On von Neumann equivalence and group approximation properties

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Abstract. The notion of von Neumann equivalence (vNE), which encapsulates both measure equivalence and W^*-equivalence, was introduced recently by Ishan, Peterson, and Ruth (2019). They have shown that many analytic properties, such as amenability, property (T), the Haagerup property, and proper proximality are preserved under von Neumann equivalence. In this article, we expand on the list of properties that are stable under von Neumann equivalence, and prove that weak amenability, weak Haagerup property, and the approximation property (AP) are von Neumann equivalence invariants. In particular, we get that AP is stable under measure equivalence. Furthermore, our techniques give an alternate proof for vNE-invariance of the Haagerup property.

1. Introduction

Two infinite countable discrete groups Γ and Λ are *measure equivalent* if there is a σ-finite measure space (Ω, m) with a measurable, measure-preserving action of $\Gamma \times \Lambda$, so that both the actions $\Gamma \curvearrowright (\Omega, m)$ and $\Lambda \curvearrowright (\Omega, m)$ admit finite-measure fundamental domains $Y, X \subset \Omega$:

$$\Omega = \bigsqcup_{\gamma \in \Gamma} \gamma Y = \bigsqcup_{\lambda \in \Lambda} \lambda X.$$

This notion was introduced by Gromov in [16, §0.5.E] in analogy with the topological notion of quasi-isometry for finitely generated groups and is fundamental in modern ergodic theory, especially in the study of measured group theory and orbit equivalence. This has been investigated intensively over the years through an entire array of techniques and many groundbreaking results have been discovered [12,13,24,25,27]. Moreover, striking connections with classical invariants in group theory have been established [14, 15].

Two groups Γ and Λ are *W^*-equivalent* if they have isomorphic group von Neumann algebras, $(L\Gamma \simeq L\Lambda)$. This concept plays a key role in the classification of von Neumann algebras and has been studied with great success over the last decades: Connes' work on amenability [8], Jones' finite index theory [23], Voiculescu's free probability [31,32], and more recently Popa's deformation/rigidity theory [28].

Over the years, many discoveries hinted towards the notion of W^*-equivalence being somewhat analogous to measure equivalence as both equivalence relations preserve many of the same "approximation type" properties. Both equivalence relations are rather coarse, especially in the amenable case (there is a single W^*-equivalence class of infinite amenable ICC countable groups, i.e., groups in which every non-trivial element has infinite conjugacy class, and a single measure equivalence class of infinite amenable groups); both behave well with respect to approximation properties (or lack thereof), thus preserving the Haagerup property, weak amenability or Kazdhan property (T) for groups. These similarities led Shlyakhtenko to ask whether measure equivalence implied W^*-equivalence in the setting of ICC groups. It was shown in [7] that this is not the case, although the converse implication of whether W^*-equivalence implies measure equivalence is still open. In [5, Corollary 6.2], were provided infinitely many measure equivalent groups that pairwise are not W^*-equivalent. More extremely, the main results in [1, 6, 19] can be used to provide infinitely many measure equivalent groups that are in fact W^*-superrigid (i.e., their W^*-equivalence classes coincide with their group isomorphism classes).

Recently, in [20], the following natural generalization of measure equivalence was introduced in the non-commutative setting. A *fundamental domain* for the action $\Gamma \curvearrowright^\sigma \mathcal{M}$ of a discrete group Γ on a von Neumann algebra $\mathcal{M}$ is a projection $p \in \mathcal{M}$ such that $\sum_{\gamma \in \Gamma} \sigma_\gamma(p) = \mathbf{1}$, where the sum converges in the strong operator topology. Two groups Γ and Λ are *von Neumann equivalent*, denoted by $\Gamma \sim_{\mathrm{vNE}} \Lambda$, if there exists a semi-finite von Neumann algebra $\mathcal{M}$ with a faithful normal semi-finite trace Tr, and trace-preserving commuting actions $\Gamma \curvearrowright \mathcal{M}$, $\Lambda \curvearrowright \mathcal{M}$ such that the actions of Γ and Λ individually admit a finite-trace fundamental domain. Von Neumann equivalence encapsulates both measure equivalence and W^*-equivalence. The former is straightforward to see by restricting to the case when $\mathcal{M}$ is abelian. To see the latter, let $\theta: L\Gamma \to L\Lambda$ be a $*$-isomorphism, and consider $\mathcal{M} = \mathcal{B}(\ell^2 \Lambda)$, where one has a trace-preserving action $\sigma: \Gamma \times \Lambda \to \mathrm{Aut}(\mathcal{M})$ given by $\sigma_{(s,t)}(T) = \theta(\lambda_s)\rho_t T \rho_t^* \theta(\lambda_s^*)$, where $\rho: \Lambda \to \mathcal{U}(\ell^2 \Lambda)$ is the right regular representation of Λ, and $\lambda: \Gamma \to \mathcal{U}(\ell^2 \Gamma)$ is the left regular representation of Γ. One can now see that the rank-one projection onto the subspace $\mathbb{C}\delta_e$ is a common fundamental domain for the actions of both Γ and Λ.

