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Existence and ergodicity for the two-dimensional stochastic Allen–Cahn–Navier–Stokes equations

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Abstract. We study in this article a stochastic version of a coupled Allen–Cahn–Navier–Stokes model in a two-dimensional bounded domain. The model consists of the Navier–Stokes equations for the velocity, coupled with an Allen–Cahn model for the order (phase) parameter, both endowed with suitable boundary conditions. We prove the existence of solutions via a semigroup approach. We also obtain the existence and uniqueness of an invariant measure via coupling methods.

1. Introduction

We study the existence and ergodicity of the stochastic Allen–Cahn–Navier–Stokes equations (AC-NSEs)

$$\begin{cases} \mathrm{d}\boldsymbol{u} + [-\nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p]\,\mathrm{d}t = \mathcal{K}\mu\nabla\phi\,\mathrm{d}t + \sqrt{Q_1}\,\mathrm{d}W_1(t), \\ \operatorname{div}\boldsymbol{u} = 0, \\ \mathrm{d}\phi + [\boldsymbol{u}\cdot\nabla\phi + \mu]\,\mathrm{d}t = \sqrt{Q_2}\,\mathrm{d}W_2(t), \\ \mu = -\epsilon\Delta\phi + \alpha f(\phi) \end{cases} \tag{1.1}$$

in $(0,+\infty)\times\mathcal{O}$, subject to the boundary and initial conditions

$$\begin{cases} \boldsymbol{u} = 0, \quad \frac{\partial\mu}{\partial\eta} = 0 & \text{on } (0,+\infty)\times\partial\mathcal{O}, \\ \boldsymbol{u}(0,x) = \boldsymbol{u}_0(x), \quad \phi(0,x) = \phi_0(x) & \text{in } \mathcal{O}, \end{cases} \tag{1.2}$$

where η is the unit outward normal to the boundary $\partial\mathcal{O}$. Model (1.1) is an example of a diffuse interface model, and it is well accepted that diffuse interface models are well-known tools to describe the dynamics of complex (e.g., binary) fluid [1]. For instance, this approach is used in [4] to describe cavitation phenomenon in a flowing liquid. The model consists of the Navier–Stokes equation coupled with the phase-field system [8, 17, 18, 25]. In (1.1)–(1.2), $\mathcal{O}\subset\mathbb{R}^2$ is a bounded, open, and simply connected domain with smooth boundary $\partial\mathcal{O}$, and $\boldsymbol{u} = (\boldsymbol{u}_1,\boldsymbol{u}_2)$ denotes the fluid velocity field, ϕ is the order (phase) parameter, which represents the relative concentration of one of the fluids, p stands for the

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pressure, ν, $\mathcal{K}$ is the kinematic viscosity of the fluid and the capillarity (stress) coefficient, respectively, and $\epsilon, \alpha > 0$ are two physical parameters describing the interaction between the two phases. In particular, ϵ is related to the thickness of the interface separating the two fluids, provided that the diffuse interface between the phases has a small but non-zero thickness. The quantity μ, also called chemical potential, is the variational derivative of the following free energy functional:

$$\mathcal{F}(\phi) = \int_{\mathcal{O}} \Big(\frac{\epsilon}{2}|\nabla\phi|^2 + \alpha F(\phi)\Big)\,\mathrm{d}x, \tag{1.3}$$

where, e.g., $F(r) = \int_0^r f(\xi)\,\mathrm{d}\xi$. W_1 and W_2 are independent cylindrical Wiener processes defined in a filtered space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ taking value in appropriate Hilbert spaces H_1, H_2, respectively. Finally, Q_1 and Q_2 are linear continuous, positive, and symmetric operators on H_1 and H_2, respectively (see (2.8) below).

Herein, we prove the existence and uniqueness of a solution $(\boldsymbol{u}(t,\boldsymbol{u}_0,\phi_0),\phi(t,\boldsymbol{u}_0,\phi_0))$ of the stochastic AC-NSEs (1.1)–(1.2) and of the corresponding invariant measure on the space $H_1 \times H_2$ defined in Section 2 below. The deterministic version of the Allen–Cahn–Navier–Stokes system (1.1)–(1.2) was extensively studied in the literature (see, e.g., [18, 25], and the references therein). As noted in [6, 23, 24], stochastic partial differential equations (SPDE) can be used to describe systems that are too complex to be described deterministically, e.g., a flow of a chemical substance in a river subjected to wind and rain, an airflow around an airplane wing perturbed by the random state of the atmosphere and weather, etc. With the development of the theory of stochastic processes, systems such as the Navier–Stokes equations perturbed by noises have been widely investigated with the goal of better understanding the complex phenomenon of turbulent flow. The mathematical theory of the stochastic Navier–Stokes equations is very rich, covering a broad area of deep results on existence of solutions, dynamical system features (i.e., how the system behaves and evolves over time, including stability, attractors, long-time behavior of the solutions, etc.), ergodicity, and many more. Let us recall that the presence of noise in a model can lead to new and important phenomena. For instance, contrary to the deterministic case, it is known that the 2D Navier–Stokes system driven with a sufficiently non-degenerate noise has a unique invariant measure and hence exhibits ergodic behavior in the sense that the time average of a solution is equal to the average over all possible initial data [6]. Recently, instead of stochastic Navier–Stokes equation, many authors have also studied ergodicity for the solutions of the stochastic magneto-hydrodynamics equations (see [2]), the solution of the stochastic Boussinesq equations (see [15, 20, 21] and the references therein), and the solutions of the stochastic magneto-hydrodynamics alpha model (see [31]); and this list is not exhaustive.

Let us mention that although we drew our inspiration from [2, 21], the problem we treat here does not fall into the framework of these references. Besides the usual nonlinear term of the conventional Navier–Stokes system, the model (1.1)–(1.2) contains another (stronger) nonlinear term that results from the coupling of the convective Allen–Cahn equation and the Navier–Stokes system. Because of this fact, the analysis of the existence and uniqueness of invariant measures of the 2D stochastic AC-NSEs driven by degenerate additive noise tend to be more complicated and subtle than of the same study done in [2] or [21]. For more details, see, for instance, the proof of Proposition 3.1 in Section 3, or the derivation of the estimates (3.12), (3.28), and (3.34), just to cite a few. Furthermore, the technical method used here to derive the proof of Lemma 4.2 in Section 4 is different from that used in [2, 21], primarily due to the presence of the term f_γ in the system of equations, which is difficult to control. This makes the mathematical analysis of the problem very challenging.

The paper is organized as follows. In Section 2, we gather all the necessary tools for the operator formulation of problem (1.1)–(1.2). In Section 3, we provide the main existence and uniqueness result for (1.1)–(1.2), which is proven via an approximating regularizing scheme. In Section 4, we establish the existence of an invariant measure μ_* corresponding to the stochastic flow $t \mapsto (\boldsymbol{u}(t), \phi(t))$ and its uniqueness via coupling methods, following [2, 13, 27]. Furthermore, the uniqueness of the invariant measure implies that the flow is ergodic, i.e.,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \Psi(\boldsymbol{u}(t), \phi(t))\, \mathrm{d}t = \int_{\mathbb{Y}} \Psi\, \mathrm{d}\mu_*$$

for all $\Psi \in L^2(\mathbb{Y}; \mu_*)$ ($\mathbb{Y}$ is defined in (2.3) below), which agrees with some physical hypotheses on the AC-NSEs, which model the flow of two fluids (for instance, oil and water).

2. Functional setting and formulation of the problem

We introduce necessary definitions of functional spaces frequently used in this work. Given two Banach spaces E_1, E_2, $\mathcal{L}(E_1, E_2)$ is the space of bounded linear operators from E_1 to E_2. If X is real Hilbert space with inner product $(\cdot,\cdot)_X$, then we denote the induced norm by $\|\cdot\|_X$, while X' will indicate its (topological) dual. If E_1 and X_1 are separable Hilbert spaces, then by $L_2(E_1, X_1)$ we will denote the Hilbert space of all Hilbert–Schmidt operators from E_1 to X_1 endowed with the canonical norm $\|\cdot\|_{L_2(E_1,X_1)}$. For any $p \in [1,\infty)$ and $s \in \mathbb{R}$, we denote by $L^p(\mathcal{O})$ and $W^{s,p}(\mathcal{O})$ the usual Lebesgue and Sobolev spaces of scalar functions, respectively. If $p = 2$, we simply write $W^{s,2}(\mathcal{O}) = H^s(\mathcal{O})$. We denote by $H_0^1(\mathcal{O})$ the closure of $\mathcal{C}_0^\infty(\mathcal{O})$ in $H^1(\mathcal{O})$. We use the notations $\mathbb{L}^p(\mathcal{O})$, $\mathbb{W}^{s,p}(\mathcal{O})$, and $\mathbb{H}^s(\mathcal{O})$ to denote the spaces $[L^p(\mathcal{O})]^2$, $[W^{s,p}(\mathcal{O})]^2$, and $[H^s(\mathcal{O})]^2$, respectively.

We introduce the following spaces:

$$\begin{aligned}
\mathcal{V} &= \{\boldsymbol{v} \in [\mathcal{C}_0^\infty(\mathcal{O})]^2 \text{ such that } \operatorname{div} \boldsymbol{v} = 0\},\\
H_1 &= \text{the closure of } \mathcal{V} \text{ in } \mathbb{L}^2(\mathcal{O}),\\
V_1 &= \text{the closure of } \mathcal{V} \text{ in } [H_0^1(\mathcal{O})]^2.
\end{aligned}$$

We denote by $(\cdot,\cdot)$ and $|\cdot|$ the inner product and the norm induced by the inner product and the norm in $\mathbb{L}^2(\mathcal{O})$ on H_1, respectively. We endow H_1 with the scalar product and norm of $\mathbb{L}^2(\mathcal{O})$. As usual, we equip the space V_1 with the gradient-scalar product and the gradient-norm $|\nabla \cdot| := \|\cdot\|$, which is equivalent to the $[H_0^1(\mathcal{O})]^2$-norm (due to Poincaré's inequality).

We now define the operator A_0 by

$$A_0 \boldsymbol{u} = -\mathcal{P}\Delta \boldsymbol{u} \quad \forall \boldsymbol{u} \in D(A_0) = \mathbb{H}^2(\mathcal{O}) \cap V_1,$$

where $\mathcal{P}$ is the Leray–Helmholtz projector in $\mathbb{L}^2(\mathcal{O})$ onto H_1. Then, A_0 is a self-adjoint positive unbounded operator in H_1 which is associated with the scalar product defined above. Furthermore, $A_0^{-1} : H_1 \to H_1$ is a self-adjoint linear compact operator on H_1 and $|A_0 \cdot|$ is a norm on $D(A_0)$ that is equivalent to the $\mathbb{H}^2(\mathcal{O})$-norm.

We introduce the linear nonnegative unbounded operator on $L^2(\mathcal{O})$

$$A_1\phi = -\Delta\phi \quad \forall \phi \in D(A_1) = \{\phi \in H^2(\mathcal{O}), \partial_\eta \phi = 0, \text{ on } \partial\mathcal{O}\}, \tag{2.1}$$

and we endow $D(A_1)$ with the norm $|A_1 \cdot| + |\langle\cdot\rangle|$, which is equivalent to the usual $H^2(\mathcal{O})$-norm. For a fixed $\gamma > 0$, we define the following operator:

$$A_\gamma\phi = -\Delta\phi + \gamma\phi \quad \forall \phi \in D(A_\gamma) := \{\phi \in H^2(\mathcal{O}), \partial_\eta \phi = 0, \text{ on } \partial\mathcal{O}\}.$$

Note also that A_γ^{-1} is a compact linear operator on $L^2(\mathcal{O})$ and $|A_\gamma \cdot|$ is a norm on $D(A_\gamma)$ that is equivalent to the $H^2(\mathcal{O})$-norm.

Hereafter, we set

$$H_2 = L^2(\mathcal{O}), \quad V_2 = D(A_\gamma^{1/2}), \quad H = H_1 \times H_2, \quad V = V_1 \times V_2. \tag{2.2}$$

In order to define the variational setting for the Allen–Cahn–Navier–Stokes equations (1.1)–(1.2), we also need to introduce the bilinear operators B_0, B_1 (and their associated trilinear forms b_0, b_1) as well as the coupling mapping R_0 which are defined, from $V_1 \times D(A_0)$ into H_1, $V_1 \times D(A_\gamma)$ into H_2, and $H_2 \times D(A_\gamma^{3/2})$ into H_1, respectively. More precisely, we set

$$(B_0(\boldsymbol{u}, \boldsymbol{v}), \boldsymbol{w}) = \int_\mathcal{O} (\boldsymbol{u} \cdot \nabla)\boldsymbol{v} \cdot \boldsymbol{w} \,\mathrm{d}x = b_0(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \quad \forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in D(A_0),$$
$$(B_1(\boldsymbol{u}, \varphi), \psi) = \int_\mathcal{O} [(\boldsymbol{u} \cdot \nabla)\varphi]\psi \,\mathrm{d}x = b_1(\boldsymbol{u}, \varphi, \psi) \quad \forall \boldsymbol{u} \in V_1, \varphi, \psi \in D(A_\gamma),$$
$$(R_0(\mu, \varphi), \boldsymbol{w}) = \int_\mathcal{O} \mu[\nabla\varphi \cdot \boldsymbol{w}] \,\mathrm{d}x = b_1(\boldsymbol{w}, \varphi, \mu) \quad \forall \boldsymbol{w} \in V_1, \ \mu \in H_2, \ \varphi \in D(A_\gamma^{3/2}).$$

Let us point out that

$$R_0(\mu, \varphi) = \mathcal{P}\mu\nabla\varphi.$$

Now, we define the Hilbert spaces $\mathbb{Y}$ and $\mathbb{V}$ by

$$\mathbb{Y} = H_1 \times V_2, \quad \mathbb{V} = V_1 \times D(A_\gamma), \tag{2.3}$$

endowed with the scalar products whose associated norms are, respectively,

$$\begin{aligned} |(\boldsymbol{u},\phi)|_{\mathbb{Y}}^2 &= \mathcal{K}^{-1}|\boldsymbol{u}|^2 + \epsilon(|\nabla\phi|^2 + \gamma|\phi|^2), \quad &(\boldsymbol{u},\phi) \in \mathbb{Y}, \\ \|(\boldsymbol{u},\phi)\|_{\mathbb{V}}^2 &= \nu\mathcal{K}^{-1}\|\boldsymbol{u}\|^2 + \epsilon^2|A_\gamma\phi|^2 \quad &\forall(\boldsymbol{u},\phi) \in \mathbb{V}. \end{aligned} \tag{2.4}$$

We recall that B_0, B_1, and R_0 satisfy the following estimates (see, for instance, [14, 18, 25]):

$$\begin{aligned} \|B_0(\boldsymbol{u},\boldsymbol{v})\|_{V_1'} &\le c|\boldsymbol{u}|^{1/2}\|\boldsymbol{u}\|^{1/2}|\boldsymbol{v}|^{1/2}\|\boldsymbol{v}\|^{1/2} \quad &\forall \boldsymbol{u},\boldsymbol{v} \in V_1, \\ \|B_0(\boldsymbol{u},\boldsymbol{u})\|^2 &\le c\|\boldsymbol{u}\||A_0\boldsymbol{u}|^3 \quad &\forall \boldsymbol{u} \in D(A_0), \\ |B_1(\boldsymbol{u},\phi)|_{L^2} &\le c|\boldsymbol{u}|^{1/2}\|\boldsymbol{u}\|^{1/2}|A_\gamma^{1/2}\phi|^{1/2}|A_\gamma\phi|^{1/2} \quad &\forall \boldsymbol{u} \in V_1,\ \phi \in D(A_\gamma), \\ \|R_0(A_\gamma\phi,\rho)\|_{V_1'} &\le c|A_\gamma^{1/2}\rho|^{1/2}|A_\gamma\rho|^{1/2}|A_\gamma\phi| \quad &\forall \rho,\phi \in D(A_\gamma) \end{aligned} \tag{2.5}$$

for some positive constant $c = c(\mathcal{O},\gamma)$.

Using the previous notations, the problem (1.1)–(1.2) can be formally written in the following abstract form:

$$\begin{cases} \mathrm{d}\boldsymbol{u} + [\nu A_0\boldsymbol{u} + B_0(\boldsymbol{u},\boldsymbol{u}) - \mathcal{K}R_0(\epsilon A_\gamma\phi,\phi)]\,\mathrm{d}t = \sqrt{Q_1}\,\mathrm{d}W_1(t) & \text{in } V_1', \\ \mathrm{d}\phi + [B_1(\boldsymbol{u},\phi) + \mu_\gamma]\,\mathrm{d}t = \sqrt{Q_2}\,\mathrm{d}W_2(t) & \text{in } V_2', \\ \mu_\gamma = \epsilon A_\gamma\phi + \alpha f_\gamma(\phi), \\ \boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \phi(0) = \phi_0, \end{cases} \tag{2.6}$$

with $f_\gamma(r) = f(r) - \alpha^{-1}\epsilon\gamma r$, $\epsilon \le \alpha$.

Hereafter, we will denote by $\lambda > 0$ and $\ell > 0$ two positive constants such that

$$\lambda|\boldsymbol{v}|^2 \le \|\boldsymbol{v}\|^2, \quad \ell|A_\gamma^{1/2}\phi|^2 \le |A_\gamma\phi|^2 \quad \forall(\boldsymbol{v},\phi) \in \mathbb{V}. \tag{2.7}$$

Remark 2.1. Since $\nabla F_\gamma(\phi) = f_\gamma(\phi)\nabla\phi$, then

$$\mu_\gamma\nabla\phi = \epsilon A_\gamma\phi\nabla\phi + \alpha\nabla F_\gamma(\phi).$$

The term $\nabla F_\gamma(\phi)$ can be incorporated into the pressure gradient. Hence, we could replace $R_0(\mu_\gamma,\phi)$ by $R_0(\epsilon A_\gamma\phi,\phi)$.

Let $(\Omega,\mathcal{F},\mathbb{F} = \{\mathcal{F}_t\}_{t\ge0},\mathbb{P})$ be a filtered probability space satisfying the usual conditions (namely, it is complete, right-continuous and $\mathcal{F}_0$ contains all null sets). Let β_k^i ($k = 1,2,\ldots$, $i = 1,2$) be a sequence of real-valued one-dimensional standard Brownian motions mutually independent on $(\Omega,\mathcal{F},\mathbb{P})$. Let

$$Q_1 = A_0^{-s_0}, \quad Q_2 = A_1^{-s_0}, \quad 1/2 < s_0 < 1 \tag{2.8}$$

be a nonnegative define symmetric operator on H_1 (resp., H_2), $\{e_k^1\}_{k\ge1}$, $\{e_k^2\}_{k\ge1}$ two complete orthonormal basis of eigenfunctions of A_0, respectively, A_1 diagonalizing Q_1 and Q_2, respectively, and $\{\lambda_k^i\}_{k\ge1}$, $i = 1,2$ be the corresponding eigenvalues so that

$$Q_1e_k^1 = \lambda_k^1e_k^1, \quad Q_2e_k^2 = \lambda_k^2e_k^2 \quad \forall k \ge 1.$$

Since Q_1 (resp., Q_2) is of trace-class, it follows that

$$\begin{aligned}\operatorname{Tr} Q_1 &= \sum_{k=1}^{\infty}(Q_1 e_k^1, e_k^1)_{L^2} = \sum_{k=1}^{\infty} \lambda_k^1 < \infty,\\ \operatorname{Tr} Q_2 &= \sum_{k=1}^{\infty}(Q_2 e_k^2, e_k^2)_{L^2} = \sum_{k=1}^{\infty} \lambda_k^2 < \infty.\end{aligned} \tag{2.9}$$

We suppose furthermore that

$$\Lambda := \sum_{k=1}^{\infty} |\nabla(Q_2^{\frac{1}{2}} e_k^2)|_{L^2}^2 = \sum_{k=1}^{\infty} \lambda_k^2 |\nabla e_k^2|_{L^2}^2 < \infty. \tag{2.10}$$

The cylindrical Wiener process $W = (W_1, W_2)$ on $H = H_1 \times H_2$ has the following representation:

$$W_i = \sum_{k=1}^{+\infty} \beta_k^i e_k^i, \quad i = 1, 2.$$

Note that the dependence on the variables is as follows:

$$W_i(t, x, \omega) = \sum_{k=1}^{+\infty} \beta_k^i(t, \omega) e_k^i(x), \quad (t, \omega, x) \in \mathbb{R}^+ \times \Omega \times \mathcal{O}.$$

Now, we consider the stochastic convolution that is the mild solution of the problem

$$\begin{cases} \mathrm{d}W_{\mathcal{A}}(t) + \mathcal{A} W_{\mathcal{A}}(t)\,\mathrm{d}t = \sqrt{Q}\,\mathrm{d}W(t),\\ W_{\mathcal{A}}(0) = 0, \end{cases} \tag{2.11}$$

given by

$$W_{\mathcal{A}}(t) = \int_0^t e^{-(t-s)\mathcal{A}} \sqrt{Q}\,\mathrm{d}W(s) := (W_{A_0}(t), W_{A_\gamma}(t)),$$

where

$$\mathcal{A} = \begin{pmatrix} \nu A_0 & 0 \\ 0 & \epsilon A_\gamma \end{pmatrix}, \quad Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}.$$

In the rest of the paper, we will assume that

$$\sqrt{Q}_1 (A_0)^\delta \text{ and } \sqrt{Q}_2 (A_\gamma)^\delta \text{ are bounded operators.} \tag{2.12}$$

Then,

$$\begin{aligned} &\|e^{-\epsilon(t-s)A_\gamma} \sqrt{Q}_2\|_{L_2(H_2, D((A_\gamma)^\sigma))}^2 \\ &\quad \le \|\sqrt{Q}_2 (A_\gamma)^\delta\|_{\mathcal{L}(H_2,H_2)}^2 \|(A_\gamma)^{\sigma-\delta} e^{-\epsilon A_\gamma(t-s)}\|_{L_2(H_2,H_2)}^2 \\ &\quad = \|\sqrt{Q}_2 (A_\gamma)^\delta\|_{\mathcal{L}(H_2,H_2)}^2 \sum_{k=1}^{\infty} (\lambda_k^2)^{2(\sigma-\delta)} e^{-2\epsilon\lambda_k^2(t-s)}, \end{aligned} \tag{2.13}$$

where e_k^2 is the orthonormal basis of eigenvectors of $A_\gamma = A_1 + \gamma I$ and λ_k^2 are the eigenvalues.

Note that by Itô's isometry property of stochastic integral, we have

$$\begin{aligned}\mathbb{E}\|W_{A_\gamma}(t)\|^2_{D((A_\gamma)^\sigma)} &= \mathbb{E}\left\| \int_0^t e^{-\epsilon A_\gamma (t-s)} \sqrt{Q}_2 \,\mathrm{d}W_2(s) \right\|^2_{D((A_\gamma)^\sigma)} \\ &= \mathbb{E} \int_0^t \|e^{-\epsilon A_\gamma (t-s)} \sqrt{Q}_2\|^2_{L_2(H_2, D((A_\gamma)^\sigma))} \,\mathrm{d}s \\ &= \int_0^t \|e^{-\epsilon A_\gamma (t-s)} \sqrt{Q}_2\|^2_{L_2(H_2, D((A_\gamma)^\sigma))} \,\mathrm{d}s.\end{aligned}$$

Now, since

$$\int_0^t \sum_{k=1}^\infty (\lambda_k^2)^{2(\sigma-\delta)} e^{-2\epsilon(\lambda_k^2)(t-s)} \,\mathrm{d}s = \frac{1}{2\epsilon} \sum_{k=1}^\infty (\lambda_k^2)^{2(\sigma-\delta)-1} (1 - e^{-2\epsilon(\lambda_k^2)t}) \text{ and } \lambda_k^2 \sim ck,$$

we infer that the Gaussian process W_{A_γ} lives in $D((A_\gamma)^\sigma)$ provided that

$$\delta > \sigma.$$

By arguing similarly as in the proof of Proposition 34 (see [10, Section 5.3] for more details), it can be shown that, in this case,

$$W_{A_\gamma} \in \mathcal{C}([0,T]; D((A_\gamma)^\sigma)), \quad \mathbb{P}\text{-a.s.}$$

Hereafter, we fix

$$\sigma \in \{1/4; 1; 3/2\} \quad \text{and} \quad \delta > 3/2.$$

By the Gagliardo–Nirenberg inequality, i.e.,

$$\|x\|_{L^4(\mathcal{O})} \leq c|x|^{1/2}_{L^2(\mathcal{O})} \|x\|^{1/2}_{H^1(\mathcal{O})} \leq c|x|^{1/2}_{L^2(\mathcal{O})} |A_\gamma^{1/2} x|^{1/2}_{L^2(\mathcal{O})}, \quad x \in D(A_\gamma^{1/2}),$$

and the embedding of $D(A_\gamma^{1/4})$ in $L^4(\mathcal{O})$, we deduce that W_{A_γ} is a Gaussian process in $L^4(\mathcal{O})$. More precisely, we have

$$\mathbb{E}\|W_{A_\gamma}(t)\|^2_{L^4(\mathcal{O})} < \infty. \tag{2.14}$$

Analogously, we find

$$W_{A_0} \in \mathcal{C}([0,T]; H_1) \times L^4([0,T] \times \mathcal{O}), \quad \mathbb{P}\text{-a.s.} \tag{2.15}$$

Now, arguing similarly as previously and by making use of the Burkholder–Davis–Gundy inequality, we obtain

$$\mathbb{E} \sup_{t \in [0,T]} \|W_{A_\gamma}(t)\|^4_{D((A_\gamma)^\sigma)} \leq c \left(\int_0^t \|e^{-\epsilon(t-s)A_\gamma} \sqrt{Q}_2\|^2_{L_2(H_2, D((A_\gamma)^\sigma))} \,\mathrm{d}s \right)^2.$$

Consequently,

$$\mathbb{E} \sup_{t\in[0,T]} \|W_{A_\gamma}(t)\|^4_{D((A_\gamma)^\sigma)} < \infty \quad \text{iff (2.12) holds with } \delta > 3/2. \tag{2.16}$$

In particular, for $\sigma = 1$, we have

$$\mathbb{E} \sup_{t\in[0,T]} \|W_{A_\gamma}(t)\|^4_{D(A_\gamma)} < \infty. \tag{2.17}$$

Also thanks to the embedding of $D((A_\gamma)^{1/4})$ in $L^4(\mathcal{O})$, we deduce that, for $\sigma = 1/4$,

$$\mathbb{E} \sup_{t\in[0,T]} \|W_{A_\gamma}(t)\|^4_{L^4(\mathcal{O})} < \infty. \tag{2.18}$$

As a direct consequence of (2.16), we infer that, for $\sigma = 3/2$,

$$\mathbb{E} \sup_{t\in[0,T]} \|W_{A_\gamma}(t)\|^4_{D(A_\gamma^{3/2})} < \infty. \tag{2.19}$$

Analogously, we find

$$\mathbb{E} \sup_{t\in[0,T]} \|W_{A_0}(t)\|^4_{D(A_0)} < \infty, \tag{2.20}$$

provided that (2.12) holds and $\delta > 3/2$.

From now on, A_0 and A_γ will satisfy (2.15) and (2.20), (2.17), and (2.19), respectively.

Assumption on f

(H1) We assume that $f \in \mathcal{C}^2(\mathbb{R})$ satisfies

$$\begin{cases} \lim_{|r|\to+\infty} f'(r) > 0, \\ |f^{(i)}(r)| \le c_f(1+|r|^{2-i}) \quad \forall r \in \mathbb{R},\ i = 0,1,2, \end{cases} \tag{2.21}$$

where c_f is some positive constant.

(H2) We also assume that

$$(f(\phi), A_1\phi) \ge -\gamma_1 |A_1^{1/2}\phi|^2 \quad \forall \phi \in D(A_1^{1/2}) \tag{2.22}$$

for some constant $\gamma_1 > 0$.

Let us point out that (2.22) is satisfied if there exists a positive constant γ_2 such that

$$f'(r) \ge -\gamma_2 \quad \forall r \in \mathbb{R}. \tag{2.23}$$

3. Existence and uniqueness result for problem (2.6)

With the above framework in place, we now define the notion of local weak solutions of the stochastic Allen–Cahn–Hilliard–Navier–Stokes equations (2.6) that we will work with in this work.

Definition 3.1. Let the assumptions on f be satisfied, and let

$$S = \{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, \{\beta_k^i(t), t \geq 0, k = 1, 2, 3, \dots ; i = 1, 2\}\}$$

be a given stochastic basis and $(\boldsymbol{u}_0, \phi_0) \in H_1 \times V_2$. By the solution to problem (2.6), we mean a pair of functions $(\boldsymbol{u}(t), \phi(t)) \in L^2_W(0, T; \mathbb{V})$ such that $\mathbb{P}$-a.s.

$$\begin{cases} \boldsymbol{u}(t) + \int_0^t [\nu A_0 \boldsymbol{u} + B_0(\boldsymbol{u}, \boldsymbol{u}) - \mathcal{K} R_0(\epsilon A_\gamma \phi, \phi)]\, \mathrm{d}s = \boldsymbol{u}_0 + \sqrt{Q_1}\, W_1(t), \\ \phi(t) + \int_0^t [B_1(\boldsymbol{u}, \phi) + \mu_\gamma]\, \mathrm{d}s = \phi_0 + \sqrt{Q_2}\, W_2(t), \\ \mu_\gamma = \epsilon A_\gamma \phi + \alpha f_\gamma(\phi). \end{cases} \tag{3.1}$$

With the above definition in mind, we are now ready to formulate our main existence result in the following theorem.

