to \mathbf{Y}$ is a function $\phi \colon M \to \Lambda$ and a collection of morphisms $\{f_\mu \colon X_{\phi(\mu)} \to Y_\mu\}_{\mu \in M}$ that are compatible with the bonding morphisms. That is, for all $\mu \leq \mu' \in M$, there exists $\lambda \geq \phi(\mu), \phi(\mu')$ such that

$$f_\mu \circ p_{\phi(\mu)\lambda} = q_{\mu\mu'} \circ f_{\mu'} \circ p_{\phi(\mu')\lambda}.$$

Two morphisms $(f_\mu, \phi), (g_\mu, \psi) \colon \mathbf{X} \to \mathbf{Y}$ are *equivalent* if for all μ, there exists $\lambda \geq \phi(\mu), \psi(\mu)$ such that

$$f_\mu \circ p_{\phi(\mu)\lambda} = g_\mu \circ p_{\psi(\mu)\lambda}.$$

Definition 2.1. A *pro-category* over a category $\mathcal{C}$, denoted by pro-$\mathcal{C}$, has as objects inverse systems in $\mathcal{C}$, and as morphisms equivalence classes of system morphisms.

Observe that there is a functor from $\mathcal{C}$ to pro-$\mathcal{C}$ that sends each object X to the rudimentary system (X) indexed by a singleton and that this allows us to treat $\mathcal{C}$ as a full subcategory of pro-$\mathcal{C}$. In the interests of easing notation, we will typically omit the system notation from rudimentary systems.

If $(\Lambda, \leq)$ is a directed set and $M \subseteq \Lambda$, say that M is *cofinal* if, for every $\lambda \in \Lambda$, there is some $\mu \in M$ such that $\lambda \leq \mu$. Given any inverse system $\mathbf{X} = (\{X_\lambda\}, p_{\lambda\lambda'}, \Lambda)$ in pro-$\mathcal{C}$ and directed subset $M \subseteq \Lambda$, then

$$\mathbf{X}^M = (\{X_\mu\}, p_{\mu\mu'}, M)$$

is an inverse system and there is a restriction map of inverse systems $\mathbf{i} \colon \mathbf{X} \to \mathbf{X}^M$.

Lemma 2.2 ([18, Chapter I, §1.1, Theorem 1]). *If* $\mathbf{X} = (\{X_\lambda\}, p_{\lambda\lambda'}, \Lambda)$ *is an inverse system in* $\mathcal{C}$ *and* $M \subseteq \Lambda$ *is cofinal, then the restriction* $\mathbf{i} \colon \mathbf{X} \to \mathbf{X}^M$ *is an isomorphism in* pro-$\mathcal{C}$.

A directed set $(\Lambda, \leq)$ is *cofinite* if, for each $\lambda \in \Lambda$, the set $\{\lambda' \in \Lambda \mid \lambda' \leq \lambda\}$ is finite. All directed sets we consider are cofinite.

Lemma 2.3 ([18, Chapter I, §1.2, Lemma 2]). *Suppose* $\mathbf{X} = (\{X_\lambda\}, p_{\lambda\lambda'}, \Lambda)$ *and* $\mathbf{Y} = (\{Y_\mu\}, q_{\mu\mu'}, M)$ *are inverse systems in* $\mathcal{C}$ *and* $\mathbf{f} = (f_\mu, \phi) \colon \mathbf{X} \to \mathbf{Y}$ *is a morphism of inverse*

systems. If M is cofinite, then $\mathbf{f}$ *is equivalent to a morphism of systems* $(g_\mu, \psi)\colon \mathbf{X} \to \mathbf{Y}$ *such that* $\psi\colon M \to \Lambda$ *increases and for any* $\mu \leq \mu'$ *the following diagram commutes:*

$$
\begin{array}{ccc}
X_{\psi(\mu')} & \xrightarrow{\;g_{\mu'}\;} & Y_{\mu'} \\
{\scriptstyle p_{\psi(\mu)\psi(\mu')}}\Big\downarrow & & \Big\downarrow{\scriptstyle q_{\mu\mu'}} \\
X_{\psi(\mu)} & \xrightarrow{\;g_\mu\;} & Y_\mu.
\end{array}
$$

We use the following notation for categories. Top is the category of topological spaces and continuous maps, and Cpt is the category of compact Hausdorff spaces and continuous maps. HTop is the category whose objects are topological spaces and whose morphisms are homotopy classes of continuous maps. HCpt is defined analogously. We use Top_*, Cpt_*, HTop_*, and HCpt_* to denote the respective pointed categories, in which each space is equipped with a basepoint, and all continuous maps and homotopies preserve basepoints. Grp is the category of groups with homomorphisms. The notation $\mathcal{C}(A, B)$ is used for the set of $\mathcal{C}$-morphisms from A to B.

3. Abstract commensurators

Write $H \leq_f G$ to mean that H is a finite-index subgroup of a group G, and $H \trianglelefteq_f G$ to mean moreover that H is normal. A *partial automorphism* of a group G is an isomorphism between two finite-index subgroups of G. If $\phi_1\colon H_1 \to K_1$ and $\phi_2\colon H_2 \to K_2$ are partial automorphisms, declare ϕ_1 *equivalent* to ϕ_2, denoted by $\sim$, if there exists $H_3 \leq_f H_1 \cap H_2$ such that ϕ_1 and ϕ_2 agree on H_3.

The *abstract commensurator* of a group G is the group of equivalence classes of partial automorphisms,

$$\mathrm{Comm}(G) = \{\phi\colon H \to K \mid \phi \text{ is an isomorphism between } H, K \leq_f G\}/\!\sim.$$

The group structure on $\mathrm{Comm}(G)$ is defined by $[\phi] \circ [\psi] = [\phi' \circ \psi']$ for any $\phi \sim \phi'$ and $\psi \sim \psi'$ such that $\phi' \circ \psi'$ is defined.

For any subgroup $H \leq G$, the *relative commensurator* of H in G is defined by

$$\mathrm{Comm}_G(H) = \{g \in G \mid [H : gHg^{-1} \cap H] < \infty \text{ and } [gHg^{-1} : gHg^{-1} \cap H] < \infty\}.$$

For any group G, there is a homomorphism $G \to \mathrm{Aut}(G)$ defined by $g \mapsto c_g$, where $c_g(x) = gxg^{-1}$ for all $x \in G$. Let $\hat{\iota}\colon G \to \mathrm{Comm}(G)$ denote the composition $G \to \mathrm{Aut}(G) \to \mathrm{Comm}(G)$. The proof of the following lemma is straightforward.

Lemma 3.1. *For any group G,*

$$\mathrm{Comm}_{\mathrm{Comm}(G)}(\hat{\iota}(G)) = \mathrm{Comm}(G).$$

3.1. Commensurations as pro-automorphisms

For any abstract group G, let $\mathcal{F}(G)$ denote the collection of finite-index subgroups of G, equipped with inclusions $G_1 \to G_2$ whenever $G_1 \leq_f G_2 \leq_f G$. The intersection of any two finite-index subgroups has finite index in G, so $\mathcal{F}(G)$ is the collection of groups of an inverse system $\mathbf{G}$ in the category Grp, ordered by reverse inclusion.

Proposition 3.2. *Let G be a group and $\mathbf{G}$ be the inverse system of finite-index subgroups of G. Then there is an isomorphism*

$$\zeta \colon \mathrm{Comm}(G) \cong \mathrm{Aut}_{\text{pro-Grp}}(\mathbf{G}).$$

Proof. For any $H \leq_f G$, the system of finite-index subgroups of H is cofinal in the system of finite-index subgroups of G. By Lemma 2.2, the inverse of the restriction map is a pro-Grp isomorphism $i_{GH} \colon \mathbf{H} \to \mathbf{G}$. The isomorphism i_{GH} is represented by the map of inverse systems that, for each finite-index subgroup $K \leq G$, includes $K \cap H \to K$. These isomorphisms are functorial; if $H_1 \leq_f H_2 \leq_f G$, then

$$i_{GH_1} = i_{GH_2} \circ i_{H_2 H_1}.$$

Suppose $H, K \leq_f G$ and $\phi \colon H \to K$ is an isomorphism. The map ϕ induces a bijection between $\mathcal{F}(H)$ and $\mathcal{F}(K)$. The collection of isomorphisms $\{\phi|_{H'} \mid H' \leq_f H\}$ is a morphism of inverse systems, which induces a pro-Grp isomorphism $\phi_* \colon \mathbf{H} \to \mathbf{K}$.

The reader may verify the following two properties. First, the assignment $\phi \mapsto \phi_*$ is functorial in the sense that $(\phi \circ \psi)_* = \phi_* \circ \psi_*$ whenever the composition $\phi \circ \psi$ is defined. Second, if $\psi \colon H' \to K'$ is the restriction of some isomorphism $\phi \colon H \to K$ to a domain $H' \leq_f H$, then $\phi_* = i_{KK'} \circ \psi_* \circ i_{HH'}^{-1}$.

