

Area and Gauss–Bonnet inequalities with scalar curvature

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Abstract. The Gauss–Bonnet theorem states for any compact surface (S, g) that the quantity

$$Q_{GB}^S(S) = \int_S \text{Sc}(S, s) \, ds + \int_{\partial S} \text{mean.curv.}(\partial S, b) \, db - 4\pi\chi(S)$$

vanishes identically. Let (X, g) be a compact Riemannian manifold of dimension $n \geq 3$ with smooth boundary, associated with a continuous map $f = (f_1, \dots, f_{n-2}): X \rightarrow [0, 1]^{n-2}$, where $\text{Lip } f_i \leq d_i^{-1}$ for positive constants d_i . For a universal constant $C_n(d_i)$ depending only on d_i and n , we show that there is a compact surface Σ homologous to the f -pullback of a generic point such that each component S of Σ satisfies $Q_{GB}^X(S) \leq C_n(d_i) \cdot \text{area}(S)$, where

$$Q_{GB}^X(S) = \int_S \text{Sc}(X, s) \, ds + \int_{\partial S} \text{mean.curv.}(\partial X, b) \, db - 4\pi\chi(S).$$

As corollaries, if X has “large positive” scalar curvature, we prove in a variety of cases that if X “spreads” in $(n - 2)$ directions “distance-wise”, then it *cannot* much “spread” in the remaining 2-directions “area-wise”.

1. Statement of the main results

1.1. The Gauss–Bonnet inequality

In this paper, unless otherwise stated we denote X to be a smooth compact Riemannian manifold of dimension $n \geq 3$ with *smooth* boundary ∂X . In the following discussion, we work on triples $(X, \partial_{\text{eff}}, \partial_{\text{side}})$, where ∂_{eff} and ∂_{side} are interiorly disjoint, compact, piecewisely smooth regions of ∂X sharing a common boundary. We point out that ∂_{eff} and ∂_{side} are distinguished *artificially* due to their different use in our discussion. For convenience, we shall call ∂_{eff} *effective boundary* and ∂_{side} *side boundary*, respectively. To obtain intuition the audience may keep the following concrete example in their mind.

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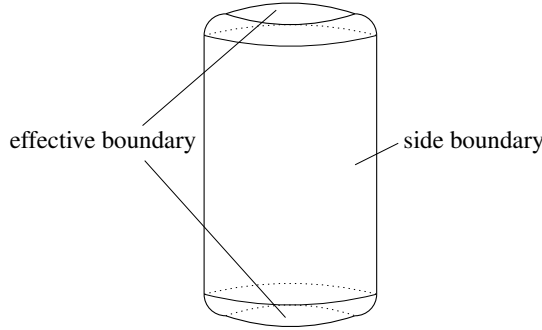


Figure 1. Side boundary and effective boundary of the smoothed 3-dimensional cylinder in Example 1.1.

Example 1.1. Denote X to be a smoothed 3-dimensional cylinder, i.e., X is topologically the product $D^2 \times [-1, 1]$ of the disk and the unit segment but with the induced smooth structure from \mathbb{R}^3 as shown in Figure 1. In our discussion we are interested in the triple

$$(D^2 \times [-1, 1], D^2 \times \{\pm 1\}, \partial D^2 \times [-1, 1]).$$

In this case, $D^2 \times \{\pm 1\}$ is considered to be effective boundary and $\partial D^2 \times [-1, 1]$ is considered to be side boundary.

Let

$$f: (X, \partial_{\text{eff}}) \rightarrow ([-1, 1]^{n-2}, \partial[-1, 1]^{n-2})$$

be a continuous map from X to the $(n - 2)$ -cube such that f sends the effective boundary to the boundary of the cube (since we focus on smooth manifolds it is convenient to imagine that the cube is smoothed). An example of such map f is the height function of the smoothed 3-dimensional cylinder above.

Denote $h \in H_2(X, \partial_{\text{side}})$ to be the relative homology class given by $h = D(f^*\omega)$, where

$$D: H^k(X, \partial_{\text{eff}}) \rightarrow H_{n-k}(X, \partial_{\text{side}})$$

is the Poincaré duality map and ω is the generator of the top cohomology group

$$H^{n-2}([-1, 1]^{n-2}, \partial[-1, 1]^{n-2}).$$

In the case when f is a smooth map, the relative homology class h has the following geometric interpretation: for any regular value $t \in \text{int}[-1, 1]^{n-2}$ of the map f , the preimage $f^{-1}(t)$ is a smooth oriented surface $\Sigma_t \subset X$ with its boundary $\partial \Sigma_t \subset \partial_{\text{side}}$ and h is just the relative homology class represented by this Σ_t . For this reason we shall call h the *f-pullback of the point class*.

Let $\partial_{\mp,i} \subset \partial[-1, 1]^{n-2}$, $i = 1, \dots, n - 2$, be the pairs of opposite faces of the $(n - 2)$ -cube. Correspondingly, we denote $\partial_{\mp,i} X \subset \partial_{\text{eff}}$ to be their f -pullbacks given by

$$\partial_{\mp,i} X = f^{-1}(\partial_{\mp,i}) \cap \partial_{\text{eff}}.$$

In the following discussion, we use d_i to denote the distances between these “faces” $\partial_{\mp,i} X$ in X , that is,

$$d_i = \text{dist}(\partial_{-,i} X, \partial_{+,i} X), \quad i = 1, \dots, n - 2.$$

Now we are ready to state our main result, where there is no, a priori, lower bound on the scalar curvature and that is proved with a use of *stable μ -bubbles*. To avoid possible regularity issue of minimizing μ -bubbles from geometric measure theory, we only focus on the case when the dimension n is no greater than seven throughout this paper.

Theorem 1.2. *With the setting above, given any $\epsilon > 0$ the relative homology class h can be represented by a smooth embedded surface $(\Sigma, \partial\Sigma) \subset (X, \partial_{\text{side}})$ such that the integrals of the scalar curvature of X over any connected component S of Σ and of the mean curvature¹ of ∂_{side} over the boundary $\Theta = \partial S$ satisfy:*

$$\begin{aligned} \int_S \text{Sc}(X, s) \, ds + 2 \int_{\Theta} \text{mean.curv}(\partial_{\text{side}}, \theta) \, d\theta \\ \leq 4\pi\chi(S) + (1 + \epsilon)C_n(d_i) \text{area}(S), \end{aligned} \tag{1.1}$$

where $\chi(S)$ is the Euler characteristics of S and

$$C_n(d_i) = \frac{4(n - 1)\pi^2}{n} \cdot \sum_{i=1}^{n-2} \frac{1}{d_i^2}.$$

Remark 1.3. We point out that the above theorem makes sense only when the scalar curvature of X and the mean curvature of ∂X are non-negative. Otherwise, we can always find a compact surface Σ in the given relative homology class such that for any component S of Σ the integral

$$\int_S \text{Sc}(X, s) \, ds$$

or

$$\int_{\Theta} \text{mean.curv}(\partial_{\text{side}}, \theta) \, d\theta$$

¹Our sign convention is such that the boundaries of *convex* domains have *positive* mean curvatures.

becomes arbitrarily negative, and so the inequality (1.1) holds trivially. Even though, such surface may be ruled out if we require its interior and boundary to be “relatively flat” around the region where the scalar curvature or the mean curvature is negative. So there may be a rough analogy of inequality (1.1) holds if we further require a curvature bound for the portion of surface and its boundary near the negative region.

1.2. Examples of corollaries: Area inequality and applications

In this subsection, we raise some corollaries deduced from Theorem 1.2 and its proof. In preparation, let us first recall the definition of iso-enlargeable manifolds from [6] (stated in a slightly different way).

Definition 1.4 (Iso-enlargeability). A Riemannian n -manifold Y is said to be *iso-enlargeable* if for any constant $d > 0$ there exist

- a compact Riemannian n -manifold U_d with smooth boundary ∂U_d ;
- a locally isometric immersion $e_d: U_d \rightarrow Y$; and
- a continuous map

$$\phi = \phi_d: (U_d, \partial U_d) \rightarrow ([0, 1]^n, \partial[0, 1]^n)$$

of non-zero degrees,

such that the distances between the pullbacks of the opposite faces of the n -cube are bounded from below by

$$\text{dist}_{U_d}(\phi^{-1}(\partial_{-,i}), \phi^{-1}(\partial_{+,i})) \geq d, \quad i = 1, \dots, n. \tag{1.2}$$

Remark 1.5. The typical examples of iso-enlargeable manifolds are those with non-positive sectional curvatures, whose universal covering allows a distance-decreasing diffeomorphism to \mathbb{R}^n . In this case, one can take U_d to be the $(2d)$ -cube in \mathbb{R}^n with the induced metric. For more examples the audience can refer to [6, p. 658].

To state the area inequality, we need to assume that the boundary ∂X of X has *non-negative mean curvature* and the scalar curvature of X is *bounded from below* by a positive constant $\sigma > 0$. The precise statement is as follows.

Corollary 1.6. *Let X_0 be a compact orientable surface, X_1 be a compact orientable iso-enlargeable manifold without boundary, and X be a compact orientable Riemannian manifold. If there is a continuous map*

$$f = (f_0, f_1): (X, \partial X) \rightarrow (X_0 \times X_1, \partial X_0 \times X_1)$$

with non-zero degree, then X contains a smoothly immersed surface $S \subset X$ such that

- S is either a topological 2-sphere without boundary or a topological disk with boundary $\partial S \subset \partial X$;
- S represents a non-zero homology class in $H_2(X, \partial X)$;
- S satisfies the area inequality

$$\text{area}(S) \leq \frac{4\pi\chi(S)}{\sigma}.$$

Sketch of the proof of Corollary 1.6. (See Section 3 for details.) Denote $\underline{X} = X_1$ and $\underline{f} = f_1: X \rightarrow \underline{X}$. Since \underline{X} is iso-enlargeable, by definition there exist local isometric immersions $e_d: U_d \rightarrow \underline{X}$ for all $d > 0$. Let $\tilde{e}_d: \tilde{U}_d \rightarrow X$ be the “ f -pullback” of e_d , that is, we denote $\tilde{e}_d: X \times \underline{X} \rightarrow X$ to be the projection map and $\tilde{U}_d \subset X \times \underline{X}$ is defined to be the set of pairs (x, \underline{x}) such that

$$\underline{f}(x) = e_d(\underline{x}) \quad \text{and} \quad \tilde{e}_d(x, \underline{x}) = x.$$

Theorem 1.2 applied to these \tilde{U}_d delivers surfaces $\tilde{S}_d \subset \tilde{U}_d$ with

$$\text{area}(\tilde{S}_d) \leq \frac{4\pi\chi(\tilde{S}_d)}{\sigma} + \varepsilon_d,$$

where $\varepsilon_d \rightarrow 0$ as $d \rightarrow \infty$. The required surface $S \subset X$ is now obtained as a (sub)limit of $\tilde{e}_d(\tilde{S}_d) \subset X$. ■

With a more careful analysis we are able to establish the following rigidity for the area inequality.

Theorem 1.7 (Compare with [18]). *Under the assumptions of Corollary 1.6, either there exists a surface S representing a non-trivial homotopy class in $\pi_2(X, \partial X)$ with*

$$\text{area}(S) < \frac{4\pi\chi(S)}{\sigma},$$

or the universal covering of X is isometric to the Riemannian product $X = S_\sigma \times \mathbb{R}^{n-2}$, where S_σ is either the 2-sphere or the hemisphere of constant curvature $\sigma/2$.

The following is another application of Theorem 1.2, where some distances d_i are kept bounded and the positive lower bound of the scalar curvature after a loss plays a role in the area inequality.

Corollary 1.8. *Let $X_i, i = 1, \dots, m$, be orientable surfaces (compact or non-compact) with inradii² $\text{inrad}(X_i) \geq d_i$. Suppose that X_0 is a compact orientable surface and*

²The *inradius* of a connected Riemannian manifold X is the supremum of the numbers R , for which there exists a closed ball $B_x(R) \subset X$, such that $B_x(R)$ is compact, does not intersect

that X is an orientable Riemannian manifold (compact or non-compact) admitting a proper continuous map

$$f = (f_0, f_1, \dots, f_m): (X, \partial X) \rightarrow (X_0 \times \dots \times X_m, \partial(X_0 \times \dots \times X_m))$$

with non-zero degree, where each $f_i: X \rightarrow X_i$ is distance-decreasing for $i = 1, \dots, m$. If the scalar curvature of X is bounded from below by

$$\text{Sc}(X) - \frac{4(n-1)\pi^2}{n} \cdot \sum_{i=1}^m \frac{1}{d_i^2} \geq \sigma_0 > 0,$$

and the boundary of X has non-negative mean curvature, then there exists a smooth surface $S \subset X$ such that

- S is a 2-sphere without boundary or a disk with boundary contained in ∂X , which represents a non-zero element in $H_2(X, \partial X)$;
- S satisfies the area inequality $\text{area}(S) \leq 4\pi\chi(S)/\sigma_0$.

Idea of the proof. Since we have $\text{inrad}(X_i) \geq d_i$, for all positive ε and d there exist compact surfaces $U_{i,\varepsilon,d}$ with boundary, locally isometric immersions

$$e_{i,\varepsilon,d}: U_{i,\varepsilon,d} \rightarrow X$$

and non-zero degree maps

$$\phi_{i,\varepsilon,d} = (\phi_{i,\varepsilon,d,1}, \phi_{i,\varepsilon,d,2}): U_{i,\varepsilon,d} \rightarrow \left[0, \frac{d_i}{1+\varepsilon}\right] \times [0, d]$$

with $\text{Lip } \phi_{i,\varepsilon,d,k} \leq 1$ for $k = 1, 2$. Then this allows an application of (a simple modification of) the previous sketch of the proof of Corollary 1.6 with $\varepsilon \rightarrow 0$ and $d \rightarrow \infty$. ■

Remark 1.9. Since we prove everything in this paper for $n = \dim(X) \leq 7$, the above theorem applies only for $m = 1$ and 2 , that makes $n = 2m + 2 \leq 6$. Not to lose $n = 7$, we may allow $X = D^2 \times S^1 \times X_1 \times X_2$, with the same assumptions on X_i and f_i , and with the same conclusion as in Corollary 1.6 (as a consequence of Proposition 2.1). This actually makes no difference in the proof.

As a further application we also investigate the case when the boundary of X has simple corners as planar polygons, where we can reduce the problem to one for Riemannian manifolds with smooth boundary based on a smoothing trick.

the boundary of X and such that it is *not contained* in a smaller ball $B_x(R - \varepsilon)$. For instance, if X is compact without boundary, then $\text{inrad}(X) = \text{diam}(X)$ and if X is complete non-compact, then $\text{inrad}(X) = \infty$.

