Z. Anal. Anwend. 43 (2024), 149–168
DOI 10.4171/ZAA/1742

Existence of solutions for a class of fractional Kirchhoff variational inequality

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Abstract. We are concerned with the following fractional Kirchhoff variational inequality:

$$(a+b[u]^2)\int_{\mathbb{R}^3}(-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}(v-u)\,dx+\int_{\mathbb{R}^3}(1+\lambda V(x))u(v-u)\,dx \\ \geq \int_{\mathbb{R}^3} f(u)(v-u)\,dx \quad \forall v\in\mathbb{K},$$

where $s\in(\frac{3}{4},1)$, $\lambda>0$. In this paper, by applying penalization techniques from Bensoussan and Lions (1978) combined with mountain pass theorem, we show the existence and concentration behavior of positive solution to the cited variational inequality. This result extend some results established by Alves, Barros and Torres [J. Math. Anal. Appl. 494 (2021)] to the fractional case.

1. Introduction

In this paper, we focus our attention on the following fractional Kirchhoff inequality, for $u\in\mathbb{K}$:

$$(a+b[u]^2)\int_{\mathbb{R}^3}(-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}(v-u)\,dx+\int_{\mathbb{R}^3}(1+\lambda V(x))u(v-u)\,dx \\ \geq \int_{\mathbb{R}^3} f(u)(v-u)\,dx \quad \forall v\in\mathbb{K}, \tag{1.1}$$

where $s\in(\frac{3}{4},1)$, $\lambda>0$,

$$[u]^2=\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(u(x)-u(y))^2}{|x-y|^{3+2s}}\,dx\,dy,$$

and the function $V:\mathbb{R}^3\to\mathbb{R}$ is a continuous potential verifying the following assumptions:

- (V_1) $V(x)\geq 0\ \forall x\in\mathbb{R}^3$;
- (V_2) there exists an open, connected and bounded domain $\Omega\subset\mathbb{R}^3$ with smooth boundary such that $\Omega:=\operatorname{int}(V^{-1}(\{0\}))\neq\emptyset$;

2020 Mathematics Subject Classification. Primary 35A15; Secondary 35J86, 49J40.
Keywords. Fractional Kirchhoff variational inequality, variational methods, critical nonlinearity.

(V$_3$) there exists $M_0 > 0$ such that the set $\mathscr{L} = \{x \in \mathbb{R}^3 : V(x) \leq M_0\}$ is nonempty and $|\mathscr{L}| < \infty$, where $|A|$ denotes the Lebesgue measure of A on $\mathbb{R}^3$.

Letting

$$E := \left\{u \in H^s(\mathbb{R}^3);\ \int_{\mathbb{R}^3} V(x)|u|^2\,dx < \infty\right\},$$

we define

$$\mathbb{K} := \{v \in E;\ v \geq \varphi \text{ a.e. in } \Omega\},$$

with $\varphi \in H^s(\mathbb{R}^3)$, $\varphi^+ \not\equiv 0$ and $\operatorname{supp}(\varphi^+) \subset \Omega$. Moreover, on the nonlinearity f, we require that

$$f(t) = \mu t^{q-1} + t^{2_s^*-1}$$

for any $t > 0$, with $4 < q < 2_s^*$, $\mu > 0$ and f vanishes in $(-\infty, 0)$.

Variational inequalities are well known in the literature of applied mathematics and lead to many applications. According to Rodrigues [32], the theory of variational inequalities born in Italy in the sixties with the work of Fichera in 1963 on the elasticity problem and the work of Stampacchia in 1964 in the frame of potential theory in connection with capacity. The classical example of a variational problem consists of an elastic membrane, with vertical displacement u on a domain Ω with $u = u_0$ along $\partial\Omega$ and it is forced to lie below some obstacle, that is, $u \leq \psi$. Then, at the equilibrium, whenever the membrane does not touch the obstacle, the elasticity provides a balance of the tension of the membrane that, geometrically, reflects into a balance of the principal curvatures of the surface described by u. On the other hand, when the membrane sticks to the obstacle, its principal curvatures are expected to adapt to those of ψ. Moreover, if an external force $-f$ is switched on, the rest configuration of the membrane will be such that the elastic tension of the membrane equilibrates the force. These physical considerations lead to the classical variational inequality

$$\int_\Omega \nabla u(\nabla v - \nabla u)\,dx \geq \int_\Omega f(x)(v(x) - u(x))\,dx \tag{1.2}$$

for any test function v, with $v \leq \psi$ and $v = u_0$ along $\partial\Omega$.

Many extensions of this problem have been considered in the literature, particularly for taking into account nonlinear elastic reactions of the membrane, non-commutative effects and nonlocal interactions [3, 4, 9, 10, 16, 25, 26, 28–30]. When replacing the local elastic reaction with a nonlocal one, with the purpose of taking into account the long-range interactions of particles, for instance, the standard Laplacian $-\Delta$ might be replaced with the fractional Laplacian $(-\Delta)^s$, equation (1.2) becomes

$$\begin{aligned}\iint_{\mathbb{R}^{2N}\setminus\mathcal{Q}} &\frac{[u(x)-u(y)][v(x)-v(y)-u(x)+u(y)]}{|x-y|^{N+2s}}\,dx\,dy\\ &\geq \int_\Omega f(x)(v(x)-u(x))\,dx,\end{aligned}$$

where $\mathcal{Q} = (\mathbb{R}^N \setminus \Omega) \times (\mathbb{R}^N \setminus \Omega)$. This kind of variational problems has been extensively studied in [9, 19, 33, 34, 37] and the references therein.

Recently, Alves, Barros and Torres Ledesma [3] pointed out there are three methods for researching variational inequalities: the nonsmooth critical point theory [12,13,23,27], the minimax principles [17,38,39] and the penalization method [8,28,29]. Moreover, they established the existence of positive solutions for the variational inequality

$$\begin{cases} u \in \mathbb{K}, \\ \displaystyle\int_{\mathbb{R}^3} \nabla u \nabla (v-u)\,dx + \int_{\mathbb{R}^3} (1+\lambda V(x)) u (v-u)\,dx \\ \qquad \displaystyle\geq \int_{\mathbb{R}^3} f(u)(v-u)\,dx \quad \forall v \in \mathbb{K}, \end{cases} \tag{1.3}$$

where the definition of these letters in this inequality is similar to our paper when $s = 1$.

Elliptic problems with critical growth like

$$-\Delta u + \lambda V(x) u = \mu u^{q-1} + u^{2^*-1} \quad \text{in } \mathbb{R}^3 \tag{1.4}$$

have been extensively studied by many authors. In fact, Alves and Barros [2] considered the existence and multiplicity for (1.4), under $V(x)$ satisfies (V_1)–(V_3). Based on the important research in [2], Alves et al. [3,4] solved (1.3) by using the penalization method under different conditions on V. Moreover, as for a class of problems where λ is large enough to get the existence result in (1.4), we refer to [7,11,14] and references therein.

