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Quantum smooth uncertainty principles for von Neumann bi-algebras

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Abstract. In this article, we prove various smooth uncertainty principles on von Neumann bi-algebras, which unify a number of uncertainty principles on quantum symmetries, such as subfactors, fusion bialgebras, etc., studied in quantum Fourier analysis. We also obtain Wigderson–Wigderson type uncertainty principles for von Neumann bi-algebras. Moreover, we give a complete answer to a conjecture proposed by A. Wigderson and Y. Wigderson.

1. Introduction

Uncertainty principles have been investigated for more than hundred years in mathematics and physics inspired by the famous Heisenberg uncertainty principle [7, 16, 25] with significant applications in information theory [3, 4].

Recently, quantum uncertainty principles on subfactors, an important type of quantum symmetries [6, 13], have been established for support and for von Neumann entropy in [11] and for Rényi entropy in [20]. These quantum uncertainty principles have been generalized on other types of quantum symmetries, such as Kac algebras [19], locally compact quantum groups [12] and fusion bialgebras [18], etc., in the unified framework of quantum Fourier analysis [10]. Such quantum inequalities were applied in the classification of subfactors [17] and as analytic obstructions of unitary categorifications of fusion rings in [18].

In 2021, A. Wigderson and Y. Wigderson [26] introduced k-Hadamard matrices, as an analog of discrete Fourier transforms, and they proved various uncertainty principles such as primary uncertainty principles, support uncertainty principles, etc. Their work unifies a number of proofs of uncertainty principles in the classical settings.

In this paper, we unify several quantum (support, entropic) uncertainty principles on quantum symmetries and we further generalize the results to various smooth supports $\mathcal{S}_\varepsilon^p$ and smooth entropies H_ε^p whose definitions are stated in the following. Let

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$\mathcal{M}$ be a finite von Neumann algebra with a normal faithful trace $\tau_{\mathcal{M}}$. The smooth support $\mathcal{S}_{\varepsilon}^{p}(x)$ of x with $p \geq 1, 0 \leq \varepsilon \leq 1$ is defined as

$$\mathcal{S}_{\varepsilon}^{p}(x) := \inf\{\tau_{\mathcal{M}}(y\mathcal{R}(x)) : y \in \mathcal{M}, 0 \leq y \leq I, \|(I-y)x\|_p \leq \varepsilon\|x\|_p\},$$

where $\|\cdot\|_p$ is the p-norm on $\mathcal{M}$, $\mathcal{R}(x)$ is the range projection of x. The smooth entropy $H_{\varepsilon}^{p}(|x|^2)$ of $|x|^2$ with $p \geq 1, 0 \leq \varepsilon \leq 1$ is defined as

$$H_{\varepsilon}^{p}(|x|^2) := \inf\{H(|y|^2) : y \in \mathcal{M}, \|x-y\|_p \leq \varepsilon\},$$

where $H(|y|^2)$ is the von Neuman entropy of $|y|^2$, i.e., $H(|y|^2) = -\tau_{\mathcal{M}}(|y|^2 \log |y|^2)$. Inspired by the notion of k-Hadamard matrices, we introduce k-transforms $\mathcal{F}$ between a pair of finite von Neumann algebras $\mathcal{A}$, $\mathcal{B}$ with normal faithful traces τ, d, respectively. Precisely, $\mathcal{F}$ is a linear map from $\mathcal{A}$ into $\mathcal{B}$ satisfying the following conditions:

(i) $\|\mathcal{F}\|_{1\to\infty} \leq 1$,

(ii) $\|\mathcal{F}^*\mathcal{F}(x)\|_\infty \geq k\|x\|_\infty$ for $x \in \mathcal{A}$.

We call their combination $(\mathcal{A}, \mathcal{B}, \tau, d, \mathcal{F})$ a von Neumann k-bi-algebra (see Definition 2.3).

In [14], Jones introduced subfactor planar algebras as an axiomatization of the standard invariant in the flavor of topological quantum field theory. The quantum Hausdorff–Young inequality related to quantum Fourier transform on subfactor planar algebras was established in [11]. The proof essentially used topological structures in planar algebras, such as Wenzl's formula [24] and the local relation. Quantum Donoho–Stark uncertainty principle and quantum Hirchman–Beckner uncertainty principle on subfactor planar algebras were subsequently established using the quantum Hausdorff–Young inequality. We axiomatize these topological structures in planar algerbas in the framework of von Neumann k-bi-algebras. More specifically, the pair of von Neumann algebras $\mathcal{A}$ and $\mathcal{B}$ are infinite-dimensional generalizations of 2-box spaces $\mathcal{P}_{2,\pm}$ of a subfactor planar algebra $\mathcal{P}$, and τ and d are corresponding to Markov traces on $\mathcal{P}_{2,\pm}$. The value k is the Jones index μ of $\mathcal{P}$ when the k-transform $\mathcal{F}$ reduces to the string Fourier transform $\mathfrak{F}_s$ from $\mathcal{P}_{2,\pm}$ into $\mathcal{P}_{2,\mp}$, which is a 90°-rotation in picture. The first inequality condition $\|\mathcal{F}\|_{1\to\infty} \leq 1$ comes from the fact $\|\mu^{1/2}\mathfrak{F}_s\|_{1\to\infty} \leq 1$, while the second is a weaker condition of $(\mu^{1/2}\mathfrak{F}_s)^*(\mu^{1/2}\mathfrak{F}_s) = kI$, which comes from the fact that $\mathfrak{F}_s$ is a unitary operator.

In this axiomatized framework, the quantum smooth uncertainty principles on von Neumann k-bi-algebras in this paper imply those for subfactor planar algebras and for the corresponding higher relative commutants of inclusions of simple C^*-algebras studied in [1].

With the smooth supports, entropies, we prove the corresponding uncertainty principles for von Neumann k-bi-algebras as follows.

Main theorem 1 (See Theorems 3.9 and 3.13). *Let the quintuple $(\mathcal{A}, \mathcal{B}, d, \tau, \mathcal{F})$ be a von Neumann k-bi-algebra. Let $x \in \mathcal{A}$ be a nonzero operator. Then*

$$\mathcal{S}_\varepsilon^1(x)\mathcal{S}_\eta^1(\mathcal{F}(x)) \geq k(1-\varepsilon)(1-\eta) \quad \forall\, \varepsilon, \eta \in [0,1].$$

Furthermore, if $\mathcal{A}$, $\mathcal{B}$ are finite dimensional and $\mathcal{F}^*\mathcal{F} = kI$, then

$$\mathcal{S}_\varepsilon^2(x)\mathcal{S}_\eta^2(\mathcal{F}(x)) \geq k(1-\varepsilon-\eta)^2 \quad \forall\, \varepsilon, \eta \in [0,1],\ \varepsilon + \eta \leq 1.$$

For a von Neumann k-bi-algebra $(\mathcal{A}, \mathcal{B}, d, \tau, \mathcal{F})$ such that $\mathcal{A}$ and $\mathcal{B}$ are finite dimensional, we set

$$\alpha = \max\left\{d(e)^{-1} : e \text{ is a projection in } \mathcal{A}\right\}$$

and

$$\beta = \max\left\{\tau(e)^{-1} : e \text{ is a projection in } \mathcal{B}\right\}.$$

Main theorem 2 (See Theorems 3.22 and 3.28). *Let the quintuple $(\mathcal{A}, \mathcal{B}, d, \tau, \mathcal{F})$ be a von Neumann k-bi-algebra. Suppose $\mathcal{A}$ and $\mathcal{B}$ are finite dimensional and $\mathcal{F}^*\mathcal{F} = kI$. Then, for any nonzero $x \in \mathcal{A}$, $\varepsilon, \eta \in [0,1]$ and $p, q \in [1,\infty]$, we have*

$$\begin{aligned}
&\frac{H_\varepsilon^p(|x|^2)}{\|x\|_2^2} + \frac{H_\eta^q(|\mathcal{F}(x)|^2)}{\|\mathcal{F}(x)\|_2^2} \\
&\qquad \geq -4\log\|x\|_2 - \frac{C_1(x)}{\|x\|_2^2} d(I)^{1-\frac{1}{p}}\varepsilon - \frac{C_2(x)}{\|\mathcal{F}(x)\|_2^2}\tau(I)^{1-\frac{1}{q}}\eta,
\end{aligned}$$

where

$$\begin{aligned}
C_1(x) &= f(\|x\| + 1 + \alpha^{1/p}), \\
C_2(x) &= f(\|\mathcal{F}(x)\| + 1 + \beta^{1/q}), \\
f(t) &= 4t\log t + 2t.
\end{aligned}$$

On the one hand, our results generalize a number of uncertainty principles for quantum symmetries in [11, 18]. On the other hand, these results are slightly stronger than uncertainty principles for k-Hadamard matrices in [26].

The primary uncertainty principle for k-Hadamard matrices plays a key role in [26] and we call this type of uncertainty principle the Wigderson–Wigderson uncertainty principle. We prove the Wigderson–Wigderson uncertainty principle for von Neumann k-bi-algebras in Theorem 2.8 and for subfactors in Theorem 3.19. In [26], A. Wigderson and Y. Wigderson proposed a conjecture on the Wigderson–Wigderson uncertainty principle for the real line $\mathbb{R}$. We give a complete answer to the conjecture. Let $\mathcal{S}(\mathbb{R})$ be the space of Schwartz functions on the real line $\mathbb{R}$.

Main theorem 3 (See Theorem 4.3). *For any nonzero $f \in \mathcal{S}(\mathbb{R})$, $q \in (1, \infty]$, define $F_q(f)$ as*

$$F_q(f) = \frac{\|f\|_q \|\hat{f}\|_q}{\|f\|_2 \|\hat{f}\|_2} = \frac{\|f\|_q \|\hat{f}\|_q}{\|f\|_2^2}.$$

Then, the following statements hold.

(i) *When $1 < q < 2$, take $1/p + 1/q = 1$, then*

$$F_q(f) \geq [p^{1/p}/q^{1/q}]^{1/2} \quad \forall\, f \in \mathcal{S}(\mathbb{R}) \setminus \{0\}.$$

(ii) *When $q > 2$, the image of F_q is $\mathbb{R}_{>0}$.*

The paper is organized as follows. In Section 2, we introduce k-transforms and von Neumann k-bi-algebras with examples from quantum Fourier analysis. We prove some basic uncertainty principles for von Neumann k-bi-algebras. In Section 3, we prove uncertainty principles on von Neumann bi-algebras for smooth support and von Neumann entropy perturbed by p-norms. We prove Wigderson–Wigderson uncertainty principles on von Neumann bi-algebras, with a better constant in the case of subfactors. In Section 4, we provide a bound for Wigderson–Wigderson uncertainty principle on the real line $\mathbb{R}$ and this answers a conjecture proposed by A. Wigderson and Y. Wigderson in [26].

2. von Neumann bi-algebras and k-transforms

In this section, we recall some basic definitions and results about von Neumann algebras. We introduce von Neumann bi-algebras with interesting examples and we prove some basic properties and uncertainty principles.

