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Radon–Nikodým theorems in non-separable Banach spaces

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Abstract. In this paper, we present two Radon–Nikodým theorems in non-separable Banach spaces using Pettis and variational McShane integrals. The first one works for a dominated additive interval multifunction $\Phi : \mathcal{I} \to ck(X)$, and the second one works for a dominated strong multimeasure $M : \mathcal{L} \to cwk(X)$, where $\mathcal{I}$ is the family of all closed non-degenerate subintervals of the interval $W = [0,1]^m \subset \mathbb{R}^m$, $\mathcal{L}$ is the family of all Lebesgue measurable subsets of W, and $cwk(X)$ ($ck(X)$) is the family of all convex weakly compact (convex compact) non-empty subsets of X.

1. Introduction and preliminaries

One of the major problems in the theory of multimeasures is that of the existence of set valued Radon–Nikodým derivatives. This issue was first considered by Debreu–Schmeidler [9] and Artstein [1], Costé [7], Costé and Pallu de la Barriere [8] and Hiai [14]. In paper [4], B. Cascales, V. Kadets, and J. Rodriguez have proved a Radon–Nikodým theorem for a dominated strong multimeasure taking convex compact values in a non-separable locally convex topological vector space (see [4, Theorem 3.1]). In paper [10], L. Di Piazza and G. Porcello have obtained a Radon–Nikodým theorem for a dominated finitely additive multimeasure or a dominated additive interval multifunction $\Phi : \mathcal{I} \to ck(X)$ using Pettis integral and [4, Theorem 3.1], where $\mathcal{I}$ is the family of all closed non-degenerate subintervals of $[0,1] \subset \mathbb{R}$ (see [10, Theorem 4.2]).

In this paper, we present two Radon–Nikodým theorems in a non-separable Banach space X using Pettis and variational McShane integrals. The first one works for a dominated additive interval multifunction $\Phi : \mathcal{I} \to ck(X)$ (see Theorem 2.6) and the second one works for a dominated strong multimeasure $M : \mathcal{L} \to cwk(X)$ (see Theorem 2.7). Theorem 2.6 improves [10, Theorem 4.2] and Theorem 2.7 improves the Banach version of [4, Theorem 3.1] for strong multimeasures defined on $\mathcal{L}$. The techniques of the proof of Theorem 2.7 can be used to the more general cases. The fact that a convex weakly compact subset of X has the Radon–Nikodým property is essential in the proof of Theorem 2.7 (cf. [2, Theorem 3.6.1]).

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Throughout, X is an arbitrary Banach space with its dual X^*. The closed unit ball of X^* is denoted by B_{X^*}. We denote by 2^X the family of all non-empty subsets of X and by $bcc(X)$ ($ck(X)$, $cwk(X)$) the subfamily of 2^X of all bounded, convex, and closed (convex compact, convex weakly compact) subsets of X. We consider on $cwk(X)$ the Minkowski addition ($A + B = \{a + b : a \in A, b \in B\}$) and the standard multiplication by scalars. By $\mathcal{H}$ we denote the subfamily of 2^X of all bounded closed subsets of X. The family $\mathcal{H}$ is a complete metric space with the Hausdorff distance, given by

$$d_{\mathcal{H}}(A, B) = \max\big\{e(A, B), e(B, A)\big\},$$

where

$$e(A, B) = \sup_{a\in A} \operatorname{dist}(a, B), \quad \operatorname{dist}(a, B) = \inf\big\{\|a - b\| : b \in B\big\}.$$

For every $C \in \mathcal{H}$ the *support function* of C is denoted by $\sigma(\cdot, C)$ and defined as follows:

$$\sigma(\cdot, C) : X^* \to \mathbb{R}, \quad \sigma(x^*, C) = \sup\big\{\langle x^*, x\rangle : x \in C\big\}.$$

Let $\alpha = (a_1, \ldots, a_m)$ and $\beta = (b_1, \ldots, b_m)$ with $-\infty < a_j < b_j < +\infty$ for $j = 1, \ldots, m$. The set $[\alpha, \beta] = \prod_{j=1}^{m}[a_j, b_j]$ is called a *closed non-degenerate interval* in $\mathbb{R}^m$. If $b_1 - a_1 = \cdots = b_m - a_m$, then $I = [\alpha, \beta]$ is called a *cube* and we set $l_I = b_1 - a_1$. We denote by $\mathcal{I}$ the family of all closed non-degenerate subintervals of $W = [0, 1]^m$. The Euclidean space $\mathbb{R}^m$ is equipped with the maximum norm. We may also find it convenient to use the symbols $B_m(t, r)$ for the open ball in $\mathbb{R}^m$ with center t and radius $r > 0$, ∂B and B^o for *boundary* and *interior* of a subset $B \subset \mathbb{R}^m$, respectively. We denote by $\mathcal{L}$ the family of all Lebesgue measurable subsets of W and by $\mathcal{B}$ the family of all Borel subsets of W. The Lebesgue measure of a set $E \in \mathcal{L}$ is denoted by $|E|$. Thus, if I is a cube, then

$$|I| = (l_I)^m.$$

The word "at almost all" always refers to the Lebesgue measure λ on W.

A map $\Gamma : W \to 2^X$ is called a *multifunction* and a map $\Phi : \mathcal{I} \to 2^X$ is said to be an *interval multifunction*. A function $f : W \to X$ is said to be a *selection* of a multifunction $\Gamma : W \to 2^X$ if $f(t) \in \Gamma(t)$ for all $t \in W$. We denote by $\mathcal{S}_\Gamma$ the family of all selections of Γ. We say that an interval multifunction $\Phi : \mathcal{I} \to 2^X$ is an *additive interval multifunction*, if for each two non-overlapping intervals $I, J \in \mathcal{I}$ with $I \cup J \in \mathcal{I}$ we have $\Phi(I \cup J) = \Phi(I) + \Phi(J)$. Two intervals I and J are said to be *non-overlapping* if $I^o \cap J^o = \emptyset$. An additive interval function $\varphi : \mathcal{I} \to X$ is said to be a *selection* of an additive interval multifunction $\Phi : \mathcal{I} \to 2^X$ if $\varphi(I) \in \Phi(I)$ for all $I \in \mathcal{I}$. We denote by $\mathcal{S}_\Phi$ the family of all additive interval selections of Φ.

The following embedding result will be useful to us (see [6, Theorems II.18 and II.19]).

Theorem 1.1. *Let $\ell_\infty(B_{X^*})$ be the Banach space of all bounded real valued functions defined on B_{X^*} endowed with the supremum norm $\|\cdot\|_\infty$. Then, the map*

$$i : cwk(X) \to \ell_\infty(B_{X^*}), \quad i(C) = \sigma(\cdot, C)$$

satisfies the following properties:

(i) $i(A+B) = i(A) + i(B)$ *for every* $A, B \in cwk(X)$,

(ii) $i(\alpha A) = \alpha \cdot i(A)$ *for every* $\alpha \geq 0$ *and every* $A \in cwk(X)$,

(iii) $d_{\mathcal{H}}(A, B) = \|i(A) - i(B)\|_\infty$ *for every* $A, B \in cwk(X)$,

(iv) $i(cwk(X))$ *is closed in* $\ell_\infty(B_{X^*})$.