Corollary 1.10. *Let X be a Riemannian manifold diffeomorphic to $X_0 \times \mathbb{R}^{n-2}$, where X_0 is a planar j -gon. Assume that X has non-negative scalar curvature and that ∂X is mean convex away from the corners. Furthermore, the dihedral angles \angle_i , $i = 1, 2, \dots, j$, satisfy*

$$\angle_i \leq \alpha_i < \pi \quad \text{and} \quad \sum_{i=1}^j (\pi - \alpha_i) > 2\pi.$$

Then there is no proper and globally Lipschitz map $\phi: X \rightarrow \mathbb{R}^{n-2}$ such that X_0 is homologous to the ϕ -pullback of a point.

Idea of the proof. One can smoothing the manifold X such that the dihedral angle inequality

$$\sum_{i=1}^j (\pi - \alpha_i) > 2\pi$$

becomes

$$\int_{\Theta=\partial S} \text{mean.curv}(\partial_{\text{side}}, \theta) \, d\theta > 2\pi$$

for any surface S homologous to X_0 with free boundary. Now the desired result follows from Gauss–Bonnet inequality (1.1) with $d_i = \infty$. (The precise argument needs technical modifications and is included in Section 3.) ■

2. Proof of Main Theorem 1.2

In this section, we give a detailed proof of our main theorem. Essentially, the proof is based on the μ -bubble method from [8] and also the idea of warped symmetrization from [9, Proposition 7.33], which was first used by Fischer-Colbrie and Schoen in [5, Proof of Theorem 4]. The consecutive use of warped symmetrization for minimal hypersurfaces appears in [6] and [16] (see also [18]), where the trick is called *torical symmetrization* by the first named author emphasizing that the ambient manifold after warped symmetrization become more symmetric, and is called *weighted minimal slicing* by Schoen and Yau with emphasis on those submanifolds produced in the symmetrization. The weighted minimal slicing can be viewed as a variant of the *dimension descent argument* of Schoen and Yau [15] through replacing conformal deformation by warped symmetrization. Torical symmetrization and weighted slicing of μ -bubbles were used recently in [7] and [3] to prove the aspherical conjecture up to dimension five. Here we shall apply the same idea to prove our Theorem 1.2.

Proof of Theorem 1.2. First notice that the consequence is obvious if the relative homology class h is trivial. In fact, we can pick up a small geodesic 2-sphere S around

some point of X . Then the inequality holds automatically since the left hand side is almost zero and the right hand side is about 8π .

So we just need to deal with the case when the relative homology class h is non-trivial. Through a slight bending for $\partial_{-,1}X$ and $\partial_{+,1}X$, without loss of generality we can assume that they intersect the rest part of boundary $\partial X - (\partial_{-,1}X \cup \partial_{+,1}X)$ in acute angles (see Figure 2). Given any small $\epsilon > 0$, it is not difficult to construct a smooth function

$$\phi_1: X \rightarrow \left[-\frac{d_1^\epsilon}{2}, \frac{d_1^\epsilon}{2}\right], \quad d_1^\epsilon := (1 - \epsilon)d_1,$$

with

$$|d\phi_1| \leq 1 \quad \text{and} \quad \phi_1^{-1}\left(\pm \frac{d_1^\epsilon}{2}\right) = \partial_{\pm,1}X.$$

Now we set the following minimizing problem. Let

$$\Omega_0 = \{x \in X : \phi_1(x) < 0\}$$

and we consider the class

$$\mathcal{C} = \{\text{Caccioppoli sets } \Omega \text{ in } X \text{ such that } \Omega \Delta \Omega_0 \Subset X - (\partial_{-,1}X \cup \partial_{+,1}X)\}.$$

We pick up the function

$$h_1: \left(-\frac{d_1^\epsilon}{2}, \frac{d_1^\epsilon}{2}\right) \rightarrow \mathbb{R}, \quad t \mapsto -\frac{2(n-1)\pi}{nd_1^\epsilon} \tan\left(\frac{\pi t}{d_1^\epsilon}\right),$$

and define

$$\mathcal{B}(\Omega) = \mathcal{H}^{n-1}(\partial^*\Omega \cap \text{int } X) - \int_X (\chi_\Omega - \chi_{\Omega_0})h_1 \circ \phi_1 \, d\mathcal{H}^n, \quad \forall \Omega \in \mathcal{C}, \quad (2.1)$$

where $\partial^*\Omega$ represents the reduced boundary of Ω , $\text{int } X$ is used to denote the interior part of X , and χ_Ω is the characteristic function for Ω .

Now we want to find a smooth minimizer Ω_1 in \mathcal{C} for the functional \mathcal{B} from the geometric measure theory. Notice that our bending of $\partial_{-,1}X$ and $\partial_{+,1}X$ causes extra corners in ∂X , we need to show that the minimizing procedure can be done away from the corner. The strategy is to modify each $\Omega \in \mathcal{C}$ to a new $\tilde{\Omega} \in \mathcal{C}$ such that $\mathcal{B}(\tilde{\Omega}) \leq \mathcal{B}(\Omega)$ and that $\partial\tilde{\Omega}$ avoids a fixed collar neighborhood of $\partial_{-,1}X \cup \partial_{+,1}X$ (independent of Ω). For simplicity we only show how to do such modification around $\partial_{+,1}X$ (since the modification around $\partial_{-,1}X$ can be done in a similar way). Since $\partial_{+,1}X$ intersects $\partial X - \partial_{+,1}X$ in acute angles, we can take a collar neighborhood \mathcal{R} such that all equidistant hypersurfaces to $\partial_{+,1}X$ in \mathcal{R} have this property. Denote \vec{N} to be the unit outward normal vector field in \mathcal{R} and H to be the mean curvature of equidistant hypersurfaces to $\partial_{+,1}X$ with respect to \vec{N} . Notice that $h_1 \circ \phi_1$ diverges to $-\infty$

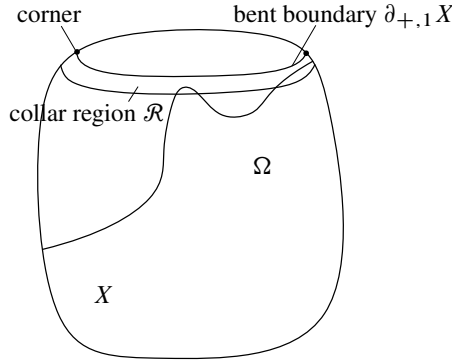


Figure 2. The modification of Ω around $\partial_{+,1}X$.

on $\partial_{+,1}X$. By passing to a smaller collar neighborhood, we can assume $H \geq h_1 \circ \phi_1$ in \mathcal{R} . Given any Ω in \mathcal{C} we take $\tilde{\Omega}$ to be $\Omega - \mathcal{R}$. Now we have

$$\begin{aligned} \mathcal{B}(\tilde{\Omega}) - \mathcal{B}(\Omega) &= \mathcal{H}^{n-1}(\partial^* \tilde{\Omega} \cap \text{int } X) - \mathcal{H}^{n-1}(\partial^* \Omega \cap \text{int } X) + \int_{\Omega \cap \mathcal{R}} h_1 \circ \phi_1 \, d\mathcal{H}^n \\ &\leq - \int_{\partial \mathcal{R} \cap \Omega} \nu \cdot \vec{N} \, d\sigma - \int_{\partial^* \Omega \cap \mathcal{R}} \nu \cdot \vec{N} \, d\sigma + \int_{\Omega \cap \mathcal{R}} h_1 \circ \phi_1 \, d\mathcal{H}^n, \end{aligned}$$

where ν is the unit outward normal of $\partial(\Omega \cap \mathcal{R})$ with respect to $\Omega \cap \mathcal{R}$. Since the equidistant hypersurfaces intersect ∂X in acute angles (i.e., $\nu \cdot \vec{N} < 0$), we have

$$\int_{\partial X \cap \partial(\Omega \cap \mathcal{R})} \nu \cdot \vec{N} \, d\sigma \leq 0.$$

Combined with the divergence theorem we see that

$$- \int_{\partial \mathcal{R} \cap \Omega} \nu \cdot \vec{N} \, d\sigma - \int_{\partial^* \Omega \cap \mathcal{R}} \nu \cdot \vec{N} \, d\sigma \leq - \int_{\Omega \cap \mathcal{R}} H \, d\mu.$$

This implies

$$\mathcal{B}(\tilde{\Omega}) - \mathcal{B}(\Omega) \leq \int_{\Omega \cap \mathcal{R}} h_1 \circ \phi_1 \, d\mu \leq 0.$$

The above discussion reduces the minimizing problem to its usual case and the geometric measure theory yields that there exists a smooth minimizer Ω_1 in \mathcal{C} for the functional \mathcal{B} , whose boundary Y_1 is a smooth embedded hypersurface with free boundary that separates $\partial_{-,1}X$ and $\partial_{+,1}X$.

Notice that a scale of ϕ_1 is homotopic to f_1 relative to $\partial_{-,1}X \cup \partial_{+,1}X$, it follows that Y_1 represents the F_1 -pullback of the point class in

$$H_{n-1}(X, \partial X - (\partial_{-,1}X \cup \partial_{+,1}X))$$

with $F_1 = f_1$. Since the relative homology class h is non-trivial, the hypersurface Y_1 has at least one component intersecting with boundary portions $\partial_{-,2}X$ and $\partial_{+,2}X$ at the same time. Let us collect all such components of Y_1 and still denote the union by Y_1 (since the rest components contribute nothing in our following construction). Now the boundary portion $\partial_{\pm,2}Y_1$ defined as the intersection of ∂Y_1 and $\partial_{\pm,2}X$ is non-empty and it is clear that

$$\text{dist}(\partial_{-,2}Y_1, \partial_{+,2}Y_1) \geq d_2$$

since the intrinsic distance is always no less than the extrinsic one.

Let $V = \psi v$ be any smooth variation vector field on Y_1 , where ψ is a smooth function on Y_1 and v is the outward unit normal vector field on Y_1 with respect to Ω_1 . It is not difficult to calculate that Y_1 satisfies the following first variation formula

$$\delta \mathcal{B}(\psi) = \int_{Y_1} (\text{mean.curv}(Y_1) - (h_1 \circ \phi_1)|_{Y_1}) \psi \, d\sigma + \int_{\partial Y_1} \psi \vec{n} \cdot \nu \, ds = 0,$$

where \vec{n} is the outward unit normal vector field of ∂Y_1 in Y_1 . This yields that the mean curvature of Y_1 is equal to $h_1 \circ \phi_1$ and Y_1 has free boundary, i.e., Y_1 intersects ∂X orthogonally along ∂Y_1 . After computing the second variation as in [8] and [12] we see

$$\begin{aligned} \delta^2 \mathcal{B}(\psi, \psi) &= \int_{Y_1} |\nabla_{Y_1} \psi|^2 - (\text{Ric}(v, v) + |A|^2 - \partial_v(h_1 \circ \phi_1)) \psi^2 \, d\sigma \\ &\quad - \int_{\partial Y_1} \text{II}_{\partial X}(v, v) \psi^2 \, ds \geq 0, \end{aligned}$$

where A is denoted to be the second fundamental form of Y_1 in X with respect to ν and $\text{II}_{\partial X}$ is the second fundamental form of ∂X in X with respect to \vec{n} . After playing Schoen–Yau’s rearranging trick [14] this implies

$$\begin{aligned} \int_{Y_1} |\nabla_{Y_1} \psi|^2 - \frac{1}{2} \left(\text{Sc}(X) - \text{Sc}(Y_1) + |\overset{\circ}{A}|^2 - \frac{4(n-1)\pi^2}{n(d_1^\epsilon)^2} \right) \psi^2 \, d\sigma \\ - \int_{\partial Y_1} \text{II}_{\partial X}(v, v) \psi^2 \, ds \geq 0, \end{aligned}$$

where $\overset{\circ}{A}$ is denoted to be the trace-free part of A and we use the relation

$$\frac{n}{n-1} h_1^2 + 2h'_1 = -\frac{4(n-1)\pi^2}{n(d_1^\epsilon)^2}$$

from our particular choice for h_1 . Following the idea of torical symmetrization, we construct a new warped product manifold as follows. Let u_1 be the first eigenfunction

of the Jacobi operator

$$\mathcal{L} = -\Delta_{Y_1} - \frac{1}{2} \left(\text{Sc}(X) - \text{Sc}(Y_1) + |\mathring{A}|^2 - \frac{4(n-1)\pi^2}{n(d_1^\epsilon)^2} \right)$$

corresponding to the Robin boundary condition

$$\frac{\partial u_1}{\partial \vec{n}} = \Pi_{\partial X}(v, v)u_1.$$

From Hopf lemma we see that u_1 is a positive function up to the boundary ∂Y_1 satisfying $\mathcal{L}u_1 \geq 0$. Now let us define $X_1 = Y_1 \times \mathbb{S}^1$ and $g_1 = g_{Y_1} + u_1^2 ds^2$. Through a straightforward calculation we see

$$\begin{aligned} \text{Sc}(X_1, (y_1, \theta)) &= \text{Sc}(Y_1, y_1) - 2u_1^{-1} \Delta_{Y_1} u_1 \\ &\geq \text{Sc}(X, y_1) - \frac{4(n-1)\pi^2}{n(d_1^\epsilon)^2}, \quad y_1 \in Y_1, \theta \in \mathbb{S}^1, \end{aligned}$$

and

$$\text{mean.curv}(\partial X_1, (b_1, \theta)) = \text{mean.curv}(X, b_1), \quad b_1 \in B_1 = \partial Y_1, \theta \in \mathbb{S}^1.$$

The rest of the proof will be completed by induction. Let $\partial_{\pm,2} X_1 = \partial_{\pm,2} Y_1 \times \mathbb{S}^1$. With these boundary portions, we can repeat our previous argument with some slight changes only in choices for functions. In fact, we replace function ϕ_1 by some

$$\phi_2: Y_1 \rightarrow \left[-\frac{d_2^\epsilon}{2}, \frac{d_2^\epsilon}{2} \right], \quad d_2^\epsilon := (1 - \epsilon)d_2,$$

with

$$|d\phi_2| \leq 1 \quad \text{and} \quad \phi_2^{-1} \left(\pm \frac{d_2^\epsilon}{2} \right) = \partial_{\pm,2} Y_1,$$

and replace function h_1 by

$$h_2: \left(-\frac{d_2^\epsilon}{2}, \frac{d_2^\epsilon}{2} \right) \rightarrow \mathbb{R}, \quad t \mapsto -\frac{2(n-1)\pi}{nd_2^\epsilon} \tan\left(\frac{\pi t}{d_2^\epsilon}\right).$$

