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On regularity and asymptotic stability for semilinear nonlocal pseudo-parabolic equations

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Abstract. We deal with a class of nonlocal pseudo-parabolic equations involving strong nonlinearities. The questions on existence, regularity and stability of solutions are addressed by using local estimates, fixed point arguments, and the relation between the Hilbert scales and fractional Sobolev spaces.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 1$, with smooth boundary $\partial\Omega$. In this paper, we consider the following problem:

$$\partial_t^{\{k\}}(u - \Delta u) - \eta\Delta u = f(u) \quad \text{in } \Omega,\ t \in (0, T], \tag{1.1}$$

$$u = 0 \quad \text{on } \partial\Omega,\ t \in (0, T], \tag{1.2}$$

$$u(0) = \xi \quad \text{in } \Omega, \tag{1.3}$$

where $\eta > 0$, f is a given nonlinear function, and $\partial_t^{\{k\}}$ denotes the nonlocal derivative of Caputo type as follows:

$$\partial_t^{\{k\}} v(t) = (k * v')(t) = \int_0^t k(t-\tau)v'(\tau)\,d\tau.$$

In this work, we make use the following assumption on the kernel function k.

(PC) *The function $k \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ is nonnegative, nonincreasing and there exists a function $m \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ such that*

$$k * m(t) = \int_0^t k(t-\tau)m(\tau)\,d\tau = 1 \quad \textit{for all } t \in (0,\infty).$$

The pair (k, m) is called the Sonine kernel [23]. The typical case is $(k, m) = (g_{1-\alpha}, g_\alpha)$ for $\alpha \in (0, 1)$, where $g_\alpha(t) = t^{\alpha-1}/\Gamma(\alpha)$. In this case, $\partial_t^{\{k\}} = \partial_t^\alpha$, the Caputo fractional derivative of order α and (1.1) is the nonlocal version of the pseudo-parabolic equation

$$u_t - \Delta u_t - \eta\Delta u = f. \tag{1.4}$$

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In [1, 5], equation (1.4) describes the infiltration of homogeneous fluids through fissured rocks. One also uses (1.4) to study the non-steady flows of second order fluids [9], the theory of the two temperatures in heat conduction [6], the monodirectional propagation of nonlinear dispersive long waves [2], etc.

Early results on qualitative theory of pseudo-parabolic equations in the linear case can be found in [13, 24, 26], where the regularity and long-time behavior of solutions were discussed. In addition, the author in [26] demonstrated a relation between solutions of pseudo-parabolic equation and parabolic equation, which showed that the solution of pseudo-parabolic equation approximates to the one of corresponding parabolic equation.

It is worth mentioning that pseudo-parabolic equations in semilinear case have attracted an extensive study. Without stress of references, we refer the reader to recent works [19,30,33], where the global existence and finite time blow-up of weak solutions were analyzed by using the so-called potential well method. It should be noted that this method was first developed for semilinear hyperbolic equations in [20] and then employed for various classes of nonlinear hyperbolic and parabolic equations, in order to address behavior of solutions depending on initial energy levels; see e.g. [7, 17, 18, 31, 32].

In order to depict the memory effect of processes modeled by (1.4), one replaces the time derivative by ∂_t^α with $\alpha \in (0, 1)$. Then (1.4) changes to

$$\partial_t^\alpha (u - \Delta u) - \eta \Delta u = f. \tag{1.5}$$

The last equation has been a subject of numerous studies. We mention some results on solvability, stability and controllability [12, 14, 16, 34] for (1.5). Recently, the authors in [21] proved the global solvability of the Cauchy problem associated with (1.5) in both bounded and unbounded domains, where the nonlinearity $f = f(u)$ takes values in Lebesgue spaces. Some existence results related to (1.5) were obtained in [28] with nonlinearity function being of polynomial and logarithmic type. Regarding a stochastic version of (1.5), the question of existence and regularity of solutions was addressed in [25]. It is worth noting that the analysis for (1.5), i.e., the special case $k(t) = g_{1-\alpha}(t)$, is based on the Mittag-Leffler functions, whose regularity is well known. In this case, it is straightforward to find the resolvents and their regular properties for the associated Cauchy problem.

Regarding problem (1.1)–(1.3), which is first introduced in this paper, employed to describe different memory effects (depending on k), the related resolvents have been unknown. We will show in Section 2 the construction of these resolvents, denoted by $\{S(t)\}$ and $\{R(t)\}$. Especially, the spatial smoothing effect of $\{R(t)\}$ is proved.

On the other hand, due to practical applications, the nonlinearity $f(u)$ may contain the advection/convection term of the form $\mathbf{H}(u) \cdot \nabla u$, where $\mathbf{H}(u)$ is a vector field (see the example in the last section). In this case, one says that $f(u)$ takes weak values, i.e., $f(u)$ belongs to a fractional Sobolev space of negative order. This situation was not addressed in cited works.

In this study, we consider problem (1.1)–(1.3) in the circumstance that the nonlinearity function $f(u)$ takes values in Hilbert scales of negative orders. This enables us to deal with the case when $f(u)$ contains the advection/convection term, thanks to the relation between the Hilbert scales and fractional Sobolev spaces (mentioned in the next section). Dealing with problem (1.1)–(1.3), we first prove the global existence and uniqueness of mild solution to (1.1)–(1.3) in a general case, where $f(\cdot)$ is locally Lipschitzian. Moreover, the asymptotic stability of solutions is proved in the case $m \notin L^1(\mathbb{R}_+)$. This will be done in Section 3. Section 4 is devoted to regularity results. We show that the obtained solution is Hölder continuous. This feature is useful for numerical schemes. Finally, we testify, in a particular case, that the mild solution and the weak solution to (1.1)–(1.3) coincide.

It should be mentioned that the questions of stability and regularity imposed in this work have not been taken into account in literature, even in the fractional case. Concerning the highlight of our work, one brings up the following:

- the unique solvability of the Cauchy problem governed by the nonlocal pseudo-parabolic equation with respect to the Sonine kernels, where the nonlinearity function is allowed to take weak values;
- the asymptotic stability of the obtained solution, which has not been addressed for the case of weak-valued nonlinearity;
- the Hölder regularity of mild solutions, which is helpful in numerical analysis;
- the agreement between the mild and weak solutions in the case that $f(u)$ takes values in H^{-1}.

2. Preliminaries

2.1. Formulation of solutions

It is known that condition (PC) ensures the complete positivity of m, i.e., the functions $s(\cdot)$ and $r(\cdot)$ obeying

$$s(t) + \lambda \int_0^t m(t-\tau)s(\tau)\,d\tau = 1, \qquad t \geq 0, \tag{2.1}$$

$$r(t) + \lambda \int_0^t m(t-\tau)r(\tau)\,d\tau = m(t), \quad t > 0, \tag{2.2}$$

take nonnegative values for each $\lambda > 0$. See [8, 29].

Denote by $s(t,\lambda)$ and $r(t,\lambda)$ the solution of (2.1) and (2.2), respectively, to emphasize the dependence on the parameter λ. The following proposition shows some important properties of these functions.

Proposition 2.1 ([27]). *For every $\lambda > 0$,*

(a) *the function $s(\cdot,\lambda)$ is nonnegative and nonincreasing. Moreover,*

$$\frac{1}{1+\lambda k(t)^{-1}} \leq s(t,\lambda) \leq \frac{1}{1+\lambda(1*m)(t)} \quad \forall t \geq 0.$$

(b) *The functions* $r(\cdot,\lambda)$ *is nonnegative and one has*

$$s(t,\lambda) = 1 - \lambda \int_0^t r(\tau,\lambda)\,d\tau = k * r(t,\lambda) \quad \forall t \geq 0,$$

so $\int_0^t r(\tau,\lambda)\,d\tau \leq \lambda^{-1} \forall t > 0$. *Moreover, if* $m(\cdot)$ *is nonincreasing, then*

$$r(t,\lambda) \leq \frac{m(t)}{1 + \lambda(1 * m)(t)} \quad \forall t > 0.$$

(c) *For each* $t > 0$, *the functions* $\lambda \mapsto s(t,\lambda)$ *and* $\lambda \mapsto r(t,\lambda)$ *are nonincreasing.*

(d) *Let* $v(t) = s(t,\lambda)v_0 + \int_0^t r(t-\tau,\lambda)g(\tau)\,d\tau$. *Then* $v(\cdot)$ *solves the problem*

$$\partial_t^{\{k\}} v(t) + \lambda v(t) = g(t), \quad v(0) = v_0,$$

where $g \in C(\mathbb{R}_+)$.