A von Neumann algebra $\mathcal{M}$ is said to be *finite* if it has a faithful normal tracial positive linear functional $\tau_{\mathcal{M}}$ (not necessarily normalized), see, e.g., [15]. We will call this linear functional as trace in the rest of the paper. We denote $\|x\|_p = \tau_{\mathcal{M}}(|x|^p)^{\frac{1}{p}}$, for $p > 0$. When $1 \leq p < \infty$, $\|\cdot\|_p$ is called the p-norm. Moreover, $\|x\|_\infty = \|x\|$, the operator norm of x. It is clear that $\|x\|_p = \|x^*\|_p = \||x|\|_p$ for $p > 0$.

The following inequalities will be used frequently in the rest of the paper.

Proposition 2.1 (Hölder's inequalities, see, for example, [11, Proposition 4.3]). *For any $x, y, z \in \mathcal{M}$, we have the following:*

(i) $|\tau_{\mathcal{M}}(xy)| \leq \|x\|_p \|y\|_q$, *where* $1 \leq p, q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$;

(ii) $|\tau_{\mathcal{M}}(xyz)| \leq \|x\|_p \|y\|_q \|z\|_r$, *where* $1 \leq p, q, r \leq \infty$, $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$;

(iii) $\|xy\|_r \leq \|x\|_p \|y\|_q$, *where* $0 < p, q, r \leq \infty$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

Notation 2.2. Suppose $\mathcal{A}$ and $\mathcal{B}$ are two finite von Neumann algebras with traces d and τ, respectively. Let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ be a linear map. For any $0 < p, q \leq \infty$, define

$$\|\mathcal{F}\|_{q \to p} := \sup\{\|\mathcal{F}(x)\|_p : x \in \mathcal{A}, \|x\|_q = 1\}.$$

Definition 2.3. Suppose $\mathcal{A}$ and $\mathcal{B}$ are two finite von Neumann algebras with traces d and τ, respectively. For $k > 0$, a *k-transform* $\mathcal{F}$ from $\mathcal{A}$ into $\mathcal{B}$ is a linear map such that $\|\mathcal{F}\|_{1\to\infty} \leq 1$ and $\|\mathcal{F}^*\mathcal{F}(x)\|_\infty \geq k\|x\|_\infty$ for any $x \in \mathcal{A}$. We call the quintuple $(\mathcal{A}, \mathcal{B}, d, \tau, \mathcal{F})$ a *von Neumann k-bi-algebra*. Here, $\mathcal{F} : L^2(\mathcal{A}) \to L^2(\mathcal{B})$ is assumed to be bounded and $\mathcal{F}^*$ is the adjoint operator.

Example 2.4. The definition of the k-transform is inspired by the definition of the k-Hadamard matrix of A. Wigderson and Y. Wigderson (see [26, Definition 2.2]). In particular, a k-Hadamard matrix $\mathcal{F}$ can be extended to a von Neumann k-bi-algebra $(\mathcal{A}, \mathcal{B}, d, \tau, \mathcal{F})$, such that $\mathcal{A}$ and $\mathcal{B}$ are finite-dimensional abelian von Neumann algebras, d and τ are counting measures.

Example 2.5. Let the quintuple $(\mathcal{A}, \mathcal{B}, d, \tau, \mathcal{F})$ be a fusion bialgebra (see Definition 2.12 in [18]), where $\mathcal{A}$ and $\mathcal{B}$ are finite-dimensional von Neumann algebras with traces d and τ, respectively, and $\mathcal{A}$ is abelian, and $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ is unitary with respect to the L^2 inner products. By the quantum Hausdorff–Young inequality $\|\mathcal{F}\|_{1\to\infty} = 1$, (see [18, Theorem 4.5]), we have that $(\mathcal{A}, \mathcal{B}, d, \tau, \mathcal{F})$ is a von Neumann 1-bi-algebra.

Example 2.6. Suppose $\mathcal{P}_\bullet$ is an irreducible subfactor planar algebra with finite Jones index (See Definition on page 4 in [13]) μ, $\mu > 0$. Let $\mathrm{Tr}_{n,\pm}$ be the unnormalized Markov trace of $\mathcal{P}_{n,\pm}$, for $n \in \mathbb{N}$, and $\mathfrak{F}_s : \mathcal{P}_{n,+} \to \mathcal{P}_{n,-}$ be the string Fourier transform, which is unitary with respect to the L^2 inner products. Then, by the quantum Hausdorff–Young inequality, (see [11, Theorems 4.8 and 7.3]), we have that for any $n \in \mathbb{N}$, $2 \leq p \leq \infty$ and $1/p + 1/q = 1$,

$$\|\mathfrak{F}_s\|_{q\to p} = \mu^{\frac{1}{p}-\frac{1}{2}}.$$

Therefore, $\|\mu^{1/2}\mathfrak{F}_s\|_{1\to\infty} = 1$ and the quintuple $(\mathcal{P}_{n,+}, \mathcal{P}_{n,-}, \mathrm{Tr}_{n,+}, \mathrm{Tr}_{n,-}, \mu^{1/2}\mathfrak{F}_s)$ is a von Neumann μ-bi-algebra.

Remark 2.7. The quantum Hausdorff–Young inequality, [11, Theorem 7.3], also applies to reducible subfactor planar algebras, and in that case $\mu^{1/2}$ is replaced by a certain constant δ_0 (see the last sentence on page 301 in [11] for the definition.) Then, $(\mathcal{P}_{n,+}, \mathcal{P}_{n,-}, \mathrm{Tr}_{n,+}, \mathrm{Tr}_{n,-}, \delta_0\mathfrak{F}_s)$ is a von Neumann δ_0^2-bi-algebra.

In [26], Wigderson and Wigderson proved what they call the primary uncertainty principles (see [26, Theorem 2.3]) for any k-Hadamard matrix A,

$$\|v\|_1\|Av\|_1 \geq k\|v\|_\infty\|Av\|_\infty, \quad v \in \mathbb{C}^n, \tag{2.1}$$

which is the fundamental result of that paper. We call Inequality (2.1) as Wigderson–Wigderson uncertainty principle. In this paper, we establish the quantum version of Wigderson–Wigderson uncertainty principle for von Neumann k-bi-algebras. When a von Neumann k-bi-algebra is obtained from Example 2.4, then, our theorem implies Theorem 2.3 in [26].

Theorem 2.8 (The quantum Wigderson–Wigderson uncertainty principle). *Let the quintuple $(\mathcal{A}, \mathcal{B}, d, \tau, \mathcal{F})$ be a von Neumann k-bi-algebra. For any $x \in \mathcal{A}$, we have*

$$\|x\|_1 \|\mathcal{F}(x)\|_1 \geq k\|x\|_\infty \|\mathcal{F}(x)\|_\infty.$$

Proof. When $1 \leq p, q \leq \infty$ and $1/p + 1/q = 1$, we have that $\|\mathcal{F}^*\|_{p\to q} = \|\mathcal{F}\|_{p\to q}$, because

$$\begin{aligned}
\|\mathcal{F}\|_{p\to q} &= \sup\big\{\|\mathcal{F}(x)\|_q : x \in \mathcal{A}, \|x\|_p = 1\big\} \\
&= \sup\big\{|\tau(\mathcal{F}(x)y^*)| : x \in \mathcal{A}, y \in \mathcal{B}, \|x\|_p = 1, \|y\|_p = 1\big\} \\
&= \sup\big\{|d(x(\mathcal{F}^*(y))^*)| : x \in \mathcal{A}, y \in \mathcal{B}, \|x\|_p = 1, \|y\|_p = 1\big\} \\
&= \sup\big\{\|\mathcal{F}^*(y)\|_q : y \in \mathcal{B}, \|y\|_p = 1\big\} \\
&= \|\mathcal{F}^*\|_{p\to q}.
\end{aligned}$$

This implies that $\|\mathcal{F}^*\|_{1\to\infty} = \|\mathcal{F}\|_{1\to\infty} \leq 1$. Then, for any $x \in \mathcal{A}$, we have

$$\|\mathcal{F}(x)\|_\infty \leq \|x\|_1, \quad k\|x\|_\infty \leq \|\mathcal{F}^*\mathcal{F}(x)\|_\infty \leq \|\mathcal{F}(x)\|_1.$$

Multiplying the above two inequalities, we obtain

$$\|x\|_1 \|\mathcal{F}(x)\|_1 \geq k\|x\|_\infty \|\mathcal{F}(x)\|_\infty.$$

This completes the proof of the theorem. ∎

Using the primary uncertainty principle, A. Wigderson and Y. Wigderson further prove the Donoho–Stark uncertainty principle for arbitrary k-Hadamard matrices (see [26, Theorem 3.2]). In this paper, we prove the Donoho–Stark uncertainty principle for von Neumann k-bi-algebras using the quantum Wigderson–Wigderson uncertainty principle. Firstly, let us recall the notion of the support in a finite von Neumann algebra.

Definition 2.9. Let $\mathcal{M}$ be a finite von Neumann algebra with a trace $\tau_{\mathcal{M}}$. For any $x \in \mathcal{M}$, let $\mathcal{R}(x)$ be the range projection of x. The *support* $\mathcal{S}(x)$ of x is defined as $\tau_{\mathcal{M}}(\mathcal{R}(x))$. When $\mathcal{M}$ is a matrix algebra, $\mathcal{S}(x)$ is the rank of x.

The support has been used in the quantum Donoho–Stark uncertainty principles on quantum symmetries such as subfactors and fusion rings, see [11, Theorem 5.2] and [18, Theorem 4.8], respectively. We generalize the Donoho–Stark uncertainty principles to von Neumann k-bi-algebras.

Theorem 2.10 (Quantum Donoho–Stark uncertainty principle). *Let the quintuple $(\mathcal{A}, \mathcal{B}, d, \tau, \mathcal{F})$ be a von Neuman k-bi-algebra. Then, for any nonzero operator $x \in \mathcal{A}$, we have*

$$\mathcal{S}(x)\mathcal{S}(\mathcal{F}(x)) \geq k.$$

Proof. We already have, from Theorem 2.8, that for any nonzero $x \in \mathcal{A}$,

$$\|x\|_1 \|\mathcal{F}(x)\|_1 \geq k\|x\|_\infty \|\mathcal{F}(x)\|_\infty.$$

Thus, all we need is to bound the 1-norm by the support of x, which can be implemented through Hölder's inequality, for any $x \in \mathcal{A}$,

$$\|x\|_1 = \|\mathcal{R}(x)x\|_1 \leq \|x\|_\infty \|\mathcal{R}(x)\|_1 = \|x\|_\infty \mathcal{S}(x).$$

Applying this bound to both x and $\mathcal{F}(x)$, we obtain the result. ∎

Remark 2.11. Our theorem is a generalization of the Donoho–Stark uncertainty principle in [5] and some variations.

(i) In Example 2.4 and Theorem 2.10 implies Theorem 3.2 in [26].

(ii) In Example 2.5 and Theorem 2.10 implies Theorem 4.8 in [18].

