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Multiple homoclinic solutions for nonsmooth second-order differential systems

Mohsen Timoumi

Abstract. In the present paper, we obtain infinitely many pairs of homoclinic solutions for a class of nonsmooth second-order differential systems when the energy functional associated is not continuously differentiable and does not satisfy the Palais–Smale condition.

1. Introduction

Consider the following second-order differential system:

$$\ddot{u}(t) + q(t)\dot{u}(t) + \nabla V(t, u(t)) = 0, \quad t \in \mathbb{R}, \tag{$\mathcal{DV}$}$$

where $q \in C(\mathbb{R}, \mathbb{R})$ and $V: \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a continuous function, differentiable in the second variable with continuous derivative $\nabla V(t, x) = \frac{\partial V}{\partial x}(t, x)$. As usual, we say that a solution $u \in C^2(\mathbb{R}, \mathbb{R}^N)$ of ($\mathcal{DV}$) is homoclinic (to 0) if $u(t) \to 0$ and $\dot{u}(t) \to 0$ as $|t| \to \infty$. Moreover, if $u(t) \neq 0$, u is called a nontrivial homoclinic solution.

When $q(t) = 0$, formally, system ($\mathcal{DV}$) reduces to the following classical Hamiltonian system:

$$\ddot{u}(t) + \nabla V(t, u(t)) = 0, \quad t \in \mathbb{R}. \tag{$\mathcal{HS}$}$$

Over the forty past years, with the aid of critical point theory and variational methods (see for example [16]), the existence and multiplicity of homoclinic solutions for ($\mathcal{HS}$) have been extensively investigated in the literature when $V(t, x)$ takes the form

$$V(t, x) = -\frac{1}{2}L(t)x \cdot x + W(t, x) \tag{1.1}$$

with $L \in C(\mathbb{R}, \mathbb{R}^{N^2})$ and $W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$; see for example [1–3, 7, 8, 10–15, 17–22, 28, 29, 31–33, 35–37], but we do not even try to review the large bibliography. Here, "$\cdot$" denotes the usual inner product in $\mathbb{R}^N$ and the associated norm will be denoted by $|\,.\,|$.

For the general case where $q(t) \neq 0$, in the last two decades, the existence and multiplicity of homoclinic solutions for ($\mathcal{DV}$) have been studied by a few mathematicians via

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critical point theory and variational methods; see [4, 5, 9, 23–27, 30, 34, 38] and the references listed therein. In all the previous papers, the potential V takes the form (1.1) where L and W satisfy suitable conditions, and the energy functional associated to system ($\mathcal{D}\mathcal{V}$) defined on a well-chosen convenient space by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}} e^{Q(t)}[|\dot{u}(t)|^2 + L(t)u(t)\cdot u(t)]\,dt - \int_{\mathbb{R}} e^{Q(t)}W(t,u(t))\,dt$$

is continuously differentiable. The difficulties encountered in all these papers are the Sobolev embedding compactness problem and the Palais–Smale condition problem. To escape from it, several authors have imposed suitable coercivity conditions on L and growth constraints on ∇W for which $I \in C^1(E,\mathbb{R})$ and the critical points of I on E are exactly the homoclinic solutions of system ($\mathcal{D}\mathcal{V}$). Note that the conditions used in the well-known papers do not cover some nonlinearity like

$$V(t,x) = -\frac{1}{2}\Big[1 + \frac{1}{2}\cos\Big(\frac{1}{|x|^\gamma}\Big)\Big]|x|^2 + d(t)|x|^\sigma, \tag{1.2}$$

where $0 < \gamma < 1$, $d \in C(\mathbb{R},\mathbb{R}^+) \cap L^\alpha(\mathbb{R})$ for $1 \le \alpha \le \frac{2}{2-\sigma}$ and $d \not\equiv 0$. It is easy to see that $V \in C^1(\mathbb{R},\mathbb{R}^N)$. However, let $q \in C(\mathbb{R},\mathbb{R})$ be such that $Q(t) = \int_0^t q(s)\,ds$ is bounded; then for $u \in H^1_Q(\mathbb{R})$ and v an indefinitely differentiable function from $\mathbb{R}$ into $\mathbb{R}^N$ with compact support, the derivative of the energy functional J associated to ($\mathcal{D}\mathcal{V}$),

