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The mapping class group of a nonorientable surface is quasi-isometrically embedded in the mapping class group of the orientation double cover

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Abstract. Let N be a connected nonorientable surface with or without boundary and punctures, and $j\colon S \to N$ be the orientation double covering. It has previously been proved that j induces an embedding $\iota\colon \mathrm{Mod}(N) \hookrightarrow \mathrm{Mod}(S)$ with one exception. In this paper, we prove that the injective homomorphism ι is a quasi-isometric embedding. The proof is based on the semihyperbolicity of $\mathrm{Mod}(S)$, which has already been established. We also prove that the embedding $\mathrm{Mod}(F') \hookrightarrow \mathrm{Mod}(F)$ induced by an inclusion of a pair of possibly nonorientable surfaces $F' \subset F$ is a quasi-isometric embedding.

1. Introduction

Let $S = S_{g,p}^b$ be the connected orientable surface of genus g with b boundary components and p punctures, and let $N = N_{g,p}^b$ be the connected nonorientable surface of genus g with b boundary components and p punctures. In the case where $b = 0$ (resp. $p = 0$), we drop the subscript b (resp. p) from $S_{g,p}^b$ and $N_{g,p}^b$. For example, $N_{g,0}^0$ is simply denoted by N_g. If we are not interested in whether a given surface is orientable or not, we denote the surface by F. The *mapping class group* $\mathrm{Mod}(F)$ of F is the group of isotopy classes of homeomorphisms of F which are orientation-preserving if F is orientable and preserve ∂F pointwise. Recall that if $H \subset G$ is a pair of finitely generated groups with word metrics d_H and d_G (induced by finite generating sets), then the *distortion* of H in G is defined as

$$\delta_H^G(n) := \max\{d_H(1,h) \mid h \in H \text{ with } d_G(1,h) \le n\}.$$

This function is independent of the choice of word metrics d_H and d_G up to Lipschitz equivalence. In addition, there exists a constant K such that $\delta_H^G(n) \le Kn$ if and only if the inclusion $H \subset G$ is a quasi-isometric embedding. The subgroup H is said to be *undistorted* (or *quasi-isometrically embedded*) in G if this condition is satisfied; otherwise, we

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say that H is *distorted*. The distortions of various subgroups in the mapping class groups of orientable surfaces have been extensively investigated. For example, the mapping class groups of subsurfaces are undistorted according to Masur–Minsky [15, Theorem 6.12] and Hamenstädt [9, Proposition 4.1]. Farb–Lubotzky–Minsky [7] proved that groups generated by Dehn twists along disjoint curves are undistorted. Moreover, Rafi–Schleimer [18] proved that an orbifold covering map of orientable surfaces induces a quasi-isometric embedding between the mapping class groups. For examples of distorted subgroups of mapping class groups, see Broaddus–Farb–Putman [4], Cohen [5], and Kuno–Omori [12], where it is proved that the Torelli group $\mathcal{I}_g^b$ is distorted in $\mathrm{Mod}(S_g^b)$. Moreover, it has been proved by Hamenstädt–Hensel [10] that the handlebody group is exponentially distorted in the mapping class group of the boundary surface.

The mapping class group of a nonorientable surface $N_{g,p}^b$ embeds in the mapping class group of the orientation double cover $S_{g-1,2p}^{2b}$ as the subgroup consisting of mapping classes that commute with the action of the deck group (see Lemma 2.7). In this paper, we prove Theorems 1.1 and 1.2 below by using the semihyperbolicity of the mapping class group of orientable surfaces, independently established by Durham–Minsky–Sisto [6, Corollary D] and Haettel–Hoda–Petyt [8, Corollary 3.11].

Theorem 1.1. *For all but $(g, p, b) = (2, 0, 0)$, the embedding of the mapping class group $\mathrm{Mod}(N_{g,p}^b)$ into the mapping class group of its orientation double cover $\mathrm{Mod}(S_{g-1,2p}^{2b})$ is undistorted.*

Theorem 1.2. *The mapping class group of a connected nonorientable surface is semihyperbolic.*

Let F be a connected (orientable or nonorientable) surface. We say that a subsurface $F' \subset F$ is *admissible* if F' is a closed subset of F. For an admissible subsurface $F' \subset F$, we have a natural homomorphism

$$\mathrm{Mod}(F') \to \mathrm{Mod}(F).$$

As is well known, Paris–Rolfsen in [17, Corollary 4.2] and Stukow in [19, Corollary 3.8] proved that, if every connected component of $F - \mathrm{Int}(F')$ has a negative Euler characteristic, then the homomorphism $\mathrm{Mod}(F') \to \mathrm{Mod}(F)$ is injective. Here, $\mathrm{Int}(F')$ is the interior of F'. From the work of Masur–Minsky [15] and Hamenstädt's unpublished paper [9], it follows that the above injective homomorphism is undistorted when the underlying surfaces are orientable. We generalize this result as follows.