(iii) In Example 2.6 and Theorem 2.10 implies Theorem 5.2 in [11].

The Meshulam's uncertainty principle [21] for a finite non-abelian group G is a special case of Examples 2.5 and 2.6. More precisely, take $\mathcal{A} := L^\infty(G)$, functions on G, with the discrete measure d. Let π be the left regular representation of G, and $\mathcal{B} := \mathcal{L}G$, the group algebra of G acting on $L^2(G)$ with the (unnormalized) trace τ. The Fourier transform $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ is defined as

$$\mathcal{F}(x) := \sum_g x(g)\pi(g) \quad \forall x \in \mathcal{A}.$$

Then, $\mathcal{F}^*\mathcal{F}(x) = |G|x$ and $\|\mathcal{F}\|_{1\to\infty} = 1$, because $\pi(g)$ is a unitary and

$$\|\mathcal{F}(x)\| \leq \sum_g |x(g)| = \|x\|_1.$$

Therefore, $\mathcal{F}$ is a $|G|$-transform and we obtain the Meshulam's uncertainty principle,

$$\mathcal{S}(x)\mathcal{S}(\mathcal{F}(x)) \geq |G| \quad \forall x \neq 0.$$

In this case, $\mathcal{S}(\mathcal{F}(x))$ is the rank of $\mathcal{F}(x)$.

Furthermore, in Example 2.6, $|G|$ is replaced by the Jones index δ^2 of the subfactor which takes values in

$$\left\{4\cos^2\frac{\pi}{n}, n = 3, 4, 5, \ldots\right\} \cup [4, \infty],$$

the remarkable Jones-index theorem in [14]. The quantum Donoho–Stark uncertainty principle

$$\mathcal{S}(x)\mathcal{S}(\mathcal{F}(x)) \geq \delta^2$$

is established in [11, Main Theorem 1]. The inequality is sharp and the extremizers are characterized as bishifts of biprojections in [11, Main Theorem 2].

An entropic uncertainty principle was proved in [11, Main Theorems 1 and 2] as well, which could be regarded as a quantum analog of Hirschman–Beckner uncertainty principle [2, 8]. We will study these uncertainty principles up to small perturbations in Section 3.

3. Quantum smooth uncertainty principles

In this section, we prove a series of smooth uncertainty principles for von Neumann bi-algebras. We firstly prove the quantum smooth support uncertainty principles in Section 3.1. Then, we proceed to prove quantum Wigderson–Wigderson uncertainty principles for general p-norms, $1 \leq p \leq \infty$, and give an example concerning the quantum Fourier transform on subfactor planar algebras in Section 3.2. Finally, we also prove quantum smooth Hirschman–Becker uncertainty principles in Section 3.3.

3.1. Quantum smooth support uncertainty principles

We firstly introduce a new smooth support which is slightly different from the classical smooth support.

Definition 3.1. Let $\mathcal{M}$ be a finite von Neumann algebra with a trace $\tau_{\mathcal{M}}$. Let $\varepsilon \in [0, 1]$ and $p \in [1, \infty]$. For any element $x \in \mathcal{M}$, we define the (p, ε) smooth support to be

$$\mathcal{S}_\varepsilon^p(x) = \inf\left\{\tau_{\mathcal{M}}(H\mathcal{R}(x)) : H \in \mathcal{M}, 0 \leq H \leq I, \|(I - H)x\|_p \leq \varepsilon\|x\|_p\right\},$$

where $\mathcal{R}(x)$ is the range projection of x.

Remark 3.2. Since the set

$$\mathcal{S}(\varepsilon, p, x) := \left\{H \in \mathcal{M} : 0 \leq H \leq I, \|(I - H)x\|_p \leq \varepsilon\|x\|_p\right\}$$

is compact in the weak operator topology and the trace is normal, there exists an $H_0 \in \mathcal{S}$ such that $\mathcal{S}_\varepsilon^p(x) = \tau_{\mathcal{M}}(H_0\mathcal{R}(x))$.

Remark 3.3. Take $\varepsilon = 0$, then $(I - H)x = 0$ and this implies $H\mathcal{R}(x) = \mathcal{R}(x)$. In this case, $\mathcal{S}_0^p(x) = \mathcal{S}(x)$ for every $p \in [1, \infty]$.

Besides Definition 3.1, there are at least three other natural candidate notions of smooth support.

Definition 3.4. Let $\mathcal{M}$ be a finite von Neumann algebra with a trace $\tau_{\mathcal{M}}$. Let $\varepsilon \in [0,1]$ and $p \in [1,\infty]$. For any element $x \in \mathcal{M}$, define

$$\begin{aligned} f_1(\varepsilon,p,x) &:= \inf\{\tau_{\mathcal{M}}(\mathcal{R}(y)) : y \in \mathcal{M}, \|x-y\|_p \leq \varepsilon\|x\|_p\}, \\ f_2(\varepsilon,p,x) &:= \inf\{\tau_{\mathcal{M}}(\mathcal{R}(Hx)) : H \in \mathcal{M}, 0 \leq H \leq I, \|(I-H)x\|_p \leq \varepsilon\|x\|_p\}, \\ f_3(\varepsilon,p,x) &:= \inf\{\tau_{\mathcal{M}}(\mathcal{R}(Qx)) : Q \in \mathcal{M}, Q=Q^*=Q^2, \|(I-Q)x\|_p \leq \varepsilon\|x\|_p\}. \end{aligned}$$

Proposition 3.5. *For any $x \in \mathcal{M}$, we have*

$$\mathcal{S}_\varepsilon^p(x) \leq f_1(\varepsilon,p,x) = f_2(\varepsilon,p,x) = f_3(\varepsilon,p,x).$$

Proof. It is clear that $f_1(\varepsilon,p,x) \leq f_2(\varepsilon,p,x) \leq f_3(\varepsilon,p,x)$.

For any $y \in \mathcal{M}$, we claim that

$$\tau_{\mathcal{M}}(\mathcal{R}(\mathcal{R}(y)x)) \leq \tau_{\mathcal{M}}(\mathcal{R}(y)), \quad \|(I-\mathcal{R}(y))x\|_p \leq \|x-y\|_p.$$

If the claim holds, then $f_3(\varepsilon,p,x) \leq f_1(\varepsilon,p,x)$. Since $\mathcal{R}(\mathcal{R}(y)x) \leq \mathcal{R}(y)$, the first inequality holds.

Next, we prove the second inequality in the claim. It is enough to prove that $|x - \mathcal{R}(y)x| \leq |x-y|$. Since $\sqrt{\cdot}$ is an operator-monotone function, it reduces to prove $(x-\mathcal{R}(y)x)^*(x-\mathcal{R}(y)x) \leq (x-y)^*(x-y)$. For any normal state ρ on $\mathcal{M}$, by the Cauchy–Schwarz inequality, we have

$$\begin{aligned} 2|\rho(y^*x)| &= 2|\langle x,y\rangle_\rho| \\ &= 2|\langle \mathcal{R}(y)x, y\rangle_\rho| \\ &\leq 2\langle \mathcal{R}(y)x, \mathcal{R}(y)x\rangle_\rho^{\frac{1}{2}} \langle y,y\rangle_\rho^{\frac{1}{2}} \\ &\leq \rho(x^*\mathcal{R}(y)x) + \rho(y^*y). \end{aligned}$$

Therefore,

$$\rho(x^*y) + \rho(y^*x) \leq \rho(x^*\mathcal{R}(y)x) + \rho(y^*y).$$

Rearranging the above inequality, we obtain

$$\rho((x-\mathcal{R}(y)x)^*(x-\mathcal{R}(y)x)) \leq \rho((x-y)^*(x-y)).$$

Thus,

$$(x-\mathcal{R}(y)x)^*(x-\mathcal{R}(y)x) \leq (x-y)^*(x-y).$$

The claim holds and we have $f_1(\varepsilon,p,x) = f_2(\varepsilon,p,x) = f_3(\varepsilon,p,x)$.

For any $H \in \mathcal{M}$, $0 \leq H \leq I$, we have

$$\tau_{\mathcal{M}}(\mathcal{R}(x)H) \leq \tau_{\mathcal{M}}(|\mathcal{R}(x)H|) \leq \tau_{\mathcal{M}}(\|\mathcal{R}(x)H\|\mathcal{R}(Hx)) \leq \tau_{\mathcal{M}}(\mathcal{R}(Hx)).$$

The first inequality is true by Hölder's inequality. The second one uses the fact that $|y^*| \leq \|y\|\mathcal{R}(y)$, $y \in \mathcal{M}$. The last inequality is due to $\|\mathcal{R}(x)H\| \leq 1$. So, we have

$$\mathcal{S}_\varepsilon^p(x) \leq f_2(\varepsilon, p, x).$$

In summary, the statement holds. ■

In [26], A. Wigderson and Y. Wigderson introduced the following smooth support for the finite-dimensional and abelian case.

Definition 3.6 (See [26, Definition 3.15]). Let $\mathcal{M} = \mathbb{C}^n$, $n \in \mathbb{N}^*$, and $\tau_\mathcal{M}$ be the counting measure. Let $\varepsilon \in [0, 1]$ and $p \in [1, \infty]$. For an operator $x \in \mathcal{M}$, the (p, ε) support-size of x is defined to be

$$|\mathrm{supp}_\varepsilon^p(x)| = \min\left\{\tau_\mathcal{M}(Q) : Q \in \mathcal{M}, Q = Q^* = Q^2, \|(I - Q)x\|_p \leq \varepsilon\|x\|_p\right\}.$$

Remark 3.7. When $\mathcal{M}$ is finite dimensional and abelian and $\tau_\mathcal{M}$ is the counting measure, then $f_3(\varepsilon, p, x)$ is equal to $|\mathrm{supp}_\varepsilon^p(x)|$. In this case, $\mathcal{S}_\varepsilon^p(x) \leq |\mathrm{supp}_\varepsilon^p(x)|$.

Lemma 3.8. *For any $x \in \mathcal{M}$, we have $\mathcal{S}_\varepsilon^p(x)$ is continuous with respect to ε.*

Proof. When $0 < c < 1$, take an $H \in \mathcal{M}$ such that

$$\mathcal{S}_\varepsilon^p(x) = \tau_\mathcal{M}(H\mathcal{R}(x)), \quad \|(I - H)x\|_p \leq \varepsilon\|x\|_p, \quad 0 \leq H \leq I.$$

Let $H' = I - c(I - H)$, then $0 \leq H' \leq I$. Moreover, we have

$$\tau_\mathcal{M}(H'\mathcal{R}(x)) = (1 - c)\tau_\mathcal{M}(\mathcal{R}(x)) + c\mathcal{S}_\varepsilon^p(x), \quad \|(I - H')x\|_p \leq c\varepsilon\|x\|_p.$$

Therefore,

$$\mathcal{S}_\varepsilon^p(x) \leq \mathcal{S}_{c\varepsilon}^p(x) \leq (1 - c)\tau_\mathcal{M}(\mathcal{R}(x)) + c\mathcal{S}_\varepsilon^p(x). \tag{3.1}$$

So,

$$\lim_{c \to 1^-} \mathcal{S}_{c\varepsilon}^p(x) = \mathcal{S}_\varepsilon^p(x).$$

When $c > 1$, replacing c by c^{-1} and ε by $c\varepsilon$ in Inequality (3.1), we have

$$\mathcal{S}_{c\varepsilon}^p(x) \leq \mathcal{S}_\varepsilon^p(x) \leq \left(1 - \frac{1}{c}\right)\tau_\mathcal{M}(\mathcal{R}(x)) + \frac{1}{c}\mathcal{S}_{c\varepsilon}^p(x).$$

So,

$$\lim_{c \to 1^+} \mathcal{S}_{c\varepsilon}^p(x) = \mathcal{S}_\varepsilon^p(x).$$

From the above discussions, $\mathcal{S}_\varepsilon^p(x)$ is continuous with respect to ε. ■

We have the following quantum L^1 smooth support uncertainty principle.

