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Cluster of vortex helices in the incompressible three-dimensional Euler equations

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Abstract. In an inviscid and incompressible fluid in dimension 3, we prove the existence of several helical filaments, or vortex helices, collapsing into each other.

1. Introduction

An ideal incompressible homogeneous fluid of density ρ in three-dimensional space in a time interval $(0, T)$ is governed by the Euler equations

$$\begin{aligned} \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} &= -\frac{1}{\rho}\nabla p \quad \text{in } \mathbb{R}^3 \times [0, T), \\ \operatorname{div} \mathbf{u} &= 0 \qquad\qquad \text{in } \mathbb{R}^3 \times [0, T), \end{aligned} \tag{1.1}$$

supplemented by the initial velocity $\mathbf{u}(\cdot, 0) = u_0(x)$. Here $\mathbf{u}: \mathbb{R}^3 \times [0, T) \to \mathbb{R}^3$ is the velocity field and $p: \mathbb{R}^3 \times [0, T) \to \mathbb{R}$ is the pressure, determined by the incompressibility condition. We will consider constant density $\rho = 1$. For a solution $\mathbf{u}$ of (1.1), its vorticity is defined as $\vec{\omega} = \operatorname{curl} \mathbf{u}$. Then $\vec{\omega}$ solves the Euler system in *vorticity form* (1.1),

$$\begin{aligned} \vec{\omega}_t + (\mathbf{u} \cdot \nabla)\vec{\omega} &= (\vec{\omega} \cdot \nabla)\mathbf{u} \qquad \text{in } \mathbb{R}^3 \times (0, T), \\ \mathbf{u} = \operatorname{curl} \vec{\psi}, \quad -\Delta\vec{\psi} &= \vec{\omega} \qquad\qquad\quad \text{in } \mathbb{R}^3 \times (0, T), \\ \vec{\omega}(\cdot, 0) &= \operatorname{curl} u_0 \qquad\quad\ \text{in } \mathbb{R}^3. \end{aligned} \tag{1.2}$$

Vortex filaments are solutions of the Euler equations whose vorticity is concentrated in a small tube near an evolving imaginary smooth curve $\Gamma(t)$ embedded in the entire $\mathbb{R}^3$ so that the associated velocity field vanishes as the distance to the curve goes to infinity. Da Rios [13] in 1906, and Levi-Civita [30] in 1908, found that, if the vorticity concentrates smoothly and symmetrically in a small tube of size $\varepsilon > 0$ around a smooth curve for a certain interval of time, then it is possible to compute the instantaneous velocity of the curve to leading order. These computations suggest that the curve should evolve by its

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binormal flow, with a large velocity of order $|\log \varepsilon|$. If $\Gamma(t)$ is parametrized as $\gamma(s,t)$, where s designates its arc-length parameter, then $\gamma(s,t)$ asymptotically obeys a law of the form

$$\gamma_t = 2\kappa|\log \varepsilon|(\gamma_s \times \gamma_{ss})$$

as $\varepsilon \to 0$, or scaling $t = |\log \varepsilon|^{-1}\tau$,

$$\gamma_\tau = 2\kappa(\gamma_s \times \gamma_{ss}) = 2\kappa c \mathbf{b}_{\Gamma(\tau)}. \tag{1.3}$$

Here κ is the *circulation* of the velocity field on the boundary of sections to the filament, which is assumed to be a constant independent of ε. In addition, $\mathbf{t}_{\Gamma(\tau)}$, $\mathbf{n}_{\Gamma(\tau)}$, $\mathbf{b}_{\Gamma(\tau)}$ are the usual tangent, normal, and binormal unit vectors to $\Gamma(\tau)$, and c its curvature. See [32, 35] for a complete discussion on this topic.

Jerrard and Seis [24] rigorously proved Da Rios' formal computation, conditional upon knowing that the vorticity of a solution remains concentrated around some curve. In particular, they consider a solution to (1.2) whose vorticity $\vec{\omega}_\varepsilon(x,t)$ satisfies, as $\varepsilon \to 0$,

$$\vec{\omega}_\varepsilon(\cdot, |\log \varepsilon|^{-1}\tau) - \delta_{\Gamma(\tau)}\mathbf{t}_{\Gamma(\tau)} \rightharpoonup 0, \quad 0 \le \tau \le T, \tag{1.4}$$

where $\Gamma(\tau)$ is a sufficiently regular curve and $\delta_{\Gamma(\tau)}$ denotes a uniform Dirac measure on the curve. They proved that in these circumstances the curve $\Gamma(\tau)$ does indeed evolve by the law (1.3). See [25] and its references for results on the flow (1.3).

The *vortex filament conjecture* [6, 24] refers to the question of existence of true solutions of (1.2) that satisfy (1.4) near a given curve $\Gamma(\tau)$ that evolves by the binormal flow (1.3). This is an open question, except for very special cases.

A known solution of the binormal flow (1.3) that does not change its form in time is a circle $\Gamma(\tau)$ with radius R translating with constant speed equal to $\frac{2}{R}$ along its axis. Solutions to (1.2) whose vorticity is concentrated in a circular vortex filament are known as vortex rings, and the study of these objects dates back to Helmholtz and Kelvin's work. In 1894, Hill found an explicit axially symmetric solution of (1.2) supported in a sphere (Hill's vortex ring). Fraenkel's result [22] provided the first construction of a vortex ring concentrated around a torus with fixed radius and a small, nearly singular section $\varepsilon > 0$, traveling with constant speed $\sim |\log \varepsilon|$, rigorously establishing the vortex filament conjecture for the case of traveling rings. Vortex rings have been analyzed in larger generality in [1, 19, 23, 34].

Another known solution of the binormal flow (1.3) that does not change its form in time is the *rotating-translating helix*. It is a circular helix of radius R and pitch $h > 0$, which rotates with constant speed $\frac{2\kappa h}{(h^2+R^2)^{3/2}}$ and translates vertically in the direction of their axis of symmetry with constant speed $\frac{2\kappa R^2}{(h^2+R^2)^{3/2}}$. Solutions to the Euler equations (1.2) whose vorticity is concentrated in a helical vortex filament are known as vortex helices, and the description of these objects started with the works of Joukowsky [28], Da Rios [14] and Levi-Civita [31]. In [17] the authors provided the first construction of a vortex helix concentrated in an ε-tubular neighborhood of a rotating-translating helix evolving by binormal flow, establishing the vortex filament conjecture for the case of rotating-translating helices. In [17] the authors also find a solution to (1.2) with several vortex helices, rotating-translating with comparable but different speeds. Traveling-rotating helical vortices with small cross-section to the three-dimensional incompressible Euler equations in an infinite pipe were constructed in [10] and [9].

In this paper we are concerned with solutions to the Euler equations (1.2) consisting of several vortex helices which are rotating-translating with almost the same speed. They are global-in-time solutions to (1.2) and their vorticity has at main order the shape of several helical filaments which collapse into each other. We call this phenomenon a cluster of vortex helices. Let us be more precise.

Let N be a given integer. For any $i = 1, \dots, N$ consider points (a_i, b_i) in $\mathbb{R}^2$, numbers σ_i and β_i, and define the evolving curves Γ_j parametrized by

$$\gamma_i(s,\tau) = \begin{pmatrix} a_i \cos\left(\frac{s-\sigma_i\tau}{\sqrt{h^2+R_i^2}}\right) - b_i \sin\left(\frac{s-\sigma_i\tau}{\sqrt{h^2+R_i^2}}\right) \\ a_i \sin\left(\frac{s-\sigma_i\tau}{\sqrt{h^2+R_i^2}}\right) + b_i \cos\left(\frac{s-\sigma_i\tau}{\sqrt{h^2+R_i^2}}\right) \\ \frac{hs+\beta_i\tau}{\sqrt{h^2+R_i^2}} \end{pmatrix} \in \mathbb{R}^3, \tag{1.5}$$

where $h > 0$ is a positive constant and

$$R_i = \sqrt{a_i^2 + b_i^2}.$$

At any instant τ, the curves $s \to \gamma_i(s,\tau)$ are circular helices of radius R_i and common pitch h, parametrized by arc length. Their curvature is $\frac{R_i}{R_i^2+h^2}$ and their torsion $\frac{h}{R_i^2+h^2}$. When time evolves, the curves rotate with constant speed $\frac{\sigma_i}{\sqrt{h^2+R_i^2}}$ around the z-axis in $\mathbb{R}^3$ and at the same time translate vertically with constant speed $\frac{\beta_i}{\sqrt{h^2+R_i^2}}$. At time $\tau = 0$ each curve γ_i passes through the point $(a_i, b_i, 0)$ in $\mathbb{R}^3$ (take $s = 0$). A direct computation gives that $\gamma_i(s,\tau)$ evolves by the binormal flow (1.3) with circulation κ_i provided the speeds σ_i and β_i are chosen to be

$$\sigma_i = \frac{2\kappa_i h}{R_i^2 + h^2}, \quad \beta_i = \frac{2\kappa_i R_i^2}{R_i^2 + h^2}.$$

Each $\gamma_i(s,\tau)$ can be recovered from $\gamma_i(s,0)$ by a rotation and a vertical translation

$$\gamma_i(s,\tau) = Q_{-\frac{\sigma_i}{\sqrt{R_i^2+h^2}}\tau}\gamma(s,0) + \left(0, 0, \frac{\beta_i}{\sqrt{R_i^2+h^2}}\tau\right)^{\mathsf{T}}, \tag{1.6}$$

where $Q_\theta = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$, or equivalently from $\gamma_i(0,0) = (a_i, b_i, 0)^{\mathsf{T}}$ by means of

$$\gamma_i(s,\tau) = Q_{\frac{s-\sigma_i\tau}{\sqrt{R_i^2+h^2}}}(a_i, b_i, 0)^{\mathsf{T}} + \left(0, 0, \frac{hs+\beta_i\tau}{\sqrt{R_i^2+h^2}}\right)^{\mathsf{T}}.$$

We will identify the base point $(a_i, b_i, 0)$ of each helix simply with (a_i, b_i). Helical filaments with comparable but different speeds as in [17] have vorticity with

$$\vec{\omega}_\varepsilon(\cdot, |\log\varepsilon|^{-1}\tau) - \sum_{i=1}^{N} \delta_{\Gamma_i(\tau)} \mathbf{t}_{\Gamma_i(\tau)} \rightharpoonup 0, \quad 0 \le \tau \le T, \text{ as } \varepsilon \to 0, \tag{1.7}$$

so that $\gamma_i(0,0) = (a_i, b_i)$ satisfy

$$\mathrm{dist}((a_i, b_i), (a_j, b_j)) > \delta, \quad \text{as } \varepsilon \to 0, \text{ for all } i \neq j,$$

for some fixed $\delta > 0$, independent of ε.

We are interested in colliding helical filaments. Let $r_0 > 0$ be a fixed number and assume that the points (a_i, b_i) have the form, for all $i = 1, \dots, N$,

$$(a_i, b_i) = (r_0, 0) + Q_i, \quad \text{with } |Q_i| \to 0 \text{ as } \varepsilon \to 0. \tag{1.8}$$

Since $(a_i, b_i) \to (r_0, 0)$ as $\varepsilon \to 0$, for all i, the evolving helices γ_i in (1.5) shrink into each other as $\varepsilon \to 0$. The purpose of this paper is to establish the existence of a solution to (1.2) whose vorticity satisfies (1.7), with colliding helical filaments Γ_i in the sense of (1.8).

We find that the points Q_i need to converge to 0 at a precise rate in terms of ε. Let us be more precise. Assume

$$(a_i, b_i) = (r_0 + s, 0) + \frac{\mathbf{P}_i}{|\log\varepsilon|}, \quad \text{for } i := 1, \dots, N, \tag{1.9}$$

as $\varepsilon \to 0$, for some constant s and points $\mathbf{P}_i$ satisfying

$$|s| < \delta \frac{\log|\log\varepsilon|}{|\log\varepsilon|}, \quad \delta < |\mathbf{P}_i| < \delta^{-1},$$

for some $\delta > 0$ small, and independent of ε. The points $\mathbf{P}_i$ are at a uniform distance d (independent of ε) from each other,

$$d = \min_{i \neq j} |\mathbf{P}_i - \mathbf{P}_j| > 0, \tag{1.10}$$

and the set $\{\mathbf{P}_1, \dots, \mathbf{P}_N\}$ is symmetric with respect to their first component, in the sense that

$$\mathbf{P} = (p_1, p_2) \in \{\mathbf{P}_1, \dots, \mathbf{P}_N\} \Leftrightarrow (p_1, -p_2) \in \{\mathbf{P}_1, \dots, \mathbf{P}_N\}. \tag{1.11}$$

Writing

$$\mathbf{P}_i = \left(\frac{\mathbf{P}_{i,1}}{\sqrt{h^2 + r_0^2}}, \mathbf{P}_{i,2} \right), \tag{1.12}$$

the points $\mathbf{P}_i$ have the form $\mathbf{P}_i^0 + \mathbf{Q}_i$ where $(\mathbf{P}_1^0, \dots, \mathbf{P}_N^0)$ satisfy the balancing equations, for $i = 1, \dots, N$,

$$\begin{aligned} &\sum_{j \neq i} \kappa_j \frac{(\mathbf{P}_{i,1} - \mathbf{P}_{j,1})}{|\mathbf{P}_i - \mathbf{P}_j|^2} = \left(\kappa_i \frac{h r_0}{2\sqrt{(h^2 + r_0^2)^3}} - \alpha \frac{h r_0}{4\sqrt{h^2 + r_0^2}} \right), \\ &\sum_{j \neq i} \kappa_j \frac{(\mathbf{P}_{i,2} - \mathbf{P}_{j,2})}{|\mathbf{P}_i - \mathbf{P}_j|^2} = 0, \end{aligned} \tag{1.13}$$

where α is the constant defined by

$$\alpha = \frac{2}{h^2 + r_0^2} \frac{\sum_{i=1}^N \kappa_i^2}{\sum_{i=1}^N \kappa_i}, \tag{1.14}$$

and the $\mathbf{Q}_i$ are small perturbations. We say that a point $\mathbf{P}_i^0$ of the form (1.12) is a *non-degenerate* solution to (1.13) if the linearization of system (1.13) has only one element in its kernel, the one originating from the symmetry assumption (1.11). This non-degeneracy is a necessary condition used to find $\mathbf{P}_i$ as a small perturbation of $\mathbf{P}_i^0$. We will make this definition more precise in Section 8.

We prove the following result.

Theorem 1.1. *Let $h > 0$, $r_0 > 0$, $\kappa_1, \dots, \kappa_N$ be given numbers such that $\sum_{i=1}^N \kappa_i \neq 0$. Suppose there exists a* non-degenerate *solution $(\mathbf{P}_1^0, \dots, \mathbf{P}_N^0)$ of the form (1.12) to system (1.13), satisfying (1.11) and (1.10). Let $\Gamma_j(\tau)$ be the helices parametrized by equation (1.5), for $j = 1, \dots, N$, with (a_i, b_i) given by (1.9). Then there exist $s^* \in \mathbb{R}$, points $\mathbf{Q}_1, \dots, \mathbf{Q}_N$, and a smooth solution $\vec{\omega}_\varepsilon(x, t)$ to (1.2), defined for $t \in (-\infty, \infty)$, such that*

$$(a_i, b_i) = (r_0 + s^*, 0) + \frac{\mathbf{P}_i}{|\log \varepsilon|}, \quad \mathbf{P}_i = \mathbf{P}_i^0 + \mathbf{Q}_i, \quad |s^*|, |\mathbf{Q}_i|, \lesssim \frac{\log|\log \varepsilon|}{|\log \varepsilon|}$$

and for all τ,

$$\vec{\omega}_\varepsilon(x, \tau|\log \varepsilon|^{-1}) - \sum_{j=1}^N \kappa_j \delta_{\Gamma_j(\tau)} \mathbf{t}_{\Gamma_j(\tau)} \rightharpoonup 0 \quad \text{as } \varepsilon \to 0,$$

in the sense of distributions.

Configurations of points $(\mathbf{P}_1^0, \dots, \mathbf{P}_N^0)$ that satisfy (1.13) and the assumptions of Theorem 1.1 are known in the literature. For instance, letting $N = n + m$ and $\kappa_i = 1$ for $i = 1, \dots, m$ and $\kappa_i = -1$ for $i = m + 1, \dots, n + m$, explicit non-degenerate solutions to (1.13) when

$$(m, n) \in \mathbb{S} := \{(2, 1), (3, 2), (4, 3), (5, 4), (6, 5)\}$$

are described in [2] (see Figure 1 for examples). In this case a direct computation gives

$$\alpha = \frac{2(m + n)}{(h^2 + r_0^2)(m - n)},$$

from which we deduce that m must be different from n. Other constructions of admissible configurations can be found in [2, 3].

Our construction takes advantage of the invariance under helical symmetry of the Euler equations as discussed in [4,5,7,17,20,21,27,37]. This invariance and the assumption that the velocity field $\mathbf{u}$ in (1.1) is orthogonal to the helical symmetry lines imply that solutions to problem (1.2) can be found by solving a transport equation in 2 dimensions. For a point

$$x = (x_1, x_2, x_3) \in \mathbb{R}^3, \quad x = (x', x_3), \quad x' \in \mathbb{R}^2,$$

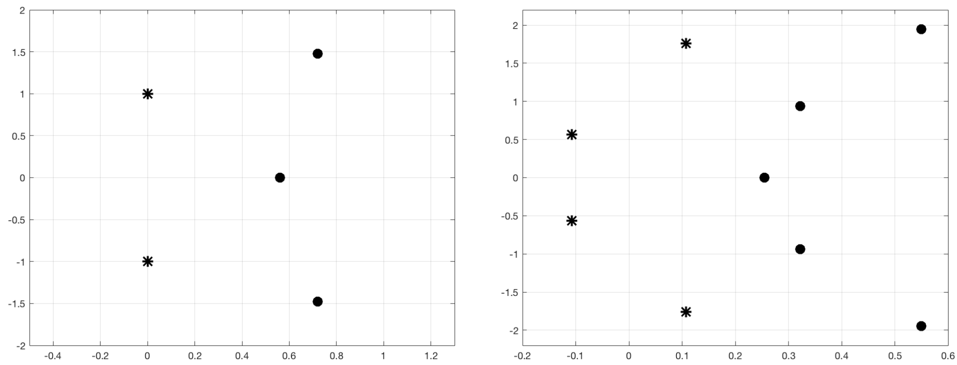

Figure 1. Configuration examples $(m,n)=(3,2)$ (left) and $(m,n)=(5,4)$ (right).

consider the scalar transport equation for $w(x',t)$,

$$\begin{cases} w_t + \nabla^\perp\psi\cdot\nabla w = 0 & \text{in } \mathbb{R}^2\times(0,T), \\ -\operatorname{div}(K\nabla\psi) = w & \text{in } \mathbb{R}^2\times(0,T), \end{cases} \tag{1.15}$$

where $(a,b)^\perp = (b,-a)$ and $K(x_1,x_2)$ is the matrix

$$K(x_1,x_2) = \frac{1}{h^2+x_1^2+x_2^2}\begin{pmatrix} h^2+x_2^2 & -x_1x_2 \\ -x_1x_2 & h^2+x_1^2 \end{pmatrix}.$$

Then there exists a vector field $\mathbf{u}=(u_1,u_2,u_3)$ with helical symmetry

$$\mathbf{u}(Q_\theta x', x_3+h\theta) = \begin{pmatrix} Q_\theta(u_1,u_2) \\ u_3+h\theta \end{pmatrix} \quad \forall\theta\in\mathbb{R},\ \forall x=(x',x_3)\in\mathbb{R}^3,$$

such that

$$\vec{\omega}(x,t) = \frac{1}{h}w(Q_{-\frac{x_3}{h}}x',t)\begin{pmatrix} Q_{\frac{\pi}{2}}x' \\ h \end{pmatrix}, \quad x=(x',x_3), \tag{1.16}$$

satisfies the Euler equation (1.2). Here, Q_θ is the rotation matrix in the plane (x_1,x_2) as defined in (1.6). The derivation of (1.15) can be found for instance in [17,20,21].