Theorem 1.3. *Let F be a connected (orientable or nonorientable) surface and $F' \subset F$ be an admissible connected subsurface such that every connected component of $F - \mathrm{Int}(F')$ has a negative Euler characteristic. Then, the embedding $\mathrm{Mod}(F') \hookrightarrow \mathrm{Mod}(F)$ is undistorted.*

2. Preliminaries

In this section, we show that the centralizer of every element in the (extended) mapping class group of an orientable surface is quasi-convex in the (extended) mapping class group and introduce the result that orientation double covers induce embeddings of mapping class groups. A quasi-geodesic, bounded, and equivariant bicombing σ for a finitely generated group G is called a *semihyperbolic structure*. We write $\sigma(1, g)$ for the combing line (discrete path) from the identity to the element $g \in G$. For definitions of quasi-geodesic, bounded and equivariant bicombing, see Alonso–Bridson [1] or the textbook [3] due to Bridson–Haefliger. A finitely generated group is said to be *semihyperbolic* if it admits a semihyperbolic structure.

Definition 2.1. Let G be a group with a semihyperbolic structure σ. Then, a subgroup $H \leq G$ is said to be σ-quasi-convex if there exists a constant $k \geq 0$ such that $\sigma(1, h)$ is contained in k-neighborhood of H for all $h \in H$.

Note that a finite index subgroup of a group with a semihyperbolic structure σ is σ-quasi-convex. This fact will be frequently used throughout this paper.

In a semihyperbolic group, every σ-quasi-convex subgroup is undistorted and inherits semihyperbolicity from the ambient group.

Lemma 2.2 ([1, Lemma 7.2 and Theorem 7.3]). *Let G be a finitely generated group with a semihyperbolic structure σ and let H be a σ-quasi-convex subgroup of G. Then, H is finitely generated, semihyperbolic and undistorted in G.*

Lemma 2.3 ([1, Corollary 7.6]). *Let G be a finitely generated group with a semihyperbolic structure σ. Then, the centralizer of any finite subset of G is σ-quasi-convex.*

A key ingredient of our proof for Theorem 1.1 is that the (extended) mapping class groups of orientable hyperbolic surfaces are semihyperbolic.

Lemma 2.4 ([6, Corollary D], [8, Corollary 3.11]). *For any orientable hyperbolic surface S of finite type, the mapping class group* $\mathrm{Mod}(S)$ *and the extended mapping class group* $\mathrm{Mod}^{\pm}(S)$ *of S are semihyperbolic. Here,* $\mathrm{Mod}^{\pm}(S)$ *is the group consisting of the isotopy classes of homeomorphisms on S that preserve ∂S pointwise.*

Remark 2.5. We note that the mapping class groups of the orientable surfaces of non-negative Euler characteristic are also semihyperbolic. There are seven orientable surfaces of non-negative Euler characteristic, S_0, $S_{0,1}$, S_0^1, $S_{0,2}$, S_0^2, $S_{0,1}^1$, and S_1. The mapping class groups of these surfaces are virtually free, and so they are semihyperbolic. In these cases, $\mathrm{Mod}^{\pm}(S)$ is also virtually free, and thus semihyperbolic.

In conclusion, we have the following lemma.

Lemma 2.6. *Let S be an orientable surface of finite type. The centralizer of any finite subset of the (extended) mapping class group of S is undistorted and semihyperbolic.*

Let $j\colon S_{g-1,2p} \to N_{g,p}$ be the orientation double covering of a nonorientable surface and $J\colon S_{g-1,2p} \to S_{g-1,2p}$ the deck transformation.