We now define ζ. Given any $[\phi] \in \mathrm{Comm}(G)$ represented by an isomorphism $\phi \colon H \to K$, let $\zeta([\phi]) = i_{GK} \circ \phi_* \circ i_{GH}^{-1}$. To see that ζ is well defined, suppose $\phi \colon H \to K$ is an isomorphism and $\psi \colon H' \to K'$ is the restriction of ϕ to some $H' \leq_f H$. Then by the above observations, we have

$$
\begin{aligned}
i_{GK'} \circ \psi_* \circ i_{GH'}^{-1} &= (i_{GK} \circ i_{KK'}) \circ \psi_* \circ (i_{GH} \circ i_{HH'})^{-1} \\
&= i_{GK} \circ (i_{KK'} \circ \psi_* \circ i_{HH'}^{-1}) \circ i_{GH}^{-1} \\
&= i_{GK} \circ \phi_* \circ i_{GH}^{-1}.
\end{aligned}
$$

The fact that ζ is a homomorphism follows from the functoriality of the assignment $\phi \mapsto \phi_*$.

To see that ζ is injective, suppose a partial automorphism $\phi \colon H \to K$ satisfies

$$\zeta([\phi]) = \mathbf{id} \in \mathrm{Aut}_{\text{pro-Grp}}(\mathbf{G}).$$

This implies that $\phi_* \circ i_{GH}^{-1}$ and i_{GK}^{-1} are equivalent maps of inverse systems $\mathbf{G} \to \mathbf{K}$. Because i_{GH}^{-1} and i_{GK}^{-1} are restriction maps, by definition of equivalence there is a finite-index subgroup $H_0 \leq_f H$ such that $\phi|_{H_0} = \mathrm{id}_{H_0}$. Therefore, $[\phi]$ is trivial in $\mathrm{Comm}(G)$.

To see that ζ is surjective, suppose $[\mathbf{f}] \in \mathrm{Aut}_{\text{pro-Grp}}(\mathbf{G})$ is represented by a collection of homomorphisms $\{f_\lambda \colon G_{\rho(\lambda)} \to G_\lambda \mid G_\lambda \in \mathcal{F}(G)\}$, where $\rho \colon \Lambda \to \Lambda$ is a function of the indexing set of $\mathcal{F}(G)$. Because $[\mathbf{f}]$ is an isomorphism, it is both a monomorphism and an epimorphism [18, Chapter II, §2.2, Theorem 6]. Because $[\mathbf{f}]$ is a monomorphism, there are an index $\lambda_0 \in \Lambda$ and a subgroup $H_0 \leq_f G_{\rho(\lambda_0)}$ such that $f_{\lambda_0}|_{H_0}$ is injective [18, Chapter II, §2.1, Theorem 2]. Because $[\mathbf{f}]$ is an epimorphism, there is a finite-index subgroup $K_0 \leq_f G$ such that $K_0 \leq_f f_{\lambda_0}(H_0)$ [18, Chapter II, §2.1, Theorem 4]. Let $H = H_0$ and $K = f_{\lambda_0}(H)$. Then $f_0 = f_{\lambda_0}|_H \colon H \to K$ is an isomorphism between finite-index subgroups of G.

It remains to check that $[i_{GK} \circ (f_0)_* \circ i_{GH}^{-1}] = [\mathbf{f}]$. Let $f' = i_{GK} \circ (f_0)_* \circ i_{GH}^{-1}$ be the endomorphism of the inverse system $\mathcal{F}(G)$, with associated indexing function $\rho' \colon \Lambda \to \Lambda$. By definition, for any $\lambda \in \Lambda$, we have $G_{\rho'(\lambda)} = f_{\lambda_0}^{-1}(G_\lambda \cap K)$ and $f'_\lambda = f_{\lambda_0}|_{G_{\rho'(\lambda)}}$.

Consider any $G_\lambda \in \mathcal{F}(G)$ and let $\lambda' \in \Lambda$ be the index such that $G_{\lambda'} = G_\lambda \cap K$. Because $G_{\lambda'} \leq_f G_\lambda$, the fact that f is a morphism of inverse systems implies that there is a subgroup $S_1 \leq_f G$ such that $f_\lambda S_1 = f_{\lambda'}|_{S_1}$. Similarly, because $G_{\lambda'} \leq G_{\lambda_0}$, there is some $S_2 \leq_f G$ such that $f_{\lambda'}|_{S_2} = f_{\lambda_0}|_{S_2}$. Now let $D = S_1 \cap S_2 \cap G_{\rho'(\lambda)} \leq_f G$. Combining the above observations, we have $f_\lambda|_D = f'_\lambda|_D$, which completes the proof. ∎

4. Full solenoids over unloopable spaces

The limit space of a sequence of finite-sheeted, regular covering maps appeared early in the study of homogeneous topological spaces as a source of examples and counterexamples. McCord [19] gave a general account of the structure of such a space, not requiring coverings to be finite-sheeted, which he called a *solenoidal space*. Elsewhere in the literature, these spaces have simply been called *solenoids*, while some authors reserve the name solenoid for inverse limits of systems of finite-sheeted coverings of closed manifolds, or of only the circle (the original solenoids of van Dantzig and Vietoris [27, 28]). Limits of sequences of finite-sheeted covers, which are not necessarily regular, are often called *weak solenoids* [21]. We are interested in the limit of the inverse system of *all* finite-sheeted covers of a space, which we call the *full solenoid* (see Definition 4.2). After introducing the definitions with care, we review McCord's work in this setting.

Definition 4.1. A topological space X is *unloopable* if it is path-connected, locally path-connected, and semi-locally simply connected.[1] Equivalently, X is unloopable if it has a locally path-connected universal cover.

We will typically work in categories whose objects are pointed topological spaces. When there is no chance of confusion, we will use the same symbol to denote the basepoints in different spaces.

---

[1] McCord calls such spaces *nice* [19]. We borrow our terminology from Bourbaki, which uses the term *dèlaçables*.

Convention. When working with an unloopable pointed topological space $(X, *)$, we fix a pointed universal cover $(\widetilde{X}, *) \to (X, *)$ once and for all. The universal cover comes equipped with a left action $\pi_1(X, *) \times \widetilde{X} \to \widetilde{X}$ written $(g, x) \mapsto gx$. We then realize the Galois correspondence between connected covers of $(X, *)$ and subgroups of its fundamental group by setting $(X_H, *) = H \backslash (\widetilde{X}, *)$ for a subgroup $H \leq \pi_1(X, *)$. We will refer to constructions as *canonical* if they are canonical for a given choice of universal cover.

Definition 4.2. Given an unloopable pointed topological space $(X, *)$, the *full solenoid* over $(X, *)$, denoted by $(\widehat{X}, *)$, is the inverse limit of the system of connected pointed finite-sheeted covers of $(X, *)$ in Top_*, where the bonding maps are the quotient covering maps $X_{H_1} \to X_{H_2}$ corresponding to subgroup inclusion $H_1 \leq H_2$:

$$(\widehat{X}, *) = \varprojlim_{H \leq_f \pi_1(X, *)} (X_H, *).$$

The path-components of $(\widehat{X}, *)$ are called *leaves*.

Example 4.3. The name "full solenoid" comes from McCord's terminology "solenoidal space", inspired by the case $X = S^1$. For each natural number n, the circle has one isomorphism class of covering space X_n, which is often visualized as an n-fold coil over the circle. The inverse limit of all X_n is the full solenoid $\widehat{S^1}$ over the circle. It is a fiber bundle over the circle with fibers homeomorphic to the Cantor set. The fiber structure is transverse to a foliation by uncountably many path components. Each path component is the image of the universal cover $\mathbb{R}$ under an injective continuous map $\mathbb{R} \to \widehat{S^1}$, and this image is dense. The full solenoid $\widehat{S^1}$ is therefore visualized as a compact coil wound in some complicated, dense way. (These properties are instances of the results described in Theorem 4.5 below.)

Given an unloopable pointed space $(X, *)$ with full solenoid $(\widehat{X}, *)$, let $(\mathbf{X}, *)$ denote the inverse system of finite-sheeted covers of $(X, *)$ with respect to a given universal cover. For any finite-index subgroup $H \leq_f \pi_1(X, *)$, there is a projection map $p_H \colon (\widehat{X}, *) \to (X_H, *)$. The collection of all such p_H determines a map of inverse systems, hence a morphism $\mathbf{p} \colon (\widehat{X}, *) \to (\mathbf{X}, *)$ in pro-Top_*.