Rotating solutions to problem (1.15) with constant speed α have the form

$$w(x',\tau) = W(Q_{\alpha\tau}x'), \quad \psi(x',\tau) = \Psi(Q_{\alpha\tau}x'). \tag{1.17}$$

Let $\tilde{x} = P_{\alpha\tau}x'$. In terms of (W,Ψ), the second equation in (1.15) becomes

$$-\operatorname{div}_{\tilde{x}}(K(\tilde{x})\nabla_{\tilde{x}}\Psi) = W,$$

and the first equation gets the form

$$\nabla_{\tilde{x}}W\cdot\nabla_{\tilde{x}}^\perp\Big(\Psi - \alpha|\log\varepsilon|\frac{|\tilde{x}|^2}{2}\Big) = 0. \tag{1.18}$$

See [17] for details. We now observe that if $W(\tilde{x}) = F(\Psi(\tilde{x}) - \alpha|\log\varepsilon|\frac{|\tilde{x}|^2}{2})$, for some function F, then automatically (1.18) holds. We conclude that if Ψ is a solution to

$$-\operatorname{div}(K(\tilde{x})\nabla_{\tilde{x}}\Psi) = F\Big(\Psi - \frac{\alpha}{2}|\log\varepsilon|\,|\tilde{x}|^2\Big) \quad \text{in } \mathbb{R}^2, \tag{1.19}$$

for some function F, and W is given by

$$W(\tilde{x}) = F\Big(\Psi(\tilde{x}) - \alpha|\log\varepsilon|\frac{|\tilde{x}|^2}{2}\Big),$$

then (w, ψ) defined by (1.17) is a solution for (1.15).

We now notice that a solution to (1.19) such that

$$F\Big(\Psi - \frac{\alpha}{2}|\log\varepsilon|\,|\tilde{x}|^2\Big) - \kappa_i\delta_{(a_i,b_i)} \rightharpoonup 0, \quad \text{as } \varepsilon \to 0,$$

gives a solution $\vec{\omega}(x,t)$ to (1.2) of the form (1.16) with the property that

$$\vec{\omega}(x,t) - \kappa_i\delta_{\Gamma_i}\mathbf{t}_{\Gamma_i} \rightharpoonup 0, \quad \text{as } \varepsilon \to 0,$$

and Γ_i defined as in (1.5) with

$$\frac{\sigma_i}{\kappa_i} \to \alpha h, \qquad \frac{\beta_i}{\kappa_i} \to \frac{2r_0^2}{r_0^2 + h^2}, \qquad \text{as } \varepsilon \to 0.$$

The proof of Theorem 1.1 is reduced to finding a non-linear function F and a solution Ψ to (1.19) such that

$$W(x) := F\Big(\Psi - \frac{\alpha}{2}|\log\varepsilon|\,|\tilde{x}|^2\Big) \sim \sum_{i=1}^{N}\kappa_j\delta_{(a_i,b_i)}, \quad \text{as } \varepsilon \to 0, \tag{1.20}$$

where $(a_i, b_i) \to (r_0, 0)$ for all i. We build such a solution by means of elliptic singular perturbation techniques. For $N = 1$ we recover the result for a single helical filament in [17]. The multiple helical filaments constructed in [17] correspond to a configuration where the centers of the helices maintain a constant distance between each other, and their motion is independent of the presence of the others. However, when dealing with colliding helices, more refined estimates are necessary to control the strong interactions between them. To achieve this, we introduced a new change of variables to obtain a unified expression of the differential operator that is independent of the individual helix. Desingularization of point vortices for the Euler equations in dimension 2 has been treated in [16, 33, 36].

Solutions concentrated near helices in the Euler equations and also other PDE settings have been built in [9–12, 15, 26, 38]. Solutions to the Euler equations (1.2) concentrated around several vortex rings which are collapsing into each other are known in the literature. The first result is due to Buffoni [8], who constructed co-axial vortex rings (sets homeomorphic to solid tori) moving along their common axis at the same propagation speed. These rings are nested in the sense that the convex hull of one ring contains the subsequent ring and at the same time the two rings do not intersect. A more recent result is contained in [3], where they also find the formal law for the dynamics of the centers of a family of clustering rings, which turns out to be the same law (1.13) that governs the helical clustering phenomena. On the other hand the elliptic problems governing clusters of vortex rings and clusters of helical filaments are different, and require a different analysis, which is reflected in particular in the far field region, also called the outer region. The same law of motion also governs the interaction of multiple vortex rings in other contexts, like in the Gross–Pitaevskii equation [2, 26]. The law for the interaction of nearly parallel vortex filaments has been studied in [29, 39]. Our configuration of multiple helices falls out of the framework considered in these papers.

Helical flow can be understood as an interpolation between no-swirl axisymmetric flow and two-dimensional flow. This interpolation explains the analogy between phenomena related to a single vortex ring and a single translating-rotating helix. However, this analogy breaks down at the level of multiple filaments, as demonstrated by the construction in [17] of several helices with centers localized at the vertices of a regular polygon – a scenario unattainable with multiple rings. Analogies and differences between multiple vortex rings and multiple helical filaments seem less clear. In this context, our result demonstrates that this analogy persists for multiple filaments in the case of a cluster of vortex rings and rotating-translating helices, provided this occurs far from the origin.

As we already discussed, Theorem 1.1 follows from proving the existence of a function Ψ and a non-linearity F to solve (1.19) and (1.20). This is what the rest of the paper is devoted to. In Section 2 we find a smooth stream function Ψ solving approximately

$$-\operatorname{div}(K(x)\nabla\Psi) \sim \sum_{i=1}^{N} \kappa_i \delta_{(a_i,b_i)},$$

in coherence with the expectation (1.20). In Section 3 we choose the non-linearity F. It will be reminiscent of $f(s) = e^s$ and the Liouville equation $\Delta u + e^u = 0$ in $\mathbb{R}^2$ will be used as a limit problem to describe the profile of the helical filaments, near the centers of the vortex helices (see (3.7)–(3.8)). We define a first approximate solution in Section 3, and estimate the error of approximation in Section 4. After the approximate solution is built, we proceed to find an actual solution close to the approximation. The actual solution is found using the inner-outer gluing method, which has been used in several other contexts. References for problems related to inviscid incompressible fluids are [16] for the problem of point vortex desingularization for the Euler equations in dimension 2, [18] for the leapfrogging of vortex rings, and also [17]. Since the interaction among different helices is strong (as their relative distance is small), it is relevant for us to pose the inner-outer gluing method so that these interactions can be controlled. Section 6 contains two basic elliptic linear theories which are at the core of the resolution of the inner-outer scheme. Sections 7 and 8 are devoted to finding an actual solution to the problem, where the choice of the centers of the helices to solve (1.13) plays a central role.

2. Finding the approximate stream function

Let $h > 0$, $r_0 > 0$, $\kappa_1, \dots, \kappa_N$ be given numbers, and define α to be the constant defined in (1.14).

The rest of the paper is devoted to finding a solution Ψ to the semi-linear elliptic equation

$$\nabla \cdot (K \nabla \Psi) + F\Big(\Psi - \frac{\alpha}{2} |\log \varepsilon|\, |x|^2\Big) = 0 \quad \text{in } \mathbb{R}^2. \tag{2.1}$$

More precisely, we look for a non-linear function F and a solution Ψ to (2.1) with the property that, if we set

$$W(x) = F\Big(\Psi - \frac{\alpha}{2} |\log \varepsilon|\, |x|^2\Big),$$

then

$$W(x) \sim 8\pi \sum_{j=1}^{N} \kappa_j \delta_{P_j}, \quad \text{as } \varepsilon \to 0, \tag{2.2}$$

for some $P_j \in \mathbb{R}^2$. The points P_j are assumed to be close to each other and collapse to the same point, as $\varepsilon \to 0$. We assume they have the form

$$P_j = (r_0 + s, 0) + \frac{\widehat{P}_j}{|\log \varepsilon|}, \quad P = (P_1, \dots, P_N), \quad \widehat{P} = (\widehat{P}_1, \dots, \widehat{P}_N). \tag{2.3}$$

We assume the following bounds on s and $\widehat{P}$:

$$\|\widehat{P}\| \lesssim 1, \quad |s| \lesssim \frac{\log|\log \varepsilon|}{|\log \varepsilon|} \tag{2.4}$$

In other words, we look for the stream function Ψ to have the asymptotic behavior

$$\Psi(x) \sim \sum_{j=1}^{N} \kappa_j \Psi_j(x), \quad \text{with } -\nabla \cdot (K \nabla \Psi_j) \sim 8\pi \delta_{P_j}, \text{ as } \varepsilon \to 0.$$

We expect each function Ψ_j to be, locally around P_j, an approximate Green's function for the operator $\nabla \cdot (K \nabla \cdot)$ in $\mathbb{R}^2$.

This section is devoted to analyzing the approximate Green's function for the operator $\nabla \cdot (K \nabla \cdot)$ in $\mathbb{R}^2$ and to constructing an approximate stream function for the N-helical filaments.

2.1. Approximate Green's function for the operator $\nabla \cdot (K \nabla \cdot)$ in $\mathbb{R}^2$

The purpose of this subsection is to find an explicit regular function which locally around a point $P = (a, b) \in \mathbb{R}^2$ satisfies approximately

$$-\nabla \cdot (K \nabla \Psi) = 8\pi \delta_P. \tag{2.5}$$

For this purpose we need to understand the structure of the operator in divergence form

$$L := -\nabla \cdot (K\nabla), \quad \text{where } K = \frac{1}{h^2 + x_1^2 + x_2^2} \begin{pmatrix} h^2 + x_2^2 & -x_1 x_2 \\ -x_1 x_2 & h^2 + x_1^2 \end{pmatrix}, \tag{2.6}$$

when evaluated around a given point P. We will show that, after an ad hoc change of variable, the operator L will look like the usual Laplace operator in $\mathbb{R}^2$ when considered in a neighborhood of P.

Let us introduce the change of variables

$$x_1 - a = \frac{ah}{R\sqrt{h^2 + R^2}} z_1 - \frac{b}{R} z_2, \quad x_2 - b = \frac{bh}{R\sqrt{h^2 + R^2}} z_1 + \frac{a}{R} z_2,$$

where $R = \sqrt{a^2 + b^2}$. This is equivalent to saying

$$z_1 = \frac{\sqrt{h^2 + R^2}}{hR} [a(x_1 - a) + b(x_2 - b)], \quad z_2 = \frac{1}{R} [-b(x_1 - a) + a(x_2 - b)].$$

We will also use the matrix notation

$$x - P = A[P]z, \quad A[P] = \begin{pmatrix} \frac{ah}{R\sqrt{h^2+R^2}} & -\frac{b}{R} \\ \frac{bh}{R\sqrt{h^2+R^2}} & \frac{a}{R} \end{pmatrix}. \tag{2.7}$$

Proposition 2.1. *In the z-variable, the operator L takes the form*

$$L = \Delta_z + B, \tag{2.8}$$

where

$$\begin{aligned} B = &\Big(\frac{h^2(R^2 - r^2) + z_2^2(h^2 + R^2)}{(h^2 + r^2)h^2} \Big) \partial_{z_1 z_1} \\ &+ \frac{1}{(h^2 + r^2)} \left(\left(z_1 \frac{h}{\sqrt{h^2 + R^2}} + R \right)^2 - r^2 \right) \partial_{z_2 z_2} \\ &- 2 \frac{\sqrt{h^2 + R^2}}{h(h^2 + r^2)} z_2 \left(z_1 \frac{h}{\sqrt{h^2 + R^2}} + R \right) \partial_{z_1 z_2} \\ &- \frac{z_1(h^2 + R^2) + Rh\sqrt{h^2 + R^2}}{h^2(h^2 + r^2)} \Big(\frac{2h^2}{h^2 + r^2} + 1 \Big) \partial_{z_1} \\ &- \frac{z_2}{h^2 + r^2} \Big(\frac{2h^2}{h^2 + r^2} + 1 \Big) \partial_{z_2}. \end{aligned} \tag{2.9}$$

Here $r = r(z)$ is

$$r^2 = |x|^2 = R^2 + 2R \frac{h}{\sqrt{h^2 + R^2}} z_1 + q_2(z), \quad \textit{with } q_2(z) = \frac{h^2}{h^2 + R^2} z_1^2 + z_2^2.$$

Proof. The operator L is explicitly given by

$$L = \frac{h^2 + x_2^2}{h^2 + r^2}\partial_{x_1x_1} + \frac{h^2 + x_1^2}{h^2 + r^2}\partial_{x_2x_2} - 2\frac{x_1x_2}{h^2 + r^2}\partial_{x_1x_2} - \frac{x_1}{h^2 + r^2}\Big(\frac{2h^2}{h^2 + r^2} + 1\Big)\partial_{x_1} - \frac{x_2}{h^2 + r^2}\Big(\frac{2h^2}{h^2 + r^2} + 1\Big)\partial_{x_2}. \tag{2.10}$$

Indeed, using the notation $K = \left(\begin{smallmatrix} K_{11} & K_{12} \\ K_{12} & K_{22}\end{smallmatrix}\right)$, we get

$$\begin{aligned} L = \nabla\cdot(K\nabla\cdot) &= K_{11}\partial^2_{x_1} + K_{22}\partial^2_{x_2} + 2K_{12}\partial^2_{x_1x_2} \\ &\quad + (\partial_{x_1}K_{11} + \partial_{x_2}K_{12})\partial_{x_1} + (\partial_{x_2}K_{22} + \partial_{x_1}K_{12})\partial_{x_2} \\ &= \frac{h^2 + x_2^2}{h^2 + r^2}\partial_{x_1x_1} + \frac{h^2 + x_1^2}{h^2 + r^2}\partial_{x_2x_2} - 2\frac{x_1x_2}{h^2 + r^2}\partial_{x_1x_2} \\ &\quad + \Big(\partial_{x_1}\Big(\frac{h^2 + x_2^2}{h^2 + r^2}\Big) - \partial_{x_2}\Big(\frac{x_1x_2}{h^2 + r^2}\Big)\Big)\partial_{x_1} \\ &\quad + \Big(\partial_{x_2}\Big(\frac{h^2 + x_1^2}{h^2 + r^2}\Big) - \partial_{x_1}\Big(\frac{x_1x_2}{h^2 + r^2}\Big)\Big)\partial_{x_2}, \end{aligned}$$

where $r = |x|$. Formula (2.10) follows directly from the facts that

$$\partial_{x_1}\Big(\frac{h^2 + x_2^2}{h^2 + r^2}\Big) - \partial_{x_2}\Big(\frac{x_1x_2}{h^2 + r^2}\Big) = -\frac{x_1}{h^2 + r^2}\Big(\frac{2h^2}{h^2 + r^2} + 1\Big)$$

and

$$\partial_{x_2}\Big(\frac{h^2 + x_1^2}{h^2 + r^2}\Big) - \partial_{x_1}\Big(\frac{x_1x_2}{h^2 + r^2}\Big) = -\frac{x_2}{h^2 + r^2}\Big(\frac{2h^2}{h^2 + r^2} + 1\Big).$$

Formulas (2.8) and (2.9) are consequences of the following straightforward computations:

$$\begin{aligned} \partial_{x_1} &= \frac{a}{R}\frac{\sqrt{h^2 + R^2}}{h}\partial_{z_1} - \frac{b}{R}\partial_{z_2}, \\ \partial_{x_2} &= \frac{b}{R}\frac{\sqrt{h^2 + R^2}}{h}\partial_{z_1} + \frac{a}{R}\partial_{z_2}, \\ \partial_{x_1x_1} &= \frac{a^2}{R^2}\frac{(h^2 + R^2)}{h^2}\partial_{z_1z_1} + \frac{b^2}{R^2}\partial_{z_2z_2} - 2\frac{ab}{R^2}\frac{\sqrt{h^2 + R^2}}{h}\partial_{z_1z_2}, \\ \partial_{x_2x_2} &= \frac{b^2}{R^2}\frac{(h^2 + R^2)}{h^2}\partial_{z_1z_1} + \frac{a^2}{R^2}\partial_{z_2z_2} + 2\frac{ab}{R^2}\frac{\sqrt{h^2 + R^2}}{h}\partial_{z_1z_2}, \\ \partial_{x_1x_2} &= \frac{ab}{R^2}\frac{(h^2 + R^2)}{h^2}\partial_{z_1z_1} - \frac{ab}{R^2}\partial_{z_2z_2} + \frac{a^2 - b^2}{R^2}\frac{\sqrt{h^2 + R^2}}{h}\partial_{z_1z_2}. \end{aligned}$$ ■

The operator B in (2.9), Proposition 2.1, becomes a small perturbation of the Laplacian, when we restrict our attention to a small region around the point P, that in the z-variable can be described with $|z| < \delta$, for a fixed δ small. Indeed, in this region the operator B has the form

$$\begin{aligned} B = &\Big(-2\frac{Rh}{(h^2+R^2)^{3/2}}z_1 + O(|z|^2)\Big)\partial_{z_1 z_1} + O(|z|^2)\partial_{z_2 z_2} \\ &- \Big(2\frac{R}{h\sqrt{h^2+R^2}}z_2 + O(|z|^2)\Big)\partial_{z_1 z_2} \\ &- \Big(\frac{R}{h\sqrt{h^2+R^2}}\Big(\frac{2h^2}{h^2+R^2}+1\Big) + O(|z|)\Big)\partial_{z_1} \\ &- \Big(\frac{z_2}{h^2+R^2}\Big(\frac{2h^2}{h^2+R^2}+1\Big) + O(|z|^2)\Big)\partial_{z_2}. \end{aligned}$$