By developing an analysis of fundamental domains for actions of groups on semi-finite von Neumann algebra, and combining it with the language of operator spaces and Hilbert C^*-modules, we introduced a general procedure in [20] for inducing actions and unitary representations in this setting. We used this to show that many analytic properties, such as amenability, property (T), the Haagerup property, and proper proximality are preserved under von Neumann equivalence.

Invariance of the Cowling–Haagerup constant under measure equivalence was proved in [22], where the author extended the result of Cowling and Zimmer [10] from orbit equivalence to measure equivalence. In this article, we extend this result to von Neumann equivalence and prove the stability of (finite) Cowling–Haagerup constant (and hence weak amenability) under von Neumann equivalence. In the process, we discover an alternate proof for the invariance of the Haagerup property under von Neumann equiv-

alence. We also prove that the weak Haagerup property [26], and the approximation property (AP) [17] are also preserved under von Neumann equivalence. In particular, AP is preserved under measure equivalence, a result that has not yet appeared in the literature.

Theorem 1.1. *Weak amenability, weak Haagerup property, approximation property, and the Haagerup property are von Neumann equivalence invariants.*

On the one hand, if one makes an attempt to adapt proofs from measure equivalence setting to vNE-setting, one immediately runs into an obstruction stemming from the absence of the notion of "cocycle". This makes the problem of proving stability of any property under vNE challenging and often requires novel ideas. On the other hand, limitation of the induction techniques developed in [20] to tackle Theorem 1.1 lies in the lack of characterization of these properties (except the Haagerup property) solely in terms of actions, and of unitary representations. Consequently, to prove Theorem 1.1 we introduce a general induction procedure for inducing Herz–Schur multipliers via von Neumann equivalence from Λ to Γ. Our approach is inspired by Jolissaint's work in [22], where some similar results are shown in the setting of measure equivalence.

Using Popa and Vaes' result [29, Theorem 5.1], Cartan rigidity results are established in [2, Theorem 4.17] for groups that are both properly proximal and weakly amenable. Combining these with Theorem 1.1 above and [20, Theorem 1.3], we immediately obtain the following.

Corollary 1.2. *Suppose Γ is countable discrete group which is properly proximal and weakly amenable. If Λ is any group von Neumann equivalent to Γ, then Λ is Cartan-rigid, i.e., $L^\infty(X, \mu) \rtimes \Lambda$ admits $L^\infty(X, \mu)$ as its unique Cartan subalgebra, up to unitary conjugacy, for any free ergodic p.m.p. action $\Lambda \curvearrowright (X, \mu)$.*

2. Preliminaries and notations

We set up the notations and collect some facts that will be needed in this article. For facts regarding semi-finite von Neumann algebras, we refer the reader to [30] for their proofs, whereas those regarding multipliers on a group can be found in [9, 21, 22]. The identity element in a von Neumann algebra $\mathcal{M}$ will be denoted by **1**.

2.1. Semi-finite traces

A *trace* on a von Neumann algebra $\mathcal{M}$ is a function Tr on the positive cone $\mathcal{M}_+$ with values in the extended reals $[0, \infty]$ satisfying the following conditions:

(i) $\mathrm{Tr}(x + y) = \mathrm{Tr}(x) + \mathrm{Tr}(y)$, $x, y \in \mathcal{M}_+$,

(ii) $\mathrm{Tr}(\alpha x) = \alpha \, \mathrm{Tr}(x)$, $\alpha \geq 0$, $x \in \mathcal{M}_+$,

(iii) $\mathrm{Tr}(x^*x) = \mathrm{Tr}(xx^*)$, $x \in \mathcal{M}$.