$$J(u) = \frac{1}{2}\int_{\mathbb{R}} e^{Q(t)}|\dot{u}(t)|^2\,dt + \int_{\mathbb{R}} e^{Q(t)}K(t,u(t))\,dt - \int_{\mathbb{R}} e^{Q(t)}W(t,u(t))\,dt,$$

is

$$\begin{aligned} J'(u)v = {} & \int_{\mathbb{R}} e^{Q(t)}\dot{u}(t)\cdot\dot{v}(t)\,dt + \int_{\mathbb{R}} e^{Q(t)}u(t)\cdot v(t)\,dt \\ & + \frac{1}{2}\int_{\mathbb{R}} e^{Q(t)}\cos(|u(t)|^{-\gamma})u(t)\cdot v(t)\,dt \\ & + \frac{\gamma}{4}\int_{\mathbb{R}} e^{Q(t)}|u(t)|^{-\gamma}\sin(|u(t)|^{-\gamma})u(t)\cdot v(t)\,dt \\ & - \sigma\int_{\mathbb{R}} e^{Q(t)}d(t)|u(t)|^{\sigma-2}u(t)\cdot v(t)\,dt. \end{aligned}$$

Let

$$w(t) = \frac{1}{1 + |t|^{\frac{1}{2-\gamma}}}, \quad u(t) = (w(t), 0, \dots, 0), \quad v(t) = (w(t)\sin(w^{-\gamma}(t)), 0, \dots, 0).$$

An easy computation shows that $u, v \in H^1_Q(\mathbb{R})$. On the other hand, we have

$$\begin{aligned} & \int_{\mathbb{R}} e^{Q(t)}|u(t)|^{-\gamma}\sin(|u(t)|^{-\gamma})u(t)\cdot v(t)\,dt \\ & \quad = \int_{\mathbb{R}} e^{Q(t)}|w(t)|^{2-\gamma}\sin^2(|w(t)|^{-\gamma})\,dt \end{aligned}$$

$$
\begin{aligned}
&\geq m_0 \int_{\mathbb{R}} |w(t)|^{2-\gamma} \sin^2(|w(t)|^{-\gamma})\, dt \\
&= 2m_0 \frac{2-\gamma}{\gamma} \int_1^{\infty} s^{-\frac{1}{\gamma}} (s^{\frac{1}{\gamma}} - 1)^{1-\gamma} \sin^2(s)\, ds \\
&= +\infty.
\end{aligned}
$$

Therefore, J is not continuously differentiable on $H^1_Q(\mathbb{R})$.

In this note and for the first time, we are interested in the existence of infinitely many pairs of homoclinic solutions for ($\mathcal{DV}$) when the function V satisfies some conditions, which covers the cases as in (1.2). More precisely, we will study the cases when the quadratic form $\frac{1}{2}L(t)x \cdot x$ is replaced by a general nonsmooth function $K(t,x)$ and no growth constraints are imposed on ∇V. To the best of our knowledge, it seems that no similar results are obtained in the literature for nonsmooth damped vibration systems. Taking $V(t,x) = -K(t,x) + W(t,x)$, where $K, W : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are continuous functions, differentiable in the second variable with continuous derivatives respectively $\nabla K(t,x)$ and $\nabla W(t,x)$, we obtain the following results.

Theorem 1.1. *Assume that q and W satisfy*

(Q) $q \in C(\mathbb{R},\mathbb{R})$ *and*

$$Q(t) = \int_0^t q(s)\, ds$$

is bounded from below with $m_0 = \inf_{t\in\mathbb{R}} e^{Q(t)}$;

(H_1) *there exist constants* $1 < \nu \leq 2$ *and* $a > 0$ *such that*

$$K(t,x) \geq a|x|^{\nu} \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N;$$

(H_2) *there exist* $\sigma \in]1,\nu[$, $1 \leq \alpha \leq \frac{2}{2-\sigma}$ *and* $d \in L^{\alpha}_Q(\mathbb{R},\mathbb{R}^+)$ *such that*

$$|W(t,x)| \leq d(t)|x|^{\sigma} \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N;$$

(H_3) $V(t,-x) = V(t,x)\ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N$;

(H_4) *there exist constants* $\tau \in]1,2[$ *and* $l \in \mathbb{R}^*_+ \cup \{+\infty\}$ *such that*

$$\lim_{|x|\to 0} \frac{V(t,x)}{|x|^{\tau}} = l \quad \text{uniformly in } t \in \mathbb{R}.$$

