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On rank one and Weyl–von Neumann theorem for multiplicative perturbations of unitary operators

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Abstract. For multiplicative perturbations of unitary operators, it is presented a version of Weyl–von Neumann theorem and a sufficient conditions for generic (in the intensity parameter) singular continuous spectrum under unitary rank one perturbations.

1. Introduction

We are interested in the spectral properties of multiplicative perturbations

$$U \mapsto UX \tag{1.1}$$

of unitary operators U, on a (complex and infinite-dimensional) Hilbert separable space $\mathcal{H}$, with also unitary perturbing X. This is a *right* perturbation, and $U \mapsto XU$ is a *left* one.

The main physical motivation comes from time τ-periodically kicked quantum Hamiltonians (A and B are self-adjoint operators)

$$A + B \sum_{j \in \mathbb{Z}} \delta(t - \tau n)$$

whose Floquet operator, from just before a kick to just before the next one, is $e^{-i\tau A} e^{-iB}$; see, for instance, [3]. In (1.1), one immediately identifies $U = e^{-i\tau A}$ and $X = e^{-iB}$.

In a previous work [1], the present authors have shown that there is no nontrivial generalization of the multiplicative version of Birman–Krein theorem [2] on preservation of absolutely continuous spectrum under certain perturbations. The original version of Birman–Krein is for additive perturbations, but from this, the multiplicative version follows; that is, the absolutely continuous parts of the unitary operators U and UX (or XU) are unitarily equivalent if $X = \mathbf{1} + W$ with trace class W.

In this note, we present multiplicative versions of two important known results for additive self-adjoint perturbations. First, a version of Weyl–von Neumann theorem [6] and, second, a version of a result on the generic presence of singular continuous spectrum for rank one perturbations due to del Rio, Makarov, and Simon [7].

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Recall that, according to Weyl–von Neumann theorem, given a self-adjoint operator A and $\varepsilon > 0$, there is a self-adjoint operator S with Hilbert–Schmidt norm $\|S\|_{\mathrm{HS}} < \varepsilon$ such that $A + S$ has pure point spectrum. Our conclusion will be similar (see Theorem 2.2): given a unitary operator U, there exists another unitary operator $X = 1 + W$ with $\|W\|_{\mathrm{HS}} < \varepsilon$ so that the perturbation UX (or XU) has pure point spectrum.

The other set of results culminate in the following (let $\sigma(A)$ denote the spectrum of the linear operator A). Given a singular (i.e., with no absolutely continuous spectrum) unitary operator U, with $\{e^{it} \mid a < t < b\} \subset \sigma(U)$, and a unitary rank one perturbation $X_\lambda = e^{i\lambda P_\phi}$, $0 \le \lambda < 2\pi$, with P_ϕ the projection onto the one-dimensional subspace generated by the cyclic vector ϕ, then, for generic (i.e., dense G_δ set) of intensities λs, the perturbed operator UX_λ has purely singular continuous spectrum in $\{e^{it} \mid a < t < b\}$. This will be a consequence of Theorem 3.5.

Section 2 presents general remarks on multiplicative perturbations of unitary operators, then Theorem 2.2 and its proof. In Section 3, after a suitable preparation, one finds Theorem 3.5 and its proof.

2. Multiplicative perturbations

If we have a unitary operator X, it is convenient to write it in the form $X = e^{iY}$, with Y a bounded self-adjoint operator. Then,

$$X = e^{iY} = \sum_{j=0}^{\infty} \frac{(iY)^j}{j!} = \mathbf{1} + \sum_{j=1}^{\infty} \frac{(iY)^j}{j!} = \mathbf{1} + W,$$

where $W = \sum_{j=1}^{\infty} \frac{(iY)^j}{j!}$. Thus, we can write

$$UX = U(\mathbf{1} + W) = U + UW.$$

Remark 2.1. If the operator $X = \mathbf{1} + W$ is unitary, one has

$$\mathbf{1} = (\mathbf{1} + W)(\mathbf{1} + W)^* = \mathbf{1} + W + W^* + W^*W,$$
$$\mathbf{1} = (\mathbf{1} + W)^*(\mathbf{1} + W) = \mathbf{1} + W^* + W + WW^*;$$

then $W^* + W + WW^* = W^* + W + W^*W = 0$, and it follows that

$$W^*W = WW^*,$$

so W is a normal operator. But this condition is not sufficient for $X = \mathbf{1} + W$ to be unitary; for example, if $W = \pm\mathbf{1}$, then X would not be unitary (it is necessary that $\sigma(W) \subset \{e^{it} - 1 \mid t \in \mathbb{R}\}$).