Lemma 2.7 ([2, Theorem 1], [23, Lemma 3], [13, Theorem 1.1]). *For all but $(g, p, b) = (1, 0, 0), (2, 0, 0)$, the orientation double covering j induces an injective homomorphism $\iota\colon \mathrm{Mod}(N^b_{g,p}) \hookrightarrow \mathrm{Mod}(S^{2b}_{g-1,2p})$. Moreover, the image of $\mathrm{Mod}(N^b_{g,p})$ given by ι consists of the isotopy classes of orientation-preserving homeomorphisms of $S^{2b}_{g-1,2p}$ that commute with J.*

Remark 2.8. Note that $\mathrm{Mod}(N_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ is never embedded in $\mathrm{Mod}(S_1) = \mathrm{SL}(2, \mathbb{Z})$, because every finite subgroup of $\mathrm{SL}(2, \mathbb{Z})$ is cyclic.

3. Quasi-isometrically embedded subsurface mapping class groups

In this section, we prove Theorems 1.1, 1.2 and 1.3. Theorem 1.3 can be reduced to the following lemma.

Lemma 3.1. *Let F be a connected orientable or nonorientable surface and $F' \subset F$ be an admissible connected subsurface. Suppose that every connected component of $F - \mathrm{Int}(F')$ has a negative Euler characteristic. Then, there exists a finite-index subgroup H of $\mathrm{Mod}(F)$ such that the natural injection $\mathrm{Mod}(F') \cap H \hookrightarrow H$ is undistorted.*

To prove Lemma 3.1, we prepare the following lemmas.

Lemma 3.2. *Let F be a connected orientable or nonorientable surface of genus g with $b \geq 1$ boundary components and p punctures. We assume that $b + p \geq 4$ if F is orientable and $g = 0$. We also assume that $g + b + p \geq 4$ if F is nonorientable. Then, there exists a pair $\{\alpha_1, \alpha_2\}$ of essential simple closed curves satisfying the following properties:*

(1) *If F is nonorientable, then the closed curves α_1, α_2 are two-sided.*

(2) *$F - (\mathrm{Int}(U(\alpha_1)) \cup \mathrm{Int}(U(\alpha_2)))$ is a disjoint union of some copies of N^1_1, S^1_0, $S^1_{0,1}$, and S^2_0. Here, we denote the regular neighborhood of α_i by $U(\alpha_i)$, $i = 1, 2$. Moreover, for every component A in $F - (\mathrm{Int}(U(\alpha_1)) \cup \mathrm{Int}(U(\alpha_2)))$ which is homeomorphic to S^2_0, it holds that $\partial A \cap \partial F \neq \emptyset$.*

Proof. Suppose that F is orientable. Then, the curve complex of F has infinite diameter (see Masur–Minsky [14, Theorem 1.1]). In particular, it has diameter at least 3. This implies that F has a pair $\{\alpha_1, \alpha_2\}$ of essential simple closed curves such that every component of $F - (\mathrm{Int}(U(\alpha_1)) \cup \mathrm{Int}(U(\alpha_2)))$ has no simple closed curve which is essential in F. The pair satisfies condition (2) in Lemma 3.2. We can reduce the case where F is nonorientable to the case where F is orientable by replacing punctures with crosscaps. Then, the pair of closed curves satisfies condition (1). ∎

Let F be a surface. A closed curve β on F is called *peripheral* if β is isotopic to a component of ∂F. A two-sided closed curve α on F is called *generic* if α bounds neither a disk nor a Möbius strip and is not peripheral. Let $\mathcal{T}(F)$ denote the subgroup of $\mathrm{Mod}(F)$, called the *twist subgroup*, generated by Dehn twists along two-sided closed curves which are either peripheral or generic on F. Note that $\mathcal{T}(F)$ is a finite-index subgroup of $\mathrm{Mod}(F)$ (see Appendix A).

Lemma 3.3. *We have the following:*

(1) $\mathcal{T}(N_{1,1}^1) \cong \mathbb{Z}$*, and its generator is a Dehn twist along a unique peripheral closed curve.*

(2) $\mathcal{T}(N_1^2) \cong \mathbb{Z}^2$*, and its generators are Dehn twists along peripheral closed curves.*

(3) $\mathcal{T}(N_2^1) \cong \mathbb{Z}^2$*, and its generators are a Dehn twist along a unique peripheral closed curve and a Dehn twist along a unique generic closed curve on* N_2^1*.*