The construction of $(\widehat{X}, *)$ is independent of the choice of universal cover in the following senses. First, any two universal covers of an unloopable pointed space $(X, *)$ are homeomorphic by a unique morphism of pointed covering spaces. Any such homeomorphism induces homeomorphisms between covering spaces $(X_H, *)$ compatible with the covering maps. It follows that $(\widehat{X}, *)$ is well defined up to homeomorphism commuting with the projection $(\widehat{X}, *) \to (X, *)$. Second, for a given choice of universal cover, the collection $\{(X_H, *) \mid H \leq_f \pi_1(X, *)\}$ contains exactly one representative from each equivalence class of connected, finite-sheeted pointed cover of $(X, *)$. Therefore, the full solenoid $(\widehat{X}, *)$ is homeomorphic to the inverse limit of the collection of *all* connected, finite-sheeted pointed covers of $(X, *)$.

McCord [19] described the structure of inverse limits of *sequences* of regular covers of an unloopable base space. The results of his original paper apply to inverse limits of general directed systems of regular covers of an unloopable base space with essentially no change to the arguments. The system of finite-sheeted, regular covers of $(X, *)$ is cofinal in the system of finite-sheeted covers of $(X, *)$, so McCord's results apply to full solenoids over unloopable spaces.

Definition 4.4. Given an unloopable pointed topological space $(X, *)$, the *baseleaf* of $(\widehat{X}, *)$ is the image of the canonical map $\ell\colon (\widetilde{X}, *) \to (\widehat{X}, *)$ induced by the covering maps $\widetilde{X} \to X_H$ for $H \leq_f \pi_1(X, *)$.

For any unloopable pointed topological space $(X, *)$, the collection of finite quotients $\pi_1(X, *)/H$ for $H \trianglelefteq_f \pi_1(X, *)$ forms an inverse system in Grp under the quotient maps $\pi_1(X, *)/H \to \pi_1(X, *)/K$ whenever $H \leq K$. The profinite completion is

$$\widehat{\pi_1(X, *)} = \varprojlim_{H \trianglelefteq_f \pi_1(X,*)} \pi_1(X, *)/H.$$

The actions $\pi_1(X, *)/H \curvearrowright X_H$ determine an effective topological group action $\widehat{\pi_1(X, *)} \curvearrowright \widehat{X}$ [19, Lemma 5.2].

There is a continuous surjective map $\Pi\colon \widehat{\pi_1(X, *)} \times \widetilde{X} \to \widehat{X}$ defined by $\Pi(t, x) = t\ell(x)$, where ℓ is the canonical baseleaf map. There is a continuous left action of $\pi_1(X, *)$ on $\widehat{\pi_1(X, *)} \times \widetilde{X}$ defined by

$$g \cdot (\gamma, x) = (\gamma \iota(g)^{-1}, gx),$$

where $\iota\colon \pi_1(X, *) \to \widehat{\pi_1(X, *)}$ is the canonical map. McCord proves that $\Pi\colon \widehat{\pi_1(X, *)} \times \widetilde{X} \to \widehat{X}$ is a (generalized) covering map whose covering transformations are precisely those homeomorphisms given by the action of $\pi_1(X, *)$.

Let $\widehat{\pi_1(X, *)} \times_{\pi_1(X,*)} \widetilde{X}$ denote the quotient of $\widehat{\pi_1(X, *)} \times \widetilde{X}$ by the action of $\pi_1(X, *)$. We summarize the consequences of the above in the following theorem.

Theorem 4.5 ([19, Theorems 5.5, 5.6, 5.8 and 5.12]). *Suppose $(X, *)$ is an unloopable pointed topological space.*

(1) *There is a homeomorphism*

$$(\widehat{\pi_1(X, *)} \times_{\pi_1(X,*)} \widetilde{X}, [\mathrm{id}, *]) \cong (\widehat{X}, *).$$

   The canonical baseleaf map $\ell\colon \widetilde{X} \to \widehat{X}$ is the composition of $\widetilde{X} \to \{\mathrm{id}\} \times \widetilde{X}$ with the quotient.

(2) *Let $K = \ell(\widetilde{X})$ be the baseleaf. A subset of $\widehat{X}$ is a path component if and only if it is of the form γK for some $\gamma \in \widehat{\pi_1(X, *)}$.*

(3) *The baseleaf map $\ell\colon \widetilde{X} \to \widehat{X}$ descends to a map $N \backslash \widetilde{X} \to \widehat{X}$ which is a bijection onto its image, where $N = \bigcap_{H \trianglelefteq_f \pi_1(X,*)} H$ is the residual finiteness kernel of $\pi_1(X, *)$.*

(4) *The canonical projection $(\widehat{X}, *) \to (X, *)$ is a principal $\widehat{\pi_1(X, *)}$-bundle.*

Corollary 4.6. *Suppose $(X, *)$ is an unloopable pointed topological space with residually finite fundamental group. The baseleaf map $\ell \colon \widetilde{X} \to \widehat{X}$ is injective with dense image.*

5. The shape of solenoids

Our first main result, Theorem 5.6, is proved using the language of shape theory. The natural approach to shape for this setting is that of inverse systems, as described by Mardešić and Segal [18]. The inverse system approach to shape associates certain inverse systems, known as expansions, to an object in a category.

Definition 5.1 ([18, Chapter I, §2.1, p. 19]). Given a category $\mathcal{T}$ and an object X in $\mathcal{T}$, a pro-$\mathcal{T}$ morphism $\mathbf{p} \colon X \to \mathbf{X}$ is a $\mathcal{T}$*-expansion*[2] of X if $\mathbf{X}$ is an object of pro-$\mathcal{T}$ and $\mathbf{p}$ satisfies a universal property: for any pro-$\mathcal{T}$ morphism $\mathbf{h} \colon X \to \mathbf{Y}$ with $\mathbf{Y}$ in pro-$\mathcal{T}$, there is a unique pro-$\mathcal{T}$ morphism $\mathbf{f}$ making the following diagram commute:

$$
\begin{array}{ccc}
X & \xrightarrow{\ \mathbf{h}\ } & \mathbf{Y}. \\
{\scriptstyle \mathbf{p}}\downarrow & \nearrow_{\mathbf{f}} & \\
\mathbf{X} & &
\end{array}
$$

As we are primarily interested in morphisms, the following representation property of expansions is central.

Fact 5.2 ([18, Chapter I, §2.3, p. 25]). Given $\mathcal{T}$-expansions $\mathbf{p} \colon X \to \mathbf{X}$ and $\mathbf{q} \colon Y \to \mathbf{Y}$ and a $\mathcal{T}$ morphism $f \colon X \to Y$, there exists a unique pro-$\mathcal{T}$ morphism $\mathbf{f}$ such that the following square commutes in pro-$\mathcal{T}$:

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\
{\scriptstyle \mathbf{p}}\downarrow & & \downarrow{\scriptstyle \mathbf{q}} \\
\mathbf{X} & \xrightarrow{\ \mathbf{f}\ } & \mathbf{Y}.
\end{array}
$$

It follows from uniqueness that the function

$$
\mathcal{T}(X, Y) \to \text{pro-}\mathcal{T}(\mathbf{X}, \mathbf{Y})
$$

is natural in the sense that it respects composition; if $f \circ g = h$ in $\mathcal{T}$, then $\mathbf{f} \circ \mathbf{g} = \mathbf{h}$ in pro-$\mathcal{T}$. In general, though, the assignment may be neither injective nor surjective [18, Chapter I, §2.3, Remark 9].

Specializing to full solenoids over unloopable pointed compact Hausdorff spaces, we are able to apply these facts to study self-homotopy equivalences in light of the following theorem.

---

[2] Mardešić and Segal also treat the more general notion of a $\mathcal{P}$-expansion for a full subcategory $\mathcal{P}$ of $\mathcal{T}$. We only need to consider $\mathcal{T}$-expansions.

Theorem 5.3 ([18, Chapter I, §5.4, Theorem 13]). *Suppose* $(\mathbf{X}, *)$ *is an inverse system of pointed compact Hausdorff spaces. If* $(X_\infty, *)$ *is the inverse limit of* $(\mathbf{X}, *)$ *in* Cpt_*, *then* $\mathbf{p}: (X_\infty, *) \to (\mathbf{X}, *)$ *is an* HTop_**-expansion of* $(X_\infty, *)$.

The proof of Theorem 5.6 relies on the following proposition. Compare, for example, to improvability results of Geoghegan and Krasinkiewicz [13].

Proposition 5.4. *Suppose* $(X, *)$ *and* $(Y, \star)$ *are unloopable pointed compact Hausdorff spaces with full solenoids* $(\hat{X}, *)$ *and* $(\hat{Y}, \star)$. *Let* $(\mathbf{X}, *)$ *and* $(\mathbf{Y}, \star)$ *be the inverse systems of connected, finite-sheeted pointed covers. Then there is a natural bijection*

$$\mathsf{HCpt}_*(\hat{X}, \hat{Y}) \to \text{pro-}\mathsf{HCpt}_*(\mathbf{X}, \mathbf{Y}).$$

Lemma 5.5. *Suppose that* $(X, *)$ *and* $(Y, \star)$ *are unloopable pointed compact Hausdorff spaces. Let* $(\mathbf{X}, *)$ *and* $(\mathbf{Y}, \star)$ *be the inverse systems of connected, finite-sheeted pointed covers. Then for every morphism* $[\mathbf{f}] \in \text{pro-}\mathsf{HCpt}_*(\mathbf{X}, \mathbf{Y})$, *there is a representative* $\mathbf{g}$ *that has a limit in* Cpt_*

Proof. Suppose that the inverse system $\mathbf{X}$ is indexed by Λ with bonding morphisms $p_{\lambda_2 \lambda_1}: X_{\lambda_1} \to X_{\lambda_2}$, and let $\mathbf{Y}$ be indexed by M with bonding morphisms $q_{\mu_2 \mu_1}: Y_{\mu_1} \to Y_{\mu_2}$. Both Λ and M are cofinite directed sets with unique minimum elements. Let $m \in M$ denote the minimum index, so that $Y_m = Y$.