Equation (2.5) thus becomes

$$-(\Delta + B)\psi = 8\pi\delta_0, \quad \psi(z) = \Psi(P + A[P]z). \tag{2.11}$$

We now choose a regularization of the Green's function of the Laplace operator Δ_z as a starting point for the construction of the approximate regularization to (2.11). The regularization we choose is a radial solution of the Liouville equation

$$\Delta u + e^u = 0 \text{ in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} e^u < \infty. \tag{2.12}$$

All solutions to (2.12) that are radially symmetric with respect to the origin are given by

$$\Gamma_{\mu\varepsilon}(z) - 2\log\varepsilon\mu, \quad \text{where } \Gamma_{\varepsilon\mu}(z) = \log\frac{8}{(\varepsilon^2\mu^2 + |z|^2)^2}$$

for any value of the constants ε and $\mu > 0$. Indeed, we have

$$-\Delta\Gamma_{\varepsilon\mu} = \varepsilon^2\mu^2 e^{\Gamma_{\varepsilon\mu}} = \frac{1}{\varepsilon^2\mu^2}U\Big(\frac{z}{\varepsilon\mu}\Big), \quad \text{with } U(y) = \frac{8}{(1+|y|^2)^2}.$$

Hence

$$-\Delta\Gamma_{\varepsilon\mu} \rightharpoonup 8\pi\delta_0, \quad \text{as } \varepsilon\mu \to 0.$$

Proposition 2.2. *For any $\mu > 0$, we define the approximate regularization for (2.11) as*

$$\begin{aligned} \psi_\mu(z) &= \Gamma_{\varepsilon\mu}(z)(1 + c_1 z_1 + c_2|z|^2) + \frac{4R^3}{h(h^2+R^2)^{\frac{3}{2}}}H_1(z), \\ c_1 &= \frac{1}{2}\frac{Rh}{(h^2+R^2)^{\frac{3}{2}}}, \\ c_2 &= \frac{R^2}{8(h^2+R^2)^2}\Big(\frac{2h^2}{h^2+R^2}+1\Big), \end{aligned} \tag{2.13}$$

and H_1 solves

$$\Delta_z(H_1) + \frac{\mathrm{Re}(z^3)}{(\varepsilon^2\mu^2 + |z|^2)^2} = 0.$$

We have that

$$L(\psi_\mu)(z) = \Delta\Gamma_{\varepsilon\mu} + \frac{4R(3h^2+R^2)}{h(h^2+R^2)^{\frac{3}{2}}} \frac{\varepsilon^2\mu^2 z_1}{(\varepsilon^2\mu^2+|z|^2)^2} + E_3(z),$$

where E_3 is a smooth function, which is uniformly bounded as $\varepsilon\mu \to 0$, for $|z| < \delta$ and any $\delta > 0$ small.

In the original variables x, the function ψ_μ in (2.13) reads

$$\Psi_{\mu,P}(x) = \psi_\mu(A[P]^{-1}(x-P)), \tag{2.14}$$

where $A[P]$ is the matrix introduced in (2.7). The function $\Psi_{P,\mu}(x)$ is smooth and represents a good approximate Green's function for the operator $\nabla \cdot (K\nabla \cdot)$ in $\mathbb{R}^2$. We will use it as a building block for the construction of a solution to (2.1).

Proof of Proposition 2.2. Let $\delta > 0$ be small. We compute

$$\begin{aligned} B[\Gamma_{\varepsilon\mu}] = &-\frac{2Rh}{(h^2+R^2)^{\frac{3}{2}}} z_1\partial_{z_1z_1}\Gamma_{\varepsilon\mu} - \frac{2R}{h\sqrt{h^2+R^2}} z_2\partial_{z_1z_2}\Gamma_{\varepsilon\mu} \\ &- \frac{R}{h\sqrt{h^2+R^2}}\Big(1+\frac{2h^2}{h^2+R^2}\Big)\partial_{z_1}\Gamma_{\varepsilon\mu} + E_1, \end{aligned}$$

where E_1 is a smooth function, uniformly bounded for $\varepsilon\mu$ small, in a bounded region for $|z| < \delta$.

We take advantage of the explicit expression of $\Gamma_{\varepsilon\mu}$ to find

$$\partial_{z_1}\Gamma_{\varepsilon\mu}(z) = -\frac{4z_1}{\varepsilon^2\mu^2+|z|^2}, \quad z_1\partial_{z_1z_1}\Gamma_{\varepsilon\mu}(z) = -\frac{4z_1}{\varepsilon^2\mu^2+|z|^2} + \frac{8z_1^3}{(\varepsilon^2\mu^2+|z|^2)^2},$$
$$z_2\partial_{z_1z_2}\Gamma_{\varepsilon\mu}(z) = \frac{8z_2^2z_1}{(\varepsilon^2\mu^2+|z|^2)^2}.$$

Using that

$$z_1z_2^2 = \frac{|z|^2z_1}{4} - \frac{\mathrm{Re}(z^3)}{4}, \quad z_1^3 = \frac{3|z|^2z_1}{4} + \frac{\mathrm{Re}(z^3)}{4},$$

we obtain

$$\begin{aligned} &-\frac{2Rh}{(h^2+R^2)^{3/2}} z_1\partial_{z_1z_1}\Gamma_{\varepsilon\mu} - \frac{2R}{h\sqrt{h^2+R^2}} z_2\partial_{z_1z_2}\Gamma_{\varepsilon\mu} \\ &\quad - \frac{R}{h\sqrt{h^2+R^2}}\Big(1+\frac{2h^2}{h^2+R^2}\Big)\partial_{z_1}\Gamma_{\varepsilon\mu} \\ &= \Big[\frac{8hR}{(h^2+R^2)^{\frac{3}{2}}} + \frac{4R(3h^2+R^2)}{h(h^2+R^2)^{\frac{3}{2}}}\Big]\frac{z_1}{\varepsilon^2\mu^2+|z|^2} \\ &\quad - \frac{16hR}{(h^2+R^2)^{\frac{3}{2}}}\frac{z_1^3}{(\varepsilon^2\mu^2+|z|^2)^2} - \frac{16R}{h(h^2+R^2)^{\frac{1}{2}}}\frac{z_1z_2^2}{(\varepsilon^2\mu^2+|z|^2)^2} \end{aligned}$$

$$= \frac{4hR}{(h^2+R^2)^{\frac{3}{2}}} \frac{z_1}{\varepsilon^2\mu^2+|z|^2} + \frac{4R^3}{h(h^2+R^2)^{\frac{3}{2}}} \frac{\mathrm{Re}(z^3)}{(\varepsilon^2\mu^2+|z|^2)^2} + \frac{4R(4h^2+R^2)}{h(h^2+R^2)^{\frac{3}{2}}} \frac{\varepsilon^2\mu^2 z_1}{(\varepsilon^2\mu^2+|z|^2)^2}.$$

We can modify the function $\Gamma_{\varepsilon\mu}$ to eliminate part of the above error. We define

$$\psi_1(z) = \Gamma_{\varepsilon\mu}(z)(1+c_1 z_1), \quad \text{with } c_1 = \frac{1}{2}\frac{Rh}{(h^2+R^2)^{\frac{3}{2}}}. \tag{2.15}$$

Since

$$(\partial_{z_1z_1}+\partial_{z_2z_2})(c_1 z_1\Gamma_{\varepsilon\mu}) = -8c_1\frac{z_1}{\varepsilon^2\mu^2+|z|^2} - 8c_1\frac{\varepsilon^2\mu^2 z_1}{(\varepsilon^2\mu^2+|z|^2)^2},$$

with this choice of c_1, we get

$$L(\psi_1)(z) = \Delta\Gamma_{\varepsilon\mu} + \frac{4R^3}{h(h^2+R^2)^{\frac{3}{2}}} \frac{\mathrm{Re}(z^3)}{(\varepsilon^2\mu^2+|z|^2)^2} + \frac{4R(3h^2+R^2)}{(h^2+R^2)^{\frac{3}{2}}} \frac{\varepsilon^2\mu^2 z_1}{(\varepsilon^2\mu^2+|z|^2)^2} + c_1 B(z_1\Gamma_{\varepsilon\mu}) + E_1,$$

where E_1 is an explicit function, which is smooth in the variable z and uniformly bounded, as $\varepsilon\mu \to 0$.

We now use the fact that $z_2\partial_{z_2}\Gamma_{\varepsilon\mu}(z) = -4z_2^2/(\varepsilon^2\mu^2+|z|^2)$ to write

$$\begin{aligned} c_1 B(z_1\Gamma_{\varepsilon\mu}) &= c_1 z_1 B(\Gamma_{\varepsilon\mu}) - c_1\frac{4Rh}{(h^2+R^2)^{3/2}} z_1\partial_{z_1}\Gamma_{\varepsilon\mu} - c_1\frac{2R}{h\sqrt{h^2+R^2}} z_2\partial_{z_2}\Gamma_{\varepsilon\mu} \\ &\quad - c_1\frac{R}{h\sqrt{h^2+R^2}}\Big(1+\frac{2h^2}{h^2+R^2}\Big)\Gamma_{\varepsilon\mu} + \bar{E}_1 \\ &= -\frac{R^2}{2(h^2+R^2)^2}\Big(\frac{2h^2}{h^2+R^2}+1\Big)\Gamma_{\varepsilon\mu} + E_2, \end{aligned}$$

where E_2 is another explicit function, smooth in the variable z and uniformly bounded, as $\varepsilon\mu \to 0$.

Combining these computations we obtain that the function ψ_1 introduced in (2.15) satisfies

$$\begin{aligned} L(\psi_1)(z) &= \Delta\Gamma_{\varepsilon\mu} + \frac{4R^3}{h(h^2+R^2)^{\frac{3}{2}}} \frac{\mathrm{Re}(z^3)}{(\varepsilon^2\mu^2+|z|^2)^2} + \frac{4R(3h^2+R^2)}{h(h^2+R^2)^{\frac{3}{2}}} \frac{\varepsilon^2\mu^2 z_1}{(\varepsilon^2\mu^2+|z|^2)^2} \\ &\quad - \frac{R^2}{2(h^2+R^2)^2}\Big(\frac{2h^2}{h^2+R^2}+1\Big)\Gamma_{\varepsilon\mu} + E_1 + E_2, \end{aligned}$$

where E_1 and E_2 are explicit functions, smooth in the variable z and uniformly bounded, as $\varepsilon\mu \to 0$.

Our next step is to introduce a further modification to ψ_1 to eliminate the two terms

$$-\frac{R^2}{2(h^2+R^2)^2}\Big(\frac{2h^2}{h^2+R^2}+1\Big)\Gamma_{\varepsilon\mu} \quad \text{and} \quad \frac{4R^3}{h(h^2+R^2)^{\frac{3}{2}}}\frac{\mathrm{Re}(z^3)}{(\varepsilon^2\mu^2+|z|^2)^2}.$$

For the first one, we observe that

$$\begin{aligned}\Delta(c_2|z|^2\Gamma_{\varepsilon\mu}) &- \frac{R^2}{2(h^2+R^2)^2}\Big(\frac{2h^2}{h^2+R^2}+1\Big)\Gamma_{\varepsilon\mu}\\ &= \Big(4c_2-\frac{R^2}{2(h^2+R^2)^2}\Big(\frac{2h^2}{h^2+R^2}+1\Big)\Big)\Gamma_{\varepsilon\mu}\\ &\quad + 2c_2 z\cdot\nabla\Gamma_{\varepsilon\mu}+c_2|z|^2\Delta\Gamma_{\varepsilon\mu}\end{aligned}$$

and choose c_2 as

$$c_2=\frac{R^2}{8(h^2+R^2)^2}\Big(\frac{2h^2}{h^2+R^2}+1\Big).$$

To correct the second term, we introduce

$$h_1(s)=s^3\int_s^1\frac{dx}{x^7}\int_0^x\frac{\eta^7}{(\varepsilon^2\mu^2+\eta^2)^2}\,d\eta.$$

It solves

$$h_1''+\frac{1}{s}h_1'-\frac{9}{s^2}h_1+\frac{s^3}{(\varepsilon^2\mu^2+s^2)^2}=0,$$

it is smooth and uniformly bounded as $\varepsilon\to 0$, and $h_1(s)=O(s^3)$, as $s\to 0$. Writing $z=|z|e^{i\theta}$, we have that

$$H_1(z):=h_1(|z|)\cos 3\theta \quad \text{solves} \quad \Delta_z(H_1)+\frac{\mathrm{Re}(z^3)}{(\varepsilon^2\mu^2+|z|^2)^2}=0. \qquad \blacksquare$$

2.2. Approximate stream function for N-helical filaments

In this section we introduce the approximate stream function for N helical filaments. It is built as a superposition of the building blocks introduced in (2.14).

Let N be a fixed integer and consider N points $P_1,\dots,P_N$ of the form (2.3) and satisfying (2.4). For any $j=1,\dots,N$, we write

$$P_j=(a_j,b_j),$$

we fix positive constants μ_j, and we define

$$\Psi_j(x)=\Psi_{P_j,\mu_j}(x),$$

where Ψ_{P_j,μ_j} is defined in (2.14). Hence, for each j we have

$$L(\Psi_j)=\Delta\Gamma_{\varepsilon\mu_j}+\frac{4R_j(3h^2+R_j^2)}{h(h^2+R_j^2)^{\frac{3}{2}}}\frac{\varepsilon^2\mu_j^2 z_1}{(\varepsilon^2\mu_j^2+|z|^2)^2}+E_{3,j}(z), \qquad (2.16)$$

where

$$R_j = \sqrt{a_j^2 + b_j^2}, \quad z = A[P_j]^{-1}(x - P_j).$$

The stream function of N-helical filaments looks at main order as a superposition of stream functions Ψ_j associated to each helical filament.

Since the relative distance of the points P_j is of order $|\log \varepsilon|^{-1}$, we multiply each Ψ_j by a cut-off function to get

$$\eta_0(x) \sum_{j=1}^{N} \kappa_j \Psi_j(x),$$

where

$$\eta_0(x) = \eta(|x - (r_0, 0)|), \tag{2.17}$$

with η a fixed smooth function with

$$\eta(s) = 1 \text{ for } s \leq \frac{1}{2}, \quad \eta(s) = 0 \text{ for } s \geq 1. \tag{2.18}$$

Using the notation introduced in (2.16), we have

$$L\left(\eta_0 \sum_{j=1}^{N} \kappa_j \Psi_j\right) = \eta_0 \sum_{j=1}^{N} \kappa_j \left(\Delta \Gamma_{\varepsilon \mu_j} + \frac{4R_j(3h^2 + R_j^2)}{h(h^2 + R_j^2)^{\frac{3}{2}}} \frac{\varepsilon^2 \mu_j^2 z_1}{(\varepsilon^2 \mu_j^2 + |z|^2)^2}\right) + g(x),$$

where

$$g(x) = \eta_0 \sum_{j=1}^{N} \kappa_j E_{3,j}(z) + \sum_{j=1}^{N} \kappa_j [L(\eta_0 \Psi_j) - \eta_0 L(\Psi_j)].$$

The function g has compact support and satisfies

$$\|g(x)\|_{L^\infty(\mathbb{R}^2)} \leq C$$

for some positive constant.

It is convenient to slightly modify the ansatz $\eta_0 \sum_{j=1}^{N} \kappa_j \Psi_j$, adding a term which is defined globally in the entire space $\mathbb{R}^2$ to cancel $g(x)$. Let $H_{2\varepsilon}(x)$ solve

$$L(H_{2\varepsilon}) + g = 0 \quad \text{in } \mathbb{R}^2. \tag{2.19}$$

For a smooth function $h(x)$ satisfying the decay condition

$$\|h\|_\nu := \sup_{x \in \mathbb{R}^2} (1 + |x|)^\nu |h(x)| < +\infty,$$

for some $\nu > 2$, there exists a solution $\psi(x)$ to the problem

$$L(\psi) + h = 0 \quad \text{in } \mathbb{R}^2,$$

which is of class $C^{1,\beta}(\mathbb{R}^2)$ for any $0<\beta<1$, and defines a linear operator $\psi = \mathcal{T}^o(g)$ of g and satisfies the bound

$$|\psi(x)| \le C\|h\|_\nu(1+|x|^2),$$

for some positive constant C. The proof of this fact can be found in Proposition 6.1.

Using this result we obtain that the solution $H_{2\varepsilon}$ to (2.19) satisfies the estimate

$$|H_{2\varepsilon}(x)| \le C(1+|x|^2).$$

In addition, observe that such a solution is given up to the addition of a constant. We define the function $H_{2\varepsilon}(x)$ to be the one which furthermore satisfies

$$H_{2\varepsilon}((x_0,0)) = 0.$$

With this is mind, we get to the definition of a first approximate stream function for N-helical filaments

$$\Psi_0(x) = \eta_0(x)\sum_{j=1}^N \kappa_j \Psi_j(x) + H_{2\varepsilon}(x), \tag{2.20}$$

so that

$$L(\Psi_0) = \eta_0 \sum_{j=1}^N \kappa_j\Big(\Delta\Gamma_{\varepsilon\mu_j} + \frac{4R_j(3h^2+R_j^2)}{h(h^2+R_j^2)^{\frac{3}{2}}}\frac{\varepsilon^2\mu_j^2(z_j)_1}{(\varepsilon^2\mu_j^2+|z_j|^2)^2}\Big), \tag{2.21}$$

where $z_j = A[P_j]^{-1}(x-P_j)$, $z_j = ((z_j)_1,(z_j)_2)$. We recall that the definition of η_0 is given in (2.17). The function Ψ_0 is defined in the whole of $\mathbb{R}^2$ and it is smooth. Recalling that $\Psi_j(x) = \Psi_{P_j,\mu_j}(x)$, we observe that its definition depends on certain parameters: the points $P_1,\ldots,P_N$ and the scaling positive parameters $\mu_1,\ldots,\mu_N$. We now proceed to define the non-linearity F in (2.1) and the first approximate solution to (2.1). More specifically, we will define the scaling parameters μ_i as functions of the points P_i, assuming P_i have the form (2.3)–(2.4).