Our main result in this section is the following theorem.

Theorem 2.2. *Let U be a unitary operator in $\mathcal{H}$. Given $\varepsilon > 0$, there exists a unitary operator $X = \mathbf{1} + W$, with $\|W\|_{\mathrm{HS}} < \varepsilon$, such that the perturbed operator*

$$U \mapsto UX$$

has pure point spectrum. It also holds true for left perturbations $U \mapsto XU$. (Left and right perturbations are in general different.)

First, we prove a Weyl–von Neumann version for additive perturbations of unitary operators.

Theorem 2.3. *Given a unitary operator U and $\varepsilon > 0$, there exists a unitary operator V on $\mathcal{H}$ with pure point spectrum such that*

$$\|U - V\|_{\mathrm{HS}} < \varepsilon.$$

Proof. Write the unitary operator $U = e^{iT}$, with T self-adjoint and bounded; by the usual Weyl–von Neumann result for self-adjoint operators, there exists a bounded self-adjoint operator B with $\|B\|_{\mathrm{HS}} < \varepsilon$ and $T + B$ is pure point. It follows that $V = e^{i(T+B)}$ is unitary and pure point.

The next ingredient is a version of the Duhamel formula [6]

$$V - U = e^{i(T+B)} - e^{iT} = -i \int_0^1 e^{iT(1-u)} B e^{iu(T+B)} \,\mathrm{d}u.$$

By using the inequality

$$\|TS\|_{\mathrm{HS}} \le \|T\| \|S\|_{\mathrm{HS}}, \tag{2.1}$$

it follows that

$$\begin{aligned}
\|V - U\|_{\mathrm{HS}} &\le \int_0^1 \|e^{iT(1-u)} B e^{iu(T+B)}\|_{\mathrm{HS}} \,\mathrm{d}u \\
&\le \int_0^1 \|e^{iT(1-u)}\| \, \|B\|_{\mathrm{HS}} \, \|e^{iu(T+B)}\| \,\mathrm{d}u \\
&\le \|B\|_{\mathrm{HS}} < \varepsilon.
\end{aligned}$$

This completes the proof since V is a pure point operator. ∎

Proof of Theorem 2.2. By Theorem 2.3, given $0 < \delta < 1$, there exists a unitary and pure point operator V such that $Q = U - V$ satisfies $\|Q\|_{\mathrm{HS}} < \delta$. Thus,

$$U = V + Q = V(\mathbf{1} + V^{-1}Q);$$

by inequality (2.1), $\|V^{-1}Q\|_{\mathrm{HS}} < \delta < 1$, and it follows that $(\mathbf{1} + V^{-1}Q)$ is invertible (in norm).

Write $X = (\mathbf{1} + V^{-1}Q)^{-1}$; hence,

$$UX = V,$$

concluding that X is unitary (since U and V are). By writing the operator X as the series

$$X = \mathbf{1} + \sum_{j=1}^{\infty} (-V^{-1}Q)^j,$$

one finds that

$$\|X - \mathbf{1}\|_{\mathrm{HS}} \leq \sum_{j=1}^{\infty} \|V^{-1}Q\|_{\mathrm{HS}}^j \leq \sum_{j=1}^{\infty} \delta^j = \frac{\delta}{1-\delta}.$$

To complete the proof, it is enough to pick δ such that $\frac{\delta}{1-\delta} < \varepsilon$ and identify

$$W = \sum_{j=1}^{\infty} (-V^{-1}Q)^j.$$

For a left perturbation

$$U \mapsto XU,$$

it is enough to consider

$$U = V + Q = (\mathbf{1} + QV^{-1})V,$$

with $X = (\mathbf{1} + QV^{-1})^{-1}$, and identify $W = \sum_{j=1}^{\infty} (-QV^{-1})^j$. ■

3. Unitary rank one perturbations

Let ϕ be a normalized vector in $\mathcal{H}$ that is cyclic for the unitary operator U, that is, the closure

$$\overline{\mathrm{Lin}\{U^j\phi \mid j \in \mathbb{Z}\}} = \mathcal{H},$$

$U^0 = \mathbf{1}$. Let $P_\phi(\cdot) = \langle \phi, \cdot \rangle \phi$ (which is self-adjoint and idempotent) denote the projection onto the subspace generated by ϕ, and for real λ consider

$$X_\lambda := e^{i\lambda P_\phi} = \mathbf{1} + W$$

with

$$W\xi = (e^{i\lambda} - 1)\langle \phi, \xi \rangle \phi = (e^{i\lambda} - 1) P_\phi(\xi).$$

In fact,

$$e^{i\lambda P_\phi} = \sum_{j=0}^{\infty} \frac{(i\lambda P_\phi)^j}{j!} = \mathbf{1} + \sum_{j=1}^{\infty} \frac{(i\lambda P_\phi)^j}{j!} \overset{P_\phi = P_\phi^2}{=} \mathbf{1} + (e^{i\lambda} - 1) P_\phi.$$

Note that there is a periodicity in the intensity parameter λ, and it suffices to consider $0 \leq \lambda < 2\pi$.