The equation related to the variational inequality (1.1) is given by

$$(a + b[u]^2)(-\Delta)^s u + (1+\lambda V(x)) u = \mu |u|^{q-2} u + |u|^{2_s^*-2} u. \tag{1.5}$$

The Kirchhoff part of problem (1.5) is due to the work of Kirchhoff [24], in which, in 1883, he studied the hyperbolic equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{p_0}{h} + \frac{E}{2L} \int_0^L |u_x|^2\,dx \right) u_{xx} = 0$$

that extends the classical D'Alembert wave equation, by considering the effects of the changes in the length of the strings during the vibrations. As $s \in (\frac{3}{4}, 1)$, the fractional Laplacian $(-\Delta)^s$ is defined by

$$(-\Delta)^s \Psi(x) = C(3,s)\,\mathrm{P.V.} \int_{\mathbb{R}^3} \frac{\Psi(x) - \Psi(y)}{|x-y|^{3+2s}}\,dy, \quad \Psi \in \mathcal{S}(\mathbb{R}^3),$$

where P.V. stands for the Cauchy principle value and $\mathcal{S}$ is the Schwartz space of rapidly decaying functions. The operator $(-\Delta)^s$ can be also defined via Fourier transform and viewed as a pseudo-differential operator of symbol $|\xi|^{2s}$, see [5], that is,

$$\mathcal{F}((-\Delta)^s u)(\xi) = |\xi|^{2s} \mathcal{F}(u)(\xi) \quad \text{for } \xi \in \mathbb{R}^3,$$

where $\mathcal{F}$ denotes the Fourier transform, i.e., for function w in the Schwartz class,

$$\mathcal{F}(w)(\xi) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^N} e^{-i\xi x} w(x)\,dx.$$

When $s = 1$, Alimohammady et al. [1] studied the multiplicity of positive solutions for a Kirchhoff type problem, under conditions (V_1)–(V_3) and $a = 1$ in (1.5). Besides, Fan [15] proved multiple positive solutions of a Kirchhoff type problem on a bounded domain, when there are competing potentials in (1.5) and $s = 1$. For more Kirchhoff problems, we refer to [21,22] and references therein.

When $s \in (0, 1)$, in [18], Fiscella and Valdinoci researched the existence of solutions in (1.5), when there is no potential and replace the nonlinearity with $\lambda f(u)$ in (1.5). Moreover, Chen considered existence of a fractional p-Kirchhoff type problem, when there is λ in the generalized Choquard nonlinearity. For more fractional Kirchhoff equation, we refer to [6,31,36] and so on. Moreover, Frites and Moussaoui in [20] and Zuo et al. in [40] have explored the variational Kirchhoff inequality based on the nonsmooth critical point theory due to Szulkin.

Since we did not find in the literature any paper dealing with the existence of non-negative solutions for problem (1.1) in $\mathbb{R}^N$, motivated by the previous exposition and by the ideas of [1,4], in the present paper, we intend to prove that (1.1) has a nontrivial weak solution. To the best of our knowledge, it is the first time to study fractional Kirchhoff variational inequality by using the penalization method.

The main result of this paper can be stated as follows.

Theorem 1.1. *Suppose that V satisfies (V_1)–(V_3). Then there exist $\lambda_* > 0$, $\mu_* > 0$ and $b_* > 0$, problem (1.1) has at least one nontrivial weak solution u_λ for $\lambda \geq \lambda_*$, $\mu \geq \mu_*$ and $b < b_*$. Furthermore, for any sequence $\lambda_n \to +\infty$, there exists a subsequence, still denoted by $\{\lambda_n\}$, such that $\{u_{\lambda_n}\}$ converges strongly in $H^s(\mathbb{R}^3)$ to a function u with $u = 0$ a.e. in Ω^c, where u is a solution of the following variational inequality:*

$$\begin{aligned}\left(a + b\int_Q \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}\,dx\,dy\right)&\\ \times \int_Q \frac{(u(x)-u(y))((v-u)(x)-(v-u)(y))}{|x-y|^{3+2s}}\,dx\,dy&\\ + \int_\Omega u(v-u)\,dx&\\ \geq \int_\Omega (\mu|u|^{q-2} + |u|^{2_s^*-2})u(v-u)\,dx,&\end{aligned}$$

where $Q := \mathbb{R}^6 \setminus (\Omega^c \times \Omega^c)$ and, for every $v \in \tilde{\mathbb{K}}$, where

$$\tilde{\mathbb{K}} = \{v \in H_0^s(\Omega);\ v \geq \varphi \text{ a.e. in } \Omega\}.$$

In order to prove Theorem 1.1, we use penalization techniques due to Bensoussan and Lions [8], that is, considering problem (1.1), we introduce the penalized problem defined as

$$\begin{aligned}(a + b[u]^2)(-\Delta)^s u + (1 + \lambda V(x))u - \frac{1}{\varepsilon}(\varphi - u)^+\chi_\Omega&\\ = \mu(u^+)^{q-1} + (u^+)^{2_s^*-1} \quad \text{in } \mathbb{R}^3,&\end{aligned} \tag{1.6}$$

where $\epsilon > 0$. Associated to (1.6), we have the energy functional

$$J_{\lambda,\varepsilon}(u) = \frac{1}{2}\|u\|_\lambda^2 + \frac{b}{4}[u]^4 + \frac{1}{2\varepsilon}\int_\Omega [(\varphi - u)^+]^2\,dx - \int_{\mathbb{R}^3} F_+(u)\,dx,$$

where

$$\begin{aligned} P\colon E_\lambda &\to E'_\lambda, \\ u &\to \langle P(u), v\rangle = -\int_\Omega (\varphi - u)^+ v\,dx \end{aligned}$$

is the penalization operator. Then, by applying mountain pass arguments, we study the existence of at least one critical point of the functional $I_{\lambda,\epsilon}$ which is a weak solution of penalized problem (1.6). By taking $\epsilon = \frac{1}{n}$ with n large enough, we denote this mountain pass solution as u_n and we show that (u_n) is bounded in E_λ; hence, up to a subsequence, there is $u \in E_\lambda$ such that

$$u_n \rightharpoonup u \quad \text{in } E_\lambda.$$

This limit is such that $P(u) = 0$ and hence a weak solution of problem (1.1).

Remark 1.2. Compared with the previous results, Theorem 1.1 can be regarded as an extension of [3, Theorem 1.1] under the fractional Kirchhoff situation. Due to the Kirchhoff term and the critical term, we cannot ensure in a standard way that weak limits of a bounded Palais–Smale sequence of the energy functional are critical points of it. Besides, if we change this inequality into an equation, a penalization term will be generated naturally. Therefore, we need to get some restrictions on b for our compactness.

The paper is organized as follows. In Section 2, we give some preliminaries about definitions and properties of the function space. In Section 3, we study the penalized problem (1.6). In Section 4, we get Theorem 1.1.

Notation.
- The letter C changes from line to line.
- $B_r(x)$ denotes the ball in $\mathbb{R}^3$ centered at $x \in \mathbb{R}^3$ with radius r.
- Γ^c means $\mathbb{R}^3 \setminus \Gamma$, where $\Gamma \subset \mathbb{R}^3$.
- $|\cdot|_r$ means the norm in $L^r(\mathbb{R}^3)$ and $|u|_{r(M)} := (\int_M |u|^r\,dx)^{\frac{1}{r}}$, where $M \subset \mathbb{R}^3$ and $u \in L^r(M)$.