3. Choice of the non-linearity and construction of the first approximate solution

In this section we define a non-linearity F in (2.1) with the property that the vorticity W, defined as

$$W(x) = F\Big(\Psi - \frac{\alpha}{2}|\log\varepsilon|\,|x|^2\Big),$$

satisfies (2.2), namely

$$W(x) \sim 8\pi\sum_{j=1}^N \kappa_j\delta_{P_j}, \quad \text{as } \varepsilon\to 0.$$

For this purpose we first identify the form of the function

$$\Psi_0 - \frac{\alpha}{2}|\log\varepsilon|\,|x|^2$$

near each point P_j, and we choose the scaling parameters μ_j in terms of the points $P_1,\dots,P_N$. It will turn out that a convenient choice for μ_j gives their size of the order

$$\log\mu_j \sim \log|\log\varepsilon|, \quad \text{as } \varepsilon\to 0.$$

3.1. Choice of μ_j in the definition of Ψ_0

We now define the scaling parameters μ_i to eliminate part of the zero-mode term of the expression for

$$\Psi_0(x) - \frac{\alpha}{2}|\log\varepsilon|\,|x|^2$$

when evaluated around a point P_i.

Take $\delta>0$ to be a fixed positive number and consider the inner region around P_i to be given by

$$|A_i^{-1}(x-P_i)| < \frac{\delta}{|\log\varepsilon|}, \tag{3.1}$$

where A_i is the matrix introduced in (2.7). We take $\delta \le \frac{\sqrt{h^2+r_0^2}}{h}\frac{d}{4}$, where d was fixed in (1.10), so that for $x\in\mathbb{R}^2$ satisfying (3.1) then $\eta_0(x)=1$. To represent a point in this region it is convenient to use the change of variables

$$x-P_i = A_i z, \quad z=\varepsilon\mu_i y. \tag{3.2}$$

Hence

$$|z| < \frac{\delta}{|\log\varepsilon|}.$$

Proposition 3.1. *Let*

$$2\kappa_i\log\mu_i = \sum_{j\neq i}\kappa_j\log\frac{8}{|A_j^{-1}(P_i-P_j)|^4} \times\left(1+c_{1,j}[A_j^{-1}(P_i-P_j)]_1 + c_{2,j}|A_j^{-1}(P_i-P_j)|^2\right)+H_{2\varepsilon}(P_i). \tag{3.3}$$

Then

$$\frac{1}{\kappa_i}\left(\Psi_0(x)-\frac{\alpha}{2}|\log\varepsilon|\,|x|^2\right) = (1+c_{1,i}\varepsilon\mu_i y_1 + c_{2,i}\varepsilon^2\mu_i^2|y|^2)\Gamma_0(y) - \frac{\alpha}{2\kappa_i}|\log\varepsilon|\,|P_i|^2 - 4\log\varepsilon - 2\log\mu_i$$

$$
\begin{aligned}
&+\varepsilon y_1\mu_i\Bigg[|\log\varepsilon|\left(4c_{1,i}-\alpha\frac{hR_i}{\kappa_i\sqrt{h^2+R_i^2}}\right)-4c_{1,i}\log\mu_i\\
&\qquad-\sum_{j\neq i}\frac{\kappa_j}{\kappa_i}4\frac{[A_j^{-1}(P_i-P_j)\cdot]_1}{|A_j^{-1}(P_i-P_j)|^2}\big(1+c_{1,j}[A_j^{-1}(P_i-P_j)]_1\\
&\qquad\qquad+c_{2,j}|A_j^{-1}(P_i-P_j)|^2\big)\\
&\qquad+\sum_{j\neq i}\frac{\kappa_j}{\kappa_i}\log\frac{8}{|A_j^{-1}(P_i-P_j)|^4}\{c_{1,j}+2c_{2,j}[A_j^{-1}(P_i-P_j)]_1\}\\
&\qquad+\frac{1}{\kappa_i}(A_i(1,0)^{\mathsf T})\cdot\nabla H_{2\varepsilon}(P_i)\Bigg]\\
&+\varepsilon y_2\mu_i\Bigg[-\sum_{j\neq i}\frac{\kappa_j}{\kappa_i}4\frac{[A_j^{-1}(P_i-P_j)\cdot]_2}{|A_j^{-1}(P_i-P_j)|^2}\big(1+c_{1,j}[A_j^{-1}(P_i-P_j)]_1\\
&\qquad\qquad+c_{2,j}|A_j^{-1}(P_i-P_j)|^2\big)\\
&\qquad+\sum_{j\neq i}2\frac{\kappa_j}{\kappa_i}\log\frac{8}{|A_j^{-1}(P_i-P_j)|^4}c_{2,j}[A_j^{-1}(P_i-P_j)]_2\\
&\qquad+\frac{1}{\kappa_i}(A_i(0,1)^{\mathsf T})\cdot\nabla H_{2\varepsilon}(P_i)\Bigg]\\
&+O(\log(|A_j^{-1}(P_i-P_j)|)|y|^2\varepsilon^2\mu_i^2),\quad \textit{for } |z|<\frac{\delta}{|\log\varepsilon|}.
\end{aligned}
$$

Observe that, since the points $P=(P_1,\dots,P_N)$ satisfy (2.3)–(2.4) we recognize that

$$
\log\mu_i^2=\log(|\log\varepsilon|)m_i(P),\quad \text{as } \varepsilon\to 0,
$$

where $m_i(P)$ are smooth functions, which are uniformly bounded together with their derivatives, as $\varepsilon\to 0$. We define

$$
\mu=\max_{i=1,\dots,N}\mu_i. \tag{3.4}
$$

Proof of Proposition 3.1. Fix $i\in\{1,\dots,N\}$ and let $A_i=A[P_i]$ be the matrix defined in (2.7). Under our assumptions on the points P_i in (2.3)–(2.4), and recalling that

$$
P_i=(r_0+s,0)+\frac{1}{|\log\varepsilon|}\widehat{P}_i,\quad P_i=(a_i,b_i),\ \widehat{P}_i=(\hat a_i,\hat b_i),
$$

we have

$$
A_i^{-1}=\begin{pmatrix}\frac{a_i\sqrt{h^2+R_i^2}}{R_ih} & \frac{b_i\sqrt{h^2+R_i^2}}{R_ih}\\ -\frac{b_i}{R_i} & \frac{a_i}{R_i}\end{pmatrix}=\begin{pmatrix}\frac{\sqrt{h^2+r_0^2}}{h} & 0\\ 0 & 1\end{pmatrix}+\frac{\log|\log\varepsilon|}{|\log\varepsilon|}\tilde A_i,
$$

and for $i\neq j$,

$$
A_j^{-1}A_i=\begin{pmatrix}1&0\\0&1\end{pmatrix}+\frac{\log|\log\varepsilon|}{|\log\varepsilon|}I_{ij},
$$

where $\tilde{A}_i$ and I_{ij} are 2×2 matrices whose entrances are smooth functions of $(s,\hat{a}_i,\hat{b}_i)$, and $(s,\hat{a}_i,\hat{b}_i,\hat{a}_j,\hat{b}_j)$ respectively, which are uniformly bounded as $\varepsilon\to 0$.

In the region $|z|<\frac{\delta}{|\log\varepsilon|}$, where z is introduced in (3.2), we have the following expansion for Ψ_0, as $\varepsilon\to 0$:

$$
\begin{aligned}
&\Psi_0(x)-\frac{\alpha}{2}|\log\varepsilon|\,|x|^2\\
&\quad=\kappa_i\Gamma_0(y)-\frac{\alpha}{2}|\log\varepsilon|\,|P_i|^2-4\kappa_i\log\varepsilon\mu_i\\
&\qquad-4\kappa_i\log\varepsilon\mu_i(c_{1,i}y_1\varepsilon\mu_i+c_{2,i}\varepsilon^2\mu_i^2|y|^2)+\kappa_i\frac{4R_i^3}{h(h^2+R_i^2)^{3/2}}H_{1i}(\varepsilon\mu_i y)\\
&\qquad-\alpha|\log\varepsilon|\frac{R_ih}{\sqrt{h^2+R_i^2}}\varepsilon\mu_i y_1-\frac{\alpha}{2}|\log\varepsilon|\varepsilon^2\mu_i^2|A_iy|^2\\
&\qquad+\kappa_i(c_{1,i}\varepsilon\mu_iy_1+c_{2,i}\varepsilon^2\mu_i^2|y|^2)\Gamma_0(y)+H_{2\varepsilon}(\varepsilon\mu_iA_iy+P_i)\\
&\qquad+\sum_{j\neq i}\kappa_j\log\frac{8}{|A_j^{-1}(P_i-P_j)|^4}\big(1+c_{1,j}[A_j^{-1}(P_i-P_j)]_1\\
&\qquad\qquad+c_{2,j}|A_j^{-1}(P_i-P_j)|^2\big)\\
&\qquad-\sum_{j\neq i}\kappa_j 4\frac{A_j^{-1}(P_i-P_j)\cdot A_j^{-1}A_iy}{|A_j^{-1}(P_i-P_j)|^2}\varepsilon\mu_i\big(1+c_{1,j}[A_j^{-1}(P_i-P_j)]_1\\
&\qquad\qquad+c_{2,j}|A_j^{-1}(P_i-P_j)|^2\big)\\
&\qquad+\sum_{j\neq i}\kappa_j\log\frac{8}{|A_j^{-1}(P_i-P_j)|^4}\{c_{1,j}[A_i^{-1}A_jy]_1\\
&\qquad\qquad+2c_{2,j}A_j^{-1}(P_i-P_j)\cdot A_j^{-1}A_iy\}\varepsilon\mu_i\\
&\qquad+O\big(\log(|A_j^{-1}(P_i-P_j)|)|A_j^{-1}A_iy|^2\varepsilon^2\mu_i^2\big),
\end{aligned}
\tag{3.5}
$$

where the constants $c_{1,j}$, $c_{2,j}$ are defined as in (2.13), namely

$$
\begin{aligned}
c_{1,j}&=\frac{1}{2}\frac{R_jh}{(h^2+R_j^2)^{\frac{3}{2}}},\\
c_{2,j}&=\frac{R_j^2}{8(h^2+R_j^2)^2}\Big(\frac{2h^2}{h^2+R_j^2}+1\Big),
\end{aligned}
\tag{3.6}
$$

with $R_j=\sqrt{a_j^2+b_j^2}$.

To get expansion (3.5) we have used that for small z, $|z|<\frac{\delta}{|\log\varepsilon|}$, one has

$$
\begin{aligned}
&\Gamma_{\varepsilon\mu_j}(A_j^{-1}[A_iz-(P_j-P_i)])\\
&\quad=\log\frac{8}{(\varepsilon^2\mu_j^2+|A_j^{-1}[A_iz-(P_j-P_i)]|^2)^2}\\
&\quad=\log\frac{8}{|A_j^{-1}(P_j-P_i)|^4}\\
&\qquad-2\log\Big(1-\frac{2A_j^{-1}(P_j-P_i)\cdot A_j^{-1}A_iz}{|A_j^{-1}(P_j-P_i)|^2}+\frac{|A_j^{-1}A_iz|^2+\varepsilon^2\mu_j^2}{|A_j^{-1}(P_j-P_i)|^2}\Big)
\end{aligned}
$$

$$= \log \frac{8}{|A_j^{-1}(P_j - P_i)|^4} + 4\frac{A_j^{-1}(P_j - P_i)}{|A_j^{-1}(P_j - P_i)|^2} \cdot A_j^{-1} A_i z + O\Big(\frac{\varepsilon^2 \mu_j^2 + |A_j^{-1} A_i z|^2}{|A_j^{-1}(P_j - P_i)|^2}\Big)$$

as $\varepsilon \to 0$.

Observe that in the region we are considering, we have the validity of the following expansion:

$$H_{2\varepsilon}(P_i + \varepsilon\mu_i A_i y) = H_{2\varepsilon}(P_i) + \varepsilon\mu_i (A_i y) \cdot \nabla H_{2\varepsilon}(P_i) + O(\varepsilon^2 \mu_i^2 |y|^2),$$
$$H_{1i}(\varepsilon\mu_i y) = O(\varepsilon^3 \mu_i^3 |y|^3), \quad \text{as } \varepsilon \to 0.$$

We now define the scaling parameters μ_i, to eliminate part of the zero-mode term of the expression in (3.5). Inserting (3.3) into (3.5) we conclude the proof of the proposition. ■

3.2. Choice of the non-linearity F

We now choose the non-linearity F in (2.1) which gives a vorticity W satisfying (2.2). We let

$$F\Big(\Psi - \frac{\alpha}{2}|\log\varepsilon|\,|x|^2\Big) = \sum_{j=1}^{N} \varepsilon^{2 - \frac{\alpha}{2\kappa_j} R_j^2} \kappa_j F_j\Big(\frac{1}{\kappa_j}\Big(\Psi - \frac{\alpha}{2}|\log\varepsilon|\,|x|^2\Big)\Big), \tag{3.7}$$

where

$$F_j(s) = \eta^j(s) f(s), \quad \text{where } f(s) = e^s. \tag{3.8}$$

Here η^j are smooth cut-off functions defined as follows.

Consider the boundary of the inner region around P_i, as defined in (3.1). Using the variable y in (3.2), this boundary is defined by $|y| = \delta/(\mu_i \varepsilon |\log\varepsilon|)$. On this boundary we have

$$\frac{1}{\kappa_i}\Big(\Psi_0(x) - \frac{\alpha}{2}|\log\varepsilon|\,|x|^2\Big) = -\frac{\alpha R_i^2}{2\kappa_i}|\log\varepsilon| + 4\log|\log\varepsilon| + 2\log\mu_i + \log 8 - 4\log\delta + o(1),$$

where $o(1)$ is with respect to $\varepsilon \to 0$. Then we choose the cut-off function η^i such that

$$\eta^i(s) = \begin{cases} 1 & \text{for } s \geq -\dfrac{\alpha R_i^2}{2\kappa_i}|\log\varepsilon| + 4\log|\log\varepsilon| + 2\log\mu_i + \log 8 + 2d_{i,\varepsilon}, \\ 0 & \text{for } s \leq -\dfrac{\alpha R_i^2}{2\kappa_i}|\log\varepsilon| + 4\log|\log\varepsilon| + 2\log\mu_i + \log 8 + d_{i,\varepsilon}, \end{cases} \tag{3.9}$$

for suitable $d_{i,\varepsilon} = -4\log\delta + o(1)$ so that

$$\eta^i\Big(\frac{1}{\kappa_i}\Big(\Psi_0(x) - \frac{\alpha}{2}|\log\varepsilon|\,|x|^2\Big)\Big) = 1 \quad \text{for } |A_i^{-1}(x - P_i)| \leq \frac{\delta^2}{|\log\varepsilon|}$$

and

$$\eta^i\Big(\frac{1}{\kappa_i}\Big(\Psi_0(x)-\frac{\alpha}{2}|\log\varepsilon|\,|x|^2\Big)\Big)=0 \quad \text{for } |A_i^{-1}(x-P_i)|\geq \frac{\delta}{|\log\varepsilon|}.$$

Here, δ is independent of ε as in (3.1) and can be taken smaller if needed.

4. Estimate of the error function

Let us define the error function to be

$$S[\Psi](x)=L(\Psi)+\sum_{j=1}^N \varepsilon^{2-\frac{\alpha}{2\kappa_j}R_j^2}\kappa_j F_j\Big(\frac{1}{\kappa_j}\Big(\Psi-\frac{\alpha}{2}|\log\varepsilon|\,|x|^2\Big)\Big). \tag{4.1}$$

A solution to (2.1) would correspond to a smooth function Ψ such that

$$S[\Psi](x)=0,\quad x\in\mathbb{R}^2.$$

Our purpose is to estimate

$$S[\Psi_0](x)\quad \text{for } x\in\mathbb{R}^2,$$

where Ψ_0 is the approximate stream function for the N-helical filaments introduced in (2.20).

Proposition 4.1. *Let $\delta>0$ be given. There exists $C>0$ such that*

$$|S(\Psi_0)|\leq C\frac{\varepsilon^{1+\sigma}}{1+|x|^\nu} \text{ in the region } |A_i^{-1}(x-P_i)|>\frac{\delta}{|\log\varepsilon|},\quad \forall i=1,\dots,N, \tag{4.2}$$

for some $\nu>2$ and $\sigma\in(0,1)$. For $i=1,\dots,N$, in the region

$$|A_i^{-1}(x-P_i)|<\frac{\delta}{|\log\varepsilon|},$$

we have

$$\varepsilon^2\mu_i^2|S(\Psi_0)|\leq C\frac{\varepsilon\mu_i\log|\log\varepsilon|}{(1+|y|^{2+a})} \tag{4.3}$$

for some $a\in(0,1)$. In (4.3), y is the scaled variable in the inner regions $x-P_i=A_iz$, $z=\varepsilon\mu_i y$.

Proof. In order to prove this, we will first analyze $S[\Psi_0]$ in regions that are close to each vortex point P_j, and then in the region which is far from all the points $P_1,\dots,P_N$. Let us be more precise.

We split the inner region around P_i, as described in (3.1), into two parts:

$$|A_i^{-1}(x-P_i)|\leq\frac{\delta^2}{|\log\varepsilon|}\quad\text{and}\quad \frac{\delta^2}{|\log\varepsilon|}\leq|A_i^{-1}(x-P_i)|\leq\frac{\delta}{|\log\varepsilon|}.$$

Assume first that $|A_i^{-1}(x-P_i)| \le \frac{\delta^2}{|\log\varepsilon|}$. According to (3.9), we have $\eta^i = 1$ and $\eta^j = 0$ for $j \ne i$, so that the non-linear term in the expression of $S(\Psi_0)$ becomes

$$
\begin{aligned}
&\sum_{j=1}^{N} \varepsilon^{2-\frac{\alpha}{2\kappa_j}R_j^2}\kappa_j F_j\Big(\frac{1}{\kappa_j}\Big(\Psi_0 - \frac{\alpha}{2}|\log\varepsilon|\,|x|^2\Big)\Big)\\
&\quad = \varepsilon^{2-\frac{\alpha}{2\kappa_i}R_i^2}\kappa_i f\Big(\frac{1}{\kappa_i}\Big(\Psi_0(x) - \frac{\alpha}{2}|\log\varepsilon|\,|x|^2\Big)\Big)\\
&\quad = \frac{\kappa_i}{\varepsilon^2\mu_i^2}U(y)e^{[c_{1,i}\varepsilon\mu_i y_1 + c_{2,i}\varepsilon^2\mu_i^2|y|^2]\Gamma_0(y)}e^{\varepsilon[\mathcal{A}_{1,i}(P)y_1\mu_i + \mathcal{A}_{2,i}(P)y_2\mu_i]}\\
&\qquad \times \exp[O(\log(|\log\varepsilon|)|y|^2\varepsilon^2\mu_i^2)]\\
&\quad = \frac{\kappa_i}{\varepsilon^2\mu_i^2}U(y)\big[1 + \varepsilon\mu_i y_1(c_{1,i}\Gamma_0(y) + \mathcal{A}_{1,i}(P)) + \varepsilon\mu_i y_2\mathcal{A}_{2,i}(P)\\
&\qquad\qquad + \varepsilon^2\mu_i^2 c_{2,i}|y|^2\Gamma_0(y) + O(\log(|\log\varepsilon|)|y|^2\varepsilon^2\mu_i^2)\big],
\end{aligned}
\tag{4.4}
$$

with

$$
\begin{aligned}
\mathcal{A}_{1,i}(P) &:= |\log\varepsilon|\Big(4c_{1,i} - \alpha\frac{hR_i}{\kappa_i\sqrt{h^2+R_i^2}}\Big) - 4c_{1,i}\log\mu_i\\
&\quad - 4\sum_{j\ne i}\frac{\kappa_j}{\kappa_i}\frac{[A_j^{-1}(P_i-P_j)]_1}{|A_j^{-1}(P_i-P_j)|^2} + \sum_{j\ne i}\frac{\kappa_j}{\kappa_i}\log\frac{8}{|A_j^{-1}(P_i-P_j)|^4}c_{1,j} + \mathbf{Y}_1(P),\\
\mathcal{A}_{2,i}(P) &:= -\sum_{j\ne i}\frac{\kappa_j}{\kappa_i}4\frac{[A_j^{-1}(P_i-P_j)]_2}{|(A_j^{-1}(P_i-P_j))|^2} + \mathbf{Y}_2(P),
\end{aligned}
$$

where $\mathbf{Y}_1(P)$ and $\mathbf{Y}_2(P)$ are smooth functions, uniformly bounded as $\varepsilon \to 0$ for points $P = (P_1, \ldots, P_N)$ satisfying (2.3)–(2.4).