Theorem 3.9 (The quantum L^1 smooth support uncertainty principle). *Let the quintuple $(\mathcal{A}, \mathcal{B}, d, \tau, \mathcal{F})$ be a von Neumann k-bi-algebra and $x \in \mathcal{A}$ be a nonzero operator. For any $\varepsilon, \eta \in [0,1]$, we have*

$$\mathcal{S}_\varepsilon^1(x)\mathcal{S}_\eta^1(\mathcal{F}(x)) \geq k(1-\varepsilon)(1-\eta).$$

Proof. Take a positive operator H in $\mathcal{A}$ such that

$$\mathcal{S}_\varepsilon^1(x) = d(\mathcal{R}(x)H), \quad \|(I-H)x\|_1 \leq \varepsilon\|x\|_1, \quad 0 \leq H \leq I.$$

By Hölder's inequality, we have

$$d(|x^*|(I-H)) \leq d(|x^*(I-H)|) = \|(I-H)x\|_1 \leq \varepsilon\|x\|_1.$$

Thus,

$$\begin{aligned}\mathcal{S}_\varepsilon^1(x) &= d(\mathcal{R}(x)H) \\ &= \frac{1}{\|x\|_\infty} d(\|x\|_\infty \mathcal{R}(x)H) \\ &\geq \frac{1}{\|x\|_\infty} d(|x^*|H) = \frac{1}{\|x\|_\infty} d(|x^*|) - \frac{1}{\|x\|_\infty} d(|x^*|(I-H)) \\ &\geq \frac{\|x\|_1}{\|x\|_\infty}(1-\varepsilon).\end{aligned}$$

Repeating the above process for $\mathcal{F}(x)$, we obtain

$$\mathcal{S}_\eta^1(\mathcal{F}(x)) \geq \frac{\|\mathcal{F}(x)\|_1}{\|\mathcal{F}(x)\|_\infty}(1-\eta).$$

Multiplying these two inequalities, we have

$$\mathcal{S}_\varepsilon^1(x)\mathcal{S}_\eta^1(\mathcal{F}(x)) \geq \frac{\|x\|_1}{\|x\|_\infty} \cdot \frac{\|\mathcal{F}(x)\|_1}{\|\mathcal{F}(x)\|_\infty}(1-\varepsilon)(1-\eta) \geq k(1-\varepsilon)(1-\eta).$$

The second inequality uses Theorem 2.8, the quantum Wigderson–Wigderson uncertainty principle. ∎

Remark 3.10. We can obtain Theorem 2.10, the quantum Donoho–Stark uncertainty principle, from Theorem 3.9 by assuming $\varepsilon = \eta = 0$.

Applying Theorem 3.9 to the quantum Fourier transform on subfactor planar algebras, we obtain the following corollary.

Corollary 3.11. *Suppose $\mathcal{P}_\bullet$ is an irreducible subfactor planar algebra with finite Jones index μ. Let $\mathfrak{F}_s$ be the Fourier transform from $\mathcal{P}_{n,\pm}$ onto $\mathcal{P}_{n,\mp}$. Then, for any nonzero n-box $x \in \mathcal{P}_{n,\pm}$, we have*

$$\mathcal{S}_\varepsilon^1(x)\mathcal{S}_\eta^1(\mathfrak{F}_s(x)) \geq \mu(1-\varepsilon)(1-\eta).$$

When $p = 2$ in Definition 3.1, we are able to choose a positive contraction H in the abelian *-subalgebra generated by $|x^*|$ such that the $(2, \varepsilon)$ support-size is exactly the trace of H. More precisely, we have the following proposition.

Proposition 3.12. *Suppose $\mathcal{M}$ is a finite von Neumann algebra with a trace $\tau_{\mathcal{M}}$. Let $x \in \mathcal{M}$, and let $\mathcal{N}$ be the abelian von Neumann subalgebra generated by $|x^*|$ in $\mathcal{M}$. For any $\varepsilon \in [0, 1]$, we have*

$$\mathcal{S}_\varepsilon^2(x) = \min\left\{\tau_{\mathcal{M}}(H) : H \in \mathcal{N}, 0 \le H \le \mathcal{R}(x), \|(I - H)x\|_2 \le \varepsilon\|x\|_2\right\}.$$

Proof. Let Φ be the trace-preserving conditional expectation from $\mathcal{M}$ onto $\mathcal{N}$. For any $H \in \mathcal{M}$, $0 \le H \le I$. Take $H' = \Phi(H)\mathcal{R}(x)$, then

$$\tau_{\mathcal{M}}(H') = \tau_{\mathcal{M}}(\Phi(H)\mathcal{R}(x)) = \tau_{\mathcal{M}}(\mathcal{R}(x)H),$$

and $H' \in \mathcal{N}$ and $0 \le H' \le \mathcal{R}(x)$.

Note that any pure state ρ on $\mathcal{N}$ is multiplicative, so $\rho(|\Phi(y)|^2) = |\rho \circ \Phi(y)|^2$, for any $y \in \mathcal{M}$. Moreover, $\rho \circ \Phi$ is a state on $\mathcal{M}$, by the Cauchy–Schwarz inequality, $|\rho \circ \Phi(y)|^2 \le \rho \circ \Phi(|y|^2)$. So, $\rho(|\Phi(y)|^2) \le \rho(\Phi(|y|^2))$, and therefore $|\Phi(y)|^2 \le \Phi(|y|^2)$.

Take $y = I - H$, then

$$\begin{aligned}\|(I - H')x\|_2^2 &= \|\Phi(I - H)\mathcal{R}(x)x\|_2^2 \\ &= \tau_{\mathcal{M}}(|\Phi(I - H)|^2|x^*|^2) \\ &\le \tau_{\mathcal{M}}(\Phi(|I - H|^2)|x^*|^2) \\ &= \tau_{\mathcal{M}}(|I - H|^2|x^*|^2) \\ &= \|(I - H)x\|_2^2.\end{aligned}$$

Therefore, the statement holds. ■

We have the following quantum L^2 smooth support uncertainty principle, in the special case of finite-dimensional algebras $\mathcal{A}$, $\mathcal{B}$ and unitary k-transform.

Theorem 3.13 (The quantum L^2 smooth support uncertainty principle). *Let the quintuple $(\mathcal{A}, \mathcal{B}, d, \tau, \mathcal{F})$ be a von Neumann k-bi-algebra. Suppose $\mathcal{A}$ and $\mathcal{B}$ are finite dimensional and $\mathcal{F}^*\mathcal{F} = kI$. For any nonzero operator $x \in \mathcal{A}$, we have*

$$\mathcal{S}_\varepsilon^2(x)\mathcal{S}_\eta^2(\mathcal{F}(x)) \ge k(1 - \varepsilon - \eta)^2 \quad \forall\, \varepsilon, \eta \in [0, 1],\ \varepsilon + \eta \le 1.$$

Proof. Take $W = \mathcal{F}/\sqrt{k}$, then $W^*W = I$. Since the definition of $\mathcal{S}_\eta^2$ is invariant under rescaling, we have that $\mathcal{S}_\eta^2(W(x)) = \mathcal{S}_\eta^2(\mathcal{F}(x))$.

Let $x = |x^*|U$ and $y = W(x) = |y^*|V$ be the polar decompositions, where U and V are the polar parts in $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $\mathcal{A}_0$ be the abelian von Neumann subalgebra of $\mathcal{A}$ generated by $|x^*|$ and $\mathcal{B}_0$ be the abelian von Neumann subalgebra of $\mathcal{B}$ generated by $|y^*|$. Let Φ be the trace-preserving conditional expectation from $\mathcal{B}$ onto $\mathcal{B}_0$ and $M = \Phi \mathcal{R}_{V^*} W \mathcal{R}_U$, where $\mathcal{R}_{V^*}$ and $\mathcal{R}_U$ are the right multiplications by V^* and U, respectively. Then, M is a linear operator from $\mathcal{A}_0$ into $\mathcal{B}_0$ such that $M|x^*| = |y^*|$. Let $\{e_i\}_{i=1}^n$ and $\{f_j\}_{j=1}^m$ be mutually orthogonal minimal projections in $\mathcal{A}_0$ and $\mathcal{B}_0$ such that $\sum_{i=1}^n e_i = I_{\mathcal{A}}$ and $\sum_{j=1}^m f_j = I_{\mathcal{B}}$. The linear operator M is a $m \times n$ matrix $(a_{ij})_{1\leq i\leq m, 1\leq j\leq n}$ with $|a_{ij}| \leq d(e_j)/\sqrt{k}$ by $\|W\|_{1\to\infty} \leq 1/\sqrt{k}$.