Definition 1.2. We say that an additive interval multifunction $\Phi : \mathcal{I} \to cwk(X)$ is *strongly absolutely continuous (sAC)* if for every $\varepsilon > 0$ there exists $\eta_\varepsilon > 0$ such that for every finite collection π of pairwise non-overlapping subintervals in $\mathcal{I}$, we have

$$\sum_{I\in\pi} |I| < \eta_\varepsilon \Rightarrow \sum_{I\in\pi} d_{\mathcal{H}}\big(\Phi(I), \{\theta\}\big) < \varepsilon,$$

where θ is the *zero vector* in X.

Replacing the last inequality with $d_{\mathcal{H}}(\sum_{I\in\pi} \Phi(I), \{\theta\}) < \varepsilon$, we obtain the notion AC for Φ.

Definition 1.3. Given a point $t \in W$, we set

$$\mathcal{I}(t) = \big\{I \in \mathcal{I} : t \in I, I \text{ is a cube}\big\}.$$

We say that an additive interval function $\varphi : \mathcal{I} \to X$ has the *cubic derivative* at the point t, if there exists a vector $\varphi'_c(t) \in X$ such that

$$\lim_{\substack{I\in\mathcal{I}(t)\\ |I|\to 0}} \|\Delta\varphi(t, I) - \varphi'_c(t)\| = 0, \quad \left(\Delta\varphi(t, I) = \frac{\varphi(I)}{|I|}\right),$$

where $\varphi'_c(t)$ is said to be the *cubic derivative* of φ at t.

Given a sequence (B_n) of subsets of X, we write $\sum_n B_n$ to denote the set of all elements of X which can be written as the sum of an unconditionally convergent series $\sum_n x_n$, where $x_n \in B_n$ for every $n \in \mathbb{N}$.

Definition 1.4. A mapping $M : \mathcal{L} \to 2^X$ is said to be a strong multimeasure if the following hold:

(i) $M(\emptyset) = \{\theta\}$,

(ii) for each sequence (E_n) of pairwise disjoint members of $\mathcal{L}$, we have

$$M\Big(\bigcup_n E_n\Big) = \sum_n M(E_n).$$

A strong multimeasure $M : \mathcal{L} \to 2^X$ is said be λ-*continuous*, if $M(Z) = \{\theta\}$ whenever $Z \subset W$ satisfies $|Z| = 0$. A countable additive vector measure $m : \mathcal{L} \to X$ is said to be a *selection* of M if $m(E) \in M(E)$ for every $A \in \mathcal{L}$. We denote by $\mathcal{S}_M$ the family of all countable additive selections of M. The strong multimeasure M is said to be of *bounded variation* if $|M|(W) < +\infty$, where $|M|(W) = \sup \sum_i \|M(E_i)\|$ and supremum is taken over all finite partitions (E_i) of W in $\mathcal{L}$ and

$$\|C\| = \sup\{\|x\| : x \in C\}, \quad (C \in \mathcal{H}).$$

Since $i(cwk(X))$ $(i(ck(X)))$ is a closed cone of $\ell_\infty(B_{X^*})$, we obtain by embedding theorem (Theorem 1.1) that a mapping $M : \mathcal{L} \to cwk(X)$ $(ck(X))$ is a strong multimeasure if and only if $M^\infty = i \circ M$ is a countable additive vector measure. In this case,

$$i\Big(\sum_n M(E_n)\Big) = \sum_n M^\infty(E_n),$$

whenever (E_n) is a sequence of pairwise disjoint members of $\mathcal{L}$ (see [5, Lemma 2.3]). For the concept of multimeasure, we refer to [13, Chapter 7] and references therein.

Definition 1.5. A multifunction $\Gamma : W \to bcc(X)$ is called *Pettis integrable* in $cwk(X)$ $(ck(X))$ if the following hold:

(i) $\sigma(x^*, \Gamma(\cdot))$ is Lebesgue integrable for every $x^* \in X^*$,

(ii) for each $E \in \mathcal{L}$ there is $C_E \in cwk(X)$ $(C_E \in ck(X))$ such that

$$\sigma(x^*, C_E) = \int_E \sigma(x^*, \Gamma(t))d\lambda \quad \text{for every } x^* \in X^*.$$

We call C_E the *Pettis integral* of Γ over E and set $(P)\int_E \Gamma(t)d\lambda = C_E$. It is well known that the mapping $M : \mathcal{L} \to cwk(X)$ $(ck(X))$ defined by $M(E) = (P)\int_E \Gamma(t)d\lambda$ is a λ-continuous strong multimeasure.

The Pettis integral for multifunctions was first considered by Castaing and Valadier [6, Chapter V] and has been widely studied in papers [3, 12, 18, 19]. The notion of Pettis integrable function $f : W \to X$ as can be found in the literature (see [11, 17, 22, 23]) corresponds to Definition 1.5 for $\Gamma(t) = \{f(t)\}$ when the integral $(P)\int_E \Gamma(t)d\lambda$ is a singleton. For definition and properties of Bochner integral, we refer to [11].

A pair (I, t) of an interval $I \in \mathcal{I}$ and a point $t \in W$ is called an $\mathcal{M}$*-tagged interval* in W. A finite collection $\pi = \{(I_i, t_i) : i = 1, \dots, p\}$ of $\mathcal{M}$-tagged intervals in W is called an $\mathcal{M}$*-partition* of W if $\{I_i : i = 1, \dots, p\}$ is a collection of pairwise non-overlapping intervals in $\mathcal{I}$ and $\bigcup_{(I,t)\in\pi} I = W$. A positive function $\delta : W \to (0, +\infty)$ is called a *gauge* on W. We say that an $\mathcal{M}$-partition π of W is δ-fine if for each $(I, t) \in \pi$ we have $I \subset B_m(t, \delta(t))$.

We now recall the definitions of McShane integrability and variational McShane integrability (or strong McShane integrability) of functions defined on W and taking values in X, cf. [22, Definitions 3.2.1 and 3.6.2].

Definition 1.6. A function $f : W \to X$ is said to be McShane integrable if there exists $I_f \in X$ with the following property: for every $\varepsilon > 0$ there exists a gauge δ on W such that for every δ-fine $\mathcal{M}$-partition π of W we have

$$\left\| I_f - \sum_{(I,t)\in\pi} f(t)|I| \right\| < \varepsilon.$$

We write then $I_f = (M)\int_W f(t)d\lambda$. The function f is said to be McShane integrable over $E \in \mathcal{L}$, if the function $f\mathbb{1}_E$ is McShane integrable, where $\mathbb{1}_E$ is the characteristic function of E. In this case, we write

$$(M)\int_E f(t)d\lambda = (M)\int_W f(t)\mathbb{1}_E(t)d\lambda.$$

Definition 1.7. A function $f : W \to X$ is said to be variationally McShane integrable (or strongly McShane integrable), if there is an additive interval function $\varphi : \mathcal{I} \to X$ with the following property: for every $\varepsilon > 0$ there exists a gauge δ on W such that for every δ-fine $\mathcal{M}$-partition π of W we have

$$\sum_{(I,t)\in\pi} \left\| f(t)|I| - \varphi(I) \right\| < \varepsilon.$$

In this case, the additive interval multifunction φ is called the variational McShane primitive of f.