A trace Tr is said to be *faithful* if $\mathrm{Tr}(x) > 0$ for any non-zero $x \in \mathcal{M}_+$, *semi-finite* if for every non-zero $x \in \mathcal{M}_+$, there exists a non-zero $y \in \mathcal{M}_+$, $y \le x$ with $\mathrm{Tr}(y) < \infty$, *finite* if $\mathrm{Tr}(\mathbf{1}) < \infty$, and *normal* if $\mathrm{Tr}(\sup_i x_i) = \sup_i \mathrm{Tr}(x_i)$ for every bounded increasing net $\{x_i\}$ in $\mathcal{M}_+$. A separable von Neumann algebra $\mathcal{M}$ is *semi-finite* if and only if it admits a faithful normal semi-finite trace. If $\mathcal{M}$ is a semi-finite von Neumann algebra with a faithful normal semi-finite trace Tr, we set $\mathfrak{n}_{\mathrm{Tr}} = \{x \in \mathcal{M} \mid \mathrm{Tr}(x^*x) < \infty\}$, and $\mathfrak{m}_{\mathrm{Tr}} = \{\sum_{j=1}^n x_j^* y_j \mid x_j, y_j \in \mathfrak{n}_{\mathrm{Tr}}, 1 \le j \le n\}$. Both $\mathfrak{n}_{\mathrm{Tr}}$ and $\mathfrak{m}_{\mathrm{Tr}}$ are ideals in $\mathcal{M}$, and the trace Tr extents to a $\mathbb{C}$-valued linear functional on $\mathfrak{m}_{\mathrm{Tr}}$, which is called the *definition ideal* of Tr.

We let $L^1(\mathcal{M}, \mathrm{Tr})$ denote the completion of $\mathfrak{m}_{\mathrm{Tr}}$ under the norm $\|a\|_1 = \mathrm{Tr}(|a|)$, and then the bilinear form $\mathcal{M} \times \mathfrak{m}_{\mathrm{Tr}} \ni (x, a) \mapsto \mathrm{Tr}(xa)$ extends to the duality between $\mathcal{M}$ and $L^1(\mathcal{M}, \mathrm{Tr})$ so that we may identify $L^1(\mathcal{M}, \mathrm{Tr})$ with $\mathcal{M}_*$. We also let $L^2(\mathcal{M}, \mathrm{Tr})$ denote the Hilbert space completion of $\mathfrak{n}_{\mathrm{Tr}}$ under the inner product $\langle a, b \rangle_{\mathrm{Tr}} = \mathrm{Tr}(b^*a)$. Left multiplication of $\mathcal{M}$ on $\mathfrak{n}_{\mathrm{Tr}}$ then induces a normal faithful representation of $\mathcal{M}$ in $\mathcal{B}(L^2(\mathcal{M}, \mathrm{Tr}))$, which is called the *standard representation*.

2.2. Multipliers on discrete groups and associated multiplier algebras

Let Γ be an infinite countable discrete group. We will denote by $c_0(\Gamma)$ the space of all complex-valued functions on Γ vanishing at infinity, i.e., $f \in c_0(\Gamma)$ if for every $\varepsilon > 0$, there exists a finite set $F \subset \Gamma$ such that $|f(s)| < \varepsilon$ for all $s \in \Gamma \setminus F$. The space of all bounded complex-valued functions on Γ will be denoted by $\ell^\infty \Gamma$, and $c_{00}(\Gamma)$ will denote the space of all finitely supported functions on Γ. For a subset $E \subset \Gamma$, we denote the characteristic function of E by $\mathbf{1}_E$.

The *Fourier–Stieltjes algebra* of Γ, denoted by $B(\Gamma)$, is the set of all coefficient functions of unitary representations of Γ, that is, for every $\varphi \in B(\Gamma)$, there exist a unitary representation $(\pi, \mathcal{H})$ of Γ and vectors $\xi, \eta \in \mathcal{H}$ such that $\varphi(s) = \langle \pi(s)\xi, \eta \rangle$ for every $s \in \Gamma$. It is a Banach algebra with respect to the norm

$$\|\varphi\|_B = \inf \|\xi\| \|\eta\|,$$

where the infimum is taken over all representations of φ as above.

The *Fourier algebra* of Γ, denoted by $A[\Gamma]$, is the set of all coefficient functions associated to the left regular representation of Γ. It is the norm closure of the algebra of finitely supported functions in the algebra $B(\Gamma)$.

A *Herz–Schur multiplier* on Γ is a function $\varphi \colon \Gamma \to \mathbb{C}$ for which there exist a Hilbert space $\mathcal{H}$ and bounded functions $\xi, \eta \colon \Gamma \to \mathcal{H}$ such that

$$\varphi(t^{-1}s) = \langle \xi(s), \eta(t) \rangle, \quad s, t \in \Gamma.$$

The set $B_2(\Gamma)$ of all Herz–Schur multipliers on Γ is a Banach algebra with respect to the pointwise product and to the norm

$$\|\varphi\|_{B_2} = \inf \|\xi\|_\infty \|\eta\|_\infty,$$

where the infimum is taken over all representations of φ as above. It turns out that $B_2(\Gamma)$ is a dual space, and the predual $Q(\Gamma)$ of $B_2(\Gamma)$ is obtained by completing $\ell^1\Gamma$ in the norm

$$\|\varphi\|_Q = \sup \left\{ \left| \sum_{s\in\Gamma} \varphi(s)u(s) \right| \,\Big|\, u \in B_2(\Gamma), \|u\|_{B_2} \leq 1 \right\}$$

(see, [11, 18]).