Proof. According to [16, Proposition 17], $\mathrm{Mod}(N_{1,1}^1) \cong \mathbb{Z}$ and is generated by a boundary slide s. As the square of s is isotopic to a Dehn twist along a unique peripheral closed curve on $N_{1,1}^1$, the twist subgroup $\mathcal{T}(N_{1,1}^1)$ is generated by a Dehn twist. To obtain an isomorphism $\mathbb{Z}^2 \to \mathcal{T}(N_1^2)$, we use the capping homomorphism $\mathrm{Mod}(N_1^2) \to \mathrm{Mod}(N_{1,1}^1)$ induced by gluing N_1^2 with a punctured disk along a boundary component C of N_1^2. Then, the kernel of the capping homomorphism is generated by a Dehn twist along a closed curve isotopic to C. Additionally, the image of a Dehn twist along a peripheral closed curve on N_1^2 which is not isotopic to C is s^2. Hence, $\mathcal{T}(N_{1,0}^2)$ is freely generated by Dehn twists along those peripheral closed curves. By [16, Propositions 22], we have $\mathrm{Mod}(N_2^1) \cong \mathbb{Z} \rtimes \mathbb{Z}$. In addition, the first copy of $\mathbb{Z}$ is generated by a Dehn twist along a unique generic closed curve on N_2^1, and the second copy is generated by a crosscap slide y. As the square of y is isotopic to a Dehn twist along a peripheral closed curve on $N_{2,0}^1$, $\mathcal{T}(N_2^1)$ is freely generated by those Dehn twists. ∎

The next lemma asserts that the mapping class group of any "essential" subsurface, excepting a few examples, is virtually isomorphic to a direct factor of a σ-quasi-convex subgroup of the ambient mapping class group.

Lemma 3.4. *Let F be a connected orientable or nonorientable surface and $F' \subset F$ be an admissible connected subsurface which is not an annulus. Suppose that* $\mathrm{Mod}(F') \neq 1$ *and that every connected component of* $F - \mathrm{Int}(F')$ *has a negative Euler characteristic. Then, there exist mapping classes $\varphi_1, \ldots, \varphi_l \in \mathcal{T}(F)$ such that a finite-index subgroup of $\bigcap_{i=1}^{l} Z_{\mathcal{T}(F)}(\varphi_i)$ is isomorphic to* $(\mathrm{Mod}(F') \cap \mathcal{T}(F)) \times \mathbb{Z}^r$.

Here, $Z_{\mathcal{T}(F)}(\varphi_i)$ is the centralizer of φ_i in $\mathcal{T}(F)$, and the free abelian rank r given in Lemma 3.4 is equal to the sum of the number of boundary components of F which are not contained in F' and the number of connected components of $F - \mathrm{Int}(F')$ which are homeomorphic to a one-holed Klein bottle.

Proof of Lemma 3.4. Let $F_1, \ldots, F_n$ be the connected components of $F - \mathrm{Int}\, F'$. We denote the genus of F_i, the number of boundary components of F_i, and the number of punctures of F_i by $g(F_i)$, $b(F_i)$, and $p(F_i)$, respectively. As the Euler characteristic of F_i is negative, F_i satisfies exactly one of the following conditions:

(a) F_i is orientable and either $g(F_i) \geq 1$ or $b(F_i) + p(F_i) \geq 4$.

(b) F_i is orientable, $g(F_i) = 0$, and $b(F_i) + p(F_i) = 3$.

(c) F_i is nonorientable and $g(F_i) + b(F_i) + p(F_i) \geq 4$.

(d) F_i is nonorientable and $g(F_i) + b(F_i) + p(F_i) = 3$.

If F_i satisfies condition (a) or (c), we have a pair P_i of essential closed curves which fills F_i in the sense of Lemma 3.2. We define a set of closed curves A_i to be a union of P_i and the set of closed curves of F_i which are parallel to $\partial F'$. In the case where F_i satisfies condition (b) or (d), the set A_i is defined to be the set of closed curves of F_i which are parallel to $\partial F'$.

Set $\varphi_\alpha := [T_\alpha]$ for each $\alpha \in A := \bigcup_{i=1}^n A_i$, and set

$$B_i := \begin{cases} \langle [T_\beta] \mid \beta \in \partial F \cap \partial F_i \rangle & \text{if } F_i \not\cong N_2^1, \\ \langle [T_\gamma] \mid \gamma \text{ is a two-sided generic closed curve} \rangle & \text{if } F_i \cong N_2^1. \end{cases}$$