The function $f : W \to X$ is variationally McShane integrable with the primitive φ, if and only if f is Bochner integrable (cf. [22, Theorem 5.1.4]). In this case,

$$\varphi(I) = (M)\int_I f(t)d\lambda = (B)\int_I f(t)d\lambda \quad \text{for every } I \in \mathcal{I}.$$

Definition 1.8. Let $\varphi : \mathcal{I} \to X$ be an additive interval function and let $t \in W^o$. We set $\mathcal{I}^o(t) = \{I \in \mathcal{I}(t) : t \in I^o\}$. If $I \in \mathcal{I}^o(t)$, then we write

$$\mathcal{I}^o(t, I) = \left\{J \in \mathcal{I}^o(t) : J \subset I\right\}$$

and define a partial ordering $\preceq_t$ on $\mathcal{I}^o(t)$ by saying that $I' \preceq_t I''$ if and only if $I' \supset I''$. Then, $(\mathcal{I}^o(t), \preceq_t)$ is a directed set. For the concepts of nets and subnets we refer to [16]. We now define

$$L_\varphi(t) = \bigcap_{I\in\mathcal{I}^o(t)} \overline{L_\varphi(t, I)}, \tag{1.1}$$

where

$$L_\varphi(t, I) = \left\{\Delta\varphi(t, J) \in X : J \in \mathcal{I}^o(t, I)\right\}$$

and $\overline{L_\varphi(t, I)}$ is the closure of $L_\varphi(t, I)$. By [16, Theorem 7, page 72] it follows that $L_\varphi(t)$ is the set of all limit points of the net $(\Delta\varphi(t, I))_{I\in\mathcal{I}^o(t)}$.

Definition 1.9. Let $\Phi : \mathcal{I} \to cwk(X)$ be an additive interval multifunction and let $t \in W^o$. For each $I \in \mathcal{I}^o(t)$ we write

$$\Delta\Phi(t, I) = \frac{\Phi(I)}{|I|}, \quad A_\Phi(t, I) = \bigcup_{J\in\mathcal{I}^o(t,I)} \Delta\Phi(t, J)$$

and define

$$L_\Phi(t) = \bigcap_{I \in \mathcal{I}^o(t)} \overline{L_\Phi(t, I)}, \tag{1.2}$$

where $L_\Phi(t, I)$ is the convex hull of the set $A_\Phi(t, I)$, i.e.,

$$L_\Phi(t, I) = \mathrm{co}(A_\Phi(t, I)),$$

and $\overline{L_\Phi(t, I)}$ is the closure of the convex set $L_\Phi(t, I)$ in X.

Assume that $x_t \in X$ is a $\sigma(X, X^*)$-limit point of a net $(x_I)_{I \in \mathcal{I}^o(t)}$ with $x_I \in \Delta\Phi(t, I)$, where $\sigma(X, X^*)$ is the weak topology in X. Then

$$x_t \in \overline{L_\Phi(t, I)}^{\sigma(X,X^*)} \quad \text{for every } I \in \mathcal{I}^o(t),$$

and since by [20, Proposition 8, page 34] or [21, Corollary 2, page 65] we have

$$\overline{L_\Phi(t, I)} = \overline{L_\Phi(t, I)}^{\sigma(X,X^*)},$$

it follows that $x_t \in L_\Phi(t)$.

2. The main results

The main results are Theorem 2.6 and Theorem 2.7. Let us start with a few auxiliary lemmas.

Lemma 2.1. *Let $\varphi : \mathcal{I} \to X$ be an additive interval function and let $C \in \mathcal{I}(t)$. Assume that*

- *φ is sAC,*
- *$C \subset W^o$.*

Then, given $0 < \varepsilon < 1$ there exists $C_\varepsilon \in \mathcal{I}^o(t)$ with $C_\varepsilon \supset C$ such that

$$\left\| \Delta\varphi(t, C) - \Delta\varphi(t, C_\varepsilon) \right\| < \varepsilon.$$

Proof. Let us consider the case when t is a boundary point of C, since if $C \in \mathcal{I}^o(t)$, then $C_\varepsilon = C$. Since $C \subset W^o$ is a cube there exist $a = (a_1, \ldots, a_m) \in W^o$ and $r > 0$ such that $C = \prod_{i=1}^m [a_i - r, a_i + r]$. Hence, for each $s > 1$ we have $C(s) = \prod_{i=1}^m [a_i - r.s, a_i + r.s] \supset C$ and t is the interior point of $C(s)$. Since φ is sAC, the following hold.

- There exists $\eta_\varepsilon > 0$ such that for each finite collection π of pairwise non-overlapping subintervals in $\mathcal{I}$, we have

$$\sum_{J \in \pi} |J| < \eta_\varepsilon \Rightarrow \sum_{J \in \pi} \|\varphi(J)\| < \frac{\varepsilon |C|}{2}. \tag{2.1}$$

- By [15, Lemma 2.3] that there exists $c > |W|$ such that $\|\varphi(I)\| \leq c$ for all $I \in \mathcal{I}$.

Since $\frac{\varepsilon}{2c}|C| < \frac{\varepsilon}{2} < 1$ we can choose a cube $C(s) \subset W$ such that

$$0 < |C(s)| - |C| < \gamma_\varepsilon = \min\left(\frac{\varepsilon}{2c}|C(s)| \cdot |C|, \eta_\varepsilon\right).$$

We are going to prove that $C_\varepsilon = C(s)$ is the required cube.

We have $C_\varepsilon \supset C$, $C_\varepsilon \in \mathcal{I}^o(t)$ and there exists a finite collection π_ε of pairwise non-overlapping subintervals in $\mathcal{I}$ such that

$$C \cup J_{\pi_\varepsilon} = C_\varepsilon, \quad \left(J_{\pi_\varepsilon} = \sum_{J \in \pi_\varepsilon} J\right)$$

and $C^o \cap J^o = \emptyset$ for all $J \in \pi_\varepsilon$. Thus, $|J_{\pi_\varepsilon}| = \sum_{J \in \pi_\varepsilon} |J|$ and $|C_\varepsilon| = |C| + |J_{\pi_\varepsilon}|$, and since $|J_{\pi_\varepsilon}| < \eta_\varepsilon$ we obtain by (2.1) that

$$\sum_{J \in \pi_\varepsilon} \|\varphi(J)\| < \frac{\varepsilon|C|}{2}. \tag{2.2}$$

Note that

$$\begin{aligned}
\left\|\Delta\varphi(t, C) - \Delta\varphi(t, C_\varepsilon)\right\| &= \left\|\frac{\varphi(C)}{|C|} - \frac{\varphi(C) + \sum_{J \in \pi_\varepsilon} \varphi(J)}{|C_\varepsilon|}\right\| \\
&= \left\|\frac{\varphi(C)}{|C|} - \frac{\varphi(C) + \sum_{J \in \pi_\varepsilon} \varphi(J)}{|C| + |J_{\pi_\varepsilon}|}\right\| \\
&= \left\|\frac{\varphi(C)(|C| + |J_{\pi_\varepsilon}|) - \varphi(C) + \sum_{J \in \pi_\varepsilon} \varphi(J)|C|}{|C| \cdot (|C| + |J_{\pi_\varepsilon}|)}\right\| \\
&= \left\|\frac{\varphi(C)|J_{\pi_\varepsilon}| - |C| \sum_{J \in \pi_\varepsilon} \varphi(J)}{|C| \cdot |C_\varepsilon|}\right\| \\
&\leq \frac{|J_{\pi_\varepsilon}| \cdot \|\varphi(C)\|}{|C| \cdot |C_\varepsilon|} + \frac{\sum_{J \in \pi_\varepsilon} \|\varphi(J)\|}{|C_\varepsilon|}.
\end{aligned}$$