By Lemma 2.3, any given morphism $[\mathbf{f}] \in \text{pro-}\mathsf{HCpt}_*(\mathbf{X}, \mathbf{Y})$ is represented by an order-preserving function $\phi: M \to \Lambda$ and a collection of continuous maps $f_\mu: (X_{\phi(\mu)}, *) \to (Y_\mu, \star)$ such that if $\mu \le \mu'$, then there is a pointed homotopy $f_\mu \circ p_{\phi(\mu)\phi(\mu')} \sim q_{\mu\mu'} \circ f_{\mu'}$. To construct the desired morphism of inverse systems, let $g_m = f_m: X_{\phi(m)} \to Y_m$. Given any $\mu \ge m$, the existence of the pointed homotopy $f_m \circ p_{\phi(m)\phi(\mu)} \sim q_{m\mu} \circ f_\mu$ guarantees that $f_m \circ p_{\phi(m)\phi(\mu)}$ uniquely lifts to a map $g_\mu: X_{\phi(\mu)} \to Y_\mu$ satisfying the equality of continuous maps $f_m \circ p_{\phi(m)\phi(\mu)} = q_{m\mu} \circ g_\mu$.

By construction, for any $\mu \in M$ we have pointed homotopies

$$q_{m\mu} \circ f_\mu \sim f_m \circ p_{\phi(m)\phi(\mu)} \sim q_{m\mu} \circ g_\mu.$$

The homotopy lifting property provides a pointed homotopy $f_\mu \sim g_\mu$. It follows that the collection of maps (g_μ) defines a map $\mathbf{g}$ of inverse systems in HCpt_* and $[\mathbf{g}] = [\mathbf{f}] \in \text{pro-}\mathsf{HCpt}_*(\mathbf{X}, \mathbf{Y})$.

In fact, it follows from uniqueness of liftings that we may consider $\mathbf{g}$ as a morphism of systems in Cpt_*. To see this, consider indices $\mu \le \mu' \in M$. Then

$$q_{m\mu'} \circ g_{\mu'} = f_m \circ p_{\phi(m)\phi(\mu')} = (f_m \circ p_{\phi(m)\phi(\mu)}) \circ p_{\phi(\mu)\phi(\mu')} = q_{m\mu} \circ (g_\mu \circ p_{\phi(\mu)\phi(\mu')}),$$

while on the other hand

$$q_{m\mu'} \circ g_{\mu'} = q_{m\mu} \circ (q_{\mu\mu'} \circ g_{\mu'}).$$

Then the uniqueness of lifts implies $g_\mu \circ p_{\phi(\mu)\phi(\mu')} = q_{\mu\mu'} \circ g_{\mu'}$.

For each $\lambda \in \Lambda$, let $p_\lambda \colon \widehat{X} \to X_\lambda$ be the system of projections. For each μ, define $\widehat{g_\mu} \colon \widehat{X} \to Y_\mu$ by $\widehat{g_\mu} = g_{\phi(\mu)} \circ p_{\phi(\mu)}$. These determine a continuous function $\widehat{g} \colon \widehat{X} \to \widehat{Y}$, the limit of $\mathbf{g}$. ∎

Proof of Proposition 5.4. By Theorem 5.3, both $(\widehat{X}, *) \to (\mathbf{X}, *)$ and $(\widehat{Y}, \star) \to (\mathbf{Y}, \star)$ are HTop_*-expansions. It follows from Fact 5.2 that there is a natural function

$$\mathsf{HCpt}_*(\widehat{X}, \widehat{Y}) \to \text{pro-}\mathsf{HTop}_*(\mathbf{X}, \mathbf{Y}).$$

Because $(\mathbf{X}, *)$ and $(\mathbf{Y}, \star)$ are systems of compact Hausdorff spaces and HCpt_* is a full subcategory of HTop_*, this determines a natural function

$$\mathsf{HCpt}_*(\widehat{X}, \widehat{Y}) \to \text{pro-}\mathsf{HCpt}_*(\mathbf{X}, \mathbf{Y}).$$

By Lemma 5.5, this map is surjective.

Now suppose $f, g \colon (\widehat{X}, *) \to (\widehat{Y}, \star)$ are pointed continuous maps whose pointed homotopy classes map to the same element $[\mathbf{h}]$ of pro-$\mathsf{HCpt}_*(\mathbf{X}, \mathbf{Y})$. Let $[\mathbf{q}] \colon \widehat{Y} \to \mathbf{Y}$ be the expansion morphism. Since f and g both map to $\mathbf{h}$, by Fact 5.2, the morphisms $\mathbf{q} \circ f$ and $\mathbf{q} \circ g$ are representatives of the same element of pro-$\mathsf{HCpt}_*(\widehat{X}, \mathbf{Y})$. Consequently, the maps $q \circ f$ and $q \circ g$ are pointed homotopic as maps $(\widehat{X}, *) \to (Y, \star)$, where $q \colon \widehat{Y} \to Y$ is the projection.

By Theorem 4.5, the map q is a fiber bundle over a compact Hausdorff space and therefore a fibration [24, §2.7, Corollary 14]. Therefore, the pointed homotopy $q \circ f \sim q \circ g$ lifts to a homotopy $f \sim F$ for some $F \colon \widehat{X} \to \widehat{Y}$. Because fibers of q are totally disconnected, q has the unique path lifting property [24, §2.2, Theorem 5]. This implies that both F and the homotopy $f \sim F$ are pointed. Moreover, because the baseleaf of $\widehat{X}$ is path-connected, F and g agree on the baseleaf of $\widehat{X}$. Indeed, if γ is a based path of $\widetilde{X}$ starting at the basepoint, then $q \circ F(\gamma) = q \circ g(\gamma)$, so that $F(\gamma)$ is the unique lift of $q \circ g(\gamma)$, and therefore equal to $g(\gamma)$. But the baseleaf is dense in $\widehat{X}$, so we conclude that $F = g$ as pointed continuous maps. ∎

Combining these facts, we arrive at a topological description of the abstract commensurator of $\pi_1(X, *)$. For any pointed topological space $(Y, \star)$, let $\mathcal{E}(Y, \star)$ be the automorphism group of $(Y, \star)$ in HTop_*. Elements of $\mathcal{E}(Y, \star)$ are equivalence classes of pointed homotopy equivalences, modulo pointed homotopy.

Theorem 5.6. *Suppose $(X, *)$ is an unloopable pointed compact Hausdorff space. Then there is a homomorphism*

$$C \colon \mathcal{E}(\widehat{X}, *) \to \text{Comm}(\pi_1(X, *)).$$

If X is homotopy equivalent to an aspherical CW complex, this map is an isomorphism.

Proof. Let $(\mathbf{X}, *)$ be the full system of finite-sheeted pointed covers of $(X, *)$. By Proposition 5.4, there is a natural bijection

$$\mathsf{HCpt}_*(\widehat{X}, \widehat{X}) \to \text{pro-}\mathsf{HCpt}_*(\mathbf{X}, \mathbf{X}).$$

A standard characterization of pro-morphisms is the natural bijection [18, Chapter I, §1.1, p. 8, Remark 4],

$$\mathsf{pro\text{-}HCpt}_*((\mathbf{X}, *), (\mathbf{X}, *)) \simeq \varprojlim_H \varinjlim_K \mathsf{HCpt}_*(X_H, X_K),$$

where H and K range over all finite-index subgroups of $\pi_1(X, *)$. Since π_1 is functorial, there is a natural function

$$\varprojlim_H \varinjlim_K \mathsf{HCpt}_*(X_H, X_K) \rightarrow \varprojlim_H \varinjlim_K \mathsf{Grp}(\pi_1(X_H, *), \pi_1(X_K, *)).$$

When X is homotopy equivalent to an aspherical CW complex, each X_H is also homotopy equivalent to an aspherical CW complex, and so this function is a term-by-term bijection by the Whitehead theorem.