2. Abstract setting and preliminary results

In this preliminary section, we fix the notation and we recall some technical results. For $s \in (\frac{3}{4}, 1)$, we define the fractional Sobolev space $H^s(\mathbb{R}^3)$ as

$$H^s(\mathbb{R}^3) := \{u \in L^2(\mathbb{R}^3) : [u] < \infty\},$$

where $[u]$ is the so-called Gagliardo semi-norm of u. It is well known that $H^s(\mathbb{R}^3)$ is a Hilbert space, under the norm

$$\|u\|^2 := \iint_{\mathbb{R}^6} \frac{|u(x) - u(y)|^2}{|x - y|^{3+2s}}\,dx\,dy + \int_{\mathbb{R}^3} |u|^2\,dx.$$

We denote by $D^{s,2}(\mathbb{R}^3)$ the completion of $C_c^\infty(\mathbb{R}^3)$ as

$$D^{s,2}(\mathbb{R}^3) = \{u \in L^{2_s^*}(\mathbb{R}^3) : [u] < \infty\}.$$

When $\lambda > 0$, we denote by E_λ the Hilbert space

$$E_\lambda := \left\{u \in H^s(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2(x)\,dx < \infty\right\},$$

endowed with the norm

$$\|u\|_\lambda^2 := a[u]^2 + \int_{\mathbb{R}^3} (1 + \lambda V(x))u^2\,dx.$$

The scalar product in E_λ is, for $v \in E_\lambda$,

$$\langle u, v\rangle_\lambda := a \iint_{\mathbb{R}^6} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{3+2s}}\,dx\,dy + \int_{\mathbb{R}^3} (1 + \lambda V(x))uv\,dx.$$

By [5], since $E_\lambda \subset H^s(\mathbb{R}^3)$, we can obtain the embedding $E_\lambda \to L^r(\mathbb{R}^3)$ is continuous for all $r \in [2, 2_s^*]$ and locally compact for all $r \in [1, 2_s^*)$. The following Sobolev inequality can be found in [5, Theorem 1.1.8]: for all $u \in D^{s,2}(\mathbb{R}^3)$, there exists $S > 0$ such that

$$S|u|_{2_s^*}^2 \le [u]^2.$$

As in [4, 8], we get the penalized problem of (1.1),

$$\begin{aligned}(a + b[u]^2)(-\Delta)^s u + (1 + \lambda V(x))u - \frac{1}{\varepsilon}(\varphi - u)^+ \chi_\Omega \\ = \mu(u^+)^{q-1} + (u^+)^{2_s^*-1} \quad \text{in } \mathbb{R}^3,\end{aligned} \tag{2.1}$$

where $u^+ = \max\{u, 0\}$ and $\varepsilon > 0$ is the penalization parameter. Let

$$\langle P(u), v\rangle = -\int_\Omega (\varphi - u)^+ v\,dx.$$

We know P is the penalty operator and $\frac{1}{\varepsilon}\int_\Omega (\varphi - u)^+ v\,dx$ is the penalization term. The associated energy functional associated to problem (2.1) is

$$J_{\lambda,\varepsilon}(u) = \frac{1}{2}\|u\|_\lambda^2 + \frac{b}{4}[u]^4 + \frac{1}{2\varepsilon}\int_\Omega [(\varphi - u)^+]^2\,dx - \int_{\mathbb{R}^3} F_+(u)\,dx,$$

where

$$F_+(t) = \int_0^t f_+(s)\,ds \quad \text{and} \quad f_+(t) = \mu(t^+)^{q-1} + (t^+)^{2_s^*-1} \quad \forall t \in \mathbb{R}.$$

It is easy to check that $J_{\lambda,\varepsilon} \in C^1(E_\lambda, \mathbb{R})$ and its differential is defined as, for any $v \in E_\lambda$,

$$\begin{aligned}J'_{\lambda,\varepsilon}(u)v = \langle u, v\rangle_\lambda + b[u]^2 \iint_{\mathbb{R}^6} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{3+2s}}\,dx\,dy \\ - \frac{1}{\varepsilon}\int_\Omega (\varphi - u)^+ v\,dx - \int_{\mathbb{R}^3} f_+(u)v\,dx.\end{aligned}$$

3. The penalized problem

We stress that, by the definition of f, $J_{\lambda,\varepsilon}$ possesses a mountain pass geometry [35].

Lemma 3.1. (i) *There exist constants $r, \rho > 0$, independent of λ and ϵ, such that*

$$J_{\lambda,\epsilon}(u) \geq \rho \quad \textit{for } \|u\|_\lambda = r.$$

(ii) *There is $e \in E_\lambda$ with $\|e\|_\lambda > r$ and $J_{\lambda,\epsilon}(e) < 0$.*

Proof. (i) By Sobolev embeddings, we derive that

$$J_{\lambda,\epsilon}(u) \geq \frac{1}{2}\|u\|_\lambda^2 - \frac{C_1\mu}{q}\|u\|_\lambda^q - \frac{C_2}{2_s^*}\|u\|_\lambda^{2_s^*}.$$

As $4 < q < 2_s^*$, if we choose $r > 0$ satisfying

$$r < \min\Big\{\Big(\frac{3\cdot 2_s^*}{8C_2}\Big)^{\frac{1}{2_s^*-2}}, \Big(\frac{3q}{8C_1\mu}\Big)^{\frac{1}{q-2}}\Big\},$$

we obtain

$$J_{\lambda,\epsilon}(u) \geq \frac{1}{8}r^2 := \rho \quad \text{for } \|u\|_\lambda = r.$$

(ii) Since $\operatorname{supp}(\varphi^+) \subset \Omega$, then we have

$$J_{\lambda,\varepsilon}(\varphi^+) \leq \frac{1}{2}\|\varphi^+\|^2 + \frac{C}{4}\|\varphi^+\|^4 \leq \frac{1}{2}\|\varphi\|^2 + \frac{C}{4}\|\varphi\|^4,$$

where $C := \frac{b}{a}$. Choose $\|\varphi\|$ small enough such that

$$\frac{1}{2}\|\varphi\|^2 + \frac{C}{4}\|\varphi\|^4 < \rho,$$

which implies that $J_{\lambda,\epsilon}(\varphi^+) < \rho$ and

$$\begin{aligned} J_{\lambda,\epsilon}(t\varphi^+) &= \frac{t^2}{2}\|\varphi^+\|_\lambda^2 + \frac{bt^4}{4}[\varphi^+]^4 \\ &\quad - \frac{t^q\mu}{q}\int_\Omega |\varphi^+|^q\,dx - \frac{t^{2_s^*}}{2_s^*}\int_\Omega |\varphi^+|^{2_s^*}\,dx \quad \text{for } t > 1. \end{aligned}$$

As $4 < q < 2_s^*$, it follows that $\lim_{t\to\infty} J_{\lambda,\varepsilon}(t\varphi^+) = -\infty$. Hence, taking $e := (1+t_0)\varphi^+$ for t_0 large enough, we get $\|e\|_\lambda > r$ and $J_{\lambda,\varepsilon}(e) < 0$. ∎

From [35, Theorem 1.15], there is a $(PS)_c$ sequence $\{u_n\} \subset E_\lambda$ such that $J_{\lambda,\epsilon}(u_n) \to c_{\lambda,\epsilon}$ and $J'_{\lambda,\epsilon}(u_n) \to 0$, where $c_{\lambda,\epsilon}$ is the mountain pass level characterized by

$$c_{\lambda,\epsilon} = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J_{\lambda,\epsilon}(\gamma(t)),$$

and

$$\Gamma = \{\gamma \in C([0,1], E_\lambda) : \gamma(0) = \varphi^+ \text{ and } \gamma(1) = e\}.$$

We can see the following results.