By Proposition 3.12, we can find two positive operators H in $\mathcal{A}_0$ and K in $\mathcal{B}_0$ such that

$$\begin{aligned} 0 \leq H \leq \mathcal{R}(x), &\quad 0 \leq K \leq \mathcal{R}(y), \\ \|(I-H)x\|_2 \leq \varepsilon \|x\|_2, &\quad \|(I-K)y\|_2 \leq \eta \|y\|_2, \\ d(H) = \mathcal{S}_\varepsilon^2(x), &\quad \tau(K) = \mathcal{S}_\eta^2(y). \end{aligned}$$

By direct computations, we have

$$H = \sum_{i=1}^n \frac{d(e_i H)}{d(e_i)} e_i, \quad K = \sum_{j=1}^m \frac{\tau(f_j K)}{\tau(f_j)} f_j.$$

Let $\widetilde{M} = KMH$, then $\widetilde{M}$ is a linear operator from $\mathcal{A}_0$ into $\mathcal{B}_0$. For any $v \in \mathcal{A}_0$, $v = \sum_{i=1}^n v_j e_j$, $v_j \in \mathbb{C}$, we have

$$\begin{aligned} \|\widetilde{M} v\|_2^2 &= \sum_{i=1}^m \tau(f_i) \left| \frac{\tau(f_i K)}{\tau(f_i)} \sum_{j=1}^n a_{ij} \frac{d(e_j H)}{d(e_j)} v_j \right|^2 \\ &\leq \sum_{i=1}^m \frac{\tau(f_i K)^2}{\tau(f_i)} \sum_{j=1}^n |a_{ij}|^2 \frac{d(e_j H)^2}{d(e_j)^3} \sum_{j=1}^n d(e_j)|v_j|^2 \\ &= \sum_{i=1}^m \frac{\tau(f_i K)^2}{\tau(f_i)} \sum_{j=1}^n |a_{ij}|^2 \frac{d(e_j H)^2}{d(e_j)^3} \|v\|_2^2 \\ &\leq \frac{1}{k} \sum_{i=1}^m \frac{\tau(f_i K)^2}{\tau(f_i)} \sum_{j=1}^n \frac{d(e_j H)^2}{d(e_j)} \|v\|_2^2 \\ &\leq \frac{d(H)\tau(K)}{k} \|v\|_2^2. \end{aligned}$$

The first inequality is true by the Cauchy–Schwarz inequality and the second one uses the fact that $|a_{ij}| \leq d(e_j)/\sqrt{k}$ and the third one is due to $\tau(f_i K) \leq \tau(f_i)$ and $d(e_j H) \leq d(e_j)$ because K, H are contractions. This implies

$$\|\widetilde{M}\|_{2\to 2} \leq \sqrt{d(H)\tau(K)} \Big/ \sqrt{k} = \sqrt{\mathcal{S}_\varepsilon^2(x)\mathcal{S}_\eta^2(y)} \Big/ \sqrt{k}. \tag{3.2}$$

For the lower bound of $\widetilde{M}$, we firstly observe that

$$\begin{aligned}\|M(I-H)|x^*|\|_2 &= \|\Phi\mathcal{R}_{V^*}W\mathcal{R}_U(I-H)|x^*|\|_2\\ &\leq \|\mathcal{R}_{V^*}W\mathcal{R}_U(I-H)|x^*|\|_2\\ &= \|(I-H)|x^*|\|_2.\end{aligned}$$

Since K is a contraction, so

$$\|KM(I-H)|x^*|\|_2 \leq \|M(I-H)|x^*|\|_2 \leq \|(I-H)|x^*|\|_2.$$

Therefore, we have

$$\begin{aligned}\|M|x^*| - \widetilde{M}|x^*|\|_2 &= \|M|x^*| - KMH|x^*|\|_2\\ &= \|(I-K)M|x^*| + KM(I-H)|x^*|\|_2\\ &\leq \|(I-K)|y^*|\|_2 + \|(I-H)|x^*|\|_2\\ &\leq (\varepsilon+\eta)\||x^*|\|_2.\end{aligned}$$

This implies

$$\begin{aligned}\|\widetilde{M}|x^*|\|_2 &\geq \|M|x^*|\|_2 - (\varepsilon+\eta)\||x^*|\|_2\\ &= \||y^*|\|_2 - (\varepsilon+\eta)\||x^*|\|_2\\ &= (1-\varepsilon-\eta)\||x^*|\|_2. \end{aligned}\tag{3.3}$$

Finally, combining equations (3.2) and (3.3) we see that

$$\mathcal{S}_\varepsilon^2(x)\mathcal{S}_\eta^2(\mathcal{F}(x)) \geq k(1-\varepsilon-\eta)^2.$$

This completes the proof of the theorem. ■

Remark 3.14. When $\mathcal{F}$ is a k-Hadamard matrix, A. Wigderson and Y. Wigderson proved the following results (see [26, Theorems 3.17 and 3.20]):

(i) for any $x \in \mathcal{M}$,

$$|\mathrm{supp}_\varepsilon^1(x)||\mathrm{supp}_\eta^1(\mathcal{F}(x))| \geq k(1-\varepsilon)(1-\eta) \quad \forall\, \varepsilon, \eta \in [0,1];$$

(ii) if $\mathcal{F}^*\mathcal{F} = kI$, then for any $x \in \mathcal{M}$,

$$|\mathrm{supp}_\varepsilon^2(x)||\mathrm{supp}_\eta^2(\mathcal{F}(x))| \geq k(1-\varepsilon-\eta)^2 \quad \forall\, \varepsilon, \eta \in [0,1],\ \varepsilon+\eta \leq 1.$$

By Remark 3.7, we have

$$\begin{aligned}|\mathrm{supp}_\varepsilon^1(x)||\mathrm{supp}_\eta^1(\mathcal{F}(x))| &\geq \mathcal{S}_\varepsilon^1(x)\mathcal{S}_\eta^1(\mathcal{F}(x)) \geq k(1-\varepsilon)(1-\eta),\\ |\mathrm{supp}_\varepsilon^2(x)||\mathrm{supp}_\eta^2(\mathcal{F}(x))| &\geq \mathcal{S}_\varepsilon^2(x)\mathcal{S}_\eta^2(\mathcal{F}(x)) \geq k(1-\varepsilon-\eta)^2.\end{aligned}$$

So, Theorems 3.9 and 3.13 imply Theorems 3.17 and 3.20 in [26].

When $\mathcal{F}$ is a k-Hadamard matrix, Theorems 3.9 and 3.13 are strictly stronger than Theorems 3.17 and 3.20 in [26]. We construct the following example.

Example 3.15. Let $\mathcal{A} = \mathcal{B} = \mathbb{C} \oplus \mathbb{C}$ and $d(f) = \tau(f) = f(0) + f(1)$, $f \in \mathbb{C}^2$. Take $x = (1, 1) \in \mathbb{C}^2$ and $\varepsilon = \eta = 1/3$. Then, $|\mathrm{supp}_\varepsilon^1(x)| = |\mathrm{supp}_\varepsilon^2(x)| = 2$ while $\mathcal{S}_\varepsilon^1(x) = \mathcal{S}_\varepsilon^2(x) = 4/3$. Let $\mathcal{F} = I$ be the 1-transform, we have

$$4 = |\mathrm{supp}_\varepsilon^1(x)||\mathrm{supp}_\eta^1(\mathcal{F}(x))| > \mathcal{S}_\varepsilon^1(x)\mathcal{S}_\eta^1(\mathcal{F}(x)) = \frac{16}{9},$$
$$4 = |\mathrm{supp}_\varepsilon^2(x)||\mathrm{supp}_\eta^2(\mathcal{F}(x))| > \mathcal{S}_\varepsilon^2(x)\mathcal{S}_\eta^2(\mathcal{F}(x)) = \frac{16}{9}.$$

Applying Theorem 3.13 to the quantum Fourier transform on subfactor planar algebras, we obtain the following corollary.

Corollary 3.16. *Suppose $\mathcal{P}_\bullet$ is an irreducible subfactor planar algebra with finite Jones index μ. Let $\mathfrak{F}_s$ be the Fourier transform from $\mathcal{P}_{n,\pm}$ onto $\mathcal{P}_{n,\mp}$. Then, for any nonzero n-box $x \in \mathcal{P}_{n,\pm}$, we have*

$$\mathcal{S}_\varepsilon^2(x)\mathcal{S}_\eta^2(\mathfrak{F}_s(x)) \geq \mu(1-\varepsilon-\eta)^2 \quad \forall\, \varepsilon, \eta \in [0,1],\ \varepsilon + \eta \leq 1.$$

3.2. Quantum Wigderson–Wigderson uncertainty principle

In this section, we prove the quantum (p, q)-Wigderson–Wigderson uncertainty principle for von Neumann k-bi-algebras for $1/p + 1/q = 1$, and for quantum Fourier transform on subfactor planar algebras for any $0 < p, q \leq \infty$.

We prove the quantum Hausdorff–Young inequality for k-transforms using the standard interpolation method.

Theorem 3.17. *Let the quintuple $(\mathcal{A}, \mathcal{B}, d, \tau, \mathcal{F})$ be a von Neumann k-bi-algebra such that $\mathcal{F}^*\mathcal{F} = kI$. For any $x \in \mathcal{A}$, we have*

$$\|\mathcal{F}(x)\|_p \leq k^{\frac{1}{p}}\|x\|_q,$$

where $2 \leq p \leq \infty$ and $1/p + 1/q = 1$.

Proof. Note that

$$\|\mathcal{F}(x)\|_\infty \leq \|x\|_1, \quad \|\mathcal{F}(x)\|_2 = \sqrt{k}\|x\|_2.$$

Applying the Riesz–Thorin interpolation theorem (see [22, Theorem IX.17]), we have that $\|\mathcal{F}(x)\|_p \leq k^{\frac{1}{p}}\|x\|_q$. ∎

Then, we have the following quantum (p, q)-Wigderson–Wigderson uncertainty principles for k-transforms, which generalize Theorem 2.8.

Theorem 3.18. *Let the quintuple $(\mathcal{A}, \mathcal{B}, d, \tau, \mathcal{F})$ be a von Neumann k-bi-algebra such that $\mathcal{F}^*\mathcal{F} = kI$. For any $x \in \mathcal{A}$, we have*

$$\|x\|_q \|\mathcal{F}(x)\|_q \geq k^{1-\frac{2}{p}} \|x\|_p \|\mathcal{F}(x)\|_p,$$

where $2 \leq p \leq \infty$ and $1/p + 1/q = 1$.

Proof. By Theorem 3.17, we have

$$\|\mathcal{F}(x)\|_p \leq k^{\frac{1}{p}} \|x\|_q.$$

For the adjoint operator $\mathcal{F}^*$, we have

$$\|\mathcal{F}^*\|_{1\to\infty} \leq 1, \quad \|\mathcal{F}^*\|_{2\to 2} = \sqrt{k}.$$

Applying the same process in Theorem 3.17 to $\mathcal{F}^*$, we also have

$$\|\mathcal{F}^*\mathcal{F}(x)\|_p \leq k^{\frac{1}{p}} \|\mathcal{F}(x)\|_q.$$

Multiplying the above two inequalities, we obtain

$$\|x\|_q \|\mathcal{F}(x)\|_q \geq k^{-\frac{2}{p}} \|\mathcal{F}^*\mathcal{F}(x)\|_p \|\mathcal{F}(x)\|_p = k^{1-\frac{2}{p}} \|x\|_p \|\mathcal{F}(x)\|_p.$$

This completes the proof of the theorem. ∎

Next, we introduce the quantum (p, q)-Wigderson–Wigderson uncertainty principle for quantum Fourier transform for any $0 < p, q \leq \infty$, based on the norm of quantum Fourier transform computed in [20].

Theorem 3.19 (The norm of quantum Fourier transform). *Suppose $\mathcal{P}_\bullet$ is an irreducible subfactor planar algebra. Let $\mathfrak{F}_s$ be the Fourier transform from $\mathcal{P}_{2,\pm}$ onto $\mathcal{P}_{2,\mp}$. Let $x \in \mathcal{P}_{2,\pm}$ be a 2-box and $0 < p, q \leq \infty$. Then*

$$K\left(\frac{1}{p}, \frac{1}{q}\right)^{-1} \|x\|_q \leq \|\mathfrak{F}_s(x)\|_p \leq K\left(\frac{1}{q}, \frac{1}{p}\right) \|x\|_q.$$

We refer the readers to Appendix A for the specific definition of the function $K(\frac{1}{p}, \frac{1}{q}) = \|\mathfrak{F}_s\|_{p\to q}$.