Thus,

$$\left\|\Delta\varphi(t, C) - \Delta\varphi(t, C_\varepsilon)\right\| \leq \frac{|J_{\pi_\varepsilon}| \cdot \|\varphi(C)\|}{|C| \cdot |C_\varepsilon|} + \frac{\sum_{J \in \pi_\varepsilon} \|\varphi(J)\|}{|C_\varepsilon|} = A + B. \tag{2.3}$$

Since $|J_{\pi_\varepsilon}| < \frac{\varepsilon}{2c}|C_\varepsilon| \cdot |C|$ and $\|\varphi(C)\| \leq c$ we obtain

$$A = \frac{|J_{\pi_\varepsilon}| \cdot \|\varphi(C)\|}{|C| \cdot |C_\varepsilon|} < \frac{\|\varphi(C)\|}{|C| \cdot |C_\varepsilon|} \frac{\varepsilon}{2c} |C_\varepsilon| \cdot |C| = \|\varphi(C)\| \frac{\varepsilon}{2c} \leq \frac{\varepsilon}{2}. \tag{2.4}$$

By (2.2) we have also

$$B = \frac{\sum_{J \in \pi_\varepsilon} \|\varphi(J)\|}{|C_\varepsilon|} < \frac{\varepsilon|C|}{2|C_\varepsilon|} < \frac{\varepsilon}{2}.$$

The last result together with (2.3) and (2.4) yields

$$\left\|\Delta\varphi(t,C)-\Delta\varphi(t,C_\varepsilon)\right\| = A + B < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

and this ends the proof. ■

The next lemma characterizes the cubic derivative in terms of a convergent net.

Lemma 2.2. *Let $\varphi : \mathcal{I} \to X$ be an additive interval function and let $t \in W^o$. Then, the following statements are equivalent:*

(i) *φ has the cubic derivative $\varphi'_c(t) = z$,*

(ii) *the net $(\Delta\varphi(t,I))_{I\in\mathcal{I}^o(t)}$ converges to z.*

Proof. (i)⇒(ii) Assume that (i) holds and let $\varepsilon > 0$. Then, there exists $\eta_\varepsilon > 0$ such that for each $I \in \mathcal{I}(t)$ we have

$$|I| < \eta_\varepsilon \Rightarrow \|\Delta\varphi(t,I) - z\| < \varepsilon.$$

Since $t \in W^o$ there exists $I_{\eta_\varepsilon} \in \mathcal{I}^o(t)$ such that $|I_{\eta_\varepsilon}| < \eta_\varepsilon$ Hence, for each $I \in \mathcal{I}^o(t) \subset \mathcal{I}(t)$ we have

$$I \subset I_{\eta_\varepsilon} \Rightarrow |I| \le |I_{\eta_\varepsilon}| < \eta_\varepsilon \Rightarrow \|\Delta\varphi(t,I) - z\| < \varepsilon.$$

This means that the net $(\Delta\varphi(t,I))_{I\in\mathcal{I}^o(t)}$ converges to z.

(ii)⇒(i) Assume that (ii) holds, and let $0 < \varepsilon < 1$. Then, there exists $I_0 \in \mathcal{I}^o(t)$ such that for each $I \in \mathcal{I}^o(t)$, we have

$$I_0 \preceq_t I \Rightarrow \|\Delta\varphi(t,I) - z\| < \frac{\varepsilon}{2}. \tag{2.5}$$

Since $t = (t_1,\ldots,t_m)$ is the interior point of I_0, there exists $r > 0$ such that $B_m(t,r) = \prod_{i=1}^m (t_i - r, t_i + r) \subset I_0$. Choose $0 < \eta_\varepsilon < r^m$ and fix an arbitrary cube $C \in \mathcal{I}(t)$ with $|C| < \eta_\varepsilon$. Since $l_C < r$, it follows that $C \subset B_m(t,r)$.

If $C \in \mathcal{I}^o(t)$, then $I_0 \preceq_t C$ and, consequently, we obtain by (2.5) that

$$\|\Delta\varphi(t,C) - z\| < \frac{\varepsilon}{2} < \varepsilon. \tag{2.6}$$

It remains to consider the case when t is a boundary point of C. Applying Lemma 2.1 with C and $\prod_{i=1}^m [t_i - r, t_i + r]$ instead of W there exists $C_\varepsilon \in \mathcal{I}^o(t)$ with $C \subset C_\varepsilon \subset \prod_{i=1}^m [t_i - r, t_i + r]$ such that

$$\|\Delta\varphi(t,C) - \Delta\varphi(t,C_\varepsilon)\| < \frac{\varepsilon}{2},$$

and since $I_0 \preceq_t C_\varepsilon$, we obtain by (2.5) that

$$\|\Delta\varphi(t,C) - z\| \le \|\Delta\varphi(t,C) - \Delta\varphi(t,C_\varepsilon)\| + \|\Delta\varphi(t,C_\varepsilon) - z\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since C was arbitrary, the last result together with (2.6) yields that for each $C \in \mathcal{I}(t)$, we have

$$|C| < \eta_\varepsilon \Rightarrow \|\Delta\varphi(t, C) - z\| < \varepsilon.$$

Then,

$$\lim_{\substack{I \in \mathcal{I}(t) \\ |I| \to 0}} \|\Delta\varphi(t, I) - z\| = 0.$$

This means that $\varphi_c'(t)$ exists and $\varphi_c'(t) = z$, and this ends the proof. ∎

By [15, Theorem 2.8], we have that if an additive interval function $\varphi : \mathcal{I} \to X$ is absolutely continuous and has the cubic derivative at almost all $t \in W$, then there exists a variationally McShane integrable function $f : W \to X$ which is the Radon–Nikodým derivative of φ with respect to λ. In the following lemma, we replace the existence of the cubic derivative $\varphi_c'(t)$ with the existence of limit points of the net $(\Delta\varphi(t, I))_{I \in \mathcal{I}^o}$.

Lemma 2.3. *Let $\varphi : \mathcal{I} \to X$ be an additive interval function and let $f : W \to X$ be a function. Assume that*

- *φ is sAC,*
- *$f(t) \in L_\varphi(t)$ at almost all $t \in W^o$, where $L_\varphi(t)$ is defined by (1.1).*

Then, f is variationally McShane integrable with

$$\varphi(I) = (M) \int_I f(t) d\lambda \quad \text{for every } I \in \mathcal{I}. \tag{2.7}$$

Proof. By hypothesis, there exists $Z \subset W$ with $|Z| = 0$ such that $f(t) \in L_\varphi(t)$ for all $t \in W^o \setminus Z$. We first prove that f is Pettis integrable. To see this fix an arbitrary $x^* \in X^*$. Since $\langle x^*, \varphi \rangle$ is sAC, by [15, Lemma 2.4] there exists a Lebesgue integrable function $g : W \to \mathbb{R}$ such that

$$\langle x^*, \varphi(I) \rangle = \int_I g(t) d\lambda \quad \text{for every } I \in \mathcal{I},$$

and there exists $Z^{x^*} \subset W$ with $|Z^{x^*}| = 0$ such that

$$\lim_{\substack{I \in \mathcal{I}(t) \\ |I| \to 0}} \left| \langle x^*, \Delta\varphi(t, I) \rangle - g(t) \right| = 0 \quad \text{for every } t \in W \setminus Z^{x^*}. \tag{2.8}$$

