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The Dirichlet problem of translating mean curvature equations

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Abstract. In this paper, we study the Dirichlet problem of translating mean curvature equations over a domain via topological restrictions. Its main difficulty is the non-existence of a C^0 a priori estimate for their classical solutions except few cases. Inspired by the work of Miranda and Giusti, we define a generalized solution of these Dirichlet problems and establish its general existence. We propose a non-closed-minimal (NCM) condition on the underlying domain. When the domain is mean convex and NCM, the generalized solution with continuous boundary data is the classical smooth solution. Moreover, the NCM assumption cannot be removed by a hemisphere example.

1. Introduction

There are intimate connections between the Dirichlet problem of many nonlinear elliptic equations and the geometry of the domain Ω. For example, see [11, 12] on their geometric assumptions on Ω to solve the Dirichlet problem of a class of prescribed mean curvature equations. In this paper we add one more example into these connections in the case of translating mean curvature equations.

Fix $\alpha \geq 0$. Let Ω be a bounded Lipschitz domain in an n-dimensional complete Riemannian manifold N with a metric σ. Suppose the graph of a C^2 function u is minimal in the conformal product manifold $N \times \mathbb{R}$ with the metric $e^{2\alpha r/n}(\sigma + \mathrm{d}r^2)$. Then u satisfies the following equation:

$$\text{(1.1)} \quad \mathrm{H}_\alpha(u) = 0 \quad \text{on } \Omega, \qquad \text{where } \mathrm{H}_\alpha(u) = -\mathrm{div}\Big(\frac{\mathrm{D}u}{\sqrt{1+|\mathrm{D}u|^2}}\Big) + \frac{\alpha}{\sqrt{1+|\mathrm{D}u|^2}}.$$

Here, div and D are the divergence and the gradient on N, respectively. We call (1.1) a translating mean curvature equation (TMCE).

The word "translating" comes from the fact that when N is the Euclidean space $\mathbb{R}^n$, the graph of u satisfying $\mathrm{H}_n(u) = 0$ is a translating soliton to the mean curvature flow in $\mathbb{R}^{n+1}$.

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To describe the Dirichlet problem of a TMCE on general Riemannian manifolds, we propose a topological condition upon the domain as follows.

Definition 1.1. Fix $n \geq 2$. Suppose Ω is an n-dimensional bounded Riemannian manifold with Lipschitz boundary. We say that Ω has the non-closed-minimal (NCM) property if its closure does not contain any Caccioppoli set E such that its (essential) boundary ∂E is a closed, embedded and minimal hypersurface with a singular set S of Hausdorff dimension at most $n-8$. For $n = 8$, we require that S is a collection of isolated points.

By Theorem 28.1 in [22], the description of S coincides with the singular set in (Λ, λ)-minimizing perimeter (almost minimizing currents with codimension one). By the maximum principle of stationary varifolds [17], all bounded domains in Euclidean spaces, Hyperbolic spaces, all domains in the hemisphere (except itself) have the NCM property. Some nontrival examples of domains with the NCM property can be found in the works of Kasue [19] and Agostiniani, Fogagnolo and Mazzieri [1]. The NCM assumption is similar to the assumptions in Theorem 2.5 of [29] by White, and in Assumption 2.8 of [5] by De Lellis and Ramic.

We say that a C^2 domain is mean convex if the mean curvature of its boundary is nonnegative, i.e., $\mathrm{div}(\vec{v}) \geq 0$ for its outward normal vector $\vec{v}$. The main result of this paper is stated as follows.

Theorem 1.2. *Suppose Ω is a C^2 bounded mean convex domain with the NCM property in an n-dimensional ($n \geq 2$) Riemannian manifold, and fix $\alpha > 0$. Then the Dirichlet problem of the TMCE*

$$\mathrm{H}_\alpha(u) = 0 \quad on\ \Omega, \quad u = \psi \quad in\ \partial\Omega, \tag{1.2}$$

admits a unique solution in $C^2(\Omega) \cap C(\bar{\Omega})$ for any continuous function ψ on $\partial\Omega$.

Remark 1.3. By Serrin [25], the mean convex assumption cannot be removed if we want to solve (1.2) for any continuous boundary data even in the case of $\alpha = 0$.

The NCM assumption also cannot be removed. In Theorem 6.1 of [10], we showed that when Ω is the upper hemisphere S^n_+ and $\alpha \geq n \geq 2$, no classical C^2 solution to (1.2) exists for any continuous boundary data. The boundary of S^n_+ is a closed embedded minimal hypersurface. We call such example *the hemisphere example*. See also Theorem B.4 in Appendix B.

An advantage of the NCM assumption is that it is independent of the choice of α. Thus, it is much better than the assumption that there is a C^2 subsolution to (1.2) for each α. But in Euclidean spaces (see Theorem 10 in [30]), such existence of the global subsolution in $\mathbb{R}^{n+1}$ can be easily obtained by ODE and thus it makes the derivation of Theorem 1.2 much easier.

Remark 1.4. Theorem 1.2 plays an very important role in the study of the minimal graph and the area minimizing problem in general conformal product manifolds given by $N \times (-\infty, A)$ with the metric $\phi^2(r)(\sigma + \mathrm{dr}^2)$. See [10].

Remark 1.5. In the case $n = 1$, the Dirichlet problem (1.2) is essentially different from the one for $n \geq 2$. We need more restrictions on α to get the conclusion in Theorem 1.2 when $n = 1$. Moreover, the NCM assumption is not well-defined on the real line $\mathbb{R}$. For more details, see Appendix C.

Let Q_α be the set $N \times \mathbb{R}$ equipped with the metric $e^{2\alpha r/n}(\sigma + \mathrm{dr}^2)$. We say that the set $\Omega \times \mathbb{R}$ in Q_α is *a conformal cone*. The main motivation of this paper comes from the Plateau problem, that is, the search of surfaces taking least area in Riemannian manifolds with prescribed boundary data. In the setting of Q_α, the first step is to find minimal graphs with prescribed graphical boundary in $\partial\Omega \times \mathbb{R}$, which is equivalent to solve the Dirichlet problem (1.2).

In the case $\alpha = 0$, the Dirichlet problem (1.2) is completely solved when Ω is bounded and mean convex (see [2, 11, 14, 18, 25], etc.) without the NCM restriction. When N is the Euclidean space $\mathbb{R}^n$, the Dirichlet problem (1.2) was easily solved for bounded mean convex domains by White [28] and Wang [28] (see also Ma [21]). This is related to translating solitons and the type II singularity of mean curvature flows [16]. A much more general form of (1.2) was already considered by Casteras, Heinonen and Holopainen in Theorem 1.1 of [4] with a lower bound restriction on the Ricci curvature of Ω.

In view of the hemisphere example in Theorem B.4, the main difficulty in Theorem 1.2 is the unknown L^∞ bound of its classical solution in $C^2(\Omega) \cap C(\bar{\Omega})$. On the other hand, by calibration, the graph of such classical solution takes the least area among smooth hypersurfaces in $\bar{\Omega} \times \mathbb{R}$ with respect to the metric $e^{2\alpha r/n}(\sigma + \mathrm{dr}^2)$. As a result, we use the idea of the generalized solution theory to the Dirichlet problem of minimal surface equations by Miranda [23, 24] and Giusti [12–14]. Such theory does not require an a priori estimate of the solution, and makes use of the area minimizing property via a minimizing process of a functional on bounded variation (BV) functions.

The idea to conclude Theorem 1.2 can be described in the following four steps.

(1) Define an area functional $\mathfrak{F}_\alpha(u, \Omega)$ among BV functions with respect to the metric $e^{2\alpha r/n}(\sigma + \mathrm{dr}^2)$ (see Definition 3.1).

(2) Establish the equivalence between the minimizing problem of $\mathfrak{F}_\alpha(u, \Omega)$ and the minimizing perimeter problem (see Theorem 4.7).

(3) Find a graph of a BV function which minimizes the perimeter of the boundary $\{(x, \psi(x)) : x \in \partial\Omega\}$ in the closed set $\bar{\Omega} \times [-k, k]$. Then, letting $k \to +\infty$, we define a generalization solution to the Dirichlet problem (1.2) with boundary data $\psi(x)$ (see Definition 5.1). Such solution may take the infinity value over Ω.

(4) Study the property of the set when the generalized solution takes infinity values. In particular, when Ω is mean convex and ψ is continuous, the boundary of those infinity sets are minimal embedded hypersurfaces with a singular set S of Hausdorff dimension at most $n - 8$. In the case $n = 8$, S is a collection of isolated points. See Theorem 6.8. By the NCM assumption, we conclude the proof of Theorem 1.2.

The paper is organized as follows. In Section 2, we collect some preliminary facts on BV functions. In Section 3, we discuss various properties of the conformal area functional $\mathfrak{F}_\alpha(u, \Omega)$. In Section 4, we show the relationship between the perimeter of subgraphs in Q_α and the area functional $\mathfrak{F}_\alpha(u, \Omega)$. In Section 5, we establish the existence of generalized solutions to the Dirichlet problem of the TMCE (1.2). In Section 6, we investigate the properties of the infinity sets of generalized solutions. In Section 7, we prove Theorem 1.2.

In Appendix A, we record a decomposition result for Radon measures in Riemannian manifolds. This is used to prove the C^∞ approximation of $\mathfrak{F}_\alpha(u, \Omega)$ (Theorem 3.5). In Appendix B, we record some results on mean curvature equations in sufficiently small balls and the hemisphere example.

2. BV functions in Riemannian manifolds

In this section, we discuss BV functions and related definitions in Riemannian manifolds. We define the convolution of functions and vector fields in an open ball for later use. The main references are [3], [8], [14], [22] and Chapter 1 of [26].

2.1. BV functions

Let (M, g) be a Riemannian manifold. Let $\langle \cdot, \cdot \rangle$ be its inner product. Write div and dvol for the divergence and the volume of M, respectively. Suppose Ω is an open set in M. Let $T_0\Omega$ be the collection of smooth vector fields with compact support in Ω. Let $\mathcal{H}^k$ denote the k-dimensional Hausdorff measure on M.

Definition 2.1. Let $u \in L^1(\Omega)$. We define

$$|\mathrm{D}u|_M(\Omega) := \sup\left\{ \int_\Omega u \operatorname{div}(X) \,\mathrm{dvol},\, X \in T_0\Omega,\ \langle X, X\rangle \le 1 \right\}.$$

If $|\mathrm{D}u|_M(\Omega) < \infty$, we say that u has bounded variation or $u \in \mathrm{BV}(\Omega)$.

Remark 2.2. If $u \in C^1(\Omega)$, the divergence theorem implies that

$$\int_\Omega u \operatorname{div}(X) \,\mathrm{dvol} = -\int_\Omega \langle X, \nabla u\rangle \,\mathrm{dvol}$$

for any $X \in T_0\Omega$, where ∇u is the gradient of u in M.

The definition in (2.1) induces a Radon measure on Ω. If there is no confusion concerning the ambient manifold, we usually omit the lower index in $|\mathrm{D}u|_M(\Omega)$ and just write it as $|\mathrm{D}u|(\Omega)$.

Now we continue to define the perimeter as follows.

Definition 2.3. For a Borel set E, let λ_E be its characteristic function. We call $|\mathrm{D}\lambda_E|(\Omega)$ the perimeter of E in Ω, written as $\mathrm{P}(E, \Omega)$.

If E has locally finite perimeter in Ω, that is, $\mathrm{P}(E, \Omega') < \infty$ for each bounded open set $\Omega' \subset\subset \Omega$ (i.e., $\lambda_E \in \mathrm{BV}_{\mathrm{loc},M}(\Omega)$), then E is called a Caccioppoli set.

The following two classical theorems on BV functions are very useful.

Theorem 2.4. *Suppose $\{u_j\}_{j=1}^\infty \in \mathrm{BV}(\Omega)$ converges to u in $L^1(\Omega)$ as $j \to +\infty$. Then*

$$|\mathrm{D}u|(\Omega) \le \lim_{j\to+\infty} \inf |\mathrm{D}u_j|(\Omega).$$

Theorem 2.5. *Suppose that* Ω *is a bounded Lipschitz boundary in* N *and that there is a sequence* $\{u_i\}_{i=1}^{\infty}$ *in* $\mathrm{BV}(\Omega)$ *satisfying*

$$\int_{\Omega} |u_i| \,\mathrm{dvol} + |\mathrm{D}u_i|(\Omega) \le c \quad \textit{for any } i,$$

where c *is a fixed constant. Then there is a* $u(x)$ *in* $\mathrm{BV}(\Omega)$ *such that a subsequence of* $\{u_i\}_{i=1}^{\infty}$ *converges to* u *in* $L^1(\Omega)$.

2.2. Radon measures

Now we record the connection between BV functions and Radon measures. For more details we refer to [26], Chapter 1.

Definition 2.6. Let X be a locally compact Hausdorff measure. A Radon measure on X is an outer measure ν on X having the following three properties:

(1) ν is Borel regular and $\nu(K) < \infty$ for any compact set $K \subset X$,
(2) $\nu(A) = \inf\{\nu(U) : U \text{ open}, A \subset U\}$ for each subset $A \subset X$,
(3) $\nu(U) = \sup\{\nu(K) : K \text{ compact}, K \subset U\}$ for each open U in X.

For any set X, we denote the set of non-negative continuous functions $f : X \to [0, \infty)$ with compact support by $K_+(X)$.

Theorem 2.7 (Remark 4.3 in [26]). *Suppose* X *is a locally compact Hausdorff space and* $\lambda : K_+(X) \to [0, \infty)$ *satisfies* $\lambda(cf) = c\lambda(f)$, $\lambda(f + h) = \lambda(f) + \lambda(h)$ *for any constant* $c \ge 0$ *and* $f, g \in K_+(\Omega)$. *Then there is a Radon measure* ν *on* X, *given by*

$$\nu(U) := \sup\big\{\lambda(f), f \in K_+(X), \mathrm{supp}(f) \subset\subset U, f \le 1\big\} \quad \textit{for all open } U \subset X, \tag{2.1}$$

such that

$$\lambda(f) = \int_X f \,\mathrm{d}\nu \quad \textit{for all } f \in K_+(\Omega). \tag{2.2}$$

Here $\mathrm{supp}(f)$ *is the closure of* $\{x : f(x) > 0\}$.

Suppose $u(x) \in \mathrm{BV}(\Omega)$. We set a nonnegative functional $\lambda_u : K^+(\Omega) \to [0, +\infty)$ as

$$\lambda_u(h) = \sup\Big\{\int u \,\mathrm{div}(X) \,\mathrm{dvol}, X \in T_0\Omega, \langle X, X\rangle \le h^2\Big\}$$

for every $h \in K^+(\Omega)$. It is clear that

$$\lambda_u(ch) = c\lambda_u(h), \quad \lambda_u(h + h_1) = \lambda_u(h) + \lambda_u(h_1),$$

where c is any positive constant and $h, h_1 \in K^+(\Omega)$.

Theorem 2.8. *Let* Ω *be an open set in a Riemannian manifold* M. *Suppose that* $u \in \mathrm{BV}_{\mathrm{loc},M}(\Omega)$. *Let* $|\mathrm{D}u|$ *be the Radon measure induced by the variation of* u.

(1) *If f is a bounded non-negative measurable function in $L^1(|\mathrm{D}u|,\Omega)$, then*

$$\int_{\Omega'} f\, d|\mathrm{D}u| = \sup\Big\{\int_{\Omega'} u\,\mathrm{div}(X)\,\mathrm{dvol},\, X\in T_0\Omega',\, \langle X,X\rangle \le f^2\Big\}$$

for any open set $\Omega' \subset\subset \Omega$.

(2) *We have that*

$$\int_{\Omega} u\,\mathrm{div}(X)\,\mathrm{dvol} = -\int_{\Omega}\langle X,\nu\rangle\,\mathrm{d}|\mathrm{D}u|,$$

where $\langle \nu,\nu\rangle = 1$ a.e. $|\mathrm{D}u|$ for any $X\in T_0\Omega$.

Proof. The existence and definition of $|\mathrm{D}u|$ are from Theorem 2.7. Then $|\mathrm{D}u|$ is a Radon measure. Similar to the proof of Theorem 5.10 in [26], Chapter 1, there is a monotone non-negative increasing sequence $\{f_j\}_{j=1}^{\infty}$ such that each $f_j\in K^+(\Omega)$, $f_j\le f$ and $\{f_j\}_{j\ge 1}$ converges to f in $L^1(|\mathrm{D}u|,\Omega)$. Let Ω' be any open set satisfying $\Omega'\subset\subset\Omega$. By (2.2),

$$\int_{\Omega'} f_j\,\mathrm{d}|\mathrm{D}u| = \sup\Big\{\int_{\Omega'} u\,\mathrm{div}(X)\,\mathrm{dvol},\, X\in T_0\Omega',\, \langle X,X\rangle \le f_j^2\Big\}.$$

Letting $j\to+\infty$ on both sides yields the conclusion (1). The conclusion (2) is from the Riesz representation theorem (see Theorem 4.1 in [26]). ∎

Next we define the trace of BV functions.

Definition 2.9. Suppose Ω is a Lipschitz domain in M. There is a bounded linear map $\mathcal{T}:\mathrm{BV}(\Omega)\to L^1(\partial\Omega)$ such that for any $u\in\mathrm{BV}(\Omega)$,

$$\int_{\Omega} u\,\mathrm{div}(X)\,\mathrm{dvol} = -\int_{\Omega}\langle \nu, X\rangle\,\mathrm{d}|\mathrm{D}u| + \int_{\partial\Omega}\mathcal{T}u\langle X,\gamma\rangle\,\mathrm{d}\mathcal{H}^n.$$

Here $\dim M = n+1$, and γ is the inward normal vector of $\partial\Omega$. We call $\mathcal{T}u$ the trace of $u(x)$ on $\partial\Omega$.

Remark 2.10. The proof of the existence of $\mathcal{T}u$ is exactly the same as that of Lemma 2.4 in [14] with the application of the C^∞ approximation of BV functions in Theorem 3.5 (3).

The trace is very useful to compute the variation of BV functions on the boundary of Lipschitz domains. A direct application of the above definition yields the following.

Lemma 2.11. *Let Ω_1 and Ω_2 be two Lipschitz domains in M and let Γ be a measurable set in $\partial\Omega_1\cap\partial\Omega_2$. We denote $\Omega_1\cup\Omega_2$ by Ω. Suppose $u(x)\in\mathrm{BV}_{\mathrm{loc},M}(\Omega)$. Then*

$$|\mathrm{D}u|(\Gamma) = \int_{\Gamma}|\mathcal{T}_1u-\mathcal{T}_2u|\,\mathrm{d}\mathcal{H}^n.$$

Here $\dim M = n+1$, and $\mathcal{T}_i$ is the trace of u in Ω_i on Γ for $i=1,2$.