We also use the following Gronwall type inequality.

Proposition 2.2 ([15]). *Let* v *be a nonnegative function satisfying*

$$v(t) \leqslant s(t,\lambda)v_0 + \int_0^t r(t-\tau,\lambda)[\alpha v(\tau) + \beta(\tau)]\,d\tau, \quad t \geq 0,$$

for $\lambda > 0$, $\alpha > 0$, $v_0 \geq 0$ *and* $\beta \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$. *Then*

$$v(t) \leqslant s(t,\lambda-\alpha)v_0 + \int_0^t r(t-\tau,\lambda-\alpha)\beta(\tau)\,d\tau.$$

Particularly, if β *is constant and* $\alpha < \lambda$, *then*

$$v(t) \leqslant s(t,\lambda-\alpha)v_0 + \frac{\beta}{\lambda-\alpha}(1 - s(t,\lambda-\alpha)).$$

We are now in a position to give a representation of a solution to (1.1)–(1.3). Let $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote the inner product and the norm, respectively, in $L^2(\Omega)$, that is,

$$\langle u,v\rangle = \int_\Omega u(x)v(x)\,dx, \quad \|u\| = \langle u,u\rangle^{\frac{1}{2}} \quad \text{for } u,v \in L^2(\Omega).$$

Let $\{(\lambda_n, e_n)\}$ be the eigensystem of the Laplacian $-\Delta$ associated with the Dirichlet boundary condition, where $\{e_n\}$ is a orthonormal basis of $L^2(\Omega)$, i.e.,

$$-\Delta e_n = \lambda_n e_n \text{ in } \Omega, \quad e_n = 0 \text{ on } \partial\Omega, \quad \|e_n\| = 1.$$

Then one can find a solution of (1.1)–(1.3) as follows:

$$u(t) = \sum_{n=1}^{\infty} u_n(t)e_n.$$

Using this formula in (1.1)–(1.3), one gains

$$(1+\lambda_n)\partial_t^{\{k\}} u_n(t) + \eta\lambda_n u_n(t) = f_n(u(t)),$$
$$u_n(0) = \xi_n := \langle \xi, e_n\rangle,$$

where $f_n(u(t)) = \langle f(u(t)), e_n\rangle$. Employing Proposition 2.1 (d), we get

$$u_n(t) = s(t,\theta_n)\xi_n + \int_0^t (1+\lambda_n)^{-1} r(t-\tau,\theta_n) f_n(u(\tau))\,d\tau, \quad \theta_n = \frac{\eta\lambda_n}{1+\lambda_n}.$$

Therefore, we have the following representation:

$$u(t) = S(t)\xi + \int_0^t R(t-\tau) f(u(\tau))\,d\tau, \tag{2.3}$$

$$S(t) = \sum_{n=1}^{\infty} s(t,\theta_n)\langle\cdot, e_n\rangle e_n, \tag{2.4}$$

$$R(t) = \sum_{n=1}^{\infty} (1+\lambda_n)^{-1} r(t,\theta_n)\langle\cdot, e_n\rangle e_n; \tag{2.5}$$

here we use the notation $\langle\cdot,\cdot\rangle$ for both inner products and dual pairs, if no confusion arises.

2.2. Properties of resolvents

For $\varrho \in \mathbb{R}$, we define the space $\mathbb{H}^\varrho$ as

$$\mathbb{H}^\varrho = \left\{ v = \sum_{i=1}^{\infty} v_n e_n : \sum_{i=1}^{\infty} \lambda_n^\varrho v_n^2 < \infty \right\}.$$

Then $\mathbb{H}^\varrho$ is a Hilbert space endowed with the norm $\|v\|_{\mathbb{H}^\varrho} := \left(\sum_{i=1}^\infty \lambda_n^\varrho v_n^2\right)^{\frac{1}{2}}$. In addition, for $\varrho > 0$, we can identify the dual space of $\mathbb{H}^\varrho$ with $\mathbb{H}^{-\varrho}$. Note that $\mathbb{H}^0 = L^2(\Omega)$, and the family $\{\mathbb{H}^\varrho\}_{\varrho\in\mathbb{R}}$ is said to be the Hilbert scales of $L^2(\Omega)$.

Lemma 2.3. *Let $\{S(t)\}$ and $\{R(t)\}$ be the families of linear operators defined by (2.4) and (2.5), respectively. Then*

(a) *for each $v \in \mathbb{H}^\mu$, we have*

$$\|S(t)v\|_{\mathbb{H}^\mu} \le s(t,\theta_1)\|v\|_{\mathbb{H}^\mu}, \quad \theta_1 = \frac{\eta\lambda_1}{1+\lambda_1},$$

for all $t \ge 0$.

(b) *For each $T > 0$ and $g \in C([0,T];\mathbb{H}^{\mu-2})$, we have*

$$\left\| \int_0^t R(t-\tau) g(\tau)\,d\tau \right\|_{\mathbb{H}^\mu}^2 \le \theta_1^{-1} \int_0^t r(t-\tau,\theta_1)\|g(\tau)\|_{\mathbb{H}^{\mu-2}}^2\,d\tau.$$

Proof. (a) Observing that

$$\|S(t)v\|^2_{\mathbb{H}^\mu} = \sum_{n=1}^{\infty} s(t,\theta_n)^2 v_n^2 \lambda_n^\mu, \quad v_n = \langle v, e_n\rangle,$$

and $\{\theta_n\}$ is increasing, we have

$$\|S(t)v\|^2_{\mathbb{H}^\mu} \le s(t,\theta_1)^2 \sum_{n=1}^{\infty} v_n^2 \lambda_n^\mu = s(t,\theta_1)^2 \|v\|^2_{\mathbb{H}^\mu}.$$

(b) We see that

$$\begin{aligned}\left\|\int_0^t R(t-\tau)g(\tau)\,d\tau\right\|^2_{\mathbb{H}^\mu} &= \sum_{n=1}^{\infty}\left(\int_0^t (1+\lambda_n)^{-1} r(t-\tau,\theta_n) g_n(\tau)\,d\tau\right)^2 \lambda_n^\mu \\ &\le \sum_{n=1}^{\infty} \lambda_n^{\mu-2}\left(\int_0^t r(t-\tau,\theta_n) g_n(\tau)\,d\tau\right)^2.\end{aligned}$$

Using the Hölder inequality, we have

$$\begin{aligned}\left(\int_0^t r(t-\tau,\theta_n) g_n(\tau)\,d\tau\right)^2 &\le \int_0^t r(t-\tau,\theta_n)\,d\tau \int_0^t r(t-\tau,\theta_n) g_n^2(\tau)\,d\tau \\ &\le \theta_n^{-1}\int_0^t r(t-\tau,\theta_n) g_n^2(\tau)\,d\tau \\ &\le \theta_1^{-1}\int_0^t r(t-\tau,\theta_1) g_n^2(\tau)\,d\tau;\end{aligned}$$

here we utilized Proposition 2.1 (b)–(c). Therefore,

$$\begin{aligned}\left\|\int_0^t R(t-\tau)g(\tau)\,d\tau\right\|^2_{\mathbb{H}^\mu} &\le \theta_1^{-1}\int_0^t r(t-\tau,\theta_1) \sum_{n=1}^{\infty} \lambda_n^{\mu-2} g_n^2(\tau)\,d\tau \\ &= \theta_1^{-1}\int_0^t r(t-\tau,\theta_1)\|g(\tau)\|^2_{\mathbb{H}^{\mu-2}}\,d\tau.\end{aligned}$$

The lemma is proved. ∎

We are now in a position to recall some notions and facts related to the regularity of the family $\{S(t)\}$.