Of course, since ϕ_2 is only a function on Y_1 , the right way is to view ϕ_2 as an \mathbb{S}^1 -invariant function on X_1 when we set the minimizing problem on X_1 . In a similar way we obtain a smooth minimizer Ω_2 for functional \mathcal{B} in (2.1) after replacements, whose boundary Y_2 is a smooth embedded hypersurface with free boundary separating $\partial_{-,2} X_1$ and $\partial_{+,2} X_1$. By limiting the class \mathcal{C} to those \mathbb{S}^1 -invariant Cacciopoli sets we can require that the hypersurface Y_2 has \mathbb{S}^1 -invariance and this yields $Y_2 = \widehat{Y}_2 \times \mathbb{S}^1$ for some hypersurface \widehat{Y}_2 in Y_1 . By tracking the relative homology class of \widehat{Y}_2 carefully, we see that \widehat{Y}_2 is homologous to the F_2 -pullback of the point class with

$$F_2 = (f_1, f_2): X \rightarrow [-1, 1]^2.$$

As before, we can construct a new warped product manifold X_2 by warping Y_2 and \mathbb{S}^1 . In particular, we have $X_2 = \hat{Y}_2 \times T^2$,

$$\text{Sc}(X_2, (\hat{y}_2, \Theta)) \geq \text{Sc}(X, \hat{y}_2) - \frac{4(n-1)\pi^2}{n} \left(\frac{1}{(d_1^\epsilon)^2} + \frac{1}{(d_2^\epsilon)^2} \right), \quad \hat{y}_2 \in \hat{Y}_2, \Theta \in T^2,$$

and

$$\text{mean.curv}(\partial X_2, (\hat{b}_2, \Theta)) = \text{mean.curv}(X, \hat{b}_2), \quad \hat{b}_2 \in \hat{B}_2 = \partial \hat{Y}_2, \Theta \in T^2.$$

In the following, we continue above procedure until all pairs of $\partial_{\pm, i}$ of $[-1, 1]^{n-2}$ are handled. Finally, we obtain a smooth embedded surface $\Sigma = \hat{Y}_{n-2}$ in X representing the relative homology class h associated with the warped product manifold

$$(X_{n-2}, g_{n-2}) = (\Sigma \times T^{n-2}, g_\Sigma + u_1^2 ds_1^2 + \dots + u_{n-2}^2 ds_{n-2}^2) \quad (2.2)$$

satisfying

$$\text{Sc}(X_{n-2}, (s, \Theta)) \geq \text{Sc}(X, s) - \frac{4(n-1)\pi^2}{n} \sum_{i=1}^{n-2} \frac{1}{(d_i^\epsilon)^2}, \quad s \in \Sigma, \Theta \in T^{n-2}, \quad (2.3)$$

and

$$\text{mean.curv}(\partial X_{n-2}, (b, \Theta)) = \text{mean.curv}(\partial X, b), \quad b \in B = \partial \Sigma, \Theta \in T^{n-2}. \quad (2.4)$$

From (2.2) one can compute

$$\text{Sc}(X_{n-2}, (s, \Theta)) = \text{Sc}(\Sigma, s) - 2 \sum_{i=1}^{n-2} u_i^{-1} \Delta_\Sigma u_i - 2 \sum_{1 \leq i < j \leq n-2} \langle \nabla_\Sigma \log u_i, \nabla_\Sigma \log u_j \rangle$$

and

$$\text{mean.curv}(\partial X_{n-2}, (b, \Theta)) = \kappa_{\partial \Sigma}(b) + \sum_{i=1}^{n-2} \nu(\log u_i),$$

where $\kappa_{\partial \Sigma}$ is the geodesic curvature of boundary curve $\partial \Sigma$ in Σ with respect to the unit outer normal ν . For any connected component S of Σ , we have

$$\begin{aligned} & \int_S \text{Sc}(X_{n-2}, (s, \Theta)) \, d\mu + 2 \int_{\partial S} \text{mean.curv}(\partial X_{n-2}, (b, \Theta)) \, d\sigma \\ &= \int_S \text{Sc}(S, s) \, ds + 2 \int_{\partial S} \kappa_{\partial S}(b) \, db \\ & \quad - \sum_{i=1}^{n-2} \int_S |\nabla_S \log u_i|^2 \, d\sigma - \int_S \left| \sum_{i=1}^{n-2} \nabla_S \log u_i \right|^2 \, d\sigma \leq 4\pi \chi(S). \end{aligned} \quad (2.5)$$

The desired estimate comes from (2.3) and (2.4) substituted into (2.5). ■

Combining the torical symmetrization technique for μ -bubbles as well as that for minimal hypersurfaces, a similar argument leads to the following

Proposition 2.1. *Let $f: X \rightarrow [-1, 1]^{n-k-2} \times T^k$ be a continuous map such that it sends $X - \partial_{\text{side}}$ to the boundary of $[-1, 1]^{n-k-2} \times T^k$. Then for any $\epsilon > 0$, the pull-back homology class h can be represented by a smooth surface $\Sigma \subset X$, the boundary of which is contained in ∂_{side} and such that all connected components S of Σ satisfy:*

$$\int_S \text{Sc}(X, s) \, ds + 2 \int_{\Theta} \text{mean.curv}(\partial_{\text{side}}, \theta) \, d\theta \leq 4\pi\chi(S) + (1 + \epsilon)C_n(d_i) \text{area}(S),$$

where $\chi(S)$ is the Euler characteristics of S and

$$C_n(d_i) = \frac{4(n - 1)\pi^2}{n} \cdot \sum_{i=1}^{n-k-2} \frac{1}{d_i^2}.$$

3. Proofs of corollaries

In this section, we present detailed proofs for those corollaries mentioned in the introduction. The proofs below basically follow the line of those sketches from introduction but some technical modifications also need to be included.

Proof of Corollary 1.6. We divide the proof into two steps.

Step 1. First let us show that for any $\epsilon > 0$ there is a smooth surface $S \subset X$ representing a non-trivial homology class in $H_2(X, \partial X)$, that is either a sphere without boundary or a disk with boundary $\partial S \subset \partial X$ such that

$$\text{area}(S) \leq \frac{4\pi\chi(S)}{\sigma} + \epsilon.$$

Denote $\underline{X} = X_1$. Since \underline{X} is iso-enlargeable, given any $d > 0$ there is a compact manifold U_d with non-empty boundary associated with a locally isometric immersion $e_d: U_d \rightarrow \underline{X}$ and a proper continuous map $\phi_d: U_d \rightarrow [-1, 1]^{n-2}$ with non-zero degree. Now let us pull back the isometric immersion $e_d: U_d \rightarrow \underline{X}$ along the map $\underline{f} = f_1: X \rightarrow \underline{X}$. Define

$$\tilde{U}_d = \{(x, u) \in X \times U_d : \underline{f}(x) = e_d(u)\}$$

and

$$\tilde{e}_d: \tilde{U}_d \rightarrow X, \quad (x, u) \mapsto x.$$

It is not difficult to show that \tilde{U}_d is a differentiable manifold the map \tilde{e}_d is an immersion, and so we can pull back the metric g of X on \tilde{U}_d such that \tilde{e}_d becomes an isometric immersion. Denote

$$\tilde{f}: \tilde{U}_d \rightarrow U_d, \quad (x, u) \mapsto u.$$

Then we have the following commutative diagram:

$$\begin{array}{ccc} \tilde{U}_d & \xrightarrow{\tilde{f}} & U_d \\ \tilde{e}_d \downarrow & & \downarrow e_d \\ X & \xrightarrow{f} & \underline{X}. \end{array}$$

Notice that

$$\partial(X \times U_d) = (\partial X \times U_d) \cup (X \times \partial U_d).$$

Corresponding we have the decomposition $\partial\tilde{U}_d = \partial_{\text{eff}} \cup \partial_{\text{side}}$, where

$$\partial_{\text{side}} = \partial\tilde{U}_d \cap (\partial X \times U_d) \quad \text{and} \quad \partial_{\text{eff}} = \partial\tilde{U}_d \cap (X \times \partial U_d).$$

In particular, we have $\tilde{f}: (\tilde{U}_d, \partial_{\text{eff}}) \rightarrow (U_d, \partial U_d)$ and $\tilde{e}_d: (\tilde{U}_d, \partial_{\text{side}}) \rightarrow (X, \partial X)$.

Now we point out that \tilde{U}_d has a definite size. Without loss of generality, we may assume that the map $f: X \rightarrow \underline{X}$ is smooth. From compactness of X , the map f has bounded Lipschitz norm independent of d and the same thing holds for the lifted map \tilde{f} . After a rescaling of \underline{X} , we just need to deal with the case when \tilde{f} is distance-decreasing. Denote

$$\hat{f} = \phi_d \circ \tilde{f}: (\tilde{U}_d, \partial_{\text{eff}}) \rightarrow ([-1, 1]^{n-2}, \partial[-1, 1]^{n-2}),$$

and let

$$\partial_{\mp, i} \tilde{U}_d = \hat{f}^{-1}(\partial_{\mp, i}) \cap \partial_{\text{eff}}.$$

From (1.2) and the distance decreasing property of \tilde{f} , we have

$$d_i := \text{dist}_{\tilde{U}_d}(\partial_{-, i} \tilde{U}_d, \partial_{+, i} \tilde{U}_d) \geq d, \quad i = 1, 2, \dots, n - 2.$$

Next we would like to apply Theorem 1.2 to the composed map \hat{f} . Denote h to be the \hat{f} -pullback of the point class. Fix a small positive constant ϵ . Then it follows from Theorem 1.2 that h can be represented by a smooth embedded surface Σ possibly with boundary $\partial\Sigma \subset \partial_{\text{side}}$ such that any of its components S satisfies

$$\begin{aligned} & \int_S \text{Sc}(X, s) \, ds + 2 \int_{\partial S} \text{mean.curv}(\partial_{\text{side}}, \theta) \, d\theta \\ & \leq 4\pi\chi(S) + (1 + \epsilon)C_n(d_i) \cdot \text{area}(S). \end{aligned}$$

Combined with the facts

$$\text{Sc}(X) \geq \sigma > 0 \quad \text{and} \quad \text{mean.curv}(\partial X) \geq 0,$$

we conclude

$$\text{area}(S) \leq 4\pi\chi(S) \left(\sigma - \frac{4(n-1)(n-2)\pi^2}{nd^2}(1+\epsilon) \right)^{-1} = \frac{4\pi\chi(S)}{\sigma} + \epsilon_d$$

for all d large enough, where ϵ_d is an error term converging to 0 as $d \rightarrow +\infty$.

It rests to show that there is at least one component S_d of Σ such that the image $\tilde{e}_d(S_d)$ represents a non-trivial relative homology class in $H_2(X, \partial X)$. For our purpose, we consider the map

$$\tilde{F} = (f_0 \circ \tilde{e}_d, \tilde{f}): (\tilde{U}_d, \partial_{\text{side}}) \rightarrow (X_0 \times U_d, \partial X_0 \times U_d).$$

From the commutative diagram above as well as the construction of \tilde{U}_d , by counting the number of regular points at one regular value of \tilde{F} it is easy to verify the fact that $\deg \tilde{F} = \deg f \neq 0$. Denote $\hat{e}_d = (\text{id}, e_d): X_0 \times U_d \rightarrow X_0 \times \underline{X}$. Notice that

$$(\hat{e}_d)_*(\tilde{F}_*(h)) = (\deg \tilde{F} \cdot \deg \phi_d)([X_0]) \neq 0 \in H_2(X_0 \times \underline{X}, \partial X_0 \times \underline{X}),$$

so we conclude that the relative homology class $(\tilde{e}_d)_*(h)$ is non-zero in $H_2(X, \partial X)$. In particular, we can pick up a component S_d of Σ whose image $\tilde{e}_d(S_d)$ represents a non-trivial relative homology class in $H_2(X, \partial X)$.

Step 2. Now we would like to complete the proof by taking the limit of S_d up to a subsequence. In general, surfaces S_d may not have a uniform bound on its mean curvature, but this can be overcome through a slight modification with an idea from the work [19]. In fact, we can use the flexible choices for functions h_i in the proof of Theorem 1.2 to guarantee that all surfaces S_d stay in the region with bounded prescribed mean curvature. First notice that, given any $d > 0$, we can construct smooth functions

$$h_{i,d}: \left(-\frac{d}{2}, \frac{d}{2}\right) \rightarrow \mathbb{R}, \quad i = 1, 2, \dots, n-2,$$

such that

(c1) $h'_{i,d}(t) < 0$ and

$$\lim_{t \in \mp d/2} h_{i,d}(t) = \pm\infty.$$

(c2) the quantity

$$\sum_{i=1}^{n-2} \left(2h'_{i,d}(t_i) + \frac{n}{n-1} h^2_{i,d}(t_i) \right), \quad -\frac{d}{2} \leq t_i \leq \frac{d}{2},$$

is positive outside $[-1, 1]^{n-2}$ and no less than $-\bar{\epsilon}_d$ globally with $\bar{\epsilon}_d \rightarrow 0$ as $d \rightarrow +\infty$.

(c3) for any i , the function $h_{i,d}$ converges to zero smoothly in every compact subset of \mathbb{R} as $d \rightarrow +\infty$.

Recall that we have

$$\hat{f} = (\hat{f}_1, \dots, \hat{f}_{n-2}): (\tilde{U}_d, \partial_{\text{eff}}) \rightarrow ([-1, 1]^{n-2}, \partial[-1, 1]^{n-2}),$$

where \tilde{U}_d satisfies

$$d_i := \text{dist}_{\tilde{U}_d}(\partial_{-,i}\tilde{U}_d, \partial_{+,i}\tilde{U}_d) \geq d, \quad i = 1, 2, \dots, n - 2.$$

We now use the proof of Theorem 1.2 rather than its statement. From the above distance estimate we can construct a map

$$\hat{f}_d = (\hat{f}_{1,d}, \hat{f}_{2,d}, \dots, \hat{f}_{n-2,d}): (\tilde{U}_d, \partial_{\text{eff}}) \rightarrow \left(\left[-\frac{d}{2}, \frac{d}{2} \right]^{n-2}, \partial \left[-\frac{d}{2}, \frac{d}{2} \right]^{n-2} \right)$$

such that each component map $\hat{f}_{i,d}$ satisfies $\text{Lip } \hat{f}_{i,d} \leq 1$ and the \hat{f}_d -pullback of the point class is the same as \hat{f} -pullback of the point class. After replacing functions h_i by $h_{i,d}$ in the proof of Theorem 1.2, we can still find desired surfaces S_d with

$$\text{area}(S_d) \leq 4\pi\chi(S_d) \left(\sigma + \sum_{i=1}^{n-2} \left(2h'_{i,d}(t_i) + \frac{n}{n-1} h^2_{i,d}(t_i) \right) \circ \hat{f}_d|_{S_d} \right)^{-1}. \quad (3.1)$$

In particular, we conclude from (c2) that $\text{area}(S_d) \leq 4\pi\chi(S_d)/\sigma + \epsilon_d$, where

$$\epsilon_d = 8\pi((\sigma - \bar{\epsilon}_d)^{-1} - \sigma^{-1}) \rightarrow 0 \text{ as } d \rightarrow +\infty.$$

Now we analyze the more delicate behavior of these surfaces S_d and the discussion can be divided into the following two cases:

(i) If some S_d does not intersect the region

$$K_d = \hat{f}_d^{-1}([-1, 1]^{n-2}),$$

it follows from (c2) and the estimate (3.1) that we have the modified estimate

$$\text{area}(S_d) < \frac{4\pi\chi(S_d)}{\sigma}. \quad (3.2)$$

This already delivers the desired surface and we are done.