A conformal manifold is defined as follows.

Definition 2.12. Let (M,g) be a Riemannian manifold. Let $\varphi(x)>0$ be a smooth positive function on M. A conformal manifold M_φ is the smooth manifold M with the metric $\varphi^2(x)g$.

Theorem 2.13. *Suppose* Ω *is an open set in* M *and* $u \in \mathrm{BV}_{\mathrm{loc},M}(\Omega)$. *Then*

$$|\mathrm{D}u|_{M_\varphi}(\Omega) = \int_\Omega \varphi^n \,\mathrm{d}|\mathrm{D}u|, \tag{2.3}$$

where $\dim M = n+1$, *and* $|\mathrm{D}u|_{M_\varphi}$ *and* $|\mathrm{D}u|$ *are the Radon measures induced in Theorem* 2.8 *in the manifolds* M_φ *and* M, *respectively.*

Remark 2.14. Note that the metric g of M can be written as $\varphi^{-2}\varphi^2 g$. A consequence of Theorem 2.13 is that $u \in \mathrm{BV}_{\mathrm{loc},M}(\Omega)$ if and only if $u \in \mathrm{BV}_{\mathrm{loc},M_\varphi}(\Omega)$.

It is easy to see that the formula in (2.3) holds for any Borel set $A \subset \Omega$.

Proof. Let div_φ and dvol_φ be the divergence and the volume of M_φ, respectively. Then $\mathrm{dvol}_\varphi = \varphi^{n+1}\,\mathrm{dvol}$, where dvol is the volume form of M. By the definition of the divergence, see [20], p. 423, we have

$$\begin{aligned}\mathrm{div}_\varphi(X)\,\mathrm{dvol}_\varphi = d(X \llcorner \mathrm{dvol}_\varphi) &= (\varphi^{n+1}\,\mathrm{div}(X) + (n+1)\varphi^n\langle X, \nabla\varphi\rangle)\,\mathrm{dvol}\\ &= \mathrm{div}(\varphi^{n+1}X)\,\mathrm{dvol},\end{aligned} \tag{2.4}$$

where $\nabla\varphi$ is the gradient of φ in M. By Theorem 2.8 (2), we have

$$\begin{aligned}|\mathrm{D}u|_{M_\varphi}(\Omega) &= \sup\Big\{\int_\Omega u\,\mathrm{div}_\varphi(X)\,\mathrm{dvol}_\varphi : \varphi^2\langle X, X\rangle \le 1, X \in T_0\Omega\Big\}\\ &= \sup\Big\{\int_\Omega u\,\mathrm{div}(X')\,\mathrm{dvol} : \langle X', X'\rangle \le \varphi^{2n}, X' \in T_0\Omega\Big\}\\ &= \int_\Omega \varphi^n\,\mathrm{d}|\mathrm{D}u|.\end{aligned}$$

The proof is complete. ∎

Remark 2.15. Indeed, from (2.4), we obtain

$$\mathrm{div}_\varphi(X) = \frac{1}{\varphi^{n+1}}\,\mathrm{div}(\varphi^{n+1}X).$$

2.3. The convolution of functions and vector fields

Now we consider how to approximate a function and a smooth vector field in a sufficiently small normal embedded ball in a Riemannian manifold.

Definition 2.16. Fix any point p in a Riemannian manifold M. Let $\exp_p$ be the exponential map near p. In the following, we identify T_pM with $\mathbb{R}^n$. There is a Euclidean ball $B_r(0)$ centered at 0 in $\mathbb{R}^n$ such that $\exp_p\colon B_r(0) \to B_r(p) \subset M$ is a diffeomorphism. Via the exponential map, we can identify $B_r(p)$ with $B_r(0)$. Moreover, the metric of M is represented as

$$g = g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j,$$

with the coordinates in $\mathbb{R}^n$. Such ball $B_r(p)$ is called a normal (open) ball.

Let $\varphi(x)$ be a symmetric smooth mollifier in $\mathbb{R}^n$, i.e., $\varphi(x) = \varphi(-x)$, $\varphi(x)$ has a compact support in the Euclidean unit ball $B_n(1)$ and

$$\int_{\mathbb{R}^n} \varphi(x)\,\mathrm{d}x = 1,$$

where dx is the standard Euclidean volume in $\mathbb{R}^n$.

Suppose W is an open set in $\mathbb{R}^n$. Let $h(x)$ denote a measurable function on W and let X denote a tangent vector field on W, written as

$$X = X^i \frac{\partial}{\partial x_i}, \tag{2.5}$$

where $\{\partial/\partial x_i\}_{i=1}^n$ is the standard orthonormal coordinate vector fields in $\mathbb{R}^n$.

Definition 2.17. Let $\sigma > 0$ be a sufficiently small positive constant. Then $\varphi_\sigma * h(x)$, the convolution of $h(x)$, is given by

$$\varphi_\sigma * h(x) = \int_{\mathbb{R}^n} \frac{1}{\sigma^n}\,\varphi\Big(\frac{x-y}{\sigma}\Big)\,h(y)\,\mathrm{d}y, \quad x \in W, \tag{2.6}$$

where we extend $h(x)$ outside W as $h(x) = 0$ for $x \notin W$. For X in (2.5), $\varphi_\sigma * X(x)$ is defined as

$$\varphi_\sigma * X(x) := \varphi_\sigma * X^i \frac{\partial}{\partial x_i}, \quad x \in W.$$

A useful property about the convolution is

$$\int_{\mathbb{R}^n} u(x)\,\varphi_\sigma * h(x)\,\mathrm{dx} = \int_{\mathbb{R}^n} h(x)\,\varphi_\sigma * u(x)\,\mathrm{dx}.$$

Theorem 2.18. *Let B be a normal open ball in a Riemannian manifold with a metric $g = g_{ij}\,\mathrm{dx^i}\,\mathrm{dx^j}$. Let f be a nonnegative continuous function on B. Let h be a continuous function on B and let X be a smooth vector field satisfying*

$$h^2 + \langle X, X\rangle \le f^2 \quad \text{in } B,$$

where $\langle\cdot,\cdot\rangle$ is the inner product determined by g. Then, for any $\varepsilon > 0$ and any compact set $K \subset B$, there exists $\sigma_0 = \sigma_0(f, K, g, \varepsilon)$ such that for all $\sigma < \sigma_0$,

$$h'^2(x) + \langle Y, Y\rangle(x) \le (f(x) + \varepsilon)^2, \quad x \in K,$$

where $\det(g) := \det(g_{ij})$*, and*

$$h' := \frac{1}{\sqrt{\det(g)}}\,\varphi_\sigma * \big(\sqrt{\det(g)}\,h\big), \tag{2.7}$$

$$Y := \frac{1}{\sqrt{\det(g)}}\,\varphi_\sigma * \big(\sqrt{\det(g)}\,X\big). \tag{2.8}$$

Proof. Let σ_1 be a positive constant less than the Euclidean distance between $\partial\Omega$ and K. Since K is compact, for all $\sigma < \sigma_1/2$, the function h' in (2.7) and the tangent vector Y in (2.8) are well defined for $x \in K \subset B$.

Let ε' be a small constant, to be determined later. For any $x_0 \in K$, there is a positive constant $\sigma_2 = \sigma_2(f, g, K, \varepsilon') < \sigma_1$ such that for all $\sigma < \sigma_2/2$ and $y, y' \in B_{x_0}(2\sigma)$,

$$\frac{1}{1+\varepsilon'} g_{ij}(y') \le g_{ij}(y) \le (1+\varepsilon') g_{ij}(y'), \tag{2.9}$$

$$\max_{y,y' \in B_{2\sigma}(x_0)} \frac{\sqrt{\det(g)}(y')}{\sqrt{\det(g)}(y)} \le 1 + \varepsilon', \tag{2.10}$$

$$f(y) \le f(y') + \varepsilon' \quad \text{for } y, y' \in B_{2\sigma}(x_0). \tag{2.11}$$

Here $B_{2\sigma}(x_0)$ is the Euclidean ball of x_0 with radius 2σ in B. By Definition 2.17 and (2.5), we have

$$Y^i = \frac{1}{\sqrt{\det(g)}} \varphi_\sigma * (\sqrt{\det(g)}\, X^i) \quad \text{and} \quad Y = Y^i \frac{\partial}{\partial x_i}. \tag{2.12}$$

Fix any point $y \in B_{x_0}(\sigma)$. With a rotation we can assume that $g_{ij}(y) = \sigma_{ik}\sigma_{kj}$, where (σ_{ik}) is a positive definite matrix. By (2.8), (2.9), (2.10) and (2.12), for any $\sigma < \sigma_1/2$, we have

$$\begin{aligned} g_{ij}(y) Y^i Y^j(y) &= \frac{1}{\det(g)(y)} (\varphi_\sigma * (\sqrt{\det(g)}\, \sigma_{ik} X^i))^2(y) \\ &\le (1+\varepsilon')^2 (\varphi_\sigma * (\sigma_{ik} X^i)^2)(y) \\ &= (1+\varepsilon')^2 (\varphi_\sigma * (g_{ij}(y) X^i X^j)(y) \\ &\le (1+\varepsilon')^3 \varphi_\sigma * (g_{ij} X^i X^j)(y). \end{aligned} \tag{2.13}$$

By (2.7), a similar derivation implies that

$$(h')^2(y) \le (1+\varepsilon')^3 \varphi_\sigma * h^2. \tag{2.14}$$

Combining (2.13) with (2.14) and using (2.11), we obtain

$$\begin{aligned} (h')^2(y) + g_{ij} Y^i Y^j(y) &\le (1+\varepsilon')^2 \varphi_\sigma * (h^2 + g_{ij} X^i X^j)(y) \\ &\le (1+\varepsilon')^3 \varphi_\sigma * f^2(y) \\ &\le (1+\varepsilon')^3 (f(y) + \varepsilon')^2. \end{aligned}$$

Because K is compact, we can choose ε' small enough such that $(1+\varepsilon')^3(f(y)+\varepsilon')^2 \le (f(y)+\varepsilon)^2$ for all $y \in B_\sigma(x_0)$ and $x_0 \in K$. For such fixed ε', define $\sigma_0 = \sigma_2(f, g, K, \varepsilon')$. Thus, for any $x_0 \in K$, $y \in B_\sigma(x_0)$ and $\sigma < \sigma_0/2$, we have

$$(h')^2(y) + g_{ij} Y^i Y^j(y) \le f(y) + \varepsilon.$$

The proof is complete. ∎

The following technical result will be very useful in the next section.

Lemma 2.19. *Let B be a normal open ball with a metric $g = g_{ij}\,\mathrm{dx^i\,dx^j}$. Suppose u is in $\mathrm{BV}(B)$ and $q(x)$ is a smooth function with compact support in B. Let X be a smooth vector field on B satisfying $\langle X, X\rangle \leq 1$. Then for any $\varepsilon > 0$, there is a positive constant $\sigma_0 = \sigma_0(u, g, q) > 0$ such that for all $\sigma \in (0, \sigma_0)$,*

$$\int_B \varphi_\sigma * (qu)\,\mathrm{div}(X)\,\mathrm{dvol} \leq \int_B u\,\mathrm{div}(qY_\sigma)\,\mathrm{dvol} - \int_B u\langle X, \nabla q\rangle\,\mathrm{dvol} + \varepsilon, \tag{2.15}$$

where

$$Y_\sigma = \frac{1}{\sqrt{\det(g)}}\,\varphi_\sigma * (\sqrt{\det(g)}X)$$

and ∇q denotes the gradient of q on B. And we assume $X = 0$ outside B.

Proof. Note that $\mathrm{dvol} = \sqrt{\det(g)}\,\mathrm{dx}$, where dvol and dx are the volume form of B with respect to g and the Euclidean metric, respectively. Moreover,

$$\mathrm{div}(X)\,\mathrm{dvol} = \mathrm{div}_{\mathbb{R}^n}(\sqrt{\det(g)}\,X)\,\mathrm{dx},$$

where div and $\mathrm{div}_{\mathbb{R}^n}$ are the divergence of B and $\mathbb{R}^n$, respectively. We also view B as an open set in the Euclidean space $\mathbb{R}^n$. Thus, $\varphi_\sigma * (qu)$ is well defined if we choose sufficiently small σ. Then

$$\begin{aligned}
\int_B \varphi_\sigma * (qu)\,\mathrm{div}(X)\,\mathrm{dvol} &= \int_{\mathbb{R}^n} \varphi_\sigma * (qu)\,\mathrm{div}_{\mathbb{R}^n}(\sqrt{\det(g)}\,X)\,\mathrm{dx} \\
&= \int_{\mathbb{R}^n} q(x)u(x)\,\mathrm{div}_{\mathbb{R}^n}(\varphi_\sigma * (\sqrt{\det(g)}X))(x)\,\mathrm{dx} \\
&= \int_B u(x)\,\mathrm{div}(qY_\sigma)\,\mathrm{dvol} - \int_B u\langle Y_\sigma, \nabla q\rangle\,\mathrm{dvol},
\end{aligned}$$

where

$$Y_\sigma = \frac{1}{\sqrt{\det(g)}}\,\varphi_\sigma * (\sqrt{\det(g)}X).$$

As a result, we have

$$\begin{aligned}
\int_B u\langle Y_\sigma, \nabla q\rangle\,\mathrm{dvol} &= \int_{\mathbb{R}^n} u\,g_{ij}\,\varphi_\sigma * (\sqrt{\det(g)}X^i)\nabla^j q\,\mathrm{dx} \\
&= \int_{\mathbb{R}^n} X^i\varphi_\sigma * (u\,g_{ij}\nabla^j q)\sqrt{\det(g)}\,\mathrm{dx}.
\end{aligned}$$

Since the set $\{X \in TB : \langle X, X\rangle \leq 1\}$ is a compact set, there is a $\sigma_0 = \sigma_0(u, g, q) > 0$, independent of any X satisfying $\langle X, X\rangle \leq 1$, such that

$$\begin{aligned}
-\int_{\mathbb{R}^n} X^i\varphi_\sigma * (u\,g_{ij}\nabla^j q)\sqrt{\det(g)}\,\mathrm{dx} &\leq -\int_{\mathbb{R}^n} X^i u\,g_{ij}\nabla^j q\sqrt{\det(g)}\,\mathrm{dx} + \varepsilon \\
&= -\int_B u\langle X, \nabla q\rangle\,\mathrm{dvol} + \varepsilon
\end{aligned}$$

for all $\sigma \in (0, \sigma_0)$. Combining the above two inequalities together, we obtain (2.15). The proof is complete. ∎

3. Area functionals and their C^∞ approximation

In this section, we define product area functionals and conformal area functionals. Then we obtain their C^∞ approximation properties in Theorem 3.5. Our proof depends on a decomposition result of Radon measures from the Besicovitch covering theorem in Riemannian manifolds (see Theorem A.4).

3.1. The conformal area functional

Throughout this section let (N, σ) denote a complete Riemannian manifold. For any $\alpha > 0$, we denote $(N \times \mathbb{R}, e^{2\alpha r/n}(\sigma + \mathrm{d}r^2))$ by Q_α and $(N \times \mathbb{R}, \sigma + \mathrm{d}r^2)$ by Q.

We write div and dvol for the divergence and volume form of N, respectively. Let Ω be an open bounded set in N. Let $C_0(\Omega)$ and $T_0(\Omega)$ denote the sets of all smooth functions and smooth vector fields with compact supports in Ω, respectively.

Definition 3.1. Let $u(x)$ be a measurable function on Ω. The product area functional $\mathfrak{F}(u, \Omega)$ is defined by

$$\mathfrak{F}(u,\Omega) :\equiv \sup\Big\{ \int_\Omega (h + u\,\mathrm{div}(X))\,\mathrm{dvol} : \\ h \in C_0(\Omega), X \in T_0\Omega, h^2 + \langle X, X\rangle \le 1\Big\}. \tag{3.1}$$

Let $\alpha > 0$ be a fixed a constant. The conformal area product functional $\mathfrak{F}_\alpha(u, \Omega)$ is defined by

$$\mathfrak{F}_\alpha(u,\Omega) :\equiv \sup\Big\{ \int_\Omega e^{\alpha u}\Big(h + \frac{1}{\alpha}\mathrm{div}(X)\Big)\,\mathrm{dvol} : \\ h \in C_0(\Omega), X \in T_0\Omega, h^2 + \langle X, X\rangle \le 1\Big\}.$$

The geometric motivation of the above two functionals is to generalize the area of the graph of C^1 functions in corresponding manifolds.

Remark 3.2. If u is a C^2 function on Ω that is the critical point of $\mathfrak{F}_\alpha(u, \Omega)$, then u satisfies the translating mean curvature equation $\mathrm{H}_\alpha(u) = 0$ on Ω, where

$$\mathrm{H}_\alpha := -\mathrm{div}\Big(\frac{\mathrm{D}u}{\sqrt{1 + |\mathrm{D}u|^2}}\Big) + \frac{\alpha}{\sqrt{1 + |\mathrm{D}u|^2}}.$$

Set $K^+(\Omega)$ as the set of all nonnegative functions with compact support in Ω. For any $f \in K^+(\Omega)$, we define two nonnegative functionals:

$$\lambda_{u,0}(f) :\equiv \sup\Big\{ \int_\Omega (h + u\,\mathrm{div}(X))\,\mathrm{dvol} : \\ h \in C_0(\Omega), X \in T_0\Omega, h^2 + \langle X, X\rangle \le f^2(x)\Big\}, \tag{3.2}$$

$$\lambda_{u,\alpha}(f) :\equiv \sup\Big\{ \int_\Omega e^{\alpha u}\Big(h + \frac{1}{\alpha}\mathrm{div}(X)\Big)\,\mathrm{dvol} : \\ h \in C_0(\Omega), X \in T_0\Omega, h^2 + \langle X, X\rangle \le f^2(x)\Big\}, \tag{3.3}$$

for any $\alpha > 0$. It is clear that both $\lambda_{u,0}(\cdot)$ and $\lambda_{u,\alpha}(\cdot)$ are linear on $K^+(\Omega)$. Rewriting the definitions of $\mathfrak{F}(u,\Omega)$ and $\mathfrak{F}_\alpha(u,\Omega)$, we obtain the following equivalent definitions:

$$\mathfrak{F}(u,\Omega) :\equiv \sup\{\lambda_{u,0}(f) : f \in K^+(\Omega), f \le 1\},$$
$$\mathfrak{F}_\alpha(u,\Omega) :\equiv \sup\{\lambda_{u,\alpha}(f) : f \in K^+(\Omega), f \le 1\}.$$

In fact, the above two formulas are true for any open set in Ω.