Definition 2.1. Let Γ be a countable discrete group.

(1) ([4]) We say that Γ has the *Haagerup property* if there exists a net $\{\varphi_i\}$ of normalized (i.e., $\varphi_i(e) = 1$ for every i), positive definite functions on Γ such that $\varphi_i \in c_0(\Gamma)$ for every i, and $\varphi_i \to 1$ pointwise.

(2) ([9]) We say that Γ is *weakly amenable* if there exists a net $\{\varphi_i\}$ of functions in $c_{00}(\Gamma)$ such that $\varphi_i \to 1$ pointwise and $\sup_i \|\varphi_i\|_{B_2} \leq C < \infty$. The *Cowling–Haagerup constant* $\Lambda_{\mathrm{cb}}(\Gamma)$ is the infimum of all constants $C < \infty$ for which such a net $\{\varphi_i\}$ exists.

(3) ([26]) We say that Γ has the *weak Haagerup property* if there exists a net $\{\varphi_i\}$ in $B_2(\Gamma) \cap c_0(\Gamma)$ such that $\sup_i \|\varphi_i\|_{B_2} \leq C < \infty$ and $\varphi_i \to 1$ pointwise. The *weak Haagerup constant* $\Lambda_{\mathrm{wcb}}(\Gamma)$ is the infimum of all constants $C < \infty$ for which such a net $\{\varphi_i\}$ exists.

(4) ([17]) We say that Γ has the *approximation property* if there exists a net $\{\varphi_i\}$ of functions in $c_{00}(\Gamma)$ such that $\varphi_i \to 1$ in the $\sigma(B_2(\Gamma), Q(\Gamma))$-topology.

Remark 2.2. It is easy to see that every finitely supported function on Γ can be realized as a coefficient of the left regular representation and hence $c_{00}(\Gamma) \subset A[\Gamma]$. Therefore, by [9, Proposition 1.1], Γ is weakly amenable if and only if there exists a net $\{\varphi_i\}$ in the Fourier algebra $A[\Gamma]$ such that $\varphi_i \to 1$ pointwise and $\sup_i \|\varphi_i\|_{B_2} < \infty$.

Remark 2.3. The inclusion map from $B(\Gamma)$ into $B_2(\Gamma)$ is a contraction (see [11, Corollary 1.8]), and so the $\sigma(B_2(\Gamma), Q(\Gamma))$-closure of any subset E of $B(\Gamma)$ contains the closure of E in the $B(\Gamma)$-norm. Hence Γ has AP if and only if the constant function 1 is in the $\sigma(B_2(\Gamma), Q(\Gamma))$-closure of $A[\Gamma]$ in $B_2(\Gamma)$.

Since $A[\Gamma] \subset B_2(\Gamma) \cap c_0(\Gamma)$, one always has $\Lambda_{\mathrm{wcb}}(\Gamma) \leq \Lambda_{\mathrm{cb}}(\Gamma)$, and a weakly amenable group has the weak Haagerup property. Similarly, as normalized, positive definite functions are Herz–Schur multipliers of norm one, it follows that if Γ has the Haagerup property, then it has the weak Haagerup property and $\Lambda_{\mathrm{wcb}}(\Gamma) = 1$. It is well known that all weakly amenable groups have AP, and there are non-weakly amenable groups with the AP as well (see [17]).

2.3. Actions on semi-finite von Neumann algebras

If Λ is a discrete group and $\Lambda \curvearrowright^\sigma \mathcal{M}$ is an action that preserves the trace Tr, then restricted to $\mathfrak{n}_{\mathrm{Tr}}$, the action is isometric with respect to $\|\cdot\|_2$ and hence gives a unitary representation in $\mathcal{U}(L^2(\mathcal{M}, \mathrm{Tr}))$, which is called the *Koopman representation* and denoted by

$\sigma^0 \colon \Lambda \to \mathcal{U}(L^2(\mathcal{M}, \mathrm{Tr}))$. Note that, considering $\mathcal{M} \subset \mathcal{B}(L^2(\mathcal{M}, \mathrm{Tr}))$ via the standard representation, we have that the action $\sigma \colon \Lambda \to \mathrm{Aut}(\mathcal{M}, \mathrm{Tr})$ becomes unitarily implemented via the Koopman representation, i.e., for $x \in \mathcal{M}$ and $s \in \Lambda$, we have $\sigma_s(x) = \sigma_s^0 x \sigma_{s^{-1}}^0$.

Definition 2.4 ([20]). Let $\Lambda \curvearrowright^{\sigma} \mathcal{M}$ be an action of a discrete group Λ on a von Neumann algebra $\mathcal{M}$. A *fundamental domain* for the action is a projection $p \in \mathcal{M}$ so that $\{\sigma_s(p)\}_{s \in \Lambda}$ gives a partition of unity.