Definition 2.4 ([22]). Let $m \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ be a function of subexponential growth, i.e.,

$$\int_0^\infty |m(t)|e^{-\epsilon t}\,dt < \infty \quad \text{for every } \epsilon > 0.$$

Denote by $\hat{m}$ the Laplace transform of m.

(i) Suppose that $\hat{m}(z) \neq 0$ for all $\mathrm{Re}(z) > 0$. For $\vartheta > 0$, m is said to be ϑ-*sectorial* if $|\arg \hat{m}(z)| \le \vartheta$ for all $\mathrm{Re}(z) > 0$.

(ii) For given $l \in \mathbb{N}$, m is called *l-regular* if there exists a constant $c > 0$ such that

$$|z^n \hat{m}^{(n)}(z)| \le c|\hat{m}(z)| \quad \text{for all } \operatorname{Re}(z) > 0,\ 0 \le n \le l.$$

Lemma 2.5 ([15]). *Assume that the kernel function m is 2-regular and ϑ-sectorial for some $\vartheta < \pi$. Then the resolvent family $S(\cdot)$ is differentiable on $(0,\infty)$. In addition, it holds that*

$$\|S'(t)\| \le \frac{M}{t}, \quad t \in (0,\infty),$$

for some $M \ge 1$.

2.3. Embeddings of fractional Sobolev spaces

Denote by $W^{r,p}(\Omega)$, $r \ge 0, 1 \le p < \infty$, the fractional Sobolev space of order r based on $L^p(\Omega)$ (see e.g. [10, 11]). Put

$$W_0^{r,p}(\Omega) := \overline{C_c^\infty(\Omega)}^{W^{r,p}(\Omega)}, \quad H^r(\Omega) := W^{r,2}(\Omega), \quad H_0^r(\Omega) := W_0^{r,2}(\Omega).$$

We assume that Ω is sufficiently smooth such that $C_c^\infty(\Omega)$ is dense in $H^r(\Omega)$ with $0 < r < 1/2$, which ensures $H_0^r(\Omega) = H^r(\Omega)$ (see [3, Corollary 8.10.1]). It is known that (see, e.g. [4])

$$\mathbb{H}^r = \begin{cases} H_0^r(\Omega), & 0 \le r < 1/2, \\ H_{00}^{1/2}(\Omega) \subsetneqq H_0^{1/2}(\Omega), & r = 1/2, \\ H_0^r(\Omega), & 1/2 < r \le 1, \\ H_0^1(\Omega) \cap H^r(\Omega), & 1 < r \le 2, \end{cases}$$

in which $H_{00}^{1/2}(\Omega)$ is the Lions–Magenes space, i.e.,

$$H_{00}^{1/2}(\Omega) = \left\{ u \in H^{1/2}(\Omega) : \int_\Omega \frac{|u(x)|^2}{(\operatorname{dist}(x,\partial\Omega))^2} dx < \infty \right\}.$$

So one has the following embeddings.

Lemma 2.6. *Let $H^{-r}(\Omega)$ be the duality of $H_0^r(\Omega)$ for $r \ge 0$. If $0 \le r \le r' \le 2$, then*

$$\mathbb{H}^{r'} \hookrightarrow \mathbb{H}^r \hookrightarrow H^r(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-r}(\Omega) \hookrightarrow \mathbb{H}^{-r} \hookrightarrow \mathbb{H}^{-r'}.$$

We also recall the following embeddings.

Lemma 2.7 ([3, Theorem 8.12.6]). *For given $1 \le p, p' \le \infty$, $0 \le r, r' < \infty$ and $r' - \frac{d}{p'} \ge r - \frac{d}{p}$, it holds that $W^{r',p'}(\Omega) \hookrightarrow W^{r,p}(\Omega)$.*

Combining Lemma 2.7 and Lemma 2.6, we gain the embeddings as follows.

Lemma 2.8. *It holds that*

(a) $L^p(\Omega) \hookrightarrow H^r(\Omega) \hookrightarrow \mathbb{H}^r$ *if* $\left\{-\frac{N}{2} < r \le 0, p \ge \frac{2N}{N-2r}\right\}$.

(b) $\mathbb{H}^r \hookrightarrow H_0^r(\Omega) \hookrightarrow L^p(\Omega)$ *if* $\left\{0 \le r < \frac{N}{2}, 1 \le p \le \frac{2N}{N-2r}\right\}$.

3. Solvability and stability

In order to deal with problem (1.1)–(1.3), we make the following assumption.

(F) *The nonlinearity function f in* (1.1) *induces a mapping*

$$f\colon \mathbb{H}^{\mu} \to \mathbb{H}^{\mu-2}, \quad \mu \in (0,2],$$

obeying that $f(0) = 0$ and that

$$\|f(v_1) - f(v_2)\|_{\mathbb{H}^{\mu-2}} \le L_f(r)\|v_1 - v_2\|_{\mathbb{H}^{\mu}}$$

for all $v_1, v_2 \in \mathbb{B}_r := \{v \in \mathbb{H}^{\mu} : \|v\|_{\mathbb{H}^{\mu}} \le r\}$; here L_f is a nonnegative function such that

$$L_f^* := \limsup_{r\to 0} L_f(r) < \frac{\theta_1}{\sqrt{2}}.$$

Based on representation (2.3), we give the following definition of a mild solution for (1.1)–(1.3).

Definition 3.1. A function $u \in C([0,T];\mathbb{H}^{\mu})$ is said to be a mild solution to problem (1.1)–(1.3) on $[0,T]$ if and only if

$$u(t) = S(t)\xi + \int_0^t R(t-\tau) f(u(\tau))\,d\tau$$

for any $t \in [0,T]$.

Theorem 3.2. *Let assumption* (F) *hold. Then there exists $\delta > 0$ such that problem* (1.1)–(1.3) *has a unique mild solution on* $[0,T]$, *provided $\|\xi\|_{\mathbb{H}^{\mu}} \leqslant \delta$.*

Proof. For $u \in C([0,T];\mathbb{H}^{\mu})$, with the norm $\|u\|_{\infty} := \sup_{t\in[0,T]}\|u(t)\|_{\mathbb{H}^{\mu}}$, let Φ be the mapping defined by

$$\Phi(u)(t) = S(t)\xi + \int_0^t R(t-\tau) f(u(\tau))\,d\tau \quad \text{for } t \in [0,T].$$

We refer to this mapping as *the solution operator*. Observe that

$$\begin{aligned}
\|\Phi(u)(t)\|_{\mathbb{H}^{\mu}}^2 &\le 2\|S(t)\xi\|_{\mathbb{H}^{\mu}}^2 + 2\left\|\int_0^t R(t-\tau) f(u(\tau))\,d\tau\right\|_{\mathbb{H}^{\mu}}^2 \\
&\le 2s(t,\theta_1)^2\|\xi\|_{\mathbb{H}^{\mu}}^2 + 2\theta_1^{-1}\int_0^t r(t-\tau,\theta_1)\|f(u(\tau))\|_{\mathbb{H}^{\mu-2}}^2\,d\tau,
\end{aligned}$$

thanks to Lemma 2.3 (b). Assume that $u(t) \in \mathbb{B}_r$ for all $t \in [0,T]$. Then, using (F) and the fact that $s(t,\theta_1) \le 1$, one gains

$$\begin{aligned}
\|\Phi(u)(t)\|_{\mathbb{H}^{\mu}}^2 &\le 2s(t,\theta_1)\|\xi\|_{\mathbb{H}^{\mu}}^2 + 2\theta_1^{-1}\left(\int_0^t r(t-\tau,\theta_1)\,d\tau\right) L_f(r)^2 r^2 \\
&= 2s(t,\theta_1)\|\xi\|_{\mathbb{H}^{\mu}}^2 + 2\theta_1^{-2}(1 - s(t,\theta_1)) L_f(r)^2 r^2 \\
&= 2s(t,\theta_1)[\|\xi\|_{\mathbb{H}^{\mu}}^2 - \theta_1^{-2} L_f(r)^2 r^2] + 2\theta_1^{-2} L_f(r)^2 r^2.
\end{aligned}$$