By Theorem 2.7, as in the case of BV functions, the above two formulas naturally induce two Radon measures.

Theorem 3.3. *Let Ω be an open set in N. Suppose u is in* $\mathrm{BV}_{\mathrm{loc},N}(\Omega)$ *such that $\mathfrak{F}(u,\Omega')$ is finite for any bounded open set $\Omega' \subset\subset \Omega$. Then*

(1) *there is a unique Radon measure μ_0 on Ω such that $\mu_0(\Omega') = \mathfrak{F}(u,\Omega')$,*

(2) *there is a unique Radon measure μ_α on Ω such that $\mu_\alpha(\Omega') = \mathfrak{F}_\alpha(u,\Omega')$.*

Proof. Applying Theorem 2.7 on $\lambda_{u,0}(\cdot)$ in (3.2) and $\lambda_{u,\alpha}(\cdot)$ in (3.3), we obtain the existence of μ_0 and μ_α. The two conclusions follow from (2.1). ■

Similar to the case of BV functions, the semicontinuous property is also valid for $\mathfrak{F}(u,\Omega)$ and $\mathfrak{F}_\alpha(u,\Omega)$.

Theorem 3.4. *Let Ω be a bounded open domain in N. Let u be in $L^1(\Omega)$ such that $\mathfrak{F}(u,\Omega)$ and $\mathfrak{F}_\alpha(u,\Omega)$ are finite. Let $\{u_k\}_{k=1}^\infty$ be a sequence in $L^1(\Omega)$.*

(1) *Suppose $\{u_k\}_{k=1}^\infty$ converges to u in $L^1(\Omega)$. Then*

$$\mathfrak{F}(u,\Omega) \le \liminf_{k\to\infty} \mathfrak{F}(u_k,\Omega).$$

(2) *Suppose $\{e^{\alpha u_k}\}_{k=1}^\infty$ converges to $e^{\alpha u}$ in $L^1(\Omega)$. Then*

$$\mathfrak{F}_\alpha(u,\Omega) \le \liminf_{k\to\infty} \mathfrak{F}_\alpha(u_k,\Omega).$$

(3) *Suppose u is in* $\mathrm{BV}(\Omega)$. *Then the following estimate holds*:

$$\max\{|\mathrm{D}u|(\Omega), \mathrm{vol}(\Omega)\} \le \mathfrak{F}(u,\Omega) \le \mathrm{vol}(\Omega) + |\mathrm{D}u|(\Omega),$$

where $\mathrm{vol}(\Omega)$ *denotes the volume of N.*

Proof. The conclusion (1) and (2) follow from the definitions of $\mathfrak{F}(u,\Omega)$ and $\mathfrak{F}_\alpha(u,\Omega)$, respectively. The left inequality in (3) is obtained by letting $h \equiv 0$ or $X \equiv 0$ and taking the supremum in (3.1). The right inequality in (3) is directly from the definition of BV functions. ■

3.2. The C^∞ approximation

In this subsection, we show the C^∞ approximation property of $|\mathrm{D}u|(\Omega)$, $\mathfrak{F}(u,\Omega)$ and $\mathfrak{F}_\alpha(u,\Omega)$ as follows.

Theorem 3.5. *Let Ω be a bounded domain in N and suppose $u(x)$ is in* $\mathrm{BV}(\Omega)$.

(1) *Then there is a sequence* $\{u_k\}_{k=1}^{\infty}$ *in* $C^{\infty}(\Omega)$ *such that* $\{u_k\}_{k=1}^{\infty}$ *converges to* u *in* $L^1(\Omega)$ *and*

$$\lim_{k\to\infty} |\mathrm{D}u_k|(\Omega) = |\mathrm{D}u|(\Omega).$$

(2) *There is a sequence* $\{u_k\}_{k=1}^{\infty}$ *in* $C^{\infty}(\Omega)$ *such that* u_k *converges to* u *in* $L^1(\Omega)$ *and*

$$\lim_{k\to\infty} \mathfrak{F}(u_k,\Omega) = \mathfrak{F}(u,\Omega).$$

(3) *In addition, suppose* $\alpha > 0$ *and that* $\mathfrak{F}_\alpha(u,\Omega)$ *is finite. Then there is a sequence* $\{u_k\}_{k=1}^{\infty}$ *in* $C^{\infty}(\Omega)$ *such that* $\{e^{\alpha u_k}\}_{k=1}^{\infty}$ *converges to* $e^{\alpha u}$ *in* $L^1(\Omega)$ *and*

$$\lim_{k\to\infty} \mathfrak{F}_\alpha(u_k,\Omega) = \mathfrak{F}_\alpha(u,\Omega).$$

Remark 3.6. The proof in Theorem 1.17 in [14] requires the existence of a well-defined symmetric mollifiers on Ω. See, for example, the definition of Ω_k and (1.12)–(1.14) in [14], p. 15. Such existence needs that the domain is contained in a large simply connected domain in Euclidean spaces to define the distance function. This is not true for arbitrary bounded domains in Riemannian manifolds. For the construction the sequence of smooth functions converging to $u(x)$, we need Theorem A.4 to decompose a bounded domain into sufficiently small domains with a reasonable decomposition of $|\mathrm{D}u|$.

Proof of Theorem 3.5. Our proof is divided into three cases.

The case of $\mathfrak{F}(u,\Omega)$.

Because u is in $\mathrm{BV}(\Omega)$ and Ω is bounded, $\mathfrak{F}(u,\Omega)$ is finite. According to Theorem 3.3, there is a Radon measure μ_0 in Ω satisfying $\mu_0(\Omega') = \mathfrak{F}(u,\Omega')$ for any open set $\Omega' \subset \Omega$.

By Theorem A.4, there is a collection of normal open balls $\{B_i\}_{i=1}^{\infty}$ such that $\Omega \subset \bigcup_{i=1}^{\infty} B_i$, and there is an integer $\kappa(\varepsilon) > 0$ such that $\{B_1,\dots,B_n\}_{i=1}^{\kappa(\varepsilon)}$ is a pairwise disjoint collection satisfying the estimate

$$\mu_0(\Omega) - \varepsilon \le \sum_{i=1}^{\kappa(\varepsilon)} \mu_0(B_i) \le \mu_0(\Omega), \qquad \sum_{i=\kappa(\varepsilon)+1}^{\infty} \mu_0(B_i) \le \varepsilon. \tag{3.4}$$

Thus, there is a partition of unity $\{q_i(x)\}_{i=1}^{\infty}$ subordinate to the above open cover. Namely, $q_i \in C_0^{\infty}(B_i)$, $0 \le q_i \le 1$ and $\sum_{i=1}^{\infty} q_i = 1$ on Ω. Fix any smooth function $\tilde{h}$ and any vector field X with compact supports in Ω satisfying

$$\tilde{h}^2 + \langle X, X\rangle \le 1 \quad \text{on } \Omega. \tag{3.5}$$

By the definition of the open normal ball, we can view B_i as an open set in $\mathbb{R}^n$ with the metric $g = g_{kl}\,\mathrm{dx^k}\,\mathrm{dx^l}$. With the coordinate frame $\{\partial/\partial x_k\}_{k=1}^{n}$ on $\mathbb{R}^n$, X can be written as $X^k\,\partial/\partial x_k$ on each B_i. Moreover, on B_i, we have

$$\tilde{h}^2 + g_{kl} X^k X^l \le 1.$$

Choose any fixed positive number ε. For each $i > 0$, one can choose $\sigma_i > 0$ such that the support of $\varphi_{\sigma_i} * (uq_i)$ (the convolution in (2.6)) is contained in B_i and in such a way that

$$\int_{B_i} |\varphi_{\sigma_i} * (uq_i) - uq_i|\,\mathrm{dvol} \le \frac{\varepsilon}{2^i}, \tag{3.6}$$

that

$$(q_i\tilde{h})^2 + (q_i)^2\langle Y_{\sigma_i}, Y_{\sigma_i}\rangle \le 1+\varepsilon \tag{3.7}$$

(by Theorem 2.18), and that

$$\int_{B_i} \varphi_{\sigma_i} * (uq_i)\operatorname{div}(X))\,\mathrm{dvol} \le \int_{B_i} u\operatorname{div}(q_i Y_{\sigma_i})\,\mathrm{dvol} - \int_{B_i} u\langle X, \nabla q_i\rangle\,\mathrm{dvol} + \frac{\varepsilon}{2^i} \tag{3.8}$$

(by Lemma 2.19). Here,

$$Y^k_{\sigma_i} = \frac{\varphi_{\sigma_i} * (\sqrt{\det(g)}X^k)}{\sqrt{\det(g)}} \quad\text{and}\quad Y_{\sigma_i} = Y^k_{\sigma_i}\partial_k.$$

Define u_ε as

$$u_\varepsilon = \sum_{i=1}^{\infty} \varphi_{\sigma_i} * (uq_i).$$

Now (3.6) implies that

$$\int_\Omega |u_\varepsilon - u|\,\mathrm{dvol} \le \sum_{i=1}^{\infty}\int_{B_i} |\varphi_{\sigma_i} * (uq_i) - uq_i|\,\mathrm{dvol} \le \varepsilon. \tag{3.9}$$

Thus, (3.8) implies

$$\begin{aligned}
&\int_{B_i} (\tilde{h}q_i + \varphi_{\sigma_i} * (uq_i)\operatorname{div}(X))\,\mathrm{dvol}\\
&\qquad \le \int_{B_i} (\tilde{h}q_i + u\operatorname{div}(q_i Y_{\sigma_i}))\,\mathrm{dvol} - \int_{B_i} u\langle X, \nabla q_i\rangle\,\mathrm{dvol} + \frac{\varepsilon}{2^i}\\
&\qquad \le (1+\varepsilon)\mu_0(B_i) - \int_{B_i} u\langle X, \nabla q_i\rangle\,\mathrm{dvol} + \frac{\varepsilon}{2^i}.
\end{aligned}$$

Combining the partition of unity with (3.4) yields

$$\int_\Omega (\tilde{h} + u_\varepsilon \operatorname{div}(X))\,\mathrm{dvol} \le (1+\varepsilon)\sum_{i=1}^{\infty}\mu_0(B_i) + \varepsilon \le (1+\varepsilon)(\mu_0(\Omega)+\varepsilon) + \varepsilon. \tag{3.10}$$

Here,

$$\sum_{i=1}^{\infty}\int_{B_i} u\langle X, \nabla q_i\rangle\,\mathrm{dvol} = 0,$$

because $\sum_{i=1}^{\infty} q_i(x) = 1$ on Ω. Taking the supremum for all $\tilde{h}$, X satisfying (3.5), we conclude that

$$\mathfrak{F}(u_\varepsilon, \Omega) \le (1+\varepsilon)(\mathfrak{F}(u, \Omega) + \varepsilon) + \varepsilon. \tag{3.11}$$

Now take a sequence $\varepsilon_k \to 0$ as $k \to \infty$. By (3.9) and (3.11), we obtain a smooth sequence $\{u_{\varepsilon_k}\}_{k=1}^{\infty}$ such that $\{u_{\varepsilon_k}\}_{k=1}^{\infty}$ converges to u in L^1 and

$$\lim_{k\to\infty}\sup \mathfrak{F}(u_{\varepsilon_k}, \Omega) \le \mathfrak{F}(u, \Omega).$$

By Theorem 3.4, we have $\lim_{k\to\infty}\inf \mathfrak{F}(u_{\varepsilon_k}, \Omega) \ge \mathfrak{F}(u, \Omega)$. Therefore, the limit holds and $\{u_{\varepsilon_k}\}_{k=1}^{\infty}$ is the desirable sequence. We obtain the conclusion (1).

The case of $|\mathrm{D}u|(\Omega)$.

The proof of the conclusion (1) is similar to that of the conclusion (2) if in the whole derivation we take $\tilde{h} \equiv 0$. From Definition 2.1, we obtain the conclusion.

The case of $\mathfrak{F}_\alpha(u, \Omega)$.

The idea to derive the conclusion (3) is also similar to the proof of the conclusion (2) except from the construction of the approximating sequence.

By Theorem 3.3, there is a unique Radon measure μ_α on Ω such that $\mu_\alpha(\Omega') = \mathfrak{F}_\alpha(u, \Omega')$ for any open $\Omega' \subset\subset \Omega$. Moreover, $\mu_\alpha(\Omega)$ is finite. Fix $\varepsilon > 0$. By Theorem A.4, there is a collection of open sets $\{B_i\}_{i=1}^\infty$ such that each B_i is an open normal ball in Ω, with $\Omega \subset \bigcup_{i=1}^\infty B_i$, and there is an integer $N(\varepsilon)$ such that $\{B_1, \ldots, B_n\}_{i=1}^{N(\varepsilon)}$ is a pairwise disjoint collection with the estimate

$$\mu_\alpha(\Omega) - \varepsilon \le \sum_{i=1}^{N(\varepsilon)} \mu_\alpha(B_i) \le \mu_\alpha(\Omega), \qquad \sum_{i=N(\varepsilon)+1}^{\infty} \mu_\alpha(B_i) \le \varepsilon. \tag{3.12}$$

Choose h in $C_0(\Omega)$ and X in $T_0\Omega$ satisfying

$$h^2 + \langle X, X\rangle \le 1.$$

Each B_i can be viewed as an open set in $\mathbb{R}^n$ with the metric $g = g_{kl}\,\mathrm{d}x^k\,\mathrm{d}x^l$.

On each B_i assume $X = X^k\partial_k$. Then

$$h^2 + g_{kl}X^kX^l \le 1.$$

Let $\{q_i(x)\}_{i=1}^\infty$ be a partition of unity subordinate to the cover $\{B_i\}_{i=1}^\infty$. For each $i > 0$, we can choose $\sigma_i > 0$ such that

$$\int_{B_i} |\varphi_{\sigma_i} * (e^{\alpha u}q_i) - e^{\alpha u}q_i|\,\mathrm{dvol} \le \frac{\varepsilon}{2^i}, \tag{3.13}$$

$$(q_ih')^2 + (q_i)^2 g_{kl}Y^k_{\sigma_i}Y^l_{\sigma_i} \le 1 + \varepsilon, \tag{3.14}$$

$$\begin{aligned}&\int_{B_i} \varphi_{\sigma_i} * (e^{\alpha u}q_i)\,\mathrm{div}(X))\,\mathrm{dvol}\\ &\qquad \le \int_{B_i} e^{\alpha u}\,\mathrm{div}(q_iY_{\sigma_i})\,\mathrm{dvol} - \int_{B_i} e^{\alpha u}\langle X, \nabla q_i\rangle\,\mathrm{dvol} + \frac{\varepsilon}{2^i}.\end{aligned} \tag{3.15}$$

Here,

$$Y^k_{\sigma_i} = \frac{\varphi_{\sigma_i} * (\sqrt{\det(g)}X^k)}{\sqrt{\det(g)}}, \quad h' = \frac{\varphi_{\sigma_i} * (\sqrt{\det(g)}h)}{\sqrt{\det(g)}} \quad \text{and} \quad Y_{\sigma_i} = Y^k_{\sigma_i}\partial_k.$$

The proof of the above arguments is similar to that of the case of $\mathfrak{F}(u, \Omega)$, just replacing u with $e^{\alpha u}$. In particular, (3.15) is from Lemma 2.19.

Now we define u_ε as

$$e^{\alpha u_\varepsilon} = \sum_{i=1}^\infty \varphi_{\sigma_i} * (e^{\alpha u}q_i).$$

This is well defined because the right-hand is a finite positive summation at any point x in Ω. According to (3.13), we have

$$\int_\Omega |e^{\alpha u_\varepsilon} - e^{\alpha u}|\,\mathrm{dvol} \le \sum_{i=1}^{\infty} \int_{B_i} |\varphi_{\sigma_i} * (e^{\alpha u} q_i) - e^{\alpha u} q_i|\,\mathrm{dvol} \le \varepsilon.$$

Replacing u with $e^{\alpha u}$ in (3.8) and repeating the same reasoning, we obtain

$$\int_{B_i} \varphi_{\sigma_i} * (e^{\alpha u} q_i)\,\mathrm{div}(X))\,\mathrm{dvol} = \int_{\mathbb{R}^n} (e^{\alpha u}\,\mathrm{div}(q_i Y_{\sigma_i}) - e^{\alpha u}\langle Y_{\sigma_i}, \nabla q_i\rangle)\,\mathrm{dvol}. \tag{3.16}$$

On the other hand,

$$\int_{B_i} \varphi_{\sigma_i} * (e^{\alpha u} q_i)\,h\,\mathrm{dvol} = \int_{B_i} \varphi_{\sigma_i} * (e^{\alpha u} q_i)\,h\sqrt{\det(g)}\,\mathrm{dx} = \int_{B_i} \varphi_i e^{\alpha v} h'\,\mathrm{dvol}, \tag{3.17}$$

where $h' = \frac{\psi_{\sigma_i} * (\sqrt{\det(g)}h)}{\sqrt{\det(g)}}$. By (3.16) and (3.17), we compute

$$\begin{aligned}
\int_\Omega e^{\alpha u_\varepsilon}\Big(h + \frac{1}{\alpha}\,\mathrm{div}(X)\Big)\,\mathrm{dvol} &= \sum_{i=1}^{\infty} \int_{B_i} \varphi_{\sigma_i} * (e^{\alpha u} q_i)\Big(h + \frac{1}{\alpha}\,\mathrm{div}(X)\Big)\,\mathrm{dvol} \\
&\le \sum_{i=1}^{\infty} \Big\{ \int_{B_i} e^{\alpha u}\Big(q_i h' + \frac{1}{\alpha}\,\mathrm{div}(q_i Y_{\sigma_i})\Big)\,\mathrm{dvol} - \frac{1}{\alpha}\int_{B_i} e^{\alpha u}\langle X, \nabla q_i\rangle \mathrm{dvol} + \frac{1}{\alpha}\frac{\varepsilon}{2^i}\Big\} \\
&\le (1+\varepsilon)\sum_{i=1}^{\infty} \mu_\alpha(B_i) + \frac{\varepsilon}{\alpha}\cdot
\end{aligned}$$

The first term in the last inequality comes from (3.14) and the definition of $\mathfrak{F}_\alpha(u, B_i)$. The second term is due to (3.15) and the fact $\sum_{i=1}^{\infty} q_i \equiv 1$ on Ω. Putting the assumption (3.12) into the above estimate yields that

$$\int_\Omega e^{\alpha u_\varepsilon}\Big(h + \frac{1}{\alpha}\,\mathrm{div}(X)\Big)\,\mathrm{dvol} \le (1+\varepsilon)(\mu_\alpha(\Omega)+\varepsilon) + \frac{\varepsilon}{\alpha} = (1+\varepsilon)(\mathfrak{F}_\alpha(u,\Omega)+\varepsilon) + \frac{\varepsilon}{\alpha}\cdot$$

Now we arrive at a similar position as in (3.10) when we show the conclusion (1). Thus, a similar derivation yields the conclusion (1).