Theorem 3.1. *Let $T > 0$ be a fixed positive time. Problem* (2.6) *has a unique solution $(\boldsymbol{u}, \phi)$ in the sense of Definition* 3.1. *Moreover,*

(i) $(\boldsymbol{u}(t), \phi(t)) \in \mathcal{C}(0, T; H_1 \times V_2)$, *$\mathbb{P}$-a.s;*

(ii) *the map $H_1 \times H_2 \to L^\infty(0, T; H_1 \times V_2) \cap L^2(0, T; \mathbb{V})$, $(\boldsymbol{u}_0, \phi_0) \mapsto (\boldsymbol{u}(t), \phi(t))$ is continuous $\mathbb{P}$-a.s.*

To prove Theorem 3.1, we introduce the following translated unknown processes:

$$\boldsymbol{v}(t) = \boldsymbol{u}(t) - W_{A_0}(t), \quad \psi(t) = \phi(t) - W_{A_\gamma}(t),$$

where $(\boldsymbol{u}, \phi)$ is the solution to (2.6).

One can easily check that the deterministic functions $\boldsymbol{v}$ and ψ satisfy

$$\begin{cases} \boldsymbol{v}' + \nu A_0 \boldsymbol{v} + B_0(\boldsymbol{v}, \boldsymbol{v}) + B_0(\boldsymbol{v}, W_{A_0}) + B_0(W_{A_0}, \boldsymbol{v}) - \mathcal{K} R_0(\epsilon A_\gamma \psi, \psi) \\ \quad = -B_0(W_{A_0}, W_{A_0}) + \mathcal{K} R_0(\epsilon A_\gamma W_{A_\gamma}, \psi) + \mathcal{K} R_0(\epsilon A_\gamma \psi, W_{A_\gamma}) \\ \qquad + \mathcal{K} R_0(\epsilon A_\gamma W_{A_\gamma}, W_{A_\gamma}), \\ \psi' + B_1(\boldsymbol{v}, \psi) + B_1(\boldsymbol{v}, W_{A_\gamma}) + B_1(W_{A_0}, \psi) + \epsilon A_\gamma \psi + \epsilon \gamma W_{A_\gamma} \\ \quad = -B_1(W_{A_0}, W_{A_\gamma}) - \alpha f_\gamma(\psi + W_{A_\gamma}), \\ \boldsymbol{v}(0) = \boldsymbol{u}_0, \quad \psi(0) = \phi_0, \end{cases} \tag{3.2}$$

where the derivatives $\boldsymbol{v}'$ and ψ' are taken in the sense of vectorial V_1' (resp., V_2') valued distributions on $(0, T)$ or, equivalently, a.e. on $[0, T]$.

We will now prove that problem (3.2) is well defined, and it is taking a considerable part of this paper.

Proposition 3.1. *Let $T > 0$ be a fixed positive time. Suppose that $(\boldsymbol{u}_0, \phi_0) \in H_1 \times V_2$. Then, there is a unique solution $(\boldsymbol{v}, \psi) \in L^2_W(0, T; \mathbb{V})$ to* (3.2) *such that $\mathbb{P}$-a.s. $(\boldsymbol{v}, \psi) : [0, T] \to \mathbb{V}'$ is absolutely continuous on $[0, T]$ and $\mathbb{P}$-a.s.*

(i) $\frac{\mathrm{d}\boldsymbol{v}(t)}{\mathrm{d}t} \in L^2(0, T; V_1')$, $\frac{\mathrm{d}\psi(t)}{\mathrm{d}t} \in L^2(0, T; (D(A_\gamma))')$,

(ii) $\boldsymbol{v} \in \mathcal{C}(0, T; H_1)$ *and* $\psi \in \mathcal{C}(0, T; V_2)$.

Proof of Proposition 3.1. Let $\varepsilon > 0$ be fixed. We consider the following approximate problem:

$$\begin{cases} \boldsymbol{v}_\varepsilon' + \nu A_0 \boldsymbol{v}_\varepsilon + \Psi_\varepsilon^1(\boldsymbol{v}_\varepsilon) + B_0(\boldsymbol{v}_\varepsilon, W_{A_0}) + B_0(W_{A_0}, \boldsymbol{v}_\varepsilon) \\ \qquad -\mathcal{K}\Psi_\varepsilon^2(\boldsymbol{v}_\varepsilon, \psi_\varepsilon) - \mathcal{K} R_0(\epsilon A_\gamma W_{A_\gamma}, \psi_\varepsilon) - \mathcal{K} R_0(\epsilon A_\gamma \psi_\varepsilon, W_{A_\gamma}) \\ \quad = -B_0(W_{A_0}, W_{A_0}) + \mathcal{K} R_0(\epsilon A_\gamma W_{A_\gamma}, W_{A_\gamma}), \\ \psi_\varepsilon' + \epsilon A_\gamma \psi_\varepsilon + \Psi_\varepsilon^3(\boldsymbol{v}_\varepsilon, \psi_\varepsilon) + B_1(\boldsymbol{v}_\varepsilon, W_{A_\gamma}) + B_1(W_{A_0}, \psi_\varepsilon) \\ \quad = -\alpha f_\gamma(\psi_\varepsilon + W_{A_\gamma}) - B_1(W_{A_0}, W_{A_\gamma}) - \epsilon\gamma W_{A_\gamma}, \\ \boldsymbol{v}_\varepsilon(0) = \boldsymbol{u}_0, \quad \psi_\varepsilon(0) = \phi_0, \end{cases} \tag{3.3}$$

$\mathbb{P}$-a.s. and a.e. $t \in [0, T]$. Here,

$$\Psi_\varepsilon^1(\boldsymbol{v}) = \begin{cases} B_0(\boldsymbol{v}, \boldsymbol{v}) & \text{if } \|\boldsymbol{v}\| \le 1/\varepsilon, \\ \frac{B_0(\boldsymbol{v}, \boldsymbol{v})}{\varepsilon^2 \|\boldsymbol{v}\|^2} & \text{if } \|\boldsymbol{v}\| > 1/\varepsilon, \end{cases}$$

$$\Psi_\varepsilon^2(\boldsymbol{v}, \psi) = \begin{cases} R_0(\epsilon A_\gamma \psi, \psi) & \text{if } \|\boldsymbol{v}\| + |A_\gamma \psi| \le 1/\varepsilon, \\ \frac{R_0(\epsilon A_\gamma \psi, \psi)}{\varepsilon^2 (\|\boldsymbol{v}\| + |A_\gamma \psi|)^2} & \text{if } \|\boldsymbol{v}\| + |A_\gamma \psi| > 1/\varepsilon, \end{cases}$$

$$\Psi_\varepsilon^3(\boldsymbol{v}, \psi) = \begin{cases} B_1(\boldsymbol{v}, \psi) & \text{if } \|\boldsymbol{v}\| + |A_\gamma \psi| \le 1/\varepsilon, \\ \frac{B_1(\boldsymbol{v}, \psi)}{\varepsilon^2 (\|\boldsymbol{v}\| + |A_\gamma \psi|)^2} & \text{if } \|\boldsymbol{v}\| + |A_\gamma \psi| > 1/\varepsilon. \end{cases}$$

Now, in order to prove that (3.3) is well defined, we will use the standard Galerkin method used in the deterministic case (see, for instance, [25]). Since the injection $H_1 \times H^1(\mathcal{O}) \subset V_1 \times D(A_\gamma)$ is compact, let $\{(w_i, \phi_i), i = 1, 2, 3, \ldots\} \subset V_1 \times D(A_\gamma)$ be an orthonormal basis of $H_1 \times H^1(\mathcal{O})$, where $\{w_i, i = 1, 2, \ldots\}$, $\{\phi_i, i = 1, 2, \ldots\}$ are eigenvectors of A_0 and A_γ, respectively. We set $\mathbb{V}_n = \text{span}\{(w_1, \phi_1), \ldots, (w_n, \phi_n)\}$, and we look for $(\boldsymbol{v}_\varepsilon^n, \psi_\varepsilon^n) \in \mathbb{V}_n$ solution to the ordinary differential equations

$$\begin{cases} \frac{\mathrm{d}\boldsymbol{v}_\varepsilon^n}{\mathrm{d}t} + A_0 \boldsymbol{v}_\varepsilon^n + \mathcal{P}_n^1 \Psi_\varepsilon^1(\boldsymbol{v}_\varepsilon^n) + \mathcal{P}_n^1 B_0(\boldsymbol{v}_\varepsilon^n, W_{A_0}) + \mathcal{P}_n^1 B_0(W_{A_0}, \boldsymbol{v}_\varepsilon^n) \\ \qquad -\mathcal{P}_n^1 \Psi_\varepsilon^2(\boldsymbol{v}_\varepsilon^n, \psi_\varepsilon^n) - \mathcal{P}_n^1 R_0(A_\gamma W_{A_\gamma}, \psi_\varepsilon^n) - \mathcal{P}_n^1 R_0(A_\gamma \psi_\varepsilon^n, W_{A_\gamma}) \\ \quad = -\mathcal{P}_n^1 B_0(W_{A_0}, W_{A_0}) + \mathcal{P}_n^1 R_0(A_\gamma W_{A_\gamma}, W_{A_\gamma}), \\ \frac{\mathrm{d}\psi_\varepsilon^n}{\mathrm{d}t} + A_\gamma \psi_\varepsilon^n + \mathcal{P}_n^2 \Psi_\varepsilon^3(\boldsymbol{v}_\varepsilon^n, \psi_\varepsilon^n) + \mathcal{P}_n^2 B_1(\boldsymbol{v}_\varepsilon^n, W_{A_\gamma}) + \mathcal{P}_n^2 B_1(W_{A_0}, \psi_\varepsilon^n) \\ \quad = -f_\gamma(\psi_\varepsilon^n + W_{A_\gamma}) - \mathcal{P}_n^2 B_1(W_{A_0}, W_{A_\gamma}) - \gamma \mathcal{P}_n^2 W_{A_\gamma}, \\ \boldsymbol{v}_\varepsilon^n(0) = \mathcal{P}_n^1 \boldsymbol{u}_0, \quad \psi_\varepsilon^n(0) = \mathcal{P}_n^2 \phi_0, \end{cases} \tag{3.4}$$

where $(\mathcal{P}_n^1, \mathcal{P}_n^2) : H_1 \times L^2(\mathcal{O}) \to \mathbb{V}_n$ is the orthogonal projection; and for the sake of simplicity, we set $\nu = \mathcal{K} = \epsilon = \alpha = 1$. It is classical that, $\mathbb{P}$-a.s., there exists a unique $(\boldsymbol{v}_\varepsilon^n, \psi_\varepsilon^n)$ in $\mathcal{C}(0, T; \mathbb{Y})$ and by taking the scalar product in H_1 of $(3.4)_1$ with $\boldsymbol{v}_\varepsilon^n$, then taking the scalar product in $L^2(\mathcal{O})$ of (3.4)$_2$ with $A_\gamma \psi_\varepsilon^n$,

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(|\boldsymbol{v}_\varepsilon^n|^2 + |A_\gamma^{1/2}\psi_\varepsilon^n|^2) + \|\boldsymbol{v}_\varepsilon^n\|^2 + |A_\gamma \psi_\varepsilon^n|^2 + (B_0(\boldsymbol{v}_\varepsilon^n, W_{A_0}), \boldsymbol{v}_\varepsilon^n) \\
&- (R_0(A_\gamma W_{A_\gamma}, \psi_\varepsilon^n), \boldsymbol{v}_\varepsilon^n) + (B_1(W_{A_0}, \psi_\varepsilon^n), A_\gamma \psi_\varepsilon^n) \\
&\quad = -(B_0(W_{A_0}, W_{A_0}), \boldsymbol{v}_\varepsilon^n) + (R_0(A_\gamma W_{A_\gamma}, W_{A_\gamma}), \boldsymbol{v}_\varepsilon^n) - (A_\gamma^{1/2} f_\gamma(\psi_\varepsilon^n + W_{A_\gamma}), A_\gamma^{1/2}\psi_\varepsilon^n) \\
&\qquad - (B_1(W_{A_0}, W_{A_\gamma}), A_\gamma \psi_\varepsilon^n) - \gamma(W_{A_\gamma}, A_\gamma \psi_\varepsilon^n). \qquad (3.5)
\end{aligned}
$$

Now, by making use of the Hölder, Ladyzhenskaya, and suitable Young inequalities, we find

$$
\begin{aligned}
|(B_0(\boldsymbol{v}_\varepsilon^n, W_{A_0}), \boldsymbol{v}_\varepsilon^n)| &= |-(B_0(\boldsymbol{v}_\varepsilon^n, \boldsymbol{v}_\varepsilon^n), W_{A_0})| \\
&\le \|\boldsymbol{v}_\varepsilon^n\|_{\mathbb{L}^4(\mathcal{O})} \|\boldsymbol{v}_\varepsilon^n\| \|W_{A_0}\|_{\mathbb{L}^4(\mathcal{O})} \\
&\le c(\mathcal{O}) |\boldsymbol{v}_\varepsilon^n|^{\frac{1}{2}} \|\boldsymbol{v}_\varepsilon^n\|^{\frac{3}{2}} \|W_{A_0}\|_{\mathbb{L}^4(\mathcal{O})} \\
&\le \frac{1}{6}\|\boldsymbol{v}_\varepsilon^n\|^2 + c(\mathcal{O})\|W_{A_0}\|_{\mathbb{L}^4(\mathcal{O})}^4 |\boldsymbol{v}_\varepsilon^n|^2. \qquad (3.6)
\end{aligned}
$$

By the Hölder, the Gagliardo–Nirenberg, and the Young inequalities together with the embedding of $\mathbb{H}^1(\mathcal{O})$ in $\mathbb{L}^4(\mathcal{O})$, we obtain

$$
\begin{aligned}
|(R_0(A_\gamma W_{A_\gamma}, \psi_\varepsilon^n), \boldsymbol{v}_\varepsilon^n)| &\le |A_\gamma W_{A_\gamma}| \|\nabla \psi_\varepsilon^n\|_{\mathbb{L}^4(\mathcal{O})} \|\boldsymbol{v}_\varepsilon^n\|_{\mathbb{L}^4(\mathcal{O})} \\
&\le c(\mathcal{O}) |A_\gamma W_{A_\gamma}| |A_\gamma \psi_\varepsilon^n| |\boldsymbol{v}_\varepsilon^n|^{\frac{1}{2}} \|\boldsymbol{v}_\varepsilon^n\|^{\frac{1}{2}} \\
&\le \frac{1}{6}\|\boldsymbol{v}_\varepsilon^n\|^2 + \frac{1}{10}|A_\gamma \psi_\varepsilon^n|^2 + c(\mathcal{O})|A_\gamma W_{A_\gamma}|^4 |\boldsymbol{v}_\varepsilon^n|^2. \qquad (3.7)
\end{aligned}
$$

Using the Hölder and the Gagliardo–Nirenberg inequalities once again, we have

$$
\begin{aligned}
|(B_1(W_{A_0}, \psi_\varepsilon^n), A_\gamma \psi_\varepsilon^n)| &\le \|W_{A_0}\|_{\mathbb{L}^4(\mathcal{O})} \|\nabla \psi_\varepsilon^n\|_{\mathbb{L}^4(\mathcal{O})} |A_\gamma \psi_\varepsilon^n| \\
&\le c(\mathcal{O}) \|W_{A_0}\|_{\mathbb{L}^4(\mathcal{O})} |\nabla \psi_\varepsilon^n|^{\frac{1}{2}} |A_\gamma \psi_\varepsilon^n|^{\frac{3}{2}} \\
&\le \frac{1}{10}|A_\gamma \psi_\varepsilon^n|^2 + c(\mathcal{O})\|W_{A_0}\|_{\mathbb{L}^4(\mathcal{O})}^4 |\nabla \psi_\varepsilon^n|^2, \qquad (3.8)
\end{aligned}
$$

where we used the Young inequality.

Thanks to the Hölder and the Young inequalities, we find

$$
\begin{aligned}
&|(-B_0(W_{A_0}, W_{A_0}), \boldsymbol{v}_\varepsilon^n)| + |(R_0(A_\gamma W_{A_\gamma}, W_{A_\gamma}), \boldsymbol{v}_\varepsilon^n)| \\
&\quad = |(B_0(W_{A_0}, \boldsymbol{v}_\varepsilon^n), W_{A_0})| + |(R_0(A_\gamma W_{A_\gamma}, W_{A_\gamma}), \boldsymbol{v}_\varepsilon^n)| \\
&\quad \le c(\mathcal{O})\|W_{A_0}\|_{\mathbb{L}^4(\mathcal{O})}^2 \|\boldsymbol{v}_\varepsilon^n\| + c(\mathcal{O})|A_\gamma W_{A_\gamma}|^2 \|\boldsymbol{v}_\varepsilon^n\| \\
&\quad \le \frac{1}{6}\|\boldsymbol{v}_\varepsilon^n\|^2 + c(\mathcal{O})\|W_{A_0}\|_{\mathbb{L}^4(\mathcal{O})}^4 + c(\mathcal{O})|A_\gamma W_{A_\gamma}|^4. \qquad (3.9)
\end{aligned}
$$

Using the Hölder and the Young inequalities in conjunction with the embedding of $\mathbb{H}^1(\mathcal{O})$ in $\mathbb{L}^4(\mathcal{O})$, we obtain

$$\begin{aligned}
|(B_1(W_{A_0}, W_{A_\gamma}), A_\gamma \psi_\varepsilon^n)| &\leq c(\mathcal{O})\|W_{A_0}\|_{\mathbb{L}^4(\mathcal{O})}|A_\gamma W_{A_\gamma}||A_\gamma \psi_\varepsilon^n| \\
&\leq \frac{1}{10}|A_\gamma \psi_\varepsilon^n|^2 + c(\mathcal{O})\|W_{A_0}\|^2_{\mathbb{L}^4(\mathcal{O})}|A_\gamma W_{A_\gamma}|^2 \\
&\leq \frac{1}{10}|A_\gamma \psi_\varepsilon^n|^2 + c(\mathcal{O})\|W_{A_0}\|^4_{\mathbb{L}^4(\mathcal{O})} + c(\mathcal{O})|A_\gamma W_{A_\gamma}|^4. \quad (3.10)
\end{aligned}$$

Thanks to the Cauchy–Schwarz and the Young inequalities, we infer that

$$\begin{aligned}
|\gamma(W_{A_\gamma}, A_\gamma \psi_\varepsilon^n)| \leq \gamma|W_{A_\gamma}||A_\gamma \psi_\varepsilon^n| &\leq \frac{1}{10}|A_\gamma \psi_\varepsilon^n|^2 + c\gamma^2|W_{A_\gamma}|^2 \\
&\leq \frac{1}{10}|A_\gamma \psi_\varepsilon^n|^2 + c\gamma|A_\gamma^{1/2} W_{A_\gamma}|^2. \quad (3.11)
\end{aligned}$$

Let us proceed to the third term on the right-hand side of (3.5). One has

$$\begin{aligned}
&-((A_\gamma^{1/2}\psi_\varepsilon^n + A_\gamma^{1/2}W_{A_\gamma})f_\gamma'(\psi_\varepsilon^n + W_{A_\gamma}), A_\gamma^{1/2}\psi_\varepsilon^n) \\
&\quad = -((A_\gamma^{1/2}\psi_\varepsilon^n + A_\gamma^{1/2}W_{A_\gamma})f'(\psi_\varepsilon^n + W_{A_\gamma}), A_\gamma^{1/2}\psi_\varepsilon^n) \\
&\quad + \gamma(A_\gamma^{1/2}\psi_\varepsilon^n + A_\gamma^{1/2}W_{A_\gamma}, A_\gamma^{1/2}\psi_\varepsilon^n).
\end{aligned}$$

Using the Cauchy–Schwarz and the Young inequalities, we deduce that

$$\begin{aligned}
\gamma(A_\gamma^{1/2}\psi_\varepsilon^n + A_\gamma^{1/2}W_{A_\gamma}, A_\gamma^{1/2}\psi_\varepsilon^n) &\leq \gamma|A_\gamma^{1/2}\psi_\varepsilon^n|^2 + \gamma|A_\gamma^{1/2}\psi_\varepsilon^n||A_\gamma^{1/2}W_{A_\gamma}| \\
&\leq \frac{3\gamma}{2}|A_\gamma^{1/2}\psi_\varepsilon^n|^2 + \frac{\gamma}{2}|A_\gamma^{1/2}W_{A_\gamma}|^2.
\end{aligned}$$

In light of (2.23), we have

$$-(A_\gamma^{1/2}\psi_\varepsilon^n f'(\psi_\varepsilon^n + W_{A_\gamma}), A_\gamma^{1/2}\psi_\varepsilon^n) = -\int_{\mathcal{O}} f'(\psi_\varepsilon^n + W_{A_\gamma})|A_\gamma^{1/2}\psi_\varepsilon^n|^2\,\mathrm{d}x \leq \gamma_2|A_\gamma^{1/2}\psi_\varepsilon^n|^2.$$

From $(2.21)_2$ together with the Gagliardo–Nirenberg inequality, we infer that

$$\begin{aligned}
&|-(A_\gamma^{1/2}W_{A_\gamma}f'(\psi_\varepsilon^n + W_{A_\gamma}), A_\gamma^{1/2}\psi_\varepsilon^n)| \\
&\leq \int_{\mathcal{O}} |f'(\psi_\varepsilon^n + W_{A_\gamma})||A_\gamma^{1/2}W_{A_\gamma}||A_\gamma^{1/2}\psi_\varepsilon^n|\,\mathrm{d}x \\
&\leq c_f\int_{\mathcal{O}} (1 + |\psi_\varepsilon^n + W_{A_\gamma}|)|A_\gamma^{1/2}W_{A_\gamma}||A_\gamma^{1/2}\psi_\varepsilon^n|\,\mathrm{d}x \\
&\leq c_f|A_\gamma^{\frac{1}{2}}W_{A_\gamma}||A_\gamma^{\frac{1}{2}}\psi_\varepsilon^n| + c_f(\|\psi_\varepsilon^n\|_{L^3(\mathcal{O})} + \|W_{A_\gamma}\|_{L^3(\mathcal{O})})\|A_\gamma^{\frac{1}{2}}W_{A_\gamma}\|_{\mathbb{L}^3(\mathcal{O})}\|A_\gamma^{1/2}\psi_\varepsilon^n\|_{\mathbb{L}^3(\mathcal{O})} \\
&\leq c_f|A_\gamma^{1/2}W_{A_\gamma}||A_\gamma^{1/2}\psi_\varepsilon^n| \\
&\quad + c_f\gamma^{-\frac{1}{3}}(1+\gamma^{-1})^{\frac{1}{2}}c(\mathcal{O})|A_\gamma^{1/2}\psi_\varepsilon^n|^{\frac{5}{3}}|A_\gamma^{1/2}W_{A_\gamma}|^{\frac{2}{3}}|A_\gamma W_{A_\gamma}|^{\frac{1}{3}}|A_\gamma\psi_\varepsilon^n|^{\frac{1}{3}} \\
&\quad + c_f\gamma^{-\frac{1}{3}}(1+\gamma^{-1})^{\frac{1}{2}}c(\mathcal{O})|A_\gamma^{1/2}W_{A_\gamma}||A_\gamma W_{A_\gamma}|^{\frac{1}{3}}|A_\gamma^{1/2}\psi_\varepsilon^n|^{\frac{2}{3}}|A_\gamma\psi_\varepsilon^n|^{\frac{1}{3}}.
\end{aligned}$$

Using now the Young inequality, we find

$$
\begin{aligned}
&|-(A_\gamma^{1/2} W_{A_\gamma} f'(\psi_\varepsilon^n + W_{A_\gamma}), A_\gamma^{1/2}\psi_\varepsilon^n)| \\
&\le \frac{1}{10}|A_\gamma \psi_\varepsilon^n|^2 + \frac{c_f}{2}|A_\gamma^{1/2}\psi_\varepsilon^n|^2 + \frac{c_f}{2}|A_\gamma^{1/2} W_{A_\gamma}|^2 \\
&\quad + c(\mathcal{O}) c_f^{\frac{6}{5}} \gamma^{-\frac{2}{5}}(1+\gamma^{-1})^{\frac{3}{5}} |A_\gamma^{1/2} W_{A_\gamma}|^{\frac{4}{5}} |A_\gamma W_{A_\gamma}|^{\frac{2}{5}} |A_\gamma^{1/2}\psi_\varepsilon^n|^2 \\
&\quad + c_f^{\frac{6}{5}} \gamma^{-\frac{2}{5}}(1+\gamma^{-1})^{\frac{3}{5}} c(\mathcal{O}) |A_\gamma^{1/2} W_{A_\gamma}|^{\frac{6}{5}} |A_\gamma W_{A_\gamma}|^{\frac{2}{5}} |A_\gamma^{1/2}\psi_\varepsilon^n|^{\frac{4}{5}} \\
&\le \frac{1}{10}|A_\gamma \psi_\varepsilon^n|^2 + \frac{c_f}{2}|A_\gamma^{1/2} W_{A_\gamma}|^2 + c_f^2 \gamma^{-\frac{2}{3}}(1+\gamma^{-1}) |A_\gamma^{1/2} W_{A_\gamma}|^2 |A_\gamma W_{A_\gamma}|^{\frac{2}{3}} \\
&\quad + \Big[\gamma_2 + \frac{c_f}{2} + c(\mathcal{O}) c_f^{\frac{6}{5}} \gamma^{-\frac{2}{5}}(1+\gamma^{-1})^{\frac{3}{5}} |A_\gamma^{1/2} W_{A_\gamma}|^{\frac{4}{5}} |A_\gamma W_{A_\gamma}|^{\frac{2}{5}} + c(\mathcal{O})\Big] |A_\gamma^{1/2}\psi_\varepsilon^n|^2.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&-(A_\gamma^{1/2} f_\gamma(\psi_\varepsilon^n + W_{A_\gamma}), A_\gamma^{1/2}\psi_\varepsilon^n) \\
&\le \frac{1}{10}|A_\gamma \psi_\varepsilon^n|^2 + \frac{\gamma + c_f}{2}|A_\gamma^{1/2} W_{A_\gamma}|^2 + c_f^2 \gamma^{-\frac{2}{3}}(1+\gamma^{-1}) |A_\gamma^{1/2} W_{A_\gamma}|^2 |A_\gamma W_{A_\gamma}|^{\frac{2}{3}} \\
&\quad + \Big[\gamma_2 + \frac{3\gamma}{2} + \frac{c_f}{2} + c(\mathcal{O}) c_f^{\frac{6}{5}} \gamma^{-\frac{2}{5}}(1+\gamma^{-1})^{\frac{3}{5}} |A_\gamma^{1/2} W_{A_\gamma}|^{\frac{4}{5}} |A_\gamma W_{A_\gamma}|^{\frac{2}{5}} + c(\mathcal{O})\Big] \\
&\quad \times |A_\gamma^{1/2}\psi_\varepsilon^n|^2. \qquad (3.12)
\end{aligned}
$$

Collecting now the estimates (3.6)–(3.12) and inserting all of them in (3.5), we obtain the following differential inequality:

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(|\boldsymbol{v}_\varepsilon^n|^2 + |A_\gamma^{1/2}\psi_\varepsilon^n|^2) + \|\boldsymbol{v}_\varepsilon^n\|^2 + |A_\gamma \psi_\varepsilon^n|^2 \\
&\le c\|W_{A_0}\|_{\mathbb{L}^4(\mathcal{O})}^4 + c|A_\gamma W_{A_\gamma}|^4 + c(c_f+1)|A_\gamma^{1/2} W_{A_\gamma}|^2 + c c_f^2 |A_\gamma^{1/2} W_{A_\gamma}|^2 |A_\gamma W_{A_\gamma}|^{\frac{2}{3}} \\
&\quad + c(\|W_{A_0}\|_{\mathbb{L}^4(\mathcal{O})}^4 + |A_\gamma W_{A_\gamma}|^4)|\boldsymbol{v}_\varepsilon^n|^2 \\
&\quad + c\Big[1 + \|W_{A_0}\|_{\mathbb{L}^4(\mathcal{O})}^4 + c_f + c_f^{\frac{6}{5}} |A_\gamma^{1/2} W_{A_\gamma}|^{\frac{4}{5}} |A_\gamma W_{A_\gamma}|^{\frac{2}{5}}\Big] |A_\gamma^{1/2}\psi_\varepsilon^n|^2 \qquad (3.13)
\end{aligned}
$$

for some positive constant $c = c(\mathcal{O}, \gamma, \gamma_2)$.