Note that, if $p \in \mathcal{M}$ is a fundamental domain, then we obtain a normal inclusion $\theta_p \colon \ell^\infty \Lambda \to \mathcal{M}$ by $\theta_p(f) = \sum_{s \in \Lambda} f(s) \sigma_s(p)$. Moreover, this embedding is equivariant with respect to the Λ-actions, where $\Lambda \curvearrowright \ell^\infty \Lambda$ is the canonical left action given by $L_\lambda(f)(s) = f(\lambda^{-1}s)$. Indeed,

$$
\sigma_\lambda(\theta_p(f)) = \sum_{s \in \Lambda} f(s) \sigma_{\lambda s}(p) = \sum_{t \in \Lambda} f(\lambda^{-1}t) \sigma_t(p)
$$
$$
= \sum_{t \in \Lambda} (L_\lambda(f))(t) \sigma_t(p) = \theta_p(L_\lambda(f)).
$$

Conversely, if $\theta \colon \ell^\infty \Lambda \to \mathcal{M}$ is an equivariant embedding, then $\theta(\delta_e)$ gives a fundamental domain.

Definition 2.5 ([20]). Let Γ and Λ be countable discrete groups. A *von Neumann coupling* between Γ and Λ consists of a semi-finite von Neumann algebra $\mathcal{M}$ with a faithful normal semi-finite trace Tr and a trace-preserving action $\Gamma \times \Lambda \curvearrowright \mathcal{M}$ such that there exist finite-trace fundamental domains q and p for the Γ- and Λ-actions, respectively. The index of the von Neumann coupling is the ratio $\mathrm{Tr}(q)/\mathrm{Tr}(p)$ and is denoted by $[\Gamma : \Lambda]_{\mathcal{M}}$. We say that Γ and Λ are *von Neumann equivalent* and write $\Gamma \sim_{\mathrm{vNE}} \Lambda$ if there exists a von Neumann coupling between them.

An element $x \in \mathcal{M}$ is *compact*, if for every $\varepsilon > 0$, there exists a projection $p \in \mathcal{M}$ such that $\|xp\| < \varepsilon$ and $\mathbf{1} - p$ is finite. If $\mathcal{M}$ is a semi-finite von Neumann algebra with a faithful normal semi-finite trace Tr, and $p \in \mathcal{M}$ is a finite-trace projection, then the map $x \mapsto \mathrm{Tr}(xp)$ is continuous in the weak operator topology.

Lemma 2.6. *Suppose $\mathcal{M}$ is a semi-finite von Neumann algebra with a faithful normal semi-finite trace Tr, $x \in \mathcal{M}$ is compact, and $\{p_i\}$ is a net of finite-trace projections such that $p_i \to 0$ in the weak operator topology. If $\{\mathrm{Tr}(p_i)\}$ is uniformly bounded, then $\mathrm{Tr}(xp_i) \to 0$.*

Proof. Given $\varepsilon > 0$, there exists a projection $q \in \mathcal{M}$ such that $\|xq\| < \varepsilon$ and $\mathrm{Tr}(\mathbf{1} - q) < \infty$. Since $p_i \to 0$ in the weak operator topology, we get that $xp_i \to 0$ in the weak operator topology and hence $\mathrm{Tr}(p_i x(\mathbf{1} - q)) \to 0$. Moreover, we have that $\{\mathrm{Tr}(p_i)\}$ is uniformly bounded, say by C, whence it follows that

$$
\limsup_i |\mathrm{Tr}(xp_i)| \leq \limsup_i \big(\|xq\| |\mathrm{Tr}(p_i)| + |\mathrm{Tr}(p_i x(\mathbf{1} - q))| \big) \leq \varepsilon C.
$$

Since $\varepsilon > 0$ was arbitrary, the proof is complete. $\blacksquare$

3. Inducing Herz–Schur multipliers

We present the proof of Theorem 1.1 in this section. The proof relies on the following lemma which is inspired by [22, Lemma 2.1].

Lemma 3.1. *Let $\mathcal{M}$ be a semi-finite von Neumann algebra with a faithful normal semi-finite-trace* Tr*, and let* $\Lambda \curvearrowright^{\sigma} (\mathcal{M}, \mathrm{Tr})$ *be a trace-preserving action with a finite-trace fundamental domain* p*. Suppose* $\Gamma \curvearrowright^{\tilde{\sigma}} (\mathcal{M}, \mathrm{Tr})$ *is another trace-preserving action that commutes with the* Λ*-action. For* $\varphi \in \ell^{\infty}\Lambda$*, define* $\widehat{\varphi} \colon \Gamma \to \mathbb{C}$ *by*

$$\widehat{\varphi}(\gamma) := \frac{1}{\mathrm{Tr}(p)} \mathrm{Tr}(\tilde{\sigma}_{\gamma}(\theta_p(\varphi))p) = \frac{1}{\mathrm{Tr}(p)} \mathrm{Tr}(\theta_p(\varphi)\tilde{\sigma}_{\gamma^{-1}}(p)), \quad \gamma \in \Gamma,$$

where $\theta_p \colon \ell^{\infty}\Lambda \hookrightarrow \mathcal{M}$ *is the* Λ*-equivariant embedding.*