Lemma 3.2. *The* $(\mathrm{PS})_{c_{\lambda,\epsilon}}$ *sequence* $\{u_n\}$ *of* $J_{\lambda,\epsilon}$ *is bounded in* E_λ.

Proof. Let $\{u_n\} \subset E_\lambda$ be a $(\mathrm{PS})_{c_{\lambda,\epsilon}}$ sequence for $J_{\lambda,\epsilon}$, that is,

$$J_{\lambda,\epsilon}(u_n) \to c \quad \text{and} \quad J'_{\lambda,\epsilon}(u_n) \to 0 \quad \text{as } n \to +\infty.$$

Since $[(\varphi - u_n)^+]^2 + (\varphi - u_n)^+ u_n \geq (\varphi - u_n)^+ \varphi$, as $q > 4$, then

$$\frac{1}{2\varepsilon}\int_\Omega [(\varphi - u_n)^+]^2\,dx + \frac{1}{q\varepsilon}\int_\Omega (\varphi - u_n)^+ u_n\,dx \geq \frac{1}{q\varepsilon}\int_\Omega (\varphi - u_n)^+ \varphi\,dx.$$

Therefore, from Hölder inequality and Sobolev embedding,

$$\begin{aligned} J_{\lambda,\epsilon}(u_n) - \frac{1}{q}J'_{\lambda,\epsilon}(u_n)u_n &\geq \Big(\frac{1}{2} - \frac{1}{q}\Big)\|u_n\|_\lambda^2 + \Big(\frac{1}{4} - \frac{1}{q}\Big)b[u_n]^4 \\ &\quad + \frac{1}{q\varepsilon}\int_\Omega (\varphi - u_n)^+\varphi\,dx + \Big(\frac{1}{q} - \frac{1}{2_s^*}\Big)\int_{\mathbb{R}^3} |u_n^+|^{2_s^*}\,dx \\ &\geq \Big(\frac{1}{2} - \frac{1}{q}\Big)\|u_n\|_\lambda^2 - \frac{1}{q\varepsilon}\int_\Omega (|\varphi| + |u_n|)|\varphi|\,dx \\ &\geq \frac{q-2}{2q}\|u_n\|_\lambda^2 - \frac{1}{q\varepsilon}|\varphi|_2^2 - \frac{1}{q\varepsilon}|\varphi|_2\|u_n\|_\lambda, \end{aligned}$$

which yields

$$\frac{q-2}{2q}\|u_n\|_\lambda^2 \leq c_{\lambda,\epsilon} + \frac{1}{q\varepsilon}|\varphi|_2^2 + o_n(1) + \Big(o_n(1) + \frac{1}{q\varepsilon}|\varphi|_2\Big)\|u_n\|_\lambda.$$

Thus, $\{u_n\}$ is bounded in E_λ. ∎

Lemma 3.3. *Given* $\tau > 0$, *there is* $\mu_* = \mu_*(\tau) > 0$ *such that*

$$c_{\lambda,\epsilon} < \frac{q-2}{2q}S^{\frac{3}{2s}} - \tau \quad \textit{for all } \lambda > 0,\ \epsilon > 0 \quad \textit{and} \quad \mu \geq \mu_*.$$

Proof. Firstly, we fix the path $\gamma \in \Gamma$,

$$\begin{aligned} \gamma\colon [0,1] &\to E_\lambda, \\ t &\mapsto \gamma(t) = (1 + tt_0)\varphi^+, \end{aligned}$$

where t_0 is defined as in Lemma 3.1. Since $(1 + tt_0)\varphi^+ \geq \varphi$, we get

$$\int_\Omega [(\varphi - (1 + tt_0)\varphi^+)^+]^2\,dx = 0,$$

and then

$$\begin{aligned} J_{\lambda,\epsilon}(\gamma(t)) &= \frac{(1+tt_0)^2}{2}\|\varphi^+\|_\lambda^2 + \frac{b(1+tt_0)^4}{4}[\varphi^+]^4 \\ &\quad - \frac{\mu(1+tt_0)^q}{q}\int_\Omega |\varphi^+|^q\,dx - \frac{(1+tt_0)^{2_s^*}}{2_s^*}\int_\Omega |\varphi^+|^{2_s^*}\,dx \\ &\leq \frac{(1+tt_0)^2}{2}\|\varphi^+\|^2 + \frac{b(1+tt_0)^4}{4}[\varphi^+]^4 - \frac{\mu(1+tt_0)^q}{q}\int_\Omega |\varphi^+|^q\,dx. \end{aligned}$$

Taking $A := \frac{1}{2}\|\varphi^+\|^2$, $B := \frac{b[\varphi^+]^4}{4}$ and $C := \frac{1}{q}\int_{\mathbb{R}^3}|\varphi^+|^q\,dx$, then we consider the following function:

$$g_\mu(t) := At^2 + Bt^4 - \mu Ct^q \quad \text{for all } t \in [0,\infty).$$

From the second derivative of $g_\mu(t)$, for any $\mu > 0$, $g_\mu(t)$ is bounded up. Let the maximum point of g_μ be $t_{\max} > 0$; thus

$$\begin{aligned} c_{\lambda,\epsilon} &\le \max_{t\ge 0} J_{\lambda,\epsilon}(\gamma(t)) \\ &\le \max_{t\ge 0}\left\{\frac{(1+tt_0)^2}{2}\|\varphi^+\|^2 + \frac{b(1+tt_0)^4}{4}[\varphi^+]^4 - \frac{\mu(1+tt_0)^q}{q}\int_\Omega |\varphi^+|^q\,dx\right\} \\ &= \max_{t\ge 0} g_\mu(t) = At_{\max}^2 + Bt_{\max}^4 - \mu Ct_{\max}^q < \frac{q-2}{2q}S^{\frac{3}{2s}} - \tau \end{aligned}$$

for all $\lambda > 0$, $\epsilon > 0$ and $\mu \ge \mu_*$, where μ_* is large enough. ∎

Lemma 3.4. *$\{u_n^+\}$ is also a* $(\mathrm{PS})_{c_{\lambda,\epsilon}}$ *sequence of $J_{\lambda,\epsilon}$.*