Then system ($\mathcal{DV}$) *possesses infinitely many pairs of nontrivial homoclinic solutions.*

Theorem 1.2. *Assume that* (Q), (H_1), (H_2), (H_3) *and the following condition are satisfied:*

$$\lim_{|x|\to 0} \frac{V(t,x)}{|x|^2} = +\infty \quad \text{uniformly in } t \in \mathbb{R}. \tag{H_4'}$$

Then system ($\mathcal{DV}$) *possesses infinitely many pairs of nontrivial homoclinic solutions.*

Theorem 1.3. *Assume that* (Q), (H$_1$), (H$_2$), (H$_3$) *and the following conditions are satisfied:*

(H$'_1$) *there exist positive constants* b, R *such that* $K(t,x) \leq b|x|^\nu \ \forall t \in \mathbb{R}$, $|x| \leq R$;

(H$_5$) *there exist constants* $\tau \in]1,\nu[$, $l \in \mathbb{R}^*_+ \cup \{+\infty\}$, $t_0 \in \mathbb{R}$ *and* $r > 0$ *such that*

$$\lim_{|x|\to 0} \frac{W(t,x)}{|x|^\tau} = l \quad \textit{uniformly in } t \in]t_0 - r, t_0 + r[.$$

Then system $(\mathcal{DV})$ *possesses infinitely many pairs of nontrivial homoclinic solutions.*

Theorem 1.4. *Assume that* (Q), (H$_1$), (H$'_1$), (H$_2$), (H$_3$) *and the following condition are satisfied:*

(H$'_5$) *there exist constants* $t_0 \in \mathbb{R}$ *and* $r > 0$ *such that*

$$\lim_{|x|\to 0} \frac{W(t,x)}{|x|^\nu} = +\infty \quad \textit{uniformly in } t \in]t_0 - r, t_0 + r[.$$

Then system $(\mathcal{DV})$ *possesses infinitely many pairs of nontrivial homoclinic solutions.*

Remark 1.1. If $Q(t) = \int_0^t q(s)\,ds \to +\infty$ as $|t| \to \infty$, an homoclinic solution of $(\mathcal{DV})$ is called a fast homoclinic solution.

Remark 1.2. In assumptions (H$_1$)–(H$_5$), (H$'_1$) and (H$'_5$), the nonlinearity ∇V does not verify any growth constraints, so the energy functional associated to $(\mathcal{DV})$ is continuous but neither continuously differentiable nor does it satisfy the Palais–Smale condition as we saw above.

2. Preliminaries

In order to prove our main results, we recall some definitions and basic results. Let X be a Banach space and X' its dual space. The weak convergence in X is denoted by "$\rightharpoonup$". Let J be a functional defined on X. Then J is said to be weakly sequentially lower semi-continuous if $\liminf_{n\to\infty} J(u_n) \geq J(u)$ for any $u \in X$ and $(u_n) \subset X$ satisfying $u_n \rightharpoonup u$.

Let J be a continuous functional defined on X and let E be a dense subspace of X; we say that J is E-differentiable if

(a) for all $u \in X$ and $v \in E$, the derivative of J at u in the direction v, denoted by $\langle J'(u), v\rangle$, exists, that is,

$$\langle J'(u), v\rangle = \lim_{s\to 0} \frac{J(u+sv) - J(u)}{s};$$

(b) the mapping J' satisfies that

(i) $v \mapsto \langle J'(u), v\rangle$ is linear in E for all $u \in X$,

(ii) $u \mapsto \langle J'(u), v\rangle$ is continuous in X for all $v \in E$, that is,

$$\langle J'(u_n), v\rangle \to \langle J'(u), v\rangle \quad \text{as } u_n \to u \quad \text{in } X.$$

A point $u \in X$ is said to be a critical point of J if $|J'(u)| = 0$, where

$$|J'(u)| = \sup\{\langle J'(u), v\rangle / v \in E,\ \|v\| = 1\}$$

and $\|\,.\,\|$ denotes the norm in X.

Now we are in position to recall a variant of Clark's Theorem due to [6].