The proof of Theorem 3.5 is complete. ∎

4. Miranda's observation

In this section, we study the relationship between area functionals and the corresponding perimeter in conformal product manifolds.

We still assume that (N, σ) is a complete Riemannian manifold. For any $\alpha > 0$, Q_α denotes $(N \times \mathbb{R}, e^{2\alpha r/n}(\sigma + \mathrm{dr}^2))$ and Q denotes $(N \times \mathbb{R}, \sigma + \mathrm{dr}^2)$. For a function $u(x)$ over a domain Ω, its subgraph is the set $\{(x,t) : x \in \Omega, t < u(x)\}$.

Our purpose is to show Miranda's observation, which says that the conformal functional $\mathfrak{F}_\alpha(u, \cdot)$ corresponds to the perimeter and if a BV function locally minimizes $\mathfrak{F}_\alpha(\cdot, \Omega)$, its subgraph locally minimizes its perimeter in $\Omega \times \mathbb{R}$.

4.1. Some conventions on the perimeter

Recall that the perimeter of a Borel set E in any open domain is $|D\lambda_E|(\Omega)$, where λ_E is the characteristic function of E. In the following we also write it as $\mathrm{P}(E,\Omega)$ with some lower index.

The perimeter and other properties of a Caccioppoli set are unchanged if we make alternations of Lebesgue measure zero. In other words, we only concern about the equivalence classes of a Caccioppoli set.

The following result extends Proposition 3.1 in [14] with exactly the same proof via the Nash embedding theorem.

Proposition 4.1. *Let M be a complete Riemannian manifold with dimension n $(n \geq 2)$. For any x in M, let $B_r(x)$ be the ball in M centered at x with radius r. Let $\mathrm{inj}(x)$ denote the injectivity radius of x, i.e., the supremum of r such that $B_r(x)$ is an embedded normal ball in M. If E is a Borel set in M, there exists a Borel set $\tilde{E}$ equivalent to E (that is, differs only by a set of $\mathcal{H}^n$ measure zero) such that*

$$0 < \mathcal{H}^n(\tilde{E} \cap B_\rho(x)) < \mathrm{vol}(B_\rho(x)) \tag{4.1}$$

for all x in $\partial\tilde{E}$ and ρ in $(0, \mathrm{inj}(x))$. Here $\mathcal{H}^n$ is the n-dimensional Hausdorff measure.

In the remainder of this paper, we always assume (4.1) holds for any Caccioppoli set.

4.2. The perimeter of subgraphs

The following result extends Theorem 14.6 of [14] in Euclidean space into Q and Q_α. It is the first fact of Miranda's observation.

Theorem 4.2. *Suppose Ω is a bounded Lipschitz domain. Let $u(x)$ be a measurable function on Ω and let U be its subgraph.*

1. *If u is in $\mathrm{BV}(\Omega)$, then $\mathfrak{F}(u,\Omega) = \mathrm{P}(U, \Omega \times \mathbb{R})$.*
2. *If $\alpha > 0$ and $e^{\alpha u}$ is in $\mathrm{BV}(\Omega)$, then $\mathfrak{F}_\alpha(u,\Omega) = \mathrm{P}_\alpha(U, \Omega \times \mathbb{R})$.*

Here P and P_α denote the perimeter of Q and Q_α, respectively.

One side of the equality above is easily obtained via the semicontinuous property.

Lemma 4.3. *Take the assumptions and notation in Theorem 4.2. Then*

$$\mathrm{P}_\alpha(U, \Omega \times \mathbb{R}) \leq \mathfrak{F}_\alpha(u,\Omega) \quad \textit{and} \quad \mathrm{P}(U, \Omega \times \mathbb{R}) \leq \mathfrak{F}(u,\Omega).$$

Proof. We only show the case of Q_α for $\alpha > 0$ because the same derivation will yield the case of Q.

Assume u is in $C^1(\Omega)$. Thus, its subgraph U has a C^1 boundary. By [14], the perimeter of a Caccioppoli set is just the volume of its boundary. Namely,

$$\mathrm{P}_\alpha(U, \Omega \times \mathbb{R}) = \int_\Omega e^{\alpha u}\sqrt{1 + |Du|^2}\,\mathrm{dvol}.$$

This gives that

$$\mathrm{P}_\alpha(U, \Omega \times \mathbb{R}) = \mathfrak{F}_\alpha(u,\Omega).$$

Next suppose $e^{\alpha u}$ belongs to $\mathrm{BV}(\Omega)$. Thus, $\mathfrak{F}_\alpha(u,\Omega)$ is bounded. By Theorem 3.5, there exists a smooth sequence $\{u_i\}_{i=1}^\infty$ in $C^\infty(\Omega)$ such that $\{e^{\alpha u_i}\}_{i=1}^\infty$ converges to $e^{\alpha u(x)}$ in $L^1(\Omega)$ with

$$\lim_{i\to\infty}\mathfrak{F}_\alpha(u_i,\Omega)=\mathfrak{F}_\alpha(u,\Omega).$$

Let U_i be the subgraph of u_i for any i. It is easy to see that $\{\lambda_{U_i}\}_{i=1}^\infty$ locally converges to λ_U in $L^1(\Omega\times\mathbb{R})$. By the semicontinuity in Theorem 2.4, we obtain

$$\mathrm{P}_\alpha(U,\Omega\times\mathbb{R})\le\lim_{i\to\infty}\inf\mathrm{P}_\alpha(U_j,\Omega\times\mathbb{R})=\lim_{i\to+\infty}\inf\mathfrak{F}_\alpha(u_i,\Omega)=\mathfrak{F}_\alpha(u,\Omega).$$

The proof is complete. ∎

Now we are ready to show Theorem 4.2. Our proof is similar to that of Theorem 14.6 in [14].

Proof. Our proof is divided into two cases: (a) $u(x)$ is uniformly bounded and (b) $u(x)$ is in the general case. In the following, $\alpha\ge 0$.

Suppose we are in the first case. There is a $T>0$ such that $-T\le u(x)\le T$ on Ω. Choose $h\in C_0(\Omega)$ and $X\in T_0(\Omega)$ satisfying

$$e^{-2\alpha(T+1)/n}(h^2+\langle X,X\rangle)\le 1, \tag{4.2}$$

where $\langle\ ,\ \rangle$ is the inner product on N.

Let $\eta(r)$ be a smooth function on $\mathbb{R}$ with its support in $[-(T+1),\sup_\Omega u+1]$ such that $\eta\equiv 1$ in $[-T,\sup_\Omega u]$ and $|\eta(r)|\le 1$. Let $\eta_1(r)$ be a smooth function with a compact support on $\mathbb{R}$ satisfying $\eta_1(r)=e^{-\alpha(r+T+1)/n}$ on $[-(T+1),(T+1)]$. Now we define a smooth vector field

$$X'=\eta_1(r)\eta(r)(h\partial_r+X),$$

with compact support on Q_α. Moreover, X' satisfies

$$\langle X',X'\rangle_{Q_\alpha}=e^{-2\alpha(T+1)/n}\eta^2(r)(h^2+\langle X,X\rangle)\le 1.$$

Here $\langle\cdot,\cdot\rangle_{Q_\alpha}$ is the inner product of Q_α.

Let dvol_α and dvol_N be the volume forms of Q_α and N, respectively. They are related as follows:

$$\mathrm{dvol}_\alpha=e^{\alpha(n+1)r/n}\,\mathrm{dvol}_N\,\mathrm{dr}. \tag{4.3}$$

We denote the subgraph of $u(x)$ by U. Note that X' has compact support in $\Omega\times\mathbb{R}$. Thus, the definition of the perimeter implies

$$\mathrm{P}_\alpha(U,\Omega\times\mathbb{R})\ge\int_{\Omega\times\mathbb{R}}\lambda_U\,\mathrm{div}_\alpha(X')\,\mathrm{dvol}_\alpha. \tag{4.4}$$

From the definition of $\eta_1(r)$ and $\eta(r)$, we have

$$X'\llcorner\,\mathrm{dvol}_\alpha=e^{-\alpha(T+1)/n}\eta(r)\,e^{\alpha r}((-1)^n h(x)\,\mathrm{dvol}_N+X\llcorner\,\mathrm{dvol}_N\,\mathrm{dr}). \tag{4.5}$$

This yields that

$$\operatorname{div}_\alpha(X')\,\mathrm{dvol}_\alpha = \mathrm{d}(X' \llcorner \mathrm{dvol}_\alpha) = e^{-\alpha(T+1)/n}\,(\eta(r)\,e^{\alpha r})'\,h(x)\,\mathrm{dvol}_N\,\mathrm{dr} + e^{-\alpha(T+1)/n} e^{\alpha r}\,\eta(r)\,\operatorname{div}(X)\,\mathrm{dvol}_N\,\mathrm{dr}. \tag{4.6}$$

We observe that

$$\int_{-\infty}^{u(x)} (\eta(r)\,e^{\alpha r})'\,\mathrm{dr} = e^{\alpha u(x)}.$$

Therefore, we conclude

$$\int_{-\infty}^{u(x)} \eta(r) e^{\alpha r}\,\mathrm{dr} = \begin{cases} e^{\alpha u(x)}/\alpha + C, & \alpha > 0, \\ u(x) + C, & \alpha = 0, \end{cases}$$

where C is a fixed constant. Denoting

$$\mathrm{F}_{\mathrm{in}} = \int_{\Omega\times\mathbb{R}} \lambda_U \operatorname{div}_\alpha(X')\,\mathrm{dvol}_{\alpha},$$

we obtain

$$\mathrm{F}_{\mathrm{in}} = \begin{cases} \int_\Omega e^{-\alpha(T+1)/n}\Big(e^{\alpha u(x)} h(x) + \frac{e^{\alpha u(x)}}{\alpha}\operatorname{div}(X)\Big)\,\mathrm{dvol}_N, & \alpha > 0, \\ \int_\Omega (h(x) + u(x)\operatorname{div}(X))\,\mathrm{dvol}_N, & \alpha = 0. \end{cases}$$

Combining (4.4) with (4.2), we conclude that

$$\mathrm{P}_\alpha(U, \Omega\times\mathbb{R}) \geq \sup_{h,X \text{ satisfying (4.2)}} \mathrm{F}_{\mathrm{in}} = \mathfrak{F}_\alpha(u,\Omega), \qquad \alpha \geq 0,$$

and $\mathrm{P}(U, \Omega\times\mathbb{R}) \geq \mathfrak{F}(u,\Omega)$ for $|u| \leq T$. By Lemma 4.3, we conclude Theorem 4.2 in the case that u is uniformly bounded .

Next we show the general conclusion when $u(x)$ is not bounded. For any $T > 1$, define

$$u_T(x) := \max\{\min(u(x), T), -T\}.$$

Since $u \in L^1(\Omega)$ and Ω is bounded, u_T converges to $u(x)$ in $L^1(\Omega)$ as $T \to +\infty$. Let U_T be the subgraph of u_T. By Proposition 2.8 in [14], there are two important facts about these subgraphs. By Lemma 2.11, the variation of λ_{U_T} in Q and Q_α take the following forms:

$$|\mathrm{D}\lambda_{U_T}|_Q(\Omega\times\{\pm T\}) = \int_{\Omega_T} \mathrm{dvol}_N, \tag{4.7}$$

$$|\mathrm{D}\lambda_{U_T}|_{Q_\alpha}(\Omega\times\{\pm T\}) = \int_{\Omega_T} e^{\alpha u(x)}\,\mathrm{dvol}_N. \tag{4.8}$$

Here Ω_T is the set $\{x \in \Omega : |u(x)| \geq T\}$. In the case of Q, the fact that $u \in L^1(\Omega)$ and Ω is bounded implies that $\mathrm{vol}(\Omega_T)$ converges to 0 as $T \to +\infty$. A similar derivation yields

$$\lim_{T\to\infty} \int_{\Omega_T} e^{\alpha u(x)}\,\mathrm{dvol}_N = 0$$

because $e^{\alpha u(x)}$ belongs to $L^1(\Omega)$. Notice that

$$\mathrm{P}_\alpha(U, \Omega \times (-T, T)) = \mathrm{P}_\alpha(U_T, \Omega \times (-T, T)).$$

By (4.8), the decomposition of $\mathrm{P}_\alpha(U_T, \Omega \times \mathbb{R})$ implies that

$$\lim_{T\to\infty} \mathrm{P}_\alpha(U_T, \Omega \times \mathbb{R}) = \mathrm{P}_\alpha(U, \Omega \times \mathbb{R}).$$

Thus, applying the first case, we arrive at

$$\mathrm{P}_\alpha(U, \Omega \times \mathbb{R}) = \lim_{T\to+\infty} \mathrm{P}_\alpha(U_T, \Omega \times \mathbb{R}) = \lim_{T\to+\infty} \mathfrak{F}_\alpha(u_T, \Omega) \geq \mathfrak{F}_\alpha(u, \Omega).$$

By Lemma 4.3, we conclude

$$\mathrm{P}_\alpha(U, \Omega \times \mathbb{R}) = \mathfrak{F}_\alpha(u, \Omega)$$

whenever $e^{\alpha u(x)}$ is in $\mathrm{BV}(\Omega)$. This is the conclusion (2).

The proof of the conclusion (1) also follows a similar derivation via (4.7). Our proof is complete. ■

As an application, a comparison result between $\mathfrak{F}_\alpha(u, \Omega)$ and $|\mathrm{D}u|(\Omega)$ is stated as follows.

Corollary 4.4. *Suppose u is a measurable function on Ω with $|u| \leq T$ such that $\mathfrak{F}_\alpha(u, \Omega)$ is finite. Then*

$$\mathfrak{F}_\alpha(u, \Omega) \geq e^{-\alpha T} \max\{\mathrm{vol}(\Omega), |\mathrm{D}u|(\Omega)\}.$$

Proof. Let U denote the subgraph of u in $\Omega \times \mathbb{R}$. By Theorem 2.13 and Theorem 4.2, we have

$$\begin{aligned}\mathfrak{F}_\alpha(u, \Omega) = \mathrm{P}_\alpha(U, \Omega \times \mathbb{R}) &= \int_{\Omega\times\mathbb{R}} e^{\alpha r}\, \mathrm{d}|\mathrm{D}\lambda_U|_Q \\ &\geq e^{-\alpha T}\mathrm{P}(U, \Omega \times \mathbb{R}) = e^{-\alpha T}\mathfrak{F}(u, \Omega).\end{aligned}$$

For the inequality, we use the fact that the support of the Radon measure $|\mathrm{D}\lambda_U|_Q$ is contained in the graph of u. The conclusion follows from Theorem 3.4 (3). ■

4.3. The minimizing perimeter property

In this subsection, we will show the second fact of Miranda's observation. According to Giusti [13], Miranda [23] first observes this phenomenon in the case of product manifolds. Now we generalize it into the case of Q_α.

The following result is similar to Theorem 14.8 in [14].

Lemma 4.5. *Let Ω be an open domain in N. Let $F \subset Q_\alpha$ be a Caccioppoli set satisfying for a.e. x in Ω, $\lambda_F(x, t) = 0$ for all $t > T_x$ and $\lambda_F(x, t) = 1$ for all $t < -T_x$, where T_x is a positive constant depending on x. Then the function $\omega(x)$ satisfying*

$$e^{\alpha\omega(x)} = \alpha \lim_{k\to+\infty} \Big(\int_{-k}^{k} e^{\alpha t} \lambda_F(x, t)\, \mathrm{d}t \Big)$$

is well defined and

$$\mathfrak{F}_\alpha(w,\Omega) \le \mathrm{P}_\alpha(F, \Omega \times \mathbb{R}).$$

Here P_α *denotes the perimeter of* Q_α*.*

Proof. By the assumption, $w(x)$ is well defined a.e. on Ω.

Suppose $h \in C_0(\Omega)$ and $X \in T_0(\Omega)$ satisfy

$$h^2 + \langle X, X\rangle \le 1, \tag{4.9}$$

where $\langle\cdot,\cdot\rangle$ is the inner product in N.

Let $\eta(r)$ be a smooth function such that $0 \le \eta(r) \le 1$ with compact support in $\mathbb{R}$. Set $X' = e^{-\alpha r/n}\eta(r)(X + h(r)\partial_r)$. Then $\langle X', X'\rangle_\alpha \le 1$, where $\langle\cdot,\cdot\rangle_\alpha$ denotes the inner product of Q_α. By Definition 2.1, we have

$$\mathrm{P}_\alpha(F, \Omega\times\mathbb{R}) \ge \int_{\Omega\times\mathbb{R}} \lambda_F(x,r)\,\mathrm{div}_\alpha(X')\,\mathrm{dvol}_\alpha, \tag{4.10}$$

where div_α and dvol_α are the divergence and the volume form of Q_α, respectively. Arguing as in (4.3), (4.4), (4.5) and (4.6), we obtain

$$\mathrm{div}_\alpha(X')\,\mathrm{dvol}_\alpha = \left\{(e^{\alpha r}\eta(r))'h(x) + e^{\alpha r}\eta(r)\,\mathrm{div}_N(X)\right\}\mathrm{dvol}_N\,\mathrm{dr},$$

where div_N is the divergence of Ω. Thus, expanding (4.10) gives

$$\begin{aligned}\mathrm{P}_\alpha(F, \Omega\times\mathbb{R}) \ge &\int_\Omega h(x)\left\{\int_{-\infty}^{\infty}(e^{\alpha r}\eta(r))'\lambda_F(x,r)\,\mathrm{dr}\right\}\mathrm{dvol}_N\\ &+\int_\Omega \mathrm{div}(X)\left\{\int_{-\infty}^{\infty} e^{\alpha r}\eta(r)\lambda_F(x,r)\,\mathrm{dr}\right\}\mathrm{dvol}_N.\end{aligned}$$

Replacing $\eta(t)$ with a sequence

$$\left\{\eta_k(t) : 0 \le \eta_k \le 1 \text{ with compact support}\right\}_{k=1}^{\infty}$$

which converges locally uniformly to the constant function 1 on $\mathbb{R}$ as $k \to +\infty$, we obtain

$$\begin{aligned}\mathrm{P}_\alpha(F, \Omega\times\mathbb{R}) \ge &\int_\Omega h(x)\left\{\int_{-\infty}^{\infty}\alpha e^{\alpha r}\lambda_F(x,r)\,\mathrm{dr}\right\}\mathrm{dvol}_N\\ &+\int_\Omega \mathrm{div}(X)\left\{\int_{-\infty}^{\infty} e^{\alpha r}\lambda_F(x,r)\,\mathrm{dr}\right\}\mathrm{dvol}_N\\ =&\int_\Omega e^{\alpha\omega(x)}\Big(h(x) + \frac{1}{\alpha}\,\mathrm{div}(X)\Big)\,\mathrm{dvol}_N.\end{aligned}$$

Here we use the condition of λ_F. Now taking the supremum of all h and X satisfying (4.9) and applying the definition of $\mathfrak{F}_\alpha(\cdot,\Omega)$, one sees that

$$\mathrm{P}_\alpha(F, \Omega\times\mathbb{R}) \ge \mathfrak{F}_\alpha(\omega,\Omega).$$

The proof is complete. ∎

Definition 4.6. Let I be any fixed open interval in $\mathbb{R}$ and let $\Omega \subset\subset \mathcal{B}$ be two bounded Lipschitz domains in N. We say that a Caccioppoli set E in $\mathcal{B} \times I$ has the least perimeter in $\bar{\Omega} \times I$ if for any Caccioppoli F subject to $F\Delta E \subset\subset \bar{\Omega} \times I$, i.e., there is a closed interval $[a,b] \subset\subset I$ such that $F\Delta E \subset \bar{\Omega} \times [a,b]$, we have that

$$\mathrm{P}_\alpha(E, \mathcal{B} \times I) \leq \mathrm{P}_\alpha(F, \mathcal{B} \times I).$$

Here P_α is the perimeter of Q_α.