In the expression of $\mathcal{A}_{1,i}$ the term

$$
-4c_{1,i}\log\mu_i + \sum_{j\ne i}\frac{\kappa_j}{\kappa_i}\log\frac{8}{|A_j^{-1}(P_i-P_j)|^4}c_{1,j}
$$

is a smooth function of the points P_j which can be described as $\log|\log\varepsilon|\mathbf{Y}_1(P)$, where $\mathbf{Y}_1(P)$ denotes again a smooth function, uniformly bounded as $\varepsilon \to 0$ for points $P = (P_1, \ldots, P_N)$ satisfying (2.3)–(2.4).

In addition, if we insert the definition of $c_{1,i}$ as given in (3.6), we write $\mathcal{A}_{1,i}(P)$ as

$$
\begin{aligned}
\mathcal{A}_{1,i}(P) &= |\log\varepsilon|\Big(2\frac{hR_i}{\sqrt{(h^2+R_i^2)^3}} - \alpha\frac{hR_i}{\kappa_i\sqrt{h^2+R_i^2}}\Big)\\
&\quad - \sum_{j\ne i}\frac{\kappa_j}{\kappa_i}4\frac{[A_j^{-1}(P_i-P_j)]_1}{|A_j^{-1}(P_i-P_j)|^2} + \log|\log\varepsilon|\mathbf{Y}_1(P),
\end{aligned}
\tag{4.5}
$$

where again $\mathbf{Y}_1(P)$ denotes an explicit smooth function, uniformly bounded as $\varepsilon \to 0$ for points $P = (P_1, \ldots, P_N)$ satisfying (2.3)–(2.4).

For later purposes it is relevant to observe that

$$\mathcal{A}_{1,i} = \log|\log\varepsilon|\bar{\mathbf{Y}}_1(P), \quad \mathcal{A}_{2,i} = |\log\varepsilon|\bar{\mathbf{Y}}_2(P), \tag{4.6}$$

under the assumption that the points P_i satisfy (2.3)–(2.4). As before, $\bar{\mathbf{Y}}_1(P)$ and $\bar{\mathbf{Y}}_2(P)$ denote explicit smooth functions, uniformly bounded as $\varepsilon \to 0$ for points $P = (P_1, \ldots, P_N)$ satisfying (2.3)–(2.4).

A direct computation shows that in the region $\frac{\delta^2}{|\log\varepsilon|} \le |A_i^{-1}(x - P_i)| \le \frac{\delta}{|\log\varepsilon|}$ we have

$$\sum_{j=1}^N \varepsilon^{2-\frac{\alpha}{2\kappa_j}R_j^2}\kappa_j F_j\Big(\frac{1}{\kappa_j}\Big(\Psi_0 - \frac{\alpha}{2}|\log\varepsilon|\,|x|^2\Big)\Big) = \varepsilon^{2-\frac{\alpha}{2\kappa_i}R_i^2} f\Big(\frac{1}{\kappa_i}\Big(\Psi_0(x) - \frac{\alpha}{2}|\log\varepsilon|\,|x|^2\Big)\Big)$$
$$= O(\varepsilon^2\mu_i^2|\log\varepsilon|^4).$$

On the other hand, we recall from (2.21) that

$$L(\Psi_0) = \eta_0 \sum_{j=1}^N \kappa_j\Big(\Delta\Gamma_{\varepsilon\mu_j} + \frac{4R_j(3h^2 + R_j^2)}{h(h^2+R_j^2)^{\frac{3}{2}}}\frac{\varepsilon^2\mu_j^2(z_j)_1}{(\varepsilon^2\mu_j^2 + |z_j|^2)^2}\Big), \tag{4.7}$$

where $z_j = A[P_j]^{-1}(x - P_j)$, $z_j = ((z_j)_1, (z_j)_2)$. For i fixed, in the variable $z = A[P_i]^{-1}(x - P_i)$, the above expression becomes

$$\begin{aligned} L(\Psi_0) &= \eta_0\kappa_i\Big(\Delta\Gamma_{\varepsilon\mu_i} + \frac{4R_i(3h^2+R_i^2)}{h(h^2+R_i^2)^{\frac{3}{2}}}\frac{\varepsilon^2\mu_j^2 z_1}{(\varepsilon^2\mu_i^2 + |z|^2)^2}\Big) \\ &\quad + \eta_0\sum_{j\neq i}^N \kappa_j\Big(\frac{8\varepsilon^2\mu_j^2}{(\varepsilon^2\mu_j^2 + |A_j^{-1}A_i z + A_j^{-1}(P_j - P_i)|^2)^2} \\ &\qquad + \frac{4R_j(3h^2+R_j^2)}{h(h^2+R_i^2)^{\frac{3}{2}}}\frac{\varepsilon^2\mu_j^2[A_j^{-1}A_i z + A_j^{-1}(P_j-P_i)]_1}{(\varepsilon^2\mu_j^2 + |A_j^{-1}A_i z + A_j^{-1}(P_j - P_i)|^2)^2}\Big). \end{aligned}$$

Therefore, for the *inner part* $|A_i^{-1}(x - P_i)| < \delta/|\log\varepsilon|$, we have

$$L(\Psi_0) = -\frac{\kappa_i}{\varepsilon^2\mu_i^2}\Big[U(y) - \frac{4R_i(3h^2+R_i^2)}{h(h^2+R_i^2)^{\frac{3}{2}}}\frac{\varepsilon\mu_i y_1}{(1+|y|^2)^2}\Big] + O(\varepsilon^2\mu^2|\log\varepsilon|^4), \tag{4.8}$$

where y is the variable introduced in (3.2),

$$x - P_i = \varepsilon\mu_i A_i y.$$

Combining (4.4) and (4.8), we conclude that in the region $|A_i^{-1}(x - P_i)| < \delta^2/|\log\varepsilon|$ we have

$$\begin{aligned} S(\Psi_0) = \frac{\kappa_i}{\varepsilon^2\mu_i^2}U(y)\Big[&\varepsilon\mu_i y_1\Big(c_{1,i}\Gamma_0(y) + \frac{R_i(3h^2+R_i^2)}{2h(h^2+R_i^2)^{\frac{3}{2}}} + \mathcal{A}_{1,i}(P)\Big) \\ &+ \varepsilon\mu_i y_2\mathcal{A}_{2,i}(P) + \varepsilon^2\mu_i^2 c_{2,i}|y|^2\Gamma_0(y) \\ &+ O(\log(|\log\varepsilon|)|y|^2\varepsilon^2\mu_i^2)\Big] \\ + O(\varepsilon^2\mu^2|\log\varepsilon|^4), \end{aligned} \tag{4.9}$$

with μ given by (3.4), and S by (4.1).

Consequently, in the region $\frac{\delta^2}{|\log\varepsilon|} \le |A_i^{-1}(x-P_i)| \le \frac{\delta}{|\log\varepsilon|}$, we have

$$S(\Psi_0)(x) = O(\varepsilon^2\mu^2|\log\varepsilon|^4).$$

Estimate (4.3) uses the above estimate, and (4.5)–(4.6).

Let us consider now the region defined by

$$|A_i^{-1}(x-P_i)| > \frac{\delta}{|\log\varepsilon|}, \quad \forall i.$$

In this outer region, all cut-off functions η^i are zero, see (3.9), hence $S[\Psi_0](x) = L(\Psi_0)$. A direct inspection of (4.7) gives

$$|L(\Psi_0)| \le C\frac{\varepsilon^2\mu^2}{1+|x|^\nu}$$

for some C independent of ε, $\nu > 2$ and μ is defined in (3.4). This concludes the proof of Proposition 4.1. ∎

Estimates (4.2) and (4.3) will be crucial to carry on the inner-outer gluing procedure leading to an exact solution of (2.1). This is what we discuss next.

5. The inner-outer gluing system

This section describes the inner-outer gluing scheme to find an actual solution to (2.1). We look for a solution $\Psi(x)$ of the equation

$$S[\Psi] := L[\Psi] + F(\Psi) = 0 \quad \text{in } \mathbb{R}^2, \tag{5.1}$$

where

$$F(\Psi) = \sum_{i=1}^N \varepsilon^{2-\frac{\alpha}{2\kappa_i}R_i^2}\kappa_i\eta^i f\left(\frac{1}{\kappa_i}\left(\Psi - \frac{\alpha}{2}|\log\varepsilon|\,|x|^2\right)\right), \quad f(u) = e^u.$$

Consider the approximate solution $\Psi_0(x)$ in (2.20). The function Ψ_0 is defined in terms of scaling parameters $\mu_1,\dots,\mu_N$ given by formula (3.3) and points $P_1,\dots,P_N$ satisfying (2.3)–(2.4). We refer to Sections 2 and 3 for the construction of Ψ_0. We look for a solution Ψ to (5.1) of the form

$$\Psi(x) = \Psi_0(x) + \varphi(x), \tag{5.2}$$

where φ is "smaller" than Ψ_0. The inner-outer gluing procedure starts with choosing φ of the form

$$\varphi(x) = \sum_{i=1}^N \eta_i(x)\phi_i(y) + \psi(x). \tag{5.3}$$

Here,

$$\eta_i = \eta\Big(\frac{|\log\varepsilon|\,|A_i^{-1}(x-P_i)|}{\delta_1}\Big)$$

for some $\delta_1 < \delta^2$, with δ fixed in (3.1) and η defined in (2.18), and y denotes the scaling variable

$$y = \frac{A_i^{-1}(x-P_i)}{\varepsilon\mu_i}.$$

In terms of $\varphi(x)$, and using the decomposition (5.3), problem (5.1) takes the form

$$\begin{aligned}
&S(\Psi_0+\varphi) = 0 \quad \text{in } \mathbb{R}^2,\\
&S(\Psi_0+\varphi) = \sum_{i=1}^N \eta_i\Big[L[\phi_i] + F'(\Psi_0)(\phi_i+\psi) + S(\Psi_0) + N_0\Big(\sum_{i=1}^N \eta_i\phi_i + \psi\Big)\Big]\\
&\qquad + L[\psi] + \Big(1-\sum_{i=1}^N \eta_i\Big)\Big[F'(\Psi_0)\psi + E_0 + N_0\Big(\sum_{i=1}^N \eta_i\phi_i+\psi\Big)\Big]\\
&\qquad + \sum_{i=1}^N (L[\eta_i\phi_i] - \eta_i L[\phi_i]),
\end{aligned}$$

where

$$N_0(\varphi) = F(\Psi_0+\varphi) - F(\Psi_0) - F'(\Psi_0)\varphi,$$

with

$$F'(\Psi_0) = \sum_{i=1}^N \varepsilon^{2-\frac{\alpha}{2\kappa_i}R_i^2}\eta^i f'\Big(\frac{1}{\kappa_i}\Big(\Psi_0 - \frac{\alpha}{2}|\log\varepsilon|\,|x|^2\Big)\Big).$$

Thus Ψ given by (5.2)–(5.3) solves (5.1) if $(\phi,\psi) := (\phi_1,\dots,\phi_N,\psi)$ satisfies the system of equations

$$\begin{aligned}
&L[\phi_i] + F'(\Psi_0)(\phi_i+\psi) + S(\Psi_0) + N_0\Big(\sum_{i=1}^N \eta_i\phi_i + \psi\Big)\\
&\quad = 0, \quad \text{for } |A_i^{-1}(x-P_i)| < \frac{2\delta_1}{|\log\varepsilon|},
\end{aligned} \tag{5.4}$$

and

$$\begin{aligned}
&L[\psi] + \Big(1-\sum_{i=1}^N \eta_i\Big)\Big[F'(\Psi_0)\psi + E_0 + N_0\Big(\sum_{i=1}^N \eta_i\phi_i + \psi\Big)\Big]\\
&\quad + \sum_{i=1}^N (L[\eta_i\phi_i] - \eta_i L[\phi_i]) = 0 \quad \text{in } \mathbb{R}^2.
\end{aligned} \tag{5.5}$$

We will refer to problem (5.4) as the *inner problem* and to (5.5) as the *outer problem*.

Let us write (5.4) in terms of the variable $y = \frac{A_i^{-1}(x-P_i)}{\varepsilon\mu_i}$. From (2.8) we have

$$L[\phi_i] = \frac{1}{\varepsilon^2\mu_i^2}[\Delta_y\phi_i + \bar{B}_i(y)[\phi_i]],$$

where $\bar{B}_i(y) = \varepsilon^2\mu_i^2 B(\varepsilon\mu_i y)$ and B is the operator given by (2.9). Using estimate (4.4), we get

$$\varepsilon^2\mu_i^2 F'(\Psi_0) = e^{\Gamma_0(y)} + b_i(y), \quad \text{with } \Gamma_0(y) = \log\frac{8}{(1+|y|^2)^2},$$

where

$$b_i(y) = e^{\Gamma_0(y)}\big[\varepsilon\mu_i y_1(c_{1,i}\Gamma_0(y) + \mathcal{A}_{1,i}(P)) + \varepsilon\mu_i y_2\mathcal{A}_{2,i}(P) + \varepsilon^2\mu_i^2 c_{2,i}|y|^2\Gamma_0(y) + O(\log(|\log\varepsilon|)|y|^2\varepsilon^2\mu_i^2)\big]. \tag{5.6}$$

Consequently, using (4.5)–(4.6), we have

$$b_i(y) = O\Big(\frac{\varepsilon\mu_i\log|\log\varepsilon|}{1+|y|^{2+a}}\Big), \tag{5.7}$$

so the term $\log|\log\varepsilon|$ assumes that the points P_i satisfy (2.3)–(2.4).

By estimates (4.3), we obtain, in the region $|y| < \frac{2\delta_1}{\mu_i\varepsilon|\log\varepsilon|}$,

$$\widetilde{E}_i := \varepsilon^2\mu_i^2 S[\Psi_0] = O\Big(\frac{\varepsilon\mu_i}{1+|y|^{2+a}}\log|\log\varepsilon|\Big), \tag{5.8}$$

for $a \in (0,1)$. Similarly, using estimate (5.7) for b_i, we get the expansion

$$\mathcal{N}_i(\varphi) := \varepsilon^2\mu_i^2 N_0(\varphi) = \frac{1}{\kappa_i}(e^{\Gamma_0(y)} + b_i(y))\varphi^2. \tag{5.9}$$

Then, multiplying the inner problem (5.4) by $\varepsilon^2\mu_i^2$, we get

$$\Delta_y\phi_i + e^{\Gamma_0}\phi = -B_i[\phi_i] - H_i(\phi,\psi) \quad \text{in } B_R, \tag{5.10}$$

where $R = \frac{2\delta_1}{\varepsilon\mu_i|\log\varepsilon|}$,

$$H_i(\phi,\psi) = \mathcal{N}_i\left(\sum_{i=1}^N \eta_i\phi_i + \psi\right) + \widetilde{E}_i + (e^{\Gamma_0} + b_i)\psi$$

and

$$B_i[\phi_i] = \bar{B}_i(y)[\phi_i] + b_i(y)\phi_i. \tag{5.11}$$

Note that

$$\bar{B}_i(y) = \Big(-2\frac{R_i h}{(h^2+R_i^2)^{3/2}}(\varepsilon\mu_i y_1) + O(|\varepsilon\mu_i y|^2)\Big)\partial_{y_1y_1} + O(|\varepsilon\mu_i y|^2)\partial_{y_2y_2} - \left(2\frac{R_i}{h\sqrt{h^2+R_i^2}}\varepsilon\mu_i y_2 + O(|\varepsilon\mu_i y|^2)\right)\partial_{y_1y_2}$$

$$
\begin{aligned}
&-\Big(\frac{\varepsilon\mu_i R_i}{h\sqrt{h^2+R_i^2}}\Big(\frac{2h^2}{h^2+R_i^2}+1\Big)+O((\varepsilon\mu_i)^2|y|)\Big)\partial_{y_1}\\
&-\Big(\frac{(\varepsilon\mu_i)^2 y_2}{h^2+R_i^2}\Big(\frac{2h^2}{h^2+R_i^2}+1\Big)+O((\varepsilon\mu_i)^3|y|^2)\Big)\partial_{y_2}.
\end{aligned} \tag{5.12}
$$

The idea is to solve equation (5.10), coupled with the outer problem (5.5) in such a way that ϕ_i has the size of the error $\widetilde{E}_i$ with two powers less of decay in y, say

$$
(1+|y|)|D_y\phi_i(y)|+|\phi_i(y)|\le \frac{C\varepsilon\mu_i \log|\log\varepsilon|}{1+|y|^a}.
$$

Recall that the basic linear operator $\Delta_y\phi+e^{\Gamma_0}\phi$ in (5.10) has a three-dimensional kernel generated by the bounded functions

$$
Z_i(y)=\frac{\partial\Gamma_0}{\partial y_i},\quad i=1,2,\qquad Z_0(y)=2+y\cdot\nabla\Gamma_0(y). \tag{5.13}
$$

This fact suggests that the solvability of (5.10) within the expected topologies depends on whether the right-hand side has components in the directions spanned by the Z_i. Instead of solving (5.10) directly, we will solve the auxiliary projected problem instead, for $i=1,\dots,N$,

$$
\Delta_y\phi_i+e^{\Gamma_0}\phi_i+B_i(\phi_i)+H_i(\phi,\psi)=\sum_{j=1}^{2}c_{ij}e^{\Gamma_0(y)}Z_j \quad \text{in } B_R, \tag{5.14}
$$

for some constants c_{ij}. We solve (5.14) coupled with the outer problem (5.5), which can be written as

$$
L[\psi]+G(\psi,\phi)=0 \quad \text{in } \mathbb{R}^2, \tag{5.15}
$$

where

$$
G(\psi,\phi)=V(x)\psi+N^0(\phi)+E^0(x)+\sum_{i=1}^{N}A_i[\phi_i], \tag{5.16}
$$

with

$$
\begin{aligned}
V(x)&=\Big(1-\sum_{i=1}^{N}\eta_i\Big)F'(\Psi_0),\quad N^0(\varphi)=\Big(1-\sum_{i=1}^{N}\eta_i\Big)N_0(\varphi),\\
E^0(x)&=\Big(1-\sum_{i=1}^{N}\eta_i\Big)S[\Psi_0],\qquad A_i[\phi_i]=L[\eta_i]\phi_i+K_{\ell j}(x)\partial_{x_\ell}\eta_i\partial_{x_j}\phi_i,
\end{aligned}
$$

where $K_{\ell j}$ are the coefficients of the matrix K defining the differential operator L; see (2.6). By (4.2), the following bounds hold:

$$
\begin{gathered}
|V(x)|\le O((\varepsilon\mu)^2|\log\varepsilon|^4),\quad |N^0(\varphi)|\le O((\varepsilon\mu)^2|\log\varepsilon|^4|\varphi|^2),\\
|E^0(x)|\le O(\varepsilon^{1+b}).
\end{gathered} \tag{5.17}
$$

In order to find a solution of (5.1), we will need to solve

$$c_{ij} = c_{ij}[\phi, \psi] = 0 \quad \text{for } i = 1, \dots, N,\ j = 1, 2.$$

This can be achieved properly choosing s^* and $\hat{P}_1, \dots, \hat{P}_N$ in the form of the points P_j as given in (2.3), under the bounds (2.4). In Section 6 we will establish linear results that are the basic tools to solve system (5.14)–(5.15). Section 7 is devoted to solving (5.14)–(5.15) by means of a fixed point scheme, and Section 8 to adjusting the points to get $c_{ij} = 0$ for all $i = 1, \dots, N$, $j = 1, 2$.