Lemma 2.1. *Let X be a separable and reflexive Banach space with norm $\|\,.\,\|$ and let E be a dense subspace of X. Assume that J is a continuous functional defined on X which is E-differentiable. Suppose that J satisfies the following conditions:*

(A_1) *J is an even functional, i.e., $J(-u) = J(u)$ for every $u \in X$, and it is bounded from below;*

(A_2) *if $u \in X$, $(u_n) \subset X$, $|J'(u_n)| \to 0$ and $u_n \rightharpoonup u$ as $n \to \infty$, then $|J'(u)| = 0$;*

(A_3) *J is weakly sequentially lower semicontinuous;*

(A_4) *the set $\{u \in X / J(u) \le J(0)\}$ is bounded in X;*

(A_5) *for every positive integer k, there exist a k-dimensional subspace X_k of X and $\rho_k > 0$ such that $\sup_{X_k \cap S_{\rho_k}} J < J(0)$, where $S_\rho = \{u \in X / \|u\| = \rho\}$.*

Then J has infinitely many pairs of critical points $(\pm u_k)_{k\in\mathbb{N}}$ satisfying

$$J(\pm u_k) \le J(0), \quad u_k \ne 0 \text{ for } k \in \mathbb{N} \quad \text{and} \quad u_k \rightharpoonup 0 \text{ as } k \to \infty.$$

Remark 2.1. Assumption (A_2) can be deduced from the following assumption:

(A_2') if $u \in X$, $(u_n) \subset X$ and $u_n \rightharpoonup u$ in X as $n \to \infty$, then

$$\langle J'(u_n), v\rangle \to \langle J'(u), v\rangle \quad \forall v \in E.$$

Therefore, the result of Lemma 2.1 is true if assumption (A_2) is replaced by (A_2').

In the following, we shall use $L^2_Q(\mathbb{R})$ to denote the Hilbert space of measurable functions from $\mathbb{R}$ into $\mathbb{R}^N$ under the inner product

$$\langle u, v\rangle_{L^2_Q} = \int_{\mathbb{R}} e^{Q(t)} u(t).v(t)\, dt$$

and the induced norm

$$\|u\|_{L^2_Q} = \left(\int_{\mathbb{R}} e^{Q(t)} |u(t)|^2\, dt\right)^{\frac{1}{2}}.$$

Similarly, $L^s_Q(\mathbb{R})$ $(2 \le s < \infty)$ denotes the Banach space of functions on $\mathbb{R}$ with values in $\mathbb{R}^N$ under the norm

$$\|u\|_{L^s_Q} = \left(\int_{\mathbb{R}} e^{Q(t)} |u(t)|^s\, dt\right)^{\frac{1}{s}}$$

and $L^\infty_Q(\mathbb{R})$ denotes the Banach space of functions on $\mathbb{R}$ with values in $\mathbb{R}^N$ under the norm

$$\|u\|_{L^\infty_Q} = \text{ess}\sup\{e^{\frac{Q(t)}{2}} |u(t)| / t \in \mathbb{R}\}.$$

Let

$$H_Q^1(\mathbb{R},\mathbb{R}^N) = \{u \in L_Q^2(\mathbb{R})/\dot{u} \in L_Q^2(\mathbb{R})\}.$$

Then $H_Q^1(\mathbb{R})$ equipped with the following inner product and norm is a Hilbert space:

$$\langle u, v\rangle = \int_{\mathbb{R}} e^{Q(t)}[\dot{u}(t)\cdot\dot{v}(t) + u(t)\cdot v(t)]\,dt, \quad u, v \in H_Q^1(\mathbb{R}),$$
$$\|u\| = \langle u, u\rangle^{\frac{1}{2}}, \qquad u \in H_Q^1(\mathbb{R}).$$

It is well known that $H_Q^1(\mathbb{R})$ is continuously embedded into $L_Q^s(\mathbb{R},\mathbb{R}^N)$ for all $2 \le s \le \infty$, and then there exists a constant $\eta_s > 0$ such that

$$\|u\|_{L_Q^s} \le \eta_s\|u\| \quad \forall u \in H_Q^1(\mathbb{R}).$$

3. Proof of theorems

Consider the functional J associated with equation ($\mathcal{DV}$) defined on the space $X = H_Q^1(\mathbb{R})$, introduced in Section 2, by

$$J(u) = \frac{1}{2}\int_{\mathbb{R}} e^{Q(t)}|\dot{u}(t)|^2\,dt + \int_{\mathbb{R}} e^{Q(t)}K(t,u(t))\,dt - \int_{\mathbb{R}} e^{Q(t)}W(t,u(t))\,dt.$$