(a) *If* $\varphi \in B_2(\Lambda)$ *is a Herz–Schur multiplier on* Λ*, then* $\widehat{\varphi}$ *is a Herz–Schur multiplier on* Γ *and* $\|\widehat{\varphi}\|_{B_2} \leq \|\varphi\|_{B_2}$*. Moreover, if* φ *is positive definite, then so is* $\widehat{\varphi}$*.*

(b) *If* $\Gamma \curvearrowright^{\tilde{\sigma}} (\mathcal{M}, \mathrm{Tr})$ *is mixing, i.e., the Koopman representation* $\Gamma \curvearrowright^{\tilde{\sigma}^0} L^2(\mathcal{M}, \mathrm{Tr})$ *is mixing, and if* $\varphi \in c_0(\Lambda)$*, then* $\widehat{\varphi} \in c_0(\Gamma)$*. In particular, if* $\varphi \in B_2(\Lambda) \cap c_0(\Lambda)$*, then* $\widehat{\varphi} \in B_2(\Gamma) \cap c_0(\Gamma)$*.*

Proof. Since p is a finite-trace fundamental domain, it follows that $\widehat{\varphi}$ is well defined and $\|\widehat{\varphi}\|_{\infty} \leq \|\varphi\|_{\infty}$. Let $\xi, \eta \colon \Lambda \to \mathcal{H}_0$ be bounded functions from Λ into a Hilbert space $\mathcal{H}_0$ such that $\varphi(t^{-1}s) = \langle \xi(s), \eta(t) \rangle$, $s, t \in \Lambda$.

Let $\mathcal{H} = L^2(\mathcal{M}, \mathrm{Tr}) \bar{\otimes} \mathcal{H}_0$. For $\gamma \in \Gamma$, we have

$$\sum_{s \in \Lambda} \|\tilde{\sigma}_{\gamma}(\sigma_s(p))p\|_2^2 \|\xi(s)\|^2 \leq \|\xi\|_{\infty}^2 \sum_{s \in \Lambda} \mathrm{Tr}(p\tilde{\sigma}_{\gamma}(\sigma_s(p))) = \|\xi\|_{\infty}^2 \mathrm{Tr}(p) < \infty.$$

Therefore, $\widehat{\xi}, \widehat{\eta} \colon \Gamma \to L^2(\mathcal{M}, \mathrm{Tr}) \bar{\otimes} \mathcal{H}_0$ given below are well defined,

$$\widehat{\xi}(\gamma) := \frac{1}{\sqrt{\mathrm{Tr}(p)}} \sum_{s \in \Lambda} \tilde{\sigma}_{\gamma}(\sigma_s(p))p \otimes \xi(s),$$

$$\widehat{\eta}(\gamma) := \frac{1}{\sqrt{\mathrm{Tr}(p)}} \sum_{t \in \Lambda} \tilde{\sigma}_{\gamma}(\sigma_t(p))p \otimes \eta(t).$$

One has, for every $\gamma \in \Gamma$,

$$\|\widehat{\xi}(\gamma)\|^2 = \frac{1}{\mathrm{Tr}(p)} \sum_{s \in \Lambda} \langle \xi(s), \xi(s) \rangle \, \mathrm{Tr}(\sigma_s(p)\tilde{\sigma}_{\gamma^{-1}}(p))$$

$$= \frac{1}{\mathrm{Tr}(p)} \sum_{s \in \Lambda} \|\xi(s)\|^2 \, \mathrm{Tr}(\sigma_s(p)\tilde{\sigma}_{\gamma^{-1}}(p))$$

$$\leq \frac{1}{\mathrm{Tr}(p)} \|\xi\|_{\infty}^2 \sum_{s \in \Lambda} \mathrm{Tr}(\sigma_s(p)\tilde{\sigma}_{\gamma^{-1}}(p))$$

$$= \frac{1}{\mathrm{Tr}(p)} \|\xi\|_{\infty}^2 \, \mathrm{Tr}\left(\sum_{s \in \Lambda} \sigma_s(p)\tilde{\sigma}_{\gamma^{-1}}(p)\right) = \|\xi\|_{\infty}^2.$$