Proof. Let $u_n^- = \min\{u_n, 0\}$, then $u_n = u_n^+ + u_n^-$ and

$$-\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{u_n^+(x)u_n^-(y) + u_n^+(y)u_n^-(x)}{|x-y|^{3+2s}}\,dx\,dy \ge 0. \tag{3.1}$$

By Lemma 3.2, $\{u_n^-\}$ is bounded, then from (3.1),

$$\begin{aligned} o_n(1) &= J'_{\lambda,\varepsilon}(u_n)u_n^- \\ &= \langle u_n, u_n^-\rangle_\lambda + b[u_n]^2\iint_{\mathbb{R}^6}\frac{(u_n(x)-u_n(y))(u_n^-(x)-u_n^-(y))}{|x-y|^{3+2s}}\,dx\,dy \\ &\quad - \frac{1}{\varepsilon}\int_\Omega(\varphi-u_n)^+u_n^-\,dx \\ &\ge \|u_n^-\|_\lambda^2, \end{aligned}$$

which implies that $\|u_n^-\|_\lambda^2 = o_n(1)$. Up to a subsequence, we get

$$\begin{aligned} u_n^- &\rightharpoonup 0 \quad \text{in } E_\lambda, \\ u_n^- &\to 0 \quad \text{in } L^r_{\mathrm{loc}}(\mathbb{R}^3) \quad \text{for } r \in [1, 2_s^*), \\ u_n^-(x) &\to 0 \quad \text{a.e. in } \mathbb{R}^3. \end{aligned}$$

Then we note that $\|u_n\|_\lambda = \|u_n^+\|_\lambda + o_n(1)$. Besides, by Lebesgue dominated convergence theorem,

$$\int_\Omega[(\varphi-u_n)^+]^2\,dx = \int_\Omega[(\varphi-u_n^+)]^2\,dx + o_n(1).$$

Then

$$
\begin{aligned}
J_{\lambda,\varepsilon}(u_n) &= \frac{1}{2}\|u_n\|_\lambda^2 + \frac{b}{4}[u_n]^4 + \frac{1}{2\varepsilon}\int_\Omega [(\varphi - u_n)^+]^2\,dx \\
&\quad - \frac{\mu}{q}\int_{\mathbb{R}^3} |u_n^+|^q\,dx - \frac{1}{2_s^*}\int_{\mathbb{R}^3} |u_n^+|^{2_s^*}\,dx \\
&= \frac{1}{2}\|u_n^+\|_\lambda^2 + \frac{b}{4}[u_n^+]^4 + \frac{1}{2\varepsilon}\int_\Omega [(\varphi - u_n^+)^+]^2\,dx \\
&\quad - \frac{\mu}{q}\int_{\mathbb{R}^3} |u_n^+|^q\,dx - \frac{1}{2_s^*}\int_{\mathbb{R}^3} |u_n^+|^{2_s^*}\,dx + o_n(1) \\
&= J_{\lambda,\varepsilon}(u_n^+) + o_n(1).
\end{aligned}
$$

Hence

$$
J_{\lambda,\varepsilon}(u_n^+) = c_{\lambda,\varepsilon} + o_n(1).
$$

Following, we are going to show $J'_{\lambda,\varepsilon}(u_n^+) = o_n(1)$. Now let $v \in E_\lambda$ with $\|v\|_\lambda \le C$ for some $C > 0$. By Lebesgue dominated convergence theorem, we have

$$
\int_\Omega (\varphi - u_n)^+ v\,dx = \int_\Omega (\varphi - u_n^+)^+ v\,dx + o_n(1),
$$

and

$$
\int_{\mathbb{R}^3} (1 + \lambda V(x))u_n^- v\,dx = o_n(1).
$$

Thus,

$$
\begin{aligned}
&\iint_{\mathbb{R}^6} \frac{(u_n^-(x) - u_n^-(y))(v(x) - v(y))}{|x-y|^{3+2s}}\,dx\,dy \\
&\qquad = \langle u_n^-, v\rangle_\lambda - \int_{\mathbb{R}^3} (1 + \lambda V(x))u_n^- v\,dx = o_n(1).
\end{aligned}
$$

Therefore, from Hölder inequality and Lemma 3.2, we obtain

$$
\begin{aligned}
&J'_{\lambda,\varepsilon}(u_n)v \\
&\quad = \langle u_n^+, v\rangle_\lambda + \langle u_n^-, v\rangle_\lambda + b[u_n]^2 \iint_{\mathbb{R}^6} \frac{(u_n(x) - u_n(y))(v(x) - v(y))}{|x-y|^{3+2s}}\,dx\,dy \\
&\qquad - \frac{1}{\varepsilon}\int_\Omega (\varphi - u_n)^+ v\,dx - \mu\int_{\mathbb{R}^3} (u_n^+)^{q-1} v\,dx - \int_{\mathbb{R}^3} (u_n^+)^{2_s^*-1} v\,dx \\
&\quad = \langle u_n^+, v\rangle_\lambda + \langle u_n^-, v\rangle_\lambda \\
&\qquad + b([u_n^+]^2 + o_n(1))\left(\iint_{\mathbb{R}^6} \frac{(u_n^+(x) - u_n^+(y))(v(x) - v(y))}{|x-y|^{3+2s}}\,dx\,dy + o_n(1)\right) \\
&\qquad - \frac{1}{\varepsilon}\int_\Omega (\varphi - u_n^+)^+ v\,dx - \mu\int_{\mathbb{R}^3} (u_n^+)^{q-1} v\,dx - \int_{\mathbb{R}^3} (u_n^+)^{2_s^*-1} v\,dx + o_n(1) \\
&\quad = J'_{\lambda,\varepsilon}(u_n^+)v + \langle u_n^-, v\rangle_\lambda + o_n(1),
\end{aligned}
$$

from where we get

$$|J'_{\lambda,\varepsilon}(u_n^+)v| \leq |J'_{\lambda,\varepsilon}(u_n)v| + |\langle u_n^-, v\rangle_\lambda| + o_n(1)$$
$$\leq \|J'_{\lambda,\varepsilon}(u_n)\|_{E'_\lambda}\|v\|_\lambda + \|u_n^-\|_\lambda\|v\|_\lambda + o_n(1).$$

Therefore, since $J'_{\lambda,\varepsilon}(u_n) = o_n(1)$, we conclude that

$$\|J'_{\lambda,\varepsilon}(u_n^+)\|_{E'_\lambda} = o_n(1). \qquad \blacksquare$$

As a byproduct of the above lemma, we obtain the clearly upper boundedness of $\{u_n\}$.