(ii) Otherwise each S_d intersects with the compact region K_d . Thanks to the fact $\text{Sc}(X) \geq \sigma > 0$, it follows from the torical band estimate in [6] that all surfaces S_d have their diameters bounded by a universal constant D depending only on σ . Recall from the symmetrization procedure in the proof of Theorem 1.2 that we obtain the slicing

$$S_d = \Sigma_{n-2,d} \subset \Sigma_{n-3,d} \subset \dots \subset \Sigma_{1,d} \subset \Sigma_{0,d} = \tilde{U}_d,$$

where $\Sigma_{i,d} \times T^{i-1}$ is a minimizing μ -bubble in $\Sigma_{i-1,d} \times T^{i-1}$ of the functional

$$\mathcal{B}(\Omega) = \mathcal{H}^{n-1}(\partial^* \Omega) - \int_X (\chi_\Omega - \chi_{\Omega_0}) h_{i,d} \circ \widehat{f}_{i,d} \, d\mathcal{H}^n. \tag{3.3}$$

Now we are ready to investigate the convergence of surfaces S_d . Take a fixed point p_d in S_d . From the diameter estimate above we just need to focus on the local slicing

$$S_d = \Sigma_{n-2,d} \subset B_{2D}^{\Sigma_{n-3,d}}(p_d) \subset \dots \subset B_{2D}^{\Sigma_{1,d}}(p_d) \subset B_{2D}^{\Sigma_{0,d}}(p_d), \tag{3.4}$$

where $B_{2D}^{\Sigma_{i,d}}(p_d)$ is the geodesic $(2D)$ -ball in $\Sigma_{i,d}$ centered at p_d . Since the Lipschitz norm of each $\widehat{f}_{i,d}$ is no greater than one, we see that $\Sigma_{i,d}$ lies in the region

$$\widehat{f}_{i,d}^{-1}(-2D - 1, 2D + 1),$$

and so the mean curvatures of the warped hypersurface $\Sigma_{i,d} \times T^{i-1}$ in $\Sigma_{i-1,d} \times T^{i-1}$, which is equal to $h_{i,d} \circ \widehat{f}_{i,d}$, are bounded by some δ_d with $\delta_d \rightarrow 0$ as $d \rightarrow +\infty$ due to (c3). Notice that the C^1 -norm of the mean curvatures of $\Sigma_{i,d} \times T^{i-1}$ is also uniformly bounded combined with the fact $\text{Lip } \widehat{f}_{i,d} \leq 1$.

We claim that up to a subsequence the local slicing (3.4) immersed into X by $\widetilde{e}_d: \widetilde{U}_d \rightarrow X$ will converge to a weighted area-minimizing slicing (see [16])

$$S = \Sigma_{n-2} \subset \Sigma_{n-3} \subset \dots \subset \Sigma_1 \subset \Sigma_0$$

in the pointed $C^{2,\alpha}$ -graphical sense. This will be done by induction and let us start with the convergence of $\widetilde{e}_d(B_{2D}^{\Sigma_{1,d}})$. Since $B_{2D}^{\Sigma_{1,d}}$ are all local minimizers of functional (3.3) in \widetilde{U}_d and also the mean curvature is uniformly bounded, it follows from a similar argument to [17] that $B_{2D}^{\Sigma_{1,d}}$ has uniformly bounded curvature estimates. Combined with the uniform C^1 -norm of the mean curvature, $B_{2D}^{\Sigma_{1,d}}$ can be locally written as graphs with uniform $C^{2,\alpha}$ -norm and so $\widetilde{e}_d(B_{2D}^{\Sigma_{1,d}})$ converges to a minimal hypersurface $\Sigma_1 \subset X$ in the pointed $C^{2,\alpha}$ -graphical sense. Next let us deal with the convergence of $\widetilde{e}_d(B_{2D}^{\Sigma_{2,d}})$. Recall that the submanifolds $B_{2D}^{\Sigma_{2,d}}$ come from minimizing functional (3.3) in $(\Sigma_{1,d} \times \mathbb{S}^1, g_{1,d} + u_{1,d}^2 dt_1^2)$, where $u_{1,d}$ is the first eigenfunction of the Jacobi operator with the Robin boundary condition. In particular, $u_{1,d}$ is a positive solution to

$$-\Delta_{\Sigma_{1,d}} u_{1,d} - \frac{1}{2} \left(\text{Sc}(X) - \text{Sc}(\Sigma_{1,d}) + |\mathring{A}|^2 - \frac{4(n-1)\pi^2}{n(d_1^\epsilon)^2} \right) u_{1,d} = \lambda_{1,d} u_{1,d}.$$

Notice that the first eigenvalue $\lambda_{1,d}$ is non-negative. On the other hand, $\lambda_{1,d}$ cannot exceed $\bar{\epsilon}_d$ as well, otherwise we still obtain (3.2) and return to the case (i). Up to a scaling, we can assume $u_{1,d}$ takes value one at the point p_d . Then the Harnack inequality combined with the Schauder estimate yields that $u_{i,d}$ has uniformly

bounded $C^{3,\alpha}$ -norm in $B_{2D}^{\Sigma_1,d}$. In particular, the warped manifold $B_{2D}^{\Sigma_1,d} \times \mathbb{S}^1$ has uniformly bounded curvature (notice that $B_{2D}^{\Sigma_1,d}$ has uniformly bounded curvature from the Gauss equation). After repeating the argument above, we see that $\tilde{e}_d(B_{2D}^{\Sigma_2,d}) \times \mathbb{S}^1$ converges to $\Sigma_2 \times \mathbb{S}^1$ in the pointed $C^{2,\alpha}$ -graphical sense for some minimal hypersurface $\Sigma_2 \subset \Sigma_1$. Now the claim follows from induction.

From the $C^{2,\alpha}$ graphical convergence above we conclude that the surface S is a smooth compact surface representing a non-trivial homology class in $H_2(X, \partial X)$, which also satisfies

$$\text{area}(S) \leq \frac{4\pi\chi(S)}{\sigma}.$$

This completes the proof. ■

Before we give the proof of the Rigidity Theorem 1.7, we have to introduce the following useful generalization of the Bourguignon–Kazdan–Warner small deformation theorem to manifolds with boundaries. A new feature here is that we need to focus on those deformations increasing the metric since we are dealing with geometric inequalities rather than topological obstructions.

Proposition 3.1. *Let (X, g) be a compact Riemannian manifold with*

- $\text{Sc}(X) \geq \sigma$ for some constant σ , and
- $\text{mean.curv.}(\partial X) \geq 0$.

Then one of the following happens:

- (i) (X, g) has non-negative Ricci curvature and convex boundary, or
- (ii) *there are a smooth metric $g' \geq g$ and a smooth positive function u on X such that the warped metric*

$$\bar{g} = g' + u^2 ds^2 \text{ on } \bar{X} = X \times \mathbb{S}^1$$

satisfies

$$\text{Sc}(\bar{X}) > \sigma \quad \text{and} \quad \text{mean.curv.}(\partial \bar{X}) \geq 0.$$

Proof. In this proof, we will show that (X, g) has non-negative Ricci curvature and convex boundary under the hypothesis that (ii) does not happen. The proof will be divided into two steps:

Step 1. X has constant scalar curvature σ and vanishing mean curvature. Let us apply the contradiction argument and suppose that X does not have constant scalar curvature σ and vanishing mean curvature. In order to obtain a contradiction to our hypothesis we shall construct a smooth metric $g' \geq g$ and a smooth positive function u on X such that the warped metric $\bar{g} = g' + u^2 ds^2$ on $\bar{X} = X \times \mathbb{S}^1$ satisfies $\text{Sc}(\bar{X}) > \sigma$ and $\text{mean.curv.}(\partial \bar{X}) \geq 0$.

Let us take $g' = g$ and search for the desired function u . Consider the following functional

$$Q(v) = \frac{\int_X |\nabla v|^2 + \frac{1}{2} \text{Sc}(X)v^2 \, d\mu + \int_{\partial X} \text{mean.curv.}(\partial X)v^2 \, d\sigma}{\int_X v^2 \, d\mu} \tag{3.5}$$

defined on the space $C^\infty(X)$. Clearly we have $Q(v) \geq \sigma/2$ from the facts $\text{Sc}(X) \geq \sigma$ and $\text{mean.curv.}(\partial X) \geq 0$. Take the first eigenvalue λ and the first eigenfunction u of the functional Q with

$$\int_X u^2 \, d\mu = 1.$$

Then we have

$$-\Delta_g u + \frac{\text{Sc}(X)}{2}u = \lambda u \text{ in } X \text{ with } \lambda \geq \frac{\sigma}{2},$$

and

$$\frac{\partial u}{\partial \nu} + \text{mean.curv.}(\partial X)u = 0 \text{ on } \partial X,$$

where ν is the outer unit normal of ∂X in X . It is standard that the first eigenfunction u is positive everywhere. Taking $v = u$ in (3.5), we see

$$\lambda = \int_X |\nabla u|^2 + \frac{1}{2} \text{Sc}(X)u^2 \, d\mu + \int_{\partial X} \text{mean.curv.}(\partial X)u^2 \, d\sigma.$$

As a result, if the scalar curvature $\text{Sc}(X)$ is strictly greater than σ at some point or the mean curvature of ∂X is strictly positive somewhere, then λ is strictly greater than $\sigma/2$. From a straightforward calculation, it follows that $\bar{X} = X \times S^1$ equipped with $g + u^2 ds^2$ satisfies

$$\text{Sc}(\bar{X}, \bar{g}) = \text{Sc}(X, g) - \frac{2\Delta_g u}{u} = 2\lambda > \sigma$$

and

$$\text{mean.curv.}(\partial \bar{X}) = \text{mean.curv.}(\partial X) + \frac{\partial \log u}{\partial \nu} = 0.$$

This leads to a contradiction to the hypothesis that (ii) does not happen.

Step 2. X has non-negative Ricci curvature and convex boundary. Again we use the contradiction argument and work with X whose Ricci curvature is not non-negative or its boundary is not convex. As before, we shall construct a smooth metric $g' \geq g$ and a smooth positive function u on X such that the warped metric $\bar{g} = g' + u^2 ds^2$ on $\bar{X} = X \times S^1$ satisfies $\text{Sc}(\bar{X}) > \sigma$ and $\text{mean.curv.}(\partial \bar{X}) \geq 0$.

Usually the choice of such g' and u comes from a standard deformation argument, but since our manifold has non-empty boundary and also we want the metric to

increase, we do a careful analysis here. Let us take h to be a fixed symmetric 2-tensor on M and investigate the family of metrics

$$g_t = g - 2th.$$

As before, we denote λ_t to be the first eigenvalue of functional (3.5) with respect to the metric g_t and u_t to be the corresponding first eigenfunction with

$$\int_X u_t^2 \, d\mu_t = \text{vol}(X, g)$$

and satisfying the Robin boundary condition

$$\frac{\partial u}{\partial \nu} + \text{mean.curv.}(\partial X)u = 0 \text{ on } \partial X.$$

Due to the facts $\text{Sc}(X) \equiv \sigma$ and $\text{mean.curv.}(\partial X) \equiv 0$, we see $\lambda_0 = \sigma$ and $u_0 \equiv 1$ at the time $t = 0$. Recall

$$\lambda_t = \text{vol}(X, g)^{-1} \left(\int_X |\nabla_t u_t|^2 + \frac{1}{2} \text{Sc}(X, g_t) u_t^2 \, d\mu_t + \int_{\partial X} \text{mean.curv.}(\partial X, g_t) u_t^2 \, d\sigma_t \right).$$

After taking derivative along t and substituting $u_0 \equiv 1$, we obtain

$$\begin{aligned} \text{vol}(X, g)\lambda'_t(0) &= \frac{1}{2} \int_X \frac{\partial}{\partial t} \Big|_{t=0} \text{Sc}(X, g_t) \, d\mu_g \\ &\quad + \int_{\partial X} \frac{\partial}{\partial t} \Big|_{t=0} \text{mean.curv.}(\partial X, g_t) \, d\sigma. \end{aligned} \tag{3.6}$$

In the following, we are going to use the next lemma.

Lemma 3.2. *Let $\{g_t\}_{-\epsilon < t < \epsilon}$ be a smooth family of metrics on a compact manifold X with $g'_t(0) = -2h$. Then we have*

$$\begin{aligned} &\frac{1}{2} \int_X \frac{\partial}{\partial t} \Big|_{t=0} \text{Sc}(X, g_t) \, d\mu_g + \int_{\partial X} \frac{\partial}{\partial t} \Big|_{t=0} \text{mean.curv.}(\partial X, g_t) \, d\sigma_g \\ &= \int_X \langle h, \text{Ric}(X) \rangle \, d\mu_g + \int_{\partial X} \langle h, A(\partial X) \rangle \, d\sigma_g, \end{aligned}$$

where $A(\partial X)$ is the second fundamental form of ∂X in (X, g_0) with respect to the outer unit normal.