Now we can conclude the second conclusion of Miranda's observation in Q_α mentioned in the introduction as follows. For the case $\alpha = 0$, see Lemma 14.7 in [14].

Theorem 4.7. *Let $\Omega \subset\subset \mathcal{B}$ be two open domains and $\alpha > 0$ a fixed positive constant. Let I be a bounded open interval. Let u be a measurable function on $\mathcal{B}$ satisfying $u(x) \in I$ for each $x \in \mathcal{B}$. Suppose $\mathfrak{F}_\alpha(u, \mathcal{B})$ is finite, satisfying*

$$\mathfrak{F}_\alpha(u, \mathcal{B}) \leq \mathfrak{F}_\alpha(v, \mathcal{B}) \tag{4.11}$$

whenever their subgraphs U and V have the relation $U\Delta V \subset\subset \bar{\Omega} \times I$. Then U has the least perimeter in $\bar{\Omega} \times I$.

Proof. Let F be a Caccioppoli set satisfying $F\Delta U \subset\subset \Omega \times I$. Since U is a subgraph of $u(x)$, F should satisfy the condition in Lemma 4.5. Let $w(x)$ be the function defined as in Lemma 4.5. Thus, $w(x)$ is contained in I for any x in $\mathcal{B}$. Let W be the subgraph graph of w.

Due to the definition of F, we have $W\Delta U \subset\subset \bar{\Omega} \times I$. By (4.11), we conclude that

$$\mathrm{P}_\alpha(U, \mathcal{B} \times \mathbb{R}) = \mathfrak{F}_\alpha(u, \mathcal{B}) \leq \mathfrak{F}_\alpha(w, \mathcal{B}) \leq \mathrm{P}_\alpha(F, \mathcal{B} \times \mathbb{R}).$$

Since $F\Delta U \subset\subset \bar{\Omega} \times I$, we obtain

$$\mathrm{P}_\alpha(U, \mathcal{B} \times I) \leq \mathrm{P}_\alpha(F, \mathcal{B} \times I).$$

By Definition 4.6, U has the least perimeter in $\bar{\Omega} \times I$.

This proof is complete. ∎

5. Existence of the generalized solutions

In this section, we propose a generalized solution to the Dirichlet problem of the TMCE in (1.2). Our idea is to extend the Miranda–Giusti generalized solution theory in [14] into the translating mean curvature equation case.

Throughout this section let (N, σ) be a Riemannian manifold and let Q_α be the conformal product manifold $N \times \mathbb{R}$ with the metric $e^{2\alpha r/n}(\sigma + \mathrm{d}r^2)$. Here n is the dimension of N. The definition of the generalized solution in our setting is given as follows.

Definition 5.1. Suppose Ω and $\mathcal{B}$ are two bounded Lipschitz domains in N with $\Omega \subset\subset \mathcal{B}$. Let u and ψ be two functions which may take infinity values, and let U and ∂si be their subgraphs over $\mathcal{B}$, respectively. Suppose ∂si is a Caccioppoli set in Q_α.

We say that u is a generalized solution to the Dirichlet problem of the TMCE in (1.2) with boundary data ψ if U coincides with ∂si outside $\Omega \times \mathbb{R}$ and U has the least perimeter in $\bar{\Omega} \times \mathbb{R}$.

Remark 5.2. For the term "the least perimeter", see Definition 4.6. In the following, we always call the above $u(x)$ a generalized solution for short if there is no confusion.

The main result of this section is stated as follows.

Theorem 5.3. *Let $\Omega \subset\subset \mathcal{B}$ be two bounded, Lipschitz open domain in N. Let ψ be any measurable function which may take infinity values in $\mathcal{B}$ such that its subgraph ∂si is a Caccioppoli set in $\mathcal{B} \times \mathbb{R} \subset Q_\alpha$. Then there is a generalized solution to the Dirichlet problem* (1.2) *with the boundary data ψ.*

Remark 5.4. The restriction on $\psi(x)$ is not restrictive.

Suppose $\psi(x) \in L^1(\partial\Omega)$ and $\partial\Omega$ is Lipschitz. With a similar derivation as in Proposition 2.15 of [14], there is a function, still denoted by $\psi(x)$, in $\mathrm{BV}(\mathcal{B})$ such that its trace on $\partial\Omega$ from Ω and $\mathcal{B} \setminus \bar{\Omega}$ are $\psi(x)$. Thus, its subgraph ∂si is a Caccioppoli set in Q_α.

Another interesting fact is that if $\psi(x)$ is continuous in $\partial\Omega$, the construction in Proposition 2.15 of [14] implies that $\psi(x)$ is continuous in $\mathcal{B} \setminus \Omega$.

Note that the proof for the above facts in the case of Riemannian manifolds does not give essential differences since all of them hold in the local sense.

Proof of Theorem 5.3. For any $k > 0$, we set

$$\psi_k(x) := \min\{k, \max\{\psi(x), -k\}\}.$$

Let ∂si_k be the subgraph of ψ_k. Let P_α denote the perimeter of Q_α.

From our definition, it is easy to see that

$$\mathrm{P}_\alpha(\partial si_k, \mathcal{B} \times (-k,k)) = \mathrm{P}_\alpha(\partial si, \mathcal{B} \times (-k,k)) < \infty$$

for any $k > 0$. By Proposition 2.8 in [14], we have the following estimate:

$$|\mathrm{D}\lambda_{\partial si_k}|_{Q_\alpha}(\mathcal{B} \times \{\pm k\}) \le e^{\alpha k}\mathrm{vol}_N(\mathcal{B}).$$

Here vol_N is the volume of N. Let λ_E denote the characteristic function for any Borel set E. Because $\lambda_{\partial si_k}$ is a constant outside $\mathcal{B} \times [-k,k]$,

$$\mathrm{P}_\alpha(\partial si_k, \mathcal{B} \times \mathbb{R}) = |\mathrm{D}\lambda_{\partial si_k}|_{Q_\alpha}(\mathcal{B} \times \{\pm k\}) + \mathrm{P}_\alpha(\partial si_k, \mathcal{B} \times (-k,k)) < \infty$$

is finite.

By Theorem 4.2 (2), this implies that $\mathfrak{F}_\alpha(\psi_k, \mathcal{B})$ is finite. By Corollary 4.4, the fact that $|\psi_k(x)| \le k$ yields $\psi_k \in \mathrm{BV}(\mathfrak{B})$.

We consider the following minimizing problems for each positive integer k:

$$\alpha_k := \min\big\{\mathfrak{F}_\alpha(u, \mathfrak{B}) : u \in \mathrm{BV}(\mathcal{B}), |u| \le k, u = \psi_k \text{ on } \mathfrak{B} \setminus \bar{\Omega}\big\}. \tag{5.1}$$

By Corollary 4.4, α_k is finite for each k. Let $\{u_{j,k}\}_{j=1}^\infty$ be the sequence in $\mathrm{BV}(\Omega)$ satisfying $|u_j| \le k$, $u = \psi_k$ on $\mathfrak{B} \setminus \Omega$ such that

$$\lim_{j\to+\infty} \mathfrak{F}_\alpha(u_{j,k}, \mathfrak{B}) = \alpha_k.$$

Again by Corollary 4.4, we have

$$\max\{|\mathrm{D}u_{j,k}|(\mathfrak{B}) : j = 1, \ldots, \infty\} \leq C(k, \alpha_k).$$

By the compactness of BV functions, there is a subsequence of $\{u_{j,k}\}_{j=1}^{\infty}$, still denoted $\{u_{j,k}\}_{j=1}^{\infty}$, such that $u_{j,k} \to u_k$ in $L^1(\mathfrak{B})$ as $j \to +\infty$. Note that $\{e^{\alpha u_{j,k}}\}_{j=1}^{\infty}$ also converges to $e^{\alpha u_k}$ in $L^1(\mathcal{B})$. By the semicontinuous property of $\mathfrak{F}_\alpha(u, \mathfrak{B})$, we have

$$\alpha_k \leq \mathfrak{F}_\alpha(u_k, \mathfrak{B}) \leq \lim_{j\to\infty} \inf \mathfrak{F}_\alpha(u_j, \mathfrak{B}) = \alpha_k,$$

with the property

$$|u_k| \leq k, \quad u_k = \psi_k \quad \text{on } \mathfrak{B} \setminus \Omega.$$

Then we conclude that for each positive integer k,

$$\mathfrak{F}_\alpha(u_k, \mathfrak{B}) = \alpha_k.$$

Now let U_k be the subgraph of u_k in $\mathfrak{B} \times \mathbb{R} \subset Q_\alpha$ for each k. Combining (5.1) with Theorem 4.7, we conclude that U_k has the least perimeter in $\bar{\Omega} \times (-k, k)$.

Fix any $T > 0$. Let E_k be the set $\partial si_k \cup \Omega \times (-T, T)$. Thus, $E_k \Delta U_k \subset\subset \bar{\Omega} \times (-k, k)$ whenever $T < k$. Consequently,

$$\begin{aligned}
\mathrm{P}_\alpha(U_k, \mathcal{B} \times (-T, T)) &\leq \mathrm{P}_\alpha(U_k, \mathcal{B} \times (-k, k)) \\
&\leq \mathrm{P}_\alpha(E_k, \mathcal{B} \times (-T, T)) + \mathrm{vol}_\alpha(\partial(\Omega \times (-T, T))) \\
&= \mathrm{P}_\alpha(\partial hi, \mathcal{B} \times (-T, T)) + \mathrm{vol}_\alpha(\partial(\Omega \times (-T, T))).
\end{aligned}$$

Here vol_α is the volume of Q_α.

Notice that the last line above is a positive constant only depending on T. Thus, the perimeter of the sequence $\{U_k\}_{k=1}^{\infty}$ is uniformly bounded on each open $\mathcal{B} \times (-T, T)$ for each $T > 0$. Arguing as in Lemma 16.3 of [14], we can extract a subsequence, still denoted by u_k, converging almost everywhere to a measurable function $u(x)$ in $\mathcal{B}$. Note that $u(x)$ may take the infinity value. Let U be the subgraph of $u(x)$. It is clear that U coincides with ∂si, the subgraph of $\psi(x)$ outside $\bar{\Omega} \times \mathbb{R}$.

Fix any $k_0 > 0$. By the definition of u_k and ψ_k, for any $k > k_0$, U_k coincides with U in $(\mathcal{B} \setminus \bar{\Omega}) \times (-k_0, k_0)$, which is ∂si. Thus, the condition (b) of Lemma 5.5 below is satisfied. Moreover, the condition (a) and (c) are obviously satisfied. By Lemma 5.5, U has the least perimeter in $\bar{\Omega} \times (-k_0, k_0)$. Because we chose arbitrary $k_0 > 0$, U has the least perimeter in $\bar{\Omega} \times \mathbb{R}$.

It follows that $u(x)$ is the desirable generalized solution of the Dirichlet problem (1.2) subject to the boundary data $\psi(x)$. The proof is complete. ∎

We say that a sequence of Borel sets $\{U_k\}_{k=1}^{\infty}$ converges locally to U as $k \to +\infty$ in an open set Ω if $\lim_{k\to+\infty} \int_A |\lambda_{U_k} - \lambda_U| \,\mathrm{dvol} = 0$ for any compact set A in Ω. Here λ denotes the characteristic function.

Lemma 5.5. *Let $\Omega, \mathcal{B}$ be two bounded domains with the property $\Omega \subset\subset \mathcal{B}$. Let I be any bounded open interval. Suppose the following hold*:

(a) $\{u_k(x)\}_{k=1}^{\infty}$ *is a sequence of measurable functions on* $\mathcal{B}$ *which takes possible infinity values such that their subgraphs* $\{U_k\}_{k=1}^{\infty}$ *converge locally to* U *in* $\mathcal{B}\times I$.

(b) *In* Q_α, $\lim_{k\to+\infty} \mathrm{P}_\alpha(U_k,(\mathcal{B}\setminus\bar{\Omega})\times I) = \mathrm{P}_\alpha(U,(\mathcal{B}\setminus\bar{\Omega})\times I)$.

(c) *For each* $k>0$, U_k *has the least perimeter with respect to the variation in* $\bar{\Omega}\times I\subset Q_\alpha$.

Then U *has the least perimeter in* $\bar{\Omega}\times I\subset Q_\alpha$.

Proof. From conditions (a) and (b), Theorem 2.11 in [14] implies that for any compact set A in $\partial\Omega\times\mathbb{R}$,

$$\lim_{k\to+\infty}\int_A |\mathcal{T}_1 U_k - \mathcal{T}_1 U|\, d\mathcal{H}^n = 0. \tag{5.2}$$

Here $\mathcal{H}^n$ is the n-dimensional Hausdorff measure in Q_α, and $\mathcal{T}_1$ is the trace on $\partial\Omega\times\mathbb{R}$ from $(\mathcal{B}\setminus\bar{\Omega})\times\mathbb{R}$.

We write I for (a,b). Suppose F is a Caccioppoli set satisfying $F\Delta U\subset\subset\bar{\Omega}\times(a,b)$. Then there is a sufficiently small $\varepsilon_0>0$ such that $F\Delta U\subset\subset\bar{\Omega}\times(a+\varepsilon_0,b-\varepsilon_0)$,

$$|\mathrm{D}\lambda_{U_k}|_{Q_\alpha}(\mathcal{B}\times\{\kappa\}) = 0,\quad |\mathrm{D}\lambda_U|_{Q_\alpha}(\mathcal{B}\times\{\kappa\}) = 0 \tag{5.3}$$

and (by condition (a))

$$\lim_{k\to+\infty}\int_{\Omega\times\{\kappa\}} |\mathcal{T}_2 U_k - \mathcal{T}_2 U|\, d\mathcal{H}^n = 0, \tag{5.4}$$

where $\mathcal{T}_2$ is the trace on $\Omega\times\{\kappa\}$, κ is equal to $a+\varepsilon_0$ or $b-\varepsilon_0$. By (5.3), there is no difference on the trace on $\Omega\times\{\kappa\}$ from its upper side or its down side. Moreover, with (5.3) the condition (b) gives that

$$\lim_{k\to+\infty} \mathrm{P}_\alpha(U_k,(\mathcal{B}\setminus\bar{\Omega})\times(a+\varepsilon_0,b-\varepsilon_0)) = \mathrm{P}_\alpha(U,(\mathcal{B}\setminus\bar{\Omega})\times(a+\varepsilon_0,b-\varepsilon_0)). \tag{5.5}$$

For a proof, see Proposition 1.13 in [14]. Now define F_k as

$$F_k = \begin{cases} F & \text{in } \Omega\times(a+\varepsilon_0,b-\varepsilon_0), \\ U_k & \text{outside } \Omega\times(a+\varepsilon_0,b-\varepsilon_0). \end{cases} \tag{5.6}$$

The above definition implies that $F_k\Delta U_k\subset\subset\bar{\Omega}\times I$. By condition (c), we have

$$\mathrm{P}_\alpha(U_k,\mathcal{B}\times I)\le \mathrm{P}_\alpha(F_k,\mathcal{B}\times I).$$

Let $\mathcal{T}_3$ be the trace on $\partial\Omega\times I$ from $\Omega\times I$. By Lemma 2.11 and (5.3), we have

$$\begin{aligned}
&\mathrm{P}_\alpha(U_k,\mathcal{B}\times(a+\varepsilon_0,b-\varepsilon_0))\\
&\quad\le \mathrm{P}_\alpha(F,\Omega\times(a+\varepsilon_0,b-\varepsilon_0)) + \mathrm{P}_\alpha(U_k,(\mathcal{B}\setminus\bar{\Omega})\times(a-\varepsilon_0,b+\varepsilon_0))\\
&\qquad + \int_{\partial\Omega\times(a+\varepsilon_0,b-\varepsilon_0)} |\mathcal{T}_1\lambda_{U_k} - \mathcal{T}_3\lambda_F|\, d\mathcal{H}^n\\
&\qquad + \sum_{\kappa=a+\varepsilon_0,b-\varepsilon_0}\int_{\Omega\times\{\kappa\}} |\mathcal{T}_2\lambda_{U_k} - \mathcal{T}_2\lambda_U|\, d\mathcal{H}^n.
\end{aligned}$$

We consider the limit as $k \to \infty$. By (5.4), the limit of the fourth term above is 0. By (5.2), the limit of the third term is

$$\int_{\partial\Omega\times(a+\varepsilon_0,b-\varepsilon_0)} |\mathcal{T}_1\lambda_U - \mathcal{T}_3\lambda_F|\, d\mathcal{H}^n.$$

With the above conclusions, applying (5.5) and (5.6), the semicontinuous property gives that

$$\begin{aligned} \mathrm{P}_\alpha(U, \mathcal{B}\times(a+\varepsilon_0, b-\varepsilon_0)) \le\ & \mathrm{P}_\alpha(F, \Omega\times(a+\varepsilon_0, b-\varepsilon_0)) \\ & + \mathrm{P}_\alpha(U, (\mathcal{B}\setminus\bar{\Omega})\times(a-\varepsilon_0, b+\varepsilon_0)) \\ & + \int_{\partial\Omega\times(a+\varepsilon_0,b-\varepsilon_0)} |\mathcal{T}_1\lambda_U - \mathcal{T}_3\lambda_F|\, d\mathcal{H}^n \\ =\ & \mathrm{P}_\alpha(F, \mathcal{B}\times(a+\varepsilon_0, b-\varepsilon_0)). \end{aligned}$$

Because $F\Delta U \subset\subset \bar{\Omega}\times(a+\varepsilon_0, b-\varepsilon_0)$, we have

$$\mathrm{P}_\alpha(U, \mathcal{B}\times I) \le \mathrm{P}_\alpha(F, \mathcal{B}\times I).$$

Recall that we choose F arbitrarily satisfying $F\Delta U \subset\subset \bar{\Omega}\times I$. Thus, U has the least perimeter in $\bar{\Omega}\times I$. The proof is complete. ■

6. Regularity of the infinity sets

After obtaining the existence of a generalized solution in (1.2), it is necessary to describe the regularity of sets that the generalized solution takes infinity values. Such regularity is a preliminary to deduce the NCM condition under which those generalized solutions give the classical solutions as those in [13, 14] in the next section.