Note that $B_i \cong \mathbb{Z}$ when $F_i \cong N_2^1$. Consider the subgroup $(\mathrm{Mod}(F')B_1 \cdots B_n) \cap \mathcal{T}(F)$ of $\mathcal{T}(F)$. We first show that $(\mathrm{Mod}(F')B_1 \cdots B_n) \cap \mathcal{T}(F)$ splits as a direct product. Pick a component C of $\partial F_i \cap \partial F'$. As F' is not an annulus and $\mathrm{Mod}(F') \neq 1$, F' has an essential proper arc whose endpoints are contained in C. In addition, as the Euler characteristic of F_i is negative, F_i' also has an essential proper arc whose endpoints are contained in C. These facts imply that there exists a two-sided essential closed curve γ_C in F such that γ_C intersects C non-trivially in minimal position and is disjoint from $\partial F_i - \{C\}$. Moreover, when $F_i \cong N_2^1$, we can choose γ_C to be disjoint from a unique two-sided generic closed curve on F_i (see Figure 1). As all elements in B_i commute with $[T_{\gamma_C}]$, we have $\mathrm{Mod}(F') \cap B_i = 1$. Therefore, $(\mathrm{Mod}(F')B_1 \cdots B_n) \cap \mathcal{T}(F) = (\mathrm{Mod}(F') \cap \mathcal{T}(F))B_1 \cdots B_n \cong (\mathrm{Mod}(F') \cap \mathcal{T}(F)) \times \mathbb{Z}^r$, where r is the sum of the free abelian ranks of $B_1, \ldots, B_n$ and is equal to the sum of the number of boundary components of F which

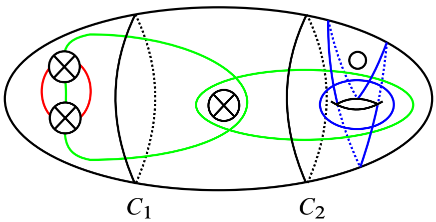

Figure 1. This picture illustrates an example of an admissible embedding of $F' = N_1^2$ into $F = N_5^1$ in Lemma 3.4. The exterior of F' is a disjoint union of $F_1 = N_2^1$ and $F_2 = S_1^2$. The red curve is a unique generic closed curve in F_1, and the pair of blue curves is a surface filling pair of F_2 in the sense of Lemma 3.2. The green curves show γ_{C_i}, where C_i is a component of $\partial F_i \cap \partial F'$.

are not contained in F' and the number of connected components of $F - \mathrm{Int}\, F'$ which are homeomorphic to N_2^1. In addition, it is clear that

$$(\mathrm{Mod}(F')B_1 \cdots B_n) \cap \mathcal{T}(F) \subset \cap_{\alpha \in A} Z_{\mathcal{T}(F)}(\varphi_\alpha).$$

To simplify the notation, we denote $\bigcap_{\alpha \in A} Z_{\mathcal{T}(F)}(\varphi_\alpha)$ (resp. $(\mathrm{Mod}(F')B_1 \cdots B_n) \cap \mathcal{T}(F)$) by Z (resp. L).