Let $E = \mathcal{D}(\mathbb{R})$ be the space of indefinitely differentiable functions from $\mathbb{R}$ into $\mathbb{R}^N$ with compact support. Then J is E-differentiable and

$$\langle J'(u), v\rangle = \int_{\mathbb{R}} e^{Q(t)}\dot{u}(t)\cdot\dot{v}(t)\,dt + \int_{\mathbb{R}} e^{Q(t)}\nabla K(t,u(t))\cdot v(t)\,dt$$
$$- \int_{\mathbb{R}} e^{Q(t)}\nabla W(t,u(t))\cdot v(t)\,dt \quad \forall u \in X,\ v \in E.$$

Step 1. J is even. If $\alpha = 1$, one gets

$$\begin{aligned}
J(u) &\ge \frac{1}{2}\int_{\mathbb{R}} e^{Q(t)}|\dot{u}(t)|^2\,dt + a\int_{\mathbb{R}} e^{Q(t)}|u(t)|^\nu\,dt - \|u\|_{L^\infty}^\sigma \int_{\mathbb{R}} e^{Q(t)}d(t)\,dt \\
&\ge \frac{1}{2}\int_{\mathbb{R}} e^{Q(t)}|\dot{u}(t)|^2\,dt + a\Big(\frac{\sqrt{m_0}}{\eta_\infty}\Big)^{2-\nu}\|u\|^{\nu-2}\int_{\mathbb{R}} e^{Q(t)}|u(t)|^2\,dt \\
&\quad - \Big(\frac{\eta_\infty}{\sqrt{m_0}}\Big)^\sigma\|u\|^\sigma\int_{\mathbb{R}} e^{Q(t)}d(t)\,dt \\
&\ge \min\Big\{\frac{1}{2}, a\Big(\frac{\sqrt{m_0}}{\eta_\infty}\Big)^{2-\nu}\|u\|^{\nu-2}\Big\}\|u\|^2 - \Big(\frac{\eta_\infty}{\sqrt{m_0}}\Big)^\sigma\int_{\mathbb{R}} e^{Q(t)}d(t)\,dt\,\|u\|^\sigma. \qquad (3.1)
\end{aligned}$$

If $1 < \alpha \le \frac{2}{2-\sigma}$, then $\frac{\sigma\alpha}{\alpha-1} \ge 2$ and we have

$$\begin{aligned}
J(u) &\ge \frac{1}{2}\int_{\mathbb{R}} e^{Q(t)}|\dot{u}(t)|^2\,dt + a\int_{\mathbb{R}} e^{Q(t)}|u(t)|^\nu\,dt \\
&\quad - \Big(\int_{\mathbb{R}} e^{Q(t)}d^\alpha(t)\,dt\Big)^{\frac{1}{\alpha}}\Big(\int_{\mathbb{R}} e^{Q(t)}|u(t)|^{\frac{\sigma\alpha}{\alpha-1}}\,dt\Big)^{\frac{\alpha-1}{\alpha}}
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1}{2}\int_{\mathbb{R}} e^{Q(t)}|\dot{u}(t)|^2\,dt + a\Big(\frac{\sqrt{m_0}}{\eta_\infty}\Big)^{2-\nu}\|u\|^{\nu-2}\int_{\mathbb{R}} e^{Q(t)}|u(t)|^2\,dt \\
&\qquad - \|d\|_{L_Q^\alpha}\eta^\sigma_{\frac{\sigma\alpha}{\alpha-1}}\|u\|^\sigma \\
&\geq \min\Big\{\frac{1}{2}, a\Big(\frac{\sqrt{m_0}}{\eta_\infty}\Big)^{2-\nu}\|u\|^{\nu-2}\Big\}\|u\|^2 - \Big(\int_{\mathbb{R}} e^{Q(t)} d^\alpha(t)\,dt\Big)^{\frac{1}{\alpha}}\eta^\sigma_{\frac{\sigma\alpha}{\alpha-1}}\|u\|^\sigma. \quad (3.2)
\end{aligned}
$$

For $\|u\| \geq (2a)^{\frac{1}{2-\nu}}\frac{\sqrt{m_0}}{\eta_\infty}$, inequalities (3.1) and (3.2) imply, for a positive constant c_1,

$$
J(u) \geq a\Big(\frac{\sqrt{m_0}}{\eta_\infty}\Big)^{2-\nu}\|u\|^\nu - c_1\|u\|^\sigma,
$$

Therefore, J is coercive and bounded from below because $\sigma < \nu$. Hence (A_1) and (A_4) are satisfied.