Taking $r^* > 0$ such that $2\theta_1^{-2} L_f(r^*)^2 < 1$, we see that

$$\|\Phi(u)(t)\|_{\mathbb{H}^\mu} \le r^* \quad \text{for } u(t) \in \mathbb{B}_{r^*},$$

provided that $\|\xi\|_{\mathbb{H}^\mu} \le \delta := \theta_1^{-1} L_f(r^*) r^*$. We have shown that $\Phi(B_{r^*}) \subset B_{r^*}$, where

$$B_{r^*} := \{u \in C([0, T]; \mathbb{H}^\mu) : \|u\|_\infty \le r^*\}.$$

It remains to show that Φ is a contraction mapping on B_{r^*}. Indeed, for $u_1, u_2 \in B_{r^*}$, we have

$$\|\Phi(u_1)(t) - \Phi(u_2)(t)\|^2_{\mathbb{H}^\mu} \le \theta_1^{-1} \int_0^t r(t-\tau, \theta_1) \|f(u_1(\tau)) - f(u_2(\tau))\|^2_{\mathbb{H}^{\mu-2}}\, d\tau,$$

thanks to Lemma 2.3 (b). Then, using (F) again, one gets

$$\begin{aligned}\|\Phi(u_1)(t) - \Phi(u_2)(t)\|^2_{\mathbb{H}^\mu} &\le \theta_1^{-1} \int_0^t r(t-\tau, \theta_1) L_f(r^*)^2 \|u_1(\tau) - u_2(\tau)\|^2_{\mathbb{H}^\mu}\, d\tau \\ &\le \theta_1^{-2} L_f(r^*)^2 \|u_1 - u_2\|^2_\infty \\ &\le \frac{1}{2}\|u_1 - u_2\|^2_\infty \quad \text{for all } t \in [0, T],\end{aligned}$$

which implies that Φ is a contraction. ■

Remark 3.3. When f is globally Lipschitzian, i.e., $L_f(r) = L_f^* > 0$ for all $r > 0$, one can prove the existence and uniqueness of a mild solution to (1.1)–(1.3), regardless of the assumption $f(0) = 0$ and the smallness of initial data. This can be done by the same reasoning as in [15].

It should be noted that the resolvent $S(t)$ has no smoothing effect, which implies that the solution $u(t)$, in general, cannot be more regular than the initial datum. Indeed, we claim that there exists $\xi \in \mathbb{H}^\mu$ such that $S(t)\xi \notin \mathbb{H}^\gamma$ for any $\gamma > \mu$. Let

$$\nu = \frac{1}{2} + \frac{\gamma}{N}, \quad \xi = \sum_{n=1}^\infty n^{-\nu} e_n.$$

Since $\lambda_n \sim Cn^{2/N}$ as $n \to \infty$ ($C > 0$), we have

$$\lambda_n^\mu n^{-2\nu} \sim C^\mu n^{-2(\nu - \frac{\mu}{N})} = C^\mu n^{-1-\frac{2(\gamma-\mu)}{N}}.$$

Then

$$\|\xi\|^2_{\mathbb{H}^\mu} = \sum_{n=1}^\infty \lambda_n^\mu n^{-2\nu} < \infty,$$

i.e., $\xi \in \mathbb{H}^\mu$. Now we estimate

$$\|S(t)\xi\|^2_{\mathbb{H}^\gamma} = \sum_{n=1}^\infty \lambda_n^\gamma s(t, \theta_n)^2 n^{-2\nu}. \tag{3.1}$$

Using Proposition 2.1 (a) yields

$$s(t,\theta_n) \geq \frac{1}{1+\theta_n k(t)^{-1}} \geq \frac{1}{1+\eta k(t)^{-1}}.$$

So

$$\begin{aligned}
\lambda_n^{\gamma} s(t,\theta_n)^2 n^{-2\nu} &\geq \frac{1}{(1+\eta k(t)^{-1})^2}\lambda_n^{\gamma} n^{-2\nu} \\
&\sim \frac{C^{\gamma}}{(1+\eta k(t)^{-1})^2} n^{-2(\nu-\frac{\gamma}{N})} \\
&= \frac{C^{\gamma}}{(1+\eta k(t)^{-1})^2} n^{-1},
\end{aligned}$$

which deduces the divergence of series (3.1).

We now consider the asymptotic stability of a solution to (1.1)–(1.3) in the sense of Lyapunov.

Theorem 3.4. *Assume that* (F) *holds. Let u^* be the solution of* (1.1) *with respect to the initial datum ξ^*, obtained by Theorem* 3.2*. If $m \notin L^1(\mathbb{R}_+)$, then u^* is asymptotically stable.*

Proof. Take r^* and $\delta = \theta_1^{-1} L_f(r^*) r^*$ from the proof of Theorem 3.2. In view of this proof, it is easily seen that u^* is uniquely defined on $[0,T]$ for any $T > 0$.

Let u be the solution of (1.1) with respect to the initial datum $\xi \in \mathbb{B}_\delta$. Then $u \in B_{r^*}$. Moreover, one sees that

$$\begin{aligned}
\|u(t)-u^*(t)\|_{\mathbb{H}^\mu}^2 &\leq 2s(t,\theta_1)\|\xi-\xi^*\|_{\mathbb{H}^\mu}^2 \\
&\quad + 2\theta_1^{-1}\int_0^t r(t-\tau,\theta_1)\|f(u(\tau)) - f(u^*(\tau))\|_{\mathbb{H}^{\mu-2}}^2\, d\tau \\
&\leq 2s(t,\theta_1)\|\xi-\xi^*\|_{\mathbb{H}^\mu}^2 \\
&\quad + 2\theta_1^{-1}\int_0^t r(t-\tau,\theta_1) L_f(r^*)^2 \|u(\tau)-u^*(\tau)\|_{\mathbb{H}^\mu}^2\, d\tau
\end{aligned}$$

for any $t > 0$. Employing Proposition 2.2 yields

$$\|u(t)-u^*(t)\|_{\mathbb{H}^\mu}^2 \leq 2s(t,\theta_1 - 2\theta_1^{-1}L_f(r^*)^2)\|\xi-\xi^*\|_{\mathbb{H}^\mu}^2$$

for any $t > 0$.

As $m \notin L^1(\mathbb{R}_+)$, it follows from Proposition 2.1 (a) that $s(t,\theta_1 - 2\theta_1^{-1}L_f(r^*)^2) \to 0$ as $t \to \infty$. We get the conclusion as desired. ■

In the next theorem, we prove the existence of an absorbing set for solutions of (1.1) in the case that f is globally Lipschitzian.

Theorem 3.5. *Assume that f is globally Lipschitzian, i.e.,*

$$\|f(v_1) - f(v_2)\|_{\mathbb{H}^{\mu-2}} \le L_f^* \|v_1 - v_2\|_{\mathbb{H}^\mu}$$

for all $v_1, v_2 \in \mathbb{H}^\mu$. If $L_f^* < \frac{1}{2}\theta_1$ and $m \notin L^1(\mathbb{R}_+)$, then there exists a bounded absorbing set for solutions of (1.1).