Now, we have the multiplicative rank one perturbation

$$U_\lambda = UX_\lambda = U(\mathbf{1} + (e^{i\lambda} - 1)P_\phi). \tag{3.1}$$

To simplify statements, denote by μ^λ the spectral measure of the pair (U_λ, ϕ). Note that since ϕ is cyclic for U, it is also cyclic for U_λ for all $\lambda \in \mathbb{R}$.

We are interested in relating the perturbation (3.1) to the Cauchy transform $F(z)$ of a Borel measure μ on $[0, 2\pi)$, defined for complex numbers $|z| \neq 1$ as

$$F(z) = \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \mathrm{d}\mu(t).$$

In case of μ^λ, we denote $F_\lambda(z) = \langle \phi, (U_\lambda + z\mathbf{1})(U_\lambda - z\mathbf{1})^{-1}\phi \rangle$.

Important results (see [9]) that relate nontangential limits of this transform to the singular μ_{s}^λ and absolutely continuous parts of μ^λ are summarized in the following theorem.

Theorem 3.1. (1) *The limit* $\lim_{r\to 1} F_\lambda(re^{it})$ *exists for Lebesgue a.e.* $t \in [0, 2\pi)$, *and if*

$$\mathrm{d}\mu^\lambda(t) = f(t)\frac{\mathrm{d}t}{2\pi} + \mathrm{d}\mu_{\mathrm{s}}^\lambda(t),$$

then $f(t) = \Re(F_\lambda(e^{it}))$.

(2) t_0 *is an eigenvalue of* U_λ *if and only if*

$$\lim_{r\to 1}(1-r)\Re(F_\lambda(re^{it_0})) \neq 0,$$

and, in general, $\lim_{r\uparrow 1}(1-r)\Re(F_\lambda(re^{it_0})) = \mu^\lambda(\{t_0\})$.

(3) μ_{s}^λ *is supported on* $\{t \mid \lim_{r\uparrow 1} F_\lambda(re^{it}) = \infty\}$.

Consider now the Borel transform $R_\lambda(z)$ associated with the unitary operator U_λ, which is given by

$$R_\lambda(z) = \langle \phi, (U_\lambda - z\mathbf{1})^{-1}\phi \rangle = \int_0^{2\pi} \frac{\mathrm{d}\mu^\lambda(t)}{e^{it} - z},$$

and it has a simple relation to the Cauchy transform $F_\lambda(z) = 1 + 2zR_\lambda(z)$. After taking expectation values, with ϕ, of the second resolvent identity, one obtains a unitary analog of the so-called Aronszajn–Krein formula; that is,

$$R_\lambda(z) = \frac{R_0(z)}{e^{i\lambda} + z(e^{i\lambda} - 1)R_0(z)},$$

which is [4, equation (9)] (note that we have a rather different notation from [4]). This is also interesting since one may obtain results for the perturbed operator U_λ from asymptotic limits of $R_0(z)$, as in Proposition 3.2 ahead, where, from this formula, one has conditions for the divergence of R_λ in terms of R_0 only.

By following Combescure [4], introduce $B(x)$ and $G(x)$ by

$$B(x) = \left[\int_0^{2\pi} \mathrm{d}\mu(t)\Big(\sin^2\Big(\frac{x-t}{2}\Big)\Big)^{-1}\right]^{-1} = \frac{1}{G(x)}.$$

For a general unitary operator V, with cyclic vector ϕ and spectral measure ν, we have corresponding quantities $B_V(z)$ and $G_V(z)$ (just integrate with respect to ν). With such notation, [4, Proposition 1] implies the following proposition.