Step 2. Let $u_n \rightharpoonup u$ in X and $v \in E$. Then

$$
\begin{aligned}
&\int_{\mathbb{R}} e^{Q(t)}\dot{u}_n(t)\cdot\dot{v}(t)\,dt + \int_{\mathbb{R}} e^{Q(t)}u_n(t)\cdot v(t)\,dt \\
&\qquad \to \int_{\mathbb{R}} e^{Q(t)}\dot{u}(t)\cdot\dot{v}(t)\,dt + \int_{\mathbb{R}} e^{Q(t)}u(t)\cdot v(t)\,dt. \quad (3.3)
\end{aligned}
$$

Since $v \in \mathcal{D}(\mathbb{R})$, by the Lebesgue convergence theorem,

$$
\begin{aligned}
&-\int_{\mathbb{R}} e^{Q(t)}u_n(t)\cdot v(t)\,dt - \int_{\mathbb{R}} e^{Q(t)}\nabla V(t,u_n(t))\cdot v(t)\,dt \\
&\qquad \to -\int_{\mathbb{R}} e^{Q(t)}u(t)\cdot v(t)\,dt - \int_{\mathbb{R}} e^{Q(t)}\nabla V(t,u_n(t))\cdot v(t)\,dt. \quad (3.4)
\end{aligned}
$$

Combining (3.3) and (3.4) yields $\langle J'(u_n), v\rangle \to \langle J'(u), v\rangle$. Therefore, condition (A_2') holds for J.

Step 3. Moreover, if $u_n \rightharpoonup u$ in X, then by [18, Theorem 1.6], we have

$$
\liminf_{n\to\infty}\frac{1}{2}\int_{\mathbb{R}} e^{Q(t)}|\dot{u}_n(t)|^2\,dt \geq \frac{1}{2}\int_{\mathbb{R}} e^{Q(t)}|\dot{u}(t)|^2\,dt.
$$

Applying Fatou's lemma and using (H_1) leads to

$$
\liminf_{n\to\infty}\int_{\mathbb{R}} e^{Q(t)}K(t,u_n(t))\,dt \geq \int_{\mathbb{R}} e^{Q(t)}K(t,u(t))\,dt,
$$

while (H_2) implies

$$
\lim_{n\to\infty}\int_{\mathbb{R}} e^{Q(t)}W(t,u_n(t))\,dt \geq \int_{\mathbb{R}} e^{Q(t)}W(t,u(t))\,dt.
$$

Hence J satisfies (A_3). To complete the proof of our results, it remains to verify condition (A_5).

3.1. Proof of Theorem 1.1

For any $k \in \mathbb{N}$, let X_k be a k-dimensional subspace of $\mathcal{D}(\mathbb{R})$. Since all norms in a finite-dimensional space are equivalent, then for any positive integer k, there exists a positive constant γ_k such that

$$\|u\| \le \gamma_k \|u\|_{L^2_Q} \quad \forall u \in X_k.$$

By (H$_4$), for $0 < l_0 < l$, there exists a constant $R_0 > 0$ such that

$$V(t,x) \ge l_0 |x|^\tau \quad \forall t \in \mathbb{R},\ |x| \le R_0. \tag{3.5}$$

For $u \in X_k$ with $\|u\| \le \frac{R_0\sqrt{m_0}}{\eta_\infty}$, we have

$$\|u\|_{L^\infty} \le \frac{\|u\|_{L^\infty_Q}}{\sqrt{m_0}} \le \frac{\eta_\infty}{\sqrt{m_0}}\|u\| \le R_0. \tag{3.6}$$

Combining (3.5) and (3.6) yields

$$\begin{aligned} J(u) &= \frac{1}{2}\int_{\mathbb{R}} e^{Q(t)}|\dot u(t)|^2\,dt - \int_{\mathbb{R}} e^{Q(t)} V(t,u(t))\,dt \\ &\le \frac{1}{2}\|u\|^2 - l_0 \int_{\mathbb{R}} e^{Q(t)}|u(t)|^\tau\,dt \\ &\le \frac{1}{2}\|u\|^2 - l_0 \frac{1}{\gamma_k^2}\Big(\frac{\sqrt{m_0}}{\sqrt{\eta_\infty}}\Big)^{2-\tau}\|u\|^\tau. \end{aligned}$$