We shall leave the proof of this lemma to Appendix B, and here we just continue the previous proof. Clearly, equation (3.6) now becomes

$$\text{vol}(X, g)\lambda'_t(0) = \int_X \langle h, \text{Ric}(X) \rangle \, d\mu_g + \int_{\partial X} \langle h, A(\partial X) \rangle \, d\sigma_g. \tag{3.7}$$

Now we make the following discussion:

Case 1. The Ricci curvature Ric(X) is negative at some point p. From the continuity we can assume p to be an interior point of X without loss of generality. Since the Ricci tensor can be diagonalized, we can find an orthonormal frame $\{v_i\}_{i=1}^n$ at point p such that $\text{Ric}(v_i) = \mu_i v_i$ and $\mu_1 < 0$. Extend this frame to some neighborhood U of p away from the boundary ∂X , still denoted by v_i , and denote $\{\omega_i\}_{i=1}^n$ to be the dual frame. Take a non-negative cut-off function η supported in U such that η is positive at p . Let

$$h = -\eta \text{Ric}(v_1, v_1) \omega_1 \otimes \omega_1.$$

Notice that the quantity $\text{Ric}(v_1, v_1)$ is negative at point p . From continuity we can shrink the support of η such that the metric $g_t = g - 2th$ satisfies $g_t \geq g$ for all $t > 0$. Clearly, (3.7) implies

$$\lambda'_t(0) = \text{vol}(M, g)^{-1} \int_X \eta |\text{Ric}(v_1, v_1)|^2 d\mu > 0.$$

Therefore, for small positive t there is a positive smooth function u_t such that the warped metric $\bar{g} = g_t + u_t^2 ds^2$ satisfies

$$\text{Sc}(\bar{X}, \bar{g}) = \text{Sc}(X, g_t) - \frac{2\Delta_t u_t}{u_t} = 2\lambda_t > \sigma$$

and

$$\text{mean.curv.}(\partial\bar{X}, \bar{g}) = \text{mean.curv.}(\partial X, g_t) + \frac{\partial \log u_t}{\partial \nu} = 0.$$

Again this contradicts to our hypothesis that (ii) does not happen.

Case 2. The second fundamental form A(∂X) is negative at some point q on ∂X. The argument is similar to that in Case 1 and it suffices to construct appropriate choice for h . Notice that the second fundamental form $A(\partial X)$ is also diagonalizable. So we can pick up an orthonormal frame $\{v_i\}_{i=1}^{n-1}$ on ∂X at point q such that $A(v_i) = \mu_i v_i$ and $\mu_1 < 0$. Denote $v_n = \nu(q)$ and then $\{v_i\}_{i=1}^n$ forms an orthonormal frame of X . Extend this frame to a neighborhood U of q and denote $\{\omega_i\}_{i=1}^n$ is the corresponding dual frame. As before, we take η to be a non-negative cut-off function support in U that is positive at q . Moreover, we take $\zeta: [0, +\infty) \rightarrow \mathbb{R}$ to be another non-negative cut-off function such that $\zeta = 1$ around 0 and $\zeta = 0$ outside $[0, 1]$. Take a fixed smooth extension \tilde{A} of the tensor $A(\partial X)$ to U . Define

$$h = \eta \zeta \left(\frac{\text{dist}(\cdot, \partial X)}{\epsilon} \right) \tilde{A}(v_1, v_1) \omega_1 \otimes \omega_1.$$

Again from continuity we can shrink the support of η such that the metric $g_t = g - 2t\eta$ satisfies $g_t \geq g$ for all $t > 0$. Now equation (3.7) becomes

$$\lambda'_t(0) = \text{vol}(M, g)^{-1} \left(o(1) + \int_{\partial X} \eta |A(v_1, v_1)|^2 d\sigma \right) \text{ as } \epsilon \rightarrow 0.$$

Take ϵ to be small enough and we obtain $\lambda'_t(0) > 0$. The rest argument is the same as before. ■

To prove Theorem 1.7, we also need the following proposition.

Proposition 3.3. *Let X be a compact Riemannian manifold (possibly with a boundary), whose universal covering splits as $\tilde{X} = \tilde{X}_0 \times \mathbb{R}^m$ for some compact simply-connected manifold \tilde{X}_0 . Let U be a compact Riemannian manifold with non-empty boundary ∂U such that*

- $\dim U = \dim X$;
- *the boundary ∂U has a decomposition $\partial U = \partial_{\text{eff}} \cup \partial_{\text{side}}$, where ∂_{eff} and ∂_{side} are interiorly disjoint, compact, piecewisely smooth regions of ∂U sharing a common boundary;*
- *there is a locally isometric immersion $e: (U, \partial_{\text{side}}) \rightarrow (X, \partial X)$.*

Assume that $f: (U, \partial_{\text{eff}}) \rightarrow (B^k(R), \partial B^k(R))$ is a proper smooth 1-Lipschitz map, where $B^k(R)$ is the R -ball in the Euclidean k -space \mathbb{R}^k , such that the f -pullback of the point class is non-zero. If $k > m$, then $R \leq \text{const}(X)$.

Proof. We would like to show that the f -pullback of the point class is homologous to zero if R is large enough. For our purpose, let us slightly modify the map f . It is clear that we can take a Lipschitz map

$$\Phi: (B^k(R), \partial B^k(R)) \rightarrow (\mathbb{S}^k(1), p_0)$$

with $\text{Lip } \Phi \leq C_0 R^{-1}$ for some universal constant C_0 independent of R . Define $F = \Phi \circ f$, then we see $\text{Lip } F \leq C_0 R^{-1}$ and that the F -pullback of the point class is the same as the f -pullback of the point class. Furthermore, we can require that Φ maps the region outside $B^k(R/2)$ to the point p_0 . As a result, F takes the constant value p_0 inside the $(R/2)$ -neighborhood of ∂_{eff} due to the fact that $f: (U, \partial_{\text{eff}}) \rightarrow (B^k(R), \partial B^k(R))$ is 1-Lipschitz.

When R is large enough, we plan to construct a suitable homotopy from the map $F: (U, \partial_{\text{eff}}) \rightarrow (\mathbb{S}^k(1), p_0)$ to a new map $F': (U, \partial_{\text{eff}}) \rightarrow (\mathbb{S}^k(1), p_0)$, where the image of the map F' has zero measure in \mathbb{S}^k . Once this is done, we conclude that the F -pullback of the point class is homologous to zero and this leads to a contradiction.

Let us work with the universal covering \tilde{U} of U and G -invariant maps on \tilde{U} , where we denote G to be the Deck transformation group of the covering $\pi: \tilde{U} \rightarrow U$. The

benefit of doing this is that we have nice description for the geometry of \tilde{U} since the local isometry $e: (U, \partial_{\text{side}}) \rightarrow (X, \partial X)$ can be lifted to a local isometry $\tilde{e}: (\tilde{U}, \tilde{\partial}_{\text{side}}) \rightarrow (\tilde{X}, \partial \tilde{X})$, where $\tilde{\partial}_{\text{side}} \subset \partial \tilde{U}$ is denoted to be the preimage $\pi^{-1}(\partial_{\text{side}})$. Denote

$$\tilde{e} = (\tilde{e}_1, \tilde{e}_2): \tilde{U} \rightarrow \tilde{X} = \tilde{X}_0 \times \mathbb{R}^m.$$

It is clear that the first component map \tilde{e}_1 restricted to every component C of preimage $\tilde{e}_2^{-1}(y)$ for any $y \in \mathbb{R}^m$ is a locally isometric map to \tilde{X}_0 . For every such component C , if we denote

$$\partial_{C,\text{side}} = \partial C \cap \tilde{\partial}_{\text{side}},$$

then we have the local isometry

$$e_C: (C, \partial_{C,\text{side}}) \rightarrow (\tilde{X}_0, \partial \tilde{X}_0).$$

Next we try to figure out the geometry of components C so that we can obtain a nice description for the geometry of \tilde{U} . This depends heavily on the following lemma.

Lemma 3.4. *Assume that X_0 is a compact simply-connected Riemannian manifold (possibly with boundary). Let C be a compact Riemannian manifold such that*

- $\dim C = \dim X_0$;
- *the boundary ∂C admits a decomposition $\partial C = \partial_{C,\text{eff}} \cup \partial_{C,\text{side}}$, where $\partial_{C,\text{eff}}$ and $\partial_{C,\text{side}}$ are interiorly disjoint, compact, piecewisely smooth regions of ∂C sharing a common boundary;*
- *there is a locally isometric map $e_C: (C, \partial_{C,\text{side}}) \rightarrow (X_0, \partial X_0)$.*

Then there is a universal constant Q depending only on X_0 such that we have the following alternative:

- *either C is isometric to X_0 ;*
- *or $\partial_{C,\text{eff}}$ is non-empty and all points in C are contained in the Q -neighborhood of $\partial_{C,\text{eff}}$.*

Proof. This lemma follows from [13, Theorem D]. First let us introduce the definition of the width for a homotopy and state Rotman’s theorem. Let $H_\tau(t)$ be a homotopy connecting two closed curves parametrized by $t \in [0, 1]$. The width of homotopy H_τ is defined to be

$$W_{H_\tau} = \max_{t \in [0,1]} \text{Length}(H_\tau(t)).$$

With this definition, Rotman proved that: if (X_0, g) is a compact simply-connected Riemannian manifold satisfying

- sectional curvature $K \geq -1$ and diameter $d \leq D$;
- all metric balls with radius less than c are simply connected,

then there is a constant $Q_1 = Q_1(\dim X_0, D, c)$ such that for any closed curve $\gamma: [0, 1] \rightarrow X_0$ we can find a homotopy $H_\tau(t)$ of $\gamma(t)$ to a point whose width satisfies $W_{H_\tau} \leq Q_1$. Clearly the condition on sectional curvature can be guaranteed by rescaling with the sacrifice that the constant Q_1 also depends on the lower bound of sectional curvatures, which does not affect our argument below.

First we point out that the covering property can only be destroyed by $\partial_{C,\text{eff}}$. As a consequence, if $\partial_{C,\text{eff}}$ is empty, then the local isometry $e_C: (C, \partial_{C,\text{side}}) \rightarrow (X_0, \partial X_0)$ is actually a covering map. It then follows from the simply-connectedness of X_0 that C is isometric to X_0 .

Next we deal with the case when $\partial_{C,\text{eff}}$ is non-empty. We claim that all points of C are contained in the $(Q_1 + 2 \text{diam } X_0)$ -neighborhood of $\partial_{C,\text{eff}}$. Suppose otherwise that there is a point y outside the $(Q_1 + 2 \text{diam } X_0)$ -neighborhood of $\partial_{C,\text{eff}}$. Slightly perturbing the metric in C^0 -sense, we may assume that X_0 has convex boundary and correspondingly the boundary portion $\partial_{C,\text{side}}$ is convex. Let γ be a unit-speed minimizing geodesic connecting y and the boundary portion $\partial_{C,\text{eff}}$. Clearly the geodesic γ has length greater than $\text{diam } X_0$ and so its image $e_C(\gamma)$ under the local isometry e_C cannot be any minimizing geodesic in X_0 . There are two possibilities:

Case 1. There is a conjugate point of $e_C(\gamma)$ in $e_C(\gamma)$ within distance $\text{diam } X_0$. It follows that there is a non-trivial Jacobi field on the geodesic $e_C(\gamma)$. By lifting the geodesic γ also has a non-trivial Jacobi field, and this contradicts to the fact that the geodesic γ is minimizing.

Case 2. There is a cut point of $e_C(\gamma)$ in $e_C(\gamma)$ within distance $\text{diam } X_0$. Denote $x = e_C(\gamma)$ and x_c to be the first cut point of x in $e_C(\gamma)$. By definition there is another minimizing geodesic ζ connecting points x and x_c with length no greater than $\text{diam } X_0$. Denote ζ^{-1} to be the inverse path of ζ . Since X_0 is compact and simply-connected, it follows from Rotman’s theorem that the closed curve

$$\beta = \zeta^{-1} * e_C(\gamma)|_{xx_c}$$

is homotopic to a point through a homotopy H_β of closed curves with width no greater than Q_1 , which satisfies

$$H_\beta(\cdot, 0) = \beta \quad \text{and} \quad H_\beta(\cdot, 1) \equiv y' \text{ for some } y' \in X_0.$$

Recall that

$$\text{dist}(y, \partial_{C,\text{eff}}) > Q_1 + 2 \text{diam } X_0.$$

Notice that the lifting \tilde{H}_β of H_β with $\tilde{H}_\beta(0, 0) = y$ stays in $(Q_1 + 2 \text{diam } X_0)$ -neighborhood of y a priori. This means that the lifting property for H_β still holds despite of the existence of $\partial_{C,\text{eff}}$. Since H_β is a homotopy of closed curves, we see

that $\tilde{H}_\beta(0, \cdot)$ and $\tilde{H}_\beta(1, \cdot)$ are liftings of the same path $H_\beta(0, \cdot)$. Notice that they also have the same end points by considering the lifting $\tilde{H}_\beta(\cdot, 1)$. Therefore, we have

$$\tilde{H}_\beta(0, \tau) = \tilde{H}_\beta(1, \tau) \quad \text{for all } \tau \in [0, 1].$$

In particular, the closed curve β is lifted to a closed curve $\tilde{\beta}$ in C with

$$\tilde{\beta}(0) = \tilde{\beta}(1) = y.$$

A further analysis yields that $\tilde{\beta}$ can be decomposed into $\tilde{\beta} = \tilde{\zeta}^{-1} * \gamma$, where $\tilde{\zeta}^{-1}$ is a lifting of ζ^{-1} . Denote y_c to be the lifting of the cut point x_c . It turns out that y_c is a cut point on γ within distance $\text{diam } X_0$, since the inverse path of $\tilde{\zeta}^{-1}$ is another minimizing geodesic connecting y and y_c . Again this is impossible since the geodesic γ is minimizing. ■

Denote $\tilde{\partial}_{\text{eff}} = \pi^{-1}(\partial_{\text{eff}})$ and $\partial_{C, \text{eff}} = \partial C \cap \tilde{\partial}_{\text{eff}}$. It follows from Lemma 3.4 that a component C of $\tilde{e}_2^{-1}(y)$ must be isometric to \tilde{X}_0 if it is not contained in Q -neighborhood of $\partial_{C, \text{eff}} \subset \tilde{\partial}_{\text{eff}}$. Now let us collect all such \tilde{X}_0 -slices with non-empty intersection with the complement \tilde{U}_Q of Q -neighborhood of $\tilde{\partial}_{\text{eff}}$ in \tilde{U} . Since \tilde{U} is locally isometrically immersed into $\tilde{X}_0 \times \mathbb{R}^m$, the collection above induces a fiber bundle $\tilde{\xi} = (\tilde{E}, \tilde{B}, \tilde{\pi})$, whose total space $\tilde{E}(\tilde{\xi})$ is an open subset of \tilde{U} containing \tilde{U}_Q and each fiber is isometric to \tilde{X}_0 . Recall that G is the Deck transformation group of the covering $\pi : \tilde{U} \rightarrow U$. Notice that $\tilde{\partial}_{\text{eff}}$ is G -invariant, and so its Q -neighborhood and \tilde{U}_Q are also G -invariant. We point out the fact that any closed totally geodesic $(n - m)$ -submanifold in $\tilde{X}_0 \times \mathbb{R}^m$ must be some \tilde{X}_0 -slices. Let \tilde{F} be a fiber of $\tilde{\xi}$, then it is compact and totally geodesic. Since Deck transformations are isometric, we see that $\phi(\tilde{F})$ is also compact and totally geodesic for all $\phi \in G$. Notice that \tilde{e} is a local isometry, so the image $\tilde{e}(\phi(\tilde{F}))$ is some \tilde{X}_0 -slice in $\tilde{X}_0 \times \mathbb{R}^m$ due to the fact mentioned above. As a result, $\phi(\tilde{F})$ is a component C of $\tilde{e}_2^{-1}(y)$ for some $y \in \mathbb{R}^m$ intersecting \tilde{U}_Q and hence a fiber in $\tilde{\xi}$. Then we conclude that the fiber bundle $\tilde{\xi}$ is G -invariant, that is, any element in G maps $\tilde{E}(\tilde{\xi}) \subset \tilde{U}$ to itself and preserves fibers.