Let $\pi_1(\mathbf{X}, *)$ be the pro-group obtained by applying the π_1 functor to the inverse system $(\mathbf{X}, *)$. As noted above, there is a natural bijection

$$\varprojlim_H \varinjlim_K \mathsf{Grp}(\pi_1(X_H, *), \pi_1(X_K, *)) \simeq \mathsf{pro\text{-}Grp}((\pi_1(\mathbf{X}, *)), (\pi_1(\mathbf{X}, *))).$$

Combining these natural morphisms, restricting to the automorphism groups in each category, and applying Proposition 3.2, we arrive at a homomorphism of groups

$$\mathcal{E}(\widehat{X}, *) \cong \mathrm{Aut}_{\mathsf{pro\text{-}HCpt}_*}(\mathbf{X}, *) \rightarrow \mathrm{Aut}_{\mathsf{pro\text{-}Grp}}(\pi_1(\mathbf{X}, *)) \cong \mathrm{Comm}(\pi_1(X, *))$$

that is an isomorphism when X is homotopy equivalent to an aspherical CW complex. ∎

Remark 5.7. Suppose $(X, *)$ is a finite connected CW complex. Let $(\mathbf{X}, *)$ be the system of finite covers and $(\widehat{X}, *)$ be the full solenoid over $(X, *)$. Then $\pi_1(\mathbf{X}, *)$ is equal to the pro-fundamental group pro-$1_1(\widehat{X}, *)$ [18, Chapter II, §3.3].

Suppose further that $(Y, \star)$ is a pointed aspherical finite connected CW complex. The proof of Theorem 5.6 can be modified in a straightforward way to prove that there is a natural bijection of hom-sets

$$\mathsf{HTop}_*(\widehat{X}, \widehat{Y}) \simeq \mathsf{pro\text{-}Grp}(\mathsf{pro\text{-}1}_1(\widehat{X}, *), \mathsf{pro\text{-}1}_1(\widehat{Y}, \star)).$$

In particular, there is a group isomorphism

$$\mathcal{E}(\widehat{Y}, \star) \cong \mathrm{Aut}(\mathsf{pro\text{-}1}_1(\widehat{Y}, \star)).$$

6. Metric notions

We now turn to the task of relating the topology of the full solenoid $(\widehat{X}, *)$ to the geometry of the group $\pi_1(X, *)$ in the case that X is an unloopable compact geodesic metric space and $\pi_1(X, *)$ is residually finite. In this section, we summarize the basic results used to metrize $\widehat{X}$ in this case.

6.1. Metric spaces

When X is an unloopable length metric space with metric d_X, X determines a metric on $\widetilde{X}$, which will also be denoted by d_X. The metric on $\widetilde{X}$ is also a length metric [4, Proposition I.3.25], and the covering projection is a local isometry with respect to this metric. If X is additionally compact, then it follows from the metric Hopf–Rinow theorem [4, Proposition I.3.7] that both X and $\widetilde{X}$ are proper geodesic metric spaces. Moreover, $\pi_1(X)$ is finitely presented and every finitely presented group arises this way [4, Corollary I.8.11].

6.2. Quasi-isometries

For metric spaces (X, d_X) and (Y, d_Y), a map $f : X \to Y$ is a *quasi-isometry* if there are constants $K \geq 1$ and $C > 0$ such that

$$\frac{1}{K} d_X(x_1, x_2) - C \leq d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2) + C$$

for any $x_1, x_2 \in X$, and for any $y \in Y$ there is some $x \in X$ such that $d_Y(f(x), y) \leq C$. The *quasi-isometry* group of X is the group $\mathrm{QI}(X)$ of equivalence classes of quasi-isometries $f : X \to X$, where $f_1, f_2 : X \to X$ are equivalent if there is some $D \geq 0$ so that

$$d(f_1(x), f_2(x)) \leq D$$

for all $x \in X$.

If G is a finitely generated group, let $\mathrm{QI}(G)$ be the quasi-isometry group of G with respect to the word metric of some finite generating set. The quasi-isometry group is independent of the chosen finite generating set. There is a natural map $\mathrm{Comm}(G) \to \mathrm{QI}(G)$, where $[\phi] \in \mathrm{Comm}(G)$ determines a quasi-isometry of G by precomposing with any closest-point projection from G to the domain of ϕ. Whyte proved that this is injective, as recorded by Farb and Mosher [11, Proposition 7.5].

Lemma 6.1. *For any unloopable compact geodesic metric space $(X, *)$ with fundamental group $G = \pi_1(X, *)$, the orbit map $G \to (\widetilde{X}, *)$ defined by $g \mapsto g*$ induces an isomorphism*

$$\mathrm{QI}(G) \cong \mathrm{QI}(\widetilde{X}).$$

Proof. By the Milnor–Schwarz lemma [4, Proposition I.8.19], the orbit map $g \mapsto g*$ is a quasi-isometry $G \to \widetilde{X}$. This induces an isomorphism $\mathrm{QI}(G) \cong \mathrm{QI}(\widetilde{X})$ [4, Exercise I.8.16.3]. ∎

In order to relate the topology of the full solenoid $(\widehat{X}, *)$ over $(X, *)$ to the coarse geometry of $G = \pi_1(X, *)$ in Section 7, we will further require that the baseleaf map $\ell : \widetilde{X} \to \widehat{X}$ be an injective map of sets. By Corollary 4.6, this is equivalent to requiring that G be residually finite.

6.3. Profinite completions

When a group G is finitely generated, for each $n \geq 1$ there is a finite-index characteristic subgroup

$$G_{\leq n} = \bigcap_{[G:H]\leq n} H.$$

The subgroups $G_{\leq n}$ are a neighborhood basis of the identity for the topology on G induced by the pseudometric $\hat{d}$ defined by

$$\hat{d}(g_1, g_2) = \exp(-\max\{n \mid g_1 g_2^{-1} \in G_{\leq n}\})$$

if such a maximum exists, and $\hat{d}(g_1, g_2) = 0$ if $g_1 g_2^{-1} \in H$ for all $H \leq_f G$.

If G is residually finite, then $\hat{d}$ is an ultrametric, and the induced topology is Hausdorff. In this case, the profinite completion $\hat{G}$ is the metric completion of G with respect to $\hat{d}$ [14, Theorem 3.5], the natural homomorphism $\iota: G \to \hat{G}$ is an inclusion, and G acts freely by isometries on $\hat{G}$ by both left and right multiplication.

6.4. Solenoid metric

Now suppose X is an unloopable compact geodesic metric space with residually finite fundamental group $G = \pi_1(X, *)$. Then G is finitely presented, so by the above discussion there are a profinite metric $\hat{d}$ on $\hat{G}$ and a proper, geodesic metric d_X on $\tilde{X}$. Equip $\hat{G} \times \tilde{X}$ with the ℓ_∞ product metric

$$d_\infty((\gamma_1, x_1), (\gamma_2, x_2)) = \max\{\hat{d}(\gamma_1, \gamma_2), d_X(x_1, x_2)\}.$$

Both the action of G on $(\hat{G}, \hat{d})$ by right multiplication and the left action of G on $(\tilde{X}, d_X)$ are isometric. Hence the left action of G on $\hat{G} \times \tilde{X}$ defined in Section 4 is isometric with respect to d_∞.

Definition 6.2. Suppose X is an unloopable compact geodesic metric space with residually finite fundamental group $G = \pi_1(X, *)$. The *solenoid metric* σ on $\hat{X}$ is the metric induced from the quotient metric on $\hat{G} \times_G \tilde{X}$, obtained by taking the infimum of d_∞ over orbit representatives, under the identification $\hat{X} \approx \hat{G} \times_G \tilde{X}$.

6.5. Metric balls in a full solenoid

Continue to consider an unloopable compact geodesic metric space X with residually finite fundamental group $G = \pi_1(X, *)$.

Definition 6.3. The *injectivity radius* of an unloopable metric space (Y, d_Y), denoted by $\mathrm{inj.\,rad}(Y)$, is the supremum of all R such that for all $x \in \tilde{Y}$, the covering projection of $B_{d_Y}(x, R)$ is an isometry onto its image.

If X is an unloopable compact geodesic metric space, then $0 < \mathrm{inj.\,rad}(X) < \infty$. Note that for all $g \in \pi_1(X, *)$ and $x \in \tilde{X}$, the inequality $d_X(g \cdot x, x) \geq \mathrm{inj.\,rad}(X)$ holds.

Lemma 6.4. *Suppose X is an unloopable compact geodesic metric space with residually finite fundamental group $G = \pi_1(X, *)$, and suppose $0 < \varepsilon < \mathrm{inj.\,rad}(X)/4$. Then for any $\hat{x} = [(\gamma, x)] \in \hat{G} \times_G \tilde{X}$, there is an isometry*

$$B_\sigma(\hat{x}, \varepsilon) \cong B_{\hat{d}}(\gamma, \varepsilon) \times B_{d_{\tilde{X}}}(x, \varepsilon).$$

In particular, each path component of $B_\sigma(\hat{x}, \varepsilon)$ is isometrically identified with $B_{d_X}(x, \varepsilon)$.