Choosing

$$\rho_k = \min\Big\{R_0, \Big(\frac{l_0}{\gamma_k^2}\Big)^{\frac{1}{2-\tau}}\Big\}\frac{\sqrt{m_0}}{\eta_\infty},$$

we obtain

$$\sup_{u\in X_k\cap S_{\rho_k}} J(u) \le -\frac{1}{2}\Big(\frac{l_0}{\gamma_k^2}\Big)^{\frac{2}{2-\tau}}\Big(\frac{\sqrt{m_0}}{\eta_\infty}\Big)^2 < 0, \quad \text{where } S_{\rho_k} = \{u \in X : \|u\| = \rho_k\}.$$

Therefore, (A$_5$) is satisfied. According to Lemma 2.1, J possesses infinitely many pairs of critical points $\pm u_k$, $k \in \mathbb{N}$, satisfying

$$J(\pm u_k) \le J(0), \quad u_k \ne 0 \text{ for } k \in \mathbb{N} \quad \text{and} \quad u_k \rightharpoonup 0 \text{ as } k \to \infty.$$

Therefore, system ($\mathcal{DV}$) has infinitely many pairs of nontrivial homoclinic solutions.

3.2. Proof of Theorem 1.2

For any $k \in \mathbb{N}$, let X_k be as above and $M > \frac{\gamma_k^2}{2}$. By assumption (H$_4'$), there exists a constant $R_k > 0$ such that

$$V(t,x) \ge M|x|^2 \quad \forall t \in \mathbb{R},\ |x| \le R_k. \tag{3.7}$$

Let $u \in X_k$ with $\|u\| \le \frac{R_k}{\eta_\infty}\sqrt{m_0} = \rho_k$; we have $\|u\|_{L^\infty} \le R_k$. Hence (3.7) yields, for $\|u\| = \rho_k$,

$$\begin{aligned} J(u) &= \frac{1}{2}\int_{\mathbb{R}} e^{Q(t)}|\dot{u}(t)|^2\,dt - \int_{\mathbb{R}} e^{Q(t)}V(t,u(t))\,dt \\ &\le \frac{1}{2}\|u\|^2 - M\|u\|^2_{L^2_Q} \le \frac{1}{2}\|u\|^2 - \frac{M}{\gamma_k^2}\|u\|^2 \\ &\le \Big(\frac{1}{2} - \frac{M}{\gamma_k^2}\Big)\rho_k^2 < 0. \end{aligned}$$

Therefore, (A$_5$) is satisfied and we conclude as in the proof of Theorem 1.1 that system ($\mathcal{DV}$) has infinitely many pairs of nontrivial homoclinic solutions.

3.3. Proof of Theorem 1.3

For any $k \in \mathbb{N}$, let X_k be a k-dimensional subspace of $\mathcal{D}(]t_0 - r, t_0 + r[)$. As above, for any positive integer k, there exists a positive constant γ_k such that

$$\|u\| \le \gamma_k \|u\|_{L^2_Q} \quad \forall u \in X_k.$$

By (H$_5$), for $0 < l_0 < l$, there exists a constant $0 < R_1 < R$ such that

$$W(t,x) \ge l_0|x|^\tau \quad \forall t \in]t_0 - r, t_0 + r[,\ |x| \le R_1. \tag{3.8}$$

For $u \in X_k$ with

$$\|u\| = \min\Big\{R_1, \Big(\frac{l_0}{2b}\Big)^{\frac{1}{\nu-\tau}}\Big\}\frac{\sqrt{m_0}}{\eta_\infty},$$