6.1. Almost minimal set

We shall recall some basic facts on almost minimal sets for later use which generalize the concept of minimal sets. We shall see both of them share many important regularity properties. The papers of Duzzar–Steffen [6], Tamanini [27] and the book of Maggi [22] are our main references. Although their results are discussed in the Euclidean space, most of their results hold in Riemannian manifolds without any essential modification of their proofs. For example, see [26].

Definition 6.1. Let W be an open set in an $(n+1)$-dimensional Riemannian manifold M. Suppose the injectivity radius of W in M is positive, written as inj_W. Let E be a Caccioppoli set in W. We say E is a (c, β)-almost minimal set in W if

$$\mathrm{P}(E, B_\rho(x)) \le \mathrm{P}(F, B_\rho(x)) + c\rho^{n+2\beta}$$

for every point x in any compact set $A \subset W$, any Caccioppoli set $F\Delta E \subset\subset B_\rho(x)$ and any $\rho < \min\{\mathrm{inj}_W, \mathrm{dist}(x, M\setminus W)\}$. Here $\beta \in (0, 1/2]$ is a given constant, c is a positive constant depending on W and P is the perimeter of M.

The boundary ∂E (see Proposition 4.1) is called almost minimal boundary. If $c = 0$, ∂E is called the minimal boundary and E is a minimal set.

Remark 6.2. We always take the convention in Proposition 4.1. That is, for any Caccioppoli set E and all $x \in \partial E$, we have $0 < \mathcal{H}^n(E \cap B_\rho(x)) < \mathrm{vol}(B_\rho(x))$ for any sufficiently small ρ such that $B_\rho(x)$ is an embedded ball.

One important example of almost minimal boundaries is the boundary of smooth domains in their sufficiently small neighborhood. Our proof is essentially due to Example A.1 in Appendix A of [7] by Eichmair, and applies the fact that a C^2 boundary has locally bounded mean curvature.

Lemma 6.3. *Let Ω be an open set in a Riemannian manifold M. Suppose $\Gamma \subset \partial\Omega$ is a C^2 connected hypersurface in M. For each point x_0 in Γ, there exists an open set W near x_0 such that Ω is a $(c, 1/2)$-almost minimal set in W. Here c is a positive constant determined by x and Γ.*

Proof. Our proof is exactly the same as that of Example A.1 in [7] by Eichmair, based on the following two reasons. First, since Γ is C^2, we can construct a C^2 foliation near Γ and Γ is one of its slices. This gives a vector field with a bounded divergence. Second, we notice that

$$\mathbb{M}_W(\partial[[U]]) = \mathrm{P}(U, W),$$

where U is any Caccioppoli set and W is any open set, and $\mathbb{M}$ denotes the mass of an integral current $\partial[[U]]$ induced by U. ∎

Next we define the regular set of the boundary of a Caccioppoli set.

Definition 6.4. Suppose F is a Caccioppoli set in a Riemannian manifold G. Define the regular set

$$\mathrm{reg}(\partial F) := \{x \in \partial F : \exists \rho > 0 \text{ such that } \partial F \llcorner B_\rho(x) \text{ is a } C^{1,\beta} \text{ graph for some } \beta \in (0,1)\}.$$

The singular set of ∂F is its complement in ∂F, written as $\mathrm{sing}(\partial F)$.

The following two facts about almost minimal boundaries are standard.

Theorem 6.5. *Fix $n \geq 2$. Let F be a (c, β)-almost minimal set in an open domain Ω with dimension $n + 1$.*

(1) (*See Theorem* 1 *in* [27], *Theorem* 5.6 *in* [6], *and Theorem* 28.1 *in* [22]). *The Hausdorff dimension of* $\mathrm{sing}(\partial F)$ *is at most $n - 7$. In the case $n = 7$,* $\mathrm{sing}(\partial F)$ *consists of isolated points.*

(2) *For any compact set $K \subset F$, there exists $r_0 := r_0(K) > 0$ such that for all $r \in (0, r_0)$, we have $B_{r_0}(x) \subset \Omega$ and*

$$\mathcal{H}^{n+1}(F \cap B_r(x)) \geq C r^{n+1} \quad \textit{for all } x \in K \cap \partial F,$$

where C is a positive constant only depending on r_0 and the metric g on K. Here $\mathcal{H}^{n+1}$ is the $(n+1)$-dimensional Hausdorff measure in Ω.

The proof of the conclusion (2) above is exactly the same as that of Proposition 5.14 in [14] if we take r_0 as small as possible. Thus, we skip the details here.

6.2. The property of the infinity sets

Recall that the mean curvature of a smooth boundary is defined as follows.

Definition 6.6. Let W be an C^2 open domain in a Riemannian manifold M. Let $\vec{v}$ be the outward normal vector of ∂W. The mean curvature of ∂W, $\mathrm{H}_{\partial W}$, is $\mathrm{div}(\vec{v})$. If $\mathrm{H}_{\partial W} \geq 0$ we say that W is mean convex.

With the above convention, the mean curvature of the unit sphere in $\mathbb{R}^{n+1}$ with respect to the normal vector is n.

For a generalized solution, we define the infinity sets as follows.

Definition 6.7. Let $\Omega \subset\subset \mathfrak{B}$ be two bounded open Lipschitz domains in a Riemannian manifold N satisfying $\Omega \subset\subset \mathcal{B}$ and let $\psi(x)$ be a measurable function taking possible infinity values on $\mathcal{B}$ such that its subgraph ∂si is a Caccioppoli set in Q_α. Let $u(x)$ be a generalized solution to the Dirichlet problem (1.2) with boundary data $\psi(x)$. Define the infinity set P_+ and P_- in Ω as follows:

$$P_+ := \{x \in \Omega : u(x) = +\infty\},$$
$$P_- := \{x \in \Omega : u(x) = -\infty\}.$$

Under the mean convex condition, the infinity sets have the minimal property in Q_α as follows.

Theorem 6.8. *Fix $n \geq 2$. Suppose $\Omega, \mathcal{B}, \psi(x)$ and $u(x)$ are as given in Definition 6.7. Moreover, assume $\psi(x)$ is continuous on $\mathcal{B}$, Ω is an n-dimensional C^2 mean convex bounded domain. Then in the closure of Ω, P_+ and P_- are two Caccioppoli sets such that their boundaries are closed, embedded and minimal hypersurfaces with a singular set of which the Hausdorff dimension is at most $n-8$. In the case $n = 8$, such singular set is a collection of isolated points.*

Remark 6.9. The above result is similar to that in [13] when Giusti considered the existence of graphs with prescribed mean curvature function. This indicates that if $P_\pm$ is not empty, then Ω does not satisfy the NCM condition in Definition 1.1 (Definition 7.1). As a result, we shall expect the NCM assumption will exclude the existence of $P_\pm$ in the above setting.

Proof of Theorem 6.8. Since the proof of the conclusions about P_+ and P_- are the same, we only present the details for P_+.

Because Ω is C^2 bounded and $\psi(x) \in C(\partial\Omega)$, from Remark 5.4, we can assume that $\psi(x)$ is uniformly bounded on $\mathcal{B} \setminus \Omega$. For any $j \in \mathbb{R}$, define $u_j(x) = u(x) - j$ on $\mathcal{B}$ and let U_j denote the subgraph of $u_j(x)$ in $\mathcal{B} \times \mathbb{R}$. Next we define $u_\infty(x)$ on $\mathcal{B}$ as $u_\infty(x) = +\infty$ on P_+ and $u_\infty(x) = -\infty$ on $\mathcal{B} \setminus P_+$. It is easy to see that the subgraph of u_∞, written as U_∞, is the set $P_+ \times \mathbb{R}$.

Fix any bounded open interval I. Letting $j \to +\infty$, a direct computation yields that

(i) U_j locally converges to U_∞ in $\mathcal{B} \times \mathbb{R}$,

(ii) $\lim_{j\to+\infty} \mathrm{P}_\alpha(U_j, (\mathcal{B} \setminus \bar{\Omega}) \times I) = 0$ and $\mathrm{P}_\alpha(U_\infty, (\mathcal{B} \setminus \bar{\Omega}) \times I) = 0$.

For any fixed constant c, consider the map $T_c\colon Q_\alpha \to Q_\alpha$ given by

$$T_c(x,r) = (x, r+c).$$

We denote the metric $e^{2r\alpha/n}(\sigma + \mathrm{dr}^2)$ by g. Observe that

$$T_c^* g = e^{2\alpha c/n} g.$$

As a result, for any open set and any Caccioppoli set F,

$$e^{\alpha j}\mathrm{P}_\alpha(F, W) = \mathrm{P}_\alpha(T_j F, T_j W) \quad \text{for each } j. \tag{6.1}$$

Recall that the subgraph U of $u(x)$ has the least perimeter in $\bar{\Omega}\times\mathbb{R}$. Fix any bounded open interval I. By the fact that $T_j(U)$ coincides with U_j and from (6.1), a direct verification shows that

(iii) for sufficiently large j, U_j has the least perimeter in $\bar{\Omega}\times I$.

From conditions (i)–(iii), Lemma 5.5 gives that U_∞ has the least perimeter in $\bar{\Omega}\times I$. This implies that the set U_∞, $P_+\times\mathbb{R}$, has the least perimeter (with respect to the metric g) in $\bar{\Omega}\times\mathbb{R}$.

Since Ω is bounded, C^2 and mean convex, there is a positive constant κ such that for the mean curvature of $\partial\Omega$, we have $|\mathrm{H}_{\partial\Omega}| \le \kappa$ on $\partial\Omega$. By Lemma 3.3 in [31], we have

$$\mathrm{H}^\alpha_{\partial\Omega\times\mathbb{R}}(x,r) = e^{-\alpha r/n}\,\mathrm{H}_{\partial\Omega}(x).$$

Here H^α means the mean curvature of $\Omega\times\mathbb{R}$ in Q_α.

Fix any point p in $\partial\Omega\times\mathbb{R}$. By Lemma 6.3, there is a bounded neighborhood W_p of p such that $\Omega\times\mathbb{R}$ is a $(\mu, 1/2)$-almost minimal set in W_p. Here μ is a constant only depending on the mean curvature of $\partial\Omega\times\mathbb{R}$. Choose any open ball B_r in W_p with radius r and any Caccioppoli set F satisfying $F\Delta(P_+\times\mathbb{R}) \subset\subset B_r$. Then

$$\mathrm{P}_\alpha(\Omega\times\mathbb{R}, B_r) \le \mathrm{P}_\alpha(F\cup(\Omega\times\mathbb{R}), B_r) + \mu r^{n+1}. \tag{6.2}$$

Since $P_+\times\mathbb{R}$ has the least perimeter in $\bar{\Omega}\times\mathbb{R}$, we obtain that

$$\mathrm{P}_\alpha(P_+\times\mathbb{R}, B_r) \le \mathrm{P}_\alpha(F\cap(\Omega\times\mathbb{R}), B_r). \tag{6.3}$$

By Lemma 15.1 in [14], the above two inequalities give that

$$\mathrm{P}_\alpha(P_+\times\mathbb{R}, B_r) \le \mathrm{P}_\alpha(F, B_r) + \mu r^{n+1}. \tag{6.4}$$

As a result $P_+\times\mathbb{R}$ is a $(\mu, 1/2)$-almost minimal set in W_p for any p in $(\partial\Omega\cap\partial P_+)\times\mathbb{R}$.

Recall that $P_+\times\mathbb{R}$ is a minimal set in $\Omega\times\mathbb{R}$. Applying Theorem 6.5 (1), the singular set of $\partial P_+\times\mathbb{R}$, $\mathrm{sing}(\partial P_+\times\mathbb{R})$, has Hausdorff dimension at most $n-7$.

Moreover, in the case $n=7$, $\mathrm{sing}(\partial P_+\times\mathbb{R})$ should consist of isolated points. In this case, if $\mathrm{sing}(\partial P_+)$ contains a point p, then the line $\{p\}\times\mathbb{R}$ belongs to $\mathrm{sing}(\partial P_+\times\mathbb{R})$. This is impossible. Thus, for $n=7$, $\mathrm{sing}(\partial P_+\times\mathbb{R})$ is also an empty set.

Define the projection $\pi\colon N\times\mathbb{R}\to N$ as $\pi(x,r)=x$. We have

$$\pi(\mathrm{sing}(\partial P_+\times\mathbb{R})) = \mathrm{sing}(\partial P_+) \quad \text{and} \quad \pi(\mathrm{reg}(\partial P_+\times\mathbb{R})) = \mathrm{reg}(\partial P_+).$$

By the Fubini theorem, $\mathrm{sing}(\partial P_+)$ is empty for $n\le 7$ and has Hausdorff dimension at most $n-8$.

Now we discuss the case of $n = 8$. The Hausdorff dimension of $\operatorname{sing}(\partial P_+)$ is zero. We follows the proof of [22], Section 28.5. Suppose there is a sequence of points $\{x_i\}_{i=1}^{\infty}$ in $\operatorname{sing}(\partial P_+)$ that converges to $x \in \operatorname{sing}(\partial P_+)$. Fix any $p_x = (x, 0)$ in $\operatorname{sing}(\partial P_+ \times \mathbb{R})$. Near x, we can view ∂P_+ as an open set in $\mathbb{R}^8$, and thus $\partial P_+ \times (-\varepsilon, \varepsilon)$, for a small positive ε, is contained in $\mathbb{R}^9$. Let $r_j = d(x, x_j)$, where d is the distance of N and $E_j = \{(p - p_x)/r_j \in \partial P_+ \times \mathbb{R}\}$. It is well known that, as $j \to +\infty$, E_j will converge to a singular minimizing cone K in $\mathbb{R}^9$ up to possibly choosing a subsequence. Now write $\mathbb{R}^9 = \{(x, r) : x \in \mathbb{R}^8, r \in \mathbb{R}\}$. Let ∂_r be the vector field in $\mathbb{R}^9$ along the r-direction. In the regular part of E_j, we have $\langle \vec{v}, \partial_r \rangle = 0$ with respect to the Euclidean space. As a result, in the regular part of K we also have $\langle \vec{v}, \partial_r \rangle = 0$. Therefore, K is $K' \times \mathbb{R}$, where K' is a singular minimizing cone in $\mathbb{R}^8$. Moreover, $((x_j, 0) - (x, 0))/r_j$ converges to $(z, 0)$ in K' with $z \neq 0 \in K$. Thus, there is a line in K' in $\mathbb{R}^8$. By Theorem 28.11 in [22], there exists a singular minimizing cone in $\mathbb{R}^7$. This is a contradiction. Thus, for $n = 8$, $\operatorname{sing}(\partial P_+)$ consists of isolated points.

In summary, the singular set of ∂P_+ has Hausdorff dimension at most $n - 8$. In the case $n = 8$, it is a collection of isolated points.

The remainder of the proof is to show that the regular part of $\partial P_+ \times \mathbb{R}$ is minimal. Let $p \in \operatorname{reg}(\partial P_+ \times \mathbb{R})$. First, we assume $p = (x, r)$ is in $\Omega \times \mathbb{R}$. Thus, $\mathrm{H}^{\alpha}_{\partial P_+ \times \mathbb{R}} = 0$ near p, where H^{α} is the mean curvature in Q_α. On the other hand,

$$\mathrm{H}^{\alpha}_{\partial P_+ \times \mathbb{R}}(p) = e^{\alpha r} \mathrm{H}_{\partial P_+}(x). \tag{6.5}$$

Thus, the regular part of ∂P_+ in Ω is embedded and minimal.

Second, assume p is in $\partial \Omega \times \mathbb{R}$. Let S_a be the local scaling centering at p, that is, $S_a(z) = p + (z - p)/a$ for any z in some ball $B_r(p)$, $a > 0$. By Theorem 9.2 in [14], $S_a((P_+ \times \mathbb{R}) \cap B_r(p))$ will converge locally to a minimizing cone with vertex at p in $\mathbb{R}^{n+1}$ as $a \to 0$. On the other hand, $\partial \Omega \times \mathbb{R}$ is C^2 and $P_+ \times \mathbb{R}$ is contained in $\Omega \times \mathbb{R}$. Thus, such minimizing cone is contained a half space in $\mathbb{R}^{n+1}$. By Theorem 15.5 in [14], it is a half space. From the Allard regularity theorem, $\partial P_+ \times \mathbb{R}$ is a $C^{1,1/2}$ graph near p.

The least perimeter property of $P_+ \times \mathbb{R}$ implies that $\mathrm{H}^{\alpha}_{\partial P_+ \times \mathbb{R}} \leq 0$ with respect to the outward normal vector of $P_+ \times \mathbb{R}$ in the Lipschitz sense. By the mean convexity of Ω, $\mathrm{H}^{\alpha}_{\partial \Omega \times \mathbb{R}} \geq 0$ near p with respect to the outward normal vector of $\Omega \times \mathbb{R}$. Since $\partial P_+ \times \mathbb{R}$ is tangent to $\partial \Omega \times \mathbb{R}$ at p, the classical maximum principle (for example, see Appendix A in [10]) implies that $\mathrm{H}^{\alpha}_{\partial P_+ \times \mathbb{R}} \equiv 0$ near p. As a result, $\partial P_+ \times \mathbb{R}$ is smooth near p. By (6.5) we obtain that ∂P_+ is embedded and minimal near x in $\partial \Omega$. This indicates that the regular part of ∂P_+ is embedded and minimal. The closeness is obvious.