Proof. Take any $(\gamma, x) \in \hat{G} \times \tilde{X}$. Consider $B = B_{d_\infty}((\gamma, x), \varepsilon) = B_{\hat{d}}(\gamma, \varepsilon) \times B_{d_X}(x, \varepsilon)$. Then for any $y \in B_{d_X}(x, \varepsilon)$ and any $g \in G$, we have $d_X(y, gy) \geq \mathrm{inj.\,rad}(X) > 4\varepsilon > 2\varepsilon$ and therefore $gy \notin B_{d_X}(x, \varepsilon)$. It follows that the ball B maps injectively into $\hat{X}$.

To see that the inclusion $B \to \hat{X}$ is isometric onto its image, it suffices to show that if $y, z \in B_{d_X}(x, \varepsilon)$ and $g \in G$ is non-trivial, then $d_X(y, gz) \geq 2\varepsilon$. This follows from the reverse triangle inequality

$$d_X(y, gz) \geq d_X(z, gz) - d_X(y, z) \geq \mathrm{inj.\,rad}(X) - 2\varepsilon > 2\varepsilon. \qquad \blacksquare$$

7. Homotopy and baseleaf quasi-isometry

In general, compactness improves the behavior of continuous functions between metric spaces. In the setting of full solenoids over spaces with residually finite fundamental groups, compactness allows one to translate from the topology of the solenoid to the coarse geometry of the baseleaf. This has been studied in the context of foliated manifolds, for example, in the work of Hurder [15, §2.4].

Definition 7.1. A function $f : (X, d_X) \to (Y, d_Y)$ on metric spaces is (L, C)-*coarsely Lipschitz* if for all $x, y \in X$

$$d_Y(f(x), f(y)) \leq L d_X(x, y) + C.$$

Given an unloopable pointed space $(X, *)$, any pointed continuous map $f : (\hat{X}, *) \to (\hat{X}, *)$ preserves the baseleaf. When $\pi_1(X, *)$ is residually finite, the baseleaf map $\ell : \tilde{X} \to \hat{X}$ is a bijection onto its image by Corollary 4.6. Identifying the set $\tilde{X}$ with its image in $\hat{X}$, say that f restricts to a function of pointed sets $f_{\tilde{X}} : (\tilde{X}, *) \to (\tilde{X}, *)$.

Lemma 7.2. *Suppose $(X, *)$ is an unloopable compact geodesic metric space with residually finite fundamental group $G = \pi_1(X, *)$. Suppose $f : (\hat{X}, *) \to (\hat{X}, *)$ is a pointed continuous function. Then for any $C > 0$, there exists $\widetilde{\mathrm{Lip}}_C(f) > 0$ such that $f_{\tilde{X}} : (\tilde{X}, d_X) \to (\tilde{X}, d_X)$ is $(\widetilde{\mathrm{Lip}}_C(f), C)$-coarsely Lipschitz.*

Proof. Since X is compact and Hausdorff, $\hat{G} \times_G \tilde{X}$ is compact. Therefore, since f is continuous, it is uniformly continuous in the σ metric on $\hat{X}$.

Take $0 < \varepsilon < \min\{\mathrm{inj.\,rad}(X)/4, C\}$ and choose $0 < \delta < \varepsilon$ such that for all $x, y \in \hat{X}$ if $\sigma(x, y) < \delta$, then $\sigma(f(x), f(y)) < \varepsilon$. Now suppose $x, y \in \tilde{X}$ and $d_X(x, y) < \delta$. This

implies $\sigma(x, y) < \delta$ by definition. Let γ be a d_X geodesic joining x to y. Since γ is a geodesic, $\gamma \subset B_{d_X}(x, \delta) \subseteq B_\sigma(x, \delta)$. Therefore, by uniform continuity,

$$f(\gamma) \subseteq B_\sigma(f(x), \varepsilon),$$

so $f(x)$ and $f(y)$ are in the same path component of $B_\sigma(f(x), \varepsilon)$. The path component of $B_\sigma(f(x), \varepsilon)$ containing $f(x)$ is simply $B_{d_X}(f(x), \varepsilon)$ by Lemma 6.4. Therefore, if $d_X(x, y) < \delta$, then $d_X(f(x), f(y)) < \varepsilon$.

Next, suppose $x, y \in \widetilde{X}$ and $M = d_X(x, y) \geq \delta$. Divide a geodesic between x and y into $N = \lfloor M/\delta \rfloor \geq 1$ length δ segments, with endpoints $x = t_0, t_1, \ldots, t_N = y$. By repeatedly applying the triangle inequality, we estimate

$$d_X(f(x), f(y)) \leq N\varepsilon \leq \frac{\varepsilon}{\delta} d_X(x, y).$$

Since $\varepsilon < C$, by setting $\widetilde{\mathrm{Lip}}_C(f) = \varepsilon/\delta$, we conclude that for all $x, y \in \widetilde{X}$

$$d_X(f(x), f(y)) < \widetilde{\mathrm{Lip}}_C(f) d_X(x, y) + C. \qquad \blacksquare$$

Lemma 7.3. *Suppose $(X, *)$ is an unloopable compact geodesic metric space with residually finite fundamental group. Given continuous pointed functions $f, g : (\widehat{X}, *) \to (\widehat{X}, *)$ such that $f \sim g$, there exists a constant C such that for all $p \in \widetilde{X}$,*

$$d_X(f_{\widetilde{X}}(p), g_{\widetilde{X}}(p)) \leq C.$$

Proof. Take $0 < \varepsilon < \mathrm{inj.\,rad}(X)/4$. Let F be a homotopy witnessing $f \sim g$. As $\widehat{X} \times [0, 1]$ is compact, F is uniformly continuous with respect to the ℓ_∞ product metric $d_\infty^{\widehat{X} \times [0,1]}$. Choose $\delta > 0$ such that for all $(x, t), (y, s) \in \widehat{X} \times [0, 1]$,

$$d_\infty^{\widehat{X} \times [0,1]}((x, t), (y, s)) < \delta \Rightarrow \sigma(F(x, t), F(y, s)) < \varepsilon.$$

Choose $0 = t_0 < t_1 < \cdots < t_L = 1$ in $[0, 1]$ such that $|t_{n+1} - t_n| < \delta$. Given any $p \in \widetilde{X}$, the image $F(\{p\} \times [t_n, t_{n+1}])$ is a path-connected subset of $B_\sigma(F(p, t_n), \varepsilon)$. By our choice of ε, Lemma 6.4 implies that $F(\{p\} \times [t_n, t_{n+1}])$ is in fact contained in $B_{d_X}(F(p, t_n), \varepsilon)$ in $(\widetilde{X}, d_X)$, thus

$$d_X(F(p, t_n), F(p, t_{n+1})) < \varepsilon.$$

Therefore, for any $p \in \widetilde{X}$, repeated application of the triangle inequality gives

$$d_X(f_{\widetilde{X}}(p), g_{\widetilde{X}}(p)) \leq L \cdot \varepsilon. \qquad \blacksquare$$

Let $\mathrm{HE}(\widehat{X}, *)$ be the collection of all pointed homotopy equivalences $f : (\widehat{X}, *) \to (\widehat{X}, *)$. Note that $\mathrm{HE}(\widehat{X}, *)$ is a monoid, and $\mathcal{E}(\widehat{X}, *)$ is the quotient group consisting of pointed homotopy classes.

Corollary 7.4. *Given $f, g \in \mathrm{HE}(\widehat{X}, *)$ such that g is a homotopy inverse of f, there exists a C such that for all $p \in \widetilde{X}$,*

$$d_X(f \circ g(p), p) \leq C \quad \text{and} \quad d_X(g \circ f(p), p) \leq C.$$

Combining this corollary with Lemma 7.2, we arrive at an exact description of the coarse behavior of pointed homotopy equivalences.

Lemma 7.5. *Suppose $(X, *)$ is an unloopable compact geodesic metric space with residually finite fundamental group. For each $f \in \mathrm{HE}(\widehat{X}, *)$, the restriction $f_{\widetilde{X}}$ is a quasi-isometry of $(\widetilde{X}, d_X)$.*

Proof. Given any $f \in \mathrm{HE}(\widehat{X}, *)$, fix a homotopy inverse g. By Corollary 7.4, there exists a C such that for all $p \in \widetilde{X}$, we have $d_X(f \circ g(p), p) \leq C$ and $d_X(g \circ f(p), p) \leq C$.

Next, by Lemma 7.2 with $L = \max\{\widetilde{\mathrm{Lip}}_C(f), \widetilde{\mathrm{Lip}}_C(g)\}$, both f and g are (L, C)-coarse Lipschitz on $(\widetilde{X}, d_X)$. This is a characterization of an (L, C)-quasi-isometry of $(\widetilde{X}, d_X)$ [9, Corollary 8.13], and we are done. ∎

Proposition 7.6. *Suppose $(X, *)$ is an unloopable compact geodesic metric space with residually finite fundamental group $G = \pi_1(X, *)$. There is a group homomorphism*

$$Q \colon \mathcal{E}(\widehat{X}, *) \to \mathrm{QI}(\widetilde{X}) \cong \mathrm{QI}(G)$$

given by $[f] \mapsto f_{\widetilde{X}}$ for a representative $f \in \mathrm{HE}(\widehat{X}, *)$.