Hence, by Lemma 2.2, we obtain that the net $(\langle x^*, \Delta\varphi(t, I) \rangle)_{I \in \mathcal{I}^o(t)}$ converges to $g(t)$ for every $t \in W^o \setminus Z^{x^*}$, i.e.,

$$\lim_{I \in \mathcal{I}^o(t)} \langle x^*, \Delta\varphi(t, I) \rangle = g(t) \quad \text{for every } t \in W^o \setminus Z^{x^*}.$$

This means that

$$L_{\langle x^*, \varphi \rangle}(t) = \{g(t)\} \quad \text{for every } t \in W^o \setminus Z^{x^*},$$

and since $\langle x^*, L_\varphi(t, I)\rangle = L_{\langle x^*,\varphi\rangle}(t, I)$ for all $I \in \mathcal{I}^o(t)$, it follows that

$$\langle x^*, f(t)\rangle \in \langle x^*, L_\varphi(t)\rangle \subset L_{\langle x^*,\varphi\rangle}(t) = \{g(t)\} \quad \text{for all } t \in W^o \setminus (Z \cup Z^{x^*}).$$

The last result together with (2.8) yields

$$\lim_{\substack{I\in\mathcal{I}(t)\\ |I|\to 0}} \langle x^*, \Delta\varphi(t, I)\rangle = \langle x^*, f(t)\rangle \quad \text{for all } t \in W^o \setminus (Z \cup Z^{x^*}).$$

Since x^* was arbitrary and φ is sAC, we obtain by [15, Lemma 2.5] that f is Pettis integrable with

$$\varphi(I) = (P)\int_I f(t)d\lambda \quad \text{for every } I \in \mathcal{I}. \tag{2.9}$$

By [15, Lemma 2.3], there exists a unique countable additive vector measure $m_\varphi : \mathcal{L} \to X$ such that m_φ is λ-continuous of bounded variation and $m_\varphi(I) = \varphi(I)$ for all $I \in \mathcal{I}$. Thus,

$$m_\varphi(E) = (P)\int_E f(t)d\lambda \quad \text{for every } E \in \mathcal{L}.$$

Thanks to [15, Lemma 2.2], the set $\varphi(\mathcal{I}) = \{\varphi(I) : I \in \mathcal{I}\}$ is a separable subset of X. If Y is the closed linear subspace spanned by $\varphi(\mathcal{I})$, then Y is also a separable subset of X. Note that by [20, Proposition 8, page 34] or [21, Corollary 2, page 65] we have $Y = \overline{Y} = \overline{Y}^{\sigma(X,X^*)}$, and since $\Delta\varphi(t, I) = \frac{\varphi(I)}{|I|} \in Y$ for all $I \in \mathcal{I}^o(t)$, we obtain that $f(t) \in Y$ at almost all $t \in W$. Thus, f is λ-essentially separably valued, and since $\langle x^*, f\rangle$ is measurable for all $x^* \in X^*$, by Pettis's measurability theorem (cf. [11, Theorem II.1.2, page 42]) it follows that f is measurable. Hence, we obtain by [19, Remark 4.1] that

$$|m_\varphi|(W) = \int_W \|f(t)\|d\lambda < +\infty.$$

Thus, the function $\|f(\cdot)\|$ is Lebesgue integrable. Therefore, by [11, Theorem II.2.2], the function f is Bochner integrable. Further, by [22, Proposition 2.3.1] and (2.9), we obtain

$$\varphi(I) = (B)\int_I f(t)d\lambda \quad \text{for every } I \in \mathcal{I}.$$

By [22, Theorem 5.1.4], we infer that f is variationally McShane integrable with the primitive φ satisfying (2.7), and the proof is complete. ∎

The next lemma characterizes Pettis integral of multifunctions.

Lemma 2.4. *Let $\Phi : \mathcal{I} \to cwk(X)$ $(ck(X))$ be an additive interval multifunction and let $\Gamma : W \to bcc(X)$ be a multifunction. Assume that Φ is AC and for each $x^* \in X^*$ we have*

$$\sigma(x^*, \Phi(I)) = \int_I \sigma(x^*, \Gamma(t))d\lambda \quad \textit{for every } I \in \mathcal{I}.$$

Then, Γ is Pettis integrable in $cwk(X)$ $(ck(X))$ with

$$\Phi(I) = (P)\int_I \Gamma(t)d\lambda \quad \textit{for every } I \in \mathcal{I}.$$

Proof. Since Φ is AC, we obtain by embedding theorem (Theorem 1.1) that $\Phi^\infty = i \circ \Phi$ is also AC. Hence, by [15, Lemma 2.3], there exists a unique countably additive λ-continuous vector measure $H^\infty : \mathcal{L} \to i(cwk(X))$ such that $\Phi^\infty(I) = H^\infty(I)$ for all $I \in \mathcal{I}$. Hence, the mapping $H : \mathcal{L} \to cwk(X)$ defined by

$$i(H(E)) = H^\infty(E) \quad \text{for every } E \in \mathcal{L}$$

is a λ-continuous strong multimeasure such that $H(I) = \Phi(I)$ for every $I \in \mathcal{I}$. Note that for each $x^* \in X^*$, we have

$$\sigma(x^*, H(I)) = \sigma(x^*, \Phi(I)) = \int_I \sigma(x^*, \Gamma(t))d\lambda \quad \text{for all } I \in \mathcal{I}.$$

It is easy to see that the family

$$\mathcal{C} = \left\{ B \in \mathcal{B} : (\forall x^* \in X^*)\left[\sigma(x^*, H(B)) = \int_B \sigma(x^*, \Gamma(t))d\lambda\right]\right\}$$

is a σ-algebra, and since $\mathcal{I} \subset \mathcal{C} \subset \mathcal{B}$ by equality $\mathcal{B} = \sigma(\mathcal{I})$, it follows that $\mathcal{C} = \mathcal{B}$, where $\sigma(\mathcal{I})$ is σ-algebra generated by $\mathcal{I}$. Thus, for each $B \in \mathcal{B}$, we have

$$\sigma(x^*, H(B)) = \int_B \sigma(x^*, \Gamma(t))d\lambda \quad \text{for all } x^* \in X^*.$$

The last result together with the fact that H is λ-continuous yields that for each $E \in \mathcal{L}$, we have

$$\sigma(x^*, H(E)) = \int_E \sigma(x^*, \Gamma(t))d\lambda \quad \text{for every } x^* \in X^*.$$

This means that Γ is Pettis integrable with $H(E) = (P)\int_E \Gamma(t)d\lambda$ for all $E \in \mathcal{L}$, and this completes the proof. ■

The following lemma shows a schematic display of the major implications involved in proving the first result Theorem 2.6.

Lemma 2.5. *Let $\Phi : \mathcal{I} \to ck(X)$ be an additive interval multifunction for which there is a set $Q \in ck(X)$ such that $\Phi(I) \subset |I|Q$ at all $I \in \mathcal{I}$. Then,*

(i) *for each $\varphi \in \mathcal{S}_\Phi$, we have $L_\varphi(t) \neq \emptyset$ for all $t \in W^o$, where $L_\varphi(t)$ is defined by (1.1),*

(ii) *for any $\varphi \in \mathcal{S}_\Phi$, a function $f_\varphi : W \to X$ such that $f_\varphi(t) = \theta$ for all $t \in \partial W$ and $f_\varphi(t) \in L_\varphi(t)$ for every $t \in W^o$ is variationally McShane integrable with the primitive φ,*

(iii) *the multifunction $\Gamma : W \to ck(X)$ defined by $\Gamma(t) = \{\theta\}$ for all $t \in \partial W$ and $\Gamma(t) = L_\Phi(t)$ for every $t \in W^o$ is Pettis integrable with $\Phi(I) = (P)\int_I \Gamma(t)d\lambda$ for all $I \in \mathcal{I}$, where $L_\Phi(t)$ is defined by (1.2).*

Proof. (i) Given $t \in W^o$ and $\varphi \in \mathcal{S}_\Phi$, we have $\Delta\varphi(t, I) \in Q$ for every $I \in \mathcal{I}^o(t)$. It follows that the net $(\Delta\varphi(t, I))_{I \in \mathcal{I}^o(t)}$ has a limit point l_φ in the compact set Q. Then, $l_\varphi \in L_\varphi(t)$ and consequently $L_\varphi(t) \neq \emptyset$.