The following theorem follows immediately from Theorem 3.19.

Theorem 3.20. *Suppose $\mathcal{P}_\bullet$ is an irreducible subfactor planar algebra. Let $\mathfrak{F}_s$ be the Fourier transform from $\mathcal{P}_{2,\pm}$ onto $\mathcal{P}_{2,\mp}$. Let $x \in \mathcal{P}_{2,\pm}$ be a 2-box and $0 < p, q \leq \infty$. Then*

$$\|x\|_q \|\mathfrak{F}_s(x)\|_q \geq K\left(\frac{1}{q}, \frac{1}{p}\right)^{-2} \|x\|_p \|\mathfrak{F}_s(x)\|_p.$$

Proof. By Theorem 3.19, we have

$$\|\mathfrak{F}_s(x)\|_p \leq K\left(\frac{1}{q}, \frac{1}{p}\right)\|x\|_q, \quad \|x\|_p \leq K\left(\frac{1}{q}, \frac{1}{p}\right)\|\mathfrak{F}_s(x)\|_q.$$

Multiplying the above two equations, we can obtain the result. ∎

3.3. Quantum Hirschman–Beckner uncertainty principle

In this subsection, we will prove the quantum (smooth) Hirschman–Beckner uncertainty principle (See Theorems 3.22 and 3.28) for von Neumann k-bi-algebras. For classical Hirschman–Beckner uncertainty principle [2,8], the Shannon entropy is used to describe the uncertainty principle on $\mathbb{R}$. For a finite von Neumann algebra with a trace, the von Neumann entropy is a noncommutative version of Shannon entropy.

Definition 3.21. Let $\mathcal{M}$ be a finite von Neumann algebra with a trace $\tau_{\mathcal{M}}$. The von Neumann entropy of $|x|^2 \in \mathcal{M}$ is defined as follows:

$$H(|x|^2) := -\tau_{\mathcal{M}}(|x|^2 \log |x|^2) = -\tau_{\mathcal{M}}(x^*x \log x^*x).$$

We have the quantum Hirschman–Beckner uncertainty principle for von Neumann k-bi-algebras.

Theorem 3.22. *Let the quintuple $(\mathcal{A}, \mathcal{B}, d, \tau, \mathcal{F})$ be a von Neumann k-bi-algebra. Suppose $\mathcal{A}$ and $\mathcal{B}$ are finite dimensional and $\mathcal{F}^*\mathcal{F} = kI$. Let x be a nonzero element in $\mathcal{A}$. Then, we have*

$$\frac{H(|x|^2)}{\|x\|_2^2} + \frac{H(|\mathcal{F}(x)|^2)}{\|\mathcal{F}(x)\|_2^2} \geq -\log \|x\|_2^2 - \log \|\mathcal{F}(x)\|_2^2 + \log k.$$

In particular, since $\|\mathcal{F}(x)\|_2^2 = k\|x\|_2^2$, we have

$$\frac{H(|x|^2)}{\|x\|_2^2} + \frac{H(|\mathcal{F}(x)|^2)}{\|\mathcal{F}(x)\|_2^2} \geq 0,$$

whenever $\|x\|_2 = 1$.

Proof. By Theorem 3.17, we have

$$\|\mathcal{F}(x)\|_q \leq k^{\frac{1}{q}}\|x\|_p,$$

where $2 \leq q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let

$$f(q) = \log \|\mathcal{F}(x)\|_q - \log \|x\|_p - \frac{1}{q}\log k,$$

then $f(q) \leq 0$ and $f(2) = 0$, which implies $f'(2) \leq 0$. Let

$$|\mathcal{F}(x)| = \sum_{i=1}^{n} \lambda_i e_i$$

be the spectral decomposition, we have

$$\begin{aligned} \frac{d}{dq}\|\mathcal{F}(x)\|_q^q\Big|_{q=2} &= \frac{d}{dq}\sum_{i=1}^{n}|\lambda_i|^q\tau(e_i)\Big|_{q=2} = \sum_{i=1}^{n}|\lambda_i|^2\log|\lambda_i|\tau(e_i) \\ &= -\frac{1}{2}H(|\mathcal{F}(x)|^2). \end{aligned}$$

Analogously,

$$\frac{d}{dq}\|x\|_p^p\Big|_{q=2} = \frac{dp}{dq}\Big|_{q=2}\frac{d}{dp}\|x\|_p^p\Big|_{p=2} = \frac{1}{2}H(|x|^2).$$

Thus,

$$\frac{d}{dq}\log\|\mathcal{F}(x)\|_q\Big|_{q=2} = -\frac{1}{4}\log\|\mathcal{F}(x)\|_2^2 - \frac{H(|\mathcal{F}(x)|^2)}{4\|\mathcal{F}(x)\|_2^2},$$

and

$$\frac{d}{dq}\log\|x\|_p\Big|_{q=2} = \frac{1}{4}\log\|x\|_2^2 + \frac{H(|x|^2)}{4\|x\|_2^2}.$$

We have

$$f'(2) = -\frac{1}{4}\log\|\mathcal{F}(x)\|_2^2 - \frac{H(|\mathcal{F}(x)|^2)}{4\|\mathcal{F}(x)\|_2^2} - \frac{1}{4}\log\|x\|_2^2 - \frac{H(|x|^2)}{4\|x\|_2^2} + \frac{1}{4}\log k.$$

Since $f'(2) \leq 0$, we obtain

$$\frac{H(|x|^2)}{\|x\|_2^2} + \frac{H(|\mathcal{F}(x)|^2)}{\|\mathcal{F}(x)\|_2^2} \geq -\log\|x\|_2^2 - \log\|\mathcal{F}(x)\|_2^2 + \log k.$$

This completes the proof. ■

Remark 3.23. Using the inequality $\log \mathcal{S}(x) \geq H(|x|^2)$ when $\|x\|_2 = 1$, we have

$$\begin{aligned} \log\mathcal{S}(\mathcal{F}(x)) &= \log\mathcal{S}(\mathcal{F}(x)/\sqrt{k}) \\ &\geq H(|\mathcal{F}(x)|^2/k) \qquad (\|\mathcal{F}(x)/\sqrt{k}\|_2 = 1) \\ &= \frac{1}{k}H(|\mathcal{F}(x)|^2) + \log k. \end{aligned}$$

So,

$$\log\mathcal{S}(x) + \log\mathcal{S}(\mathcal{F}(x)) \geq H(|x|^2) + \frac{1}{k}H(|\mathcal{F}(x)|^2) + \log k \geq \log k.$$

Exponentiating both sides of the above inequality, we could obtain

$$\mathcal{S}(x)\mathcal{S}(\mathcal{F}(x)) \geq k,$$

the quantum support uncertainty principle (see Theorem 2.10).

A natural question is to consider the perturbations of the inequality in Theorem 3.22. We firstly consider the smooth von Neumann entropy.

Definition 3.24. Let $\mathcal{M}$ be a finite von Neumann algebra. For any $x \in \mathcal{M}$, $\varepsilon \in [0, 1]$ and $p \in [1, \infty]$, the (p, ε) *smooth entropy* of $|x|^2$ is defined by

$$H_\varepsilon^p(|x|^2) := \inf\{H(|y|^2) : y \in \mathcal{M}, \|x - y\|_p \leq \varepsilon\},$$

Remark 3.25. We thank Kaifeng Bu for referring us to another smooth Rényi entropy studied by R. Renner and S. Wolf in quantum information in [23].

The von Neumann entropy is continuous with respect to the operator norm and satisfies the Lipschitz condition.

Let $x = (x_i)$, $y = (y_i) \in \mathbb{R}^n$ be two real vectors. It is said that x majorizes y (see [9, Definition 4.3.41]) if

$$\max_{1\leq i_1<\cdots<i_k\leq n} \sum_{j=1}^{k} x_{i_j} \geq \max_{1\leq i_1<\cdots<i_k\leq n} \sum_{j=1}^{k} y_{i_j}$$

for each $k = 1, \ldots, n$, with equality for $k = n$. We use $x \succeq y$ to denote x majorizes y.

Lemma 3.26. *Let A, B be two matrices in $M_n(\mathbb{C})$. Let $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$ and $\lambda_1' \geq \cdots \geq \lambda_n' \geq 0$ be the eigenvalues of $|A|$ and $|B|$, respectively. Then, we have*

$$\sum_{i=1}^{n} |\lambda_i - \lambda_i'| \leq \|A - B\|_1,$$

where the Schatten 1-norm of $A - B$ is defined by $\mathrm{Tr}(|A - B|)$ and Tr is the unnormalized trace.

Proof. Let

$$\widetilde{A} = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix},$$

$$\widetilde{B} = \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix},$$

$$\widetilde{A - B} = \begin{pmatrix} 0 & A - B \\ A^* - B^* & 0 \end{pmatrix}.$$

Let $\lambda_1'' \geq \cdots \geq \lambda_n'' \geq 0$ be the eigenvalues of $|A-B|$, then

$$\begin{aligned}\lambda(\widetilde{A}) &= (\lambda_1,\ldots,\lambda_n,-\lambda_n,\ldots,-\lambda_1),\\ \lambda(\widetilde{B}) &= (\lambda_1',\ldots,\lambda_n',-\lambda_n',\ldots,-\lambda_1'),\\ \lambda(\widetilde{A-B}) &= (\lambda_1'',\ldots,\lambda_n'',-\lambda_n'',\ldots,-\lambda_1'')\end{aligned}$$

are eigenvalues of $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{A-B}$, respectively. We have

$$\lambda(\widetilde{A})-\lambda(\widetilde{B}) = (\lambda_1-\lambda_1',\ldots,\lambda_n-\lambda_n',\lambda_n'-\lambda_n,\ldots,\lambda_1'-\lambda_1).$$

The maximum value of the sum of n components of $\lambda(\widetilde{A})-\lambda(\widetilde{B})$ is $\sum_{i=1}^n |\lambda_i-\lambda_i'|$. By [9, Theorem 4.3.37 (b)], $\lambda(\widetilde{A-B})$ majorizes $\lambda(\widetilde{A})-\lambda(\widetilde{B})$. Therefore,

$$\sum_{i=1}^n |\lambda_i-\lambda_i'| \leq \sum_{i=1}^n \lambda_i'' = \|A-B\|_1. \qquad \blacksquare$$