Proof. By Lemma 7.5, given $f \in \mathrm{HE}(\widehat{X}, *)$, the restriction $f_{\widetilde{X}}$ is a quasi-isometry of $\widetilde{X}$. The assignment $f \mapsto f_{\widetilde{X}}$ is a monoid homomorphism

$$\mathrm{HE}(\widehat{X}, *) \to \mathrm{QI}(\widetilde{X}).$$

Next, if $f, g \in \mathrm{HE}(\widehat{X}, *)$ and $f \sim g$, then by Lemma 7.3 there exists a constant C such that for all $p \in \widetilde{X}$, the distance $d_X(f(p), g(p)) \leq C$. Thus, $f_{\widetilde{X}}$ and $g_{\widetilde{X}}$ represent the same element of $\mathrm{QI}(\widetilde{X})$. Therefore, we obtain a monoid homomorphism

$$Q \colon \mathcal{E}(\widehat{X}, *) \to \mathrm{QI}(\widetilde{X}).$$

Every monoid homomorphism of groups is a group homomorphism. The conclusion follows by composing with the isomorphism $\mathrm{QI}(\widetilde{X}) \cong \mathrm{QI}(G)$ of Lemma 6.1. ∎

Theorem 7.7. *Suppose $(X, *)$ is an unloopable compact geodesic metric space with residually finite fundamental group $G = \pi_1(X, *)$. The homomorphism $Q \colon \mathcal{E}(\widehat{X}, *) \to \mathrm{QI}(G)$ factors through the map $C \colon \mathcal{E}(\widehat{X}, *) \to \mathrm{Comm}(G)$ of Theorem 5.6.*

Proof. Suppose $[f] \in \mathcal{E}(\widehat{X}, *)$. By Lemma 5.5, there is a morphism of the covering system $\{f_\alpha \colon X_{\phi(\alpha)} \to X_\alpha\}$ with limit $\widehat{f}$ homotopic to f. As in the proof of Theorem 5.6, the collection of induced maps $(f_{\alpha *})$ is a pro-Grp automorphism of the inverse system of finite-index subgroups of G.

Pick an index α and set $H_0 = \pi_1(X_{\phi(\alpha)}, *)$. As in the proof of Proposition 3.2, there are finite-index subgroups $H, K \leq_f G$ such that $\phi = f_{\alpha *}|_H \colon H \to K$ is an isomorphism. By definition, $C([f]) = [\phi] \in \mathrm{Comm}(G)$.

The inclusion of $[\phi]$ into QI(G) is the composition of nearest point projection to H and $\phi\colon H \to K$. Let $(\widetilde{X}, p) \to (X, *)$ be the universal cover of X. Under the identification QI$(G) \cong$ QI$(\widetilde{X})$ induced by the orbit map $g \mapsto gp$, the image of $[\phi]$ in QI$(\widetilde{X})$ is the map $hp \mapsto \phi(h)p$, which bijectively maps the H-orbit of the basepoint p to the K-orbit of p, precomposed with a closest-point projection $\widetilde{X} \to Hp$.

Let $\widetilde{f_\alpha}\colon (\widetilde{X}, p) \to (\widetilde{X}, p)$ be the unique lift of f_α. Since $\widehat{f}$ is the limit of f_α, for any point hp in the H-orbit of the basepoint $p \in \widetilde{X}$, we have

$$\widehat{f}(hp) = \widetilde{f_\alpha}(hp) = (f_{\alpha *}(h))p = \phi(h)p.$$

Since H and K are finite-index subgroups of G, this implies that $\widehat{f}$ is a bounded distance from the image of ϕ in QI$(\widetilde{X})$. Therefore, by Lemma 7.3, f and ϕ are bounded distance maps, and we are done. ∎

8. Hyperbolicity and the boundary realization

For a closed hyperbolic surface Σ, Odden proved the existence of an isomorphism

$$\mathrm{Comm}(\pi_1(\Sigma, *)) \cong \mathrm{Homeo}(\widehat{\Sigma}, *)/\mathrm{Homeo}_0(\widehat{\Sigma}, *)$$

by proving that both groups are isomorphic to the relative commensurator of $\pi_1(\Sigma)$ in $\mathrm{Homeo}(\partial\mathbb{H}^2)$ [20, Theorems 4.6 and 4.12]. We will prove that the abstract commensurator of any torsion-free, non-elementary hyperbolic group G is isomorphic to its relative commensurator in the group of homeomorphisms of its boundary, generalizing half of Odden's proof. We then provide a topological analog of the other half of the proof in the case when G acts on a sufficiently nice metric space X which is homotopy equivalent to a finite aspherical CW complex.

Suppose G is a Gromov hyperbolic group. Its boundary ∂G is well defined up to G-equivariant homeomorphism. There is a well-defined map QI$(G) \to \mathrm{Homeo}(\partial G)$, which is injective in the case $|\partial G| > 3$ [9, Corollary 11.115].[3] When $|\partial G| > 3$, we say G is *non-elementary*. The next lemma follows from composition with the inclusion $\mathrm{Comm}(G) \to$ QI(G) described in Section 6.

Lemma 8.1. *Suppose G is a torsion-free, non-elementary hyperbolic group. The composition $\mathrm{Comm}(G) \to$ QI$(G) \to \mathrm{Homeo}(\partial G)$ is an injective map*

$$\Phi\colon \mathrm{Comm}(G) \to \mathrm{Homeo}(\partial G).$$

A group G has the *unique root property* if $g^n = h^n$ implies $g = h$ for all $g, h \in G$ and all non-zero $n \in \mathbb{Z}$. If G has the unique root property and $\phi_1\colon H_1 \to K_1$ and $\phi_2\colon H_2 \to K_2$

---

[3]Note that there is a typo in the cited corollary. To see injectivity in the case that G is non-elementary, read comments after Lemma 11.112 and in the proof of Lemma 11.130 in the same reference.

are equivalent partial automorphisms of G, then ϕ_1 and ϕ_2 agree on $H_1 \cap H_2$ [1, Lemma 2.4]. Note that if G satisfies the unique root property then for any $H \leq_f G$ the natural map $\mathrm{Aut}(H) \to \mathrm{Comm}(G)$ is injective.

Lemma 8.2. *Suppose G is a torsion-free, non-elementary hyperbolic group. Then G has the unique root property and trivial center. In particular, the map $\hat{\iota}: G \to \mathrm{Comm}(G)$ defined in Section* 3 *is injective.*

Proof. Any non-trivial element $g \in G$ in a hyperbolic group has virtually cyclic centralizer $C_G(g)$ [4, Corollary III.$R\Gamma$.3.10]. Since G is torsion-free, $C_G(g)$ is infinite cyclic. It follows that the center of G is trivial, so the conjugation map $G \to \mathrm{Aut}(G)$ is injective. Further, it follows that G has the unique root property [1, Lemma 2.2], and so the natural map $\mathrm{Aut}(G) \to \mathrm{Comm}(G)$ is injective. ∎

Theorem 8.3. *Suppose G is a torsion-free, non-elementary hyperbolic group. Then the inclusion $\Phi: \mathrm{Comm}(G) \to \mathrm{Homeo}(\partial G)$ induces an isomorphism*

$$\mathrm{Comm}(G) \cong \mathrm{Comm}_{\mathrm{Homeo}(\partial G)}(G),$$

where $G \leq \mathrm{Homeo}(\partial G)$ is identified with its image under the injective map $\Phi \circ \hat{\iota}$.

Proof. According to Lemma 3.1, we know $\mathrm{Comm}(G) = \mathrm{Comm}_{\mathrm{Comm}(G)}(\hat{\iota}(G))$. If K is a group containing a subgroup H and ϕ is any homomorphism with domain K, then $\phi(\mathrm{Comm}_K(H)) \leq \mathrm{Comm}_{\phi(K)}(\phi(H))$. This implies

$$\Phi(\mathrm{Comm}(G)) \leq \mathrm{Comm}_{\Phi(\mathrm{Comm}(G))}(\Phi(\hat{\iota}(G))) \leq \mathrm{Comm}_{\mathrm{Homeo}(\partial G)}(G).$$

Therefore, it suffices to show that Φ has image $\mathrm{Comm}_{\mathrm{Homeo}(\partial G)}(G)$.

Define a function

$$\Psi: \mathrm{Comm}_{\mathrm{Homeo}(\partial G)}(G) \to \mathrm{Comm}(G)$$

as follows: Given $f \in \mathrm{Comm}_{\mathrm{Homeo}(\partial G)}(G)$, let $G_1 = f^{-1}Gf \cap G$ and $G_2 = fG_1f^{-1}$. According to definition, both G_1 and G_2 are finite-index subgroups of G, and conjugation $c_f: G_1 \to G_2$ is an isomorphism. Let $\Psi(f) = [c_f] \in \mathrm{Comm}(G)$.