(ii) By hypothesis, we have $f_\varphi(t) \in L_\varphi(t)$ for every $t \in W^o$. It is easy to see that Φ is sAC. Hence, φ is also sAC and, therefore, by Lemma 2.3, the function f_φ is variationally McShane integrable with the primitive φ.

(iii) Since for each $t \in W^o$ and $\varphi \in \mathcal{S}_\Phi$, we have

$$\emptyset \neq L_\varphi(t) \subset L_\Phi(t) \subset Q,$$

it follows that Γ is well defined. Let us prove that Γ satisfies (iii). To this end, fix an arbitrary $x^* \in X^*$. Since the additive interval function

$$\psi : \mathcal{I} \to \mathbb{R}, \quad \psi(I) = \sigma(x^*, \Phi(I))$$

is sAC, we obtain by [15, Lemma 2.4] that there exists a Lebesgue integrable function $g : W \to \mathbb{R}$ with

$$\psi(I) = \sigma(x^*, \Phi(I)) = \int_I g(t)d\lambda \quad \text{for every } I \in \mathcal{I}, \tag{2.10}$$

and there exists $Z^{x^*} \subset W$ with $|Z^{x^*}| = 0$ such that $\psi'_c(t)$ exists and $\psi'_c(t) = g(t)$ at all $t \in W \setminus Z^{x^*}$. Therefore, by Lemma 2.2, we obtain that the net $(\Delta\psi(t, I))_{I \in \mathcal{I}^o(t)}$ converges to $g(t)$ for every $t \in W^o \setminus Z^{x^*}$, i.e.,

$$\lim_{I \in \mathcal{I}^o(t)} \sigma(x^*, \Delta\Phi(t, I)) = \lim_{I \in \mathcal{I}^o(t)} \Delta\psi(t, I) = g(t) \quad \text{for all } t \in W^o \setminus Z^{x^*}. \tag{2.11}$$

Then, given $t \in W^o \setminus Z^{x^*}$ and $\varepsilon > 0$, there exists $I_\varepsilon \in \mathcal{I}^o(t)$ such that

$$I \in \mathcal{I}^o(t, I_\varepsilon) \Rightarrow \sigma(x^*, \Delta\Phi(t, I)) < g(t) + \varepsilon$$

and by Definition 1.9, it follows that

$$\sigma(x^*, A_\Phi(t, I_\varepsilon)) \le g(t) + \varepsilon \Rightarrow \sigma(x^*, L_\Phi(t, I_\varepsilon)) \le g(t) + \varepsilon \Rightarrow \sigma(x^*, L_\Phi(t)) \le g(t) + \varepsilon.$$

This means that

$$\sigma(x^*, \Gamma(t)) \le g(t) \quad \text{for all } t \in W^o \setminus Z^{x^*}. \tag{2.12}$$

Suppose that for some $t \in W^o \setminus Z^{x^*}$ there exists $r \in \mathbb{R}$ such that

$$\sigma(x^*, \Gamma(t)) < r \quad \text{and} \quad r < g(t).$$

By virtue of (2.11) there exists $I_r \in \mathcal{I}^o(t)$ such that

$$I \in \mathcal{I}^o(t, I_r) \Rightarrow r < \sigma(x^*, \Delta\Phi(t, I)).$$

Hence,

$$I \in \mathcal{I}^o(t, I_r) \Rightarrow \big(\exists x_I \in \Delta\Phi(t, I)\big)\big[r < \langle x^*, x_I\rangle\big],$$

and if we write $\mathcal{I}_r(t) = \mathcal{I}^o(t, I_r)$, then

$$(\forall J \in \mathcal{I}_r(t))\big[r < \langle x^*, x_J\rangle\big]. \tag{2.13}$$

Since $x_J \in \Delta\Phi(t, J) \subset Q$ for all $J \in \mathcal{I}_r(t)$, by [16, Theorem 2, page 136] follows that the net $(x_J)_{J\in\mathcal{I}_r(t)}$ has a limit point $x_t \in Q$. Then,

$$x_t \in \overline{L_\Phi(t, J)} \quad \text{for every } J \in \mathcal{I}_r(t),$$

and since

$$L_\Phi(t) = \bigcap_{I\in\mathcal{I}^o(t)} \overline{L_\Phi(t, I)} = \bigcap_{J\in\mathcal{I}_r(t)} \overline{L_\Phi(t, J)},$$

it follows that $x_t \in L_\Phi(t) = \Gamma(t)$. Hence, by (2.12), we obtain

$$\langle x^*, x_t\rangle \leq \sigma(x^*, \Gamma(t)) < r,$$

and since $\langle x^*, x_t\rangle$ is a limit point of the net $(\langle x^*, x_J\rangle)_{J\in\mathcal{I}_r(t)}$, it follows that there exists $J_r \in \mathcal{I}_r(t)$ such that

$$\langle x^*, x_{J_r}\rangle < r.$$

The last result together with (2.13) implies that

$$r < \langle x^*, x_{J_r}\rangle < r.$$

This contradiction shows that

$$\sigma(x^*, \Gamma(t)) = g(t) \quad \text{for every } t \in W^o \setminus Z^{x^*}.$$

Hence, the function $\sigma(x^*, \Gamma(\cdot))$ is Lebesgue integrable and, consequently, we obtain by (2.10) that

$$\sigma(x^*, \Phi(I)) = \int_I \sigma(x^*, \Gamma(t))d\lambda \quad \text{for every } I \in \mathcal{I}.$$

Since x^* was arbitrary, the last result holds for all $x^* \in X^*$, and since Φ is sAC we obtain that Φ is also AC. Therefore, we obtain by Lemma 2.4 that Γ is Pettis integrable with $\Phi(I) = (P)\int_I \Gamma(t)d\lambda$ for all $I \in \mathcal{I}$, and the proof is complete. ■

We are now ready to prove the first result.

Theorem 2.6. *Let $\Phi : \mathcal{I} \to ck(X)$ be an additive interval multifunction for which there is a set $Q \in ck(X)$ such that $\Phi(I) \subset |I|Q$ at all $I \in \mathcal{I}$. Then, there exists a Pettis integrable multifunction $\Gamma : W \to ck(X)$ such that*

- (i) *for each $\varphi \in \mathcal{S}_\Phi$ there exists a variationally McShane integrable function $f \in \mathcal{S}_\Gamma$ with the primitive φ,*
- (ii) *$\Phi(I) = (P)\int_I \Gamma(t)d\lambda$ for all $I \in \mathcal{I}$.*

Proof. The multifunction Γ defined by (iii) in Lemma 2.5 is Pettis integrable with

$$\Phi(I) = (P)\int_I \Gamma(t)d\lambda \quad \text{for all } I \in \mathcal{I}.$$

If $\varphi \in \mathcal{S}_\Phi$, then the function f_φ defined by (ii) in Lemma 2.5 is variationally McShane integrable with the primitive φ. Since

$$f_\varphi(t) \in L_\varphi(t) \subset L_\Phi(t) = \Gamma(t) \quad \text{for every } t \in W^o$$

and $f_\varphi(t) = \theta \in \{\theta\} = \Gamma(t)$ for all $t \in \partial W$, it follows that $f_\varphi \in \mathcal{S}_\Gamma$, and this ends the proof. ∎

The second result works for a dominated strong multimeasure $M : \mathcal{L} \to cwk(X)$. Since $ck(X) \subset cwk(X)$, it follows that Theorem 2.7 improves the Banach version of [4, Theorem 3.1] for strong multimeasures defined on $\mathcal{L}$. The technique of the proof of this theorem can be used to the more general cases.