Proposition 3.27 (Lipschitz condition). *Let $\mathcal{M}$ be a finite-dimensional von Neumann algebra with a trace $\tau_{\mathcal{M}}$. For any $x, y \in \mathcal{M}$, let $t = \max\{\|x\|, \|y\|, 1\}$, we have*

$$|H(|x|^2)-H(|y|^2)| \leq f(t)\tau_{\mathcal{M}}(I)^{1-\frac{1}{p}}\|x-y\|_p, \quad 1\leq p\leq\infty,$$

where

$$f(t) = 4t\log t + 2t.$$

Proof. Since $\mathcal{M}$ is finite dimensional, we may assume that

$$\mathcal{M} \quad = \quad \bigoplus_{i=1}^m M_{n_i}(\mathbb{C}).$$

Let Tr_i be the unnormalized trace on $M_{n_i}(\mathbb{C})$, then we have $\tau_{\mathcal{M}} = \sum_{i=1}^m \delta_i \mathrm{Tr}_i$. Suppose $x = \sum_{i=1}^m x_i$ and $y = \sum_{i=1}^m y_i$. Let $\alpha_{i1} \geq \cdots \geq \alpha_{in_i}$ and $\beta_{i1} \geq \cdots \geq \beta_{in_i}$ be eigenvalues of $|x_i|$ and $|y_i|$, respectively. Then, we have

$$\begin{aligned}|H(|y|^2)-H(|x|^2)| &= |\tau_{\mathcal{M}}(|y|^2\log|y|^2-|x|^2\log|x|^2)|\\ &\leq \sum_{i=1}^m \delta_i|\mathrm{Tr}_i(|y_i|^2\log|y_i|^2-|x_i|^2\log|x_i|^2)|\\ &= \sum_{i=1}^m \delta_i|\sum_{j=1}^{n_i}(\alpha_{ij}^2\log\alpha_{ij}^2-\beta_{ij}^2\log\beta_{ij}^2)|\\ &\leq \sum_{i=1}^m \delta_i\sum_{j=1}^{n_i}|\alpha_{ij}-\beta_{ij}|\cdot(4t\log t+2t).\end{aligned}$$

By Lemma 3.26, we have $\sum_{j=1}^{n_i} |\alpha_{ij} - \beta_{ij}| \leq \|x_i - y_i\|_1$. Then

$$|H(|y|^2) - H(|x|^2)| \leq (4t \log t + 2t) \sum_{i=1}^{m} \delta_i \|x_i - y_i\|_1 = f(t)\|x - y\|_1.$$

By Hölder's inequality, we further have

$$|H(|x|^2) - H(|y|^2)| \leq f(t)\tau_{\mathcal{M}}(I)^{1-\frac{1}{p}} \|x - y\|_p, \quad 1 \leq p \leq \infty. \qquad \blacksquare$$

For a von Neumann k-bi-algebra $(\mathcal{A}, \mathcal{B}, d, \tau, \mathcal{F})$ such that $\mathcal{A}$ and $\mathcal{B}$ are finite dimensional, we set $\alpha = \max\{d(e)^{-1} : e \text{ is a projection in } \mathcal{A}\}$ and $\beta = \max\{\tau(e)^{-1} : e \text{ is a projection in } \mathcal{B}\}$. We have the quantum smooth Hirschman–Beckner uncertainty principle.

Theorem 3.28. *Let the quintuple $(\mathcal{A}, \mathcal{B}, d, \tau, \mathcal{F})$ be a von Neumann k-bi-algebra. Suppose $\mathcal{A}$ and $\mathcal{B}$ are finite dimensional and $\mathcal{F}^*\mathcal{F} = kI$. Let x be a nonzero element in $\mathcal{A}$. For any $\varepsilon, \eta \in [0, 1]$ and $p, q \in [1, \infty]$, we have*

$$\frac{H_\varepsilon^p(|x|^2)}{\|x\|_2^2} + \frac{H_\eta^q(|\mathcal{F}(x)|^2)}{\|\mathcal{F}(x)\|_2^2} \geq -4\log\|x\|_2 - \frac{C_1(x)}{\|x\|_2^2} d(I)^{1-\frac{1}{p}}\varepsilon - \frac{C_2(x)}{\|\mathcal{F}(x)\|_2^2}\tau(I)^{1-\frac{1}{q}}\eta,$$

where $C_1(x) = f(\|x\| + 1 + \alpha^{1/p})$ and $C_2(x) = f(\|\mathcal{F}(x)\| + 1 + \beta^{1/q})$ and $f(t) = 4t\log t + 2t$.

Proof. For any $y \in \mathcal{A}$ with $\|x - y\|_p \leq \varepsilon$, we have

$$\|y\| \leq \|x\| + \|x - y\| \leq \|x\| + \alpha^{1/p}\|x - y\|_p \leq \|x\| + \alpha^{1/p}.$$

Note that $f(t)$ is positive and monotonically increasing when $t \geq 1$. From Proposition 3.27, we have

$$|H(|y|^2) - H(|x|^2)| \leq C_1(x)d(I)^{1-\frac{1}{p}}\|x - y\|_p \leq C_1(x)d(I)^{1-\frac{1}{p}}\varepsilon.$$

Thus,

$$H_\varepsilon^p(|x|^2) \geq H(|x|^2) - C_1(x)d(I)^{1-\frac{1}{p}}\varepsilon.$$

Analogously, we have

$$H_\eta^q(|\mathcal{F}(x)|^2) \geq H(|\mathcal{F}(x)|^2) - C_2(x)\tau(I)^{1-\frac{1}{q}}\eta.$$

Adding the above two equations and applying Theorem 3.22, we obtain the result. $\blacksquare$

Applying Theorem 3.28 to quantum Fourier transform on subfactor planar algebras, we have the following quantum smooth Hirschman–Beckner uncertainty principle for quantum Fourier transform.

Corollary 3.29. *Suppose x is a nonzero 2-box in an irreducible subfactor planar algebra. For any $\varepsilon, \eta \in [0,1]$ and $p, q \in [1, \infty]$, we have*

$$H_{\varepsilon}^{p}(|x|^2) + H_{\eta}^{q}(|\mathfrak{F}_s(x)|^2) \geq \|x\|_2^2(\log \mu - 4\log\|x\|_2) - C(x)(\mu^{1-\frac{1}{p}}\varepsilon + \mu^{1-\frac{1}{q}}\eta),$$

where $C(x) = f(\|x\|_2 + 2)$ and $f(t) = 4t\log t + 2t$ and μ is the Jones index.

Remark 3.30. When $p = q = 1$, then the remaining item $-C(x)(\mu^{1-\frac{1}{p}}\varepsilon + \mu^{1-\frac{1}{q}}\eta)$ in Corollary 3.29 is independent of Jones index.

Remark 3.31. We have the following statements.

(i) In Example 2.5, taking $\varepsilon = \eta = 0$, then Theorem 3.28 implies Theorem 4.9 in [18].

(ii) In Example 2.6, taking $\varepsilon = \eta = 0$, then Theorem 3.28 implies Theorem 5.5 in [11].

In [11], the minimizers of Hirschman–Beckner uncertainty principle on subfactor planar algebras were characterized as bi-shifts of biprojections (see [11, Theorems 6.4 and 6.13]). So, it is natural to ask the following inverse problem.

Problem 3.32. Find positive function $C(\varepsilon, \mu)$ for $\varepsilon, \mu > 0$ such that $\lim_{\varepsilon\to 0} C(\varepsilon, \mu) \to 0$, and for any 2-box x of any irreducible subfactor planar algebra with Jones index μ, $\|x\|_2 = 1$, if

$$H(|x|^2) + H(|\mathfrak{F}_s(x)|^2) \geq 2\log\mu - \varepsilon,$$

then $\|x - y\| \leq C(\varepsilon, \mu)$ for some bi-shift of biprojection y.

4. An answer to a conjecture of Wigderson and Wigderson

The famous Heisenberg uncertainty principle in [7] could be mathematically formulated in terms of Schwarz functions on $\mathbb{R}$ (see, e.g., [16, 25] and [26, Theorem 4.9]) as follows.

Theorem 4.1 (Heisenberg's uncertainty principle). *Let $\mathcal{S}(\mathbb{R})$ be the space of Schwartz functions. For any $f \in \mathcal{S}(\mathbb{R})$,*

$$\int_{\mathbb{R}} x^2|f(x)|^2dx \int_{\mathbb{R}} \xi^2|\hat{f}(\xi)|^2dx \geq \frac{1}{16\pi^2}\|f\|_2^2\|\hat{f}\|_2^2,$$

where

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i x\xi}dx$$

is the Fourier transform of f.

In [26], A. Wigderson and Y. Wigderson proved the following generalization of Heisenberg's uncertainty principle for arbitrary q-norm.

Theorem 4.2 (See [26, Theorem 4.11]). *For any $f \in \mathcal{S}(\mathbb{R})$, and any $1 < q \leq \infty$,*

$$\int_{\mathbb{R}} x^2 |f(x)|^2 dx \int_{\mathbb{R}} \xi^2 |\hat{f}(\xi)|^2 dx \geq 2^{-\frac{10q-8}{q-1}} \|f\|_q^2 \|\hat{f}\|_q^2.$$

In order to compare these inequalities for different q, they proposed the following conjecture.

Conjecture 1 (See [26, Conjecture 4.13]). For any nonzero $f \in \mathcal{S}(\mathbb{R})$, $q \in (1, \infty]$, define

$$F_q(f) = \frac{\|f\|_q \|\hat{f}\|_q}{\|f\|_2 \|\hat{f}\|_2} = \frac{\|f\|_q \|\hat{f}\|_q}{\|f\|_2^2}.$$

Then, the image of $F_q : \mathcal{S}(\mathbb{R}) \setminus \{0\} \to \mathbb{R}_{>0}$ is $\mathbb{R}_{>0}$ for all $q \neq 2$.

Moreover, they proved the conjecture for $q = \infty$ in [26, Theorem 4.12].

In the following theorem, we verify Conjecture 1 for $q > 2$ and disprove Conjecture 1 for $1 < q < 2$. More precisely, we have the following theorem.

Theorem 4.3. *The following statements hold.*

(i) *If $1 < q < 2$, taking $1/p + 1/q = 1$, then*

$$F_q(f) \geq [p^{1/p}/q^{1/q}]^{1/2} \quad \forall f \in \mathcal{S}(\mathbb{R}) \setminus \{0\}.$$

(ii) *If $q > 2$, then the image of F_q is $\mathbb{R}_{>0}$.*

To prove Theorem 4.3, we firstly prove a technical lemma.

Lemma 4.4. *Let $1 < p < q \leq \infty$, and define a function $F_{p,q}\colon \mathcal{S}(\mathbb{R}) \setminus \{0\} \to \mathbb{R}_{>0}$ by*

$$F_{p,q}(f) = \frac{\|f\|_q \|\hat{f}\|_q}{\|f\|_p \|\hat{f}\|_p}.$$

If there exist two sequences $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ in $\mathcal{S}(\mathbb{R}) \setminus \{0\}$ such that

$$\lim_{n\to\infty} F_{p,q}(f_n) = 0, \quad \lim_{n\to\infty} F_{p,q}(g_n) = \infty,$$

and $\lambda f_n + (1-\lambda) g_n \neq 0$ for any $\lambda \in [0, 1]$ and all $n \geq 1$, then the image of $F_{p,q}$ is all of $\mathbb{R}_{>0}$.