Integrating (3.13) in time over $[0, t]$, where $t \in [0, T]$, we deduce that

$$
\begin{aligned}
&|\boldsymbol{v}_\varepsilon^n(t)|^2 + |A_\gamma^{1/2}\psi_\varepsilon^n(t)|^2 + \int_0^t \|\boldsymbol{v}_\varepsilon^n(s)\|^2\,\mathrm{d}s + \int_0^t |A_\gamma \psi_\varepsilon^n(s)|^2\,\mathrm{d}s \\
&\quad \le c c_1 + \int_0^t k(s)(|\boldsymbol{v}_\varepsilon^n(s)|^2 + |A_\gamma^{1/2}\psi_\varepsilon^n(s)|^2)\,\mathrm{d}s \qquad (3.14)
\end{aligned}
$$

for some positive constant $c = c(\mathcal{O}, \gamma, \gamma_2, c_f)$. We note that the constant c is independent of ε and n. Here,

$$
\begin{aligned}
c_1 &= |u_0|^2 + |A_\gamma^{1/2}\phi_0|_{L^2}^2 + \int_0^T (\|W_{A_0}(s)\|_{\mathbb{L}^4(\mathcal{O})}^4 + |A_\gamma W_{A_\gamma}(s)|^4 + |A_\gamma^{1/2} W_{A_\gamma}(s)|^2)\,\mathrm{d}s \\
&\quad + \int_0^T |A_\gamma^{1/2} W_{A_\gamma}(s)|^2 |A_\gamma W_{A_\gamma}(s)|^{\frac{2}{3}}\,\mathrm{d}s,
\end{aligned}
$$

$$k(s) = c(1 + \|W_{A_0}(s)\|^4_{\mathbb{L}^4(\mathcal{O})} + |A_\gamma W_{A_\gamma}(s)|^4 + |A_\gamma^{1/2} W_{A_\gamma}(s)|^{\frac{4}{5}} |A_\gamma W_{A_\gamma}(s)|^{\frac{2}{5}}).$$

Hence, by the generalized Gronwall–Bellman lemma (see, for instance, [26, Corollary 1]), we get

$$|\boldsymbol{v}^n_\varepsilon(t)|^2 + |A_\gamma^{1/2}\psi^n_\varepsilon(t)|^2 \le cc_1 \exp\left(\int_0^t k(\tau)\,\mathrm{d}\tau\right) \quad \forall t \in [0,T]. \tag{3.15}$$

Furthermore, from (3.14) and (3.15), we infer that

$$\int_0^t \|\boldsymbol{v}^n_\varepsilon(s)\|^2\,\mathrm{d}s + \int_0^t |A_\gamma\psi^n_\varepsilon(s)|^2\,\mathrm{d}s \le cc_1 \exp\left(\int_0^t k(\tau)\,\mathrm{d}\tau\right) \quad \forall t \in [0,T]. \tag{3.16}$$

As a direct consequence of (3.15) and (3.16), we can say that (for a fixed ε) $(\boldsymbol{v}^n_\varepsilon, \psi^n_\varepsilon)$ is $\mathbb{P}$-a.s. uniformly bounded in

$$L^\infty(0,T;H_1 \times V_2) \cap L^2(0,T;V_1 \times D(A_\gamma)).$$

It then follows from the Banach–Alaoglu theorem that there exists a subsequence of $(\boldsymbol{v}^n_\varepsilon, \psi^n_\varepsilon)$, still denoted by $(\boldsymbol{v}^n_\varepsilon, \psi^n_\varepsilon)$, such that

$$\begin{aligned} (\boldsymbol{v}^n_\varepsilon, \psi^n_\varepsilon) &\to (\boldsymbol{v}_\varepsilon, \psi_\varepsilon) \quad \text{weak-star in } L^\infty(0,T;H_1 \times V_2), \\ (\boldsymbol{v}^n_\varepsilon, \psi^n_\varepsilon) &\to (\boldsymbol{v}_\varepsilon, \psi_\varepsilon) \quad \text{weak in } L^2(0,T;V_1 \times D(A_\gamma)), \end{aligned}$$

where $(\boldsymbol{v}_\varepsilon, \psi_\varepsilon) \in L^\infty(0,T;H_1 \times V_2) \cap L^2(0,T;V_1 \times D(A_\gamma))$ $\mathbb{P}$-a.s.

Furthermore, since the injection $H_1 \times V_2 \subset V_1 \times D(A_\gamma)$ is compact, we have

$$\begin{aligned} (\boldsymbol{v}^n_\varepsilon, \psi^n_\varepsilon) &\to (\boldsymbol{v}_\varepsilon, \psi_\varepsilon) \quad \text{strongly in } L^2(0,T;H_1 \times V_2), \\ (\boldsymbol{v}^n_\varepsilon, \psi^n_\varepsilon) &\to (\boldsymbol{v}_\varepsilon, \psi_\varepsilon) \quad \text{a.e., in } (0,T) \times \mathcal{O}, \end{aligned} \tag{3.17}$$

$\mathbb{P}$-a.s. Now, since the weak convergence in $L^2(0,T;V_1 \times D(A_\gamma))$ is not enough to ensure that

$$\begin{aligned} \Psi^1_\varepsilon(\boldsymbol{v}^n_\varepsilon) &\to \Psi^1_\varepsilon(\boldsymbol{v}_\varepsilon) && \text{as } n \to \infty, \\ \Psi^2_\varepsilon(\boldsymbol{v}^n_\varepsilon, \psi^n_\varepsilon) &\to \Psi^2_\varepsilon(\boldsymbol{v}_\varepsilon, \psi_\varepsilon) && \text{as } n \to \infty, \\ \Psi^3_\varepsilon(\boldsymbol{v}^n_\varepsilon, \psi^n_\varepsilon) &\to \Psi^3_\varepsilon(\boldsymbol{v}_\varepsilon, \psi_\varepsilon) && \text{as } n \to \infty; \end{aligned} \tag{3.18}$$

we need to derive stronger a priori estimates. For this, we take the inner product in H_1 of $(3.4)_1$ with $2A_0\boldsymbol{v}^n_\varepsilon$, the inner product in $L^2(\mathcal{O})$ of $(3.4)_2$ with $2A_\gamma^2\psi^n_\varepsilon$, and obtain, after adding up the corresponding equalities

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}[\|\boldsymbol{v}^n_\varepsilon\|^2 + |A_\gamma\psi^n_\varepsilon|^2] + 2|A_0\boldsymbol{v}^n_\varepsilon|^2 + 2|A_\gamma^{3/2}\psi^n_\varepsilon|^2 \\ &= -2(\Psi^1_\varepsilon(\boldsymbol{v}^n_\varepsilon), A_0\boldsymbol{v}^n_\varepsilon) - 2(B_0(\boldsymbol{v}^n_\varepsilon, W_{A_0}), A_0\boldsymbol{v}^n_\varepsilon) - 2(B_0(W_{A_0}, \boldsymbol{v}^n_\varepsilon), A_0\boldsymbol{v}^n_\varepsilon) \\ &\quad + 2(\Psi^2_\varepsilon(\boldsymbol{v}^n_\varepsilon, \psi^n_\varepsilon), A_0\boldsymbol{v}^n_\varepsilon) + 2(R_0(A_\gamma W_{A_\gamma}, \psi^n_\varepsilon), A_0\boldsymbol{v}^n_\varepsilon) \\ &\quad + 2(R_0(A_\gamma\psi^n_\varepsilon, W_{A_\gamma}), A_0\boldsymbol{v}^n_\varepsilon) - 2(\Psi^3_\varepsilon(\boldsymbol{v}^n_\varepsilon, \psi^n_\varepsilon), A_\gamma^2\psi^n_\varepsilon) \end{aligned}$$

$$
\begin{aligned}
&- 2(A_\gamma^{1/2} B_1(\boldsymbol{v}_\varepsilon^n, W_{A_\gamma}), A_\gamma^{3/2}\psi_\varepsilon^n) - 2(A_\gamma^{1/2} B_1(W_{A_0}, \psi_\varepsilon^n), A_\gamma^{3/2}\psi_\varepsilon^n)\\
&- 2(B_0(W_{A_0}, W_{A_0}), A_0\boldsymbol{v}_\varepsilon^n) + 2(R_0(A_\gamma W_{A_\gamma}, W_{A_\gamma}), A_0\boldsymbol{v}_\varepsilon^n)\\
&- 2(A_\gamma^{1/2} f_\gamma(\psi_\varepsilon^n + W_{A_\gamma}), A_\gamma^{3/2}\psi_\varepsilon^n)\\
&- 2(A_\gamma^{1/2} B_1(W_{A_0}, W_{A_\gamma}), A_\gamma^{3/2}\psi_\varepsilon^n) - 2\gamma(A_\gamma^{1/2} W_{A_\gamma}, A_\gamma^{3/2}\psi_\varepsilon^n). \qquad (3.19)
\end{aligned}
$$

If $\|\boldsymbol{v}_\varepsilon^n\| \le \frac{1}{\varepsilon}$, one has

$$
\begin{aligned}
|(\Psi_\varepsilon^1(\boldsymbol{v}_\varepsilon^n), A_0\boldsymbol{v}_\varepsilon^n)| &\le c|\boldsymbol{v}_\varepsilon^n|^{1/2}|A_0\boldsymbol{v}_\varepsilon^n|^{3/2}\|\boldsymbol{v}_\varepsilon^n\|\\
&\le c\|\boldsymbol{v}_\varepsilon^n\|^{1/2}|A_0\boldsymbol{v}_\varepsilon^n|^{3/2}\|\boldsymbol{v}_\varepsilon^n\|\\
&\le c\varepsilon^{-1}\|\boldsymbol{v}_\varepsilon^n\|^{1/2}|A_0\boldsymbol{v}_\varepsilon^n|^{3/2},
\end{aligned}
$$

and if $\|\boldsymbol{v}_\varepsilon\| > \varepsilon^{-1}$, one has

$$
\begin{aligned}
|(\Psi_\varepsilon^1(\boldsymbol{v}_\varepsilon^n), A_0\boldsymbol{v}_\varepsilon^n)| &\le \frac{c}{\varepsilon^2\|\boldsymbol{v}_\varepsilon^n\|^2}|\boldsymbol{v}_\varepsilon^n|^{1/2}|A_0\boldsymbol{v}_\varepsilon^n|^{3/2}\|\boldsymbol{v}_\varepsilon^n\|\\
&\le \frac{c}{\varepsilon^2\|\boldsymbol{v}_\varepsilon^n\|^2}\|\boldsymbol{v}_\varepsilon^n\|^{1/2}|A_0\boldsymbol{v}_\varepsilon^n|^{3/2}\|\boldsymbol{v}_\varepsilon^n\|\\
&\le c\varepsilon^{-1}\|\boldsymbol{v}_\varepsilon^n\|^{1/2}|A_0\boldsymbol{v}_\varepsilon^n|^{3/2}.
\end{aligned}
$$

Thus, in both cases, we have

$$
|(\Psi_\varepsilon^1(\boldsymbol{v}_\varepsilon^n), A_0\boldsymbol{v}_\varepsilon^n)| \le c\varepsilon^{-1}\|\boldsymbol{v}_\varepsilon^n\|^{1/2}|A_0\boldsymbol{v}_\varepsilon^n|^{3/2} \le \frac{1}{18}|A_0\boldsymbol{v}_\varepsilon^n|^2 + c\varepsilon^{-4}\|\boldsymbol{v}_\varepsilon^n\|^2, \qquad (3.20)
$$

where c is a positive constant which is independent of ε and n.

Observe now that

$$
\begin{aligned}
|(B_0(\boldsymbol{v}_\varepsilon^n, W_{A_0}), A_0\boldsymbol{v}_\varepsilon^n)| &\le c\|\boldsymbol{v}_\varepsilon^n\|^{1/2}|A_0\boldsymbol{v}_\varepsilon^n|^{3/2}\|\nabla W_{A_0}|_{\mathbb{L}^2(\mathcal{O})}\\
&\le \frac{1}{18}|A_0\boldsymbol{v}_\varepsilon^n|^2 + c\|\nabla W_{A_0}|_{\mathbb{L}^2(\mathcal{O})}^4\|\boldsymbol{v}_\varepsilon^n\|^2, \qquad (3.21)\\
|(B_0(W_{A_0}, \boldsymbol{v}_\varepsilon^n), A_0\boldsymbol{v}_\varepsilon^n)| &\le c(\mathcal{O})\|W_{A_0}\|_{\mathbb{L}^4(\mathcal{O})}\|\nabla\boldsymbol{v}_\varepsilon^n\|_{\mathbb{L}^4(\mathcal{O})}|A_0\boldsymbol{v}_\varepsilon^n|\\
&\le c(\mathcal{O})\|W_{A_0}\|_{\mathbb{L}^4(\mathcal{O})}\|\boldsymbol{v}_\varepsilon^n\|^{1/2}|A_0\boldsymbol{v}_\varepsilon^n|^{3/2}\\
&\le \frac{1}{18}|A_0\boldsymbol{v}_\varepsilon^n|^2 + c(\mathcal{O})\|W_{A_0}\|_{\mathbb{L}^4(\mathcal{O})}^4\|\boldsymbol{v}_\varepsilon^n\|^2. \qquad (3.22)
\end{aligned}
$$

Owing to the Gagliardo–Nirenberg inequality, we have

$$
\begin{aligned}
|(\Psi_\varepsilon^2(\boldsymbol{v}_\varepsilon^n, \psi_\varepsilon^n), A_0\boldsymbol{v}_\varepsilon^n)| &\le c(\mathcal{O})|A_0\boldsymbol{v}_\varepsilon^n|\|\nabla\psi_\varepsilon^n\|_{\mathbb{L}^6(\mathcal{O})}\|A_\gamma\psi_\varepsilon^n\|_{L^3(\mathcal{O})}\\
&\le c(\mathcal{O})|A_0\boldsymbol{v}_\varepsilon^n||A_\gamma\psi_\varepsilon^n|^{\frac{5}{3}}|A_\gamma^{3/2}\psi_\varepsilon^n|^{\frac{1}{3}}\\
&\le c(\mathcal{O})\varepsilon^{-1}|A_0\boldsymbol{v}_\varepsilon^n||A_\gamma\psi_\varepsilon^n|^{\frac{2}{3}}|A_\gamma^{3/2}\psi_\varepsilon^n|^{\frac{1}{3}},
\end{aligned}
$$

which holds if $\|\boldsymbol{v}_\varepsilon^n\| + |A_\gamma\psi_\varepsilon^n| \le \varepsilon^{-1}$.

Now, if $\|\boldsymbol{v}_\varepsilon^n\| + |A_\gamma \psi_\varepsilon^n| > \varepsilon^{-1}$, we obtain

$$\begin{aligned}
|(\Psi_\varepsilon^2(\boldsymbol{v}_\varepsilon^n, \psi_\varepsilon^n), A_0\boldsymbol{v}_\varepsilon^n)| &\le \frac{c(\mathcal{O})}{\varepsilon^2(\|\boldsymbol{v}_\varepsilon^n\| + |A_\gamma \psi_\varepsilon^n|)^2}|A_0\boldsymbol{v}_\varepsilon^n|\|\nabla\psi_\varepsilon^n\|_{\mathbb{L}^6(\mathcal{O})}\|A_\gamma\psi_\varepsilon^n\|_{L^3(\mathcal{O})} \\
&\le \frac{c(\mathcal{O})}{\varepsilon^2(\|\boldsymbol{v}_\varepsilon^n\| + |A_\gamma \psi_\varepsilon^n|)^2}|A_0\boldsymbol{v}_\varepsilon^n||A_\gamma\psi_\varepsilon^n|^{\frac{5}{3}}|A_\gamma^{3/2}\psi_\varepsilon^n|^{\frac{1}{3}} \\
&\le c(\mathcal{O})\varepsilon^{-1}|A_0\boldsymbol{v}_\varepsilon^n||A_\gamma\psi_\varepsilon^n|^{\frac{2}{3}}|A_\gamma^{3/2}\psi_\varepsilon^n|^{\frac{1}{3}}.
\end{aligned}$$

In conclusion, when $\|\boldsymbol{v}_\varepsilon^n\| + |A_\gamma \psi_\varepsilon^n| \le \varepsilon^{-1}$ or $\|\boldsymbol{v}_\varepsilon^n\| + |A_\gamma \psi_\varepsilon^n| > \varepsilon^{-1}$, we get

$$\begin{aligned}
|(\Psi_\varepsilon^2(\boldsymbol{v}_\varepsilon^n, \psi_\varepsilon^n), A_0\boldsymbol{v}_\varepsilon^n)| &\le c(\mathcal{O})\varepsilon^{-1}|A_0\boldsymbol{v}_\varepsilon^n||A_\gamma\psi_\varepsilon^n|^{\frac{2}{3}}|A_\gamma^{3/2}\psi_\varepsilon^n|^{\frac{1}{3}} \\
&\le \frac{1}{18}|A_0\boldsymbol{v}_\varepsilon^n|^2 + c(\mathcal{O})\varepsilon^{-2}|A_\gamma\psi_\varepsilon^n|^{\frac{4}{3}}|A_\gamma^{3/2}\psi_\varepsilon^n|^{\frac{2}{3}} \\
&\le \frac{1}{18}|A_0\boldsymbol{v}_\varepsilon^n|^2 + \frac{1}{18}|A_\gamma^{3/2}\psi_\varepsilon^n|^2 + c(\mathcal{O})\varepsilon^{-3}|A_\gamma\psi_\varepsilon^n|^2, \qquad (3.23)
\end{aligned}$$

where we have also used the Young inequality with exponents $(2, 2)$ firstly and secondly with exponents $(3/2, 3)$.

By the Hölder and the Gagliardo–Nirenberg inequalities, we infer that

$$\begin{aligned}
|(R_0(A_\gamma W_{A_\gamma}, \psi_\varepsilon^n), A_0\boldsymbol{v}_\varepsilon^n)| &\le c(\mathcal{O})|A_0\boldsymbol{v}_\varepsilon^n||A_\gamma\psi_\varepsilon^n||A_\gamma W_{A_\gamma}|^{1/2}|A_\gamma^{3/2}W_{A_\gamma}|^{1/2} \\
&\le \frac{1}{18}|A_0\boldsymbol{v}_\varepsilon^n|^2 + c(\mathcal{O})|A_\gamma W_{A_\gamma}|_{L^2(\mathcal{O})}|A_\gamma^{3/2}W_{A_\gamma}||A_\gamma\psi_\varepsilon^n|^2. \qquad (3.24)
\end{aligned}$$

By the Hölder, the Gagliardo–Nirenberg, and suitable Young's inequalities, together with the embedding of $\mathbb{H}^1(\mathcal{O})$ in $\mathbb{L}^6(\mathcal{O})$, we find

$$\begin{aligned}
|(R_0(A_\gamma \psi_\varepsilon^n, W_{A_\gamma}), A_0\boldsymbol{v}_\varepsilon^n)| &\le |A_\gamma\psi_\varepsilon^n|_{L^3(\mathcal{O})}\|\nabla W_{A_\gamma}\|_{\mathbb{L}^6(\mathcal{O})}|A_0\boldsymbol{v}_\varepsilon^n| \\
&\le c(\mathcal{O})|A_0\boldsymbol{v}_\varepsilon^n||A_\gamma W_{A_\gamma}||A_\gamma\psi_\varepsilon^n|^{1/2}|A_\gamma^{3/2}\psi_\varepsilon^n|^{1/2} \\
&\le \frac{1}{18}|A_0\boldsymbol{v}_\varepsilon^n|^2 + \frac{1}{18}|A_\gamma^{3/2}\psi_\varepsilon^n|^2 + c(\mathcal{O})|A_\gamma W_{A_\gamma}|^4|A_\gamma\psi_\varepsilon^n|^2. \qquad (3.25)
\end{aligned}$$

In the case $\|\boldsymbol{v}_\varepsilon^n\| + |A_\gamma \psi_\varepsilon^n| \le \varepsilon^{-1}$, we get

$$\begin{aligned}
&|(\Psi_\varepsilon^3(\boldsymbol{v}_\varepsilon^n, \psi_\varepsilon^n), A_\gamma^2\psi_\varepsilon^n)| \\
&\quad = |(A_\gamma^{1/2}B_1(\boldsymbol{v}_\varepsilon^n, \psi_\varepsilon^n), A_\gamma^{3/2}\psi_\varepsilon^n)| \\
&\quad \le c(\mathcal{O})\|\boldsymbol{v}_\varepsilon^n\|^{\frac{1}{2}}|A_0\boldsymbol{v}_\varepsilon^n|^{\frac{1}{2}}|A_\gamma\psi_\varepsilon^n||A_\gamma^{3/2}\psi_\varepsilon^n| + c(\mathcal{O})\|\boldsymbol{v}_\varepsilon^n\||A_\gamma\psi_\varepsilon^n|^{\frac{1}{2}}|A_\gamma^{3/2}\psi_\varepsilon^n|^{3/2} \\
&\quad \le c(\mathcal{O})\varepsilon^{-1}\|\boldsymbol{v}_\varepsilon^n\|^{\frac{1}{2}}|A_0\boldsymbol{v}_\varepsilon^n|^{\frac{1}{2}}|A_\gamma^{3/2}\psi_\varepsilon^n| + c(\mathcal{O})\varepsilon^{-1}|A_\gamma\psi_\varepsilon^n|^{\frac{1}{2}}|A_\gamma^{3/2}\psi_\varepsilon^n|^{3/2}.
\end{aligned}$$

Now, if $\|\boldsymbol{v}_\varepsilon^n\| + |A_\gamma \psi_\varepsilon^n| > \varepsilon^{-1}$, we obtain

$$\begin{aligned}
&|(\Psi_\varepsilon^3(\boldsymbol{v}_\varepsilon^n, \psi_\varepsilon^n), A_\gamma^2 \psi_\varepsilon^n)| \\
&\quad = \frac{1}{\varepsilon^2(\|\boldsymbol{v}_\varepsilon^n\| + |A_\gamma \psi_\varepsilon^n|)^2}(|B_1(\boldsymbol{v}_\varepsilon^n, \psi_\varepsilon^n), A_\gamma^2 \psi_\varepsilon^n)| \\
&\quad = \frac{1}{\varepsilon^2(\|\boldsymbol{v}_\varepsilon^n\| + |A_\gamma \psi_\varepsilon^n|)^2}|(A_\gamma^{1/2} B_1(\boldsymbol{v}_\varepsilon^n, \psi_\varepsilon^n), A_\gamma^{3/2} \psi_\varepsilon^n)| \\
&\quad \le c(\mathcal{O})\|\boldsymbol{v}_\varepsilon^n\|^{\frac{1}{2}}|A_0 \boldsymbol{v}_\varepsilon^n|^{\frac{1}{2}}|A_\gamma \psi_\varepsilon^n||A_\gamma^{3/2}\psi_\varepsilon^n| + c(\mathcal{O})\|\boldsymbol{v}_\varepsilon^n\||A_\gamma \psi_\varepsilon^n|^{\frac{1}{2}}|A_\gamma^{3/2}\psi_\varepsilon^n|^{3/2} \\
&\quad \le c(\mathcal{O})\varepsilon^{-1}\|\boldsymbol{v}_\varepsilon^n\|^{\frac{1}{2}}|A_0 \boldsymbol{v}_\varepsilon^n|^{\frac{1}{2}}|A_\gamma^{3/2}\psi_\varepsilon^n| + c(\mathcal{O})\varepsilon^{-1}|A_\gamma \psi_\varepsilon^n|^{\frac{1}{2}}|A_\gamma^{3/2}\psi_\varepsilon^n|^{3/2}.
\end{aligned}$$

So, for both cases, we derive the following estimate:

$$\begin{aligned}
&|(\Psi_\varepsilon^3(\boldsymbol{v}_\varepsilon^n, \psi_\varepsilon^n), A_\gamma^2 \psi_\varepsilon^n)| \\
&\quad \le c(\mathcal{O})\varepsilon^{-1}\|\boldsymbol{v}_\varepsilon^n\|^{\frac{1}{2}}|A_0 \boldsymbol{v}_\varepsilon^n|^{\frac{1}{2}}|A_\gamma^{3/2}\psi_\varepsilon^n| + c(\mathcal{O})\varepsilon^{-1}|A_\gamma \psi_\varepsilon^n|^{\frac{1}{2}}|A_\gamma^{3/2}\psi_\varepsilon^n|^{3/2} \\
&\quad \le \frac{1}{18}|A_0 \boldsymbol{v}_\varepsilon^n|^2 + \frac{1}{18}|A_\gamma^{3/2}\psi_\varepsilon^n|^2 + c(\mathcal{O})\varepsilon^{-4}\|\boldsymbol{v}_\varepsilon^n\|^2 + c(\mathcal{O})\varepsilon^{-4}|A_\gamma \psi_\varepsilon^n|^2. \qquad (3.26)
\end{aligned}$$

Thanks to the Agmon inequality (as found in, for example, [30, p. 52]) along with the Young inequality, we see that

$$\begin{aligned}
&|(A_\gamma^{1/2} B_1(\boldsymbol{v}_\varepsilon^n, W_{A_\gamma}), A_\gamma^{3/2}\psi_\varepsilon^n)| \\
&\quad \le c(\mathcal{O})\|\boldsymbol{v}_\varepsilon^n\|^{\frac{1}{2}}|A_0 \boldsymbol{v}_\varepsilon^n|^{\frac{1}{2}}|A_\gamma W_{A_\gamma}||A_\gamma^{3/2}\psi_\varepsilon^n| \\
&\quad \le \frac{1}{18}|A_0 \boldsymbol{v}_\varepsilon^n|^2 + \frac{1}{18}|A_\gamma^{3/2}\psi_\varepsilon^n|^2 + c(\mathcal{O})|A_\gamma W_{A_\gamma}|^4\|\boldsymbol{v}_\varepsilon^n\|^2. \qquad (3.27)
\end{aligned}$$

Once more, using the Agmon and the Gagliardo–Nirenberg inequalities, we obtain

$$\begin{aligned}
|(A_\gamma^{1/2} B_1(W_{A_0}, \psi_\varepsilon^n), A_\gamma^{3/2}\psi_\varepsilon^n)| &\le c(\mathcal{O})\|\nabla W_{A_0}\|_{\mathbb{L}^2(\mathcal{O})}|A_\gamma^{1/2}\psi_\varepsilon^n|^{1/2}|A_\gamma^{3/2}\psi_\varepsilon^n|^{3/2} \\
&\quad + c(\mathcal{O})\|W_{A_0}\|_{\mathbb{L}^4(\mathcal{O})}|A_\gamma \psi_\varepsilon^n|^{1/2}|A_\gamma^{3/2}\psi_\varepsilon^n|^{3/2} \\
&\le c(\mathcal{O})\|\nabla W_{A_0}\|_{\mathbb{L}^2(\mathcal{O})}|A_\gamma \psi_\varepsilon^n|^{1/2}|A_\gamma^{3/2}\psi_\varepsilon^n|^{3/2} \\
&\quad + c(\mathcal{O})\|W_{A_0}\|_{\mathbb{L}^4(\mathcal{O})}|A_\gamma \psi_\varepsilon^n|^{1/2}|A_\gamma^{3/2}\psi_\varepsilon^n|^{3/2} \\
&\le \frac{1}{18}|A_\gamma^{3/2}\psi_\varepsilon^n|^2 + c(\mathcal{O})\|\nabla W_{A_0}\|_{\mathbb{L}^2(\mathcal{O})}^4|A_\gamma \psi_\varepsilon^n|^2 \\
&\quad + c(\mathcal{O})\|W_{A_0}\|_{\mathbb{L}^4(\mathcal{O})}^4|A_\gamma \psi_\varepsilon^n|^2. \qquad (3.28)
\end{aligned}$$

By combining the Hölder, the Gagliardo–Nirenberg, and the Young inequalities, we deduce that

$$\begin{aligned}
|(B_0(W_{A_0}, W_{A_0}), A_0 \boldsymbol{v}_\varepsilon^n)| &\le c(\mathcal{O})|A_0 W_{A_0}|^2\|\boldsymbol{v}_\varepsilon^n\| \\
&\le c(\mathcal{O}) + c(\mathcal{O})|A_0 W_{A_0}|^4\|\boldsymbol{v}_\varepsilon^n\|^2, \qquad (3.29)
\end{aligned}$$

$$
\begin{aligned}
|(R_0(A_\gamma W_{A_\gamma}, W_{A_\gamma}), A_0 \boldsymbol{v}_\varepsilon^n)| &\le c(\mathcal{O})|A_0 \boldsymbol{v}_\varepsilon^n||A_\gamma W_{A_\gamma}|^{3/2}|A_\gamma^{3/2} W_{A_\gamma}|^{1/2} \\
&\le \frac{1}{18}|A_0 \boldsymbol{v}_\varepsilon^n|^2 + c(\mathcal{O})|A_\gamma W_{A_\gamma}|^3 |A_\gamma^{3/2} W_{A_\gamma}|, \qquad (3.30) \\
|(A_\gamma^{1/2} B_1(W_{A_0}, W_{A_\gamma}), A_\gamma^{3/2}\psi_\varepsilon^n)| &\le c(\mathcal{O})\|\nabla W_{A_0}|_{\mathbb{L}^2(\mathcal{O})}|A_\gamma W_{A_\gamma}|^{\frac12}|A_\gamma^{\frac32} W_{A_\gamma}|^{\frac12}|A_\gamma^{\frac32}\psi_\varepsilon^n| \\
&\quad + c(\mathcal{O})\|W_{A_0}\|_{\mathbb{L}^4(\mathcal{O})}|A_\gamma W_{A_\gamma}|^{\frac12}|A_\gamma^{\frac32} W_{A_\gamma}|^{\frac12}|A_\gamma^{\frac32}\psi_\varepsilon^n| \\
&\le \frac{1}{18}|A_\gamma^{3/2}\psi_\varepsilon^n|^2 + c(\mathcal{O})\|\nabla W_{A_0}|^2_{\mathbb{L}^2(\mathcal{O})}|A_\gamma W_{A_\gamma}||A_\gamma^{\frac32} W_{A_\gamma}| \\
&\quad + c(\mathcal{O})\|W_{A_0}\|^2_{\mathbb{L}^4(\mathcal{O})}|A_\gamma W_{A_\gamma}||A_\gamma^{\frac32} W_{A_\gamma}|, \qquad (3.31) \\
\gamma|(A_\gamma^{1/2} W_{A_\gamma}, A_\gamma^{3/2}\psi_\varepsilon^n)| &\le \gamma|A_\gamma^{3/2}\psi_\varepsilon^n||A_\gamma^{1/2} W_{A_\gamma}|_{L^2} \\
&\le \frac{1}{18}|A_\gamma^{3/2}\psi_\varepsilon^n|^2 + \frac{9}{2}\gamma^2|A_\gamma^{1/2} W_{A_\gamma}|^2. \qquad (3.32)
\end{aligned}
$$

We will now estimate the sixth term on the right-hand side of (3.19). We recall that $f_\gamma(r) = f(r) - \alpha^{-1}\epsilon\gamma r$, for all $r \in \mathbb{R}$. Thus,

$$
\begin{aligned}
(A_\gamma^{1/2} f_\gamma(\psi_\varepsilon^n + W_{A_\gamma}), A_\gamma^{3/2}\psi_\varepsilon^n) &= ((A_\gamma^{1/2}\psi_\varepsilon^n + A_\gamma^{1/2} W_{A_\gamma}) f_\gamma'(\psi_\varepsilon^n + W_{A_\gamma}), A_\gamma^{3/2}\psi_\varepsilon^n) \\
&= ((A_\gamma^{1/2}\psi_\varepsilon^n + A_\gamma^{1/2} W_{A_\gamma}) f'(\psi_\varepsilon^n + W_{A_\gamma}), A_\gamma^{3/2}\psi_\varepsilon^n) \\
&\quad - \alpha^{-1}\epsilon\gamma(A_\gamma^{1/2}\psi_\varepsilon^n + A_\gamma^{1/2} W_{A_\gamma}, A_\gamma^{3/2}\psi_\varepsilon^n).
\end{aligned}
$$

Now, by Cauchy–Schwarz's and Young's inequalities, we obtain

$$
\begin{aligned}
&|-\alpha^{-1}\epsilon\gamma(A_\gamma^{1/2}\psi_\varepsilon^n + A_\gamma^{1/2} W_{A_\gamma}, 2A_\gamma^{3/2}\psi_\varepsilon^n)| \\
&\quad \le \alpha^{-1}\epsilon\gamma(|A_\gamma^{1/2}\psi_\varepsilon^n| + |A_\gamma^{1/2} W_{A_\gamma}|)|A_\gamma^{3/2}\psi_\varepsilon^n| \\
&\quad \le \gamma(|A_\gamma^{1/2}\psi_\varepsilon^n| + |A_\gamma^{1/2} W_{A_\gamma}|)|A_\gamma^{3/2}\psi_\varepsilon^n| \\
&\quad \le \frac{1}{54}|A_\gamma^{3/2}\psi_\varepsilon^n|^2 + c\gamma^2(|A_\gamma^{1/2}\psi_\varepsilon^n|^2 + |A_\gamma^{1/2} W_{A_\gamma}|^2). \qquad (3.33)
\end{aligned}
$$

Thanks to the Hölder inequality together with (2.21), we deduce that

$$
\begin{aligned}
&|((A_\gamma^{1/2}\psi_\varepsilon^n + A_\gamma^{1/2} W_{A_\gamma}) f'(\psi_\varepsilon^n + W_{A_\gamma}), A_\gamma^{3/2}\psi_\varepsilon^n)| \\
&\quad \le \int_{\mathcal{O}} |f'(\psi_\varepsilon^n + W_{A_\gamma})||A_\gamma^{1/2}\psi_\varepsilon^n||A_\gamma^{3/2}\psi_\varepsilon^n| \,\mathrm{d}x \\
&\qquad + \int_{\mathcal{O}} |f'(\psi_\varepsilon^n + W_{A_\gamma})||A_\gamma^{1/2} W_{A_\gamma}||A_\gamma^{3/2}\psi_\varepsilon^n| \,\mathrm{d}x \\
&\quad \le c_f \int_{\mathcal{O}} (1 + |\psi_\varepsilon^n + W_{A_\gamma}|)|A_\gamma^{1/2}\psi_\varepsilon^n||A_\gamma^{3/2}\psi_\varepsilon^n| \mathrm{d}x \\
&\qquad + c_f \int_{\mathcal{O}} (1 + |\psi_\varepsilon^n + W_{A_\gamma}|)|A_\gamma^{1/2} W_{A_\gamma}||A_\gamma^{3/2}\psi_\varepsilon^n| \,\mathrm{d}x \\
&\quad \le c_f|A_\gamma^{1/2}\psi_\varepsilon^n||A_\gamma^{3/2}\psi_\varepsilon^n| + c_f|A_\gamma^{1/2} W_{A_\gamma}||A_\gamma^{3/2}\psi_\varepsilon^n| \\
&\qquad + c_f(\|\psi_\varepsilon^n\|_{L^\infty(\mathcal{O})} + \|W_{A_\gamma}\|_{L^\infty(\mathcal{O})})|A_\gamma^{1/2}\psi_\varepsilon^n||A_\gamma^{3/2}\psi_\varepsilon^n| \\
&\qquad + c_f(\|\psi_\varepsilon^n\|_{L^\infty(\mathcal{O})} + \|W_{A_\gamma}\|_{L^\infty(\mathcal{O})})|A_\gamma^{1/2} W_{A_\gamma}||A_\gamma^{3/2}\psi_\varepsilon^n|.
\end{aligned}
$$

The Young inequality implies

$$\begin{aligned} &c_f|A_\gamma^{1/2}\psi_\varepsilon^n||A_\gamma^{3/2}\psi_\varepsilon^n| + c_f|A_\gamma^{1/2}W_{A_\gamma}||A_\gamma^{3/2}\psi_\varepsilon^n| \\ &\le \frac{1}{54}|A_\gamma^{3/2}\psi_\varepsilon^n|^2 + cc_f^2|A_\gamma^{1/2}\psi_\varepsilon^n|^2 + cc_f^2|A_\gamma^{1/2}W_{A_\gamma}|^2 \end{aligned}$$

for some positive constant c independent of n and ε.