Lemma 3.5 ([3, Lemma 3.6]). *If $\{u_n\}$ is given by Lemma 3.2, then for any $c_{\lambda,\varepsilon} > 0$,*

$$\limsup_{n\to\infty}\|u_n\|_\lambda^2 \leq \frac{2q}{q-2}c_{\lambda,\varepsilon}.$$

Lemma 3.6. *Let $K > 0$ be independent on λ and let $\{u_n\} \subset E_\lambda$ be a $(\mathrm{PS})_c$ sequence for $J_{\lambda,\varepsilon}$ with $0 < c < K$. Then, as $\lambda_n \to \infty$, there exists $u \in H_0^s(\Omega)$ such that $u_n \rightharpoonup u$ in E_{λ_n} and $u_n \to u$ in $L^r(\mathbb{R}^3)$ for all $r \in [2, 2_s^*)$.*

Proof. Lemma 3.5 yields that

$$\|u_n\|^2 \leq \|u_n\|_{\lambda_n}^2 \leq \frac{2q}{q-2}K.$$

We assume that $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^3)$. By the Fatou lemma, we have

$$\int_{\mathbb{R}^3} V(x)|u|^2\,dx \leq \liminf_{n\to\infty}\int_{\mathbb{R}^3} V(x)|u_n|^2\,dx \leq \liminf_{n\to\infty}\frac{\|u_n\|_{\lambda_n}^2}{\lambda_n} = 0,$$

which implies that $u(x) = 0$ almost everywhere in $\mathbb{R}^3 \setminus V^{-1}(0)$. Thus, $u \in H_0^s(\Omega)$.

Let $F := \{x \in \mathbb{R}^3 : V(x) \leq M_0\}$. Then

$$\int_{F^c}|u_n|^2\,dx \leq \frac{1}{\lambda_n M_0}\int_{F^c}\lambda_n V(x)|u_n|^2\,dx \leq \frac{2q}{(q-2)\lambda_n M_0}K \to 0.$$

For any $r \in (1, \frac{2_s^*}{2})$, and fixing $r' = \frac{r}{r-1}$, as R is large enough, from (V_3), we have

$$\int_{F\cap B_R^c}|u_n - u|^2\,dx \leq |u_n - u|_{2r}^2|\mathcal{L}(F\cap B_R^c)|^{\frac{1}{r'}}$$
$$\leq C\|u_n - u\|^2|\mathcal{L}(F\cap B_R^c)|^{\frac{1}{r'}} \to 0.$$

On the other hand, $u_n \to u$ in $L^2(B_R)$. This shows that $u_n \to u$ in $L^2(\mathbb{R}^3)$ as $\lambda_n \to \infty$. Following, by using interpolation, we obtain $u_n \to u$ in $L^r(\mathbb{R}^3)$ for all $r \in [2, 2_s^*)$. $\blacksquare$

Proposition 3.7. *There exist $\lambda^* = \lambda^*(\tau) > 0$ and $b^*(\tau) > 0$ such that, for all $\lambda \geq \lambda^*$ and $b \in (0, b^*)$, $J_{\lambda,\epsilon}$ satisfies the $(\mathrm{PS})_c$ condition at the level $c \in \left(0, \frac{q-2}{2q}S^{\frac{3}{2s}} - \tau\right)$, where τ is from Lemma 3.3.*

Proof. Let $\{u_n\}$ be a $(\mathrm{PS})_c$ sequence of $J_{\lambda,\epsilon}$. From Lemmas 3.2 and 3.4, $\{u_n\}$ is a non-negative bounded sequence; hence, up to a subsequence, there are $u \in E_\lambda$ and $A \geq 0$ such that

$$\begin{aligned} u_n &\rightharpoonup u && \text{in } E_\lambda, \\ u_n &\to u && \text{in } L^r_{\mathrm{loc}}(\mathbb{R}^3) \quad \text{for any } r \in [1, 2^*_s), \\ u_n(x) &\to u(x) && \text{a.e. in } \mathbb{R}^3, \\ [u_n]^2 &\to A^2. \end{aligned}$$

Setting the functional

$$\begin{aligned} P_{\lambda,\varepsilon}(u) = \frac{1}{2}\|u\|_\lambda^2 + \frac{bA^2}{2}[u]^2 + \frac{1}{2\varepsilon}\int_\Omega [(\varphi - u)^+]^2\, dx \\ - \frac{\mu}{q}\int_{\mathbb{R}^3} u^q\, dx - \frac{1}{2^*_s}\int_{\mathbb{R}^3} u^{2^*_s}\, dx, \end{aligned}$$

we have

$$P_{\lambda,\varepsilon}(u_n) = J_{\lambda,\varepsilon}(u_n) + \frac{bA^4}{4} + o_n(1),$$

and

$$J'_{\lambda,\varepsilon}(u_n)u_n - P'_{\lambda,\varepsilon}(u_n)u_n = o_n(1).$$

Thus,

$$P_{\lambda,\varepsilon}(u_n) = c + \frac{bA^4}{4} + o_n(1) \quad \text{and} \quad P'_{\lambda,\varepsilon}(u_n)u_n = o_n(1).$$

For any $h \in C_0^\infty(\mathbb{R}^3)$, taking h as a test function in (2.1), we obtain $P'_{\lambda,\varepsilon}(u) = 0$. Therefore, $P_{\lambda,\varepsilon}(u) \geq 0$.

Now, setting $v_n := u_n - u$, firstly by Lebesgue dominated convergence theorem, we get

$$\int_\Omega [(\varphi - u_n)^+]^2\, dx = \int_\Omega [(\varphi - u)^+]^2\, dx + o_n(1).$$

Therefore, by the Brezis–Lieb Lemma in [35] and from Lemma 3.3, it follows that

$$\begin{aligned} &P_{\lambda,\varepsilon}(u_n) - P_{\lambda,\varepsilon}(u) + o_n(1) \\ &\quad = \frac{1}{2}\|v_n\|_\lambda^2 + \frac{bA^2}{2}[v_n]^2 - \frac{\mu}{q}\int_{\mathbb{R}^3} |v_n|^q\, dx - \frac{1}{2^*_s}\int_{\mathbb{R}^3} |v_n|^{2^*_s}\, dx \\ &\quad = c + \frac{bA^4}{4} - P_{\lambda,\varepsilon}(u) < \frac{q-2}{2q} S^{\frac{3}{2s}} - \tau + \frac{bA^4}{4}. \end{aligned} \tag{3.2}$$

Then, taking $b^*(\tau)$ small enough, for any $b \in (0, b^*)$, we can assume that

$$-\tau + \frac{bA^4}{4} \leq 0.$$

Thus,

$$P_{\lambda,\varepsilon}(u_n) - P_{\lambda,\varepsilon}(u) + o_n(1) < \frac{q-2}{2q} S^{\frac{3}{2s}}.$$

Besides, also by Lebesgue dominated convergence theorem, we get

$$\int_{\Omega} (\varphi - u_n)^+ u_n \, dx = \int_{\Omega} (\varphi - u)^+ u \, dx + o_n(1);$$

then we have

$$\begin{aligned} o_n(1) &= P'_{\lambda,\varepsilon}(u_n)u_n = P'_{\lambda,\varepsilon}(u_n)u_n - P'_{\lambda,\varepsilon}(u)u \\ &= \|v_n\|_\lambda^2 + bA^2[v_n]^2 - \mu \int_{\mathbb{R}^3} |v_n|^q \, dx - \int_{\mathbb{R}^3} |v_n|^{2_s^*} \, dx. \end{aligned} \tag{3.3}$$