Thus, $\|\widehat{\xi}\|_\infty \le \|\xi\|_\infty$. Similarly, $\|\widehat{\eta}\|_\infty \le \|\eta\|_\infty$. Finally, for $\gamma_1, \gamma_2 \in \Gamma$, using the fact that Γ- and Λ-actions are commuting and trace-preserving, we have

$$
\begin{aligned}
\langle \widehat{\xi}(\gamma_1), \widehat{\eta}(\gamma_2) \rangle &= \frac{1}{\mathrm{Tr}(p)} \Big\langle \sum_{s\in\Lambda} \widetilde{\sigma}_{\gamma_1}(\sigma_s(p)) p \otimes \xi(s), \sum_{t\in\Lambda} \widetilde{\sigma}_{\gamma_2}(\sigma_t(p)) p \otimes \eta(t) \Big\rangle \\
&= \frac{1}{\mathrm{Tr}(p)} \sum_{s,t\in\Lambda} \langle \xi(s), \eta(t) \rangle \, \mathrm{Tr}(p\widetilde{\sigma}_{\gamma_2}(\sigma_t(p))\widetilde{\sigma}_{\gamma_1}(\sigma_s(p))) \\
&= \frac{1}{\mathrm{Tr}(p)} \sum_{t\in\Lambda} \mathrm{Tr}\Big(p\widetilde{\sigma}_{\gamma_2}(\sigma_t(p))\widetilde{\sigma}_{\gamma_1}\Big(\sum_{s\in\Lambda} \varphi(t^{-1}s)\sigma_s(p) \Big) \Big) \\
&= \frac{1}{\mathrm{Tr}(p)} \sum_{t\in\Lambda} \mathrm{Tr}(p\widetilde{\sigma}_{\gamma_2}(\sigma_t(p))\widetilde{\sigma}_{\gamma_1}(\sigma_t(\theta_p(\varphi)))) \\
&= \frac{1}{\mathrm{Tr}(p)} \sum_{t\in\Lambda} \mathrm{Tr}(\widetilde{\sigma}_{\gamma_2^{-1}}(p)\sigma_t(p\widetilde{\sigma}_{\gamma_2^{-1}\gamma_1}(\theta_p(\varphi)))) \\
&= \frac{1}{\mathrm{Tr}(p)} \mathrm{Tr}\Big(\sum_{t\in\Lambda} \sigma_{t^{-1}}(\widetilde{\sigma}_{\gamma_2^{-1}}(p)) p\widetilde{\sigma}_{\gamma_2^{-1}\gamma_1}(\theta_p(\varphi)) \Big) \\
&= \frac{1}{\mathrm{Tr}(p)} \mathrm{Tr}(\widetilde{\sigma}_{\gamma_2^{-1}\gamma_1}(\theta_p(\varphi))p) = \widehat{\varphi}(\gamma_2^{-1}\gamma_1).
\end{aligned}
$$

Therefore, $\widehat{\varphi}$ is a Herz–Schur multiplier with $\|\widehat{\varphi}\|_{B_2} \le \|\varphi\|_{B_2}$. Furthermore, if φ is positive definite, then one can take $\eta = \xi$ and it is straightforward to see that $\widehat{\varphi}$ is positive definite on Γ as well.

Note that, if $\varphi \in c_0(\Lambda)$, then $\theta_p(\varphi)$ is compact. Since the action of Γ is mixing, $\widetilde{\sigma}_\gamma(p) \to 0$ in the weak operator topology as $\gamma \to \infty$, and (b) now follows from Lemma 2.6. ∎

Proposition 3.2. *Let $\mathcal{M}$ be a semi-finite von Neumann algebra with a faithful normal semi-finite trace Tr, and let $\Lambda \curvearrowright (\mathcal{M}, \mathrm{Tr})$ be a trace-preserving action with a finite-trace fundamental domain p. Suppose $\Gamma \curvearrowright^\sigma (\mathcal{M}, \mathrm{Tr})$ is another trace-preserving action that commutes with the Λ-action. Consider the map $\Phi: \ell^\infty \Lambda \to \ell^\infty \Gamma$ defined by $\Phi(\varphi) = \widehat{\varphi}$, where $\widehat{\varphi}$ is defined as in Lemma 3.1. Then Φ is a contractive linear map from $B_2(\Lambda)$ into $B_2(\Gamma)$, and is*

(a) *continuous on norm bounded sets with respect to the topology of pointwise convergence;*

(b) *$\sigma(B_2(\Lambda), Q(\Lambda))$-$\sigma(B_2(\Gamma), Q(\Gamma))$ continuous.*

Proof. It is clear that $\Phi: B_2(\Lambda) \to B_2(\Gamma)$ is linear, and that it is contractive follows from Lemma 3.1 (a). Moreover, $\Phi: \ell^\infty \Lambda \to \ell^\infty \Gamma$ is also a linear contraction.