We now claim that L is a finite-index subgroup of Z. To see this, consider a subset $\mathcal{S}$ of Z realizing all possible reversing patterns on orientations of closed curves in A. If there is no element of Z which reverses an orientation of a closed curve in A, we set $\mathcal{S} = \{1\}$. As A is finite, we can choose $\mathcal{S}$ to be finite. Pick an element f in Z. Then, f preserves each closed curve in A, and so there exists an element $s \in \mathcal{S}$ such that sf fixes an orientation of each closed curve in A. In the following, we prove that $sf \in L$. This immediately implies that L has finite index in Z. As sf fixes an orientation of each closed curve in A, sf can be decomposed as a product of mapping classes of the regular neighborhood $U(A)$ of A and $F - \mathrm{Int}(U(A))$. By Lemma 3.2, $F - \mathrm{Int}(U(A))$ is a disjoint union of F', outer surfaces F_i satisfying condition (b) or (d), and some copies of S_0^1, $S_{0,1}^1$, S_0^2, N_1^1. Obviously, $sf|_{F'}$ is contained in $\mathrm{Mod}(F')$. Additionally, if F_i satisfies condition (b) or (d), we have that $sf|_{F_i}$ is contained in $\mathrm{Mod}(F')B_i$ by Lemma 3.3 and the fact that $\mathrm{Mod}(F_i)$ is an abelian group freely generated by Dehn twists along peripheral closed curves if F_i satisfies condition (b). Note that the copies of S_0^2 are in one-to-one correspondence with the components of $\bigcup_{i=1}^n \partial F_i$. Hence, the restriction of sf to the copies of S_0^1, $S_{0,1}^1$, S_0^2, and N_1^1 in $F - \mathrm{Int}(U(A))$ is contained in $\mathrm{Mod}(U(\bigcup_{i=1}^n \partial F_i)) \subset \mathrm{Mod}(F')B_1 \cdots B_n$ by Alexander's theorem and Epstein's theorem [16, Proposition 5]. Therefore, we have that $sf|_{F - \mathrm{Int}\, U(A)} \in \mathrm{Mod}(F')B_1 \cdots B_n$. Furthermore, we can verify that $sf|_{U(A)}$ is contained in $\mathrm{Mod}(F')$. To see this, we use the fact that sf and $sf|_{F - \mathrm{Int}\, U(A)}$ commute with all of φ_α, $\alpha \in A$. As $sf|_{N(A)} = sf \cdot (sf|_{F - \mathrm{Int}\, U(A)})^{-1}$, the restriction $sf|_{U(A)}$ also commutes with all of φ_α, $\alpha \in A$. If F_i satisfies condition (b) or (d), the restriction of $sf|_{U(A)}$ to F_i is contained in $\mathrm{Mod}(F')$, because $A_i \subset F'$. If F_i satisfies condition (a) or (c), the restriction of $sf|_{U(A)}$ to F_i is contained in $\mathrm{Mod}(F')$, because $sf|_{U(A)}$ should be trivial on the regular neighborhood of the filling pair P_i. Therefore, $sf|_{U(A)} \in \mathrm{Mod}(F')$, and so $sf \in \mathrm{Mod}(F')B_1 \cdots B_n$. Since $sf \in \mathcal{T}(F)$, we have that $sf \in L$, as desired. ∎

Remark 3.5. Assume that $F = S_{0,p+2}$ and $F' = S_{0,p}^1$ in Lemma 3.4. If $p \geq 3$, we can conclude that the braid group on p-strands coincides with $Z([T_\beta])$ in $\mathrm{Mod}(S_{0,p+2})$ because $\mathcal{S} = \{1\}$. Here, β is a peripheral closed curve parallel to $\partial F'$.

Let us prove Theorem 1.2.

Proof of Theorem 1.2. First, we treat the case where N has no boundary. The mapping class groups $\mathrm{Mod}(N_1) = 1$ and $\mathrm{Mod}(N_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ are obviously semihyperbolic. If $N \neq N_1, N_2$, semihyperbolicity of $\mathrm{Mod}(N)$ comes from the fact that it is a finite-index subgroup of the centralizer of $[J]$ in $\mathrm{Mod}^{\pm}(S)$ (Lemma 2.6).

Next, we treat the case where N has a boundary. We obtain the nonorientable surface F without a boundary from N by attaching twice-punctured disks to the boundary components of N. Then, $\mathrm{Mod}(F)$ admits a semihyperbolic structure σ. According to Lemma 3.4, there exist mapping classes $\varphi_1, \dots, \varphi_l \in \mathcal{T}(F)$ such that a finite-index subgroup of $\bigcap_{i=1}^{l} Z_{\mathcal{T}(F)}(\varphi_i)$ is isomorphic to $\mathrm{Mod}(N) \cap \mathcal{T}(F)$. Hence, we have the sequence of subgroups

$$\mathrm{Mod}(N) \cap \mathcal{T}(F) \le \bigcap_{i=1}^{l} Z_{\mathcal{T}(F)}(\varphi_i) \le \mathcal{T}(F) \le \mathrm{Mod}(F).$$

Each term of the sequence is quasi-convex in the latter term with respect to any semihyperbolic structure for the latter term by Lemma 2.3, and therefore $\mathrm{Mod}(N) \cap \mathcal{T}(F)$ is σ-quasi-convex. As $\mathrm{Mod}(N) \cap \mathcal{T}(F)$ is a finite-index subgroup of $\mathrm{Mod}(N)$, $\mathrm{Mod}(N)$ is also σ-quasi-convex. By Lemma 2.2 (2), $\mathrm{Mod}(N)$ is semihyperbolic. ■

Let us prove Lemma 3.1 by using Theorem 1.2.

Proof of Lemma 3.1. First, we consider the case where $\mathrm{Mod}(F') = 1$. In this case, Lemma 3.1 is trivial.