6. Linear theories

This section collects two results. The first one regards the solvability of the *outer linear theory*, and the second the *inner linear theory*. They were obtained in [17]. For completeness we state them here and give a sketch of their proofs in Appendix A.

6.1. Outer linear theory

Consider the Poisson equation for the operator L,

$$L[\psi] + g(x) = 0 \quad \text{in } \mathbb{R}^2, \tag{6.1}$$

for a bounded function g. Here L is the differential operator in divergence form defined in (2.6).

We take functions $g(x)$ that satisfy the decay condition

$$\|g\|_\nu := \sup_{x\in\mathbb{R}^2} (1 + |x|)^\nu |g(x)| < +\infty,$$

where $\nu > 2$.

Proposition 6.1 ([17, Proposition 7.1]). *There exists a solution $\psi(x)$ to problem (6.1), which is of class $C^{1,\beta}(\mathbb{R}^2)$ for any $0 < \beta < 1$, that defines a linear operator $\psi = \mathcal{T}^o(g)$ of g and satisfies the bound*

$$|\psi(x)| \le C\|g\|_\nu (1 + |x|^2), \tag{6.2}$$

for some positive constant C.

6.2. Inner linear theory

In this section we consider the problem

$$\Delta\phi + e^{\Gamma_0(y)}\phi + h(y) = 0 \quad \text{in } \mathbb{R}^2. \tag{6.3}$$

For numbers $m > 2$, $0 < \beta < 1$ we consider the norms

$$\begin{aligned} \|h\|_m &= \sup_{y\in\mathbb{R}^n} (1+|y|)^m |h(y)|, \\ \|h\|_{m,\beta} &= \|h\|_m + (1+|y|)^{m+\beta}[h]_{B_1(y),\beta}, \end{aligned} \tag{6.4}$$

where we use the standard notation

$$[h]_{A,\beta} = \sup_{z_1,z_2\in A} \frac{|h(z_1)-h(z_2)|}{|z_1-z_2|^\beta},$$

and A is a subset of $\mathbb{R}^2$. We recall the definition of the functions $Z_i(y)$ in (5.13):

$$Z_i(y) = \partial_{y_i}\Gamma_0(y), \ \ i=1,2, \quad Z_0(y) = 2 + y\cdot\nabla\Gamma_0(y).$$

Lemma 6.2 ([17, Lemma 6.1]). *Given $m > 2$ and $0 < \beta < 1$, there exists a $C > 0$ and a solution $\phi = \mathcal{T}[h]$ of problem (6.3) for each h with $\|h\|_m < +\infty$ that defines a linear operator of h and satisfies the estimate*

$$\begin{aligned} &(1+|y|)|\nabla\phi(y)| + |\phi(y)| \\ &\quad \le C\Bigg[\log(2+|y|)\left|\int_{\mathbb{R}^2} hZ_0\right| + (1+|y|)\sum_{j=1}^{2}\left|\int_{\mathbb{R}^2} hZ_j\right| \\ &\qquad + (1+|y|)^{2-m}\|h\|_m\Bigg]. \end{aligned} \tag{6.5}$$

In addition, if $\|h\|_{m,\beta} < +\infty$, we have

$$\begin{aligned} &(1+|y|^{2+\beta})[D_y^2\phi]_{B_1(y),\beta} + (1+|y|^2)|D_y^2\phi(y)| \\ &\quad \le C\Bigg[\log(2+|y|)\left|\int_{\mathbb{R}^2} hZ_0\right| + (1+|y|)\sum_{j=1}^{2}\left|\int_{\mathbb{R}^2} hZ_j\right| \\ &\qquad + (1+|y|)^{2-m}\|h\|_{m,\beta}\Bigg]. \end{aligned} \tag{6.6}$$

For a fixed number $\delta > 0$ and a sufficiently large $R > 0$ we consider the equation

$$\Delta\phi + e^{\Gamma_0}\phi + B_i[\phi] + h(y) = \sum_{j=0}^{2} c_{ij}e^{\Gamma_0}Z_j \quad \text{in } B_R. \tag{6.7}$$

For a function h defined in $A\subset\mathbb{R}^2$ we denote by $\|h\|_{m,\beta,A}$ the numbers defined in (6.4) but with the sup taken with elements in A only, namely

$$\begin{aligned} \|h\|_{m,A} &= \sup_{y\in A} |(1+|y|)^m h(y)|, \\ \|h\|_{m,\beta,A} &= \sup_{y\in A}(1+|y|)^{m+\beta}[h]_{B(y,1)\cap A} + \|h\|_{m,A}. \end{aligned}$$

Let us also define, for a function of class $C^{2,\alpha}(A)$,

$$\|\phi\|_{*,m-2,A} = \|D^2\phi\|_{m,\beta,A} + \|D\phi\|_{m-1,A} + \|\phi\|_{m-2,A}. \tag{6.8}$$

In this notation we omit the dependence on A when $A = \mathbb{R}^2$. The following is the main result of this section.

Proposition 6.3 ([17, Proposition 6.1]). *There is $C > 0$ such that for all sufficiently large R and a differential operator B_i as in (5.11) with estimates (5.7) and (5.12), problem (6.7) has a solution $\phi = T_i[h]$ for certain scalars $c_{ij} = c_{ij}[h]$, that defines a linear operator of h and satisfies*

$$\|\phi\|_{*,m-2,B_R} \le C\|h\|_{m,\beta,B_R}.$$

In addition, the linear functionals c_i can be estimated as

$$c_{i0}[h] = \gamma_0 \int_{B_R} hZ_0 + O(R^{2-m})\|h\|_{m,\beta,B_R},$$
$$c_{ij}[h] = \gamma_j \int_{B_R} hZ_j + O(\varepsilon\mu_i \log|\log\varepsilon|)\|h\|_{m,\beta,B_R}, \quad j = 1, 2,$$

where $\gamma_j^{-1} = \int_{\mathbb{R}^2} e^{\Gamma_0} Z_j^2$, $j = 0, 1, 2$.

7. Solving the inner-outer gluing system

We let X^o be the Banach space of all functions $\psi \in C^{2,\beta}(\mathbb{R}^2)$ such that

$$\|\psi\|_\infty < +\infty,$$

and formulate the outer equation (5.15) as the fixed point problem in X^o,

$$\psi = \mathcal{T}^o[G(\psi, \phi)], \quad \psi \in X^o,$$

where $\mathcal{T}^o$ is defined in Proposition 6.1, while G is the operator given by (5.16).

We formulate the projected problem (5.14) as the one of finding (ϕ_i, c_{ij}), where

$$\phi_i = \phi_{i,1} + \phi_{2,i}$$

with

$$\Delta_y\phi_{i,1} + e^{\Gamma_0}\phi_{i,1} + B_i[\phi_{i,1}] + B_i[\phi_{i,2}] + H_i(\phi, \psi) = \sum_{j=0}^{2} c_{ij}e^{\Gamma_0}Z_j \quad \text{in } B_R, \tag{7.1}$$

where Z_j are given by (5.13), and

$$\Delta_y\phi_{i,2} + e^{\Gamma_0}\phi_{i,2} + c_{i0}e^{\Gamma_0}Z_0 = 0 \quad \text{in } \mathbb{R}^2. \tag{7.2}$$

Problem (7.1) is formulated using the operator T_i in Proposition 6.3, with

$$c_{ij} = c_{ij}[H_i(\phi, \psi) + B_i(\phi_{i,2})], \quad j = 0, 1, 2,$$
$$\phi_{i,1} \in X_*, \quad \phi_{i,1} = T_i(H_i(\phi, \psi) + B_i[\phi_{i,2}]),$$

where X_* is the Banach space of functions $\phi \in C^{2,\beta}(B_R)$ such that

$$\|\phi\|_{*,m-2,B_R} < \infty$$

(see (6.8)).

Problem (7.2) is formulated using the operator $\mathcal{T}$ in Lemma 6.2,

$$\phi_{i,2} = \mathcal{T}[c_{i0}[H_i(\phi, \psi) + B_i(\phi_{i,2})]e^{\Gamma_0} Z_0];$$

in fact $\phi_{i,2}$ is a radial function satisfying

$$\phi_{i,2}(y) = c_{i0}[H_i(\phi, \psi) + B_i(\phi_{i,2})]\Big(\frac{4}{3}\frac{|y|^2 - 1}{|y|^2 + 1}\log(1 + |y|^2) - \frac{8}{3}\frac{1}{|y|^2 + 1}\Big). \tag{7.3}$$

Having in mind the a priori bound in (6.5), (6.6) in Lemma 6.2, it is natural to ask that $\phi_{i,2} \in C^{2,\beta}(\mathbb{R}^2)$,

$$\|\phi\|_{**,\beta} = \sup_{y\in\mathbb{R}^2} \frac{1}{\log(1 + |y|)}\Big[(1 + |y|^{2+\beta})[D_y^2\phi]_{B_1(y),\beta} + (1 + |y|^2)|D_y^2\phi(y)| + (1 + |y|)|\nabla\phi(y)| + |\phi(y)|\Big] \tag{7.4}$$

and denote by X_{**} the Banach space of functions $\phi \in C^{2,\beta}$ with $\|\phi\|_{**,\beta} < \infty$.

Proposition 7.1. *Let $\beta \in (0, 1)$. There exist positive constants C and $\sigma_1 > 0$, functions $\psi \in X^o$, $\bar\phi_1 = (\phi_{1,1}, \dots, \phi_{1,N}) \in X_*^N$, $\bar\phi_2 = (\phi_{2,1}, \dots, \phi_{2,N}) \in X_{**}^N$, and constants c_{ij}, $i = 1, \dots, N$, $j = 1, 2$, solutions to (5.15)–(7.1)–(7.2) such that*

$$\|\psi\|_\infty \le C\varepsilon^{1+\sigma_1}, \quad \|\phi_{i,1}\|_{*,m-2,B_R} \le CR^{-1},$$
$$\|\phi_{i,2}\|_{**,\beta} \le CR^{1-m}, \quad i := 1, \dots, N, \tag{7.5}$$

with $R = 2\delta_1/(\varepsilon\mu_i|\log\varepsilon|)$.

Proof. Problem (5.14)–(5.15) consists in finding ψ, $\bar\phi_1$, $\bar\phi_2$, solutions of the fixed point problem

$$(\psi, \bar\phi_1, \bar\phi_2) = \mathcal{A}(\psi, \bar\phi_1, \bar\phi_2) \tag{7.6}$$

given by

$$\psi = \mathcal{T}^o[G(\psi, \bar\phi_1 + \bar\phi_2)], \quad \psi \in X^o,$$
$$\phi_{i,1} = T_i[H_i(\bar\phi_1 + \bar\phi_2, \psi) + B_i(\phi_{i,2})], \quad \phi_{i,1} \in X_*, \tag{7.7}$$
$$\phi_{i,2} = \mathcal{T}[c_{i0}[H_i(\bar\phi_1 + \bar\phi_2, \psi) + B_i(\phi_{i,2})]e^{\Gamma_0} Z_0], \quad \phi_{i,2} \in X_{**}.$$

Let $m > 2$ and define

$$B_M = \big\{(\psi, \bar\phi_1, \bar\phi_2) \in X^o \times X_*^N \times X_{**}^N : \\ \|\psi\|_\infty \le M\varepsilon^{1+\sigma_1},\ \|\phi_{i,1}\|_{*,m-2,B_R} \le MR^{-1}, \\ \|\phi_{i,2}\|_{**,\beta} \le MR^{1-m},\ i = 1, \dots, N\big\}, \tag{7.8}$$

for some positive constant M independent of ε and σ_1. We will solve (7.6)–(7.7) in B_M.

We first show that $\mathcal{A}(B_M) \subset B_M$. Assume that $(\psi, \bar\phi_1, \bar\phi_2) \in B_M$. We first want to show that $\mathcal{A}(\psi, \bar\phi_1, \bar\phi_2) \in B$. By definition of X_* and X_{**}, we have

$$A_i(\phi_i) \le \frac{C}{1+|x|^\nu}\Big(|\log\varepsilon|^2|\phi_{i,1}+\phi_{i,2}| + \frac{|\log\varepsilon|}{\varepsilon\mu_i}|D_y(\phi_{i,1}+\phi_{i,2})|\Big) \\ \le \frac{R^{2-m}|\log\varepsilon|^2}{1+|x|^\nu}\|\phi_{i,1}\|_{*,m-2,B_R} + \frac{|\log\varepsilon|^3}{1+|x|^\nu}\|\phi_{i,2}\|_{**,\beta}$$

for some $\nu > 2$ and for x in a subset of $B((x_0,0),1)$, and $A_i(\phi_i) = 0$ elsewhere. From (5.16) and (5.17), we get

$$|G(\psi, \bar\phi_1 + \bar\phi_2)| \\ \le \frac{C}{1+|x|^\nu}\varepsilon^{1+b}\Big(1 + \sum_{i=1}^N |\phi_{i,1}+\phi_{i,2}|^2\eta_i + |\psi|^2 + |\psi|\Big) \\ + \frac{C}{1+|x|^\nu}|\log\varepsilon|^2\sum_{i=1}^N(R^{2-m}\|\phi_{i,1}\|_{*,m-2,B_R} + |\log\varepsilon|\,\|\phi_{i,2}\|_{**,\beta}). \tag{7.9}$$

From Proposition 6.1, we get

$$\|\psi\|_\infty = \|\mathcal{T}^o(G(\psi, \bar\phi_1 + \bar\phi_2))\|_\infty \le C\varepsilon^{1+\sigma_1}, \tag{7.10}$$

where $\sigma_1 = \min\{b, m-2-\sigma\}$ for $\sigma > 0$ small. From (5.9), (5.8), we get, for some $a \in (0,1)$,

$$|H_i(\bar\phi_1 + \bar\phi_2, \psi)| \le |\widetilde E_i| + \frac{8}{1+|y|^2}\Big(\frac{1}{(1+|y|^2)} + \frac{C\varepsilon\mu_i\log|\log\varepsilon|}{(1+|y|^a)}\Big)|\psi| \\ + \frac{C}{(1+|y|)^4}\Big(|\psi|^2 + \sum_{i=1}^N(|\eta_i\phi_{i,1}|^2 + |\eta_i\phi_{i,2}|^2)\Big), \tag{7.11}$$

where

$$|\widetilde E_i| \le C\frac{\varepsilon\mu_i\log|\log\varepsilon|}{(1+|y|^{2+a})}.$$

Using the assumptions on ψ, $\phi_{i,1}$, and $\phi_{i,2}$, we get

$$\|H_i(\bar\phi_1 + \bar\phi_2, \psi)\|_{m,\beta,B_R} \le C\varepsilon\mu_i\log|\log\varepsilon| \le CR^{-1}, \tag{7.12}$$

for $m < 2 + a$. From (5.11), (5.7), and (5.12), we get

$$|B_i[\phi_{i,2}]| \le C\varepsilon\mu_i\Big(|D\phi_{i,2}| + |y|\,|D^2\phi_{i,2}| + \frac{\log|\log\varepsilon|}{1+|y|^{2+a}}|\phi_{i,2}|\Big), \tag{7.13}$$

and using again that $m < 2 + a$, we have

$$\begin{aligned}\|B_i(\phi_{i,2})\|_{m,\beta,B_R} &\le CR^{m-2}\|\phi_{i,2}\|_{**,\beta} + CR^{-1}\log|\log\varepsilon|\|\phi_{i,2}\|_{**,\beta}\\ &\le CR^{m-2}\|\phi_{i,2}\|_{**,\beta}, \le CR^{-1},\end{aligned} \tag{7.14}$$

and from Proposition 6.3, using estimates (7.12) and (7.14), we find that

$$\|\phi_{i,1}\|_{*,m-2,B_R} \le CR^{-1}. \tag{7.15}$$

Now using (4.9), (5.6), and the fact that $\phi_{i,2}$ is radial, we have

$$\int_{B_R}\tilde{E}_i Z_0 = O(\varepsilon^2\mu_i^2|\log\varepsilon|),$$

$$\int_{B_R} B_i(\phi_{i,2})Z_0 = O(\|\phi_{i,2}\|_{**,\beta}R^2(\varepsilon\mu_i)^2|\log\varepsilon|).$$

In addition, since $\int_{B_R} e^{\Gamma_0(y)}Z_0 = O(R^{-2})$ and by regularity we have $\psi(x) = \psi(P_i) + \varepsilon\mu_i A_i y\|\psi\|_\infty$ for $x \in B(P_i, \delta/|\log\varepsilon|)$, we get

$$\int_{B_R}[e^{\Gamma_0(y)} + b_i(y)]\psi Z_0 = O(\varepsilon\mu_i\|\psi\|_\infty).$$

We also have $\int_{B_R}\mathcal{N}_i(\sum_{i=1}^N \eta_i\phi_i + \psi)Z_0 = O(\sum_{i=1}^N \|\phi_{i,1}\|^2_{*,m-2,B_R})$, and consequently from the definition of H_i,

$$\int_{B_R}[H_i(\phi,\psi) + B_i(\phi_{i,2})]Z_0 = O\Big(\frac{\|\phi_{i,2}\|_{**,\beta}}{|\log\varepsilon|}\Big).$$

By Proposition 6.3, we have

$$|c_{i0}[H_i(\phi,\psi) + B_i(\phi_{i,2})]| \le CR^{2-m}\|H_i(\phi,\psi) + B_i(\phi_{i,2})\|_{m,\beta,R} + O\Big(\frac{\|\phi_{i,2}\|_{**,\beta}}{|\log\varepsilon|}\Big).$$

Then

$$|c_{i0}[H_i(\phi,\psi) + B_i(\phi_{i,2})]| \le C\|\phi_{i,2}\|_{**,\beta}, \tag{7.16}$$

while from Lemma 6.2 and (7.16) we get

$$\|\phi_{i,2}\|_{**,\beta} \le C|c_{i0}[H_i(\phi,\psi) + B_i(\phi_{i,2})]| \le CR^{1-m}. \tag{7.17}$$

Combining (7.10)–(7.15)–(7.17), we conclude that $\mathcal{A}(\psi,\phi_1,\phi_2) \in B_M$ if we choose M large enough (but independent of ε) in the definition of the set B_M in (7.8).