Proposition 3.2. *Let $\lambda \neq 0$. Then,* $\mathrm{d}\mu^\lambda$ *has an atom at the point* $x \in [0, 2\pi)$ *if and only if* $B(x) \neq 0$ *(i.e.,* $G(x) < \infty$*) and*

$$\lim_{\varepsilon \to 0} e^{i(x+i\varepsilon)} R_0(e^{i(x+i\varepsilon)}) = \frac{e^{i\lambda}}{1 - e^{i\lambda}}$$

or, equivalently,

$$\lim_{\varepsilon \to 0} F_0(e^{i(x+i\varepsilon)}) = i \cot\Big(\frac{\lambda}{2}\Big).$$

Remark 3.3. Given the relation $F_\lambda(z) = 1 + 2zR_\lambda(z)$, with $z = e^{i(x+i\varepsilon)}$, if

$$\lim_{\varepsilon \to 0} \frac{F_0(e^{i(x+i\varepsilon)}) - 1}{2} = \frac{e^{i\lambda}}{1 - e^{i\lambda}},$$

then

$$\begin{aligned}\lim_{\varepsilon \to 0} F_0(e^{i(x+i\varepsilon)}) &= \frac{2e^{i\lambda}}{1 - e^{i\lambda}} + 1 = \frac{1 + e^{i\lambda}}{1 - e^{i\lambda}} \\ &= \frac{e^{-i\frac{\lambda}{2}} + e^{i\frac{\lambda}{2}}}{e^{-i\frac{\lambda}{2}} - e^{i\frac{\lambda}{2}}} = i \cot\Big(\frac{\lambda}{2}\Big).\end{aligned}$$

We need the following results related to the spectrum $\sigma(U)$.

Theorem 3.4. *Given the unitary operator U, the set*

$$S = \{e^{ix} \mid G(x) = \infty\}$$

is a dense G_δ in $\sigma(U)$.

Theorem 3.5. $\{\lambda \mid U_\lambda$ *does not have eigenvalues in* $\sigma(U)\}$ *is a dense* G_δ *set in* $[0, 2\pi)$.

3.1. Proof of Theorem 3.4

The set $S = \{e^{ix} \mid G(x) = \infty\}$ is dense in $\sigma(U)$. To prove this, we first recall that $G(x)$ is given by

$$G(x) = \int_0^{2\pi} \mathrm{d}\mu(t)\Big(\sin^2\Big(\frac{x-t}{2}\Big)\Big)^{-1},$$

and that

$$\Re(F(re^{ix})) = \int_0^{2\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos(x-t)} \mathrm{d}\mu(t).$$

Now, suppose that $G(x) < \infty$ over an interval (a, b). Consequently,

$$\lim_{r \to 1} \Re(F(re^{ix})) = 0.$$

Then, using the fact that $\lim_{r\to 1} F(re^{ix})$ exists for Lebesgue a.e. $t \in [0, 2\pi)$ and

$$\mathrm{d}\mu(t) = f(t)\frac{\mathrm{d}t}{2\pi} + \mathrm{d}\mu_{\mathrm{s}}(t),$$

where $f(x) = \lim_{r\to 1} \Re(F(re^{ix}))$ and $\mathrm{d}\mu_s$ is supported on

$$\{x \mid \lim_{r\to 1} \Re(F(re^{ix})) = \infty\},$$

we obtain that $\mu(a,b) = 0$; that is, $(a,b) \cap \operatorname{supp}(\mathrm{d}\mu) = \emptyset$. Therefore,

$$S = \{e^{ix} \mid G(x) = \infty\}$$

is dense in $\sigma(U)$.

Next, we show that $S = \{e^{ix} \mid G(x) = \infty\}$ is a G_δ. To do this, introduce

$$G^m(x) = \int_0^{2\pi} \mathrm{d}\mu(t)\Big(\frac{1}{m^2} + \sin^2\Big(\frac{x-t}{2}\Big)\Big)^{-1},$$

which is a C^∞ function and $G(x) = \sup_m G^m(x)$. Then,

$$\begin{aligned}\{e^{ix} \mid G(x) = \infty\} &= \{e^{ix} \mid \text{for every } n, \text{ there exists } m \text{ such that } G^m(x) > n\}\\ &= \bigcap_n \bigcup_m \{e^{ix} \mid G^m(x) > n\}\end{aligned}$$

is a G_δ.

Remark 3.6. According to Proposition 3.2, only values of e^{ix} with $G(x) < \infty$ can serve as eigenvalues of U_λ. Note that if the spectrum $\sigma(U)$ is a perfect set without isolated points, Theorem 3.5 states that the points e^{iy} with $G(y) = \infty$ are locally nonenumerable within $\sigma(U)$.