Thus, we obtain the conclusion. The proof is complete. ∎

7. Existence of classical solutions

In this section, we define the NCM assumption and obtain classical solutions from a generalized solution to the Dirichlet problem of (1.2) under the mean convex and this assumption.

Definition 7.1. Suppose Ω is an n-dimensional bounded Riemannian manifold with Lipschitz boundary. We say that Ω has the non-closed-minimal (NCM) property if its closure does not contain any Caccioppoli set E such that its (essential) boundary ∂E is a closed embedded minimal hypersurface with a singular set S of which the Hausdorff dimension is at most $n-8$. For $n=8$, we require that S is a collection of isolated points.

Recall that

$$\mathrm{H}_\alpha(u) := -\mathrm{div}\Big(\frac{\mathrm{D}u}{\sqrt{1+|\mathrm{D}u|^2}}\Big) + \frac{\alpha}{\sqrt{1+|\mathrm{D}u|^2}},$$

and that the main result of this paper is Theorem 1.2.

7.1. The interior regularity

First, we show the interior regularity of locally bounded generalized solutions to the Dirichlet problem in (1.2).

Theorem 7.2. *Fix $\alpha > 0$. Suppose u is a locally bounded generalized solution to the Dirichlet problem in (1.2) on Ω with continuous boundary data ψ in $\partial\Omega$. Then u is smooth on Ω satisfying $\mathrm{H}_\alpha(u) = 0$.*

Remark 7.3. Notice that we cannot apply Theorem 14.13 in [14] because the area functional $\mathfrak{F}_\alpha(u, \Omega)$ is not convex among BV functions. Note also that changing the value of $u(x)$ on a measure zero set does not change the property of the perimeter and generalized solutions. Here we choose a representative in the equivalent class of u (different with the value in a zero-measure set) which is smooth.

Proof. Notice that our conclusion is not affected by the boundary data. Without loss of generality, we assume that

$$|u(x)| \le \mu \quad \text{on } \Omega, \tag{7.1}$$

for some positive constant μ.

Let U be the subgraph of $u(x)$ in Q_α. By definition, U locally minimizes perimeter in $\Omega\times\mathbb{R} \subset Q_\alpha$. Let n be the dimension of Ω.

Let $\mathrm{sing}(\partial U)$ be the closed singular set of ∂U in $\Omega\times\mathbb{R}$. Then, by Theorem 6.5 (1), the Hausdorff dimension of $\mathrm{sing}(\partial U)$ is at most $n-7$. Thus, the regular part of ∂U is a connected, open smooth hypersurface in $\Omega\times\mathbb{R}$. Here the smoothness follows the regularity of minimal hypersurfaces.

Let S be the projection of $\mathrm{sing}(\partial U)$ into Ω. Set $\Omega_1 = \Omega\setminus S$. Let Σ_1 be the restriction of ∂U on $\Omega_1\times\mathbb{R}$. Then Σ_1 is minimal, embedded and therefore smooth.

Lemma 7.4. *Take the assumptions as above. Then $u(x)$ is in $C^\infty(\Omega_1)$ and Σ_1 is a smooth minimal graph over Ω_1 of $u(x)$ in Q_α.*

Proof. For the convenience of computations, we work in the product manifold Q with the metric $\sigma + \mathrm{dr}^2$ instead of the conformal product manifold Q_α. Let $\vec{v}$ be the upward normal vector of Σ_1 in Q. Since $u(x)$ locally minimizes the functional $\mathfrak{F}_\alpha(v, \Omega_1)$, arguing as in Lemma 2.2 of [31], on Σ_1, $\Theta = \langle\vec{v}, \partial_r\rangle \ge 0$ satisfies

$$\Delta\Theta + (|A|^2 + \bar{\mathrm{R}}\mathrm{ic}(\vec{v},\vec{v}))\Theta - \alpha\langle\nabla\Theta, \partial_r\rangle = 0.$$

Here $\bar{\text{R}}\text{ic}$ is the Ricci curvature of Q, Δ is the Laplacian operator on Σ_1, and ∇ is the covariant derivative of Σ_1. It is easy to see that Σ_1 is connected. By the Harnack principle, $\Theta \equiv 0$ or $\Theta > 0$ on the whole Σ_1.

Now assume $\Theta \equiv 0$ on Σ_1. Let $\pi\colon \Omega \times \mathbb{R} \to \mathbb{R}$ be the standard projection $\pi(x, r) = x$. Fix a point p on Σ_1 and let V be an open neighborhood of p. Then $\mathcal{H}^{n-1}(\pi(V)) > 0$. Because $\langle \vec{v}, \partial_r \rangle = 0$ on V, ∂_r is a vector field in the tangent bundle of V. Let $\gamma(t)$ be the parameterized curve of ∂_t passing through p. This is a part of the vertical line passing through $\{(\pi(p), r) : r \in \mathbb{R}\}$. Since u is bounded, Σ_1 is also bounded. Thus, this line cannot be contained completely in Σ_1. This means $\pi(p)$ should belong to S, the projection of $\text{sing}(\partial U)$. We choose p arbitrarily. Thus, $\pi(V) \subset S$. This means $\mathcal{H}^{n-1}(S) > 0$. This is a contradiction because the Hausdorff dimension of S is at most $n - 8$.

As a result, $\Theta > 0$ on the whole Σ_1. Since Σ_1 is smooth, $u(x)$ belongs to $C^\infty(\Omega_1)$. The proof is complete. ∎

Fix any x_0 in S. We choose $r_0 > 0$ sufficiently small such that Theorem B.1 holds for any r in $(0, r_0]$. Here r satisfies that $B_r(x_0) \subset \Omega$ is a mean convex embedded ball centered at x_0 with radius r.

Let $\mathcal{T}u$ be the trace of $u(x)$ in $\partial B_{r_0}(x_0)$ both from $B_{r_0}(x_0)$ and $\Omega \setminus B_{r_0}(x_0)$.

Notice that the closed set S in Ω satisfies that $H^t(S) = 0$ for any $t > n - 8$. Since $u(x)$ is smooth over $\Omega_1 = \Omega \setminus S$, $\mathcal{T}u$ is a C^1 function on $\partial B_{r_0}(x_0) \setminus S$. Let $\{S_i\}_{i=1}^\infty$ be a sequence of closed sets in $\partial B_{r_0}(x_0)$ such that $S_{i+1} \subset S_i$ for all $i = 1, \dots, \infty$ and $\bigcap_{i=1}^\infty S_i = S$. By (7.1), we construct a sequence of smooth functions $\psi_i(x)$ on $\partial B_{r_0}(x_0)$ such that

$$\psi_i = \mathcal{T}u \quad \text{in } \partial B_{r_0}(x_0) \setminus S_i, \qquad \sup_{\partial B_{r_0}(x_0)} |\psi_i| \le 2\mu.$$

By Theorem B.1, let $\{u_i\}_{i=1}^\infty$ be the solution of the Dirichlet problem (1.2) with boundary data $\{\psi_i\}_{i=1}^\infty$ on $\partial B_{r_0}(x_0)$ in $C^2(B_{r_0}(x_0)) \cap C(\bar{B}_{r_0}(x_0))$.

By Lemma B.2 and (7.1), there is a uniformly constant C such that

$$\max_{x \in K, i=1,2,\dots} |\text{D}u_i(x)| \le C$$

on any compact set K in $B_{r_0}(x_0)$. By the classical Schauder estimate, so is the second derivative of $u_i, i = 1, 2, \dots$. Then $\{u_i\}_{i=1}^\infty$ converges to a C^2 function v on $B_{r_0}(x_0)$ in the C^2 norm up to possibly a subsequence. Moreover, v satisfies $\text{H}_\alpha(v) = 0$ in $B_{r_0}(x_0)$. This convergence implies that v is a BV function on $B_{r_0}(x_0)$ and $\{u_j\}_{j=1}^\infty$ also converges to v in $B_{r_0}(x_0)$ in the sense of BV functions. By Theorem 2.11 in [14], their traces $\{\mathcal{T}(u_j)\}_{j=1}^\infty$ will converge to $\mathcal{T}(v)$ in $L^1(\partial B_{r_0}(x_0))$. That is,

$$\mathcal{T}(v) = \mathcal{T}u \quad \text{in } L^1(\partial B_{r_0}(x_0)). \tag{7.2}$$

Let $T_t\colon N \times \mathbb{R} \to N \times \mathbb{R}$ be the vertical translation

$$T_t(x, r) = (x, r + t).$$

Let $\vec{v}$ be the upward normal vector of the graph of v in $B_{r_0}(x_0)$ with respect to the metric in Q_α. Define a unit vector field

$$X(x, v(x) - t) = e^{t\alpha/n}\, T_t^*(\vec{v})$$

for any $x \in B_{r_0}(x_0)$ and $t \in \mathbb{R}$. Let div_α be the divergence of Q_α. Then

$$\mathrm{div}_\alpha(X) = 0.$$

Let W be the domain in $\Omega \times \mathbb{R}$ enclosed by Σ_u, Σ_v (the graphs of v) and $\partial B_{r_0}(x_0) \times \mathbb{R}$. By (7.2),

$$\partial W = \Sigma_u - \Sigma_v.$$

Applying the divergence theorem on W, we have

$$\begin{aligned} 0 = \int_W \mathrm{div}_\alpha(X)\, \mathrm{dvol}_\alpha &= \int_{\Sigma_u^r} \langle X, \vec{v}_u \rangle_\alpha \, \mathrm{d}\mathcal{H}^n - \int_{\Sigma_v} \mathrm{dvol}_\alpha \\ &\le \mathcal{H}^{n-1}(\Sigma_u^r) - \mathcal{H}^{n-1}(\Sigma_v). \end{aligned} \tag{7.3}$$

Here $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure in Q_α, vol_α is the volume form of Q_α, Σ_u^r is the restriction of the regular part of Σ_u in $\bar{B}_{r_0}(x_0) \times \mathbb{R}$ and $\vec{v}_u$ is the upper normal vector field of Σ_u in Q_α.

Define the function

$$\tilde{v} = \begin{cases} v & \text{on } B_{r_0}(x_0), \\ u & \text{otherwise.} \end{cases} \tag{7.4}$$

Let $\tilde{V}$ be the subgraph of $\tilde{v}$. By (7.4) and the fact that $u(x)$ is locally bounded in $\bar{B}_{r_0}(x_0)$, we have that $U \Delta \tilde{V}$ is contained in a compact set in $\bar{\Omega} \times \mathbb{R}$. Because $u(x)$ is a generalized solution, we obtain

$$\mathrm{P}_\alpha(U, \Omega \times \mathbb{R}) \le \mathrm{P}_\alpha(\tilde{V}, \Omega \times \mathbb{R}).$$

By Theorem 4.4 in [14], the above inequality gives that

$$\mathcal{H}^{n-1}(\Sigma_u^r) \le \mathcal{H}^{n-1}(\Sigma_v).$$

Thus, the equality in (7.3) holds. As a result, $X = \vec{v}_u$ on Σ_u^r in the open set $B_{r_0}(x_0) \times \mathbb{R}$. By the uniqueness of the integral distribution of X, $\{T_t(\Sigma_v)\}_{t \in \mathbb{R}}$, $\Sigma_u^r \subset T_{t_0}(\Sigma_v)$ for some t_0. Because of $\partial \Sigma_u^r$, t_0 has to be 0 and $u \equiv v$ on $B_{r_0}(x_0) \setminus S$. Since S is a zero-measure set, we conclude that $u(x) = v(x)$ in $B_{r_0}(x_0)$. Thus, u is smooth on $B_{r_0}(x_0)$.

Notice that x_0 is arbitrarily chosen in S. This means u is smooth over Ω. The proof is complete. ■

7.2. The proof of Theorem 1.2

Notice that $\partial\Omega$ is C^2. Then, by Remark 5.4, we can extend $\psi(x)$ as a bounded BV function (still written as $\psi(x)$) on a larger bounded open set $\mathcal{B}$ such that $\Omega \subset\subset \mathcal{B}$, its subgraph is a Caccioppoli set in Q_α and its trace on $\partial\Omega$ is $\psi(x)$. By Theorem 5.3, there is a generalized solution $u(x)$ with the continuous boundary data $\psi(x)$.

Our proof is divided into two steps. The first step is to show that $u(x)$ is locally bounded. The second step is to show the boundary continuity of $u(x)$.

Lemma 7.5. *Take the assumption of Theorem 1.2. The generalized solution $u(x)$ on $\bar{\Omega}$ is locally bounded with the bounded boundary data $\psi(x)$.*

Proof. Recall that $P_\pm$ are the sets $\{x \in \bar{\Omega} : u(x) = \pm\infty\}$. Let n be the dimension of Ω. By Theorem 6.8, $P_\pm$ are two Caccioppoli sets such that their boundaries are closed embedded minimal hypersurfaces with a closed singular set S of Hausdorff dimension at most $n-8$. In the case $n = 8$, S is a collection of isolated points. By the NCM property of Ω, $P_\pm$ are equivalent to an empty set. Namely, $P_\pm$ are Lebesgue measure zero sets in the closure of Ω.

Now assume P_+ is not empty. Let x_0 be a point in P_+. Then $u(x)$ is not locally bounded in a neighborhood of x_0. Moreover, there is a sequence $\{x_j\}_{j=1}^\infty$ in Ω converging to x_0 in $\bar{\Omega}$ such that $u(x_j) > j$ for each positive integer j. For each positive integer j, define $u_j(x) = u(x) - j$ on $\mathcal{B}$. Let U_j be the subgraph of $u_j(x)$ in $\mathcal{B} \times \mathbb{R}$. Since $\{u_j(x)\}_{j=1}^\infty$ is a decreasing sequence, the following holds:

(i) λ_{U_i} locally converges to λ_{U_∞} in $\mathcal{B} \times \mathbb{R}$. Here U_∞ is a subgraph of the measurable function which takes $+\infty$ on P_+ and $-\infty$ on $\mathcal{B} \setminus P_+$.

Fix any bounded open interval I. Since $\psi(x)$ is uniformly bounded on $\mathcal{B} \setminus \bar{\Omega}$, there is a $j_0 > 0$ such that for all $j \geq j_0$,

(ii) $U_j \cap ((\bar{B} \setminus \Omega) \times I) = (\bar{B} \setminus \Omega) \times I$.

Because U_j is $T_{-j}(U)$, arguing as in (iii) in the proof of Theorem 6.8, we have

(iii) U_j has the least perimeter in $\bar{\Omega} \times I$.

Since $u_j(x_j) = u(x_j) - j > 0$, we have $(x_j, 0)$ in U_j for each j. There are two cases to be discussed: x_0 in Ω or x_0 in $\partial\Omega$.

Assume we are in the first case: x_0 in Ω. There is an $r_1 > 0$ such that $B_{r_1}((x_0, 0))$ is contained in $\Omega \times (-1, 1)$. Thus, $\{U_j\}_{j=1}^\infty$ are minimal sets in $B_{r_1}((x_0, 0))$. By Theorem 6.5 (2), we have

$$\mathrm{vol}(U_j \cap B_r((x_j, 0))) > c r^{n+1}$$

for some $c > 0$ and for all $r < r_1/2$, where c is a positive constant depending on the metric in $B_{r_1}((x_0, 0))$. Now, letting $j \to +\infty$, by (i), we obtain

$$\mathrm{vol}((P_+ \times \mathbb{R}) \cap B_r((x_0, 0))) > c r^{n+1}. \tag{7.5}$$

This gives a contradiction since P_+ is a Lebesgue zero measure set. Therefore, P_+ is the empty set.

Assume we are in the second case: x_0 in $\partial\Omega$. By Lemma 6.3, there is a sufficiently small $r_2 > 0$ such that $\Omega \times (-1, 1)$ is a $(\mu, 1/2)$-almost minimal set in $B_{r_2}((x_0, 0))$. Arguing similarly, as in (6.2), (6.3) and (6.4), $\{U_j\}_{j=j_0}^\infty$ is a sequence of $(\mu, 1/2)$-almost minimal sets in $B_{r_2}((x_0, 0))$. Now applying Theorem 6.5 (2) and arguing as in the first case, we will obtain the same contradiction as in (7.5). Thus, in this case, we still obtain that P_+ is the empty set.

A similar derivation yields that P_- is also an empty set in Ω. Thus, $u(x)$ is locally bounded. ∎

By Theorem 7.2, the generalized solution in Lemma 7.5 is smooth on Ω. Now we conclude the boundary continuity of $u(x)$ when $\psi(x)$ is continuous on $\partial\Omega$.

Lemma 7.6. *The generalized solution $u(x)$ is continuous on $\bar{\Omega}$, and it is equal to $\psi(x)$ on $\partial\Omega$.*

Proof. Suppose that $x_0 \in \partial\Omega$ and that

$$\lambda = \limsup_{x \in \Omega, x \to x_0} u(x) > \psi(x_0).$$

Then there exist a sequence $\{x_j\}$ in Ω converging to x_0 and $\lambda > 0$ such that

$$\lim_{j \to +\infty} u(x_j) = \lambda > \psi(x_0).$$

Let z_0 be the point (x_0, λ) in $\partial\Omega \times \mathbb{R}$. By Remark 5.4, we can extend $\psi(x)$ as a continuous function in $\mathcal{B} \setminus \Omega$. Here $\mathcal{B}$ is a large open set strictly containing Ω. There is an $R > 0$ such that the normal ball $B_R(z_0)$ in Q_α does not intersect the graph of $\psi(x)$.

We can view $B_R(z_0)$ as an open set in $\mathbb{R}^{n+1}$ with the induced metric from Q_α. Now we blow up $U \cap B_R(z_0)$ in $\mathbb{R}^{n+1}$ as follows:

$$U_j = \{z \in \mathbb{R}^{n+1} : j^{-1}z + z_0 \in U \cap B_R(z_0)\}.$$

Arguing similarly, as the derivation in Theorem 37.4 of [26], U_j will converge weakly to an area minimizing cone C in $\mathbb{R}^{n+1}$. Notice that $\Omega \times \mathbb{R}$ is a C^2 domain. Then C is contained in a half space in $\mathbb{R}^{n+1}$. By Theorem 15.5 in [14], C is just a closed half-space in $\mathbb{R}^{n+1}$. By the Allard regularity theorem, ∂U is $C^{1,1/2}$ near z_0 and can be written as a graph of a $C^{1,1/2}$ function over $\partial\Omega \times \mathbb{R}$ near z_0.