We next show that $\mathcal{A}$ is a contraction map in B_M. Let $\varphi^j = \sum_{i=1}^N \eta_i(\phi_{i,1}^j + \phi_{i,2}^j) + \psi^j$, for $j = 1, 2$, such that $(\psi^j, \bar\phi_1^j, \bar\phi_2^j) \in B_M$. Let $G(\varphi^j) = G(\psi^j, \bar\phi_1^j + \bar\phi_2^j)$ and observe that

$$\begin{aligned} |G(\varphi^1) - G(\varphi^2)| \le |V(x)(\psi^1 - \psi^2)| \\ + \Big(1 - \sum_{i=1}^N \eta_i\Big)|N_0(\varphi^1) - N_0(\varphi^2)| \\ + \sum_{i=1}^N |A_i[\phi_{i,1}^1 - \phi_{i,1}^2]| + \sum_{i=1}^N |A_i[\phi_{i,2}^1 - \phi_{i,2}^2]|, \end{aligned} \tag{7.18}$$

where the terms are defined in (5.16). A direct computation gives

$$\begin{aligned} &|V(x)(\psi^1 - \psi^2)| + \Big(1 - \sum_{i=1}^N \eta_i\Big)|N_0(\varphi^1) - N_0(\varphi^2)| \\ &\quad \le C(\varepsilon\mu)^2|\log\varepsilon|^4\Big(|\psi^1 - \psi^2| + \sum_{i=1}^N \eta_i^2(|\phi_{i,1}^1 - \phi_{i,1}^2|^2 + |\phi_{i,2}^1 - \phi_{i,2}^2|^2)\Big) \end{aligned}$$

and, as in (7.9),

$$\begin{aligned} |A_i[\phi_{i,1}^1 - \phi_{i,1}^2]| &\le C\big(\big|L(\eta_i)|\phi_{i,1}^1 - \phi_{i,1}^2|\big| + |K_{\ell j}\partial_{x_\ell}\eta_i\partial_{x_j}(\phi_{i,1}^1 - \phi_{i,1}^2)|\big) \\ &\le \frac{C\varepsilon^{m-2-\sigma}}{1+|x|^\nu}\|\phi_{i,1}^1 - \phi_{i,1}^2\|_{*,m-2,B_R} \end{aligned}$$

for $\sigma > 0$ small. In order to estimate $A_i[\phi_2^1 - \phi_2^2]$, we observe that

$$\Delta_y[\phi_{i,2}^1 - \phi_{i,2}^2] + f'(\Gamma_0)[\phi_{i,2}^1 - \phi_{i,2}^2] + c_{i0}^{12}e^{\Gamma_0}Z_0 = 0 \quad \text{in } \mathbb{R}^2,$$

where

$$c_{i0}^{12} = c_{i0}[H_i(\bar\phi_1^1 + \bar\phi_2^1, \psi^1) + B_i(\phi_{i,2}^1)] - c_{i0}[H_i(\bar\phi_1^2 + \bar\phi_2^2, \psi^2) + B_i(\phi_{i,2}^2)].$$

By definition,

$$\begin{aligned} c_{i0}^{12} = \int_{B_R}\Big[&B_i[\phi_{i,2}^1 - \phi_{i,2}^2] + (e^{\Gamma_0(y)} + b_0(y))(\psi^1 - \psi^2) \\ &+ \mathcal{N}_i\Big(\psi^1 + \sum_i \eta_i(\phi_{i,1}^1 + \phi_{i,2}^1)\Big) - \mathcal{N}_i\Big(\psi^2 + \sum_i \eta_i(\phi_{i,1}^2 + \phi_{i,2}^2)\Big)\Big]Z_0\,dy. \end{aligned}$$

Using (5.11) and (5.9), we get

$$\begin{aligned} |c_{i0}^{12}| \le C\Big[&\frac{1}{|\log\varepsilon|}\|\phi_{i,2}^1 - \phi_{i,2}^2\|_{**,\beta} + \|\phi_{i,2}^1 - \phi_{i,2}^2\|_{**,\beta}^2 \\ &+ \|\phi_{i,1}^1 - \phi_{i,1}^2\|_{*,m-2,\beta}^2 + \|\psi^1 - \psi^2\|_\infty^2 \\ &+ R^{-1}[\|\psi^1 - \psi^2\|_\infty + \|\phi_{i,1}^1 - \phi_{i,1}^2\|_{*,m-2,\beta} + \|\phi_{i,2}^1 - \phi_{i,2}^2\|_{**,\beta}]\Big]. \end{aligned}$$

Then by (7.3) and (7.4), we obtain

$$\|\phi_{i,2}^1 - \phi_{i,2}^2\|_{**,\beta} \le C|c_{i0}^{12}|. \tag{7.19}$$

Now we have

$$\begin{aligned} |A_i[\phi_{i,2}^1 - \phi_{i,2}^2]| &\le C\left(\left|L(\eta_i)|\phi_{i,2}^1 - \phi_{i,2}^2|\right| + |K_{\ell j}\partial_{x_\ell}\eta_i\partial_{x_j}(\phi_{i,2}^1 - \phi_{i,2}^2)|\right) \\ &\le \frac{C|\log\varepsilon|^3}{1+|x|^\nu}\|\phi_{i,2}^1 - \phi_{i,2}^2\|_{**,\beta}. \end{aligned}$$

Combining all these estimates in (7.18), we obtain, iterating (7.19), that

$$\begin{aligned} |G(\varphi^1) - G(\varphi^2)| \le \frac{C\varepsilon_0^\sigma}{1+|x|^\nu}&\left(\|\psi^1 - \psi^2\|_\infty + \sum_{i=1}^N \|\phi_{i,1}^1 - \phi_{i,1}^2\|_{*,m-2,B_R}\right) \\ &+ \frac{C}{1+|x|^\nu}\frac{1}{|\log\varepsilon|}\sum_{i=1}^N \|\phi_{i,2}^1 - \phi_{i,2}^2\|_{**,\beta}. \end{aligned}$$

From Proposition 6.1, we have

$$\begin{aligned} \|\mathcal{T}^o[G(\varphi^1)] - \mathcal{T}^o[G(\varphi^2)]\|_\infty \le C\varepsilon^{\sigma_0}&\left(\|\psi^1 - \psi^2\|_\infty + \sum_{i=1}^N \|\phi_{i,1}^1 - \phi_{i,1}^2\|_{*,m-2,B_R}\right) \\ &+ \frac{C}{|\log\varepsilon|}\sum_{i=1}^N \|\phi_{i,2}^1 - \phi_{i,2}^2\|_{**,\beta}, \end{aligned} \tag{7.20}$$

for some $\sigma_0 > 0$. Now let $T_i(\varphi^j) = T_i[H_i(\bar\phi_1^j + \bar\phi_2^j, \psi^j) + B_i(\phi_{i,2}^j)]$ for $j = 1, 2$. Then by Proposition 6.3,

$$\begin{aligned} &\|T_i(\varphi^1) - T_i(\varphi^2)\|_{*,m-2,B_R} \\ &\quad \le C\left[\|H_i(\bar\phi_1^1 + \bar\phi_2^1, \psi^1) - H_i(\bar\phi_1^2 + \bar\phi_2^2, \psi^2)\|_{m,\beta,B_R} + \|B_i(\phi_{i,2}^1 - \phi_{i,2}^2)\|_{m,\beta,B_R}\right] \\ &\quad \le C\left[\|\psi^1 - \psi^2\|_\infty + \|\phi_{i,2}^1 - \phi_{i,2}^2\|_{**,\beta}^2 + \|\phi_{i,1}^1 - \phi_{i,1}^2\|_{*,m-2,\beta}^2 + \|\psi^1 - \psi^2\|_\infty^2\right. \\ &\qquad + R^{-1}[\|\psi^1 - \psi^2\|_\infty + \|\phi_{i,1}^1 - \phi_{i,1}^2\|_{*,m-2,\beta} + \|\phi_{i,2}^1 - \phi_{i,2}^2\|_{**,\beta}] \\ &\qquad \left. + R^{m-2}\|\phi_{i,2}^1 - \phi_{i,2}^2\|_{**,\beta}\right]. \end{aligned}$$

As a consequence, using (7.19) and (7.20), we get that $\mathcal{A}$ is a contraction mapping in B_M and problem (7.6)–(7.7) has a fixed point. ∎

8. The reduced problem

In Section 7 we proved the existence of a solution $(\phi_1, \dots, \phi_N, \psi)$ to the coupled system of equations

$$\Delta_y \phi_i + f'(\Gamma_0)\phi_i + B_i(\phi_i) + H_i(\phi, \psi) = \sum_{j=1}^2 c_{ij} e^{\Gamma_0(y)} Z_j \quad \text{in } B_R,$$

for $i = 1, \ldots, N$, and

$$L\psi + G(\psi, \phi) = 0 \quad \text{in } \mathbb{R}^2.$$

The solution is described in Proposition 7.1, and estimates are contained in (7.5).

In order to obtain an actual solution to our main problem (5.1), we need to show that the reduced system

$$c_{ij} = c_{ij}[B_i(\phi_{i,2}) + H_i(\phi, \psi)] = 0 \quad \text{for } i = 1, \ldots, N,\ j = 1, 2,$$

can be solved provided the points $P_1, \ldots, P_N$ in (2.3)–(2.4) are chosen properly. From Section 7 we get that $\|B_i(\phi_{i,2}) + H_i(\phi, \psi)\|_{m,\beta,B_R} \lesssim \varepsilon\mu \log|\log \varepsilon|$. Hence from Proposition 6.3, we obtain that

$$c_{ij} = \gamma_j \int_{B_R} [H_i(\phi, \psi) + B_i(\phi_{i,2})] Z_j \, dy + \varepsilon^{1+\sigma} \mathbf{Y}(P),$$

for some $\sigma > 0$. Here, and in the rest of this section, by $\mathbf{Y}(P)$ we denote a smooth function, uniformly bounded as $\varepsilon \to 0$ for points $P = (P_1, \ldots, P_N)$ satisfying (2.3)–(2.4). The specific expression of this function changes from line to line, and even in the same line.

In addition, since $H_i(\phi, \psi) = \mathcal{N}_i(\sum_{i=1}^N \eta_i \phi_i + \psi) + \tilde{E}_i + (e^{\Gamma_0} + b_i)\psi$, using estimates (7.11) and (7.13), we find

$$\int_{B_R} \left[\mathcal{N}_i \left(\sum_{i=1}^N \eta_i \phi_i + \psi \right) + (e^{\Gamma_0} + b_i)\psi + B_i(\phi_{i,2}) \right] Z_j = \varepsilon^{1+\sigma} \mathbf{Y}(P).$$

This fact implies that solving the reduced system $c_{ij} = 0$ is equivalent to proving

$$\int_{B_R} \tilde{E}_i Z_j \, dy = \varepsilon^{1+\sigma} \mathbf{Y}(P) \quad \text{for } i = 1, \ldots, N \text{ and } j = 1, 2. \tag{8.1}$$

Formula (4.9) gives a rather explicit expression for $\tilde{E}_i$, which is used to get

$$\int_{B_R} \tilde{E}_i Z_1 \, dy = \varepsilon\mu_i\kappa_i [M F_{1,i}(P) + \log(|\log \varepsilon|) M \mathbf{Y}(P) + G_{1,i}(P)],$$

where $M = \int_{\mathbb{R}^2} U(y) y_1 Z_1(y) \, dy$,

$$F_{1,i}(P) = -\left[\log \varepsilon \left(2 \frac{R_i}{\sqrt{(1 + R_i^2)^3}} - \alpha \frac{R_i}{\kappa_i \sqrt{1 + R_i^2}} \right) + \sum_{j \neq i} \frac{\kappa_j}{\kappa_i} 4 \frac{[A_j^{-1}(P_i - P_j)]_1}{|A_j^{-1}(P_i - P_j)|^2} \right]$$

and

$$G_{1,i} = c_{1,i} \int_{\mathbb{R}^2} U \Gamma_0 y_1 Z_1 \, dy + \frac{R_i(3h^2 + R_i^2)}{2h(h^2 + R_i^2)^{\frac{3}{2}}} M + \varepsilon\mu_i \mathbf{Y}(P).$$

On the other hand,

$$\int_{B_R} \tilde{E}_i Z_2 \, dy = \varepsilon\mu_i\kappa_i M [F_{2,i}(P) + \mathbf{Y}(P) + \varepsilon\mu_i \mathbf{Y}(P)],$$

where

$$F_{2,i} = -\Big[\sum_{j\neq i} \frac{\kappa_j}{\kappa_i} 4 \frac{[A_j^{-1}(P_i - P_j)]_2}{|A_j^{-1}(P_i - P_j)|^2}\Big].$$

Now we recall the form of the points $P_1, \dots, P_N$ as in (2.3)–(2.4):

$$P_i = (a_i, b_i) = (r_0 + s, 0) + \frac{(\hat{a}_i, \hat{b}_i)}{|\log \varepsilon|}, \quad R_i = \sqrt{a_i^2 + b_i^2}$$

and define

$$\widetilde{P}_i = \big(\sqrt{h^2 + r_0^2}\,\hat{a}_i, \hat{b}_i\big).$$

Inserting this information in (8.1) we obtain that the reduced problem is

$$\Big(2\frac{hR_i}{\sqrt{(h^2+R_i^2)^3}} - \alpha \frac{hR_i}{\kappa_i \sqrt{h^2 + R_i^2}}\Big) + 4\sum_{j\neq i} \frac{\kappa_j}{\kappa_i} \frac{[(\widetilde{P}_i - \widetilde{P}_j)]_1}{|(\widetilde{P}_i - \widetilde{P}_j)|^2} = \frac{\log|\log\varepsilon|}{|\log\varepsilon|}\mathbf{Y}(P), \tag{8.2}$$

$$\sum_{j\neq i} \frac{\kappa_j}{\kappa_i} 4 \frac{[(\widetilde{P}_i - \widetilde{P}_j)]_2}{|(\widetilde{P}_i - \widetilde{P}_j)|^2} = \frac{\mathbf{Y}(P)}{|\log\varepsilon|}, \tag{8.3}$$

where again $\mathbf{Y}(P)$ denotes a generic smooth function, uniformly bounded as $\varepsilon \to 0$ for points $P = (P_1, \dots, P_N)$ satisfying (2.3)–(2.4).

The non-linear system (8.2)–(8.3) is a perturbation of the following limit problem (1.13), which for convenience we write using complex notation:

$$\sum_{j\neq i} \frac{\kappa_j}{\mathbf{P}_i - \mathbf{P}_j} = \Big(\kappa_i \frac{hr_0}{2\sqrt{(h^2 + r_0^2)^3}} - \alpha \frac{hr_0}{4\sqrt{h^2 + r_0^2}}\Big) \tag{8.4}$$

for $i = 1, \dots, N$. Here $\mathbf{P}_j = (\mathbf{P}_{j,1}, \mathbf{P}_{j,2})$ is identified with the complex number $\mathbf{P}_j = \mathbf{P}_{j,1} + i\mathbf{P}_{j,2}$.

For all $i = 1, \dots, N$, let $\mathbb{F}_i : \mathbb{C}^N \mapsto \mathbb{C}$ be the ith left-hand side in (8.4), that is,

$$\mathbb{F}_i(\mathbf{P}) = \sum_{j\neq i} \frac{\kappa_j}{\mathbf{P}_i - \mathbf{P}_j} \quad \text{for } i = 1, \dots, N,$$

and let U_i denote the right-hand side of (8.4),

$$U_i(r_0) = \Big(\kappa_i \frac{hr_0}{2\sqrt{(h^2 + r_0^2)^3}} - \alpha \frac{hr_0}{4\sqrt{h^2 + r_0^2}}\Big).$$

The point $\mathbf{P^0}$ satisfies

$$\mathbb{F}_i(\mathbf{P^0}) = U_i(r_0).$$

We can explicitly calculate the derivative of $\mathbb{F}$ at $\mathbf{P^0}$, and we get

$$d\mathbb{F}_{\mathbf{P^0}} = \begin{pmatrix} -\sum_{i=2}^N \kappa_i T_{1i} & \kappa_2 T_{12} & \cdots & \kappa_N T_{1N} \\ \kappa_1 T_{21} & -\sum_{i=1, i\neq 2}^N \kappa_i T_{2i} & \cdots & \kappa_N T_{2N} \\ \cdots & & & \\ \kappa_1 T_{N1} & \kappa_2 T_{N2} & \cdots & -\sum_{i=1}^{N-1} \kappa_i T_{Ni} \end{pmatrix},$$

where $T_{ij} = 1/(\mathbf{P}_i^0 - \mathbf{P}_j^0)^2 = T_{ji}$. A direct inspection gives that the vector $\mathbf{e}_0 = (1, \ldots, 1) \in \mathbb{C}^N$ is an element of the kernel of $d\mathbb{F}_{\mathbf{P}^0}$. The non-degeneracy assumption on the point $\mathbf{P}^0$ means precisely that this is the only element in the kernel.