Owing to the embedding of $D(A_\gamma)$ in $L^\infty(\mathcal{O})$, we obtain

$$\begin{aligned} &c_f(\|\psi_\varepsilon^n\|_{L^\infty(\mathcal{O})} + \|W_{A_\gamma}\|_{L^\infty(\mathcal{O})})|A_\gamma^{1/2}\psi_\varepsilon^n||A_\gamma^{3/2}\psi_\varepsilon^n| \\ &\quad + c_f(\|\psi_\varepsilon^n\|_{L^\infty(\mathcal{O})} + \|W_{A_\gamma}\|_{L^\infty(\mathcal{O})})|A_\gamma^{1/2}W_{A_\gamma}||A_\gamma^{3/2}\psi_\varepsilon^n| \\ &\le c(\mathcal{O})c_f|A_\gamma\psi_\varepsilon^n||A_\gamma^{1/2}\psi_\varepsilon^n||A_\gamma^{3/2}\psi_\varepsilon^n| + c(\mathcal{O})c_f|A_\gamma\psi_\varepsilon^n||A_\gamma^{1/2}W_{A_\gamma}||A_\gamma^{3/2}\psi_\varepsilon^n| \\ &\quad + c(\mathcal{O})c_f|A_\gamma W_{A_\gamma}||A_\gamma^{1/2}\psi_\varepsilon^n||A_\gamma^{3/2}\psi_\varepsilon^n| + c(\mathcal{O})c_f|A_\gamma W_{A_\gamma}||A_\gamma^{1/2}W_{A_\gamma}||A_\gamma^{3/2}\psi_\varepsilon^n| \\ &\le \frac{1}{54}|A_\gamma^{3/2}\psi_\varepsilon^n|^2 + c(\mathcal{O})c_f^2|A_\gamma^{1/2}\psi_\varepsilon^n|^2|A_\gamma\psi_\varepsilon^n|^2 + c(\mathcal{O})c_f^2|A_\gamma^{1/2}W_{A_\gamma}|^2|A_\gamma\psi_\varepsilon^n|^2 \\ &\quad + c(\mathcal{O})c_f^2|A_\gamma W_{A_\gamma}|^2|A_\gamma^{1/2}\psi_\varepsilon^n|^2 + c(\mathcal{O})c_f^2|A_\gamma W_{A_\gamma}|^2|A_\gamma^{1/2}W_{A_\gamma}|^2, \end{aligned}$$

where we used the Young inequality. Hence,

$$\begin{aligned} &|(A_\gamma^{1/2}f_\gamma(\psi_\varepsilon^n + W_{A_\gamma}), A_\gamma^{3/2}\psi_\varepsilon^n)| \\ &\le \frac{1}{18}|A_\gamma^{3/2}\psi_\varepsilon^n|^2 + c|A_\gamma^{1/2}\psi_\varepsilon^n|^2 + c|A_\gamma^{1/2}W_{A_\gamma}|^2 \\ &\quad + c(|A_\gamma^{1/2}\psi_\varepsilon^n|^2 + |A_\gamma^{1/2}W_{A_\gamma}|^2)|A_\gamma\psi_\varepsilon^n|^2 \\ &\quad + c|A_\gamma W_{A_\gamma}|^2|A_\gamma^{1/2}\psi_\varepsilon^n|^2 + c|A_\gamma W_{A_\gamma}|^2|A_\gamma^{1/2}W_{A_\gamma}|^2, \end{aligned} \tag{3.34}$$

where $c = c(\mathcal{O}, \gamma, c_f)$ is a positive constant which is independent of n and ε.

Plugging (3.20)–(3.34) into the right-hand side of (3.19), we arrive at

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}[\|\boldsymbol{v}_\varepsilon^n\|^2 + |A_\gamma\psi_\varepsilon^n|^2] + |A_0\boldsymbol{v}_\varepsilon^n|^2 + |A_\gamma^{3/2}\psi_\varepsilon^n|^2 \\ &\le c + c(\varepsilon^{-4} + |\nabla W_{A_0}|^4 + \|W_{A_0}\|_{\mathbb{L}^4(\mathcal{O})}^4 + |A_0W_{A_0}|^4 + |A_\gamma W_{A_\gamma}|^4)\|\boldsymbol{v}_\varepsilon^n\|^2 \\ &\quad + c\big(\varepsilon^{-3} + \varepsilon^{-4} + |A_\gamma W_{A_\gamma}||A_\gamma^{3/2}W_{A_\gamma}| + |A_\gamma W_{A_\gamma}|^4 + |\nabla W_{A_0}|^4 \\ &\quad + \|W_{A_0}\|_{\mathbb{L}^4(\mathcal{O})}^4 + |A_\gamma^{1/2}\psi_\varepsilon^n|^2 + |A_\gamma^{1/2}W_{A_\gamma}|^2\big)|A_\gamma\psi_\varepsilon^n|^2 \\ &\quad + c|A_\gamma W_{A_\gamma}|^3|A_\gamma^{3/2}W_{A_\gamma}| + c|A_\gamma W_{A_\gamma}|^2|A_\gamma^{1/2}\psi_\varepsilon^n|^2 + c|A_\gamma W_{A_\gamma}|^2|A_\gamma^{1/2}W_{A_\gamma}|^2 \\ &\quad + c|\nabla W_{A_0}|^2|A_\gamma W_{A_\gamma}||A_\gamma^{\frac{3}{2}}W_{A_\gamma}| + c\|W_{A_0}\|_{\mathbb{L}^4(\mathcal{O})}^2|A_\gamma W_{A_\gamma}||A_\gamma^{\frac{3}{2}}W_{A_\gamma}| \\ &\quad + c|A_\gamma^{1/2}\psi_\varepsilon^n|^2 + c|A_\gamma^{1/2}W_{A_\gamma}|^2 + c|A_\gamma^{1/2}W_{A_\gamma}|^2 \end{aligned} \tag{3.35}$$

for some $c = c(\mathcal{O}, \gamma, c_f)$.

Let us set

$$
\begin{aligned}
\mathcal{Y}_{n,\varepsilon}(t) &= \|\boldsymbol{v}_\varepsilon^n(t)\|^2 + |A_\gamma \psi_\varepsilon^n(t)|^2,\\
\mathcal{Y}_1(t) &= c(\varepsilon^{-4} + |\nabla W_{A_0}(t)|^4 + \|W_{A_0}(t)\|^4_{\mathbb{L}^4(\mathcal{O})} + |A_0 W_{A_0}(t)|^4 + |A_\gamma W_{A_\gamma}(t)|^4)\\
&\quad + c\big(\varepsilon^{-3} + \varepsilon^{-4} + |A_\gamma W_{A_\gamma}(t)||A_\gamma^{3/2} W_{A_\gamma}| + |A_\gamma W_{A_\gamma}(t)|^4 + |\nabla W_{A_0}(t)|^4\\
&\quad + \|W_{A_0}(t)\|^4_{\mathbb{L}^4(\mathcal{O})} + |A_\gamma^{1/2}\psi_\varepsilon^n(t)|^2 + |A_\gamma^{1/2} W_{A_\gamma}(t)|^2\big),\\
\mathcal{Y}_2(t) &= c\big(1 + |A_\gamma W_{A_\gamma}(t)|^3 |A_\gamma^{3/2} W_{A_\gamma}(t)| + |A_\gamma W_{A_\gamma}(t)|^2 |A_\gamma^{1/2}\psi_\varepsilon^n(t)|^2\\
&\quad + |A_\gamma W_{A_\gamma}(t)|^2 |A_\gamma^{1/2} W_{A_\gamma}(t)|^2 + |\nabla W_{A_0}(t)|^2 |A_\gamma W_{A_\gamma}(t)||A_\gamma^{3/2} W_{A_\gamma}(t)|\\
&\quad + \|W_{A_0}(t)\|^2_{\mathbb{L}^4(\mathcal{O})} |A_\gamma W_{A_\gamma}(t)||A_\gamma^{3/2} W_{A_\gamma}(t)|\\
&\quad + |A_\gamma^{1/2}\psi_\varepsilon^n(t)|^2 + |A_\gamma^{1/2} W_{A_\gamma}(t)|^2 + |A_\gamma^{1/2} W_{A_\gamma}|^2\big).
\end{aligned}
$$

Hence, we can rewrite (3.35) as follows:

$$
\frac{\mathrm{d}\mathcal{Y}_{n,\varepsilon}}{\mathrm{d}t} + |A_0 \boldsymbol{v}_\varepsilon^n|^2 + |A_\gamma^{3/2}\psi_\varepsilon^n|^2 \le \mathcal{Y}_1 \mathcal{Y}_{n,\varepsilon} + \mathcal{Y}_2. \tag{3.36}
$$

Notice that

$$
\|(\boldsymbol{v}_\varepsilon^n, \psi_\varepsilon^n)(0)\|_{\mathbb{V}} = \|(\mathcal{P}_n^1 \boldsymbol{u}_0, \mathcal{P}_n^2 \phi_0)\|_{\mathbb{V}} \le \|(\boldsymbol{u}_0, \phi_0)\|_{\mathbb{V}} < +\infty
$$

provided that $(\boldsymbol{u}_0, \phi_0) \in \mathbb{V}$. Hence, by assuming also that $(\boldsymbol{u}_0, \phi_0) \in \mathbb{V}$, and since (2.19), (2.20), and (3.16) hold true, we then derive from (3.36) by an application of the Gronwall lemma that the sequence $(\boldsymbol{u}_\varepsilon^n, \psi_\varepsilon^n)$ satisfies

$$
\|\boldsymbol{v}_\varepsilon^n(t)\|^2 + |A_\gamma \psi_\varepsilon^n(t)|^2 \le C_\varepsilon, \quad \int_0^T (|A_0 \boldsymbol{v}_\varepsilon^n(s)|^2 + |A_\gamma^{3/2}\psi_\varepsilon^n(s)|^2)\,\mathrm{d}s \le C_\varepsilon, \ \mathbb{P}\text{-a.s.,} \tag{3.37}
$$

which proves that (for a fixed positive number ε) the sequence $(\boldsymbol{v}_\varepsilon^n, \psi_\varepsilon^n)$ is uniformly bounded in $L^\infty(0,T;\mathbb{V}) \cap L^2(0,T;D(A_0) \times D(A_\gamma^{3/2}))$, $\mathbb{P}$-a.s.

Furthermore, using (3.37), we can check that

$$
\left(\frac{\mathrm{d}\boldsymbol{v}_\varepsilon^n}{\mathrm{d}t}, \frac{\mathrm{d}\psi_\varepsilon^n}{\mathrm{d}t}\right) \text{ is bounded in } L^2(0,T;\mathbb{Y}) \ \mathbb{P}\text{-a.s.} \tag{3.38}
$$

Since $D(A_0) \times D(A_\gamma^{3/2}) \subset \mathbb{V} \subset \mathbb{Y}$ with compact injections, by [22, Theorem 5.1, Chapter 1], there exists $(\boldsymbol{v}_\varepsilon, \psi_\varepsilon) \in L^\infty(0,T;\mathbb{Y}) \cap L^2(0,T;D(A_0) \times D(A_\gamma^{3/2}))$, and a subsequence of $(\boldsymbol{v}_\varepsilon^n, \psi_\varepsilon^n)$ (still) denoted by $(v_\varepsilon^n, \psi_\varepsilon^n)$ such that for all $T > 0$, we have $\mathbb{P}$-a.s.

$$
\begin{aligned}
(\boldsymbol{v}_\varepsilon^n, \psi_\varepsilon^n) &\to (\boldsymbol{v}_\varepsilon, \psi_\varepsilon) && \text{strongly in } L^2(0,T;\mathbb{V}),\\
(\boldsymbol{v}_\varepsilon^n, \psi_\varepsilon^n) &\to (\boldsymbol{v}_\varepsilon, \psi_\varepsilon) && \text{a.e., in } (0,T) \times \mathcal{O},\\
(\boldsymbol{v}_\varepsilon^n, \psi_\varepsilon^n) &\to (\boldsymbol{v}_\varepsilon, \psi_\varepsilon) && \text{weak-star in } L^\infty(0,T;\mathbb{V}),\\
(\boldsymbol{v}_\varepsilon^n, \psi_\varepsilon^n) &\to (\boldsymbol{v}_\varepsilon, \psi_\varepsilon) && \text{weakly in } L^2(0,T;D(A_0) \times D(A_\gamma^{3/2})),\\
\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{v}_\varepsilon^n, \psi_\varepsilon^n) &\to \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{v}_\varepsilon, \psi_\varepsilon) && \text{weakly in } L^2(0,T;\mathbb{Y}).
\end{aligned}
\tag{3.39}
$$

Furthermore, since $(\boldsymbol{v}_\varepsilon^n, \psi_\varepsilon^n) \to (\boldsymbol{v}_\varepsilon, \psi_\varepsilon)$ as $n \to +\infty$ in $L^2(0, T; \mathbb{V})$ for all $T > 0$, there exists a subsequence (still) denoted by $(\boldsymbol{v}_\varepsilon^n, \psi_\varepsilon^n)$ (see [5, Theorem 4.9]) such that

$$\|(\boldsymbol{v}_\varepsilon^n(\omega), \psi_\varepsilon^n(\omega))\|_{\mathbb{V}} \to (\boldsymbol{v}_\varepsilon(\omega), \psi_\varepsilon(\omega)) \quad \text{a.e. in } \mathcal{O} \tag{3.40}$$

and for a fix $\omega \in \Omega$. With these convergences (3.39)–(3.40) in hand, we derive (3.18).

From (3.18) and (3.39), we can take limits in (3.4) exactly as in [9], with reference to the proof of Theorem 7, and we obtain that $(\boldsymbol{v}_\varepsilon, \psi_\varepsilon)$ is a solution to (3.3).

The case where the initial data $(\boldsymbol{u}_0, \phi_0)$ belongs to $\mathbb{Y}\backslash\mathbb{V}$ can be done similarly as in [25] in order to prove that $(\boldsymbol{v}_\varepsilon, \psi_\varepsilon)$ is a solution to problem (3.3).

3.1. Uniform estimates in ε

Multiplying the first and second equations of (3.3) by $\mathcal{K}^{-1}\boldsymbol{v}_\varepsilon$ and $\epsilon A_\gamma \psi_\varepsilon$, respectively, adding side by side the corresponding equalities, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[\mathcal{K}^{-1}|\boldsymbol{v}_\varepsilon|^2 + \epsilon|A_\gamma^{1/2}\psi_\varepsilon|^2] + \nu\mathcal{K}^{-1}\|\boldsymbol{v}_\varepsilon\|^2 + \epsilon^2|A_\gamma\psi_\varepsilon|^2 \\
&= -\mathcal{K}^{-1}(B_0(\boldsymbol{v}_\varepsilon, W_{A_0}), \boldsymbol{v}_\varepsilon) + (R_0(\epsilon A_\gamma W_{A_\gamma}, \psi_\varepsilon), \boldsymbol{v}_\varepsilon) - (B_1(W_{A_0}, \psi_\varepsilon), \epsilon A_\gamma\psi_\varepsilon) \\
&\quad - (\mathcal{K}^{-1}B_0(W_{A_0}, W_{A_0}), \boldsymbol{v}_\varepsilon) + (R_0(\epsilon A_\gamma W_{A_\gamma}, W_{A_\gamma}), \boldsymbol{v}_\varepsilon) \\
&\quad - \alpha(A_\gamma^{1/2} f_\gamma(\psi_\varepsilon + W_{A_\gamma}), \epsilon A_\gamma^{1/2}\psi_\varepsilon) - (B_1(W_{A_0}, W_{A_\gamma}), \epsilon A_\gamma\psi_\varepsilon) \\
&\quad - \gamma\epsilon(W_{A_\gamma}, \epsilon A_\gamma\psi_\varepsilon).
\end{aligned} \tag{3.41}$$

Drawing on the same reasoning as in the proof of estimates (3.6)–(3.12), we find

$$\begin{aligned}
|\mathcal{K}^{-1}(B_0(\boldsymbol{v}_\varepsilon, W_{A_0}), \boldsymbol{v}_\varepsilon)| &\leq \frac{\nu\mathcal{K}^{-1}}{6}\|\boldsymbol{v}_\varepsilon\|^2 + c(\mathcal{O}, \nu, \mathcal{K})\|W_{A_0}\|^4_{\mathbb{L}^4(\mathcal{O})}|\boldsymbol{v}_\varepsilon|^2, \\
|(R_0(\epsilon A_\gamma W_{A_\gamma}, \psi_\varepsilon), \boldsymbol{v}_\varepsilon)| &\leq \frac{\nu\mathcal{K}^{-1}}{6}\|\boldsymbol{v}_\varepsilon\|^2 + \frac{\epsilon^2}{10}|A_\gamma\psi_\varepsilon|^2 + c(\mathcal{O}, \nu, \mathcal{K})|A_\gamma W_{A_\gamma}|^4|\boldsymbol{v}_\varepsilon|^2, \\
|(B_1(W_{A_0}, \psi_\varepsilon), \epsilon A_\gamma\psi_\varepsilon)| &\leq \frac{\epsilon^2}{10}|A_\gamma\psi_\varepsilon|^2 + c(\mathcal{O}, \epsilon)\|W_{A_0}\|^4_{\mathbb{L}^4(\mathcal{O})}|\nabla\psi_\varepsilon|^2, \\
|(B_1(W_{A_0}, W_{A_\gamma}), \epsilon A_\gamma\psi_\varepsilon)| &\leq \frac{\epsilon^2}{10}|A_\gamma\psi_\varepsilon|^2 + c(\mathcal{O})\|W_{A_0}\|^4_{\mathbb{L}^4(\mathcal{O})} + c(\mathcal{O})|A_\gamma W_{A_\gamma}|^4, \\
|\gamma\epsilon(W_{A_\gamma}, \epsilon A_\gamma\psi_\varepsilon)| &\leq \frac{\epsilon^2}{10}|A_\gamma\psi_\varepsilon|^2 + c\gamma\epsilon^2|A_\gamma^{1/2}W_{A_\gamma}|^2, \\
&|(-\mathcal{K}^{-1}B_0(W_{A_0}, W_{A_0}), \boldsymbol{v}_\varepsilon)| + |(R_0(\epsilon A_\gamma W_{A_\gamma}, W_{A_\gamma}), \boldsymbol{v}_\varepsilon)| \\
&\quad \leq \frac{\nu\mathcal{K}^{-1}}{6}\|\boldsymbol{v}_\varepsilon\|^2 + c(\mathcal{O}, \nu, \mathcal{K})\|W_{A_0}\|^4_{\mathbb{L}^4(\mathcal{O})} + c(\mathcal{O}, \epsilon)|A_\gamma W_{A_\gamma}|^4, \\
&-\alpha((A_\gamma^{1/2}\psi_\varepsilon + A_\gamma^{1/2}W_{A_\gamma})f_\gamma'(\psi_\varepsilon + W_{A_\gamma}), \epsilon A_\gamma^{1/2}\psi_\varepsilon) \\
&\quad \leq \frac{\epsilon^2}{10}|A_\gamma\psi_\varepsilon|^2 + C|A_\gamma^{1/2}W_{A_\gamma}|^2 + C|A_\gamma^{1/2}W_{A_\gamma}|^2|A_\gamma W_{A_\gamma}|^{\frac{2}{3}} \\
&\qquad + C\big[1 + |A_\gamma^{1/2}W_{A_\gamma}|^{\frac{4}{5}}|A_\gamma W_{A_\gamma}|^{\frac{2}{5}}\big]|A_\gamma^{1/2}\psi_\varepsilon|^2
\end{aligned}$$

for some positive constant $C = C(\mathcal{O}, c_f, \gamma, \alpha)$.

Consequently,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}[\mathcal{K}^{-1}|\boldsymbol{v}_\varepsilon|^2 + \epsilon|A_\gamma^{1/2}\psi_\varepsilon|^2] + \nu\mathcal{K}^{-1}\|\boldsymbol{v}_\varepsilon\|^2 + \epsilon^2|A_\gamma\psi_\varepsilon|^2 \\
&\quad \le c\|W_{A_0}\|^4_{\mathbb{L}^4(\mathcal{O})} + c|A_\gamma W_{A_\gamma}|^4 + c|A_\gamma^{1/2}W_{A_\gamma}|^2 \\
&\qquad + c|A_\gamma^{1/2}W_{A_\gamma}|^2 + c|A_\gamma^{1/2}W_{A_\gamma}|^2|A_\gamma W_{A_\gamma}|^{\frac{2}{3}} \\
&\qquad + c(\|W_{A_0}\|^4_{\mathbb{L}^4(\mathcal{O})} + |A_\gamma W_{A_\gamma}|^4)|\boldsymbol{v}_\varepsilon|^2 \\
&\qquad + c\big[1 + |A_\gamma^{1/2}W_{A_\gamma}|^{\frac{4}{5}}|A_\gamma W_{A_\gamma}|^{\frac{2}{5}} + \|W_{A_0}\|^4_{\mathbb{L}^4(\mathcal{O})}\big]|A_\gamma^{1/2}\psi_\varepsilon|^2, \qquad (3.42)
\end{aligned}$$

where $c = c(\mathcal{O}, \alpha, \gamma, \mathcal{K}, \nu, \epsilon, \gamma_2, c_f)$ is a positive constant which is independent of ε.

Integrating (3.42) in time over $[0, t]$, where $t \in [0, T]$, we arrive at

$$\begin{aligned}
&|(\boldsymbol{v}_\varepsilon(t), \psi_\varepsilon(t))|^2_{\mathbb{Y}} + \int_0^t \|(\boldsymbol{v}_\varepsilon(s), \psi_\varepsilon(s))\|^2_{\mathbb{V}}\,\mathrm{d}s \\
&\le |(u_0, \psi_0)|^2_{\mathbb{Y}} + c\int_0^T (\|W_{A_0}(s)\|^4_{\mathbb{L}^4(\mathcal{O})} + |A_\gamma W_{A_\gamma}(s)|^4 + |A_\gamma^{1/2}W_{A_\gamma}(s)|^2)\,\mathrm{d}s \\
&\quad + c\int_0^T |A_\gamma^{1/2}W_{A_\gamma}(s)|^2 + |A_\gamma^{1/2}W_{A_\gamma}(s)|^2|A_\gamma W_{A_\gamma}(s)|^{\frac{2}{3}}\,\mathrm{d}s \\
&\quad + c\int_0^t \big[1+|A_\gamma^{\frac{1}{2}}W_{A_\gamma}|^{\frac{4}{5}}|A_\gamma W_{A_\gamma}|^{\frac{2}{5}}+\|W_{A_0}\|^4_{\mathbb{L}^4(\mathcal{O})}+|A_\gamma W_{A_\gamma}|^4\big](|\boldsymbol{v}_\varepsilon|^2+|A_\gamma^{1/2}\psi_\varepsilon|^2)\,\mathrm{d}s
\end{aligned} \qquad (3.43)$$

for all $t \in [0, T]$ and with $c = c(\mathcal{O}, \alpha, \gamma, \mathcal{K}, \nu, \epsilon, \gamma_2, c_f)$. It then follows by an application of the generalized Gronwall–Bellman lemma (see [26]) that

$$|(\boldsymbol{v}_\varepsilon(t), \psi_\varepsilon(t))|^2_{\mathbb{Y}} + \int_0^t \|(\boldsymbol{v}_\varepsilon(s), \psi_\varepsilon(s))\|^2_{\mathbb{V}}\,\mathrm{d}s \le cc_2\exp\bigg(\int_0^T k_1(s)\,\mathrm{d}s\bigg), \qquad (3.44)$$

$\mathbb{P}$-a.s., where $c = c(\mathcal{O}, \alpha, \gamma, \mathcal{K}, \nu, \epsilon, \gamma_2, c_f)$,

$$\begin{aligned}
c_2 &= |(u_0, \psi_0)|^2_{\mathbb{Y}} + c\int_0^T (\|W_{A_0}(s)\|^4_{\mathbb{L}^4(\mathcal{O})} + |A_\gamma W_{A_\gamma}(s)|^4 + |A_\gamma^{1/2}W_{A_\gamma}(s)|^2)\,\mathrm{d}s \\
&\quad + c\int_0^T |A_\gamma^{1/2}W_{A_\gamma}(s)|^2 + |A_\gamma^{1/2}W_{A_\gamma}(s)|^3|A_\gamma W_{A_\gamma}(s)|\,\mathrm{d}s, \\
k_1(s) &= c(1 + |A_\gamma^{\frac{1}{2}}W_{A_\gamma}(s)|^{\frac{2}{3}}|A_\gamma W_{A_\gamma}(s)|^{\frac{2}{3}} + \|W_{A_0}(s)\|^4_{\mathbb{L}^4(\mathcal{O})} + |A_\gamma W_{A_\gamma}(s)|^4).
\end{aligned}$$

We note that the constant c is independent of ε and $\omega \in \Omega$.