Proof. As mentioned in Remark 3.3, for each $\xi \in \mathbb{H}^\mu$, there exists a unique solution u of (1.1) with $u(0) = \xi$. In addition, we have

$$\begin{aligned}\|u(t)\|^2_{\mathbb{H}^\mu} &\le 2s(t,\theta_1)\|\xi\|^2_{\mathbb{H}^\mu} + 2\theta_1^{-1}\int_0^t r(t-\tau,\theta_1)\|f(u(\tau))\|^2_{\mathbb{H}^{\mu-2}}\,d\tau \\ &\le 2s(t,\theta_1)\|\xi\|^2_{\mathbb{H}^\mu} \\ &\quad + 2\theta_1^{-1}\int_0^t r(t-\tau,\theta_1)[2(L_f^*)^2\|u(\tau)\|^2_{\mathbb{H}^\mu} + 2\|f(0)\|^2_{\mathbb{H}^{\mu-2}}]\,d\tau.\end{aligned}$$

Using Proposition 2.2 again, we obtain

$$\|u(t)\|^2_{\mathbb{H}^\mu} \le 2s(t,\theta_1 - 4\theta_1^{-1}(L_f^*)^2)\|\xi\|^2_{\mathbb{H}^\mu} + \frac{4\|f(0)\|^2_{\mathbb{H}^{\mu-2}}}{\theta_1 - 4(L_f^*)^2}$$

for all $t > 0$. Since $s(t,\theta_1 - 4\theta_1^{-1}(L_f^*)^2) \to 0$ as $t \to \infty$, there exists $T = T(\xi) > 0$ such that $2s(t,\theta_1 - 4\theta_1^{-1}(L_f^*)^2)\|\xi\|^2_{\mathbb{H}^\mu} < 1$ for all $t \ge T$. That is,

$$\|u(t)\|^2_{\mathbb{H}^\mu} \le 1 + \frac{4\|f(0)\|^2_{\mathbb{H}^{\mu-2}}}{\theta_1 - 4\theta_1^{-1}(L_f^*)^2}$$

for all $t \ge T$. Equivalently, the ball $\mathbb{B}_\rho$ with

$$\rho = \left(1 + \frac{4\|f(0)\|^2_{\mathbb{H}^{\mu-2}}}{\theta_1 - 4\theta_1^{-1}(L_f^*)^2}\right)^{\frac{1}{2}}$$

is an absorbing set for solutions of (1.1). The proof is complete. ■

4. Regularity results

4.1. Hölder regularity

Denote

$$\mathbf{V}^{\mu,\gamma}_{r,r^*} = \left\{u \in B_{r^*} : \sup_{\substack{h\in(0,T)\\ t\in(0,T-h)}} \frac{t^\gamma\|u(t+h) - u(t)\|_{\mathbb{H}^\mu}}{h^\gamma} \le r\right\},$$

where the ball B_{r^*} is taken from the proof of Theorem 3.2 and $\gamma \in (0,1)$.

Theorem 4.1. *Let the assumptions of Theorem 3.2 be satisfied. If the kernel function m is 2-regular and θ-sectorial for some $\theta \in (0, \pi)$ and*

$$\ell_1 = 3\theta_1^{-1}(L_f^*)^2 \sup_{t\in(0,T]} t^{2\gamma} \int_0^t r(t-\tau, \theta_1)\tau^{-2\gamma}\, d\tau < 1,$$

$$\ell_2 = \sup_{\substack{h\in(0,T)\\ t\in(0,T-h]}} \left(\frac{t}{h}\right)^{2\gamma} \int_t^{t+h} r(\tau, \theta_1)\, d\tau < \infty$$

hold, then the solution of (1.1)–(1.3) is Hölder continuous on $(0, T]$.

Proof. It suffices to show that the solution operator $\Phi\colon \mathbf{V}^{\mu,\gamma}_{r,r^*} \to \mathbf{V}^{\mu,\gamma}_{r,r^*}$ is contractive. In fact, we need to show $\Phi(\mathbf{V}^{\mu,\gamma}_{r,r^*}) \subset \mathbf{V}^{\mu,\gamma}_{r,r^*}$ for some $r > 0$.

Let $u \in \mathbf{V}^{\mu,\gamma}_{r,r^*}$. Then

$$\begin{aligned}\Phi(u)(t+h) - \Phi(u)(t) &= [S(t+h) - S(t)]\xi \\ &\quad + \int_0^t R(\tau)[f(u(t-\tau+h)) - f(u(t-\tau))]\, d\tau \\ &\quad + \int_t^{t+h} R(\tau) f(u(t-\tau+h))\, d\tau \\ &= M_1(t) + M_2(t) + M_3(t), \quad t > 0.\end{aligned}$$

By assumption, $S(\cdot)$ is differentiable on $(0, \infty)$. Then

$$\|M_1(t)\|_{\mathbb{H}^\mu} \le \int_t^{t+h} \|S'(\tau)\xi\|_{\mathbb{H}^\mu}\, d\tau \le M\|\xi\|_{\mathbb{H}^\mu} \int_t^{t+h} \frac{d\tau}{\tau} = M\|\xi\|_{\mathbb{H}^\mu} \ln\Big(1 + \frac{h}{t}\Big).$$

So

$$\|M_1(t)\|_{\mathbb{H}^\mu} \le M\gamma^{-1}\|\xi\|_{\mathbb{H}^\mu}\left(\frac{h}{t}\right)^\gamma \tag{4.1}$$

for any $\gamma \in (0, 1)$. Here we used the inequality $\ln(1 + b) \le \frac{x^\gamma}{\gamma}$ for $b > 0$.

Now employing Lemma 2.3 (b), we have

$$\begin{aligned}\|M_2(t)\|^2_{\mathbb{H}^\mu} &\le \theta_1^{-1} \int_0^t r(\tau, \theta_1)\|f(u(t-\tau+h)) - f(u(t-\tau))\|^2_{\mathbb{H}^{\mu-2}}\, d\tau \\ &\le \theta_1^{-1} \int_0^t r(\tau, \theta_1)(L_f^* + \epsilon)^2\|u(t-\tau+h) - u(t-\tau)\|^2_{\mathbb{H}^\mu}\, d\tau.\end{aligned}$$

Put $D_h u(t) = u(t+h) - u(t)$; then

$$\begin{aligned}\|M_2(t)\|^2_{\mathbb{H}^\mu} &\le \theta_1^{-1}h^{2\gamma}(L_f^* + \epsilon)^2 \int_0^t r(\tau, \theta_1)(t-\tau)^{-2\gamma} \frac{(t-\tau)^{2\gamma}\|D_h u(t-\tau)\|^2_{\mathbb{H}^\mu}}{h^{2\gamma}}\, d\tau \\ &\le \theta_1^{-1}h^{2\gamma}r^2(L_f^* + \epsilon)^2 \int_0^t r(\tau, \theta_1)(t-\tau)^{-2\gamma}\, d\tau,\end{aligned}$$

thanks to the formulation of $\mathbf{V}^{\mu,\gamma}_{r,r^*}$. Thus

$$\left(\frac{t}{h}\right)^{2\gamma}\|M_2(t)\|^2_{\mathbb{H}^\mu} \le \frac{1}{3}\ell_1 r^2. \tag{4.2}$$

Regarding $M_3(t)$, we see that

$$\begin{aligned}\|M_3(t)\|^2_{\mathbb{H}^\mu} &\le \theta_1^{-1}\int_t^{t+h} r(\tau,\theta_1)\|f(u(t-\tau+h))\|^2_{\mathbb{H}^{\mu-2}}\,d\tau\\ &\le \theta_1^{-1}(L_f^*+\epsilon)^2\int_t^{t+h} r(\tau,\theta_1)\|u(t-\tau+h)\|^2_{\mathbb{H}^\mu}\,d\tau\\ &\le \theta_1^{-1}(L_f^*+\epsilon)^2(r^*)^2\int_t^{t+h} r(\tau,\theta_1)\,d\tau.\end{aligned}$$

Hence

$$\left(\frac{t}{h}\right)^{2\gamma}\|M_3(t)\|^2_{\mathbb{H}^\mu} \le \ell_2\theta_1^{-1}(L_f^*+\epsilon)^2(r^*)^2. \tag{4.3}$$

Combining (4.1)–(4.3), one obtains

$$\begin{aligned}&\left(\frac{t}{h}\right)^{2\gamma}\|\Phi(u)(t+h)-\Phi(u)(t)\|^2_{\mathbb{H}^\mu}\\ &\quad\le 3\left(\frac{t}{h}\right)^{2\gamma}\sum_{i=1}^{3}\|M_i(t)\|^2_{\mathbb{H}^\mu}\\ &\quad\le 3M^2\gamma^{-2}\|\xi\|^2_{\mathbb{H}^\mu}+3\ell_2\theta_1^{-1}(L_f^*+\epsilon)^2(r^*)^2+\ell_1 r^2.\end{aligned}$$

Since $\ell_1 < 1$, we are able to choose $r > 0$ (large enough) such that

$$\left(\frac{t}{h}\right)^{2\gamma}\|\Phi(u)(t+h)-\Phi(u)(t)\|^2_{\mathbb{H}^\mu} \le r^2 \quad \forall h\in(0,T),\ t\in(0,T-h),$$

which implies $\Phi(u)\in\mathbf{V}^{\mu,\gamma}_{r,r^*}$. The proof is complete. ■

4.2. Mild solutions vs weak solutions

In this subsection, we assume that (F) holds with $\mu = 1$.