We look for a solution to (8.2)–(8.3) as a small perturbation of $\mathbf{P}^0$. Let $\mathbf{q} = (q_1, \ldots, q_N) \in \mathbb{C}^N$, and redefine $\bar{P}_j$ in complex variables: we write

$$\tilde{P}_j = \mathbf{P}_j^0 + q_j, \quad j = 1, \ldots, N,$$

and then the reduced problem (8.2)–(8.3) can be written as

$$\mathbb{F}_i(\tilde{P}) = U_i(R_i) + \tilde{\sigma}_i, \quad i = 1, \ldots, N, \tag{8.5}$$

with $\mathrm{Re}(\tilde{\sigma}_i) = \frac{\log|\log\varepsilon|}{|\log\varepsilon|}\mathbf{Y}(P)$ and $\mathrm{Im}(\tilde{\sigma}_i) = \frac{1}{|\log\varepsilon|}\mathbf{Y}(P)$, where $\mathbf{Y}(P)$ again denotes a smooth function, uniformly bounded as $\varepsilon \to 0$ for points $P = (P_1, \ldots, P_N)$ satisfying (2.3)–(2.4). We have the expansions

$$\mathbb{F}(\tilde{P}) = \mathbb{F}(\mathbf{P}^0) + d\mathbb{F}_{\mathbf{P}^0}(\mathbf{q}) + O(|\mathbf{q}|^2)$$

and

$$U_i(R_i) = U_i(r_0) + U_i'(r_0)\Big(s + \frac{\mathrm{Re}(\mathbf{P}_i^0 + q_i)}{\sqrt{r_0^2 + h^2}|\log\varepsilon|}\Big) + O\Big(\Big(\frac{|\mathbf{q}|}{|\log\varepsilon|} + |s|\Big)^2\Big) + O\Big(\frac{|s|}{|\log\varepsilon|}\Big).$$

Thus (8.5) takes the form

$$d\mathbb{F}_{\mathbf{P}^0}(\mathbf{q}) = \mathscr{G}(s, \mathbf{q}) + \frac{sh}{2}\Big[\frac{h^2 - 2r_0^2}{(h^2 + r_0^2)^{\frac{5}{2}}}\mathbf{e}_1 - \frac{\alpha}{2}\frac{h^2}{(h^2 + r_0^2)^{\frac{3}{2}}}\mathbf{e}_0\Big], \tag{8.6}$$

where $\mathbf{e}_1 = (\kappa_1, \ldots, \kappa_N)$ and $\mathscr{G}(s, \mathbf{q})$ is a smooth function, with $\mathscr{G}(0,0) = 0$, $D_\mathbf{q}\mathscr{G}(0,0) = 0$, and $D_s\mathscr{G}(0,0) = o(1)$ as $\varepsilon \to 0$. In addition, $\mathrm{Re}\,\mathscr{G}(s, \mathbf{q}) = O(\frac{\log|\log\varepsilon|}{|\log\varepsilon|})$ and $\mathrm{Im}\,\mathscr{G}(s, \mathbf{q}) = O(\frac{1}{|\log\varepsilon|})$ as $\varepsilon \to 0$, in the range for s and $\mathbf{q}$ we are considering. Since $d\mathbb{F}$ has a one-dimensional kernel, we have that the kernel of $(d\mathbb{F})^\intercal$ is also one-dimensional and it is spanned by $\mathbf{e}_1$. From the value of α given by (1.14), the projection of the right-hand side of (8.6) onto $\mathbf{e}_1$ is equal to

$$\mathscr{G} \cdot \mathbf{e}_1 - sh\frac{r_0^2 \sum \kappa_i^2}{(h^2 + r_0^2)^{\frac{5}{2}}}.$$

Now we consider the following projected problem:

$$d\mathbb{F}_{\mathbf{P}^0}(\mathbf{q}) = \mathscr{G}(s, \mathbf{q}) + \frac{sh}{2}\Big[\frac{h^2 - 2r_0^2}{(h^2 + r_0^2)^{\frac{5}{2}}}\mathbf{e}_1 - \frac{\sum \kappa_i^2}{\sum \kappa_i}\frac{h^2}{(h^2 + r_0^2)^{\frac{5}{2}}}\mathbf{e}_0\Big] - \frac{\mathscr{G} \cdot \mathbf{e}_1 - sh\frac{r_0^2 \sum \kappa_i^2}{(h^2+r_0^2)^{\frac{5}{2}}}}{\sum \kappa_i^2}\mathbf{e}_1.$$

Since $\mathbf{P}^0$ is a non-degenerate solution, we can solve the above problem uniquely in $\mathbf{q} := \mathbf{q}(s)$ using the implicit function theorem around $(\mathbf{q}, s) = (0, 0)$. From the estimates on $\mathcal{G}$, we get that $|\mathbf{q}| \lesssim \frac{\log|\log \varepsilon|}{|\log \varepsilon|}$. Using the Banach fixed point theorem, it is possible to find a solution $s = s^*$ with

$$\mathcal{G} \cdot \mathbf{e}_1 - sh \frac{r_0^2 \sum \kappa_i^2}{(h^2 + r_0^2)^{\frac{5}{2}}} = 0.$$

Since $\operatorname{Re} \mathcal{G}(s, \mathbf{q}) = O(\frac{\log|\log \varepsilon|}{|\log \varepsilon|})$ as $\varepsilon \to 0$, one has that $|s^*| \lesssim \frac{\log|\log \varepsilon|}{|\log \varepsilon|}$. We thus get a solution of (8.6),

$$P_i = \begin{pmatrix} r_0 + s^* + \frac{\operatorname{Re}(\mathbf{P}_i^0 + q_i(s^*))}{|\log \varepsilon| \sqrt{h^2 + r_0^2}} \\ \frac{\operatorname{Im}(\mathbf{P}_i^0 + q_i(s^*))}{|\log \varepsilon|} \end{pmatrix},$$

with the expected estimates. This concludes the proof of our result.

A. Proofs of Propositions 6.1 and 6.3

Proof of Proposition 6.1. To solve equation (6.1), we decompose g and ψ into Fourier modes as

$$g(x) = \sum_{j=-\infty}^{\infty} g_j(r) e^{ji\theta}, \quad \psi(x) = \sum_{j=-\infty}^{\infty} \psi_j(r) e^{ji\theta}, \quad x = re^{i\theta}.$$

Then

$$L[\psi] = \frac{1}{h^2 + r^2} \Big(\frac{h^2}{r^2} + 1 \Big) \partial_\theta^2 \psi + \frac{h^2}{r} \partial_r \Big(\frac{r}{h^2 + r^2} \partial_r \psi \Big).$$

Thus this operator decouples the Fourier modes: equation (6.1) becomes equivalent to the following infinite set of ODEs:

$$\begin{aligned} &L_k[\psi_k] + g_k(r) = 0, \quad r \in (0, \infty), \\ &L_k[\psi_k] := \frac{h^2}{r} \Big(\frac{r}{r^2 + h^2} \psi_k' \Big)' - \frac{k^2}{h^2 + r^2} \Big(\frac{h^2}{r^2} + 1 \Big) \psi_k, \quad k \in \mathbb{Z}. \end{aligned} \tag{A.1}$$

When $r \to 0$ or $r \to +\infty$, the operator L_k resembles

$$\begin{aligned} L_k[p] &\sim \frac{1}{r}(rp')' - \frac{k^2 p}{r^2} \qquad \text{as } r \to 0, \\ L_k[p] &\sim \frac{h^2}{r} \Big(\frac{1}{r} p' \Big)' - \frac{k^2 p}{r^2} \quad \text{as } r \to +\infty. \end{aligned}$$

For $k \geq 1$, L_k satisfies the maximum principle. This gives the existence of a positive function $z_k(r)$ with $L_k[z_k] = 0$ with

$$z_k(r) \sim r^k \text{ as } r \to 0, \quad z_k(r) \sim r^{\frac{1}{2}} e^{(k/h)r} \text{ as } r \to +\infty.$$

Take $k = 1$. The function

$$\bar{\psi}(r) = z_1(r) \int_r^\infty \frac{(1+s^2)\,ds}{s z_1(s)^2} \int_0^s \frac{1}{1+\rho^\nu} z_1(\rho)\rho\,d\rho$$

solves $L_1[\bar{\psi}] + \frac{1}{1+r^\nu} = 0$, and satisfies the bounds

$$\bar{\psi}(r) = O(r^2) \ \text{ as } r \to 0 \quad \text{and} \quad \bar{\psi}(r) = O(r^{-\nu+2}) \ \text{ as } r \to +\infty.$$

We take $\bar{\psi}$ as a barrier for the equation at $k = 1$. In addition, this function works as a barrier for $k \geq 2$. For $k \geq 2$, the function

$$\psi_k(r) = z_k(r) \int_r^\infty \frac{(1+s^2)\,ds}{s z_k(s)^2} \int_0^s h_k(\rho) z_k(\rho)\rho\,d\rho$$

is the unique decaying solution (A.1), and it satisfies the estimate

$$|\psi_k(r)| \leq \frac{4}{k^2} \|g\|_\nu \bar{\psi}(r),$$

since

$$|g_k(r)| \leq \frac{\|g\|_\nu}{1+r^\nu}.$$

If $k = 0$, the solution is given by the explicit formula

$$\psi_0(r) = -\int_0^r \frac{1+s^2}{h^2 s}\,ds \int_0^s g_0(\rho)\rho\,d\rho$$

and satisfies the bound

$$|\psi_0(r)| \leq C \|g\|_\nu (1+r^2).$$

The function

$$\psi(x) := \sum_{j=-\infty}^{\infty} \psi_j(r) e^{ji\theta},$$

with the ψ_k being the functions built above, clearly defines a linear operator of g and satisfies estimate (6.2). The proof is concluded. ∎

Proof of Lemma 6.2. Setting $y = re^{i\theta}$, we write

$$h(y) = \sum_{k=-\infty}^{\infty} h_k(r) e^{ik\theta}, \quad \phi(y) = \sum_{k=-\infty}^{\infty} \phi_k(r) e^{ik\theta}.$$

The equation is equivalent to

$$\mathrm{L}_k[\phi_k] + h_k(r) = 0, \quad r \in (0,\infty), \tag{A.2}$$

where

$$\mathrm{L}_k[\phi_k] = \phi_k'' + \frac{1}{r}\phi_k' + e^{\Gamma_0}\phi_k - \frac{k^2}{r^2}\phi_k.$$

Using the formula of variation of parameters, the following formula (continuously extended to $r = 1$) defines a smooth solution of (A.2) for $k = 0$:

$$\phi_0(r) = -z(r) \int_1^r \frac{ds}{sz(s)^2} \int_0^s h_0(\rho) z(\rho) \rho \, ds, \quad z(r) = \frac{r^2 - 1}{1 + r^2}.$$

Noting that $\int_0^\infty h_0(\rho) z(\rho) \rho \, ds = \frac{1}{2\pi} \int_{\mathbb{R}^2} h(y) Z_0(y) \, dy$, we see that this function satisfies

$$|\phi_0(r)| \le C \left[\log(2 + r) \left| \int_{\mathbb{R}^2} h(y) Z_0(y) \, dy \right| + (1 + r)^{2-m} \|h\|_m \right].$$

Now we observe that

$$\phi_k(r) = -z(r) \int_0^r \frac{ds}{sz(s)^2} \int_0^s h_k(\rho) z(\rho) \rho \, ds, \quad z(r) = \frac{4r}{1 + r^2}$$

solves (A.2) for $k = -1, 1$ and satisfies

$$|\phi_k(r)| \le C \left[(1 + r) \sum_{j=1}^{2} \left| \int_{\mathbb{R}^2} h(y) Z_j(y) \, dy \right| + (1 + r)^{2-m} \|h\|_m \right].$$

For $k = 2$ there is a function $z(r)$ such that $\mathcal{L}_2[z] = 0$, $z(r) \sim r^2$ as $r \to 0$ and as $r \to \infty$. For $|k| \ge 2$ we have

$$\bar{\phi}_k(r) = \frac{4}{k^2} z(r) \int_0^r \frac{ds}{sz(s)^2} \int_0^s |h_k(\rho)| z(\rho) \rho \, ds$$

is a positive supersolution for equation (A.2), hence the equation has a unique solution ϕ_k with $|\phi_k(r)| \le \bar{\phi}_k(r)$. Thus

$$|\phi_k(r)| \le \frac{C}{k^2} (1 + r)^{2-m} \|h\|_m, \quad |k| \ge 2.$$

Thus

$$\phi(y) = \sum_{k=-\infty}^{\infty} \phi_k(r) e^{ik\theta}, \quad \text{with } y = re^{i\theta},$$

defines a linear operator of functions h which is a solution of equation (6.3) which, adding up the individual estimates above, satisfies estimate (6.5). As a corollary we find that similar bounds are obtained for first and second derivatives. In fact, let us set for a large $y = Re$, $R = |y| \gg 1$, $\phi_R(z) = R^{m-2} \phi(R(e + z))$. Then in a neighborhood of y, we find

$$\Delta_z \phi_R + \frac{8R^2}{(1 + R^2 |e + z|^2)^2} \phi_R + h_R(z) = 0, \quad |z| < \frac{1}{2},$$

where $h_R(z) = R^m h(R(e + z))$. Let us set

$$\delta_i = \left| \int_{\mathbb{R}^2} h Z_i \right|, \quad i = 0, 1, 2.$$

Then from (6.5) and a standard elliptic estimate we find

$$\|\nabla_z \phi_R\|_{L^\infty(B_{\frac{1}{4}}(0))} + \|\phi_R\|_{L^\infty(B_{\frac{1}{2}}(0))} \le C\Big[\delta_0 R^{m-2}\log R + \sum_{i=1}^{2}\delta_i R^{m-1} + \|h\|_m\Big],$$

using that $\|h_R\|_{L^\infty(B_{\frac{1}{2}}(0))} \le C\|h\|_m$. Now, since $[h_R]_{B_{\frac{1}{2}}(0),\alpha} \le C\|h\|_{m,\alpha}$, from interior Schauder estimates and the bound for ϕ_R we find

$$\|D_z^2\phi_R\|_{L^\infty(B_{\frac{1}{4}}(0))} + [D_z^2\phi_R]_{B_{\frac{1}{4}}(0),\alpha} \le C\Big[\delta_0 R^{m-2}\log R + \sum_{i=1}^{2}\delta_i R^{m-1} + \|h\|_{m,\alpha}\Big].$$

From these relations, estimates (6.5) and (6.6) follow. ■

Proof of Proposition 6.3. We consider a standard linear extension operator $h \mapsto \tilde{h}$ to the entire $\mathbb{R}^2$, in such a way that the support of $\tilde{h}$ is contained in B_{2R} and $\|\tilde{h}\|_{m,\beta} \le C\|h\|_{m,\beta,B_R}$, with C independent of all large R. The operator B_i is defined in (5.11) and the coefficients are of class C^1 in the entire $\mathbb{R}^2$ and have compact support in B_{2R}. Then we consider the auxiliary problem in the entire space,

$$\Delta\phi + e^{\Gamma_0}\phi + B_i[\phi] + \tilde{h}(y) = \sum_{j=0}^{2} c_{ij} e^{\Gamma_0} Z_j \quad \text{in } \mathbb{R}^2, \tag{A.3}$$

where, assuming that $\|h\|_m < +\infty$ and ϕ is of class C^2, $c_{ij} = c_{ij}[h,\phi]$ are the scalars defined so that

$$\gamma_i \int_{\mathbb{R}^2} (B_i[\phi] + \tilde{h}(y))Z_j = c_{ij}, \quad \gamma_j^{-1} = \int_{\mathbb{R}^2} e^{\Gamma_0} Z_j^2.$$

For $j = 1,2$, we have $B_i[Z_j] = O((1+|y|)^{-2})\varepsilon\mu_i + O((1+|y|)^{-(3+a)})\varepsilon\mu_i \log|\log\varepsilon|$, by estimates (5.7) and (5.12). Similarly, for $j = 0$, we have

$$B_i[Z_0] = O((1+|y|)^{-3})\varepsilon\mu_i + O((1+|y|)^{-(2+a)})\varepsilon\mu_i \log|\log\varepsilon|.$$

Since $m > 2$, we get

$$\int_{\mathbb{R}^2} B_i[\phi]Z_j = \int_{\mathbb{R}^2} \phi\tilde{B}_i[Z_j] = O(\|\phi\|_{m-2})\varepsilon\mu_i \log|\log\varepsilon|,$$

where $\tilde{B}_i$ have same estimates as B_i mentioned above. On the other hand,

$$\int_{\mathbb{R}^2\setminus B_R} h(y)Z_0 = O(R^{2-m})\|h\|_{m,\beta,B_R}, \quad \int_{\mathbb{R}^2\setminus B_R} h(y)Z_j = O(R^{1-m})\|h\|_{m,\beta,B_R}$$

for $j = 1,2$. In addition, we readily check that

$$\|B_i[\phi]\|_{m,\beta} \le C\frac{\delta}{|\log\varepsilon|}\|\phi\|_{*,m-2,\beta},$$

where

$$\|\phi\|_{*,m-2,\beta} = \|D_y^2\phi\|_{m,\beta} + \|D_y\phi\|_{m-1} + \|\phi\|_{m-2}.$$

Let us consider the Banach space X of all $C^{2,\beta}(\mathbb{R}^2)$ functions with $\|\phi\|_{*,m-2,\beta} < +\infty$. We find a solution of (A.3) if we solve the equation

$$\phi = \mathcal{A}[\phi] + \mathcal{H}, \quad \phi \in X, \tag{A.4}$$

where

$$\mathcal{A}[\phi] = \mathcal{T}\Big[B_i[\phi] - \sum_{j=0}^{2} c_{ij}[0,\phi]e^{\Gamma_0} Z_j\Big], \quad \mathcal{H} = \mathcal{T}\Big[\tilde{h} - \sum_{j=0}^{2} c_{ij}[\tilde{h},0]e^{\Gamma_0} Z_j\Big],$$

and $\mathcal{T}$ is the operator built in Lemma 6.2. We observe that

$$\|\mathcal{A}[\phi]\|_{*,m-2,\beta} \le C\frac{\delta}{|\log\varepsilon|}\|\phi\|_{*,m-2,\beta}, \quad \|\mathcal{H}\|_{*,m-2,\beta} \le C\|h\|_{m,\beta,B_R}.$$

So we find that equation (A.4) has a unique solution that defines a linear operator of h and that satisfies

$$\|\phi\|_{*,m-2,\beta} \le C\|h\|_{m,\beta,B_R}.$$

The result of the proposition follows by just setting $T_i[h] = \phi|_{B_R}$. The proof is concluded. ∎

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