Now, we fix $\omega \in \Omega$ and select a sub-sequence $\varepsilon = \varepsilon(\omega)$ such that

$$\begin{aligned}
(\boldsymbol{v}_\varepsilon, \psi_\varepsilon)(t) &\to (\boldsymbol{v}, \psi)(t) && \text{weakly in } L^2(0, T; \mathbb{V}), \text{ weak-star in } L^\infty(0, T; \mathbb{Y}), \\
A_0\boldsymbol{v}_\varepsilon(t) &\to A_0\boldsymbol{v}(t) && \text{weakly in } L^2(0, T; V_1'), \\
A_\gamma\psi_\varepsilon(t) &\to A_\gamma\psi(t) && \text{weakly in } L^2(0, T; V_2'),
\end{aligned} \qquad (3.45)$$

and

$$\begin{aligned}
B_0(\boldsymbol{v}_\varepsilon, W_{A_0}) &\to B_0(\boldsymbol{v}, W_{A_0}) && \text{weakly in } L^2(0,T;V_1'),\\
B_0(W_{A_0}, \boldsymbol{v}_\varepsilon) &\to B_0(W_{A_0}, \boldsymbol{v}) && \text{weakly in } L^2(0,T;V_1'),\\
R_0(\epsilon A_\gamma \psi_\varepsilon, W_{A_\gamma}) &\to R_0(\epsilon A_\gamma \psi, W_{A_\gamma}) && \text{weakly in } L^2(0,T;V_1'),\\
B_1(\boldsymbol{v}_\varepsilon, W_{A_\gamma}) &\to B_1(\boldsymbol{v}, W_{A_\gamma}) && \text{weakly in } L^2(0,T;V_2'),\\
B_1(W_{A_0}, \psi_\varepsilon) &\to B_1(W_{A_0}, \psi) && \text{weakly in } L^2(0,T;V_2'),\\
f_\gamma(\psi_\varepsilon + W_{A_\gamma}) &\to f_\gamma(\psi + W_{A_\gamma}) && \text{weakly in } L^2(0,T;H_2).
\end{aligned} \tag{3.46}$$

Moreover, using the Hölder and the Gagliardo–Nirenberg inequalities, we obtain

$$\begin{aligned}
\|\psi_\varepsilon^1(\boldsymbol{v}_\varepsilon)\|_{V_1'} &\le c|\boldsymbol{v}_\varepsilon|\|\boldsymbol{v}_\varepsilon\|,\\
\|\psi_\varepsilon^2(\boldsymbol{v}_\varepsilon, \psi_\varepsilon)\|_{V_1'} &\le c_\epsilon |A_\gamma^{1/2}\psi_\varepsilon|^{1/2}|A_\gamma\psi_\varepsilon|^{3/2},\\
\|\psi_\varepsilon^3(\boldsymbol{v}_\varepsilon, \psi_\varepsilon)\|_{V_2'} &\le c\|\boldsymbol{v}_\varepsilon\||A_\gamma^{1/2}\psi_\varepsilon|.
\end{aligned}$$

Hence, in light of (3.44), we see that $\int_0^T \|\psi_\varepsilon^1(\boldsymbol{v}_\varepsilon(s))\|_{V_1'}^2\,\mathrm{d}s$, $\int_0^T \|\psi_\varepsilon^2(\boldsymbol{v}_\varepsilon(s), \psi_\varepsilon(s))\|_{V_1'}^{4/3}\,\mathrm{d}s$, and $\int_0^T \|\psi_\varepsilon^3(\boldsymbol{v}_\varepsilon(s), \psi_\varepsilon(s))\|_{V_2'}^2\,\mathrm{d}s$ are uniformly bounded with respect to ε.

Once more, we fix $\omega \in \Omega$. Then, we select a sub-sequence $\varepsilon = \varepsilon(\omega)$ such that

$$\begin{aligned}
\psi_\varepsilon^1(\boldsymbol{v}_\varepsilon(t)) &\to z_1(t) && \text{weakly in } L^2(0,T;V_1'),\\
\psi_\varepsilon^2(\boldsymbol{v}_\varepsilon(t), \psi_\varepsilon(t)) &\to z_2(t) && \text{weakly in } L^{4/3}(0,T;V_1'),\\
\psi_\varepsilon^3(\boldsymbol{v}_\varepsilon(t), \psi_\varepsilon(t)) &\to z_3(t) && \text{weakly in } L^2(0,T;V_2').
\end{aligned} \tag{3.47}$$

Hence, we have

$$\begin{cases}
\boldsymbol{v}'(t) + \nu A_0\boldsymbol{v}(t) + z_1(t) + B_0(\boldsymbol{v}(t), W_{A_0}(t)) + B_0(W_{A_0}(t), \boldsymbol{v}(t))\\
-\mathcal{K}z_2(t) - \mathcal{K}R_0(\epsilon A_\gamma W_{A_\gamma}(t), \psi(t)) - \mathcal{K}R_0(\epsilon A_\gamma\psi(t), W_{A_\gamma}(t))\\
\quad = -B_0(W_{A_0}(t), W_{A_0}(t)) + \mathcal{K}R_0(\epsilon A_\gamma W_{A_\gamma}(t), W_{A_\gamma}(t)),\\
\psi'(t) + \epsilon A_\gamma\psi(t) + z_3(t) + B_1(\boldsymbol{v}(t), W_{A_\gamma}(t)) + B_1(W_{A_0}(t), \psi(t))\\
\quad = -\alpha f_\gamma(\psi(t) + W_{A_\gamma}(t)) - B_1(W_{A_0}(t), W_{A_\gamma}(t)) - \epsilon\gamma W_{A_\gamma}(t),\\
\boldsymbol{v}(0) = \boldsymbol{u}_0, \quad \psi(0) = \phi_0,
\end{cases} \tag{3.48}$$

a.e. $t \in [0,T]$. Moreover, since $\boldsymbol{v}_\varepsilon'$ and ψ_ε' are uniformly bounded in $L^{4/3}(0,T;V_1')$ and $L^2(0,T;V_2')$, respectively, we also have that for $\varepsilon \to 0$ (see [22, Theorem 5.1])

$$\begin{aligned}
\boldsymbol{v}_\varepsilon(t,\omega) &\to \boldsymbol{v}(t,\omega) && \text{strongly in } L^2(0,T;H_1),\\
\psi_\varepsilon(t,\omega) &\to \psi(t,\omega) && \text{strongly in } L^2(0,T;D(A_\gamma^{1/2})).
\end{aligned} \tag{3.49}$$

As in [21, p. 6], we can check that

$$\begin{aligned}
&\int_0^T (\Psi_\varepsilon^1(\boldsymbol{v}_\varepsilon(t)), y(t))\,\mathrm{d}t \to \int_0^T b_0(\boldsymbol{v}(t), \boldsymbol{v}(t), y(t))\,\mathrm{d}t \quad \forall y \in \mathcal{C}([0,T]; D(A_0)),\\
&\int_0^T (\Psi_\varepsilon^3(\boldsymbol{v}_\varepsilon(t), \psi_\varepsilon(t)), \rho(t))\,\mathrm{d}t \to \int_0^T b_1(\boldsymbol{v}(t), \psi(t), \rho(t))\,\mathrm{d}t \quad \forall \rho \in \mathcal{C}([0,T]; D(A_\gamma))
\end{aligned} \tag{3.50}$$

as $\varepsilon \to 0$. Then, by (3.47)$_1$ and (3.47)$_3$, it follows that $z_1(t) = B_0(\boldsymbol{v}(t), \boldsymbol{v}(t))$ and $z_3(t) = B_1(\boldsymbol{v}(t), \psi(t))$ a.e. $t \in [0, T]$.

We will now prove that

$$z_2(t) = R_0(\epsilon A_\gamma \psi(t), \psi(t)) \quad \text{a.e. } t \in [0, T].$$

Observe now that

$$\begin{aligned}
&\int_0^T (\Psi_\varepsilon^2(\boldsymbol{v}_\varepsilon(t), \psi_\varepsilon(t)), y(t))\,\mathrm{d}t \\
&= \int_{t\in[0,T]:\|\boldsymbol{v}_\varepsilon\|+|A_\gamma\psi_\varepsilon|\le 1/\varepsilon} b_1(y(t), \psi_\varepsilon(t), \epsilon A_\gamma \psi_\varepsilon(t))\,\mathrm{d}t \\
&\quad + \int_{t\in[0,T]:\|\boldsymbol{v}_\varepsilon\|+|A_\gamma\psi_\varepsilon|> 1/\varepsilon} \frac{b_1(y(t), \psi_\varepsilon(t), \epsilon A_\gamma \psi_\varepsilon(t))}{\varepsilon^2(\|\boldsymbol{v}_\varepsilon\| + |A_\gamma\psi_\varepsilon|)^2}\,\mathrm{d}t \\
&:= J_\varepsilon^1 + J_\varepsilon^2
\end{aligned}$$

for all $y \in \mathcal{C}([0, T]; D(A_0))$.

In light of (3.44) and (3.49)$_1$, we deduce that

$$b_1(y(t), \psi_\varepsilon(t), \epsilon A_\gamma \psi_\varepsilon(t)) \to b_1(y(t), \psi(t), \epsilon A_\gamma \psi(t)) \text{ a.e. } t \in [0, T]$$

due to

$$\begin{aligned}
&|b_1(y(t), \psi_\varepsilon(t), \epsilon A_\gamma \psi_\varepsilon(t)) - b_1(y(t), \psi(t), \epsilon A_\gamma \psi(t))| \\
&\le |b_1(y(t), \psi_\varepsilon(t) - \psi(t), \epsilon A_\gamma \psi_\varepsilon(t))| + |b_1(y(t), \psi(t), \epsilon A_\gamma(\psi_\varepsilon(t) - \psi(t)))| \\
&\le c\epsilon|A_\gamma\psi_\varepsilon(t)|\|y(t)\|_{L^\infty}|\nabla(\psi_\varepsilon(t) - \psi(t))| \\
&\quad + c\epsilon|A_\gamma^{1/2}(\psi_\varepsilon(t) - \psi(t))||A_\gamma^{1/2}\psi(t)|_{\mathbb{L}^6}|\nabla y(t)|_{\mathbb{L}^3} \\
&\quad + c\epsilon\|y(t)\|_{L^\infty}|A_\gamma^{1/2}(\psi_\varepsilon(t) - \psi(t))||A_\gamma\psi(t)| \\
&\le c\epsilon|A_0 y(t)|[|A_\gamma\psi_\varepsilon(t)| + |A_\gamma\psi(t)|]|A_\gamma^{1/2}(\psi_\varepsilon(t) - \psi(t))|.
\end{aligned}$$

Furthermore, as

$$|b_1(y(t), \psi_\varepsilon(t), \epsilon A_\gamma \psi_\varepsilon(t))| \le c\epsilon\|y(t)\|_{L^\infty}(|A_\gamma\psi_\varepsilon(t)||A_\gamma^{1/2}\psi_\varepsilon(t)|),$$

we infer from the Lebesgue dominated convergence theorem that

$$J_\varepsilon^1 \to \int_0^T b_1(y(t), \psi(t), \epsilon A_\gamma \psi(t))\,\mathrm{d}t \text{ as } \varepsilon \to 0.$$

On the other hand, from (3.44), we have

$$\sup_{t\in[0,T]} \left\{\|\boldsymbol{v}_\varepsilon(t)\| + |A_\gamma\psi_\varepsilon(t)| > 1/\varepsilon\right\} \le c\varepsilon^2 \quad \mathbb{P}\text{-a.s.}$$

Consequently,

$$
\begin{aligned}
|J_\varepsilon^2| &\leq c\epsilon \int_{t\in[0,T]:\|\boldsymbol{v}_\varepsilon\|+|A_\gamma\psi_\varepsilon|>1/\varepsilon} \frac{|A_\gamma^{1/2}\psi_\varepsilon(t)|^{1/2}|A_\gamma\psi_\varepsilon(t)|^{3/2}\|y(t)\|}{\varepsilon^2(\|\boldsymbol{v}_\varepsilon(t)\|+|A_\gamma\psi_\varepsilon(t)|)^2}\,\mathrm{d}t \\
&\leq c\epsilon \int_{t\in[0,T]:\|\boldsymbol{v}_\varepsilon\|+|A_\gamma\psi_\varepsilon|>1/\varepsilon} \frac{|A_\gamma^{1/2}\psi_\varepsilon(t)|^{1/2}|A_\gamma\psi_\varepsilon(t)|^{3/2}\|y(t)\|}{\varepsilon^2|A_\gamma\psi_\varepsilon(t)|^2}\,\mathrm{d}t \\
&\leq c\epsilon \int_{t\in[0,T]:\|\boldsymbol{v}_\varepsilon\|+|A_\gamma\psi_\varepsilon|>1/\varepsilon} \frac{|A_\gamma^{1/2}\psi_\varepsilon(t)|^{1/2}\|y(t)\|}{\varepsilon^2|A_\gamma\psi_\varepsilon(t)|^{1/2}}\,\mathrm{d}t \\
&\leq c\epsilon\sqrt{\varepsilon} \to 0 \text{ as } \varepsilon \to 0.
\end{aligned}
$$

By (3.47), it follows therefore that $z_2(t) = R_0(\epsilon A_\gamma\psi(t), \psi(t))$ a.e. $t \in [0, T]$.

Hence, the pair $(\boldsymbol{v}, \psi)$ is a solution to (3.2) (for a fixed $\omega \in \Omega$). Furthermore, for each $\omega \in \Omega$, we can check that (3.48) with $z_1 = B_0(\boldsymbol{v}, \boldsymbol{v})$, $z_2 = R_0(\epsilon A_\gamma\psi, \psi)$, and $z_3 = B_1(\boldsymbol{v}, \psi)$ has at most one solution $(\boldsymbol{v}, \psi)$ with the above properties. Indeed, if $(\boldsymbol{v}_1, \psi_1)$ and $(\boldsymbol{v}_2, \psi_2)$ are two solutions to (3.48), then we can easily check that $\boldsymbol{v} = \boldsymbol{v}_1 - \boldsymbol{v}_2$ and $\psi = \psi_1 - \psi_2$ satisfy

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[\mathcal{K}^{-1}|\boldsymbol{v}|^2 + \epsilon|A_\gamma^{1/2}\psi|^2] + \nu\mathcal{K}^{-1}\|\boldsymbol{v}\|^2 + \epsilon^2|A_\gamma\psi|^2 + \mathcal{K}^{-1}(B_0(\boldsymbol{v}, \boldsymbol{v}_1), \boldsymbol{v}) \\
&+ \mathcal{K}^{-1}(B_0(\boldsymbol{v}, W_{A_0}), \boldsymbol{v}) - (R_0(\epsilon A_\gamma\psi_2, \psi), \boldsymbol{v}) - (R_0(\epsilon A_\gamma W_{A_\gamma}, \psi), \boldsymbol{v}) \\
&+ (B_1(\boldsymbol{v}_2, \psi), \epsilon A_\gamma\psi) + (B_1(W_{A_0}, \psi), \epsilon A_\gamma\psi) \\
&\quad = -\alpha[(f_\gamma(\psi_1 + W_{A_\gamma}) - f_\gamma(\psi_2 + W_{A_\gamma}), \epsilon A_\gamma\psi)].
\end{aligned}
$$

By the Hölder and the Young inequalities, we obtain

$$
\begin{aligned}
|\mathcal{K}^{-1}(B_0(\boldsymbol{v}, \boldsymbol{v}_1), \boldsymbol{v})| &\leq c(\mathcal{O})\mathcal{K}^{-1}|\boldsymbol{v}|\|\boldsymbol{v}\|\|\boldsymbol{v}_1\| \leq \frac{\nu}{8\mathcal{K}}\|\boldsymbol{v}\|^2 + c(\mathcal{O}, \nu, \mathcal{K})\|\boldsymbol{v}_1\|^2|\boldsymbol{v}|^2, \\
|\mathcal{K}^{-1}(B_0(\boldsymbol{v}, W_{A_0}), \boldsymbol{v})| &\leq c(\mathcal{O})\mathcal{K}^{-1}\|W_{A_0}\|_{\mathbb{L}^4(\mathcal{O})}|\boldsymbol{v}|^{1/2}\|\boldsymbol{v}\|^{3/2} \\
&\leq \frac{\nu}{8\mathcal{K}}\|\boldsymbol{v}\|^2 + c(\mathcal{O}, \nu, \mathcal{K})\|W_{A_0}\|_{\mathbb{L}^4(\mathcal{O})}^4|\boldsymbol{v}|^2, \\
|(R_0(\epsilon A_\gamma\psi_2, \psi), \boldsymbol{v})| &\leq c(\mathcal{O})\epsilon|A_\gamma\psi_2|\|A_\gamma^{1/2}\psi\|_{\mathbb{L}^4(\mathcal{O})}\|\boldsymbol{v}\|_{\mathbb{L}^4(\mathcal{O})} \\
&\leq c(\mathcal{O})\epsilon|A_\gamma\psi_2||A_\gamma^{1/2}\psi|^{1/2}|A_\gamma\psi|^{1/2}|\boldsymbol{v}|^{1/2}\|\boldsymbol{v}\|^{1/2} \\
&\leq \frac{\nu}{8\mathcal{K}}\|\boldsymbol{v}\|^2 + \frac{\epsilon^2}{10}|A_\gamma\psi|^2 \\
&\quad + c(\mathcal{O}, \nu, \mathcal{K}, \epsilon)|A_\gamma\psi_2|^2(|\boldsymbol{v}|^2 + \epsilon|A_\gamma^{1/2}\psi|^2).
\end{aligned}
$$

Analogously, we find

$$
\begin{aligned}
|(R_0(\epsilon A_\gamma W_{A_\gamma}, \psi), \boldsymbol{v})| &\leq \frac{\nu}{8\mathcal{K}}\|\boldsymbol{v}\|^2 + \frac{\epsilon^2}{10}|A_\gamma\psi|^2 \\
&\quad + c(\mathcal{O}, \epsilon, \mathcal{K}, \nu)|A_\gamma W_{A_\gamma}|^2(|\boldsymbol{v}|^2 + \epsilon|A_\gamma^{1/2}\psi|^2).
\end{aligned}
$$

Thanks to the Hölder and the Young inequalities, we derive that

$$\begin{aligned}|(B_1(\boldsymbol{v}_2,\psi),\epsilon A_\gamma\psi)| &\le c(\mathcal{O})\epsilon|\boldsymbol{v}_2|^{1/2}\|\boldsymbol{v}_2\|^{1/2}|A_\gamma^{1/2}\psi|^{1/2}|A_\gamma\psi|^{3/2}\\ &\le \frac{\epsilon^2}{10}|A_\gamma\psi|^2 + c(\mathcal{O},\epsilon)|\boldsymbol{v}_2|^2\|\boldsymbol{v}_2\|^2(\epsilon|A_\gamma^{1/2}\psi|^2),\\ |(B_1(W_{A_0},\psi),\epsilon A_\gamma\psi)| &\le \epsilon c(\mathcal{O})\|W_{A_0}\|_{\mathbb{L}^4(\mathcal{O})}|A_\gamma^{1/2}\psi|^{1/2}|A_\gamma\psi|^{3/2}\\ &\le \frac{\epsilon^2}{10}|A_\gamma\psi|^2 + c(\mathcal{O},\epsilon)\|W_{A_0}\|^4_{\mathbb{L}^4(\mathcal{O})}(\epsilon|A_\gamma^{1/2}\psi|^2).\end{aligned}$$

Applying the Lagrange mean value theorem to f'_γ (see [28, Corollary 2]), using also the second assumption of f (cf. (2.21)), we infer that

$$\begin{aligned}&\alpha|(f_\gamma(\psi_1+W_{A_\gamma}) - f_\gamma(\psi_2+W_{A_\gamma}),\epsilon A_\gamma\psi)|\\ &= \alpha\epsilon|(\psi f'_\gamma(\theta\psi_1+(1-\theta)\psi_2+W_{A_\gamma}),A_\gamma\psi)|\\ &\le \alpha\epsilon\int_{\mathcal{O}}|f'_\gamma(\theta\psi_1+(1-\theta)\psi_2+W_{A_\gamma})||\psi||A_\gamma\psi|\,\mathrm{d}x\\ &\le c_f\alpha\epsilon\int_{\mathcal{O}}(1+|\theta\psi_1+(1-\theta)\psi_2+W_{A_\gamma}|)|\psi||A_\gamma\psi|\,\mathrm{d}x + \alpha\epsilon|\psi||A_\gamma\psi|\\ &\le \alpha\epsilon(c_f+1)|\psi||A_\gamma\psi| + \alpha\epsilon c_f(|\psi_1|_{L^4(\mathcal{O})}+|\psi_2|_{L^4(\mathcal{O})}+|W_{A_\gamma}|_{L^4(\mathcal{O})})|\psi|_{L^4(\mathcal{O})}|A_\gamma\psi|\\ &\le \alpha\epsilon(c_f+1)|\psi||A_\gamma\psi|\\ &\quad + \alpha\epsilon c_f\gamma^{-1/2}(\gamma^{-1}+1)^{1/2}c(\mathcal{O})(|A_\gamma^{1/2}\psi_1|+|A_\gamma^{1/2}\psi_2|+|A_\gamma^{1/2}W_{A_\gamma}|)|A_\gamma^{1/2}\psi||A_\gamma\psi|.\end{aligned}$$

It then follows that

$$\begin{aligned}&\alpha|(f_\gamma(\psi_1+W_{A_\gamma}) - f_\gamma(\psi_2+W_{A_\gamma}),\epsilon A_\gamma\psi)|\\ &\quad\le \frac{\epsilon^2}{10}|A_\gamma\psi|^2 + c\alpha^2(c_f+1)^2|\psi|^2 + c(\mathcal{O})\alpha^2c_f^2\gamma^{-1}(\gamma^{-1}+1)[|A_\gamma^{1/2}\psi_1|^2\\ &\qquad + |A_\gamma^{1/2}\psi_2|^2 + |A_\gamma^{1/2}W_{A_\gamma}|^2]|A_\gamma^{1/2}\psi|^2,\end{aligned}$$

where we used the Young inequality.

From the above estimates, we derive that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|(\boldsymbol{v},\psi)|^2_{\mathbb{Y}} + \frac{1}{2}\|(\boldsymbol{v},\psi)\|^2_{\mathbb{V}}\\ &\quad\le c(1+|A_\gamma^{1/2}\psi_1|^2+|A_\gamma^{1/2}\psi_2|^2+|A_\gamma^{1/2}W_{A_\gamma}|^2+\|\boldsymbol{v}_1\|^2\\ &\qquad + \|W_{A_0}\|^4_{\mathbb{L}^4(\mathcal{O})}+|A_\gamma\psi_2|^2+|A_\gamma W_{A_\gamma}|^2+|\boldsymbol{v}_2|^2\|\boldsymbol{v}_2\|^2)|(\boldsymbol{v},\psi)|^2_{\mathbb{Y}},\end{aligned}$$

where $c = c(\mathcal{O}, c_f, \gamma, \epsilon, \alpha)$ is a positive constant. Now, by applying the Gronwall lemma, we deduce that $|(\boldsymbol{v},\psi)|^2_{\mathbb{Y}} = 0$, i.e., $(\boldsymbol{v}_1,\psi_1) = (\boldsymbol{v}_2,\psi_2)$. This implies that, for $\varepsilon \to 0$,

$$\boldsymbol{v}_\varepsilon(t) \to \boldsymbol{v}(t), \quad \psi_\varepsilon(t) \to \psi(t) \tag{3.51}$$

weakly in $L^2(0,T;V_1)$ and $L^2(0,T;D(A_\gamma))$, respectively, $\mathbb{P}$-a.s. By (3.51), it follows that $\boldsymbol{v}$ and ψ (and $\boldsymbol{v}'$ and ψ') with respect to the filtration $\mathcal{F}_t$ (because it is the case for $\boldsymbol{v}_\varepsilon$

and ψ_ε) and therefore $(\boldsymbol{v}, \psi) \in L^2_W(0, T; \mathbb{V})$ and $(\boldsymbol{v}', \psi') \in L^2_W(0, T; V_1' \times D(A_\gamma)')$. The proof of Proposition 3.1 is now complete. ∎

Let us now proceed to the proof of Theorem 3.1.

Proof of Theorem 3.1. Note that $\boldsymbol{u}_\varepsilon = \boldsymbol{v}_\varepsilon + W_{A_0}$ and $\phi_\varepsilon = \psi_\varepsilon + W_{A_\gamma}$ satisfy

$$\begin{cases} \mathrm{d}\boldsymbol{u}_\varepsilon + [\nu A_0 \boldsymbol{u}_\varepsilon + \Psi^1_\varepsilon(\boldsymbol{u}_\varepsilon) - \mathcal{K}\Psi^2_\varepsilon(\boldsymbol{u}_\varepsilon, \phi_\varepsilon)]\mathrm{d}t = \sqrt{Q_1}\,\mathrm{d}W_1(t), \\ \mathrm{d}\phi_\varepsilon + \Psi^3_\varepsilon(\boldsymbol{u}_\varepsilon, \phi_\varepsilon)\,\mathrm{d}t + \mu^\varepsilon_\gamma\,\mathrm{d}t = \sqrt{Q_2}\,\mathrm{d}W_2(t), \\ \mu^\varepsilon_\gamma = \epsilon A_\gamma \phi_\varepsilon + \alpha f_\gamma(\phi_\varepsilon), \\ \boldsymbol{u}_\varepsilon(0) = \boldsymbol{u}_0, \quad \phi_\varepsilon(0) = \phi_0, \end{cases} \tag{3.52}$$

$\mathbb{P}$-a.s. and a.e. $t \in [0, T]$.

By applying Itô's formula to the process $|\boldsymbol{u}_\varepsilon(t)|^2$ (see, for instance, [12, Theorem 4.32]), integrating the resulting equality between 0 and t, and then taking the mathematical expectation, we derive that

$$\mathbb{E}|\boldsymbol{u}_\varepsilon(t)|^2 + 2\mathbb{E}\int_0^t \nu\|\boldsymbol{u}_\varepsilon\|^2\,\mathrm{d}s - 2\mathcal{K}\mathbb{E}\int_0^t \langle \Psi^2_\varepsilon(\boldsymbol{u}_\varepsilon, \phi_\varepsilon), \boldsymbol{u}_\varepsilon\rangle\,\mathrm{d}s = |\boldsymbol{u}_0|^2 + t\,\mathrm{Tr}\,Q_1. \tag{3.53}$$

Once more, by applying the Itô formula to the process $|\phi_\varepsilon(t)|^2$, we obtain

$$\mathbb{E}|\phi_\varepsilon(t)|^2 + 2\mathbb{E}\int_0^t (\mu^\varepsilon_\gamma, \phi_\varepsilon)\,\mathrm{d}s + 2\mathbb{E}\int_0^t (\Psi^3_\varepsilon(\boldsymbol{u}_\varepsilon, \phi_\varepsilon), \phi_\varepsilon)\,\mathrm{d}s = |\phi_0|^2 + t\,\mathrm{Tr}\,Q_2. \tag{3.54}$$

Applying again the Itô formula to $|\nabla\phi_\varepsilon(t)|^2$, we further obtain

$$\mathbb{E}|\nabla\phi_\varepsilon(t)|^2 + 2\mathbb{E}\int_0^t (\mu^\varepsilon_\gamma, A_1\phi_\varepsilon)\,\mathrm{d}s + 2\mathbb{E}\int_0^t (\Psi^3_\varepsilon(\boldsymbol{u}_\varepsilon, \phi_\varepsilon), A_1\phi_\varepsilon)\,\mathrm{d}s = |\nabla\phi_0|^2 + t\Lambda. \tag{3.55}$$

Now, multiplying (3.54) by $\epsilon\gamma$ and (3.55) by ϵ, respectively, adding up side by side the resulting equations, we arrive at

$$\begin{aligned} &\mathbb{E}[\epsilon(|\nabla\phi_\varepsilon(t)|^2 + \gamma|\phi_\varepsilon(t)|^2)] + 2\mathbb{E}\int_0^t (\mu^\varepsilon_\gamma, \epsilon A_\gamma\phi_\varepsilon)\,\mathrm{d}s + 2\mathbb{E}\int_0^t (\Psi^3_\varepsilon(\boldsymbol{u}_\varepsilon, \phi_\varepsilon), \epsilon A_\gamma\phi_\varepsilon)\,\mathrm{d}s \\ &\quad = \epsilon[|\nabla\phi_0|^2 + \gamma|\phi_0|^2] + \epsilon\gamma t\,\mathrm{Tr}\,Q_2 + \epsilon t\Lambda. \end{aligned} \tag{3.56}$$

It then follows from (3.53) and (3.56) that

$$\begin{aligned} \mathbb{E}|(\boldsymbol{u}_\varepsilon(t), \phi_\varepsilon(t))|^2_{\mathbb{Y}} &+ 2\int_0^t \|(\boldsymbol{u}_\varepsilon, \phi_\varepsilon)\|^2_{\mathbb{V}}\,\mathrm{d}s \\ &= |(\boldsymbol{u}_0, \phi_0)|^2_{\mathbb{Y}} + t\mathcal{K}^{-1}\,\mathrm{Tr}\,Q_1 + \epsilon\gamma t\,\mathrm{Tr}\,Q_2 \\ &\quad + \epsilon t\Lambda - 2\epsilon\alpha\mathbb{E}\int_0^t (f_\gamma(\phi_\varepsilon(s)), A_\gamma\phi_\varepsilon(s))\,\mathrm{d}s, \end{aligned} \tag{3.57}$$

where we used $(3.52)_3$ together with the fact that

$$-\langle \Psi_\varepsilon^2(\boldsymbol{u}_\varepsilon,\phi_\varepsilon),\boldsymbol{u}_\varepsilon\rangle + (\Psi_\varepsilon^3(\boldsymbol{u}_\varepsilon,\phi_\varepsilon),\epsilon A_\gamma\phi_\varepsilon) = 0.$$

Let us proceed to estimate the nonlinear term on the right-hand side of (3.57). Indeed, from the definition of the map f_γ, we have

$$\begin{aligned}
-2\epsilon\alpha(f_\gamma(\phi_\varepsilon(s)),A_\gamma\phi_\varepsilon(s)) &= -2\epsilon\alpha(A_\gamma^{1/2}f_\gamma(\phi_\varepsilon(s)),A_\gamma^{1/2}\phi_\varepsilon(s))\\
&= -2\epsilon\alpha(A_\gamma^{1/2}\phi_\varepsilon(s)f_\gamma'(\phi_\varepsilon(s))A_\gamma^{1/2}\phi_\varepsilon(s))\\
&= -2\epsilon\alpha(A_\gamma^{1/2}\phi_\varepsilon(s)f'(\phi_\varepsilon(s))A_\gamma^{1/2}\phi_\varepsilon(s))+2\epsilon^2\gamma|A_\gamma^{1/2}\phi_\varepsilon(s)|^2\\
&= -2\epsilon\alpha\int_{\mathcal{O}} f'(\phi_\varepsilon(s))|A_\gamma^{1/2}\phi_\varepsilon(s)|^2\,\mathrm{d}s + 2\epsilon^2\gamma|A_\gamma^{1/2}\phi_\varepsilon(s)|^2,
\end{aligned}$$

from which we infer that

$$\begin{aligned}
-2\epsilon\alpha(f_\gamma(\phi_\varepsilon(s)),A_\gamma\phi_\varepsilon(s)) &\le 2\epsilon\alpha\gamma_2|A_\gamma^{1/2}\phi_\varepsilon(s)|^2 + 2\epsilon^2\gamma|A_\gamma^{1/2}\phi_\varepsilon(s)|^2\\
&\le 2\epsilon(\alpha\gamma_2+\epsilon\gamma)|A_\gamma^{1/2}\phi_\varepsilon(s)|^2, \qquad (3.58)
\end{aligned}$$

where we used (2.23) and the fact that $\epsilon \le \alpha$.