Now we are ready to construct the desired homotopy. Denote

$$\tilde{F} = F \circ \pi : (\tilde{U}, \tilde{\partial}_{\text{eff}}) \rightarrow (\mathbb{S}^k(1), p_0)$$

to be the lift of F and we consider its restriction on $\tilde{E}(\tilde{\xi}) \subset \tilde{U}$. Recall that we have $\text{Lip } F \leq C_0 R^{-1}$. Since the diameter of X_0 is bounded due to its compactness, the image of each fiber concentrates in some small geodesic ball of $\mathbb{S}^k(1)$ once R is large enough. In this case, we can conduct an averaging procedure along fibers to obtain a G -invariant map $\tilde{F}' : \tilde{E}(\tilde{\xi}) \rightarrow \mathbb{S}^k$, which is constant on each fiber. Recall that the map F takes the constant value p_0 in $(R/2)$ -neighborhood of ∂_{eff} and the same thing holds for \tilde{F} in $(R/2)$ -neighborhood of $\tilde{\partial}_{\text{eff}}$. If R is large enough, the averaged

map \tilde{F}' will take the constant value p_0 around $\partial\tilde{E}(\tilde{\xi})$. Through the constant extension we finally obtain a G -invariant map defined on the whole \tilde{U} , still denoted by \tilde{F}' . In particular, this induces a map

$$\tilde{F}': (\tilde{U}, \tilde{d}_{\text{eff}}) \rightarrow (\mathbb{S}^k(1), p_0).$$

When R is large enough, the maps \tilde{F} and \tilde{F}' are close enough in C^0 -sense. From the linear homotopy on $\mathbb{S}^k(1)$ along minimizing geodesics, we are able to construct a desired homotopy $\tilde{\Phi}$ between the maps \tilde{F} and \tilde{F}' . Notice that $\tilde{\Phi}$ is also G -invariant and so it induces a homotopy Φ between the maps F and $F' = \tilde{F}' \circ \pi^{-1}$.

Now we show that the image of F' has zero measure in \mathbb{S}^k . Equivalently, let us prove this for the image of \tilde{F}' . Since the map \tilde{F}' is constant on each fiber, there is a smooth map $\tilde{F}'_B: \tilde{B}(\tilde{\xi}) \rightarrow \mathbb{S}^k$ such that $\tilde{F}' = \tilde{F}'_B \circ \tilde{\pi}$ and so the image of \tilde{F}' is the same as that of \tilde{F}'_B . Notice that the dimension of $\tilde{B}(\tilde{\xi})$ is equal to $m < k$. So the image of \tilde{F}'_B has zero measure in \mathbb{S}^k and we complete the proof. ■

Now we prove Theorem 1.7.

Proof of Theorem 1.7. All we need to show is that if there is no compact surface S representing a non-trivial homotopy class in $\pi_2(X, \partial X)$ with

$$\text{area}(S) < \frac{4\pi\chi(S)}{\sigma},$$

then X splits as the Riemannian product $S_\sigma \times Y$, where S_σ is a sphere or a hemisphere with constant curvature $\sigma/2$.

First we point out that X cannot admit a smooth metric $g' \geq g$ and a positive smooth function u such that $(X \times \mathbb{S}^1, g' + u^2 ds^2)$ has mean convex boundary and scalar curvature $\text{Sc}(X \times \mathbb{S}^1) > \sigma$. Otherwise, it follows from the proof of Corollary 1.6 (with warped symmetrization once more) that there is a smooth surface S in (X, g') representing a non-trivial homotopy class in $\pi_2(X, \partial X)$ whose area is less than $4\pi\chi(S)/\sigma$. The area of S with respect to the metric g can only be smaller and so we obtain a contradiction. As a consequence, Proposition 3.1 yields that (X, g) has non-negative Ricci curvature and convex boundary.

Now we are going to prove the splitting of the given manifold X . Since X has non-negative Ricci curvature and convex boundary, the Cheeger–Gromoll splitting theorem (as well as its with-boundary version proved in Appendix A) yields that the universal covering \tilde{X} of X splits into $\tilde{X}_0 \times \mathbb{R}^m$, where \tilde{X}_0 is a compact simply-connected Riemannian manifold with non-negative Ricci curvature and convex boundary.

It remains to show $m \geq n - 2$. Once this is done, \tilde{X} must split into $\tilde{X}_0 \times \mathbb{R}^{n-2}$ for some compact simply-connected surface \tilde{X}_0 , since we know $\pi_2(X, \partial X) \neq 0$ from the proof of Corollary 1.6. From $\text{Sc}(\tilde{X}) \geq \sigma > 0$, we also see that \tilde{X}_0 is a 2-sphere with

area no greater than $8\pi/\sigma$ or a disk with area no greater than $4\pi/\sigma$. In both cases, the equality implies that \tilde{X}_0 is standard. Namely, \tilde{X}_0 is either the round 2-sphere or the hemi-2-sphere with constant sectional curvature $\sigma/2$.

Recall from the proof of Corollary 1.6 that for any $d > 0$ there is a compact Riemannian manifold \tilde{U}_d with $\partial\tilde{U}_d = \partial_{\text{eff}} \cup \partial_{\text{side}}$ such that

- there is a local isometry $\tilde{e}_d: (\tilde{U}_d, \partial_{\text{side}}) \rightarrow (X, \partial X)$;
- there is a smooth map

$$\hat{f}_d = (\hat{f}_{1,d}, \hat{f}_{2,d}, \dots, \hat{f}_{n-2,d}): (\tilde{U}_d, \partial_{\text{eff}}) \rightarrow \left(\left[-\frac{d}{2}, \frac{d}{2} \right]^{n-2}, \partial \left[-\frac{d}{2}, \frac{d}{2} \right]^{n-2} \right)$$

such that $\text{Lip } \hat{f}_{i,d} \leq 1$ and that the \hat{f}_d -pullback of the point class is homologically non-trivial.

By taking a sufficiently large positive constant \hat{C} independent of d , from the map \hat{f}_d we can construct a new map

$$\hat{f}'_d: (\tilde{U}_d, \partial_{\text{eff}}) \rightarrow (B^{n-2}(\hat{C}d), \partial B^{n-2}(\hat{C}d)) \quad \text{with } \text{Lip } \hat{f}'_d \leq 1$$

such that the \hat{f}'_d -pullback of the point class is the same as the \hat{f}_d -pullback of the point class and so homologically non-trivial. Since d can be arbitrarily large, it follows from Proposition 3.3 that we have $m \geq n - 2$. This completes the proof. ■

Proof of Corollary 1.8. First we point out that it suffices to deal with the case when all manifolds X and X_i are compact. Otherwise, we take X'_i to be the largest disk D_i in X_i and take

$$X' = f^{-1}(X_0 \times D_1 \times \dots \times D_m).$$

It follows from the properness of the map f that X' is compact.

Now let us focus on the compact case. In the same spirit of iso-enlargeable case, we have to find a nice description for the largeness of X_i from its inradii estimate. We claim that if a compact surface X_i has inradii d_i , then for any $\epsilon > 0$ and $d > 0$, we can find a compact manifold $U_{i,\epsilon,d}$ with non-empty boundary $\partial U_{i,\epsilon,d}$ associated with a locally isometric immersion $e_{i,\epsilon,d}: U_{i,\epsilon,d} \rightarrow X_i$ and a continuous map

$$\begin{aligned} \phi_{i,\epsilon,d} &= (\phi_{i,\epsilon,d,1}, \phi_{i,\epsilon,d,2}): (U_{i,\epsilon,d}, \partial U_{i,\epsilon,d}) \\ &\rightarrow \left(\left[0, \frac{d_i}{1+\epsilon} \right] \times [0, d], \partial \left(\left[0, \frac{d_i}{1+\epsilon} \right] \times [0, d] \right) \right) \end{aligned}$$

such that $\phi_{i,\epsilon,d}$ has non-zero degree and $\text{Lip } \phi_{i,\epsilon,d,k} \leq 1$ for $k = 1, 2$.

The construction of $U_{i,\epsilon,d}$ will be divided into the following two cases.

Case 1. X_i has no boundary. In this case, the inradii of X_i is just its diameter and so we have $\text{diam } X_i \geq d_i$. By definition we can find two points p and q satisfying $\text{dist}(p, q) \geq d_i$. From the continuity of the distance function we can find two small disks D_p and D_q centered at p and q respectively such that $\check{X}_i = X_i - (D_p \cup D_q)$ is a compact manifold with boundary components $C_p = \partial D_p$ and $C_q = \partial D_q$ satisfying

$$\text{dist}(C_p, C_q) \geq d_i(1 + \epsilon/2)^{-1}.$$

From this fact we can construct a smooth function

$$\phi_1: (\check{X}_i, C_p, C_q) \rightarrow \left(\left[0, \frac{d_i}{1 + \epsilon} \right], 0, \frac{d_i}{1 + \epsilon} \right)$$

with $\text{Lip } \phi_1 \leq 1$ from smoothing $\text{dist}(\cdot, C_p)$.

Next let us construct $U_{i,\epsilon,d}$. Take a minimizing geodesic γ connecting C_p and C_q . Then γ represents a relative homology class $[\gamma] \in H_1(\check{X}_i, \partial\check{X}_i)$. From the Poincaré duality we can take the dual class α of $[\gamma]$ in $H^1(\check{X}_i)$. It is a well-known fact that $H^1(\check{X}_i) \cong [\check{X}_i, S^1]$, where $[\check{X}_i, S^1]$ is the set of all homotopy classes of continuous maps from \check{X}_i to S^1 . As a consequence, we can take a continuous map $\phi_2: \check{X}_i \rightarrow S^1$ corresponding to α from the isomorphism above. Without loss of generality we may assume ϕ_2 is smooth and $\phi_2^{-1}(1) = \gamma$. Given any $N \in \mathbb{N}_+$, we denote

$$\pi_N: (S^1, 1) \rightarrow (S^1, 1), \quad e^{\sqrt{-1}\theta} \mapsto e^{\sqrt{-1}N\theta}.$$

Now we pull back the covering map π_N along $\phi_2: \check{X}_i \rightarrow S^1$ to obtain a N -sheeted covering $\check{\pi}_N: \check{X}_{i,N} \rightarrow \check{X}_i$. We can also lift ϕ_2 to $\phi_{2,N}: \check{X}_{i,N} \rightarrow S^1$ and we have the following commutative diagram:

$$\begin{array}{ccc} \check{X}_{i,N} & \xrightarrow{\phi_{2,N}} & S^1 \\ \check{\pi}_N \downarrow & & \downarrow \pi_N \\ \check{X} & \xrightarrow{\phi_2} & S^1. \end{array}$$

Now let us take $U_{i,\epsilon,d}$ to be the metric completion of $\check{X}_{i,N} - \phi_{2,N}^{-1}(1)$. For the convenience of the audience, we provide a more intuitive way in the following Figure 3 to see what $U_{i,\epsilon,d}$ is.

Take

$$\phi_{i,\epsilon,d,1} = \phi_1 \circ \check{\pi}_N: U_{i,\epsilon,d} \rightarrow \left[0, \frac{d_i}{1 + \epsilon} \right]$$

and

$$\phi_{i,\epsilon,d,2} = \left(\frac{d}{2\pi} \right)^{-1} \phi_{2,N}: U_{i,\epsilon,d} \rightarrow [0, d],$$

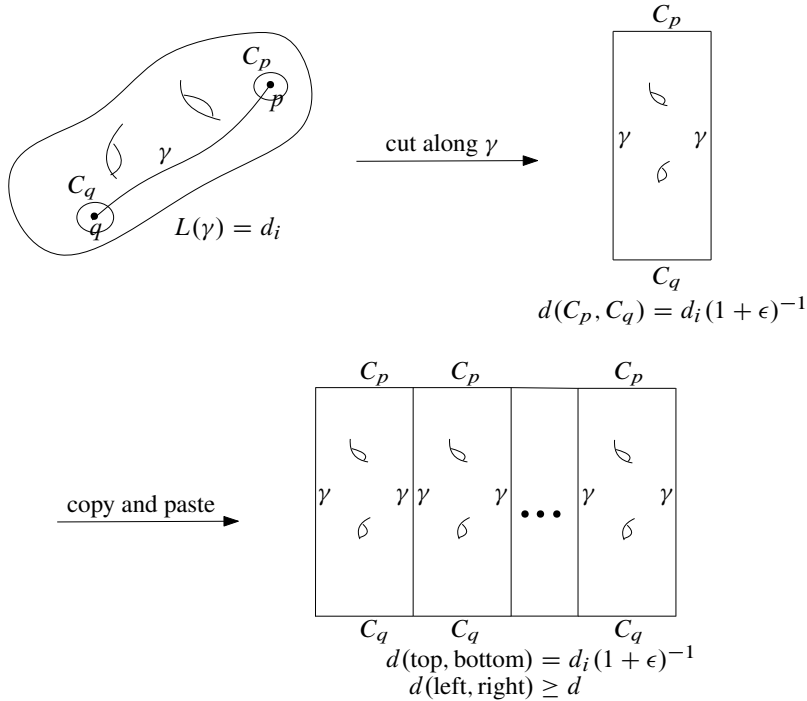


Figure 3. The construction of $U_{i,\epsilon,d}$.

where $\bar{\phi}_{2,N}$ is the extension of $\phi_{2,N}: \check{X}_{i,N} - \phi_{2,N}^{-1}(0) \rightarrow (0, 2\pi) \cong S^1 - \{1\}$. Notice that $\text{Lip } \phi_{i,\epsilon,d,1} = \text{Lip } \phi_1 \leq 1$ and

$$\text{Lip } \phi_{i,\epsilon,d,2} = \left(\frac{d}{2\pi}\right)^{-1} \cdot \frac{\text{Lip } \phi_2}{N}.$$

Once we choose N to be large enough, we can obtain $\text{Lip } \phi_{i,\epsilon,d,2} \leq 1$. From our construction it is also not difficult to verify that $\phi_{i,\epsilon,d}$ has non-zero degree, and so we complete the construction.