Definition 4.2. A function $u \in C([0,T];\mathbb{H}^1)$ is said to be a weak solution to problem (1.1)–(1.3) on $[0,T]$ if $u(0)=\xi$ and equation (1.1) holds in $\mathbb{H}^{-1}$ for every $t\in[0,T]$, i.e.,

$$\langle\partial_t^{\{k\}}u,v\rangle+\langle\partial_t^{\{k\}}\nabla u,\nabla v\rangle+\eta\langle\nabla u,\nabla v\rangle=\langle f(u),v\rangle \tag{4.4}$$

for every $v\in\mathbb{H}^1$ and $t\in[0,T]$.

Theorem 4.3. *A function $u \in C([0,T];\mathbb{H}^1)$ is a weak solution to (1.1)–(1.3) if and only if it is a mild solution.*

Proof. Let $u \in C([0,T];\mathbb{H}^1)$ be a weak solution to (1.1)–(1.3). Then the smoothness of the boundary of Ω implies that $e_n \in \mathbb{H}^2$, $n \in \mathbb{N}$. Taking $v = e_n$ in (4.4) and integrating by parts yields

$$\langle \partial_t^{\{k\}} u, e_n\rangle - \langle \partial_t^{\{k\}} u, \Delta e_n\rangle - \eta\langle u, \Delta e_n\rangle = \langle f(u), e_n\rangle,$$

which ensures that

$$(1+\lambda_n)\partial_t^{\{k\}} u_n(t) + \eta\lambda_n u_n(t) = f_n(u(t)), \quad u_n(0) = \xi_n = \langle u(0), e_n\rangle,$$

where $u_n(t) = \langle u(t), e_n\rangle$ and $f_n(u(t)) = \langle f(u(t)), e_n\rangle$. Thus

$$u_n(t) = s(t,\theta_n)\xi_n + (1+\lambda_n)^{-1}\int_0^t r(t-\tau,\theta_n) f_n(u(\tau))\,d\tau,$$

where $\theta_n = \frac{\eta\lambda_n}{1+\lambda_n}$. Put

$$\sigma_1(t) = \sum_{n=1}^{\infty} \lambda_n s(t,\theta_n)^2\xi_n^2 = \sum_{n=1}^{\infty} \sigma_1(t)[n],$$
$$\sigma_2(t) = \sum_{n=1}^{\infty} \lambda_n\left((1+\lambda_n)^{-1}\int_0^t r(t-\tau,\theta_n) f_n(u(\tau))\,d\tau\right)^2 = \sum_{n=1}^{\infty} \sigma_2(t)[n].$$

In order to show that u admits the representation in Definition 3.1, we testify the uniform convergence of the series $\sigma_1(t)$ and $\sigma_2(t)$. Indeed, one sees that

$$\sigma_1(t)[n] = \lambda_n s(t,\theta_n)^2\xi_n^2 \le \lambda_n\xi_n^2,$$

and the series $\sum_{n=1}^{\infty}\lambda_n\xi_n^2$ converges to $\|\xi\|_{\mathbb{H}^1}^2$. Then $\sigma_1(t)$ is uniformly convergent due to the Weierstrass test. Concerning $\sigma_2(t)$, we observe that

$$\begin{aligned}
\sigma_2(t)[n] &\le \lambda_n^{-1}\left(\int_0^t r(t-\tau,\theta_n)\,d\tau\right)\left(\int_0^t r(t-\tau,\theta_n)|f_n(u(\tau))|^2\,d\tau\right)\\
&\le \lambda_n^{-1}\theta_n^{-1}\int_0^t r(t-\tau,\theta_n)|f_n(u(\tau))|^2\,d\tau\\
&\le \theta_1^{-1}\int_0^t r(t-\tau,\theta_1)\lambda_n^{-1}|f_n(u(\tau))|^2\,d\tau, \qquad (4.5)
\end{aligned}$$

thanks to the Hölder inequality and Proposition 2.1 (b).

In addition, we have $t \mapsto f(u(t))$ is continuous as mapping from $[0,T]$ into $\mathbb{H}^{-1}$. So the series $\sum_{n=1}^{\infty}\lambda_n^{-1}\|f_n(u(t))\|^2$ is uniformly convergent on $[0,T]$. That is, for every $\varepsilon > 0$, one can find $N_\varepsilon \in \mathbb{N}$ such that

$$\sum_{n=N_\varepsilon}^{N_\varepsilon+p} \lambda_n^{-1}\|f_n(u(t))\|^2 < \varepsilon\theta_1^2 \quad \text{for all } p \in \mathbb{N},\ t \in [0,T].$$

Now taking (4.5) into account, we get

$$\sum_{n=N_\varepsilon}^{N_\varepsilon+p} \sigma_2(t)[n] \le \theta_1^{-1} \int_0^t r(t-\tau,\theta_1)\left(\sum_{n=N_\varepsilon}^{N_\varepsilon+p} \lambda_n^{-1}\|f_n(u(\tau))\|^2\right) d\tau < \varepsilon$$

for all $t \in [0,T]$, which implies the uniform convergence of σ_2.

Conversely, assume that $u \in C([0,T];\mathbb{H}^1)$ is a mild solution of problem (1.1)–(1.3). Then

$$u(t) = S(t)\xi + \int_0^t R(t-\tau) f(u(\tau))\, d\tau.$$

Obviously, $u(0) = \xi$. It remains to check that equation (1.1) holds in $\mathbb{H}^{-1}$. Recall that u can be represented by

$$u(t) = \sum_{n=1}^{\infty} u_n(t) e_n, \quad \text{with } u_n(t) = s(t,\theta_n)\xi_n + (1+\lambda_n)^{-1} r(\cdot,\theta_n) * f_n(u(\cdot)).$$

Then

$$\begin{aligned} k * u_n'(t) &= \frac{d}{dt}[k * (u_n - \xi_n)](t) \\ &= \frac{d}{dt}[k * (s(\cdot,\theta_n) - 1)\xi_n](t) \\ &\quad + (1+\lambda_n)^{-1} \frac{d}{dt}[k * r(\cdot,\theta_n) * f_n(u(\cdot))](t). \end{aligned}$$

Noting that

$$\frac{d}{dt}[k * (s(\cdot,\theta_n) - 1)](t) = -\theta_n s(t,\theta_n), \quad k * r(\cdot,\theta_n)(t) = s(t,\theta_n),$$

thanks to Proposition 2.1 (d) and (b), we have

$$\begin{aligned} k * u_n'(t) &= -\theta_n s(t,\theta_n)\xi_n + (1+\lambda_n)^{-1} \frac{d}{dt}[s(\cdot,\theta_n) * f_n(u(\cdot))](t) \\ &= -\theta_n s(t,\theta_n)\xi_n + (1+\lambda_n)^{-1}[f_n(u(t)) + s'(\cdot,\theta_n) * f_n(u(\cdot))(t)] \\ &= -\theta_n s(t,\theta_n)\xi_n + (1+\lambda_n)^{-1}[f_n(u(t)) - \theta_n r(\cdot,\theta_n) * f_n(u(\cdot))(t)]. \end{aligned}$$