Plugging (3.58) into the right-hand side of (3.57), we arrive at

$$\begin{aligned}
&\mathbb{E}|(\boldsymbol{u}_\varepsilon(t),\phi_\varepsilon(t))|_{\mathbb{Y}}^2 + 2\mathbb{E}\int_0^t \|(\boldsymbol{u}_\varepsilon(s),\phi_\varepsilon(s))\|_{\mathbb{V}}^2\,\mathrm{d}s\\
&\le |(\boldsymbol{u}_0,\phi_0)|_{\mathbb{Y}}^2 + t\mathcal{K}^{-1}\operatorname{Tr}Q_1 + \epsilon\gamma t\operatorname{Tr}Q_2 + \epsilon t\Lambda\\
&\quad + 2(\alpha\gamma_2+\epsilon\gamma)\mathbb{E}\int_0^t \epsilon|A_\gamma^{1/2}\phi_\varepsilon(s)|^2\,\mathrm{d}s\\
&\le |(\boldsymbol{u}_0,\phi_0)|_{\mathbb{Y}}^2 + t\mathcal{K}^{-1}\operatorname{Tr}Q_1 + \alpha\gamma t\operatorname{Tr}Q_2 + \alpha t\Lambda\\
&\quad + 2\alpha(\gamma_2+\gamma)\mathbb{E}\int_0^t |(\boldsymbol{u}_\varepsilon(s),\phi_\varepsilon(s))|_{\mathbb{Y}}^2\,\mathrm{d}s \qquad (3.59)
\end{aligned}$$

for all $t\in[0,T]$, where we have also used the fact that $\epsilon\le\alpha$.

Hence, an application of the Gronwall lemma entails that

$$\begin{aligned}
&\mathbb{E}|(\boldsymbol{u}_\varepsilon(t),\phi_\varepsilon(t))|_{\mathbb{Y}}^2\\
&\quad \le [|(\boldsymbol{u}_0,\phi_0)|_{\mathbb{Y}}^2 + T\mathcal{K}^{-1}\operatorname{Tr}Q_1 + \alpha\gamma T\operatorname{Tr}Q_2 + \alpha T\Lambda]e^{2\alpha(\gamma_2+\gamma)t},\\
&2\mathbb{E}\int_0^t \|(\boldsymbol{u}_\varepsilon(s),\phi_\varepsilon(s))\|_{\mathbb{V}}^2\,\mathrm{d}s\\
&\quad \le [|(\boldsymbol{u}_0,\phi_0)|_{\mathbb{Y}}^2 + T\mathcal{K}^{-1}\operatorname{Tr}Q_1 + \alpha\gamma T\operatorname{Tr}Q_2 + \alpha T\Lambda]e^{2\alpha(\gamma_2+\gamma)t}
\end{aligned}$$

for all $t\in[0,T]$. This implies that, for $\varepsilon\to 0$,

$$\begin{aligned}
\boldsymbol{u}_\varepsilon &\to \boldsymbol{u} = \boldsymbol{v} + W_{A_0} &&\text{weakly in } L_W^2(0,T;V_1),\\
\phi_\varepsilon &\to \phi = \psi + W_{A_\gamma} &&\text{weakly in } L_W^2(0,T;D(A_\gamma)),
\end{aligned}$$

where $(\boldsymbol{u},\phi)$ is a solution problem (2.6) or (1.1)–(1.2).

As for uniqueness, if $(\tilde{\boldsymbol{u}}(t), \widetilde{\phi}(t))$ is a solution with initial data $(\boldsymbol{u}_1, \phi_1)$, we have by (2.6) that

$$\begin{cases} \mathrm{d}\bar{\boldsymbol{u}} + \nu A_0 \bar{\boldsymbol{u}}\,\mathrm{d}t = [-(B_0(\bar{\boldsymbol{u}}, \boldsymbol{u}) + B_0(\tilde{\boldsymbol{u}}, \bar{\boldsymbol{u}})) + \mathcal{K}(R_0(\epsilon A_\gamma \widetilde{\phi}, \bar{\phi}) - R_0(\epsilon A_\gamma \bar{\phi}, \phi))]\,\mathrm{d}t, \\ \mathrm{d}\bar{\phi} + \epsilon A_\gamma \bar{\phi}\,\mathrm{d}t = (-(B_1(\bar{\boldsymbol{u}}, \phi) + B_1(\tilde{\boldsymbol{u}}, \bar{\phi})) + \alpha[f_\gamma(\phi) - f_\gamma(\widetilde{\phi})])\,\mathrm{d}t, \end{cases} \tag{3.60}$$

where we have set $\bar{\boldsymbol{u}} := \boldsymbol{u} - \tilde{\boldsymbol{u}}$ and $\bar{\phi} := \phi - \widetilde{\phi}$ and where we have also used the bilinearity of B_0, B_1, and R_0.

Now, we take the inner product of equation $(3.60)_1$ with $\mathcal{K}^{-1}\bar{u}(t)$ in H_1 and the inner product of equation $(3.60)_2$ with $\epsilon A_\gamma \bar{\phi}(t)$ in $L^2(\mathcal{O})$. Using also the orthogonality properties of b_0 and b_1 and adding the resulting equations, we infer that

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|(\bar{\boldsymbol{u}}(t), \bar{\phi}(t))|^2_{\mathbb{Y}} + \nu\mathcal{K}^{-1}\|\bar{\boldsymbol{u}}(t)\|^2 + \epsilon^2|A_\gamma\bar{\phi}(t)|^2 \\ &\quad = -\mathcal{K}^{-1}b_0(\bar{\boldsymbol{u}}, \boldsymbol{u}, \bar{\boldsymbol{u}}) + (R_0(\epsilon A_\gamma\widetilde{\phi}, \bar{\phi}), \bar{\boldsymbol{u}}) - (R_0(\epsilon A_\gamma \bar{\phi}, \phi), \bar{\boldsymbol{u}}) \\ &\qquad - b_1(\bar{\boldsymbol{u}}, \phi, \epsilon A_\gamma\bar{\phi}) - b_1(\tilde{\boldsymbol{u}}, \bar{\phi}, \epsilon A_\gamma \bar{\phi}) - \alpha\epsilon(f_\gamma(\phi) - f_\gamma(\widetilde{\phi}), A_\gamma\bar{\phi}). \end{aligned} \tag{3.61}$$

Arguing as in [18, p. 10], one has

$$\begin{aligned} |\mathcal{K}^{-1}b_0(\bar{\boldsymbol{u}}, \boldsymbol{u}, \bar{\boldsymbol{u}})| &\le \frac{\mathcal{K}^{-1}\nu}{4}\|\bar{\boldsymbol{u}}\|^2 + c\|\boldsymbol{u}\|^2|\bar{\boldsymbol{u}}|^2, \\ |b_1(\bar{\boldsymbol{u}}, \phi, \epsilon A_\gamma\bar{\phi})| &\le \frac{\nu\mathcal{K}^{-1}}{4}\|\bar{\boldsymbol{u}}\|^2 + \frac{\epsilon^2}{5}|A_\gamma\bar{\phi}|^2 + c|A_\gamma^{1/2}\phi|^2|A_\gamma\phi|^2|\bar{\boldsymbol{u}}|^2, \\ |b_1(\tilde{\boldsymbol{u}}, \bar{\phi}, \epsilon A_\gamma\bar{\phi})| &\le \frac{\epsilon^2}{5}|A_\gamma\bar{\phi}|^2 + c|\tilde{\boldsymbol{u}}|^2\|\tilde{\boldsymbol{u}}\|^2|A_\gamma^{1/2}\bar{\phi}|^2, \\ |(R_0(\epsilon A_\gamma\widetilde{\phi}, \bar{\phi}), \bar{\boldsymbol{u}})| &\le \frac{\nu\mathcal{K}^{-1}}{4}\|\bar{\boldsymbol{u}}\|^2 + \frac{\epsilon^2}{5}|A_\gamma\bar{\phi}|^2 + c|A_\gamma\widetilde{\phi}|^2(|\bar{\boldsymbol{u}}|^2 + |A_\gamma^{1/2}\bar{\phi}|^2), \\ |(R_0(\epsilon A_\gamma\bar{\phi}, \phi), \bar{\boldsymbol{u}})| &\le \frac{\nu\mathcal{K}^{-1}}{4}\|\bar{\boldsymbol{u}}\|^2 + \frac{\epsilon^2}{5}|A_\gamma\bar{\phi}|^2 + c|A_\gamma\phi|^2|A_\gamma^{1/2}\phi|^2|\bar{\boldsymbol{u}}|^2, \end{aligned}$$

where c is a positive large constant possibly depending on $\mathcal{K}, \nu, \epsilon, \alpha, \mathcal{O}$.

Regarding the last term in (3.61), we apply the Lagrange mean value theorem to f_γ so as to get

$$\begin{aligned} -\alpha\epsilon(f_\gamma(\phi) - f_\gamma(\widetilde{\phi}), A_\gamma\bar{\phi}) &= -\alpha\epsilon(f_\gamma'(\phi + \theta\bar{\phi})\bar{\phi}, A_\gamma\bar{\phi}) \\ &= -\alpha\epsilon(f'(\phi + \theta\bar{\phi})\bar{\phi}, A_\gamma\bar{\phi}) + \epsilon^2\gamma|A_\gamma^{1/2}\bar{\phi}|^2, \end{aligned}$$

with $0 < \theta < 1$. Now, from $(2.21)_2$, using the Hölder and the Young inequalities, we find

$$\begin{aligned} &-\alpha\epsilon(f_\gamma(\phi) - f_\gamma(\widetilde{\phi}), A_\gamma\bar{\phi}) \\ &\quad \le c_f\alpha\epsilon\int_{\mathcal{O}}(1 + |\phi + \theta\bar{\phi}|)|\bar{\phi}||A_\gamma\bar{\phi}|\,\mathrm{d}x + \epsilon^2\gamma|A_\gamma^{1/2}\bar{\phi}|^2 \\ &\quad \le c_f\alpha\epsilon|\bar{\phi}||A_\gamma\bar{\phi}| + \epsilon^2\gamma|A_\gamma^{1/2}\bar{\phi}|^2 + c_{\mathcal{O}}c_f\alpha\epsilon|\phi|_{L^4(\mathcal{O})}|\bar{\phi}|_{L^4(\mathcal{O})}|A_\gamma\bar{\phi}| \\ &\qquad + c_{\mathcal{O}}c_f\alpha\epsilon|\bar{\phi}|^2_{L^4(\mathcal{O})}|A_\gamma\bar{\phi}| \end{aligned}$$

$$
\begin{aligned}
&\le c_f \alpha \epsilon |\overline{\phi}||A_\gamma \overline{\phi}| + \epsilon^2 \gamma |A_\gamma^{1/2} \overline{\phi}|^2 \\
&\quad + c_{\mathcal{O}} c_f \alpha \epsilon \gamma^{-1/2} (\gamma^{-1} + 1)^{\frac{1}{2}} (|A_\gamma^{1/2} \phi||A_\gamma^{1/2} \overline{\phi}| + |A_\gamma^{1/2} \overline{\phi}|^2)|A_\gamma \overline{\phi}| \\
&\le \frac{\epsilon^2}{5} |A_\gamma \overline{\phi}|^2 + 2c_f^2 \alpha^2 |\overline{\phi}|^2 + \epsilon^2 \gamma |A_\gamma^{1/2} \overline{\phi}|^2 \\
&\quad + 4(c_{\mathcal{O}} c_f \alpha)^2 \gamma^{-1} (\gamma^{-1} + 1)(|A_\gamma^{1/2} \phi|^2 |A_\gamma^{1/2} \overline{\phi}|^2 + |A_\gamma^{1/2} \overline{\phi}|^4) \\
&\le \frac{\epsilon^2}{5} |A_\gamma \overline{\phi}|^2 + (2c_f^2 \gamma^{-1} + \gamma) \alpha^2 |A_\gamma^{1/2} \overline{\phi}|^2 \\
&\quad + 4(c_{\mathcal{O}} c_f \alpha)^2 \gamma^{-1} (\gamma^{-1} + 1)(|A_\gamma^{1/2} \phi|^2 |A_\gamma^{1/2} \overline{\phi}|^2 + |A_\gamma^{1/2} \overline{\phi}|^4).
\end{aligned}
$$

Here, $c_{\mathcal{O}}$ is a positive constant depending on the domain $\mathcal{O}$.

Inserting these previous estimates into the right-hand side of (3.61), we obtain

$$
\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} |(\bar{\boldsymbol{u}}(t), \overline{\phi}(t))|_{\mathbb{Y}}^2 \le g(t) |(\bar{\boldsymbol{u}}(t), \overline{\phi}(t))|_{\mathbb{Y}}^2, \tag{3.62}
$$

where $c = c(\mathcal{O}, \alpha, \mathcal{K}, \nu, c_f, \gamma, \epsilon)$ is a positive constant and

$$
\begin{aligned}
g(t) :&= c(1 + \|\boldsymbol{u}(t)\|^2 + |\tilde{\boldsymbol{u}}(t)|^2 \|\tilde{\boldsymbol{u}}(t)\|^2 + |A_\gamma^{1/2} \phi(t)|^2 + |A_\gamma^{1/2} \phi(t)|^2 |A_\gamma \phi(t)|^2 \\
&\quad + |A_\gamma \widetilde{\phi}(t)|^2 + |A_\gamma \phi(t)|^2 |A_\gamma^{1/2} \phi(t)|^2).
\end{aligned} \tag{3.63}
$$

It then follows by applying the Gronwall lemma that

$$
|((\boldsymbol{u} - \tilde{\boldsymbol{u}})(t), (\phi - \widetilde{\phi})(t))|_{\mathbb{Y}}^2 \le |(\boldsymbol{u}_0 - \tilde{\boldsymbol{u}}_0, \phi_0 - \widetilde{\phi}_0)|_{\mathbb{Y}}^2 \exp^{\int_0^t g(s)\, \mathrm{d}s}, \quad \mathbb{P}\text{-a.s.}
$$

This completes the uniqueness of $(\boldsymbol{u}, \phi)$ as well as the continuity of

$$
(\boldsymbol{u}_0, \phi_0) \mapsto (\boldsymbol{u}(t), \phi(t)). \qquad \blacksquare
$$

4. Ergodicity

4.1. Existence of invariant measure

In this part, we aim to prove the existence of invariant measures of (2.6) by the Krylov–Bogoliubov theorem (see, e.g., [11, p. 14]). To state the main result of this section, we firstly introduce some notations and definitions. Let $C_b(\mathbb{Y})$ denote the set of all bounded continuous functions on $\mathbb{Y}$. We equip it with the norm

$$
\|\Psi\|_\infty = \sup_{X \in \mathbb{Y}} |\Psi(X)|_{\mathbb{Y}}.
$$

Then, $(C_b(\mathbb{Y}), \|\cdot\|_\infty)$ is a Banach space.

For each $t > 0$, we define the semigroup P_t associated with the solution

$$
\{(\boldsymbol{u}(t, U_0), \phi(t, U_0)) \in L_W^2(0, T; \mathbb{V}),\ U_0 = (\boldsymbol{u}_0, \phi_0) \in \mathbb{Y}\}
$$

of (2.6) by

$$P_t\Psi(U_0) = \mathbb{E}[\Psi((\boldsymbol{u}(t,U_0),\phi(t,U_0)))], \quad \Psi \in C_b(\mathbb{Y}), \quad U_0 = (\boldsymbol{u}_0,\phi_0) \in \mathbb{Y}. \tag{4.1}$$

The proof of Markov property of P_t is standard (see, for instance, [3] or [7]). We also have $P_{t+s}\Psi(.) = P_t P_s\Psi(.)$ for $t, s \geq 0$.

Denote by $\mathcal{B}(\mathbb{Y})$ the σ-field of all Borel subsets of $\mathbb{Y}$ and by $P_r(\mathbb{Y})$ the set of all probability measures defined on $(\mathbb{Y}, \mathcal{B}(\mathbb{Y}))$. Define P_t^* to be the dual semigroup of P_t given by

$$\int_{\mathbb{Y}} \Psi(x) P_t^*\mu_*(\mathrm{d}x) = \int_{\mathbb{Y}} P_t\Psi(x)\mu_*(\mathrm{d}x)$$

for $\mu_* \in P_r(\mathbb{Y})$, $t \geq 0$, and $\Psi \in C_b(\mathbb{Y})$. A measure $\mu_* \in P_r(\mathbb{Y})$ is called invariant measure if $P_t^*\mu_* = \mu_*$ for each $t \geq 0$.

Definition 4.1. A subset $\Gamma \subset P_r(\mathbb{Y})$ is said to be tight if there exists an increasing sequence (κ_n) of compact sets of $\mathbb{Y}$ such that

$$\lim_{n\to\infty} \mu_*(\kappa_n) = 1 \quad \text{uniformly on } \Gamma,$$

or, equivalently, if for any $\delta > 0$ there exists a compact set K_δ such that

$$\mu_*(K_\delta) \geq 1-\delta, \quad \mu_* \in \Gamma.$$

Now, we state the following result concerning the existence of the invariant measures of (2.6).

Theorem 4.1. *There exists an invariant measure $\mu_* \in P_r(\mathbb{Y})$ associated with the semigroup P_t satisfying*

$$\int_{\mathbb{Y}} P_t\Psi(x)\mu_*(\mathrm{d}x) = \int_{\mathbb{Y}} \Psi(x)\mu_*(\mathrm{d}x) \quad \textit{for any } t \geq 0 \textit{ and } \Psi \in C_b(\mathbb{Y}).$$

Moreover, the support of μ_* is included in $\mathbb{V}$ and

$$\int_{\mathbb{Y}} \|(x,y)\|_{\mathbb{V}}^2 \mu_*(\mathrm{d}x,\mathrm{d}y) < +\infty. \tag{4.2}$$

Proof. Firstly, let us point out the following estimate:

$$\begin{aligned}\mathbb{E}|(\boldsymbol{u}(t,U_0),\phi(t,U_0))|_{\mathbb{Y}}^2 &+ 2\mathbb{E}\int_0^t \|(\boldsymbol{u}(s,U_0),\phi(s,U_0))\|_{\mathbb{V}}^2\,\mathrm{d}s \\ &\leq c_3(\alpha,\gamma,\gamma_2,T)(|(\boldsymbol{u}_0,\phi_0)|_{\mathbb{Y}}^2 + t\mathcal{K}^{-1}\operatorname{Tr}Q_1 + \alpha\gamma t\operatorname{Tr}Q_2 + \alpha t\Lambda), \quad t \geq 0, \end{aligned} \tag{4.3}$$

which can be deduced from (2.6) by arguing as in (3.59). Here, $(\boldsymbol{u}(t,U_0),\phi(t,U_0))$ is the pathwise solution of system (2.6) starting from the initial data $U_0 = (\boldsymbol{u}_0,\phi_0)$.

Let $\Pi_t(U_0,\cdot)$ be the law of the process $(\boldsymbol{u}(t),\phi(t))$. Then, for any $\Psi \in C_b(\mathbb{Y})$, we have

$$P_t\Psi(U_0) = \int_{\mathbb{Y}} \Psi(\boldsymbol{u}_1,\phi_1)\Pi_t(U_0,\mathrm{d}x_1,\mathrm{d}\phi_1) \quad \forall(\boldsymbol{u}_1,\phi_1) \in \mathbb{Y}.$$

In order to prove the existence of an invariant measure, it is enough to only check that, in view of the Krylov–Bogoliubov theorem [11], that the set of measures

$$\mu_{*,T} := \frac{1}{T}\int_0^T \Pi_t(U_0, .)\, \mathrm{d}t, \quad T > 1,$$

is tight in $P_r(\mathbb{Y})$. We fix $U_0 = (\boldsymbol{u}_0, \phi_0) \in \mathbb{Y}$. Then, by (4.3), we have

$$\frac{1}{t}\mathbb{E}\int_0^t \|(\boldsymbol{u}(s, U_0), \phi(s, U_0))\|_{\mathbb{V}}^2\, \mathrm{d}s \le c_3(|(\boldsymbol{u}_0, \phi_0)|_{\mathbb{Y}}^2 + \mathcal{K}^{-1}\operatorname{Tr} Q_1 + \epsilon\gamma \operatorname{Tr} Q_2 + \epsilon\Lambda).$$

Let $\mathbb{B}_R$ be the ball of radius R in $\mathbb{V} = V_1 \times D(A_\gamma)$. Then, for all $R > 0$, we derive that

$$\begin{aligned}
\mu_{*,T}(\mathbb{B}_R^c) &= \frac{1}{T}\int_0^T \Pi_t(U_0, \mathbb{B}_R^c)\, \mathrm{d}t \\
&\le \frac{1}{TR^2}\int_0^T \mathbb{E}\|(\boldsymbol{u}(t, U_0), \phi(t, U_0))\|_{\mathbb{V}}^2\, \mathrm{d}t \\
&\le \frac{c_3}{R^2}(|(\boldsymbol{u}_0, \phi_0)|_{\mathbb{Y}}^2 + \mathcal{K}^{-1}\operatorname{Tr} Q_1 + \epsilon\gamma \operatorname{Tr} Q_2 + \epsilon\Lambda),
\end{aligned}$$

from which we get the tightness of $\{\mu_{*,T}\}_{T\ge 1}$. Denote by μ_* a cluster point of $\{\mu_{*,T}\}_{T\ge 1}$. Then, by integrating (4.3) on $\mathbb{Y}$ with respect to μ_*, we get (4.2). This completes the proof. ∎

4.2. The uniqueness of invariant measures

Here, we follow the approach in [2,13,27] to prove the uniqueness of the invariant measure μ_*, using the coupling method (see, e.g., [2, 13, 19, 27]). Lemmas 4.1–4.4 below are the main steps in the proof. We still denote by $(\boldsymbol{u}(t, U_0), \phi(t, U_0))$ the solution to (2.6) with initial data $U_0 = (\boldsymbol{u}_0, \phi_0) \in \mathbb{Y}$.

Lemma 4.1. *Let* $\beta = \min(\nu\lambda, \epsilon\ell) - \alpha(\gamma_2 + \gamma)$, *and we assume that*

$$\beta > 0. \tag{4.4}$$

Then, the following estimates hold:

$$\mathbb{E}|(\boldsymbol{u}(t, U_0), \phi(t, U_0))|_{\mathbb{Y}}^2 \le |U_0|_{\mathbb{Y}}^2 e^{-2\beta t} + \frac{L_1}{2\beta} \tag{4.5}$$

and

$$\mathbb{E}\int_0^t \|(\boldsymbol{u}(s, U_0), \phi(s, U_0))\|_{\mathbb{V}}^2\, \mathrm{d}s \le \frac{1}{2\beta}|U_0|_{\mathbb{Y}}^2 + \frac{L_1}{2\beta}t, \tag{4.6}$$

for all $t \in [0, T]$, *where*

$$L_1 = (\mathcal{K}^{-1}\operatorname{Tr} Q_1 + \epsilon\gamma \operatorname{Tr} Q_2 + \epsilon\Lambda).$$

Proof. Let $t \in [0, T]$ be fixed. We set for the sake of simplicity

$$\mathcal{Y}(t) = \mathbb{E}|(\boldsymbol{u}(t, U_0), \phi(t, U_0))|^2_{\mathbb{Y}}.$$

Now, as in (3.59) we can prove that

$$\mathcal{Y}(t) + 2\mathbb{E}\int_0^t \|(\boldsymbol{u}(s, U_0), \phi(s, U_0))\|^2_{\mathbb{V}}\,\mathrm{d}s \le |U_0|^2_{\mathbb{Y}} + L_1 t + 2\alpha(\gamma_2 + \gamma)\int_0^t \mathcal{Y}(s)\,\mathrm{d}s. \tag{4.7}$$

From (2.7) and (4.7), we infer that

$$\mathcal{Y}(t) + 2\beta\int_0^t \mathcal{Y}(s)\,\mathrm{d}s \le |U_0|^2_{\mathbb{Y}} + tL_1. \tag{4.8}$$

We can then conclude, through the application of the Gronwall lemma [16, Lemma 1] that

$$\mathcal{Y}(t) \le \mathcal{Y}(0)e^{-2\beta t} + L_1\int_0^t e^{-2\beta(t-\tau)}\,\mathrm{d}\tau = \mathcal{Y}(0)e^{-2\beta t} + \frac{L_1}{2\beta} - \frac{L_1}{2\beta}e^{-2\beta t}, \tag{4.9}$$

from which we get (4.5). Furthermore, (4.6) is a direct consequence of (4.8). ∎

Lemma 4.2. *Let $r_0, r_1 > 0$. Then, there exists $\kappa = \kappa(r_0, r_1)$ and $T = T(r_0, r_1) > 0$ such that for any $t \in [T(r_0, r_1), 2T(r_0, r_1)]$, $|\boldsymbol{u}_0| \le r_0$, $|A_\gamma^{1/2}\phi_0| \le r_0$, we have*

$$\mathbb{P}(|\boldsymbol{u}(t, U_0)| \le r_1, |A_\gamma^{1/2}\phi(t, U_0)| \le r_1) \ge \kappa(r_0, r_1). \tag{4.10}$$

Proof. Let $\boldsymbol{v} = \boldsymbol{u} - W_{A_0}$ and $\psi = \phi - W_{A_\gamma}$, where W_{A_0} and W_{A_γ} are mild solutions to (2.11).

Multiplying the second equation of (3.2) by $\epsilon\gamma\psi$, we obtain $\mathbb{P}$-a.s.

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\epsilon\gamma|\psi(t)|^2) + b_1(\boldsymbol{v}(t), W_{A_\gamma}(t), \epsilon\gamma\psi(t)) + (\epsilon A_\gamma\psi(t), \epsilon\gamma\psi(t))\\
&\quad = -(\epsilon\gamma W_{A_\gamma}(t), \epsilon\gamma\psi(t)) - b_1(W_{A_0}(t), W_{A_\gamma}(t), \epsilon\gamma\psi(t))\\
&\qquad - \alpha(f_\gamma(\psi(t) + W_{A_\gamma}(t)), \epsilon\gamma\psi(t)).
\end{aligned} \tag{4.11}$$

Once more, by multiplying the second equation of (3.2) by $\epsilon A_1\psi$, we get $\mathbb{P}$-a.s.

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\epsilon|\nabla\psi(t)|^2 + b_1(\boldsymbol{v}(t), \psi(t), \epsilon A_1\psi(t)) + b_1(\boldsymbol{v}(t), W_{A_\gamma}(t), \epsilon A_1\psi(t))\\
&+ b_1(W_{A_0}(t), \psi(t), \epsilon A_1\psi(t)) + (\epsilon A_1\psi(t), \epsilon A_\gamma\psi(t)) + (\epsilon\gamma W_{A_\gamma}(t), \epsilon A_1\psi(t))\\
&\quad = -b_1(W_{A_0}(t), W_{A_\gamma}(t), \epsilon A_1\psi(t)) - \alpha(f_\gamma(\psi(t) + W_{A_\gamma}(t)), \epsilon A_1\psi(t)).
\end{aligned} \tag{4.12}$$

Adding up (4.11) and (4.12) side by side, we arrive at

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\epsilon|A_\gamma^{1/2}\psi(t)|^2) + \epsilon^2|A_\gamma\psi(t)|^2\\
&= -b_1(\boldsymbol{v}(t), \psi(t), \epsilon A_\gamma\psi(t)) - b_1(\boldsymbol{v}(t), W_{A_\gamma}(t), \epsilon A_\gamma\psi(t))\\
&\quad - b_1(W_{A_0}(t), \psi(t), \epsilon A_\gamma\psi(t)) - (\epsilon\gamma W_{A_\gamma}(t), \epsilon A_\gamma\psi(t))\\
&\quad - b_1(W_{A_0}(t), W_{A_\gamma}(t), \epsilon A_\gamma\psi(t)) - \alpha(f_\gamma(\psi(t) + W_{A_\gamma}(t)), \epsilon A_\gamma\psi(t)).
\end{aligned} \tag{4.13}$$

Now, multiplying the first equation of (3.2) by $\mathcal{K}^{-1}\boldsymbol{v}$, we deduce that $\mathbb{P}$-a.s.

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{K}^{-1}|\boldsymbol{v}(t)|^2) + \nu\mathcal{K}^{-1}\|\boldsymbol{v}(t)\|^2 + \mathcal{K}^{-1}b_0(\boldsymbol{v}(t), W_{A_0}(t), \boldsymbol{v}(t))\\
&- (R_0(\epsilon A_\gamma\psi(t), \psi(t)), \boldsymbol{v}(t)) + \mathcal{K}^{-1}b_0(W_{A_0}(t), W_{A_0}(t), \boldsymbol{v}(t))\\
&\quad = (R_0(\epsilon A_\gamma W_{A_\gamma}(t), \psi(t)), \boldsymbol{v}(t)) + (R_0(\epsilon A_\gamma\psi(t), W_{A_\gamma}(t)), \boldsymbol{v}(t))\\
&\quad\quad + (R_0(\epsilon A_\gamma W_{A_\gamma}(t), W_{A_\gamma}(t)), \boldsymbol{v}(t)).
\end{aligned} \tag{4.14}$$

By adding up (4.13) and (4.14) side by side, we find that $\mathbb{P}$-a.s.