Since Ω is mean convex, by Lemma 3.3 in [31], we have

$$\mathrm{H}^{\alpha}_{\partial\Omega \times \mathbb{R}}(x, r) = e^{-\alpha r/n}\, \mathrm{H}_{\partial\Omega}(x) \geq 0,$$

with respect to the outward normal vector of $\partial\Omega \times \mathbb{R}$ in Q_α. Let $\vec{v}'$ be the normal vector of ∂U near z_0 which points outward to $(\mathcal{B} \setminus \bar{\Omega}) \times \mathbb{R}$ at z_0. The fact that U locally minimizes the perimeter in $\bar{\Omega} \times \mathbb{R} \subset Q_\alpha$ yields that the mean curvature of ∂U in Q_α is

$$\mathrm{H}^{\alpha}_{\partial U} = \mathrm{div}_{Q_\alpha}(\vec{v}') \leq 0$$

near z_0 in the Lipschitz sense. Note that ∂U is tangent to $\partial\Omega \times \mathbb{R}$ at z_0. By the classical maximum principle (see Theorem 8.19 in [11] and Appendix A in [10]) ∂U coincides with $\partial\Omega \times \mathbb{R}$ near z_0. This contradicts the fact $\lambda = \limsup_{x \in \Omega, x \to x_0} u(x)$. Thus, we conclude

$$\lim_{x \to x_0} \sup u(x) \leq \psi(x_0).$$

With a similar argument, $\lim_{x \to x_0} \inf u(x) \geq \psi(x_0)$. As a result,

$$\lim_{x \to x_0} u(x) = \psi(x_0).$$

Thus, $u(x)$ is continuous until the boundary and $u(x) = \psi(x)$ for each x in $\partial\Omega$. ∎

The existence part of Theorem 1.2 is proved by combining Lemma 7.5 and Lemma 7.6 together. The uniqueness is obvious from the maximum principle. Hence, the proof of Theorem 1.2 is complete.

A. A decomposition result of Radon measures

Now we consider a decomposition of Radon measures on Riemannian manifolds. The reason we derive it here is that bounded domains in Riemannian manifolds may not be contained in a simply connected domain. Thus, no existence of smooth mollifiers as in Euclidean spaces is available here (see Remark 3.6). The main references of this section are Chapter 1 of [26] and Section 2.8 of [9].

Throughout this section, let N be a complete Riemannian manifold with $\dim N = n$. For every point x in N, we denote the open (closed) embedded normal ball (see Definition 2.16) centered at x with radius r by $B_r(x)$ ($\bar{B}_r(x)$).

Definition A.1. Let $\mathcal{F}$ be a collection of closed normal balls such that their radius is uniformly bounded. Let A denote the set of all centers of those balls. We say that $\mathcal{F}$ covers A finely if the infimum of the radius of balls containing every point in A is 0.

The following theorem is a statement of Theorem 2.8.14 in [9] by Federer in the case of Riemannian manifolds.

Theorem A.2 (Besicovitch's covering theorem). *Let $\Omega \subset N$ be a bounded open set. There is a positive constant $\kappa = \kappa(n, \Omega)$ such that the following property holds. Let $\mathcal{F}$ be a collection of closed embedded normal balls in Ω with uniformly bounded radius. Let A be the set of all centers of these balls in $\mathcal{F}$. If $\mathcal{F}$ covers A finely, then there are κ subcollections $\{\mathcal{F}_i\}_{i=1}^{\kappa}$ of $\mathcal{F}$ such that the balls in each $\mathcal{F}_i$ are pairwise disjoint and $A \subset \bigcup_{i=1}^{\kappa} \bigcup_{\bar{B} \in \mathcal{F}_i} \bar{B}$.*

Using Theorem A.2, it is straightforward to prove the following result.

Corollary A.3. *Suppose Ω is a bounded open set in N. Let μ be a Radon measure on Ω with $\mu(\Omega) < \infty$. Let $\mathcal{F}$ be a collection of closed normal balls covering Ω finely. Then there is a countable pairwise disjoint collection of closed normal balls $\{\bar{B}_{r_j}(x_j) \in \mathcal{F} : j = 1, \ldots, \infty\}$ with $\mu(\Omega \setminus \bigcup_{j=1}^{\infty} \bar{B}_{r_j}(x_j)) = 0$.*

Next, we obtain a useful decomposition of Radon measures in Riemannian manifolds.

Theorem A.4. *Let Ω be an open bounded set in an n-dimensional Riemannian manifold N. Fix any $\varepsilon > 0$ and $r_0 > 0$. Suppose μ is a Radon measurable satisfying $\mu(\Omega) < \infty$. Then there is a collection of countable open normal balls in Ω, defined by*

$$\mathcal{B} = \big\{B_k = B_{r_k}(x_k) : k = 1, \ldots, \infty, x_k \in \Omega, r_k \leq r_0, \mu(\partial B_k) = 0\big\},$$

and a positive integer $\kappa_0 = \kappa_0(\varepsilon, n, \Omega)$ such that $\Omega \subset \bigcup_{B_k \in \mathcal{B}} B_k$ and

(1) *$\{B_1, \ldots, B_{\kappa_0}\}$ is a pairwise disjoint subcollection of $\mathcal{B}$ with*

$$\mu(\Omega) - \varepsilon \leq \sum_{k=1}^{\kappa_0} \mu(B_k) = \mu\Big(\bigcup_{k=1}^{\kappa_0} B_k\Big) \leq \mu(\Omega),$$

(2) *the subcollection $\{B_k : k = \kappa_0 + 1, \ldots, \infty\}$ of $\mathcal{B}$ satisfies*

$$\sum_{k=\kappa_0+1}^{\infty} \mu(B_k) \leq \kappa\varepsilon,$$

where $\kappa = \kappa(n, \Omega)$ is the positive integer given in Theorem A.2.

Proof. Let d be the distance given by the metric on Ω. We define a collection of closed normal balls as follows:

$$\mathcal{F} = \left\{\bar{B}_r(x) : x \in \Omega, r < \min\{r_0, d(x, \partial\Omega)\}, \mu(\partial \bar{B}_r(x)) = 0\right\}. \tag{A.1}$$

Since $\mu(\Omega) < \infty$, Fubini's theorem implies that $\mu(\partial \bar{B}_r(x)) = 0$ a.e. for any $x \in \Omega$ and $r \in (0, \min\{r_0, d(x, \partial\Omega)\})$. Thus, $\mathcal{F}$ covers Ω finely.

Fix $\varepsilon > 0$. By Corollary A.3, there is $\kappa_0 = \kappa_0(\varepsilon, n, \Omega)$ and a pairwise disjoint subcollection of closed balls $\{\bar{B}_{r_i}(x_i)\}$ in $\mathcal{F}$ such that

$$\mu(\Omega) - \frac{\varepsilon}{4} \leq \sum_{k=1}^{\kappa_0} \mu(B_{r_k}(x_k)) = \mu\Big(\bigcup_{k=1}^{\kappa_0} B_{r_k}(x_k)\Big) \leq \mu(\Omega),$$

because $\mu(\partial \bar{B}_r(x)) = 0$ for each $\bar{B}_r(x)$ in $\mathcal{F}$.

Namely, there is a pairwise disjoint collection of finite open balls

$$\{B_{r_1}(x_1), \ldots, B_{r_{\kappa_0}}(x_{\kappa_0})\} \tag{A.2}$$

satisfying

$$\mu\Big(\Omega \setminus \bigcup_{k=1}^{\kappa_0} \bar{B}_{r_k}(x_k)\Big) \leq \frac{\varepsilon}{4}. \tag{A.3}$$

Now define an open set Ω_η as

$$\Omega_\eta :\equiv \Big\{x \in \Omega : d(x, \Omega \setminus \bigcup_{k=1}^{\kappa_0} \bar{B}_{r_k}(x_k)) < \eta\Big\},$$

where η is a sufficiently small positive constant such that $\mu(\Omega_\eta) \leq \varepsilon/2$. Similarly, as in (A.1), we define a collection of closed normal balls in Ω_η as

$$\mathcal{F}_\eta = \left\{\bar{B}_r(x) : x \in \Omega_\eta, r < \min\{r_0, d(x, \partial\Omega_\eta)\}, \mu(\partial \bar{B}_r(x)) = 0\right\}.$$

By Theorem A.2, there are $\kappa = \kappa(\Omega, n)$ subcollections $\{\mathcal{F}_{\eta,k}\}_{k=1}^{\kappa}$ such that the closed balls in each $\mathcal{F}_{\eta,k}$ are pairwise disjoint and

$$\Omega_\eta \subset \bigcup_{k=1}^{\kappa} \bigcup_{\bar{B}_r(x) \in \mathcal{F}_{\eta,k}} \bar{B}_r(x).$$

Moreover, for each $k = 1, \ldots, \kappa$,

$$\sum_{\bar{B}_r(x) \in \mathcal{F}_{\eta,k}} \mu(\bar{B}_r(x)) \leq \mu(\Omega_\eta) \leq \frac{\varepsilon}{2}.$$

Note that there are only countable closed normal balls in each subcollection $\mathcal{F}_{\eta,k}$. Each ball $\bar{B}_r(x)$ in each collection $\mathcal{F}_{\eta,k}$ can be replaced with a large open ball $B_{r_x}(x)$ with $r_x < \min\{r_0, d(x, \partial\Omega_\eta), 1.5r\}$ so that

$$\sum_{\bar{B}_r(x) \in \mathcal{F}_{\eta,k}} \mu(\bar{B}_r(x)) \leq \sum_{\bar{B}_r(x) \in \mathcal{F}_{\eta,k}} \mu(B_{r_x}(x)) \leq \varepsilon. \tag{A.4}$$

This gives κ collections of open normal balls as follows:

$$\mathcal{F}'_{\eta,k} := \{B_{r_x}(x) : \bar{B}_r(x) \in \mathcal{F}_{\eta,k}, \bar{B}_r(x) \subset B_{r_x}(x) \subset \Omega_\eta\},$$

with the condition (A.4) for $k = 1, \ldots, \kappa$. Now we relabel all open balls in $\{\mathcal{F}_{\eta,k}\}_{k=1}^{\kappa}$ and list them as follows:

$$\{B_{r_k}(x_k) : k = \kappa_0 + 1, \ldots, \infty\} = \{B_{r_x}(x) : B_{r_x}(x) \in \mathcal{F}'_{\eta,k}, k = 1, \ldots, \kappa\}. \tag{A.5}$$

Obviously,

$$\Omega_\eta \subset \bigcup_{k=\kappa_0+1}^{\infty} B_{r_k}(x_k),$$

according to our definition. Condition (A.4) yields

$$\sum_{k=\kappa_0+1}^{\infty} \mu(B_{r_k}(x_k)) \leq \kappa\varepsilon. \tag{A.6}$$

Combining the open balls in (A.2) with the property (A.3) and the open balls in (A.5) with the property (A.6) together, we obtain the desirable collection of open normal balls. The proof is complete. ■

B. Some PDE results

In this section, we collect some PDE results on mean curvature equations. Let M be a Riemannian manifold with dimension $n \geq 2$ and let $B_r(x)$ denote an embedded ball centered at x with radius r.

Fix x_0 in M. When take r sufficiently small, the metric in $B_r(x_0)$ is much close to the Euclidean metric. A well-known fact is that the mean curvature of $\partial B_r(x_0)$ satisfies the estimate

$$\mathrm{H}_{\partial B_r(x_0)} = \frac{n}{r} + O(r).$$

And the Sobolev inequality also holds in $B_r(x_0)$ when r is sufficiently close to 0. Thus, following the derivations in Theorem 16.10 of [11], we have the following theorem.

Theorem B.1. *Fix $\alpha > 0$. Suppose $x_0 \in M$. Then there is a sufficiently small $r_0 > 0$ such that, for any $r \in (0, r_0]$, the Dirichlet problem*

$$\begin{cases} \operatorname{div}\Big(\dfrac{\mathrm{D}u}{\omega}\Big) = \alpha & x \in B_r(x_0), \\ u(x) = \psi(x), & x \in \partial B_r(x_0), \end{cases}$$

is uniquely solved in $C^2(B_r(x_0)) \cap C(\bar{B}_r(x_0))$ for any continuous data $\psi(x)$ on $\partial\Omega$. Here $\omega = \sqrt{1 + |\mathrm{D}u|^2}$.

The following interior curvature of mean curvature equations is classical.

Lemma B.2 (Theorem 1.10 in [15]). *For any $r > 0$ and x in M, let $u(x)$ be a C^2 function on $B_r(x)$ satisfying* $\operatorname{div}(\mathrm{D}u/\omega) = \alpha/\omega$. *Then*

$$\max_{B_{r/2}(x)} |\mathrm{D}u| \le C,$$

where C is a constant depending only on α and $\max_{B_r(x)}|u|$.

Note that the proof of Gui, Jian and Ju [15] is also valid in any Riemannian manifolds.

Assume $\psi(x)$ is in $C^3(\partial B_r(x_0))$. It is easy to see that the C^0 estimate and the boundary gradient estimate for the solution in (B.1) is from the comparison with the solution to (B.1). The interior gradient estimate is from Lemma B.2. By the classical continuous method [11], we obtain the existence in Corollary B.3 when $\psi(x)$ is in $C(\partial B_r(x))$. The general case is from the standard approximation process. In summary, as an application, we obtain the following.

Corollary B.3. *Let x_0, α and r_0 be given as in Theorem B.1. For any r in $(0, r_0)$ and $\psi(x) \in C(\partial B_r(x_0))$, the Dirichlet problem of the TMCE*

$$\begin{cases} \operatorname{div}\Big(\dfrac{\mathrm{D}u}{\omega}\Big) = \dfrac{\alpha}{\omega}, & x \in B_r(x_0), \\ u(x) = \psi(x), & x \in \partial B_r(x_0), \end{cases} \tag{B.1}$$

is uniquely solved in $C^2(B_r(x_0)) \cap C(\bar{B}_r(x_0))$.

Next we give an example to illustrate that the NCM property is necessary. Let S^n_+ be the n-dimensional upper hemisphere with the standard metric σ_n. Note that ∂S^n_+ is an $(n-1)$-dimensional unit sphere that is minimal in S^n. Thus, by Definition 7.1, S^n_+ does not have the NCM property.

Theorem B.4 (Theorem 6.1 in [10]). *For any $\alpha \ge n \ge 2$, there is no solution in $C^2(S^n_+) \cap C(\bar{S}^n_+)$ to the Dirichlet problem*

$$\begin{cases} \operatorname{div}\Big(\dfrac{\mathrm{D}u}{\omega}\Big) = \dfrac{\alpha}{\omega}, & x \in \Omega,\ \omega = \sqrt{1 + |\mathrm{D}u|^2}, \\ u(x) = \psi(x), & x \in \partial\Omega, \end{cases}$$

for any $\psi(x)$ in $C(\partial\Omega)$.

We refer to the above example as the *hemisphere* example.

C. The difference between dimension one and higher dimensions

In this section, we point out the essential difference in the Dirichlet problem of (1.2) between the case of $n = 1$ and $n \ge 2$.

In the real line $\mathbb{R}$ (i.e., $n = 1$), Ω is an open interval. With a direct computation, the Dirichlet problem (1.2) becomes

$$\frac{u_{rr}}{1 + |u_r|^2} = \alpha \quad \text{on } \Omega, \quad u = \psi \quad \text{on } \partial\Omega. \tag{C.1}$$

Here α is a fixed positive constant and u_r, u_{rr} denote the first and second derivatives of u with respect to $r \in \mathbb{R}$. Let $|\Omega|$ denote the length of Ω. Integrating on both sides of (C.1), we have the following non-existence result.

Theorem C.1. *If $\alpha|\Omega| > \pi$, there is no solution to the Dirichlet problem* (C.1) *when Ω is an open interval and ψ is any two-point function.*

In the case $n \geq 2$, the solvability of the Dirichlet problem (1.2) corresponds a totally different geometry. We just take the Dirichlet problem of (1.2) on a special class of warped product manifolds.

Fix $n \geq 2$. Let S^{n-1} be the standard sphere and consider a conical metric σ_{n-1}. Let $\phi(r)$ be an strictly increasing smooth function on $[0, \phi)$ such that

$$\phi(0) = 0, \quad \phi'(0) = 1, \quad \lim_{r \to +\infty} (\log \phi)'(r) \leq \beta,$$

for some positive $\beta > 0$. We consider a warped product manifold Q_ϕ given as follows:

$$Q_\phi := (S^{n-1} \times (0, \infty), \phi^2(r)\sigma_{n-1} + \mathrm{d}r^2).$$

Note that the condition at ϕ implies that Q_ϕ is complete when $r \to 0^+$. For $\phi(r) = r$ and $\phi(r) = \sinh(r)$, Q_ϕ is the Euclidean space $\mathbb{R}^n$ and the hyperbolic space $\mathbb{H}^n$, respectively.

Fix any $\alpha > 0$. Suppose $u = u(r)$ is a C^2 function on Q_ϕ which only depends on the parameter $r \in (0, \infty)$. Then $u(r)$ satisfies $\mathrm{H}_\alpha(u) = 0$ (see (1.2)) on the open set $S^{n-1} \times (0, +\infty)$ if and only if

$$\frac{u_{rr}}{1 + u_r^2} + (n-1)(\log \phi)'(r) u_r = \alpha, \tag{C.2}$$

with $u_r(0) = 0$ and $u(0) = C$. Here C is any given constant. For more details on computations, see the proof of Theorem A.1 in [31]. It is not hard to see that the equation (C.2) has a smooth solution $u(r)$ in $(0, \infty)$. For any bounded mean convex domain Ω in Q_ϕ, by applying the maximum principal, we obtain a uniformly bound of the solution $u_s(x)$ to the Dirichlet problem

$$-\mathrm{div}\Big(\frac{\mathrm{D}u}{\sqrt{1 + |\mathrm{D}u|^2}}\Big) + \frac{s\alpha}{\sqrt{1 + |\mathrm{D}u|^2}} = 0 \quad \text{on } \Omega, \qquad u = s\psi \quad \text{on } \partial\Omega,$$

for any continuous function $s\psi \in C(\partial\Omega)$. Here $s \in [0, 1]$. Applying the canonical continuous method in Dirichlet problems will yield the following existence result.

Theorem C.2. *Fix $n \geq 2$. Suppose Ω is a mean convex bounded C^2 domain in the Euclidean space $\mathbb{R}^n$ or the Hyperbolic space $\mathbb{H}^n$. The Dirichlet problem* (1.2) *has a unique solution in $C^2(\Omega) \cap C(\bar{\Omega})$ for any continuous boundary data.*

In summary, when $n \geq 2$, the existence of the Dirichlet problem (1.2) in Euclidean spaces depends on the solution to (C.2). Except from $\alpha > 0$, the existence of the latter problem does not put any requirement upon α.

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