Next, we consider the case where F' is an annulus. Then, Lemma 3.1 can be obtained by using semihyperbolicity of $\mathrm{Mod}(F)$, because any finitely generated abelian subgroup is undistorted in a semihyperbolic group (see Bridson–Haefliger [3, Chapter III.Γ, Theorem 4.10]).

We now consider the case where $\mathrm{Mod}(F') \neq 1$ and F' is not an annulus. By Lemma 3.4, there exist mapping classes $\varphi_1, \dots, \varphi_l \in \mathcal{T}(F)$ and a non-negative number r such that $(\mathrm{Mod}(F') \cap \mathcal{T}(F)) \times \mathbb{Z}^r$ is naturally embedded in $\bigcap_{i=1}^{l} Z_{\mathcal{T}(F)}(\varphi_i)$ as a finite-index subgroup. Let σ be a semihyperbolic structure of $\mathcal{T}(F)$. As each direct factor is quasi-convex in a given direct product, the subgroup $\mathrm{Mod}(F') \cap \mathcal{T}(F)$ is quasi-convex in $\bigcap_{i=1}^{l} Z_{\mathcal{T}(F)}(\varphi_i)$. Hence, $\mathrm{Mod}(F') \cap \mathcal{T}(F)$ is σ-quasi-convex. Thus, $\mathrm{Mod}(F') \cap \mathcal{T}(F)$ is undistorted in $\mathcal{T}(F)$. ■

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. First, we treat the case where N has no boundary. Assume that $N = N_{g,p}$ and (g, p) is neither $(1, 0)$ nor $(2, 0)$. Set $S = S_{g-1,2p}$. Let $\iota\colon \mathrm{Mod}(N) \hookrightarrow \mathrm{Mod}(S)$ be the injective homomorphism obtained in Lemma 2.7 and $J\colon S \to S$ the deck transformation. As J is orientation-reversing and has order 2, the centralizer $Z([J])$ of the mapping class in $\mathrm{Mod}^{\pm}(S)$ splits as

$$Z([J]) = I \times \langle [J] \rangle,$$

where $I = Z([J]) \cap \mathrm{Mod}(S)$ and $\langle [J] \rangle$ is cyclic of order 2. By Lemma 2.6, $I \times \langle [J] \rangle$ is undistorted in $\mathrm{Mod}^{\pm}(S)$, and I is therefore undistorted in $\mathrm{Mod}(S)$. From Lemma 2.7, we have $I = \iota(\mathrm{Mod}(N))$.

Next, we treat the case where N has a boundary. We obtain the nonorientable surface F without a boundary from N by attaching twice-punctured disks to the boundary components of N. Let S (resp. $\widetilde{F}$) be the orientation double cover of N (resp. F). We have the following commutative diagram whose homomorphisms are all injective:

$$\begin{array}{ccc} \mathrm{Mod}(S) & \longrightarrow & \mathrm{Mod}(\widetilde{F}) \\ \uparrow & \circlearrowright & \uparrow \\ \mathrm{Mod}(N) & \longrightarrow & \mathrm{Mod}(F). \end{array}$$

By Theorem 1.3, $\mathrm{Mod}(N)$ is undistorted in $\mathrm{Mod}(F)$. As F has no boundary, $\mathrm{Mod}(F)$ is undistorted in $\mathrm{Mod}(\widetilde{F})$. Thus, $\mathrm{Mod}(N)$ is undistorted in $\mathrm{Mod}(S)$. ■

Finally, we remark that for closed surfaces, hyperelliptic mapping class groups are also undistorted subgroups because they are centralizers of certain elements in the mapping class groups (see Stukow [21] for the definition of hyperelliptic mapping class groups of closed nonorientable surfaces).

A. Appendix

In this appendix, we show that the twist subgroup of the mapping class group of a surface has finite index. The twist subgroup coincides with the pure mapping class group (the subgroup consisting of the elements fixing the punctures) when the underlying surface is orientable. So we only have to show that the twist subgroup has finite index when the underlying surface is nonorientable.

Proposition A.1. *Let N be a connected nonorientable surface. Then $\mathcal{T}(N)$ is a finite index subgroup of* $\mathrm{Mod}(N)$.

Proof. Our notation is based on [11, Section 4].