Hence

$$\begin{aligned} (1+\lambda_n) k * u_n'(t) &= -\eta\lambda_n s(t,\theta_n)\xi_n - \theta_n r(\cdot,\theta_n) * f_n(u(\cdot))(t) + f_n(u(t)) \\ &= \omega_1(t)[n] + \omega_2(t)[n] + \omega_3(t)[n]. \end{aligned} \tag{4.6}$$

We will show that the series $\sum_{n=1}^{\infty} \omega_i(t)[n] e_n$ for $i = 1,2,3$ are uniformly convergent in $\mathbb{H}^{-1}$ on $[0,T]$. Regarding the series $\sum_{n=1}^{\infty} \omega_1(t)[n] e_n$, we see that

$$\lambda_n^{-1}(\omega_1(t)[n])^2 = \eta^2 \lambda_n s(t,\theta_n)^2 \xi_n^2 \le \eta^2 \lambda_n \xi_n^2 \quad \forall t \in [0,T],$$

and the series $\sum_{n=1}^{\infty} \eta^2 \lambda_n \xi_n^2$ converges to $\eta^2 \|\xi\|^2_{\mathbb{H}^1}$. This ensures the uniform convergence in $\mathbb{H}^{-1}$ of $\sum_{n=1}^{\infty} \omega_1(t)[n]e_n$ and that

$$\sum_{n=1}^{\infty} \omega_1(t)[n]e_n = \eta \Delta S(t)\xi. \tag{4.7}$$

Now considering the series $\sum_{n=1}^{\infty} \omega_2(t)[n]e_n$, one has

$$\begin{aligned}
\lambda_n^{-1}(\omega_2(t)[n])^2 &= \lambda_n^{-1}\theta_n^2 \left(\int_0^t r(t-\tau, \theta_n) f_n(u(\tau))\, d\tau \right)^2 \\
&\leq \lambda_n^{-1}\theta_n^2 \left(\int_0^t r(t-\tau, \theta_n)\, d\tau \right) \left(\int_0^t r(t-\tau, \theta_n) |f_n(u(\tau))|^2\, d\tau \right) \\
&\leq \lambda_n^{-1}\theta_n \int_0^t r(t-\tau, \theta_n) |f_n(u(\tau))|^2\, d\tau \\
&\leq \eta \int_0^t r(t-\tau, \theta_1) \lambda_n^{-1} |f_n(u(\tau))|^2\, d\tau.
\end{aligned}$$

Then the uniform convergence of the series $\sum_{n=1}^{\infty} \lambda_n^{-1}(\omega_2(t)[n])^2$ is testified by the same reasoning for $\sigma_2(t)$. It is easily seen that

$$\sum_{n=1}^{\infty} \omega_2(t)[n]e_n = \eta \Delta [R * f(u)](t). \tag{4.8}$$

Finally, we have

$$\sum_{n=1}^{\infty} \omega_3(t)[n]e_n = \sum_{n=1}^{\infty} f_n(u(t))e_n,$$

which is the decomposition of $f(u(t))$ in $\mathbb{H}^{-1}$. This together with (4.6)–(4.8) leads to

$$\begin{aligned}
k * \partial_t(u - \Delta u)(t) &= \eta \Delta S(t)\xi + \eta \Delta [R * f(u)](t) + f(u(t)) \\
&= \eta \Delta [S(t)\xi + R * f(u)(t)] + f(u(t)) \\
&= \eta \Delta u(t) + f(u(t)),
\end{aligned}$$

which is an equation in $\mathbb{H}^{-1}$. The proof is complete. ∎

An example

Let $k(t) = g_{1-\alpha}(t)e^{-\beta t}$ with $\alpha \in (0,1)$ and $\beta > 0$. Then

$$m(t) = g_\alpha(t)e^{-\beta t} + \beta \int_0^t g_\alpha(s)e^{-\beta s}\, ds$$

(see e.g. [29]). Obviously,

$$m'(t) = \frac{\alpha - 1}{\Gamma(\alpha)} t^{\alpha-2} e^{-\beta t} < 0, \quad t \in (0, \infty);$$

then $m(\cdot)$ is a decreasing function. Furthermore,

$$\hat{m}(\lambda) = \lambda^{-1}(\lambda+\beta)^{1-\alpha} \quad \text{for } \operatorname{Re}(\lambda) > 0.$$

We first check that m is $\frac{\pi}{2}$-sectorial. Indeed, if $\operatorname{Re}(\lambda) > 0$, then

$$\arg(\lambda+\beta) \in \Big(-\frac{\pi}{2}, \frac{\pi}{2}\Big), \quad \arg(\lambda+\beta)^{1-\alpha} \in \Big(-\frac{\pi}{2}, \frac{\pi}{2}\Big), \quad \arg(\lambda^{-1}) \in \Big(-\frac{\pi}{2}, \frac{\pi}{2}\Big).$$

So

$$\arg(\hat{m}(\lambda)) = \arg(\lambda^{-1}) + \arg(\lambda+\beta)^{1-\alpha} \in \Big(-\frac{\pi}{2}, \frac{\pi}{2}\Big),$$

thanks to the fact that $\arg(\lambda^{-1})$ and $\arg(\lambda+\beta)^{1-\alpha}$ have opposite signs.

Now we testify that m is 2-regular. Observe that

$$\hat{m}'(\lambda) = (1-\alpha)\lambda^{-1}(\lambda+\beta)^{-\alpha} - \lambda^{-2}(\lambda+\beta)^{1-\alpha}.$$

Then

$$\lambda\hat{m}'(\lambda) = \Big[(1-\alpha)\frac{\lambda}{\lambda+\beta} - 1\Big]\hat{m}(\lambda).$$

Since $\operatorname{Re}(\lambda) > 0$ and $\beta > 0$, it is easily seen that $\big|\frac{\lambda}{\lambda+\beta}\big| < 1$. Thus

$$|\lambda\hat{m}'(\lambda)| \le (2-\alpha)|\hat{m}(\lambda)|.$$

In addition,

$$\begin{aligned}\hat{m}''(\lambda) &= \alpha(\alpha-1)\lambda^{-1}(\lambda+\beta)^{-\alpha-1} - (1-\alpha)\lambda^{-2}(\lambda+\beta)^{-\alpha}\\ &\quad + 2\lambda^{-3}(\lambda+\beta)^{1-\alpha} - (1-\alpha)\lambda^{-2}(\lambda+\beta)^{-\alpha}.\end{aligned}$$

Then

$$\begin{aligned}\lambda^2\hat{m}''(\lambda) &= \alpha(\alpha-1)\lambda(\lambda+\beta)^{-\alpha-1} - (1-\alpha)(\lambda+\beta)^{-\alpha}\\ &\quad + 2\lambda^{-1}(\lambda+\beta)^{1-\alpha} - (1-\alpha)(\lambda+\beta)^{-\alpha}\\ &= \Big[\alpha(\alpha-1)\frac{\lambda^2}{(\lambda+\beta)^2} - 2(1-\alpha)\frac{\lambda}{\lambda+\beta} + 2\Big]\hat{m}(\lambda).\end{aligned}$$

This implies

$$|\lambda^2\hat{m}''(\lambda)| \le [(2+\alpha)(1-\alpha)+2]|\hat{m}(\lambda)|.$$

Thus m is 2-regular, which ensures the differentiability of the resolvent $S(\cdot)$ on $(0,\infty)$.