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|(\boldsymbol{v}(t), \psi(t))|^2_{\mathbb{Y}} + \nu\mathcal{K}^{-1}\|\boldsymbol{v}(t)\|^2 + \epsilon^2|A_\gamma\psi(t)|^2\\
&= -\mathcal{K}^{-1}b_0(\boldsymbol{v}(t), W_{A_0}(t), \boldsymbol{v}(t)) - \mathcal{K}^{-1}b_0(W_{A_0}(t), W_{A_0}(t), \boldsymbol{v}(t))\\
&\quad + (R_0(\epsilon A_\gamma W_{A_\gamma}(t), \psi(t)), \boldsymbol{v}(t)) + (R_0(\epsilon A_\gamma W_{A_\gamma}(t), W_{A_\gamma}(t)), \boldsymbol{v}(t))\\
&\quad - b_1(W_{A_0}(t), \psi(t), \epsilon A_\gamma\psi(t)) - (\epsilon\gamma W_{A_\gamma}(t), \epsilon A_\gamma\psi(t))\\
&\quad - b_1(W_{A_0}(t), W_{A_\gamma}(t), \epsilon A_\gamma\psi(t)) - \alpha(f_\gamma(\psi(t) + W_{A_\gamma}(t)), \epsilon A_\gamma\psi(t)).
\end{aligned} \tag{4.15}$$

Consequently,

$$\begin{aligned}
&\frac{1}{2}[e^{\delta t}|(\boldsymbol{v}, \psi)(t)|^2_{\mathbb{Y}}] + \nu\mathcal{K}^{-1}\int_0^t e^{\delta s}\|\boldsymbol{v}\|^2\,\mathrm{d}s + \epsilon^2\int_0^t e^{\delta s}|A_\gamma\psi|^2\,\mathrm{d}s\\
&= \frac{1}{2}|(\boldsymbol{u}_0, \phi_0)|^2_{\mathbb{Y}} - \mathcal{K}^{-1}\int_0^t e^{\delta s}b_0(\boldsymbol{v}, W_{A_0}, \boldsymbol{v})\,\mathrm{d}s + \int_0^t e^{\delta s}(R_0(\epsilon A_\gamma W_{A_\gamma}, \psi), \boldsymbol{v})\,\mathrm{d}s\\
&\quad - \int_0^t e^{\delta s}b_1(W_{A_0}, \psi, \epsilon A_\gamma\psi)\,\mathrm{d}s - \int_0^t e^{\delta s}\mathcal{K}^{-1}b_0(W_{A_0}, W_{A_0}, \boldsymbol{v})\,\mathrm{d}s\\
&\quad + \int_0^t e^{\delta s}(R_0(\epsilon A_\gamma W_{A_\gamma}, W_{A_\gamma}), \boldsymbol{v})\,\mathrm{d}s - \alpha\int_0^t e^{\delta s}(f_\gamma(\psi + W_{A_\gamma}), \epsilon A_\gamma\psi)\,\mathrm{d}s\\
&\quad - \int_0^t e^{\delta s}b_1(W_{A_0}, W_{A_\gamma}, \epsilon A_\gamma\psi)\,\mathrm{d}s - \int_0^t e^{\delta s}(\gamma\epsilon W_{A_\gamma}, \epsilon A_\gamma\psi)\,\mathrm{d}s + \frac{1}{2}\delta\int_0^t e^{\delta s}|(v, \psi)|^2_{\mathbb{Y}}\,\mathrm{d}s
\end{aligned} \tag{4.16}$$

for all $t \in [0, T]$, $\mathbb{P}$-a.s., where δ is a positive constant independent of t, and it will be chosen later.

Let us proceed to estimate the terms on the right-hand side of (4.16).

Using the Hölder, the Ladyzhenskaya, and the Young inequalities, we find that

$$\begin{aligned}
|\mathcal{K}^{-1}b_0(\boldsymbol{v}, W_{A_0}, \boldsymbol{v})| &\leq \mathcal{K}^{-1}\|\boldsymbol{v}\|_{\mathbb{L}^4(\mathcal{O})}\|\boldsymbol{v}\|\|W_{A_0}\|_{\mathbb{L}^4(\mathcal{O})}\\
&\leq \mathcal{K}^{-1}c(\mathcal{O})|\boldsymbol{v}|^{\frac{1}{2}}\|\boldsymbol{v}\|^{\frac{3}{2}}\|W_{A_0}\|_{\mathbb{L}^4(\mathcal{O})}\\
&\leq \frac{\nu\mathcal{K}^{-1}}{6}\|\boldsymbol{v}\|^2 + c(\mathcal{O}, \nu)\|W_{A_0}\|^4_{\mathbb{L}^4(\mathcal{O})}(\mathcal{K}^{-1}|\boldsymbol{v}|^2).
\end{aligned} \tag{4.17}$$

By the Hölder and the Young inequalities, we get

$$\begin{aligned}
&|\mathcal{K}^{-1}b_0(W_{A_0}, W_{A_0}, \boldsymbol{v})| + |(R_0(\epsilon A_\gamma W_{A_\gamma}, W_{A_\gamma}), \boldsymbol{v})| \\
&\quad \le \mathcal{K}^{-1}c(\mathcal{O})\|W_{A_0}\|^2_{\mathbb{L}^4(\mathcal{O})}\|\boldsymbol{v}\| + c(\mathcal{O})\epsilon|A_\gamma W_{A_\gamma}|^2\|\boldsymbol{v}\| \\
&\quad \le \frac{\nu\mathcal{K}^{-1}}{6}\|\boldsymbol{v}\|^2 + c(\mathcal{O},\nu,\mathcal{K})\|W_{A_0}\|^4_{\mathbb{L}^4(\mathcal{O})} + c(\mathcal{O},\mathcal{K},\nu,\epsilon)|A_\gamma W_{A_\gamma}|^4. \qquad (4.18)
\end{aligned}$$

Once more, using the Hölder and the Young inequalities together with the embedding of $\mathbb{H}^1(\mathcal{O})$ in $\mathbb{L}^4(\mathcal{O})$, we obtain

$$\begin{aligned}
|b_1(W_{A_0}, W_{A_\gamma}, \epsilon A_\gamma\psi)| &\le \epsilon c(\mathcal{O})\|W_{A_0}\|_{\mathbb{L}^4(\mathcal{O})}|A_\gamma W_{A_\gamma}||A_\gamma\psi| \\
&\le \frac{\epsilon^2}{10}|A_\gamma\psi|^2 + c(\mathcal{O})\|W_{A_0}\|^4_{\mathbb{L}^4(\mathcal{O})} + c(\mathcal{O})|A_\gamma W_{A_\gamma}|^4. \qquad (4.19)
\end{aligned}$$

Thanks to the Hölder, the Gagliardo–Nirenberg, and the Young inequalities, we obtain

$$\begin{aligned}
|(R_0(\epsilon A_\gamma W_{A_\gamma}, \psi), \boldsymbol{v})| &\le \epsilon|A_\gamma W_{A_\gamma}|\|\nabla\psi\|_{\mathbb{L}^4(\mathcal{O})}\|\boldsymbol{v}\|_{\mathbb{L}^4(\mathcal{O})} \\
&\le c(\mathcal{O})\epsilon|A_\gamma W_{A_\gamma}||A_\gamma\psi||\boldsymbol{v}|^{\frac{1}{2}}\|\boldsymbol{v}\|^{\frac{1}{2}} \\
&\le \frac{\nu\mathcal{K}^{-1}}{6}\|\boldsymbol{v}\|^2 + \frac{\epsilon^2}{10}|A_\gamma\psi_\varepsilon|^2 \\
&\quad + c(\mathcal{O},\epsilon,\nu,\mathcal{K})|A_\gamma W_{A_\gamma}|^4(\mathcal{K}^{-1}|\boldsymbol{v}|^2), \qquad (4.20)
\end{aligned}$$

where we have also used the embedding of $\mathbb{H}^1(\mathcal{O})$ in $\mathbb{L}^4(\mathcal{O})$.

One has

$$\begin{aligned}
|b_1(W_{A_0}, \psi, \epsilon A_\gamma\psi)| &\le \epsilon\|W_{A_0}\|_{\mathbb{L}^4(\mathcal{O})}\|\nabla\psi\|_{\mathbb{L}^4(\mathcal{O})}|A_\gamma\psi| \\
&\le \epsilon c(\mathcal{O})\|W_{A_0}\|_{\mathbb{L}^4(\mathcal{O})}|\nabla\psi|^{\frac{1}{2}}|A_\gamma\psi|^{\frac{3}{2}} \\
&\le \frac{\epsilon^2}{10}|A_\gamma\psi|^2 + c(\mathcal{O},\epsilon)\|W_{A_0}\|^4_{\mathbb{L}^4(\mathcal{O})}(\epsilon|A_\gamma^{1/2}\psi|^2), \qquad (4.21)
\end{aligned}$$

where we used the Hölder, the Gagliardo–Nirenberg, and the Young inequalities.

Combining the Hölder and the Young inequalities, we see that

$$\begin{aligned}
-(\epsilon\gamma W_{A_\gamma}, \epsilon A_\gamma\psi) &\le \epsilon^2\gamma|W_{A_\gamma}||A_\gamma\psi| \le \epsilon^2\gamma^{1/2}|A_\gamma^{1/2}W_{A_\gamma}||A_\gamma\psi| \\
&\le \frac{\epsilon^2}{10}|A_\gamma\psi|^2 + c\epsilon^2\gamma|A_\gamma^{1/2}W_{A_\gamma}|^2. \qquad (4.22)
\end{aligned}$$

Next, owing to (2.23), we have

$$\begin{aligned}
&-\alpha(f_\gamma(\psi + W_{A_\gamma}), \epsilon A_\gamma\psi) \\
&= -\alpha\epsilon(A_\gamma^{1/2}(\psi + W_{A_\gamma})f'(\psi + W_{A_\gamma}), A_\gamma^{1/2}\psi) + \epsilon^2\gamma(A_\gamma^{1/2}(\psi + W_{A_\gamma}), A_\gamma^{1/2}\psi) \\
&\le \epsilon\alpha(\gamma_2 + \gamma)|A_\gamma^{1/2}\psi|^2 - \alpha\epsilon(A_\gamma^{1/2}W_{A_\gamma}f'(\psi + W_{A_\gamma}), A_\gamma^{1/2}\psi) + \epsilon^2\gamma(A_\gamma^{1/2}W_{A_\gamma}, A_\gamma^{1/2}\psi) \\
&\le \epsilon\alpha(\gamma_2 + \gamma)|A_\gamma^{1/2}\psi|^2 - \alpha\epsilon(A_\gamma^{1/2}W_{A_\gamma}f'(\psi + W_{A_\gamma}), A_\gamma^{1/2}\psi) + \epsilon^2\gamma|A_\gamma^{1/2}W_{A_\gamma}||A_\gamma^{1/2}\psi| \\
&\le \epsilon\alpha(\gamma_2 + \gamma)|A_\gamma^{\frac{1}{2}}\psi|^2 - \alpha\epsilon(A_\gamma^{\frac{1}{2}}W_{A_\gamma}f'(\psi + W_{A_\gamma}), A_\gamma^{\frac{1}{2}}\psi) \\
&\quad + \epsilon^2\gamma|A_\gamma^{\frac{1}{2}}W_{A_\gamma}| + \epsilon^2\gamma|A_\gamma^{\frac{1}{2}}W_{A_\gamma}||A_\gamma^{\frac{1}{2}}\psi|^2.
\end{aligned}$$

Furthermore, thanks to the Hölder and the Gagliardo–Nirenberg inequalities in conjunction with the second assumption of (2.21), we deduce that

$$
\begin{aligned}
&-\alpha\epsilon(A_\gamma^{\frac12}W_{A_\gamma}f'(\psi+W_{A_\gamma}),A_\gamma^{\frac12}\psi)\\
&\le \alpha\epsilon c_f\int_{\mathcal{O}}(1+|\psi+W_{A_\gamma}|)|A_\gamma^{\frac12}W_{A_\gamma}||A_\gamma^{\frac12}\psi|\,\mathrm{d}x\\
&\le \alpha\epsilon c_f|A_\gamma^{\frac12}W_{A_\gamma}||A_\gamma^{\frac12}\psi|+\alpha\epsilon c_f(\|\psi\|_{L^4}+\|W_{A_\gamma}\|_{L^4})|A_\gamma^{\frac12}W_{A_\gamma}|\|A_\gamma^{\frac12}\psi\|_{\mathbb{L}^4}\\
&\le \alpha\epsilon c_f|A_\gamma^{\frac12}W_{A_\gamma}||A_\gamma^{\frac12}\psi|\\
&\quad+\alpha\epsilon c_f c_{\mathcal{O}}(|\psi|^{\frac12}\|\psi\|_{H^1}^{\frac12}+|W_{A_\gamma}|^{\frac12}\|W_{A_\gamma}\|_{H^1}^{\frac12})|A_\gamma^{\frac12}W_{A_\gamma}||A_\gamma^{\frac12}\psi|^{\frac12}\|A_\gamma^{\frac12}\psi\|_{\mathbb{H}^1}^{\frac12}\\
&\le \alpha\epsilon c_f|A_\gamma^{\frac12}W_{A_\gamma}||A_\gamma^{\frac12}\psi|\\
&\quad+\alpha c_f\gamma^{-\frac14}(1+\gamma^{-1})^{\frac12}c_{\mathcal{O}}(|A_\gamma^{\frac12}\psi|^{\frac32}|A_\gamma^{\frac12}W_{A_\gamma}|+|A_\gamma^{\frac12}W_{A_\gamma}|^2|A_\gamma^{\frac12}\psi|^{\frac12})(\epsilon^2|A_\gamma\psi|)^{\frac12}\\
&\le \frac{\epsilon^2}{10}|A_\gamma\psi|^2+\alpha^{\frac43}c_f^{\frac43}\gamma^{-\frac13}(1+\gamma^{-1})^{\frac23}c_{\mathcal{O}}(|A_\gamma^{\frac12}\psi|^2|A_\gamma^{\frac12}W_{A_\gamma}|^{\frac43}+|A_\gamma^{\frac12}W_{A_\gamma}|^{\frac83}|A_\gamma^{\frac12}\psi|^{\frac23})\\
&\quad+\alpha\epsilon c_f|A_\gamma^{\frac12}W_{A_\gamma}||A_\gamma^{\frac12}\psi|.
\end{aligned}
$$

This implies

$$
\begin{aligned}
&-\alpha\epsilon(A_\gamma^{\frac12}W_{A_\gamma}f'(\psi+W_{A_\gamma}),A_\gamma^{\frac12}\psi)\\
&\quad\le \frac{\epsilon^2}{10}|A_\gamma\psi|^2+\alpha^{\frac43}c_f^{\frac43}\gamma^{-\frac13}(1+\gamma^{-1})^{\frac23}c_{\mathcal{O}}|A_\gamma^{\frac12}W_{A_\gamma}|^3\\
&\qquad+\alpha\epsilon c_f|A_\gamma^{\frac12}W_{A_\gamma}|+\alpha\epsilon c_f|A_\gamma^{\frac12}W_{A_\gamma}||A_\gamma^{\frac12}\psi|^2\\
&\qquad+\alpha^{\frac43}c_f^{\frac43}\gamma^{-\frac13}(1+\gamma^{-1})^{\frac23}c_{\mathcal{O}}[|A_\gamma^{\frac12}W_{A_\gamma}|^{\frac43}+|A_\gamma^{\frac12}W_{A_\gamma}|^2]|A_\gamma^{\frac12}\psi|^2,
\end{aligned}
$$

where we used suitable Young's inequalities. Hence,

$$
\begin{aligned}
&-\alpha(f_\gamma(\psi+W_{A_\gamma}),\epsilon A_\gamma\psi)\\
&\le \frac{\epsilon^2}{10}|A_\gamma\psi|^2+\epsilon\alpha(\gamma_2+\gamma)|A_\gamma^{\frac12}\psi|^2+\epsilon^2\gamma|A_\gamma^{\frac12}W_{A_\gamma}|+\epsilon^2\gamma|A_\gamma^{\frac12}W_{A_\gamma}||A_\gamma^{\frac12}\psi|^2\\
&\quad+\alpha^{\frac43}c_f^{\frac43}\gamma^{-\frac13}(1+\gamma^{-1})^{\frac23}c_{\mathcal{O}}|A_\gamma^{\frac12}W_{A_\gamma}|^3+\alpha\epsilon c_f|A_\gamma^{\frac12}W_{A_\gamma}|+\alpha\epsilon c_f|A_\gamma^{\frac12}W_{A_\gamma}||A_\gamma^{\frac12}\psi|^2\\
&\quad+\alpha^{\frac43}c_f^{\frac43}\gamma^{-\frac13}(1+\gamma^{-1})^{\frac23}c_{\mathcal{O}}[|A_\gamma^{\frac12}W_{A_\gamma}|^{\frac43}+|A_\gamma^{\frac12}W_{A_\gamma}|^2]|A_\gamma^{\frac12}\psi|^2. \qquad (4.23)
\end{aligned}
$$

Plugging the estimates (4.17)–(4.23) into the right-hand side of (4.16), we find that

$$
\begin{aligned}
&\frac12 e^{\delta t}|(\boldsymbol{v},\psi)(t)|_{\mathbb{Y}}^2+\frac12\min(\nu\lambda_1,\epsilon\ell)\int_0^t e^{\delta s}|(\boldsymbol{v},\psi)|_{\mathbb{Y}}^2\,\mathrm{d}s\\
&\le \frac12|(\boldsymbol{u}_0,\phi_0)|_{\mathbb{Y}}^2\\
&\quad+\int_0^t e^{\delta s}[c(\mathcal{O},\nu,\mathcal{K})\|W_{A_0}\|_{\mathbb{L}^4(\mathcal{O})}^4+c(\mathcal{O},\mathcal{K},\nu,\epsilon)|A_\gamma W_{A_\gamma}|^4+c\epsilon^2\gamma|A_\gamma^{1/2}W_{A_\gamma}|^2]\,\mathrm{d}s
\end{aligned}
$$

$$+\int_0^t e^{\delta s}[(\epsilon^2\gamma+\alpha\epsilon c_f)|A_\gamma^{1/2}W_{A_\gamma}|+\alpha^{4/3}c_f^{4/3}\gamma^{-1/3}(1+\gamma^{-1})^{2/3}c_{\mathcal{O}}|A_\gamma^{1/2}W_{A_\gamma}|^3]\,\mathrm{d}s$$
$$+\int_0^t e^{\delta s}[c(\mathcal{O},\epsilon)\|W_{A_0}\|^4_{\mathbb{L}^4(\mathcal{O})}+c(\mathcal{O},\epsilon,\nu,\mathcal{K})|A_\gamma W_{A_\gamma}|^4+(\epsilon\gamma+\alpha c_f)|A_\gamma^{1/2}W_{A_\gamma}|$$
$$+\epsilon^{-1}\alpha^{4/3}c_f^{4/3}\gamma^{-1/3}(1+\gamma^{-1})^{2/3}c_{\mathcal{O}}(|A_\gamma^{1/2}W_{A_\gamma}|^{4/3}+|A_\gamma^{1/2}W_{A_\gamma}|^2)]|(\boldsymbol{v},\psi)|^2_{\mathbb{Y}}$$
$$+\int_0^t e^{\delta s}[\alpha(\gamma_2+\gamma)+\delta/2]|(\boldsymbol{v},\psi)|^2_{\mathbb{Y}}\,\mathrm{d}s,$$

where we used the fact that $\min(\nu\lambda_1,\epsilon\ell)|(\boldsymbol{v},\psi)|^2_{\mathbb{Y}}\le\|(\boldsymbol{v},\psi)\|^2_{\mathbb{V}}$ due to (2.7).

Observe now that, thanks to (2.15) and (2.17), we have for each $\eta>0$

$$\mathbb{P}(S_\eta)>0, \tag{4.24}$$

where

$$S_\eta=\{\omega\in\Omega:\|W_{A_0}(t)\|^2_{\mathbb{L}^4(\mathcal{O})}+|A_\gamma W_{A_\gamma}(t)|^2\le\eta,\ t\in[0,2T]\}.$$

Furthermore, let

$$\min(\nu\lambda_1,\epsilon\ell)>2\alpha(\gamma_2+\gamma). \tag{4.25}$$

Hence, for η small enough such that

$$0<\eta<\frac{1}{2\bar{c}}[\min(\nu\lambda_1,\epsilon\ell)-2\alpha(\gamma_2+\gamma)],$$

where $\bar{c}=c(\mathcal{O},\nu,\mathcal{K},\gamma,\epsilon,c_f,\alpha)$ is a positive constant, we infer that

$$e^{\delta t}|(\boldsymbol{v},\psi)(t)|^2_{\mathbb{Y}}+\min(\nu\lambda_1,\epsilon\ell)\int_0^t e^{\delta s}|(\boldsymbol{v},\psi)|^2_{\mathbb{Y}}\,\mathrm{d}s$$
$$\le|(\boldsymbol{u}_0,\phi_0)|^2_{\mathbb{Y}}+\int_0^t e^{\delta s}[2\alpha(\gamma_2+\gamma)+\delta+2\bar{c}\eta]|(\boldsymbol{v},\psi)|^2_{\mathbb{Y}}\,\mathrm{d}s$$
$$+2\int_0^t e^{\delta s}[c(\mathcal{O},\nu,\mathcal{K})+c(\mathcal{O},\mathcal{K},\nu,\epsilon)+c\epsilon^2\gamma]\eta\,\mathrm{d}s$$
$$+2\int_0^t e^{\delta s}[(\epsilon^2\gamma+\alpha\epsilon c_f)+\alpha^{4/3}c_f^{4/3}\gamma^{-1/3}(1+\gamma^{-1})^{2/3}c_{\mathcal{O}}]\eta\,\mathrm{d}s$$

for a.e. $t\in[0,2T]$, $\mathbb{P}$-a.s. on S_η. Now, choosing δ between 0 and $\min(\nu\lambda_1,\epsilon\ell)-2\alpha(\gamma_2+\gamma)-2\bar{c}\eta$, we further obtain

$$|(\boldsymbol{v},\psi)(t)|^2_{\mathbb{Y}}\le e^{-\delta t}|(\boldsymbol{u}_0,\phi_0)|^2_{\mathbb{Y}}+\frac{\eta}{\delta}[2\{c(\mathcal{O},\nu,\mathcal{K})+c(\mathcal{O},\mathcal{K},\nu,\epsilon)+c\epsilon^2\gamma\}$$
$$+2\{(\epsilon^2\gamma+\alpha\epsilon c_f)+\alpha^{4/3}c_f^{4/3}\gamma^{-1/3}(1+\gamma^{-1})^{2/3}c_{\mathcal{O}}\}]$$

for all $t\in[T,2T]$, $\mathbb{P}$-a.s. on S_η, where δ is independent of T. The latter yields for $T=T(r_0,r_1)$ large enough,

$$|\boldsymbol{u}(t)|\le r_1,\ |A_\gamma^{1/2}\phi(t)|\le r_1\quad\forall t\in[0,2T]$$

on the set S_η with positive probability. ∎

Lemma 4.3. *Let $\Psi \in C_b(\mathbb{Y})$ be such that $\|\Psi\|_\infty \leq 1$. Then, for any $t > 0$, there exists $\delta_1 > 0$ such that*

$$|P_t\Psi(x, y) - P_t\Psi(x_1, y_1)| \leq \frac{1}{2} \tag{4.26}$$

for all (x, y), $(x_1, y_1) \in \mathbb{Y}$, $|(x, y)|_{\mathbb{Y}} < \delta_1$, $|(x_1, y_1)|_{\mathbb{Y}} < \delta_1$.

Proof. Let $U = (\boldsymbol{u}, \phi)$ be the solution of (2.6) with initial value $(x, y) \in \mathbb{Y}$ and denote by DU the Gâteaux derivative of U. Denote $DU = \begin{pmatrix} D_x\boldsymbol{u} & D_y\boldsymbol{u} \\ D_x\phi & D_y\phi \end{pmatrix} = \begin{pmatrix} \chi_1 & \chi_2 \\ \chi_3 & \chi_4 \end{pmatrix}$, where D_x and D_y are Gâteaux derivatives with respect to x and y. Then,

$$\begin{cases} \chi_1' + \nu A_0\chi_1 + B_0(\chi_1, \boldsymbol{u}) + B_0(\boldsymbol{u}, \chi_1) - \mathcal{K} R_0(\epsilon A_\gamma \chi_3, \phi) - \mathcal{K} R_0(\epsilon A_\gamma \phi, \chi_3) = 0, \\ \chi_3' + B_1(\chi_1, \phi) + B_1(\boldsymbol{u}, \chi_3) + \epsilon A_\gamma \chi_3 + \alpha f_\gamma'(\phi)\chi_3 = 0, \\ \chi_1(0) = 1, \quad \chi_2(0) = 0, \end{cases} \tag{4.27}$$

and

$$\begin{cases} \chi_2' + \nu A_0\chi_2 + B_0(\chi_2, \boldsymbol{u}) + B_0(\boldsymbol{u}, \chi_2) - \mathcal{K} R_0(\epsilon A_\gamma \chi_4, \phi) - \mathcal{K} R_0(\epsilon A_\gamma \phi, \chi_4) = 0, \\ \chi_4' + B_1(\chi_2, \phi) + B_1(\boldsymbol{u}, \chi_4) + \epsilon A_\gamma \chi_4 + \alpha f_\gamma'(\phi)\chi_4 = 0, \\ \chi_3(0) = 0, \ \chi_4(0) = 1 \end{cases} \tag{4.28}$$

$\mathbb{P}$-a.s. for all $t \in [0, T]$.

Now, we take the inner product of the first equation of (4.28) with $\mathcal{K}^{-1}\chi_1(t)$ in H_1. Then, take the inner product of the second equation of (4.28) with $\epsilon A_\gamma \chi_3(t)$ in $L^2(\mathcal{O})$. Adding the resulting equations, we obtain, after obvious manipulations,

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[\mathcal{K}^{-1}|\chi_1|^2 + \epsilon|A_\gamma^{1/2}\chi_3|^2] + \nu\mathcal{K}^{-1}\|\chi_1\|^2 + \epsilon^2|A_\gamma\chi_3|^2 \\ &\quad = -\mathcal{K}^{-1}b_0(\chi_1, \boldsymbol{u}, \chi_1) + b_1(\chi_1, \chi_3, \epsilon A_\gamma\phi) \\ &\qquad - b_1(\boldsymbol{u}, \chi_3, \epsilon A_\gamma\chi_3) - \alpha(f_\gamma'(\phi)\chi_3, \epsilon A_\gamma\chi_3). \end{aligned} \tag{4.29}$$

Let us proceed with estimating all the terms on the right-hand side of (4.29). Note that

$$\begin{aligned} |\mathcal{K}^{-1}b_0(\chi_1, \boldsymbol{u}, \chi_1)| &\leq c\mathcal{K}^{-1}|\chi_1|\|\chi_1\|\|\boldsymbol{u}\| \\ &\leq \frac{\nu\mathcal{K}^{-1}}{4}\|\chi_1\|^2 + c(\nu, \mathcal{K})\|\boldsymbol{u}\|^2(\mathcal{K}^{-1}|\chi_1|^2), \end{aligned} \tag{4.30}$$

$$\begin{aligned} |b_1(\chi_1, \chi_3, \epsilon A_\gamma\phi)| &\leq c\epsilon|\chi_1|^{1/2}\|\chi_1\|^{1/2}|A_\gamma^{1/2}\chi_3|^{1/2}|A_\gamma\chi_3|^{1/2}|A_\gamma\phi| \\ &\leq \frac{\nu\mathcal{K}^{-1}}{4}\|\chi_1\|^2 + \frac{\epsilon^2}{6}|A_\gamma\chi_3|^2 + c(\epsilon, \nu, \mathcal{K})|A_\gamma\phi|^2|\chi_1||A_\gamma^{1/2}\chi_3| \\ &\leq \frac{\nu\mathcal{K}^{-1}}{4}\|\chi_1\|^2 + \frac{\epsilon^2}{6}|A_\gamma\chi_3|^2 \\ &\quad + c(\epsilon, \nu, \mathcal{K})|A_\gamma\phi|^2(\mathcal{K}^{-1}|\chi_1|^2 + \epsilon|A_\gamma^{1/2}\chi_3|^2), \end{aligned} \tag{4.31}$$

$$
\begin{aligned}
|b_1(\boldsymbol{u}, \chi_3, \epsilon A_\gamma \chi_3)| &\le c\epsilon |\boldsymbol{u}|^{1/2} \|\boldsymbol{u}\|^{1/2} |A_\gamma^{1/2}\chi_3|^{1/2} |A_\gamma \chi_3|^{3/2} \\
&\le \frac{\epsilon^2}{6} |A_\gamma \chi_3|^2 + c(\epsilon) |\boldsymbol{u}|^2 \|\boldsymbol{u}\|^2 (\epsilon |A_\gamma^{1/2}\chi_3|^2). \qquad (4.32)
\end{aligned}
$$

By (2.21), we have

$$
\begin{aligned}
|\alpha(f_\gamma'(\phi)\chi_3, \epsilon A_\gamma \chi_3)| &\le \epsilon \alpha c_f \int_{\mathcal{O}} (1+|\phi|)|\chi_3||A_\gamma \chi_3| \,\mathrm{d}x + \epsilon^2 \gamma |A_\gamma^{1/2}\chi_3|^2 \\
&\le \epsilon\alpha c_f |\chi_3||A_\gamma\chi_3| + \epsilon^2\gamma |A_\gamma^{1/2}\chi_3|^2 \\
&\quad + \epsilon \alpha c_f |\phi|_{L^4(\mathcal{O})} |\chi_3|_{L^4(\mathcal{O})} |A_\gamma \chi_3| \\
&\le \alpha c_f \epsilon |\chi_3||A_\gamma\chi_3| + \epsilon^2\gamma |A_\gamma^{1/2}\chi_3|^2 \\
&\quad + \alpha c_f c_{\mathcal{O}} \gamma^{-\frac{1}{2}} (1+\gamma^{-1})^{\frac{1}{2}} \epsilon |A_\gamma^{1/2}\phi| |A_\gamma^{1/2}\chi_3| |A_\gamma\chi_3| \\
&\le \frac{\epsilon^2}{6} |A_\gamma\chi_3|^2 + 3(\alpha c_f)^2 |\chi_3|^2 + \epsilon^2 \gamma |A_\gamma^{1/2}\chi_3|^2 \\
&\quad + 3(\alpha c_f c_{\mathcal{O}})^2 \gamma^{-1} (1+\gamma^{-1}) |A_\gamma^{1/2}\phi|^2 |A_\gamma^{1/2}\chi_3|^2,
\end{aligned}
$$

from which we infer that

$$
\begin{aligned}
&|\alpha(f_\gamma'(\phi)\chi_3, \epsilon A_\gamma \chi_3)| \\
&\le \frac{\epsilon^2}{6}|A_\gamma \chi_3|^2 + [3(\alpha c_f)^2 (\epsilon\gamma)^{-1} + \epsilon\gamma \\
&\quad + 3(\alpha c_f c_{\mathcal{O}})^2 (\epsilon\gamma)^{-1} (1+\gamma^{-1}) |A_\gamma^{1/2}\phi|^2] (\epsilon |A_\gamma^{1/2}\chi_3|^2). \qquad (4.33)
\end{aligned}
$$

Collecting all estimates (4.30)–(4.33) and inserting them on the right-hand side of (4.29), we obtain, after straightforward transformations, that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}[\mathcal{K}^{-1}|\chi_1|^2 + \epsilon |A_\gamma^{1/2}\chi_3|^2] + \min(\nu\mathcal{K}^{-1}, \epsilon^2)(\|\chi_1\|^2 + |A_\gamma\chi_3|^2) \\
&\le c(1 + \|\boldsymbol{u}\|^2 + |\boldsymbol{u}|^2\|\boldsymbol{u}\|^2 + |A_\gamma^{1/2}\phi|^2 + |A_\gamma\phi|^2)(\mathcal{K}^{-1}|\chi_1|^2 + \epsilon|A_\gamma^{1/2}\chi_3|^2), \quad (4.34)
\end{aligned}
$$

where $c = c(\nu, \mathcal{O}, \mathcal{K}, \epsilon, \alpha, c_f, \gamma)$. Hence, an application of the Gronwall lemma entails that

$$
\begin{aligned}
&\mathcal{K}^{-1}|\chi_1(t)|^2 + \epsilon |A_\gamma^{1/2}\chi_3(t)|^2 + \int_0^t (\|\chi_1(s)\|^2 + |A_\gamma \chi_3(s)|^2)\,\mathrm{d}s \\
&\le c \exp^{c\int_0^t \mathcal{Y}(s)\,\mathrm{d}s}, \quad t \in [0, T] \qquad (4.35)
\end{aligned}
$$

with $\mathcal{Y}(t) = c(1 + \|\boldsymbol{u}(t)\|^2 + |\boldsymbol{u}(t)|^2\|\boldsymbol{u}(t)\|^2 + |A_\gamma^{1/2}\phi(t)|^2 + |A_\gamma\phi(t)|^2)$.