To finish the proof, it suffices to show $(\Phi \circ \Psi)(f) = f$ for any homeomorphism $f \in \mathrm{Comm}_{\mathrm{Homeo}(\partial G)}(G)$. To this end, fix $f \in \mathrm{Comm}_{\mathrm{Homeo}(\partial G)}(G)$ and let

$$\widehat{f} = (\Phi \circ \Psi)(f).$$

Find finite-index subgroups $G_1, G_2 \leq G$ such that $fGf^{-1} = G_2$. Applying the above definitions, $\widehat{f}g\widehat{f}^{-1} = fgf^{-1}$ for any $g \in G_1$. Equivalently, for any $g \in G_1$,

$$(f^{-1}\widehat{f})g(f^{-1}\widehat{f})^{-1} = g. \tag{8.1}$$

We now appeal to basic facts about the action of a hyperbolic group on its boundary—see the survey of Kapovich and Benakli for a reference [17, §4]. Every $g \in G$ has exactly

two fixed points on ∂G, an attracting fixed point g^+ and a repelling fixed point g^-, and attracting fixed points of elements of G are dense in ∂G. Suppose $x \in \partial G$ is an attracting fixed point g^+ for $g \in G$. Replacing g with a sufficiently large positive power, we may assume $g \in G_1$. It follows from (8.1) that $f^{-1}(\widehat{f}(g^+))$ is a fixed point of g with the same dynamical properties as g^+, therefore $f^{-1}(\widehat{f}(g^+)) = g^+$. By density of the set of attracting fixed points, this implies $\widehat{f} = f$. ∎

Corollary 8.4. *Suppose $(X, *)$ is an unloopable compact geodesic metric space which is homotopy equivalent to an aspherical CW complex. Suppose $G = \pi_1(X, *)$ is a residually finite, torsion-free, non-elementary hyperbolic group. Then the map $Q \colon \mathcal{E}(\widehat{X}, *) \to \mathrm{QI}(G)$ of Proposition 7.6 induces an isomorphism*

$$\mathcal{E}(\widehat{X}, *) \cong \mathrm{Comm}_{\mathrm{Homeo}(\partial G)}(G).$$

Proof. By Theorem 7.7, the image of Q lies in $\mathrm{Comm}(G) \leq \mathrm{QI}(G)$. Because X is homotopy equivalent to an aspherical CW complex, Q defines an isomorphism $\mathcal{E}(\widehat{X}, *) \to \mathrm{Comm}(G)$ by Theorem 5.6. The result follows from Theorem 8.3. ∎

References

[1] L. Bartholdi and O. Bogopolski, On abstract commensurators of groups. *J. Group Theory* **13** (2010), no. 6, 903–922 Zbl 1262.20046 MR 2736164

[2] I. Biswas, S. Nag, and D. Sullivan, Determinant bundles, Quillen metrics and Mumford isomorphisms over the universal commensurability Teichmüller space. *Acta Math.* **176** (1996), no. 2, 145–169 Zbl 0959.32026 MR 1397561

[3] T. E. Brendle and D. Margalit, Normal subgroups of mapping class groups and the metaconjecture of Ivanov. *J. Amer. Math. Soc.* **32** (2019), no. 4, 1009–1070 Zbl 1476.57020 MR 4013739

[4] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*. Grundlehren Math. Wiss. 319, Springer, Berlin, 1999 Zbl 0988.53001 MR 1744486

[5] J. Burillo, S. Cleary, and C. E. Röver, Commensurations and subgroups of finite index of Thompson's group F. *Geom. Topol.* **12** (2008), no. 3, 1701–1709 Zbl 1190.20031 MR 2421137

[6] A. Clark and R. Fokkink, Bihomogeneity of solenoids. *Algebr. Geom. Topol.* **2** (2002), 1–9 Zbl 0995.54025 MR 1885212

[7] A. Clark, S. Hurder, and O. Lukina, Classifying matchbox manifolds. *Geom. Topol.* **23** (2019), no. 1, 1–27 Zbl 1411.37009 MR 3921315

[8] A. Clark, S. Hurder, and O. Lukina, Pro-groups and generalizations of a theorem of Bing. *Topology Appl.* **271** (2020), article no. 106986 Zbl 1432.57050 MR 4046916

[9] C. Druţu and M. Kapovich, *Geometric group theory (with an appendix by Bogdan Nica)*. Amer. Math. Soc. Colloq. Publ. 63, American Mathematical Society, Providence, RI, 2018 Zbl 1447.20001 MR 3753580

[10] J. Dyer, S. Hurder, and O. Lukina, The discriminant invariant of Cantor group actions. *Topology Appl.* **208** (2016), 64–92 Zbl 1364.37065 MR 3506971

[11] B. Farb and L. Mosher, The geometry of surface-by-free groups. *Geom. Funct. Anal.* **12** (2002), no. 5, 915–963 Zbl 1048.20026 MR 1937831

[12] R. Fokkink and L. Oversteegen, Homogeneous weak solenoids. *Trans. Amer. Math. Soc.* **354** (2002), no. 9, 3743–3755 Zbl 0998.54018 MR 1911519

[13] R. Geoghegan and J. Krasinkiewicz, Empty components in strong shape theory. *Topology Appl.* **41** (1991), no. 3, 213–233 Zbl 0762.57016 MR 1135099

[14] M. Hall, Jr., A topology for free groups and related groups. *Ann. of Math. (2)* **52** (1950), 127–139 Zbl 0045.31204 MR 36767

[15] S. Hurder, Coarse geometry of foliations. In *Geometric study of foliations (Tokyo, 1993)*, pp. 35–96, World Scientific, River Edge, NJ, 1994 MR 1363719

[16] N. V. Ivanov, Automorphism of complexes of curves and of Teichmüller spaces. *Int. Math. Res. Not. IMRN* **1997** (1997), no. 14, 651–666 Zbl 0890.57018 MR 1460387

[17] I. Kapovich and N. Benakli, Boundaries of hyperbolic groups. In *Combinatorial and geometric group theory (New York, 2000/Hoboken, NJ, 2001)*, pp. 39–93, Contemp. Math. 296, American Mathematical Society, Providence, RI, 2002 Zbl 1044.20028 MR 1921706

[18] S. Mardešić and J. Segal, *Shape theory. The inverse system approach*. North-Holland Math. Libr. 26, North-Holland Publishing, Amsterdam, 1982 Zbl 0495.55001 MR 676973

[19] M. C. McCord, Inverse limit sequences with covering maps. *Trans. Amer. Math. Soc.* **114** (1965), 197–209 Zbl 0136.43603 MR 173237

[20] C. Odden, The baseleaf preserving mapping class group of the universal hyperbolic solenoid. *Trans. Amer. Math. Soc.* **357** (2005), no. 5, 1829–1858 Zbl 1077.57017 MR 2115078

[21] J. T. Rogers, Jr. and J. L. Tollefson, Maps between weak solenoidal spaces. *Colloq. Math.* **23** (1971), 245–249 Zbl 0224.54007 MR 309046

[22] C. E. Röver, Abstract commensurators of groups acting on rooted trees. *Geom. Dedicata* **94** (2002), 45–61 Zbl 1064.20032 MR 1950873

[23] S. Smale, Differentiable dynamical systems. *Bull. Amer. Math. Soc.* **73** (1967), 747–817 Zbl 0202.55202 MR 228014

[24] E. H. Spanier, *Algebraic topology*. McGraw-Hill Book, New York, 1966 Zbl 0145.43303 MR 210112

[25] D. Sullivan, Linking the universalities of Milnor–Thurston, Feigenbaum and Ahlfors–Bers. In *Topological methods in modern mathematics (Stony Brook, NY, 1991)*, pp. 543–564, Publish or Perish, Houston, TX, 1993 Zbl 0803.58018 MR 1215976

[26] D. Sullivan, Solenoidal manifolds. *J. Singul.* **9** (2014), 203–205 Zbl 1323.57018 MR 3249058

[27] D. van Dantzig, Über topologisch homogene Kontinua. *Fund. Math.* **15** (1930), 102–125 Zbl 56.1130.01

[28] L. Vietoris, Über den höheren Zusammenhang kompakter Räume und eine Klasse von zusammenhangstreuen Abbildungen. *Math. Ann.* **97** (1927), no. 1, 454–472 Zbl 53.0552.01 MR 1512371

[29] R. F. Williams, Expanding attractors. *Inst. Hautes Études Sci. Publ. Math.* **43** (1974), 169–203 Zbl 0279.58013 MR 348794

Edgar A. Bering IV
Department of Mathematics and Statistics, San José State University, One Washington Square, MacQuarrie Hall, San José, CA 95192, USA; edgar.bering@sjsu.edu

Daniel Studenmund
Department of Mathematics and Statistics, Binghamton University, PO Box 6000, Binghamton, NY 13902-6000, USA; daniel@math.binghamton.edu