Theorem 2.7. *Let $M : \mathcal{L} \to cwk(X)$ be a strong multimeasure for which there is a set $Q \in cwk(X)$ such that $M(A) \subset |A|Q$ for all $A \in \mathcal{L}$. Then, there exists a Pettis integrable multifunction $\Gamma : W \to bcc(X)$ such that*

- (i) *for each $m \in \mathcal{S}_M$ there exists a variationally McShane integrable function $f \in \mathcal{S}_\Gamma$ with $m(I) = (M)\int_I f(t)d\lambda$ for all $I \in \mathcal{I}$,*
- (ii) *$M(E) = (P)\int_E \Gamma(t)d\lambda$ for all $E \in \mathcal{L}$.*

Proof. Let (E_i) be a finite partition of W in $\mathcal{L}$. Since

$$\sum_i \|M(E_i)\| \le \left(\sum_i |E_i|\right)\|Q\| = |W| \cdot \|Q\| < +\infty,$$

it follows that M is of bounded variation. It is easy to see that M is also λ-continuous. We now can define an additive interval multifunction as follows:

$$\Phi : \mathcal{I} \to cwk(X), \quad \Phi(I) = M(I).$$

(a) We first claim that there exists $Z \subset W^o$ with $|Z| = 0$ such that

$$L_\Phi(t) \neq \emptyset, \quad \text{for every } t \in W^o \setminus Z,$$

where $L_\Phi(t)$ is defined by 1.2. To see this, we consider a countably additive selector m of M. Then, m is λ-continuous and

$$\frac{m(E)}{|E|} \in Q \quad \text{for all } E \in \mathcal{L}\ (|E| \neq 0),$$

and since Q has the Radon–Nikodým property, it follows that there exists a Bochner integrable (= variationally McShane integrable) function $f : W \to X$ with

$$m(E) = (B)\int_E f(t)d\lambda = (M)\int_E f(t)d\lambda \quad \text{for every } E \in \mathcal{L}.$$

By [15, Theorem 2.8] the additive interval function $\varphi : \mathcal{I} \to X$ defined by $\varphi(I) = m(I)$ for all $I \in \mathcal{I}$ is sAC, $(\varphi)'_c(t)$ exists and $(\varphi)'_c(t) = f(t)$ at almost all $t \in W$. Hence, by Lemma 2.2 there exists $Z \subset W^o$ with $|Z| = 0$ such that the net $(\Delta\varphi(t, I))_{I \in \mathcal{I}^o(t)}$ converges to $f(t)$ at all $t \in W^o \setminus Z$, and since

$$L_\varphi(t) = \{f(t)\} \subset L_\Phi(t),$$

it follows that $L_\Phi(t) \neq \emptyset$ for all $t \in W^o \setminus Z$.

(b) We now claim that $L_\Phi(t)$ is a bounded subset of X for all $t \in W^o$. Indeed, by the inclusion

$$\overline{L_\Phi(t, I)} \subset Q \quad (t \in W^o, I \in \mathcal{I}^o(t)),$$

we obtain

$$L_\Phi(t) \subset Q \quad \text{for all } t \in W^o,$$

and, consequently,

$$\|L_\Phi(t)\| \leq \|Q\| < +\infty \quad \text{for every } t \in W^o.$$

(c) Finally, we claim that the multifunction

$$\Gamma : W \to bcc(X), \quad \Gamma(t) = \begin{cases} L_\Phi(t), & t \in W^o \setminus Z, \\ \{\theta\}, & t \in Z \cup \partial W, \end{cases}$$

is the required multifunction. Observe that (i) has already been obtained in the proof of (a). It remains to prove (ii). To see this fix an arbitrary $x^* \in X^*$. Since the additive interval function

$$\psi : \mathcal{I} \to \mathbb{R}, \quad \psi(I) = \sigma(x^*, \Phi(I))$$

is AC, we obtain by [15, Lemma 2.4] that there exists a Lebesgue integrable function $g : W \to \mathbb{R}$ with

$$\psi(I) = \sigma(x^*, \Phi(I)) = \int_I g(t) d\lambda \quad \text{for every } I \in \mathcal{I} \tag{2.14}$$

and there exists $Z^{x^*} \subset W$ with $|Z^{x^*}| = 0$ such that $\psi'_c(t)$ exists and $\psi'_c(t) = g(t)$ for all $t \in W \setminus Z^{x^*}$. Therefore, by Lemma 2.2, we obtain

$$\lim_{I \in \mathcal{I}^o(t)} \sigma(x^*, \Delta\Phi(t, I)) = \lim_{I \in \mathcal{I}^o(t)} \Delta\psi(t, I) = g(t) \quad \text{for all } t \in W^o \setminus Z^{x^*}. \tag{2.15}$$

The last result together with the definition of $\Gamma(t)$ yields

$$\sigma(x^*, \Gamma(t)) = \sigma(x^*, L_\Phi(t)) \leq g(t) \quad \text{for all } t \in W^o \setminus (Z \cup Z^{x^*}). \tag{2.16}$$

Suppose that for some $t \in W^o \setminus (Z \cup Z^{x^*})$ there exists $r \in \mathbb{R}$ such that

$$\sigma(x^*, \Gamma(t)) < r \quad \text{and} \quad r < g(t).$$

By virtue of (2.15) there exists $I_r \in \mathcal{I}^o(t)$ such that

$$I \subset \mathcal{I}^o(t, I_r) \Rightarrow r < \sigma(x^*, \Delta\Phi(t, I)).$$

Hence,

$$I \in \mathcal{I}^o(t, I_r) \Rightarrow (\exists x_I \in \Delta\Phi(t, I))[r < \langle x^*, x_I \rangle],$$

and if we write $\mathcal{I}_r(t) = \mathcal{I}^o(t, I_r)$, then

$$(\forall J \in \mathcal{I}_r(t))\big[r < \langle x^*, x_J \rangle\big]. \tag{2.17}$$