we have $\|u\|_{L^\infty} \le R_1$. Therefore, (3.8) and (H$_1'$) imply

$$\begin{aligned} J(u) &= \frac{1}{2}\int_{\mathbb{R}} e^{Q(t)}|\dot{u}(t)|^2\,dt + \int_{\mathbb{R}} e^{Q(t)}K(t,u(t))\,dt - \int_{\mathbb{R}} e^{Q(t)}W(t,u(t))\,dt \\ &\le \frac{1}{2}\|u\|^2 + b\int_{\mathbb{R}} e^{Q(t)}|u(t)|^\nu\,dt - l_0\int_{\mathbb{R}} e^{Q(t)}|u(t)|^\tau\,dt \\ &\le \frac{1}{2}\|u\|^2 + b\Big(\frac{\eta_\infty}{\sqrt{m_0}}\Big)^{\nu-\tau}\|u\|^{\nu-\tau}\int_{\mathbb{R}} e^{Q(t)}|u(t)|^\tau\,dt - l_0\int_{\mathbb{R}} e^{Q(t)}|u(t)|^\tau\,dt \\ &\le \frac{1}{2}\|u\|^2 - \Big[l_0 - b\Big(\frac{\eta_\infty}{\sqrt{m_0}}\Big)^{\nu-\tau}\|u\|^{\nu-\tau}\Big]\int_{\mathbb{R}} e^{Q(t)}|u(t)|^\tau\,dt \\ &\le \frac{1}{2}\|u\|^2 - \Big[l_0 - b\Big(\frac{\eta_\infty}{\sqrt{m_0}}\Big)^{\nu-\tau}\|u\|^{\nu-\tau}\Big]\frac{1}{\gamma_k^2}\Big(\frac{\sqrt{m_0}}{\eta_\infty}\Big)^{2-\tau}\|u\|^\tau \\ &\le \frac{1}{2}\|u\|^2 - \frac{l_0}{2}\frac{1}{\gamma_k^2}\Big(\frac{\sqrt{m_0}}{\eta_\infty}\Big)^{2-\tau}\|u\|^\tau. \end{aligned}$$

Since $0 < \tau < 2$, we deduce that there exists a positive constant ρ_k small enough such that $J(u) < 0$ for $u \in X_k$ with $\|u\| = \rho_k$, which is (A$_5$). Therefore, system ($\mathcal{DV}$) has infinitely many pairs of nontrivial homoclinic solutions.

3.4. Proof of Theorem 1.4

For any $k \in \mathbb{N}$, let X_k be defined as in the proof of Theorem 1.3 and let $M > b$. By assumption (H$'_5$), there exists a constant $0 < R_k < R$ such that

$$W(t,x) \geq M|x|^\nu \quad \forall t \in]t_0 - r, t_0 + r[,\ |x| \leq R_k. \tag{3.9}$$

Let $u \in X_k$ with

$$\|u\| = \inf\Big\{R_k, \Big(\frac{M}{2\gamma_k^2}\Big)^{\frac{1}{2-\nu}}\Big\}\frac{\sqrt{m_0}}{\eta_\infty} = \rho_k;$$

then we have $\|u\|_{L^\infty} \leq R_k$. Hence (3.9) and (H$'_1$) yield

$$\begin{aligned}
J(u) &= \frac{1}{2}\int_{\mathbb{R}} e^{Q(t)}|\dot{u}(t)|^2\,dt + \int_{\mathbb{R}} e^{Q(t)}K(t,u(t))\,dt - \int_{\mathbb{R}} e^{Q(t)}W(t,u(t))\,dt \\
&\leq \frac{1}{2}\int_{\mathbb{R}} e^{Q(t)}|\dot{u}(t)|^2\,dt + b\int_{\mathbb{R}} e^{Q(t)}|u(t)|^\nu\,dt - M\int_{\mathbb{R}} e^{Q(t)}|u(t)|^\nu\,dt \\
&\leq \frac{1}{2}\|u\|^2 - (M-b)\int_{\mathbb{R}} e^{Q(t)}|u(t)|^\nu\,dt \\
&\leq \frac{1}{2}\|u\|^2 - \frac{M-b}{\gamma_k^2}\Big(\frac{\sqrt{m_0}}{\eta_\infty}\Big)^{2-\nu}\|u\|^\nu \\
&\leq -\Big(\frac{M-b}{\gamma_k^2}\Big)^{\frac{2}{2-\nu}}\Big(\frac{\sqrt{m_0}}{\eta_\infty}\Big)^2 < 0.
\end{aligned}$$

Condition (A$_5$) is satisfied. As above, system ($\mathcal{DV}$) has infinitely many pairs of nontrivial homoclinic solutions.

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Mohsen Timoumi
Department of Mathematics, University of Monastir, 5000 Monastir, Tunisia;
m_timoumi@yahoo.com