Fixed $\lambda > 0$, assume $\|v_n\|_\lambda^2 + bA^2[v_n]^2 \to l_1(\lambda)$ and $\int_{\mathbb{R}^3} |v_n|^{2_s^*} \, dx \to l_2$. If $l_1(\lambda) = 0$, then $u_n \to u$ in E_λ. Thus, we may assume $l_1(\lambda) > 0$. Since Lemma 3.6, we have

$$\lim_{n\to\infty} \mu \int_{\mathbb{R}^3} |v_n|^q \, dx = o_\lambda(1).$$

Therefore, from (3.3), as $n \to \infty$, we may get $l_2 = l_2(\lambda) = l_1(\lambda) + o_\lambda(1)$ and

$$\|v_n\|^2 \le C(\|v_n\|_\lambda^q + \|v_n\|_\lambda^{2_s^*}).$$

As $q \in (4, 2_s^*)$, we get, for any $t \in \mathbb{R}$,

$$|t|^q \le \frac{1}{2C}|t|^2 + C|t|^{2_s^*}.$$

Then we find

$$\begin{aligned} l_1(\lambda) = \lim_{n\to\infty} (\|v_n\|_\lambda^2 + bA^2[v_n]^2) &\ge \lim_{n\to\infty} \|v_n\|_\lambda^2 \\ &\ge \left(\frac{1}{2C(C+1)}\right)^{\frac{2}{2_s^*-2}} := C_1 > 0, \end{aligned} \tag{3.4}$$

where $C_1 > 0$ does not depend on λ. Since $n \to \infty$, by Sobolev inequality,

$$S \le \frac{\|v_n\|_\lambda^2}{|v_n|_{2_s^*}^2} \le \frac{l_1(\lambda)}{l_2(\lambda)^{\frac{2}{2_s^*}}} = \frac{l_1(\lambda)}{(l_1(\lambda) + o_\lambda(1))^{\frac{2}{2_s^*}}}.$$

From (3.4), we have

$$\liminf_{\lambda\to\infty} l_1(\lambda) \ge S^{\frac{3}{2s}}.$$

Thus, from (3.2), we get

$$\frac{q-2}{2q} S^{\frac{3}{2s}} > \liminf_{\lambda\to\infty} \Big(\frac{1}{2} - \frac{1}{2_s^*}\Big) l_1(\lambda) \ge \frac{q-2}{2q} S^{\frac{3}{2s}},$$

which is absurd. Then we complete this proof. ■

Remark 3.8. From Lemma 3.1, Lemma 3.3 and Proposition 3.7, there exist $\mu_*, \lambda^*, b^* > 0$, for every $\mu \ge \mu_*$, $\lambda \ge \lambda^*$ and $b \in (0, b^*)$, problem (2.1) has at least one weak solution u_ε. Besides, from Lemma 3.4, u_ε is non-negative and nontrivial.

4. Proof of Theorem 1.1

From now on, making the change of notations

$$\epsilon = \frac{1}{n}, \quad u_n = u_{\frac{1}{n}}, \quad J_n = J_{\lambda,\epsilon} \quad \text{and} \quad J_n(u_n) = c_n = c_{\lambda,\epsilon},$$

for any $v \in E_\lambda$, we get

$$\begin{aligned}\langle u_n, v\rangle_\lambda + b[u_n]^2 \iint_{\mathbb{R}^6} \frac{(u_n(x)-u_n(y))(v(x)-v(y))}{|x-y|^{3+2s}}\,dx\,dy \\ - \frac{1}{\varepsilon}\int_\Omega (\varphi - u_n)^+ v\,dx \\ = \int_{\mathbb{R}^3} f(u_n)v\,dx. \end{aligned} \tag{4.1}$$

From Lemma 3.2, there is $u \in E_\lambda$ such that, up to a subsequence, $u_n \rightharpoonup u$ in E_λ.

Lemma 4.1 ([3, Lemma 3.11]). *If u is given above, then $u \in \mathbb{K}$.*

Remark 4.2. Take $v := u_n - u$ in (4.1). Since

$$\langle P(u_n), u_n - u\rangle = \langle P(u_n) - P(u), u_n - u\rangle \geq 0,$$

we may get

$$\begin{aligned}\langle u_n, u_n - u\rangle_\lambda + b[u_n]^2 \iint_{\mathbb{R}^6} \frac{(u_n(x)-u_n(y))((u_n-u)(x)-(u_n-u)(y))}{|x-y|^{3+2s}}\,dx\,dy \\ \leq \int_{\mathbb{R}^3} f(u_n)(u_n-u)\,dx. \end{aligned} \tag{4.2}$$

Lemma 4.3. *We have $u_n \to u$ in E_λ as $\lambda \to \infty$.*

Proof. Firstly, we show $u_n \to u$ in $H^s(\mathbb{R}^3)$. Assume by contradiction that

$$u_n \nrightarrow u \quad \text{in } H^s(\mathbb{R}^3),$$

that is,

$$\|u_n - u\|^2 \to B > 0.$$

Setting $v_n = u_n - u$, from [3, Lemma 3.12], we may have

$$\int_{\mathbb{R}^3} f(u_n)v_n\,dx = \int_{\mathbb{R}^3} f(v_n)v_n + \int_{\mathbb{R}^3} f(u)v_n + o_n(1).$$

Moreover, as

$$\int_{\mathbb{R}^3} f(u)v_n = o_n(1),$$

we have

$$\int_{\mathbb{R}^3} f(u_n)v_n\,dx = \int_{\mathbb{R}^3} f(v_n)v_n + o_n(1). \tag{4.3}$$

Since

$$\begin{aligned}
&\iint_{\mathbb{R}^6} \frac{(u_n(x)-u_n(y))(u(x)-u(y))}{|x-y|^{3+2s}}\,dx\,dy \\
&\quad\le \iint_{\mathbb{R}^6} \frac{|(u_n(x)-u_n(y))(u(x)-u(y))|}{|x-y|^{3+2s}}\,dx\,dy \le [u_n][u],
\end{aligned}$$

thus from (4.2), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3} f(u_n)(u_n-u)\,dx \\
&\quad\ge \langle u_n, u_n-u\rangle_\lambda \\
&\qquad + b[u_n]^2 \iint_{\mathbb{R}^6} \frac{(u_n(x)-u_n(y))((u_n-u)(x)-(u_n-u)(y))}{|x-y|^{3+2s}}\,dx\,dy \\
&\quad= \langle u_n, u_n-u\rangle_\lambda \\
&\qquad + b[u_n]^2\left([u_n]^2 - \iint_{\mathbb{R}^6} \frac{(u_n(x)-u_n(y))(u(x)-u(y))}{|x-y|^{3+2s}}\,dx\,dy\right) \\
&\quad\ge \langle u_n, u_n-u\rangle_\lambda + b[u_n]^2([u_n]^2-[u_n][u]) \\
&\quad= \langle u_n, u_n-u\rangle_\lambda + b[u_n]^3([u_n]-[u]).
\end{aligned} \tag{4.4}$$