Since $x_J \in \Delta\Phi(t, J) \subset Q$ for all $J \in \mathcal{I}_r(t)$, by [16, Theorem 2, page 136], it follows that the net $(x_J)_{J \in \mathcal{I}_r(t)}$ has a weak limit point $x_t \in Q$. Hence,

$$x_t \in \overline{L_\Phi(t, J)}^{\sigma(X, X^*)} = \overline{L_\Phi(t, J)} \quad \text{for every } J \in \mathcal{I}_r(t),$$

and since

$$L_\Phi(t) = \bigcap_{I \in \mathcal{I}^o(t)} \overline{L_\Phi(t, I)} = \bigcap_{J \in \mathcal{I}_r(t)} \overline{L_\Phi(t, J)},$$

it follows that $x_t \in L_\Phi(t) = \Gamma(t)$. Hence, by (2.16), we obtain

$$\langle x^*, x_t \rangle \leq \sigma(x^*, \Gamma(t)) < r.$$

The fact that $\langle x^*, x_t \rangle$ is a limit point of the net $(\langle x^*, x_J \rangle)_{J \in \mathcal{I}_r(t)}$ together with the last result yields that there exists $J_r \in \mathcal{I}_r(t)$ such that

$$\langle x^*, x_{J_r} \rangle < r.$$

The last result together with (2.17) implies that

$$r < \langle x^*, x_{J_r} \rangle < r.$$

This contradiction shows that

$$\sigma(x^*, \Gamma(t)) = g(t) \quad \text{for every } t \in W^o \setminus (Z \cup Z^{x^*}).$$

Hence, the function $\sigma(x^*, \Gamma(\cdot))$ is Lebesgue integrable, and consequently, we obtain by (2.14) that

$$\sigma(x^*, \Phi(I)) = \int_I \sigma(x^*, \Gamma(t)) d\lambda \quad \text{for every } I \in \mathcal{I}.$$

Since x^* was arbitrary, the last result holds for every $x^* \in X^*$.

Since the family

$$\mathcal{C} = \left\{ B \in \mathcal{B} : (\forall x^* \in X^*)\left[\sigma(x^*, M(B)) = \int_B \sigma(x^*, \Gamma(t)) d\lambda\right]\right\}$$

is a σ-algebra and since $\mathcal{I} \subset \mathcal{C} \subset \mathcal{B}$ by equality $\mathcal{B} = \sigma(\mathcal{I})$ it follows that $\mathcal{C} = \mathcal{B}$, where $\sigma(\mathcal{I})$ is σ-algebra generated by $\mathcal{I}$. Thus, for each $B \in \mathcal{B}$, we have

$$\sigma(x^*, M(B)) = \int_B \sigma(x^*, \Gamma(t))d\lambda \quad \text{for all } x^* \in X^*.$$

The last result together with the fact that M is λ-continuous yields that for each $E \in \mathcal{L}$, we have

$$\sigma(x^*, M(E)) = \int_E \sigma(x^*, \Gamma(t))d\lambda \quad \text{for every } x^* \in X^*.$$

This means that Γ is Pettis integrable with $M(E) = (P)\int_E \Gamma(t)d\lambda$ for all $E \in \mathcal{L}$, and this ends the proof. ∎

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References

[1] Z. Artstein, Set-valued measures. *Trans. Amer. Math. Soc.* **165** (1972), 103–125 Zbl 0237.28008 MR 0293054

[2] R. D. Bourgin, *Geometric aspects of convex sets with the Radon–Nikodým property*. Lecture Notes in Math. 993, Springer, Berlin, 1983 Zbl 0512.46017 MR 0704815

[3] B. Cascales, V. Kadets, and J. Rodríguez, Measurable selectors and set-valued Pettis integral in non-separable Banach spaces. *J. Funct. Anal.* **256** (2009), no. 3, 673–699 Zbl 1160.28004 MR 2484932

[4] B. Cascales, V. Kadets, and J. Rodríguez, Radon–Nikodým theorems for multimeasures in non-separable spaces. *Zh. Mat. Fiz. Anal. Geom.* **9** (2013), no. 1, 7–24

[5] B. Cascales and J. Rodríguez, Birkhoff integral for multi-valued functions. *J. Math. Anal. Appl.* **297** (2004), no. 2, 540–560 Zbl 1066.46037 MR 2088679

[6] C. Castaing and M. Valadier, *Convex analysis and measurable multifunctions*. Lecture Notes in Mathematics 580, Springer, Berlin-New York, 1977 Zbl 0346.46038 MR 0467310

[7] A. Costé, La propriété de Radon–Nikodým en intégration multivoque. *C. R. Acad. Sci. Paris Sér. A-B* **280** (1975), no. 22, A1515–A1518 Zbl 0313.28007 MR 0374371

[8] A. Costé and R. Pallu de la Barrière, Radon–Nikodým theorems for set-valued measures whose values are convex and closed. *Ann. Soc. Math. Polon.* **20** (1978), no. 2, 283–309

[9] G. Debreu and D. Schmeidler, The Radon–Nikodým derivative of a correspondence. In *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability (Univ. California, Berkeley, Calif., 1970/1971), Vol. II: Probability theory*, pp. 41–56, University of California Press, Berkeley, CA, 1972 Zbl 0263.28004 MR 0407230

[10] L. Di Piazza and G. Porcello, Radon–Nikodým theorems for finitely additive multimeasures. *Z. Anal. Anwend.* **34** (2015), no. 4, 373–389 Zbl 1331.28026 MR 3420487

[11] J. Diestel and J. J. Uhl, Jr., *Vector measures*. Mathematical Surveys 15, American Mathematical Society, Providence, RI, 1977 Zbl 0369.46039 MR 0453964

[12] K. El Amri and C. Hess, On the Pettis integral of closed valued multifunctions. *Set-Valued Anal.* **8** (2000), no. 4, 329–360 Zbl 0974.28009 MR 1802239

[13] C. Hess, Set-valued integration and set-valued probability theory: an overview. In *Handbook of measure theory, Vol. I, II*, pp. 617–673, North-Holland, Amsterdam, 2002 Zbl 1022.60011 MR 1954624

[14] F. Hiai, Radon–Nikodým theorems for set-valued measures. *J. Multivariate Anal.* **8** (1978), no. 1, 96–118 Zbl 0384.28006 MR 0583862

[15] S. B. Kaliaj, Some remarks on descriptive characterizations of the strong McShane integral. *Math. Bohem.* **144** (2019), no. 4, 339–355 Zbl 1474.28018 MR 4047341

[16] J. L. Kelley, *General topology*. D. Van Nostrand, Toronto-New York-London, 1955 Zbl 0066.16604 MR 0070144

[17] K. Musiał, Topics in the theory of Pettis integration. *Rend. Istit. Mat. Univ. Trieste* **23** (1991), no. 1, 177–262 Zbl 0798.46042 MR 1248654

[18] K. Musiał, Pettis integrability of multifunctions with values in arbitrary Banach spaces. *J. Convex Anal.* **18** (2011), no. 3, 769–810 Zbl 1245.28011 MR 2858094

[19] K. Musiał, Approximation of Pettis integrable multifunctions with values in arbitrary Banach spaces. *J. Convex Anal.* **20** (2013), no. 3, 833–870 Zbl 1284.28006 MR 3136603

[20] A. P. Robertson and W. J. Robertson, *Topological vector spaces*. Cambridge Tracts in Mathematics and Mathematical Physics, No. 53, Cambridge University Press, New York, 1964 Zbl 0123.30202 MR 0162118

[21] H. H. Schaefer and M. P. Wolff, *Topological vector spaces*. 2nd edn., Grad. Texts in Math. 3, Springer, New York, 1999 Zbl 0983.46002 MR 1741419

[22] S. Schwabik and G. Ye, *Topics in Banach space integration*. Ser. Real Anal. 10, World Scientific, Hackensack, NJ, 2005 Zbl 1088.28008 MR 2167754

[23] M. Talagrand, Pettis integral and measure theory. *Mem. Amer. Math. Soc.* **51** (1984), no. 307, ix+224 Zbl 0582.46049 MR 0756174

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