First, we assume that N has no boundary. Let $p_1, \ldots, p_n$ be the punctures of N. $\mathcal{T}(F)$ is obviously a normal subgroup of $\mathrm{PMod}(N)$, because any conjugate of a Dehn twist is a Dehn twist. So, it is enough to show that the quotient group $\mathrm{PMod}(N)/\mathcal{T}(N)$ is a finite group. Here, $\mathrm{PMod}(N)$ is the pure mapping class group of F.

Case where the genus of N is one. Let v_i be a boundary slide of p_i along a one-sided simple closed curve β_i given in [11, Section 4.1]. According to [11, Theorem 4.1], the pure mapping class group $\mathrm{PMod}(N)$ is generated by $v_1, \ldots, v_n$. As v_i^2 is a Dehn twist along the boundary of the regular neighborhood of β_i, we have $v_i^2 \in \mathcal{T}(N)$. Besides, $(v_i v_j)^2$ is a Dehn twist along a generic closed curve bounding a one-holed Möbius strip M with punctures p_i and p_j (see [11, Lemma 4.2]). Hence, we have $v_i v_j \mathcal{T}(N) = v_j^{-1} v_i^{-1} \mathcal{T}(N)$, and this set is in turn identical with $v_j v_i \mathcal{T}(N)$. Thus the quotient group $\mathrm{PMod}(N)/\mathcal{T}(N)$ is a finite abelian group.

Case where the genus of N is two. Let y be a crosscap slide of N, v_i be a boundary slide of p_i along β_i and w_i be a boundary slide of p_i along γ_i given in [11, Section 4.2]. By [11, Theorem 4.9], the quotient group $\mathrm{PMod}(N)/\mathcal{T}(N)$ is generated by the equivalent classes $y\mathcal{T}(N)$, $v_i\mathcal{T}(N)$ and $w_i\mathcal{T}(N)$. Then y^2 is a Dehn twist along a generic closed curve c bounding a one-holed Klein bottle in N, and therefore $y^2 \in \mathcal{T}(N)$. Besides, as in the case where the genus of N is one, v_i^2 and w_i^2 are both contained in $\mathcal{T}(N)$. It is easy to show that $v_i w_i$ is the product of Dehn twists along a pair of closed curves separating an annulus with the puncture p_i (see [16, Lemma 18]), and hence we have $v_i\mathcal{T}(N) = w_i\mathcal{T}(N)$. Moreover, as in the case where the genus of N is one, we have $v_i v_j\mathcal{T}(N) = v_j v_i\mathcal{T}(N)$. So we only have to see that $y\mathcal{T}(N)$ commutes with $v_i\mathcal{T}(N)$. Korkmaz proved in [11, Figure 6] that the diffeomorphism $t_c^{-1} y v_i y v_i$ induces a diffeomorphism of a three-holed sphere which fixes the boundary of the sphere pointwise. The mapping class group of a three-holed sphere is generated by Dehn twists, and so $y v_i y v_i \in \mathcal{T}(N)$. Thus, $y\mathcal{T}(N)$ commutes with $v_i\mathcal{T}(N)$ and $\mathrm{PMod}(N)/\mathcal{T}(N)$ is finite abelian.

Case where the genus of N is more than two can be treated similarly. The precise index is given in [20, Corollary 6.4]. Hence, $\mathcal{T}(N)$ is a finite index subgroup of $\mathrm{Mod}(N)$ if N has no boundary.

Next, we assume that N has a boundary. We obtain the nonorientable surface F without a boundary from N by capping the boundary components of N with once-punctured disks. Let $p_1, \ldots, p_n$ be the punctures of F which are not contained in N. Then we have the following exact sequence [22, (7.3)]:

$$1 \to \mathbb{Z}^n \to \mathrm{PMod}(N) \to \mathrm{PMod}(F, \{p_1, \ldots, p_n\}) \to 1,$$

where $\mathrm{PMod}(F, \{p_1, \ldots, p_n\})$ is the index 2^n subgroup of $\mathrm{PMod}(F)$ preserving the local orientation around the punctures $p_1, \ldots, p_n$. By restricting the above exact sequence to the twist subgroup, we can see that the sequence

$$1 \to \mathbb{Z}^n \to \mathcal{T}(N) \to \mathcal{T}(F) \to 1$$

is also exact. As $\mathrm{PMod}(F)/\mathcal{T}(F)$ is finite, so is $\mathrm{PMod}(F, \{p_1, \ldots, p_n\})/\mathcal{T}(F)$. Thus, $\mathrm{PMod}(N)/\mathcal{T}(N)$ is a finite group. ∎

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