Case 2. X_i has non-empty boundary. In this case, from the condition $\text{inrad}(X_i) \geq d_i$ we can find a point p such that $\text{dist}(p, \partial X_i) \geq d_i$. We remove a small disk D_p centered at p such that

$$\text{dist}(\partial D_p, \partial X_i) \geq d_i(1 + \epsilon/2)^{-1}.$$

Denote $\check{X}_i = X_i - D_p$ and we return to Case 1.

The rest of the proof is quite similar to that of Corollary 1.6. With these maps $\phi_{i,\epsilon,d}$, it follows from Theorem 1.2 that there exists a smooth surface $S_{\epsilon,d} \subset X$ such that

- $S_{\epsilon,d}$ is a 2-sphere without boundary or a disk with its boundary contained in ∂X , which represents a non-zero element in $H_2(X, \partial X)$;

- $S_{\epsilon,d}$ satisfies the area inequality

$$\text{area}(S_{\epsilon,d}) \leq 4\pi\chi(S_{\epsilon,d}) \left(\text{Sc}(X)|_{S_{\epsilon,d}} - \frac{4(n-1)\pi^2}{n} \left(\sum_{i=1}^m \frac{1+\epsilon}{d_i^2} + \frac{m}{d^2} \right) \right)^{-1}.$$

Now we would like to take the limit of $S_{\epsilon,d}$ as $\epsilon \rightarrow 0$ and $d \rightarrow +\infty$ to obtain a sphere or a disk S representing a non-trivial homology class in $H_2(X, \partial X)$ and satisfying $\text{area}(S) \leq 4\pi\chi(S)/\sigma_0$, where

$$\sigma_0 = \inf_X \text{Sc}(X) - \frac{4(n-1)\pi^2}{n} \left(\sum_{i=1}^m \frac{1}{d_i^2} \right).$$

With a similar modification as in the proof of Corollary 1.6, we have the following alternative: either one of $S_{\epsilon,d}$ satisfies

$$\text{area}(S_{\epsilon,d}) < \frac{4\pi\chi(S)}{\sigma_0},$$

or we have uniformly $C^{2,\alpha}$ estimate for surfaces $S_{\epsilon,d}$ so that we can take the limit S of $S_{\epsilon,d}$ up to subsequence in $C^{2,\alpha}$ graphical sense as $\epsilon \rightarrow 0$ and $d \rightarrow +\infty$. In both cases, we find the desired surface and so we complete the proof. ■

Finally, we give the proof of Corollary 1.10.

Proof of Corollary 1.10. The proof will be divided into two steps.

Step 1. First let us give a proof with the additional assumption $\text{Sc}(X) > 0$ and we will apply the contradiction argument. Suppose that the consequence is false, then there is a proper and globally Lipschitz map $\phi: X \rightarrow \mathbb{R}^{n-2}$ such that X_0 is homologous to the ϕ -pullback of the point class. Without loss of generality, we can further assume that the map ϕ is smooth and its Lipschitz constant is less than one. Since ϕ is proper, the subset $X_2 = \phi^{-1}([-1, 1]^{n-2})$ is compact and so there is a positive constant δ such that $\text{Sc}(X) \geq \delta$ in X_2 . Take a smooth and even function $\eta: \mathbb{R} \rightarrow \mathbb{R}$ such that $\eta \equiv 0$ outside $[-1, 1]$ and $-\delta < \eta < 0$ in $[-1, 1]$. Let $h_i, i = 1, \dots, n-2$, be the solution of the following ordinary differential equation

$$\frac{n}{n-1}h_i^2 + 2h_i' = \frac{\eta}{2(n-2)}, \quad h_i(0) = 0.$$

It is easy to show that h_i is a smooth odd function defined on a finite interval $[-d_0, d_0]$ and

$$\lim_{t \rightarrow \mp d_0} h_i(t) = \pm\infty.$$

In the following, we would like to apply the argument in the proof of Theorem 1.2 with functions h_i . But since the boundary ∂X has corners, we have to smoothing the

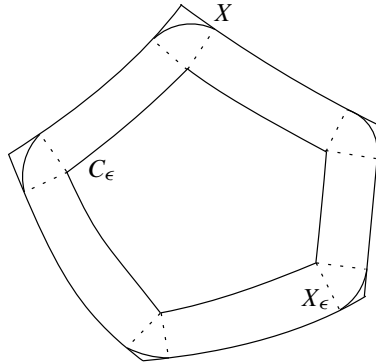


Figure 4. The construction of X_ϵ .

manifold X first. The main idea is the rounding technique from [6, p. 699]. Given any small positive constant ϵ , let us denote X_ϵ to be the ϵ -neighborhood of the ϵ -core

$$C_\epsilon = \{x \in X : \text{dist}(x, \partial X) \geq \epsilon\}.$$

As shown in Figure 4, for ϵ small C_ϵ is a Riemannian manifold with j corners close to that of X and the boundary of X_ϵ consists of two parts — one is contained in ∂X and the other is contained in the ϵ -level-set of the corners of C_ϵ .

For our purpose, we just need to focus on the compact region

$$K = \phi^{-1}([-d_0, d_0]^{n-2}).$$

Notice that the ϵ -level-set of the corners of C_ϵ in K has mean curvature $\epsilon^{-1} + O(1)$ as $\epsilon \rightarrow 0$. Denote $l = \{p\} \times \mathbb{R}^{n-2}$ for some interior point $p \in X_0$. Since ∂X_ϵ is in an ϵ -neighborhood, we see that $l \cap K$ is contained in the interior of X_ϵ for ϵ small enough. In particular, we can talk about the winding number of a closed curve in $\partial X_\epsilon \cap K$ with respect to l . Notice that any closed curve γ in ∂X_ϵ with non-zero winding number with respect to l has part of it contained in the ϵ -level-set of corners of C_ϵ with length no less than

$$\sum_{i=1}^j (\pi - \alpha_i + o(1))\epsilon \quad \text{as } \epsilon \rightarrow 0.$$

Combined with the mean curvature estimate $\epsilon^{-1} + O(1)$, we have

$$\int_\gamma \text{mean.curv}(\partial X_\epsilon, \theta) \, d\theta \geq \sum_{i=1}^j (\pi - \alpha_i) + o(1) \quad \text{as } \epsilon \rightarrow 0. \tag{3.8}$$

We point out that the boundary ∂X_ϵ of X_ϵ from the rounding technique is only $C^{1,1}$, but we can smoothing it with the help of mean curvature flow (see [4] for instance). Here we do not plan to get involved into this technical issue, but just deal with X_ϵ as a smooth Riemannian manifold. Since the mean curvature flow keeps mean convexity and does not affect much on mean curvature (away from the corners) and the length of γ , we still have the estimate (3.8) after the smoothing procedure.

Now we can repeat the argument from the proof of Theorem 1.2 on the compact domain K and conclude that there is a smooth embedded surface S homologous to $X_0 \cap X_\epsilon$ with free boundary $\partial S \subset \partial X_\epsilon \cap K$ such that each component S' of S satisfies

$$\int_{S'} \text{Sc}(X_\epsilon, s) + \sum_{i=1}^{n-2} \left(\frac{n}{n-1} h_i^2 + 2h'_i \right) (\phi(s)) ds + 2 \int_{\Theta'=\partial S'} \text{mean.curv}(\partial X_\epsilon, \theta) d\theta \leq 4\pi\chi(S').$$

Since the surface S is homologous to $X_0 \cap X_\epsilon$ which has winding number one with respect to l , there is at least one component S' such that $\partial S'$ has non-zero winding number with respect to l . Notice that the quantity

$$\text{Sc}(X_\epsilon, s) + \sum_{i=1}^{n-2} \left(\frac{n}{n-1} h_i^2 + 2h'_i \right) (\phi(s))$$

is no less than

$$\min \left\{ \frac{\delta}{2}, \min_{\phi^{-1}([-d_0, d_0]^{n-2})} \text{Sc}(X_\epsilon, s) \right\} > 0,$$

so the first integral on the left hand side is positive. For the second integral, it follows from (3.8) that

$$\int_{\Theta'=\partial S'} \text{mean.curv}(\partial X_\epsilon, \theta) d\theta \geq \sum_{i=1}^j (\pi - \alpha_i) + o(1) \quad \text{as } \epsilon \rightarrow 0.$$

If we take ϵ small enough such that

$$\sum_{i=1}^j (\pi - \alpha_i) + o(1) > 2\pi,$$

then we obtain $4\pi\chi(S') > 4\pi$, which contradicts to the fact $\chi(S') \leq 1$.

Step 2. To complete the proof, we have to show the way of deforming the scalar curvature to be positive by increasing the dihedral angles at the corners or decreasing the mean curvatures of the boundary a little bit but still keeping the mean convexity of the boundary. The discussion will be divided into two cases.

Case 2a. If the mean curvature of the boundary ∂X is positive somewhere, then we try to construct a suitable conformal deformation to increase the scalar curvature through the idea coming from [10]. Let us take an exhaustion $\{X_j\}_{j=1}^\infty$ of X such that the boundary point with positive mean curvature is contained in each X_j . Consider the functional

$$Q_j(u) = \frac{\int_{X_j} |\nabla u|^2 + \frac{n-2}{4(n-1)} \text{Sc}(X)u^2 \, d\mu + \frac{n-2}{2} \int_{\partial X \cap X_j} \text{mean.curv.}(\partial X)u^2 \, d\sigma}{\int_{X_j} u^2 \, d\mu}.$$

Clearly there is a positive constant μ_j such that $Q_j(u) \geq \mu_j$ for any function u in $C^\infty(X_j)$. As a result, we can find a positive smooth function ζ on X such that for any u in C^∞ it holds that

$$\int_X \zeta u^2 \, d\mu \leq \int_X |\nabla u|^2 + \frac{n-2}{4(n-1)} \text{Sc}(X)u^2 \, d\mu + \frac{n-2}{2} \int_{\partial X} \text{mean.curv.}(\partial X)u^2 \, d\sigma.$$

Using this inequality we are able to construct a smooth positive function v such that

$$-\Delta v + \frac{n-2}{4(n-1)} \text{Sc}(X)v > 0 \quad \text{in } X$$

and

$$\frac{\partial v}{\partial \nu} + \frac{n-2}{2} \text{mean.curv.}(\partial X)v \geq 0 \quad \text{on } \partial X.$$

As in [10, Lemma 2.9], we can further modify v to satisfy $0 < \delta \leq v \leq 1$ for some positive constant δ . Define $\bar{g} = v^{\frac{4}{n-2}}g$. Then (X, \bar{g}) has positive scalar curvature, mean convex boundary and the unchanged dihedral angles at the corners. Since the new metric \bar{g} is equivalent to the original one from our construction, any globally Lipschitz map on (X, g) keeps globally Lipschitz on (X, \bar{g}) . So the previous arguments in Step 1 can be applied to the new manifold (X, \bar{g}) to deduce a contradiction.

Case 2b. Now we assume that the boundary ∂X is minimal. In this case, a bending procedure is suggested by the first named author in his work [6, p. 701] and here we shall provide further details based on the work [11] by Lawson and Michelsohn.

As shown in Figure 5, we can localize the bending around some point p in the corner that is the intersection of faces ∂_i and ∂_j . For convenience, we extend the manifold X to X_e such that ∂_j lies in the interior of X_e and ∂_i is extended to $\partial_{i,e} \subset \partial X_e$. Take a hypersurface Σ in X_e with focal radius r_f intersecting ∂_i orthogonally such that the point p lies in the $(r_f/3)$ -neighborhood of Σ . Denote r and s to be the signed distance function to Σ and ∂_i respectively. We make the conventions that points on

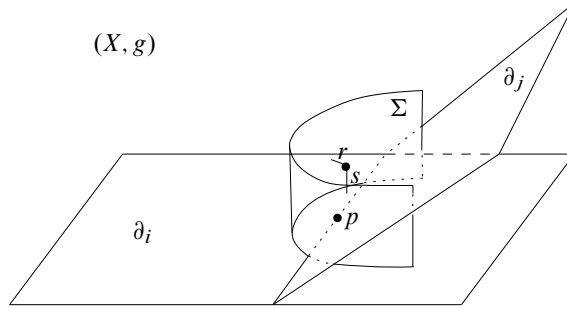


Figure 5. The geometry around corners.

the right hand side of Σ have positive r -values and that those points above ∂_i have positive s -values. Let

$$F(r, s) = s - f(r).$$

It follows from [11, p. 403] that the mean curvature of the hypersurface $\{F(r, s) = 0\}$ with respect to the unit normal vector field $\bar{v} = \nabla F/|\nabla F|$ is

$$H_F = -\frac{f''}{W^3}(1 - \langle \nabla r, \nabla s \rangle^2) - \frac{f'}{W}H_r + \frac{1}{W}H_s + \frac{f'}{W^3} \text{Hess}_r(\nabla s, \nabla s) - \frac{(f')^2}{W^3} \text{Hess}_s(\nabla r, \nabla r),$$

where H_r is the mean curvature of r -level-sets with respect to ∇r , H_s is the mean curvature of s -level-sets with respect to ∇s and

$$W = (1 + (f')^2 - 2f'\langle \nabla r, \nabla s \rangle)^{\frac{1}{2}}.$$

With respect to \bar{v} we expect the mean curvature H_F to be non-positive (and negative somewhere) for some appropriate choice of function f . The construction of the desired f is as follows. Since our bending for ∂_i only happens around p , we can assume $|\langle \nabla r, \nabla s \rangle| \leq 1/2$ a priori. Recall that r_f is the focal radius of Σ . We want to construct a non-negative smooth function $f: [-r_f/2, r_f/2] \rightarrow \mathbb{R}$ taking small values and also satisfying

$$0 \leq f' \leq \frac{1}{2} \quad \text{and} \quad f'' \geq 0.$$

Notice that we have $|H_r| \leq C$ for some universal constant C when $-r_f/2 \leq r \leq r_f/2$. Since dihedral angles \angle_{ij} are strictly less than $\pi/2$, we can find a universal constant $\Lambda = \Lambda(\angle_{ij})$ such that any point q in X around p can be connected to ∂_i through a path γ_q with length no greater than $\Lambda \cdot s(q)$. From the fact $H_s = 0$ on ∂_i , by integration along γ_q we can obtain $H_s \leq Cs$ in X when $|s|$ is less than the focal radius of $\partial_{i,e}$.