(a) Let $\varphi_i \to 0$ in $B_2(\Lambda)$ pointwise and $\|\varphi_i\|_{B_2} < C$ for every i. Since $\|\cdot\|_\infty \le \|\cdot\|_{B_2}$, after passing to a subnet if necessary, we may assume that $\varphi_i \to 0$ weak*. Since θ_p is a normal inclusion, it follows that $\theta_p(\varphi_i) \to 0$ in the weak operator topology, and therefore, $\mathrm{Tr}(\theta_p(\varphi_i)\sigma_{\gamma^{-1}}(p)) \to 0$ for every $\gamma \in \Gamma$, and hence $\widehat{\varphi}_i \to 0$ pointwise.

(b) Since $\Phi: B_2(\Lambda) \to B_2(\Gamma)$ is a linear contraction, the dual $\Phi^*: B_2(\Gamma)^* \to B_2(\Lambda)^*$ is continuous. Therefore, to prove that Φ is $\sigma(B_2(\Lambda), Q(\Lambda))$-$\sigma(B_2(\Gamma), Q(\Gamma))$ continuous, it suffices to show that Φ maps $Q(\Gamma)$ into $Q(\Lambda)$. To this end, notice that an argument similar to that given in the proof of previous part shows that $\Phi: \ell^\infty \Lambda \to \ell^\infty \Gamma$ is normal, whence it follows that the dual map Φ^* maps $\ell^1 \Gamma$ into $\ell^1 \Lambda$. Since $\ell^1 \Gamma$ and $\ell^1 \Lambda$ are dense, respectively, in $Q(\Gamma)$ and $Q(\Lambda)$, it follows that $\Phi^*(Q(\Gamma)) \subset Q(\Lambda)$. ∎

Proof of Theorem 1.1. Suppose $\Lambda_{\mathrm{cb}}(\Lambda) \leq C$, and choose a net $\{\varphi_i\}$ of finitely supported functions on Λ such that $\sup_i \|\varphi_i\|_{B_2} \leq C$, and $\varphi_i \to 1$ pointwise. It follows from Lemma 3.1 that $\widehat{\varphi}_i \in B_2(\Gamma)$ and $\|\widehat{\varphi}_i\|_{B_2} \leq C$ for every i. Since each φ_i is finitely supported, we have that $\widehat{\varphi}_i$ is a coefficient of the Koopman representation $\sigma^0: \Gamma \to \mathcal{U}(L^2(\mathcal{M}, \mathrm{Tr}))$. Moreover, the existence of a fundamental domain for Γ implies that σ^0 is a multiple of the left-regular representation [20, Proposition 4.2] and hence $\widehat{\varphi}_i \in A[\Gamma]$ for all i. From Proposition 3.2, we also have that $\widehat{\varphi}_i \to 1$ pointwise. This shows that $\Lambda_{\mathrm{cb}}(\Gamma) \leq C$, and Γ is weakly amenable.

If the net $\{\varphi_i\}$ is in $c_0(\Lambda) \cap B_2(\Lambda)$, then by Lemma 3.1(b), $\widehat{\varphi}_i \in c_0(\Gamma) \cap B_2(\Gamma)$ for every i. Now the same argument as in the previous paragraph shows that if Λ has the weak Haagerup property, then Γ has the weak Haagerup property and $\Lambda_{\mathrm{wcb}}(\Gamma) \leq \Lambda_{\mathrm{wcb}}(\Lambda)$.

Note that $\theta_p(\varphi)p = \varphi(e)p$. Therefore, if $\varphi(e) = 1$, then $\widehat{\varphi}(e) = 1$. Moreover, if φ is a normalized positive definite function, then there exist a Hilbert space $\mathcal{H}_\varphi$, a unitary representation $\pi_\varphi: \Lambda \to \mathcal{U}(\mathcal{H}_\varphi)$, and a unit vector $\xi_\varphi \in \mathcal{H}_\varphi$ such that $\varphi(s) = \langle \pi_\varphi(s)\xi_\varphi, \xi_\varphi \rangle$, $s \in \Lambda$ [3, Theorem 2.5.11]. In particular, $\|\varphi\|_\infty \leq 1$. Therefore, the arguments in the previous paragraphs show that the Haagerup property is invariant under von Neumann equivalence.

If Λ has AP, then 1 is in the $\sigma(B_2(\Lambda), Q(\Lambda))$-closure of finitely supported functions on Λ. From the first paragraph we have that if φ is a finitely supported function on Λ, then $\widehat{\varphi} \in A[\Gamma]$. Therefore, Proposition 3.2 gives that 1 is in the $\sigma(B_2(\Gamma), Q(\Gamma))$-closure of $A[\Gamma]$ inside $B_2(\Gamma)$, whence it follows that Γ has AP. ∎

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