On the other hand, it is evident that $\{e^{ix} \mid G(x) = \infty\} \subset \sigma(U)$. And Theorem 3.5 reveals that $\{e^{ix} \mid G(x) < \infty\}$ has an empty interior within $\sigma(U)$. Moreover, this interior is also empty within the circle $S^1 = \{e^{it} \mid 0 \le t < 2\pi\}$. In fact, the theorem provides a stronger result, suggesting that $\sigma(U_\lambda)$ could have an empty interior in S^1. Furthermore, if $G(x) < \infty$, it implies that the integral

$$F(x) = \int_0^{2\pi} \frac{(1-r^2) + 2ri\sin(x-t)}{1 + r^2 - 2r\cos(x-t)} \mathrm{d}\mu(t)$$

converges absolutely, and $F(x)$ is purely imaginary as $r \to 1$.

Lemma 3.7. *Let B be a subset of $\mathbb{R}$ that is nowhere dense, and let $H : B \to \mathbb{R}$ be a function satisfying, for $x < y$,*

$$\alpha(y-x) < H(y) - H(x) < \beta(y-x) \tag{3.2}$$

with fixed $\alpha, \beta > 0$. Then, the image of H is a set that is nowhere dense.

Proof. See [7, Lemma 3.2]. ∎

Theorem 3.8. *The set $S = \{F(x) \mid G(x) < \infty$ and $x \in \operatorname{supp}(\mu)\}$ is a countable union of sets that are nowhere dense in $[0, 2\pi)$ ($\operatorname{int}(\overline{S}) = \emptyset$).*

Proof. See [7, Lemma 3.1]. ∎

3.2. Proof of Theorem 3.5

Let $M : (0, 2\pi) \to I$ be the function (which is a homeomorphism), where $I = \{a + ib \mid a = 0\}$ (the imaginary axis), defined by $M(\lambda) = i\cot(\frac{\lambda}{2})$. Then, by Theorem 3.8, we have that the set

$$\left\{\lambda \mid \text{there exists } x \text{ with } G(x) < \infty,\ e^{ix} \in \sigma(U),\ M(x) = i\cot\left(\frac{\lambda}{2}\right)\right\}$$

is a countable union of nowhere dense subsets. Therefore, its complement is a dense set by the Baire category theorem. But, by Proposition 3.2, this complement is exactly

$$\{\lambda \mid U_\lambda \text{ has no eigenvalues in } \sigma(U)\},$$

which is dense. Furthermore, by [5, 8], this set is also a G_δ.

As a consequence of these results, we have the following corollary.

Corollary 3.9. *If $\{e^{it} \mid a < t < b\} \subset \sigma(U)$ and U does not have absolutely continuous spectrum, then, for a generic set of λ in $[0, 2\pi)$, U_λ has purely singular continuous spectrum in $\{e^{it} \mid a < t < b\}$.*

Proof. Combine Theorem 3.5 and the multiplicative version of Birman–Krein theorem (since $(e^{i\lambda} - 1)P_\phi$ in (3.1) is trace class). ∎

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References

[1] V. R. Bazao, C. R. de Oliveira, and P. A. Diaz, On the Birman–Krein theorem. *C. R. Math. Acad. Sci. Paris* **361** (2023), 1081–1086 Zbl 07755463 MR 4659068

[2] M. Š. Birman and M. G. Kreĭn, On the theory of wave operators and scattering operators. *Dokl. Akad. Nauk SSSR* **144** (1962), 475–478 Zbl 0196.45004 MR 0139007

[3] G. Casati and L. Molinari, "Quantum chaos" with time-periodic Hamiltonians. *Progr. Theoret. Phys. Suppl.* (1989), no. 98, 287–322 MR 1033467

[4] M. Combescure, Spectral properties of a periodically kicked quantum Hamiltonian. *J. Statist. Phys.* **59** (1990), no. 3-4, 679–690 Zbl 0713.58044 MR 1063177

[5] S. De Bièvre and G. Forni, Transport properties of kicked and quasiperiodic Hamiltonians. *J. Statist. Phys.* **90** (1998), no. 5-6, 1201–1223 Zbl 0923.47042 MR 1628308

[6] C. R. de Oliveira, *Intermediate spectral theory and quantum dynamics*. Prog. Math. Phys. 54, Birkhäuser, Basel, 2009 Zbl 1165.47001 MR 2723496

[7] R. Del Rio, N. Makarov, and B. Simon, Operators with singular continuous spectrum. II. Rank one operators. *Comm. Math. Phys.* **165** (1994), no. 1, 59–67 Zbl 1055.47500 MR 1298942

[8] B. Simon, Operators with singular continuous spectrum. I. General operators. *Ann. of Math. (2)* **141** (1995), no. 1, 131–145 Zbl 0851.47003 MR 1314033

[9] B. Simon, Analogs of the m-function in the theory of orthogonal polynomials on the unit circle. *J. Comput. Appl. Math.* **171** (2004), no. 1-2, 411–424 Zbl 1055.33010 MR 2077215

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