Proof. We define

$$h_n(\lambda) = F_{p,q}(\lambda f_n + (1-\lambda) g_n),$$

then $h_n(\lambda)$ is a continuous function for any $n \geq 1$. Thus, $h_n(\lambda)$ can take all real values between $F_{p,q}(f_n)$ and $F_{p,q}(g_n)$. The result follows immediately from the assumptions. ∎

Theorem 4.5. *Let $1 < p < q \le \infty$, and define a function $F_{p,q}\colon \mathcal{S}(\mathbb{R}) \setminus \{0\} \to \mathbb{R}_{>0}$ by*

$$F_{p,q}(f) = \frac{\|f\|_q \|\hat{f}\|_q}{\|f\|_p \|\hat{f}\|_p}.$$

When $\frac{1}{p} + \frac{1}{q} < 1$, then the image of $F_{p,q}$ is all of $\mathbb{R}_{>0}$.

Proof. We will consider two special families of Schwartz functions. Let $a > b > 0$ be real numbers. Define

$$f_{a,b}(x) = e^{-\pi((a+ib)x)^2} = e^{-\pi(a^2-b^2)x^2} e^{-2\pi i abx^2}.$$

From the definition, we see that $|f_{a,b}(x)| = e^{-\pi(a^2-b^2)x^2}$. We can compute the r-norm of $f_{a,b}$ as follows:

$$\|f_{a,b}(x)\|_r = \left(\int_{-\infty}^{\infty} e^{-r\pi(a^2-b^2)x^2} dx\right)^{\frac{1}{r}} = \left(\frac{1}{\sqrt{r(a^2-b^2)}}\right)^{\frac{1}{r}}, \quad 1 < r \le \infty.$$

When $r = \infty$, the above equality means $\|f_{a,b}(x)\|_\infty = 1$. The Fourier transform of $f_{a,b}$ is

$$\widehat{f_{a,b}}(\xi) = \frac{1}{a+bi} e^{-\pi(\xi/(a+bi))^2} = \frac{1}{a+bi} e^{-\pi\xi^2(a^2-b^2)/(a^2+b^2)^2} e^{2\pi i\xi^2 ab/(a^2+b^2)^2}.$$

In particular,

$$\left|\widehat{f_{a,b}}(\xi)\right| = \frac{1}{\sqrt{a^2+b^2}} e^{-\pi\xi^2(a^2-b^2)/(a^2+b^2)^2}.$$

Similarly, we can compute the r-norm of $\widehat{f_{a,b}}(\xi)$ as follows:

$$\|\widehat{f_{a,b}}(\xi)\|_r = \frac{1}{\sqrt{a^2+b^2}} \left(\frac{a^2+b^2}{\sqrt{r(a^2-b^2)}}\right)^{\frac{1}{r}}, \quad 1 < r \le \infty.$$

When $r = \infty$, $\|\widehat{f_{a,b}}(\xi)\|_\infty = \frac{1}{\sqrt{a^2+b^2}}$. This implies that

$$F_{p,q}(f_{a,b}) = \frac{\|f_{a,b}\|_q \|\widehat{f_{a,b}}\|_q}{\|f_{a,b}\|_p \|\widehat{f_{a,b}}\|_p} = \frac{\sqrt[p]{p}\,(a^2+b^2)^{\frac{1}{q}-\frac{1}{p}}}{\sqrt[q]{q}\,(a^2-b^2)^{\frac{1}{q}-\frac{1}{p}}}, \quad 1 < p < q \le \infty.$$

If $a > 1$ and $b = \sqrt{a^2-1}$, then we get

$$F_{p,q}(f_{a,\sqrt{a^2-1}}) = \frac{\sqrt[p]{p}}{\sqrt[q]{q}} (2a^2-1)^{\frac{1}{q}-\frac{1}{p}}.$$

So,

$$\lim_{a\to\infty} F_{p,q}(f_{a,\sqrt{a^2-1}}) = 0, \quad 1 < p < q \le \infty. \tag{4.1}$$

Next, we consider another family of functions:

$$g_c(x) = \frac{1}{\sqrt{c}} e^{-\pi(x/c)^2} + \sqrt{c} e^{-\pi(cx)^2}, \quad c > 0.$$

Both items of g_c are positive, so

$$\frac{1}{2}\Big(\frac{1}{\sqrt{c}}\|e^{-\pi(x/c)^2}\|_r + \sqrt{c}\|e^{-\pi(cx)^2}\|_r\Big) < \|g_c\|_r < \frac{1}{\sqrt{c}}\|e^{-\pi(x/c)^2}\|_r + \sqrt{c}\|e^{-\pi(cx)^2}\|_r$$

for any $1 < r \leq \infty$, namely,

$$\frac{c^{\frac{1}{r}-\frac{1}{2}} + c^{\frac{1}{2}-\frac{1}{r}}}{2\,\sqrt[2r]{r}} < \|g_c\|_r < \frac{c^{\frac{1}{r}-\frac{1}{2}} + c^{\frac{1}{2}-\frac{1}{r}}}{\sqrt[2r]{r}}.$$

Note that $\widehat{g_c} = g_c$, so we obtain an estimation of $F_{p,q}(g_c)$:

$$\frac{\sqrt[p]{p}(c^{\frac{1}{q}-\frac{1}{2}} + c^{\frac{1}{2}-\frac{1}{q}})^2}{4\,\sqrt[q]{q}(c^{\frac{1}{p}-\frac{1}{2}} + c^{\frac{1}{2}-\frac{1}{p}})^2} < F_{p,q}(g_c) < \frac{4\,\sqrt[p]{p}(c^{\frac{1}{q}-\frac{1}{2}} + c^{\frac{1}{2}-\frac{1}{q}})^2}{\sqrt[q]{q}(c^{\frac{1}{p}-\frac{1}{2}} + c^{\frac{1}{2}-\frac{1}{p}})^2}, \quad 1 < p < q \leq \infty.$$

Therefore,

$$\lim_{c\to\infty} F_{p,q}(g_c) = \begin{cases} \infty, & \dfrac{1}{p} + \dfrac{1}{q} < 1,\ 1 < p < q \leq \infty, \\ 0, & \dfrac{1}{p} + \dfrac{1}{q} > 1,\ 1 < p < q \leq \infty. \end{cases} \tag{4.2}$$

Combining equation (4.1) and equation (4.2), we have

$$\lim_{a\to\infty} F_{p,q}(f_{a,\sqrt{a^2-1}}) = 0, \quad \lim_{c\to\infty} F_{p,q}(g_c) = \infty,$$

when $\frac{1}{p} + \frac{1}{q} < 1$. Then, by Lemma 4.4, we can obtain the conclusion. ∎

In particular, take $p = 2$, then $2 < q \leq \infty$, which implies that Conjecture 1 holds for all $2 < q \leq \infty$.

Theorem 4.6. *When $1 < q < 2$, we have*

$$F_q(f) = \frac{\|f\|_q\|\hat{f}\|_q}{\|f\|_2\|\hat{f}\|_2} = \frac{\|f\|_q\|\hat{f}\|_q}{\|\hat{f}\|_2^2} \geq [p^{1/p}/q^{1/q}]^{1/2},$$

where

$$\frac{1}{p} + \frac{1}{q} = 1$$

for any $f \in \mathcal{S}(\mathbb{R}) \setminus \{0\}$.

Proof. By the sharp Hausdorff–Young inequality (see [2, Theorem 1]), we have

$$\|\hat{f}\|_p \le [q^{1/q}/p^{1/p}]^{1/2}\|f\|_q$$

for any $f \in \mathcal{S}(\mathbb{R}) \setminus \{0\}$. By Hölder's inequality, we further have

$$\|\hat{f}\|_2^2 \le \|\hat{f}\|_p\|\hat{f}\|_q.$$

Combining the above two equations, we obtain

$$F_q(f) \ge [p^{1/p}/q^{1/q}]^{1/2}\frac{\|\hat{f}\|_p\|\hat{f}\|_q}{\|\hat{f}\|_2^2} \ge [p^{1/p}/q^{1/q}]^{1/2}.$$

■

Now, we give the proof of Theorem 4.3.

Proof of Theorem 4.3. Combining Theorems 4.5 and 4.6, we can obtain the conclusion. ■

By Theorem 4.3, we have known that

$$F_q(f) \ge [p^{1/p}/q^{1/q}]^{1/2}$$

for all $f \in \mathcal{S}(\mathbb{R}) \setminus \{0\}$ when $1 < q < 2$. Let

$$C_q = \inf\{F_q(f) : f \in \mathcal{S}(\mathbb{R}) \setminus \{0\}\},$$

then $C_q \ge [p^{1/p}/q^{1/q}]^{1/2}$ by Theorem 4.6. So, it is natural to ask what the optimal constant C_q is.

Problem 4.7. Determine the constant

$$C_q = \inf\left\{F_q(f) : f \in \mathcal{S}(\mathbb{R}) \setminus \{0\}\right\}$$

when $1 < q < 2$.

A. The function $K(1/p, 1/q)$

The first quadrant is divided into three regions R_T, R_F, R_{TF} as follows:

$$\begin{aligned} R_F &:= \left\{(1/p, 1/q) \in [0,\infty]^2 : 1/p + 1/q \le 1, 1/q \le 1/2\right\},\\ R_T &:= \left\{(1/p, 1/q) \in [0,\infty]^2 : 1/p + 1/q \ge 1, 1/p \ge 1/2\right\},\\ R_{TF} &:= \left\{(1/p, 1/q) \in [0,\infty]^2 : 1/p \le 1/2, 1/q \ge 1/2\right\}.\end{aligned}$$

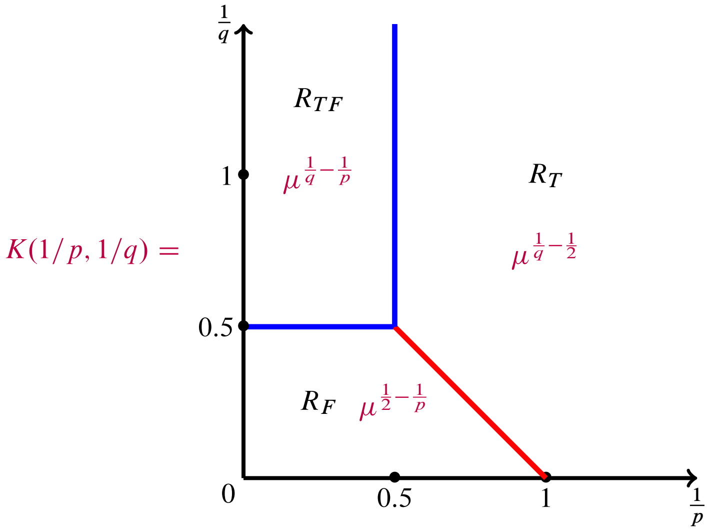

Figure 1. The norm of the Fourier transform $\mathfrak{F}_S$.

The function $K(1/p, 1/q)$ on $[0,\infty)^2$ (see Figure 1) is given by

$$K(1/p,1/q) = \begin{cases} \mu^{1/2-1/p} & \text{for } (1/p,1/q) \in R_F, \\ \mu^{1/q-1/2} & \text{for } (1/p,1/q) \in R_T, \\ \mu^{1/q-1/p} & \text{for } (1/p,1/q) \in R_{TF}. \end{cases}$$

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