Moreover, from the Brezis–Lieb Lemma in [35], we get

$$[v_n]^2 = [u_n]^2 - [u]^2 + o_n(1) \ge 0;$$

thus, $[u_n] = [u] + o_n(1)$ or $[u_n] \ge [u]$. Then by (4.3) and (4.4), we have

$$\int_{\mathbb{R}^3} f(v_n)v_n + o_n(1) \ge \langle u_n, u_n-u\rangle_\lambda = \|v_n\|_\lambda^2.$$

This is equivalent to

$$\|v_n\|^2 \le \|v_n\|_\lambda^2 \le \mu|v_n|_q^q + |v_n|_{2_s^*}^{2_s^*} + o_n(1). \tag{4.5}$$

On the other hand, since

$$\|v_n\|^2 \le \|v_n\|_\lambda^2 \le \|u_n\|_\lambda^2 + o_n(1),$$

from Lemma 3.5, we have

$$\limsup_{n\to\infty}\|v_n\|^2 < S^{\frac{3}{2s}},$$

that is,

$$B < S^{\frac{3}{2s}}. \tag{4.6}$$

Moreover, by (4.5),

$$S \leq \frac{\|v_n\|^2}{|v_n|_{2_s^*}^2} \leq \frac{\|v_n\|^2}{(\|v_n\|^2 - \mu |v_n|_q^q + o_n(1))^{\frac{2}{2_s^*}}}.$$

Taking $\lambda \to \infty$ and $n \to \infty$, from Lemma 3.6, we will see

$$S \leq \frac{\|v_n\|^2}{(\|v_n\|^2 + o_\lambda(1) + o_n(1))^{\frac{2}{2_s^*}}} = \frac{B}{B^{\frac{2}{2_s^*}}};$$

then $B \geq S^{\frac{3}{2s}}$. Because of (4.6), $S^{\frac{3}{2s}} \leq B < S^{\frac{3}{2s}}$, which shows a contradiction. Thus, $\|v_n\| \to 0$, and from (4.5), we have $\|v_n\|_\lambda \to 0$ when $n \to \infty$ and λ is large enough. ∎

Remark 4.4. For any $v \in \mathbb{K}$, taking $v - u_n$ in (4.1), from Lemmas 4.1 and 4.3, we can obtain u is a non-negative and nontrivial solution for (1.1).

To complete our concentration, we verify the following theorem.

Theorem 4.5. *Let $\{u_n\} \subset E_{\lambda_n}$ be a sequence satisfying*

$$(a + b[u_n]^2) \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u_n (-\Delta)^{\frac{s}{2}} (v - u_n)\, dx + \int_{\mathbb{R}^3} (1 + \lambda_n V(x)) u_n (v - u_n)\, dx$$
$$\geq \int_{\mathbb{R}^3} f(u_n)(v - u_n)\, dx$$

as $\lambda_n \to \infty$ and for all $v \in \mathbb{K}$. Then there are a subsequence of $\{u_n\}$, still denoted by itself, and $u \in H^s(\mathbb{R}^3)$ such that $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^3)$. Moreover,

(i) *$u = 0$ a.e. in Ω^c;*

(ii) *$\|u_n - u\|_{\lambda_n}^2 \to 0$;*

(iii) *we may obtain*

$$u_n \to u \quad \text{in } H^s(\mathbb{R}^3),$$
$$\lambda_n \int_{\mathbb{R}^3} V(x) u_n^2\, dx \to 0,$$
$$\|u_n\|_{\lambda_n}^2 \to \int_Q \frac{|u(x) - u(y)|^2}{|x - y|^{3+2s}}\, dx\, dy + \int_\Omega |u|^2\, dx,$$

where $Q := \mathbb{R}^3 \times \mathbb{R}^3 \setminus (\Omega^c \times \Omega^c)$.

(iv) *u is a solution of the variational inequality*

$$(a + b[u]_Q^2) \iint_Q \frac{(u(x) - u(y))((w - u)(x) - (w - u)(y))}{|x - y|^{3+2s}}\, dx\, dy + \int_\Omega u(w - u)\, dx$$
$$\geq \int_\Omega (\mu |u|^{q-2} + |u|^{2^*-2}) u(w - u)\, dx$$

for all $w \in \tilde{\mathbb{K}}$, where

$$\tilde{\mathbb{K}} := \{w \in H_0^s(\Omega),\ w \geq \varphi \text{ a.e. in } \Omega\}.$$

Proof. From Lemma 3.5, $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$, so up to a subsequence, there exists $u \in H^s(\mathbb{R}^3)$ such that

$$u_n \rightharpoonup u \quad \text{in } H^s(\mathbb{R}^3).$$

(i) The same process as in Lemma 3.6 works to prove $u = 0$ a.e. in Ω^c.

(ii) Since $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^3)$, if we set $v_n = u_n - u$, by the same arguments explored in the proof of Lemma 4.3, we can show that $v_n \to 0$ in $H^s(\mathbb{R}^3)$, and

$$\|v_n\|_{\lambda_n}^2 \le \int_{\mathbb{R}^3} f(v_n)v_n \, dx + o_n(1).$$

Thus, we obtain, as $n \to \infty$,

$$v_n \to 0 \quad \text{in } E_{\lambda_n}.$$

(iii) From (i)–(ii), we get

$$\lambda_n \int_{\mathbb{R}^3} V(x)|u_n|^2 \, dx = \lambda_n \int_{\mathbb{R}^3} V(x)|u_n - u|^2 \, dx \le \|u_n - u\|_{\lambda_n}^2 \to 0,$$

and

$$\|u_n\|_{\lambda_n}^2 \to \int_Q \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy + \int_\Omega |u|^2 \, dx.$$

(iv) For any $w \in \tilde{\mathbb{K}}$, we may get

$$\langle u_n, w - u_n \rangle_{\lambda_n} + b[u_n]^2 \iint_{\mathbb{R}^6} \frac{(u_n(x) - u_n(y))((w - u_n)(x) - (w - u_n)(y))}{|x - y|^{3+2s}} \, dx \, dy$$
$$\ge \int_{\mathbb{R}^3} f(u_n)(w - u_n) \, dx.$$

Thus, from (i)–(iii), taking $n \to \infty$, we obtain

$$\Big(a + b \int_Q \frac{|u(x) - u(y)|^2}{|x - y|^{3+2s}} \, dx \, dy\Big)$$
$$\times \int_Q \frac{(u_n(x) - u_n(y)((w - u_n)(x) - (w - u_n)(y)))}{|x - y|^{3+2s}} \, dx \, dy$$
$$+ \int_\Omega u(w - u) \, dx \ge \int_\Omega (\mu u^{q-2} + u^{2_s^*-2})u(w - u) \, dx.$$

■

Acknowledgments. The authors warmly thank the anonymous referees for their useful and nice comments on the paper.

Funding. S. Deng and W. Luo have been supported by National Natural Science Foundation of China, No. 11971392, and Natural Science Foundation of Chongqing, China, cstc2019jcyjjqX0022.

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Received 24 July 2023; revised 10 October 2023.

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