Moreover, we see that the Hessians of r and s are bounded as well. So we conclude that

$$H_F \leq -\frac{3}{32}f'' + C_1f' + C_2f$$

for some universal constants C_1 and C_2 , where we use $W \leq 2$ and

$$1 - \langle \nabla r, \nabla s \rangle^2 \geq \frac{3}{4}.$$

With r_1 to be determined later, we define

$$f(r) = \begin{cases} e^{-\frac{1}{(r-r_1)^2}}, & r > r_1; \\ 0, & r \leq r_1. \end{cases}$$

From a direct computation, we see

$$H_F \leq f(r) \left(-\frac{3}{8}(r-r_1)^{-6} + \frac{9}{16}(r-r_1)^{-4} + 2C_1(r-r_1)^{-3} + C_2 \right).$$

By taking r_1 to be sufficiently close to $r_f/3$, we can obtain the desired bent hypersurface $\{F(r, s) = 0\} \cap X$, whose mean curvature with respect to unit outer normal (opposite to $\bar{\nu}$) is non-negative (and is positive around p). Obviously, the bending procedure above does not change dihedral angle too much, and so it preserves the dihedral angle condition $\angle_i \leq \alpha_i < \pi$, and

$$\sum_{i=1}^j (\pi - \alpha_i) > 2\pi.$$

This reduces to the previous case and we complete the proof. ■

A. Splitting theorem for manifolds with non-negative Ricci and convex boundary

In this appendix, we point out that the original proof of Cheeger–Gromoll splitting theorem can also applied to show

Proposition A.1. *Let (X^n, g) be a complete Riemannian manifold with non-negative Ricci curvature and convex boundary. Assume there is a geodesic line in X . Then (X, g) splits as a Riemannian product (X_1, g_1) and the real line, where (X_1, g_1) also has non-negative Ricci curvature and convex boundary.*

We give a sketch of the proof.

Sketch of the proof. Since the boundary is convex, each pair of points can be connected by a length-minimizing geodesic just as in no-boundary case. This guarantees the validity of

$$\Delta \operatorname{dist}(p, \cdot) \leq \frac{n-1}{\operatorname{dist}(p, \cdot)}$$

in the distribution sense for any point $p \in X$. Denote $\gamma : (-\infty, +\infty) \rightarrow X$ to be the geodesic line. The Busemann functions B^+ and B^- are defined to be

$$B^+(x) = \lim_{t \rightarrow +\infty} (d_M(x, \gamma(t)) - t),$$

$$B^-(x) = \lim_{t \rightarrow -\infty} (d_M(x, \gamma(t)) + t).$$

Denote $B = B^+ + B^-$. As in the no-boundary case, one can show that the function B is super-harmonic in the distribution sense. Clearly, we have

$$B^+ + B^- \geq 0 \quad \text{in } X, \tag{A.1}$$

$$B^+ + B^- = 0 \quad \text{on } \gamma. \tag{A.2}$$

Now we show that B has to be a zero function. If γ is contained in the interior of X , it follows from the maximum principle that B is identical to zero in X . If γ is contained in the boundary ∂X , then we need to consider the normal derivative of B along γ . For any point x_0 , it follows from the length-minimizing property that the distance function $\operatorname{dist}(\cdot, \gamma(t))$ is differentiable at the point x_0 when $x_0 \neq \gamma(t)$. And it is clear that the normal derivative

$$\frac{\partial}{\partial \nu} \operatorname{dist}(\cdot, \gamma(t))$$

vanishes at point x_0 , where ν is the outer unit normal of ∂X . Take a smooth path $\zeta : [0, \epsilon] \rightarrow X$ with $\zeta(0) = x_0$ and $\zeta'(0) = -\nu(x_0)$. Based on the monotonicity of the Busemann functions B^+ and B^- , it is easy to see

$$\limsup_{s \rightarrow 0^+} \frac{B(\zeta(s))}{s} \leq \lim_{s \rightarrow 0^+} \frac{\operatorname{dist}(\zeta(s), \gamma(t)) - t + \operatorname{dist}(\zeta(s), \gamma(-t)) - t}{s} = 0$$

for any fixed large t . On the other hand, from (A.1) and (A.2), we have

$$\liminf_{s \rightarrow 0^+} \frac{B(\zeta(s))}{s} \geq 0.$$

This means that the normal derivative of B along γ exists and is equal to zero. From strong maximum principle we see that B is again a zero function.

As in the closed case, this implies that B^+ and B^- are smooth harmonic functions. From construction we have $|\nabla B^+| \equiv 1$ and it follows from Bochner formula

that $\nabla^2 B^+ \equiv 0$ and $\text{Ric}(X) \equiv 0$. Therefore, the function B^+ has no critical point and all level-sets of B^+ are totally geodesic. From strong maximum principle we conclude that either some component of ∂X is contained in some level-set of B^+ or that $B^+|_{\partial X}$ also has no critical point in ∂X . In the first case, X is diffeomorphic to $\partial X \times [0, +\infty)$ and the range of B^+ cannot be the entire real line. This is impossible since the image of $B^+|_{\gamma}$ is already the whole \mathbf{R} . In the second case, X splits topologically as $X_1 \times \mathbf{R}$, where each X_1 -slice corresponds to a level-set of B^+ .

Finally, we show that X also splits as a Riemannian manifold. It suffices to show that the level-sets of B^+ intersects ∂X orthogonally. Let γ' be an integral curve of vector field $\nabla B^+|_{\partial X}$ on ∂X . Denote S_t to be the level-set $\{B^+ = t\}$ diffeomorphic to X' . We investigate the function

$$d(t) = \text{dist}_{S_t}(\gamma \cap S_t, \gamma' \cap S_t).$$

If γ is in the interior of X , then $d(t) > 0$ for all t . If γ is contained in ∂X , then it is also an integral curve of vector field $\nabla B^+|_{\partial X}$ on ∂X . So either γ' coincides with γ or $d(t) > 0$ for all t . The proof will be completed if we can show the desired orthogonality from $d(t) > 0$ for all t . From a direct calculation, we see

$$\begin{aligned} d'(t) &= -\langle v, \nabla B^+ \rangle, \\ d''(t) &= -|\nabla B^+|_{\partial X}^{-4} A_{\partial X}(\nabla B^+|_{\partial X}, \nabla B^+|_{\partial X}) \leq 0. \end{aligned}$$

Since a concave function with a lower bound must be a constant, we see that ∇B^+ is orthogonal to v along γ' . From the arbitrary choice for γ' , we conclude that the level-sets of B^+ intersects ∂X orthogonally. ■

B. Proof of Lemma 3.2

The proof of Lemma 3.2 will follow from a straightforward calculation (see also [1]) and here we include a detailed calculation for completeness.

Let $\{g_t\}_{-\epsilon < t < \epsilon}$ be a smooth family of metrics on compact manifolds X with non-empty boundary ∂X . Assume $g'_t(0) = -2h$. First, we start with the following simple lemma.

Lemma B.1. *For any smooth vector field U, V and W , we have*

$$g\left(\frac{\partial}{\partial t}\Big|_{t=0} \nabla_U^{g_t} V, W\right) = -(\nabla_U h)(V, W) - (\nabla_V h)(U, W) + (\nabla_W h)(U, V). \quad (\text{B.1})$$

Proof. First, notice that we have

$$\frac{\partial}{\partial t}\Big|_{t=0} [U, V] = 0.$$

This yields

$$\frac{\partial}{\partial t} \Big|_{t=0} \nabla_U^{g_t} V = \frac{\partial}{\partial t} \Big|_{t=0} \nabla_V^{g_t} U.$$

So the derivative

$$\frac{\partial}{\partial t} \Big|_{t=0} \nabla_U^{g_t} V$$

is a tensor and we can just make a computation in an orthonormal coordinate $\{x^i\}$ with respect to the metric g . It is clear that

$$\frac{\partial}{\partial t} \Big|_{t=0} \nabla_{\partial_i}^{g_t} \partial_j = \left(\frac{\partial}{\partial t} \Big|_{t=0} \Gamma_{ij,t}^k \right) \partial_k = -(\partial_j h_{ik} + \partial_i h_{jk} - \partial_k h_{ij}) \partial_k.$$

This implies

$$g \left(\frac{\partial}{\partial t} \Big|_{t=0} \nabla_{\partial_i}^{g_t} \partial_j, \partial_k \right) = -(\nabla_{\partial_i} h)(\partial_j, \partial_k) - (\nabla_{\partial_j} h)(\partial_i, \partial_k) + (\nabla_{\partial_k} h)(\partial_i, \partial_j).$$

The desired equality (B.1) now follows from a simple linear combination on both sides. ■

We recall the following fact from [2, Theorem 1.174].

Lemma B.2. *We have*

$$\frac{\partial}{\partial t} \Big|_{t=0} \text{Sc}(X, g_t) = 2\langle h, \text{Ric}(X, g) \rangle_g - 2 \text{div}_g (\text{div}_g h - d \text{tr}_g h).$$

Next we compute the variation formula of the mean curvature of ∂X . For convenience, we take an orthonormal coordinate system $\{x^i\}_{i=1}^{n-1}$ on ∂X with respect to the induced metric from (X, g) . Denote s to be the distance function to ∂X in (X, g) . Then $\{x^i, s\}$ forms a coordinate system of X around ∂X . We are going to prove the following result.

Lemma B.3. *It holds that*

$$\begin{aligned} \frac{\partial}{\partial t} \Big|_{t=0} \text{mean.curv.}(\partial X, g_t) &= 2\langle h, A(\partial X, g) \rangle_g - \text{mean.curv.}(\partial X, g) h(\nu, \nu) \\ &\quad + 2(\text{div}_{\partial X} h)(\nu) - \text{tr}_{\partial X}(\nabla_\nu h). \end{aligned}$$

Proof. From the definition, we know

$$\text{mean.curv.}(\partial X, g_t) = g_t^{ij} A_{t,ij},$$

where A_t is the second fundamental form of ∂X in (X, g_t) with respect to the outer unit normal ν_t . This implies

$$\frac{\partial}{\partial t} \Big|_{t=0} \text{mean.curv.}(\partial X, g_t) = 2\langle h, A(\partial X, g) \rangle_g + g^{ij} \frac{\partial}{\partial t} \Big|_{t=0} A_{t,ij}.$$

Now we calculate the second term on the right hand side. From equation (B.1), we have

$$\begin{aligned} \frac{\partial}{\partial t} \Big|_{t=0} A_{t,ij} &= -\frac{\partial}{\partial t} \Big|_{t=0} g_t(\nabla_{\partial_i}^{g_t} \partial_j, \nu_t) \\ &= 2h(\nabla_{\partial_i} \partial_j, \nu) + (\nabla_{\partial_i} h)(\partial_j, \nu) \\ &\quad + (\nabla_{\partial_j} h)(\partial_i, \nu) - (\nabla_\nu h)(\partial_i, \partial_j) - g\left(\nabla_{\partial_i} \partial_j, \frac{\partial}{\partial t} \Big|_{t=0} \nu_t\right), \end{aligned}$$

where ∇ is the covariant derivative with respect to metric g and ν is the outer unit normal of ∂X in (X, g) . Since $\{x^i\}$ is an orthonormal coordinate system on ∂X , we conclude

$$g\left(\nabla_{\partial_i} \partial_j, \frac{\partial}{\partial t} \Big|_{t=0} \nu_t\right) = g(\nabla_{\partial_i} \partial_j, \nu) \left\langle \frac{\partial}{\partial t} \Big|_{t=0} \nu_t, \nu \right\rangle_g.$$

Taking the derivative of on both sides of the equation $g_t(\nu_t, \nu_t) \equiv 1$ with respect to t , we obtain

$$\left\langle \frac{\partial}{\partial t} \Big|_{t=0} \nu_t, \nu \right\rangle_g = h(\nu, \nu).$$

As a result, we see

$$g^{ij} \frac{\partial}{\partial t} \Big|_{t=0} A_{t,ij} = -\text{mean.curv.}(\partial X, g) h(\nu, \nu) + 2(\text{div}_{\partial X} h)(\nu) - \text{tr}_{\partial X}(\nabla_\nu h).$$

This complete the proof. ■

Now we are ready to prove Lemma 3.2.

Proof of Lemma 3.2. From the divergence theorem as well as Lemmas B.2 and B.3, we have

$$\begin{aligned} &\frac{1}{2} \int_X \frac{\partial}{\partial t} \Big|_{t=0} \text{Sc}(X, g_t) d\mu_g + \int_{\partial X} \frac{\partial}{\partial t} \Big|_{t=0} \text{mean.curv.}(\partial X, g_t) d\sigma \\ &= \int_X \langle h, \text{Ric}(X, g) \rangle_g d\mu_g + 2 \int_{\partial X} \langle h, A(\partial X, g) \rangle_g d\sigma_g \\ &\quad + \int_{\partial X} \nu(\text{tr}_g h) - (\text{div}_g h)(\nu) + 2(\text{div}_{\partial X} h)(\nu) - \text{tr}_{\partial X}(\nabla_\nu h) d\sigma_g \\ &\quad - \int_{\partial X} \text{mean.curv.}(\partial X, g) h(\nu, \nu) d\sigma_g. \end{aligned}$$

Let us deal with the terms in the third line. Clearly, we have

$$\begin{aligned} \nu(\text{tr}_g h) &= \nu(g^{ij} h(\partial_i, \partial_j) + h(\partial_s, \partial_s)) \\ &= -2\langle A(\partial X, g), h \rangle + \text{tr}_{\partial X}(\nabla_\nu h) + 2\langle h, A(\partial X, g) \rangle + \nu h(\partial_s, \partial_s) \\ &= \text{tr}_{\partial X}(\nabla_\nu h) + (\nabla_\nu h)(\nu, \nu) \end{aligned}$$

and

$$(\operatorname{div}_g h)(v) = (\operatorname{div}_{\partial X} h)(v) + (\nabla_v h)(v, v).$$

This yields that the integral in the third line is equal to

$$\int_{\partial X} (\operatorname{div}_{\partial X} h)(v) \, d\sigma_g.$$

We compute

$$\begin{aligned} (\operatorname{div}_{\partial X} h)(v) &= g^{ij} (\nabla_{\partial_i} h)(\partial_j, v) \\ &= g^{ij} (\nabla_{\partial_i} (h(\cdot, v))) (\partial_j) - g^{ij} h(\partial_j, \nabla_{\partial_i} v) \\ &= \operatorname{div}_{\partial X} (h(\cdot, v)|_{\partial X}) + h(v, v) \operatorname{mean.curv.}(\partial X, g) - \langle h, A(\partial X, g) \rangle. \end{aligned}$$

Finally, we arrive at

$$\begin{aligned} \frac{1}{2} \int_X \frac{\partial}{\partial t} \Big|_{t=0} \operatorname{Sc}(X, g_t) \, d\mu_g + \int_{\partial X} \frac{\partial}{\partial t} \Big|_{t=0} \operatorname{mean.curv.}(\partial X, g_t) \, d\sigma \\ = \int_X \langle h, \operatorname{Ric}(X, g) \rangle_g \, d\mu_g + \int_{\partial X} \langle h, A(\partial X, g) \rangle_g \, d\sigma_g. \end{aligned}$$

This completes the proof. ■

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