Analogously, we find that

$$
\begin{aligned}
&\mathcal{K}^{-1}|\chi_2(t)|^2 + \epsilon |A_\gamma^{1/2}\chi_4(t)|^2 + \int_0^t (\|\chi_2(s)\|^2 + |A_\gamma \chi_4(s)|^2)\,\mathrm{d}s \\
&\le c e^{c\int_0^t \mathcal{Y}(s)\,\mathrm{d}s}, \quad t \in [0, T]. \qquad (4.36)
\end{aligned}
$$

We will now give an estimate for $\mathbb{E}[\Psi(\boldsymbol{u}(t,\xi),\phi(t,\xi))-\Psi(\boldsymbol{u}(t,\xi_1),\phi(t,\xi_1))]$, with $\xi=(x,y)\in\mathbb{Y}$, $\xi_1=(x_1,y_1)\in\mathbb{Y}$. To achieve our goal, we will follow an idea of [27] (see also [2]). Let us introduce the following cut-off function:

$$\Upsilon_\delta(x_0)=\begin{cases}=1 & \text{if } x_0\in[0,\delta],\\ =0 & \text{if } x_0\in[2\delta,\infty],\\ \in[0,1] & \text{if } x_0\in[\delta,2\delta].\end{cases}$$

We have

$$\mathbb{E}[\Psi(\boldsymbol{u}(t,\xi),\phi(t,\xi))-\Psi(\boldsymbol{u}(t,\xi_1),\phi(t,\xi_1))]=J_1(t)+J_2(t)+J_3(t), \tag{4.37}$$

with

$$\begin{aligned}J_1(t)=&\,\mathbb{E}\Big[\Psi(\boldsymbol{u}(t,\xi),\phi(t,\xi))\times\Upsilon_\delta\Big(\int_0^t\|(\boldsymbol{u}(s,\xi),\phi(s,\xi))\|_{\mathbb{V}}^2\,\mathrm{d}s\Big)\Big]\\ &-\mathbb{E}\Big[\Psi(\boldsymbol{u}(t,\xi_1),\phi(t,\xi_1))\times\Upsilon_\delta\Big(\int_0^t\|(\boldsymbol{u}(s,\xi_1),\phi(s,\xi_1))\|_{\mathbb{V}}^2\,\mathrm{d}s\Big)\Big],\\ J_2(t)=&\,\mathbb{E}\Big[\Psi(\boldsymbol{u}(t,\xi),\phi(t,\xi))\times\Big(1-\Upsilon_\delta\Big(\int_0^t\|(\boldsymbol{u}(s,\xi),\phi(s,\xi))\|_{\mathbb{V}}^2\,\mathrm{d}s\Big)\Big)\Big],\end{aligned}$$

and

$$J_3(t)=-\mathbb{E}\Big[\Psi(\boldsymbol{u}(t,\xi_1),\phi(t,\xi_1))\times\Big(1-\Upsilon_\delta\Big(\int_0^t\|(\boldsymbol{u}(s,\xi_1),\phi(s,\xi_1))\|_{\mathbb{V}}^2\,\mathrm{d}s\Big)\Big)\Big].$$

Using the Chebyshev inequality and (4.6), we deduce that

$$\begin{aligned}|J_2(t)|&\le\Big(\mathbb{P}\int_0^t\|(\boldsymbol{u}(s,\xi),\phi(s,\xi))\|_{\mathbb{V}}^2\ge\delta\,\mathrm{d}s\Big)\|\Psi\|_\infty\\ &\le\frac{\|\Psi\|_\infty}{\delta}\mathbb{E}\int_0^t\|(\boldsymbol{u}(s,\xi),\phi(s,\xi))\|_{\mathbb{V}}^2\,\mathrm{d}s\\ &\le[(|(x,y)|_{\mathbb{Y}}^2+L_2)\,e^{[\beta+2\alpha(\gamma_2+\gamma)]t}+L_3t]\frac{\|\Psi\|_\infty}{2\beta\delta}.\end{aligned} \tag{4.38}$$

Similarly,

$$|J_3(t)|\le[(|(x_1,y_1)|_{\mathbb{Y}}^2+L_2)\,e^{[\beta+2\alpha(\gamma_2+\gamma)]t}+L_3t]\frac{\|\Psi\|_\infty}{2\beta\delta}. \tag{4.39}$$

Here,

$$L_2=\frac{L_1}{\beta},\quad L_3=\frac{4\alpha^2(\gamma_2+\gamma)^2L_1}{\beta[2\alpha(\gamma_2+\gamma)+\beta]}.$$

With the view to estimate the term $J_1(t)$, we rewrite it as

$$J_1(t)=\int_0^1\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbb{E}\Big[\Psi(\boldsymbol{u}(t,\xi_\tau),\phi(t,\xi_\tau))\times\Upsilon_\delta\Big(\int_0^t\|(\boldsymbol{u}(s,\xi_\tau),\phi(s,\xi_\tau))\|_{\mathbb{V}}^2\,\mathrm{d}s\Big)\Big]\mathrm{d}\tau,$$

where $\xi_\tau=\tau\xi+(1-\tau)\xi_1$, $\tau\in[0,1]$.

Hereafter, we set $h = (x - x_1, y - y_1)$ and denote by $\mathbb{A} : V_1 \times V_2 \to V_1' \times V_2'$ the canonical isomorphism of $V_1 \times V_2$ onto $V_1' \times V_2'$, and

$$\rho_\tau = \inf\left\{t > 0 : \int_0^t \|(\boldsymbol{u}(s, \xi_\tau), \phi(s, \xi_\tau))\|_{\mathbb{V}}^2 \, \mathrm{d}s \geq 2\delta\right\}.$$

Invoking now the Bismut–Elworthy formula (see [29]), we obtain

$$\begin{aligned} J_1(t) = \int_0^1 & \frac{1}{t}\mathbb{E}[\Psi(U(t, \xi_\tau)) \times \Upsilon_\delta\left(\int_0^t \|(\boldsymbol{u}(s, \xi_\tau), \phi(s, \xi_\tau))\|_{\mathbb{V}}^2 \, \mathrm{d}s\right) \\ & \times \int_0^t (Q^{-1/2} DU(s, \xi_\tau)h, \mathrm{d}W(s))] \, \mathrm{d}\tau \\ & + 2\int_0^1 \mathbb{E}[\Psi(U(t, \xi_\tau)) \times \Upsilon_\delta'\left(\int_0^t \|(\boldsymbol{u}(s, \xi_\tau), \phi(s, \xi_\tau))\|_{\mathbb{V}}^2 \, \mathrm{d}s\right) \\ & \times \int_0^t \left(1 - \frac{s}{t}\right)(\mathbb{A}U(s, \xi_\tau), DU(s, \xi_\tau)h) \, \mathrm{d}s] \, \mathrm{d}\tau. \end{aligned}$$

Then, we deduce that

$$\begin{aligned} |J_1(t)| \leq c\|\Psi\|_\infty \int_0^1 & \left[\frac{1}{t}\mathbb{E}\left(\int_0^{t\wedge\rho_\tau} |Q^{-1/2} DU(s, \xi_\tau)h|^2 \, \mathrm{d}s\right)^{1/2} \right. \\ & \left. + 2\|\Upsilon_\delta'\|_\infty \mathbb{E}\left(\int_0^{t\wedge\rho_\tau} \|\iota^h(s, \xi_\tau)\|_{V_1\times V_2}^2 \, \mathrm{d}s\right)^{1/2} \left(\int_0^t \|U(s, \xi_\tau)\|_{\mathbb{V}}^2 \, \mathrm{d}s\right)^{1/2}\right] \mathrm{d}\tau, \end{aligned}$$

where $\iota^h = (DU) \cdot h$. Now, by estimates (4.35) and (4.36), as well as the condition (2.8), we have that

$$\int_0^{t\wedge\rho_\tau} |Q^{-1/2} DU(s, \xi_\tau)h|^2 \, \mathrm{d}s \leq c|h|^2.$$

Thanks to the estimates (4.6) and (4.35)–(4.39), we get

$$\begin{aligned} & \mathbb{E}[\Psi(\boldsymbol{u}(t, \xi), \phi(t, \xi)) - \Psi(\boldsymbol{u}(t, \xi_1), \phi(t, \xi_1))] \\ & \quad \leq c(\nu, \epsilon, \mathcal{K}, T, \gamma, \gamma_2, \alpha, \lambda, \ell)\|\Psi\|_\infty \delta_1\left(\frac{\delta_1}{\delta} + 2e^{\delta\delta_1}(1 + t^{-1/2})\right) \leq \frac{1}{2} \end{aligned} \tag{4.40}$$

for all $|\xi|_{\mathbb{Y}} \leq \delta_1$, $|\xi_1|_{\mathbb{Y}} \leq \delta_1$, when δ is appropriately chosen and δ_1 is small enough. The proof of Lemma 4.3 is now complete. ∎

Remark 4.1. (1) From (4.40), we can observe that, for δ_1 small enough, the factor on the right-hand side of this inequality containing δ_1 decreases to zero.

(2) In (4.40), we can choose $\delta > 0$ to be any constant and then choose δ_1 small enough such that $\delta_1 + 2\exp^{\{\delta\delta_1\}}(1 + t^{-1}) < 3$ and $c(\nu, \epsilon, \mathcal{K}, T, \gamma, \gamma_2, \alpha, \lambda, \ell)\|\Psi\|_\infty \delta_1 \leq 1/6$.

Let $\gamma_0 = \frac{\mathrm{Tr}\, Q_1 + \epsilon\gamma\, \mathrm{Tr}\, Q_2 + \epsilon\Lambda}{2\beta}$, and for $M > 0$

$$\tau_M = \inf\{mT;\ m \in \mathbb{N} : |U(mT, \boldsymbol{u}_0, \phi_0)|^2 \geq M\gamma_0\}. \tag{4.41}$$

Lemma 4.4. *For any $T > 0$, there exists $M(T)$, $C(T)$, such that*

$$\mathbb{P}(\tau_M \geq mT) \leq C(T)e^{-\beta mT}(1 + |(\boldsymbol{u}_0, \phi_0)|_{\mathbb{Y}}^2), \tag{4.42}$$

and for $\jmath_0 < \beta$,

$$\mathbb{E}e^{\jmath_0 \tau_M} \leq C(\jmath_0, T)(1 + |(\boldsymbol{u}_0, \phi_0)|_{\mathbb{Y}}^2). \tag{4.43}$$

Proof. By (4.5) in Lemma 4.1 and the Markov property of $\{U(mT, \boldsymbol{u}_0, \phi_0)\}_{m\in\mathbb{N}}$, we find that

$$\mathbb{E}(|U((m+1)T, \boldsymbol{u}_0, \phi_0)|_{\mathbb{Y}}^2 \mid \mathcal{F}_{mT}) \leq e^{-2\beta T}|U(mT, \boldsymbol{u}_0, \phi_0)|_{\mathbb{Y}}^2 + \gamma_0. \tag{4.44}$$

Using the Chebyshev inequality, we obtain

$$\mathbb{P}(|U((m+1)T, \boldsymbol{u}_0, \phi_0)|_{\mathbb{Y}}^2 \geq M\gamma_0 \mid \mathcal{F}_{mT}) \leq \frac{1}{M\gamma_0}e^{-2\beta T}|U(mT, \boldsymbol{u}_0, \phi_0)|_{\mathbb{Y}}^2 + \frac{1}{M}. \tag{4.45}$$

Hereafter, we set

$$\begin{aligned} \widetilde{B}_m &= \{|U(\jmath T, U_0)|_{\mathbb{Y}}^2 \geq M\gamma_0;\ \jmath = 0, 1, \ldots, m\}, \\ \widetilde{\widetilde{B}}_m &= \{|U(mT, U_0)|_{\mathbb{Y}}^2 \geq M\gamma_0\}, \quad U_0 = (\boldsymbol{u}_0, \phi_0). \end{aligned}$$

Notice that

$$\widetilde{B}_{m+1} = \widetilde{B}_m \cap \widetilde{\widetilde{B}}_{m+1}.$$

Multiplying (4.45) by $1_{\widetilde{B}_m}$ and then taking the mathematical expectation on the resulting inequality, we derive that

$$\mathbb{P}(\widetilde{B}_{m+1}) \leq \frac{1}{M\gamma_0}e^{-2\beta T}\mathbb{E}(|U(mT, U_0)|_{\mathbb{Y}}^2 1_{\widetilde{B}_m}) + \frac{1}{M}\mathbb{P}(\widetilde{B}_m). \tag{4.46}$$

Similarly, we infer from (4.44) that

$$\mathbb{E}(|U((m+1)T, U_0)|_{\mathbb{Y}}^2 1_{\widetilde{B}_m}) \leq e^{-2\beta T}\mathbb{E}(|U(mT, U_0)|_{\mathbb{Y}}^2 1_{\widetilde{B}_m}) + \gamma_0\mathbb{P}(\widetilde{B}_m). \tag{4.47}$$

Let

$$e_m = \mathbb{E}(|U(mT, U_0)|_{\mathbb{Y}}^2 1_{\widetilde{B}_m}), \quad \mathbb{P}_m = \mathbb{P}(\widetilde{B}_m).$$

Therefore, from (4.46) and (4.47), one has

$$\begin{pmatrix} \mathbb{P}_{m+1} \\ e_{m+1} \end{pmatrix} \leq \begin{pmatrix} \frac{1}{M} & \frac{1}{M\gamma_0}e^{-2\beta T} \\ \gamma_0 & e^{-2\beta T} \end{pmatrix} \begin{pmatrix} \mathbb{P}_m \\ e_m \end{pmatrix}.$$

The eigenvalues of the above matrix are 0 and $\frac{1}{M} + e^{-2\beta T}$. Choosing M such that

$$\frac{1}{M} + e^{-2\beta T} = e^{-\beta T},$$

we deduce that

$$\mathbb{P}_m^2 + e_m^2 \le C(T)e^{-2m\beta T}(\mathbb{P}_0^2 + e_0^2) \le C(T)e^{-2m\beta T}(1 + |(\boldsymbol{u}_0, \phi_0)|_{\mathbb{Y}}^2).$$

Since $\widetilde{B}_m = \{\tau_M \ge mT\}$, (4.42) follows.

Moreover, for $\jmath_0 < \beta$, we have

$$\mathbb{E}[e^{\jmath_0 \tau_M}] = \sum_{n\ge 0} e^{\jmath_0 nT}\mathbb{P}(\tau_M = nT) \le \sum_{n\ge 0} C(T)e^{\jmath_0 nT}e^{-n\beta T}(1 + |(\boldsymbol{u}_0, \phi_0)|_{\mathbb{Y}}^2) \le C(T, \jmath_0)(1 + |(\boldsymbol{u}_0, \phi_0)|_{\mathbb{Y}}^2).$$

This completes the proof of Lemma 4.4. ∎

Theorem 4.2. *There is a unique invariant measure μ_* for semigroup P_t.*

The proof of Theorem 4.2 is based on the following lemma.

Lemma 4.5. *There are $c_* > 0$ and $\tilde{\jmath}_0 > 0$ such that, for any $T > 0$, $\jmath \in \mathbb{N}$, and any $\Psi \in C_b(\mathbb{Y})$,*

$$|P_{\jmath T}\Psi(c\boldsymbol{u}_0, \phi_0) - P_{\jmath T}\Psi(\boldsymbol{u}_0^1, \phi_0^1)| \le c_*\|\Psi\|_\infty e^{-\jmath\{\tilde{\jmath}_0 T\}}(1 + |(\boldsymbol{u}_0, \phi_0)|_{\mathbb{Y}}^2 + |(\boldsymbol{u}_0^1, \phi_0^1)|_{\mathbb{Y}}^2). \tag{4.48}$$

Proof. We follow the idea in [13] (see also [27] or [2]). Let $T > 0$ and $\delta_1 > 0$ be as in Lemma 4.3. Let $\widetilde{R} = \min(\delta_1, R)$, $R = M\gamma_0$, where γ_0 is defined as in Lemma 4.4, and M is chosen as in the proof of Lemma 4.4.

For notational simplicity, in the sequel, we set $U_0 = (\boldsymbol{u}_0, \phi_0)$, $U_0^1 = (\boldsymbol{u}_0^1, \phi_0^1)$ and $U(t, U_0) = (u(t, \boldsymbol{u}_0, \phi_0), \phi(t, \boldsymbol{u}_0, \phi_0))$. Hence, for any $U_0, U_0^1 \in \mathbb{B}_{\widetilde{R}}(0)$, where $\mathbb{B}_{\widetilde{R}}(0)$ denotes the ball centered at the origin of $\widetilde{R}$ radius, we have

$$\|\pi_T(U(., U_0)) - \pi_T(U(., U_0^1))\|_{TV} = \sup_{\|\Psi\|_\infty \le 1, \Psi \in C_b(\mathbb{Y})} |\mathbb{E}(\Psi(U(T, U_0))) - \mathbb{E}(\Psi(U(T, U_0^1)))| \le \frac{1}{2}, \tag{4.49}$$

where $\pi_T(U(., U_0))$ and $\pi_T(U(., U_0^1))$ are the laws of $U(T, U_0)$ and $U(T, U_0^1)$, respectively.

Then (see [2, Appendix]), there is a maximal coupling $(X_1(U_0, U_0^1), X_2(U_0, U_0^1))$ of $(U(T, U_0), U(T, U_0^1))$ which depends measurably on U_0 and U_0^1. This means that the law of $X_1(U_0, U_0^1)$ (resp., $X_2(U_0, U_0^1)$) coincides with that of $U(T, U_0)$ (resp., $U(T, U_0^1)$) and

$$\mathbb{P}(X_1(U_0, U_0^1) \ne X_2(U_0, U_0^1)) \le \frac{1}{2},$$

$$P_T\Psi(U(T, U_0)) - P_T\Psi(U(T, U_0^1)) = \mathbb{E}[\Psi(X_1(U_0, U_0^1)) - \Psi(X_2(U_0, U_0^1))].$$

Let

$$
\begin{aligned}
&(\Gamma_1^1(U_0,U_0^1),\Gamma_2^1(U_0,U_0^1))\\
&\quad=\begin{cases}(X_1^1(U_0,U_0^1),X_2^1(U_0,U_0^1)) & \text{if } U_0,U_0^1\in\mathbb{B}_{\widetilde{R}}(0),\ U_0\neq U_0^1,\\ (U(T,U_0),U(T,U_0^1)) & \text{if } U_0=U_0^1,\\ (U(T,U_0),\widetilde{U}(T,U_0^1)) & \text{otherwise,}\end{cases}
\end{aligned}
$$

where $\widetilde{U}(T,U_0^1)$ is the solution of the stochastic equation where the Wiener process W has been replaced by an independent copy $\widetilde{W}$.

We again construct iteratively the coupling $(\Gamma_1^n(U_0,U_0^1),\Gamma_2^n(U_0,U_0^1))$ of $(U(nT,U_0), U(nT,U_0^1))$ by the formula

$$
\begin{aligned}
&(\Gamma_1^{1+n}(U_0,U_0^1),\Gamma_2^{1+n}(U_0,U_0^1))\\
&\quad=(\Gamma_1^1(\Gamma_1^n(U_0,U_0^1),\Gamma_2^n(U_0,U_0^1)),\Gamma_2^1(\Gamma_1^n(U_0,U_0^1),\Gamma_2^n(U_0,U_0^1))).
\end{aligned}
$$

Then, for $U_0,U_0^1\in\mathbb{B}_{\delta_1}(0)$, one has

$$
\begin{aligned}
&|\mathbb{E}[\Psi(U(nT,U_0))]-\mathbb{E}[\Psi(U(nT,U_0^1))]|\\
&\quad=|\mathbb{E}[\Psi(\Gamma_1^n(U_0,U_0^1))]-\mathbb{E}[\Psi(\Gamma_2^n(U_0,U_0^1))]|\\
&\quad\le 2\|\Psi\|_\infty\mathbb{P}(\Gamma_1^n(U_0,U_0^1)\neq\Gamma_2^n(U_0,U_0^1)).
\end{aligned}
$$

Furthermore, we define

$$
\ell_{M,1}=\inf\{n\in\mathbb{N}:\Gamma_1^n,\Gamma_2^n\in\mathbb{B}_{\widetilde{R}}(0)\},
$$

and recursively,

$$
\ell_{M,\jmath+1}=\inf\{n>\ell_{M,\jmath}:\Gamma_1^n,\Gamma_2^n\in\mathbb{B}_{\widetilde{R}}(0)\}.
$$

Then, (4.43) can be generalized to two solutions, and we have

$$
\mathbb{E}[e^{\jmath_0\ell_{M,1}T}]\le C(\jmath_0,T)(1+|(u_0,\phi_0)|_{\mathbb{Y}}^2), \tag{4.50}
$$

and, by the Markov property,

$$
\mathbb{E}[e^{\jmath_0(\ell_{M,\jmath+1}-\ell_{M,\jmath})T}\mid\mathcal{F}_{\ell_{M,\jmath}T}]\le C(\jmath_0,T)(1+|\Gamma_1^{\ell_{M,\jmath}}|_{\mathbb{Y}}^2+|\Gamma_2^{\ell_{M,\jmath}}|_{\mathbb{Y}}^2).
$$

which implies that, for $\jmath\ge 1$,

$$
\begin{aligned}
\mathbb{E}[e^{\jmath_0\ell_{M,\jmath+1}T}]&\le C(\jmath_0,T)\mathbb{E}[e^{\jmath_0\ell_{M,\jmath}T}(1+|\Gamma_1^{\ell_{M,\jmath}}|_{\mathbb{Y}}^2+|\Gamma_2^{\ell_{M,\jmath}}|_{\mathbb{Y}}^2)]\\
&\le C(\jmath_0,T)(1+2\widetilde{R}^2)\mathbb{E}[e^{\jmath_0\ell_{M,\jmath}T}]
\end{aligned} \tag{4.51}
$$

and

$$
\mathbb{E}[e^{\jmath_0\ell_{M,\jmath}T}]\le C(\jmath_0,T)^\jmath(1+2\widetilde{R}^2)^{\jmath-1}(1+|(\boldsymbol{u}_0,\phi_0)|_{\mathbb{Y}}^2). \tag{4.52}
$$

Now, we construct a sequence of stopping times to enter inside the ball $\mathbb{B}_{\widetilde{R}}$ defined recursively by

$$\tilde{l}_{M,J+1} = \inf\{n \geq \tilde{\ell}_{M,J} : |\Gamma_1^n|_{\mathbb{Y}} \leq \widetilde{R},\ |\Gamma_2^n|_{\mathbb{Y}} \leq \widetilde{R}\}.$$

We set

$$\tilde{\ell}_0 = \inf\{J \in \mathbb{N} : \Gamma_1^{\tilde{\ell}_{M,J+1}} = \Gamma_2^{\tilde{\ell}_{M,J+1}}\}. \tag{4.53}$$

Recall, in virtue of (4.49), that

$$\mathbb{P}(\Gamma_1^{\tilde{\ell}_{M,J}} \neq \Gamma_2^{\tilde{\ell}_{M,J}}) \leq \frac{1}{2},$$

and then,

$$\mathbb{P}(\tilde{\ell}_0 > J+1 \mid \tilde{\ell}_0 > J) \leq \frac{1}{2}.$$

Writing

$$\mathbb{P}(\tilde{\ell}_0 > J+1) = \mathbb{P}(\tilde{\ell}_0 > J+1 \mid \tilde{\ell}_0 > J)\mathbb{P}(\tilde{\ell}_0 > J),$$

we obtain

$$\mathbb{P}(\tilde{\ell}_0 > J) \leq 2^{-J}.$$

So, for any $\tilde{\jmath}_0$ ($\tilde{\jmath}_0$ will be chosen later), we have

$$\begin{aligned}
\mathbb{E}[e^{\tilde{\jmath}_0 \tilde{\ell}_{M,\tilde{\ell}_0} T}] &\leq \sum_{J\geq 0} \mathbb{E}(e^{\tilde{\jmath}_0 \tilde{\ell}_{M,J} T} 1_{J=\tilde{\ell}_0}) \\
&\leq \sum_{J\geq 0} \mathbb{P}(J = \tilde{\ell}_0)^{1-\frac{\tilde{\jmath}_0}{\jmath_0}} [\mathbb{E}(e^{\jmath_0 \tilde{\ell}_{M,J} T})]^{\tilde{\jmath}_0/\jmath_0} \\
&\leq \sum_{J\geq 0} \left(\frac{1}{2}\right)^{(J-1)(1-\tilde{\jmath}_0/\jmath_0)} [C(\jmath_0, T)^J (1 + 2\widetilde{R}^2)^{J-1} (1 + |(\boldsymbol{u}_0, \phi_0)|_{\mathbb{Y}}^2)]^{\tilde{\jmath}_0/\jmath_0}.
\end{aligned}$$

We choose $\tilde{\jmath}_0$ such that

$$2^{\{-(1-\tilde{\jmath}_0/\jmath_0)\}}[C(\jmath_0, T)(1 + 2\widetilde{R}^2)]^{\tilde{\jmath}_0/\jmath_0} < 1, \tag{4.54}$$

and we get

$$\mathbb{E}[e^{\tilde{\jmath}_0 \tilde{\ell}_{M,\tilde{\ell}_0} T}] \leq C(\tilde{\jmath}_0, \jmath_0, \widetilde{R}, T)(1 + |(\boldsymbol{u}_0, \phi_0)|_{\mathbb{Y}}^2). \tag{4.55}$$

Since

$$\tilde{\tilde{\ell}}_0 = \inf\{J \in \mathbb{N} : \Gamma_1^J = \Gamma_2^J,\ \Gamma_i^J \in \mathbb{B}_{\widetilde{R}},\ i = 1, 2\} \leq \tilde{\ell}_{M,\tilde{\ell}_0} + 1,$$

we deduce that

$$\mathbb{E}[e^{\tilde{\jmath}_0 \tilde{\tilde{\ell}}_0 T}] \leq C(\tilde{\jmath}_0, \jmath_0, \widetilde{R}, T)(1 + |(\boldsymbol{u}_0, \phi_0)|_{\mathbb{Y}}^2).$$

This implies that

$$\begin{aligned}
\mathbb{P}(\Gamma_1^J \neq \Gamma_2^J) &= \mathbb{P}(e^{\tilde{\jmath}_0 \tilde{\tilde{\ell}}_0 T} \geq e^{\tilde{\jmath}_0 T J}) \\
&\leq C(\tilde{\jmath}_0, \jmath_0, \widetilde{R}, T)(1 + |(\boldsymbol{u}_0, \phi_0)|_{\mathbb{Y}}^2) e^{-\tilde{\jmath}_0 T J}.
\end{aligned}$$

Hence,

$$\begin{aligned}&|\mathbb{E}[\Psi(U(jT,U_0))]-\mathbb{E}[\Psi(U(jT,U_0^1))]|\\&\quad\leq 2\|\Psi\|_\infty\mathbb{P}(\Gamma_1^J(U_0,U_0^1)\neq\Gamma_2^J(U_0,U_0^1))\\&\quad\leq\|\Psi\|_\infty C(\tilde{j}_0,j_0,\widetilde{R},T)(1+|U_0|_{\mathbb{Y}}^2+|U_0^1|_{\mathbb{Y}}^2)e^{-\tilde{j}_0Tj}.\end{aligned}$$

This proves (4.48). ■

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