Regarding the nonlinearity, let

$$f(u) = |u|^{p-1}u + \mathbf{H}(u)\cdot\nabla u,$$

where $p > 1$ and $\mathbf{H}(u) = (h_1(u), \dots, h_N(u))$ is a vector field obeying that

$$h_i(0) = 0, \quad |h_i(v) - h_i(w)| \lesssim |v-w| \quad \forall v, w \in \mathbb{R};$$

here the notation $X \lesssim Y$ means that $X \le cY$ for some constant $c > 0$. So

$$
\begin{aligned}
|f(u) - f(v)| &\lesssim (|u|^{p-1} + |v|^{p-1})|u - v| \\
&\quad + \sum_{i=1}^{N} [|h_i(v)||\partial_{x_i}(u - v)| + |h_i(u) - h_i(v)||\partial_{x_i} u|] \\
&\lesssim (|u|^{p-1} + |v|^{p-1})|u - v| + \sum_{i=1}^{N} [|v||\partial_{x_i}(u - v)| + |u - v||\partial_{x_i} u|].
\end{aligned}
$$

Let $\nu = 2 - \mu \le 0$ and $0 < 2\mu < N \le 2 + 2\mu$. Take

$$
\tilde{r} = \frac{2N}{N + 2\nu}, \quad \tilde{p} = \frac{2N}{N - 2\mu}, \quad \tilde{q} = \frac{N}{2}.
$$

Then

$$
\frac{1}{\tilde{r}} = \frac{1}{\tilde{p}} + \frac{1}{\tilde{q}}.
$$

Now, for $u, v \in \mathbb{H}^{\mu}$, we have

$$
\||u|^{p-1} v\|_{L^{\tilde{r}}} \le \||u|^{p-1}\|_{L^{\tilde{q}}} \|v\|_{L^{\tilde{p}}} = \|u\|_{L^{(p-1)\tilde{q}}}^{p-1} \|v\|_{L^{\tilde{p}}},
$$

thanks to the generalized Hölder inequality. Assume that

$$
p \le \frac{N + 2\nu}{N - 2\mu}.
$$

Then $(p - 1)\tilde{q} \le \tilde{p}$. So $(p - 1)\tilde{q} \le \tilde{p}$, which implies

$$
\||u|^{p-1} v\|_{L^{\tilde{r}}} \lesssim \|u\|_{L^{\tilde{p}}}^{p-1} \|v\|_{L^{\tilde{p}}}.
$$

In view of Lemma 2.8, we see that $L^{\tilde{r}}(\Omega) \subset \mathbb{H}^{-\nu}$, $\mathbb{H}^{\mu} \subset L^{\tilde{p}}(\Omega)$. Hence one deduces that

$$
\||u|^{p-1} v\|_{\mathbb{H}^{-\nu}} \lesssim \|u\|_{\mathbb{H}^{\mu}}^{p-1} \|v\|_{\mathbb{H}^{\mu}}. \tag{4.9}
$$

Now taking

$$
\tilde{r} = \frac{2N}{N + 2\nu}, \quad \tilde{p} = \frac{2N}{N + 2 - 2\mu}, \quad \tilde{q} = N,
$$

one also has

$$
\frac{1}{\tilde{r}} = \frac{1}{\tilde{p}} + \frac{1}{\tilde{q}}.
$$

Using the generalized Hölder inequality again, we have

$$
\|v \partial_{x_i} u\|_{L^{\tilde{r}}} \le \|v\|_{L^{\tilde{q}}} \|\partial_{x_i} u\|_{L^{\tilde{p}}} \lesssim \|v\|_{L^{\tilde{q}}} \|u\|_{W^{1,\tilde{p}}}. \tag{4.10}
$$

Note that

$$
\mathbb{H}^{\mu} \subset W_0^{\mu,2}(\Omega) \subset L^{\frac{2N}{N-2\mu}}(\Omega) \subset L^{\tilde{q}}(\Omega).
$$

Furthermore, due to Lemma 2.7, one sees that

$$\mathbb{H}^{\mu} \subset W_0^{\mu,2}(\Omega) \subset W_0^{1,\tilde{p}}(\Omega).$$

Using these embeddings in (4.10) yields

$$\|v\partial_{x_i}u\|_{\mathbb{H}^{-\nu}} \lesssim \|v\partial_{x_i}u\|_{L^{\tilde{r}}} \lesssim \|v\|_{\mathbb{H}^{\mu}}\|u\|_{\mathbb{H}^{\mu}}. \tag{4.11}$$

Therefore, it follows from (4.9) and (4.11) that

$$\|f(u)-f(v)\|_{\mathbb{H}^{-\nu}} \lesssim (\|u\|_{\mathbb{H}^{\mu}}^{p-1} + \|v\|_{\mathbb{H}^{\mu}}^{p-1} + \|u\|_{\mathbb{H}^{\mu}} + \|v\|_{\mathbb{H}^{\mu}})\|u-v\|_{\mathbb{H}^{\mu}},$$

which ensures that (F) is fulfilled with $L_f^* = 0$.

Finally, we testify the technical conditions given in Theorem 4.1. Since $m(\cdot)$ is decreasing, we have

$$r(t,\theta_1) \le \frac{m(t)}{1+\theta_1(1*m)(t)} \le \frac{m(t)}{\theta_1^{\delta}(1*m)^{\delta}(t)},$$

thanks to Proposition 2.1 (b) and the inequality $1+b \ge b^{\delta}$ for $b>0$ and $\delta \in (0,1)$. In addition, since $m(t) \sim t^{\alpha-1}/\Gamma(\alpha)$ as $t \to 0$, one see that

$$\frac{m(t)}{(1*m)^{\delta}(t)} \sim \alpha^{\delta}\Gamma(\alpha)^{\delta-1}t^{\alpha-\alpha\delta-1} \quad \text{as } t \to 0.$$

Concerning ℓ_1, let

$$\Lambda_1(t) = t^{2\gamma}\int_0^t r(t-\tau,\theta_1)\tau^{-2\gamma}\,d\tau.$$

Then

$$\Lambda_1(t) \le t^{2\gamma}\int_0^t \frac{m(t-\tau)\tau^{-2\gamma}\,d\tau}{\theta_1^{\delta}(1*m)^{\delta}(t-\tau)} \lesssim t^{2\gamma}\int_0^t (t-\tau)^{\alpha-\alpha\delta-1}\tau^{-2\gamma}\,d\tau.$$

Recalling that $g_\alpha * g_\beta(t) = g_{\alpha+\beta}(t)$, we see that

$$\Lambda_1(t) \lesssim t^{2\gamma}g_{\alpha-\alpha\delta} * g_{1-2\gamma}(t) = t^{2\gamma}g_{\alpha-\alpha\delta+1-2\gamma}(t) \lesssim t^{\alpha-\alpha\delta},$$

which implies that $\sup_{t\in(0,T]}\Lambda_1(t)$ is finite.

Regarding ℓ_2, put

$$\Lambda_2(t,h) = \left(\frac{t}{h}\right)^{2\gamma}\int_t^{t+h} r(\tau,\theta_1)\,d\tau.$$

Then

$$\Lambda_2(t,h) \lesssim \left(\frac{t}{h}\right)^{2\gamma}\int_t^{t+h}\frac{m(\tau)\,d\tau}{(1*m)^{\delta}(\tau)} = \left(\frac{t}{h}\right)^{2\gamma}\frac{m(t+\vartheta h)h}{(1*m)^{\delta}(t+\vartheta h)}$$

for some $\vartheta \in [0,1]$, due to the mean value theorem. So

$$\Lambda_2(t,h) \lesssim \left(\frac{t}{h}\right)^{2\gamma} \frac{m(t)h}{(1*m)^\delta(t)},$$

thanks to the fact that $t \mapsto \frac{m(t)}{(1*m)^\delta(t)}$ is also a decreasing function. It follows that

$$\Lambda_2(t,h) \lesssim h^{1-2\gamma} t^{2\gamma+\alpha-1-\alpha\delta},$$

which ensures the finiteness of

$$\ell_2 = \sup_{\substack{h\in(0,T)\\ t\in(0,T-h]}} \Lambda_2(t,h),$$

provided that $1+\alpha\delta-\alpha \le 2\gamma < 1$.

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