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On the NLS dynamics for infinite energy vortex configurations on the plane

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Abstract

We derive the asymptotical dynamical law for Ginzburg-Landau vortices in the plane under the Schrödinger dynamics, as the Ginzburg-Landau parameter goes to zero. The limiting law is the well-known point-vortex system. This result extends to the whole plane previous results of [8, 13] established for bounded domains, and holds for arbitrary degree at infinity. When this degree is non-zero, the total Ginzburg-Landau energy is infinite.

1. Introduction

The purpose of this paper is to investigate the dynamics of vortices for the nonlinear Schrödinger equation on the plane, when the total degree at infinity is non zero. The equation we are interested in, also often referred to as the Gross-Pitaevskii equation, is written on $\mathbb{R}^2 \times \mathbb{R}$ as

$$(GP)_\varepsilon \qquad i\partial_t u_\varepsilon + \Delta u_\varepsilon = \frac{1}{\varepsilon^2}(|u_\varepsilon|^2 - 1)u_\varepsilon,$$

where $0 < \varepsilon < 1$ denotes a small parameter. This equation is Hamiltonian, with Hamiltonian given by the Ginzburg-Landau energy

$$E_\varepsilon(u) \equiv \int_{\mathbb{R}^2} e_\varepsilon(u) := \int_{\mathbb{R}^2} \frac{|\nabla u|^2}{2} + \frac{(1-|u|^2)^2}{4\varepsilon^2}.$$

One peculiarity of E_ε and $(GP)_\varepsilon$ is that finite energy fields do not tend to zero at infinity, but have instead to stay close to the unit circle S^1. The Gross-Pitaevskii equation possesses distinguished stationary solutions called

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vortices and labelled by an integer $d \in \mathbb{Z}^*$, which have the special form, for $z = x_1 + \imath x_2$,

$$u_{\varepsilon,d}(z) \equiv u_{\varepsilon,d}(r,\theta) = f_{\varepsilon,d}(r)\exp(id\theta) = f_{\varepsilon,d}(r)\Big(\frac{z}{|z|}\Big)^d$$

where $f_{\varepsilon,d} : \mathbb{R}^+ \to [0,1]$ satisfies $f_{\varepsilon,d}(0) = 0$, $f_{\varepsilon,d}(+\infty) = 1$, and

$$\frac{d^2}{dr^2} f_{\varepsilon,d} + \frac{1}{r}\frac{d}{dr} f_{\varepsilon,d} - \frac{d^2}{r^2} f_{\varepsilon,d} + \frac{1}{\varepsilon^2} f_{\varepsilon,d}(1 - (f_{\varepsilon,d})^2) = 0.$$

Notice that ε has the dimension of a length and that by scaling

$$f_{\varepsilon,d}(r) = f_{1,d}\Big(\frac{r}{\varepsilon}\Big).$$

In particular ε is the characteristic length scale describing the core of the vortex, and as $\varepsilon \to 0$

$$u_{\varepsilon,d}(z) \to \Big(\frac{z}{|z|}\Big)^d = \exp \imath d\theta.$$

It is known (see e.g. [12]) that $|\nabla u_{\varepsilon,d}(z)| \sim d/|z|$ as $|z| \to +\infty$, so that

$$\int |\nabla u_{\varepsilon,d}|^2 = +\infty. \tag{1.1}$$

On the other hand, the potential term remains bounded (actually $\int (1 - |u_{\varepsilon,d}|^2)^2/4\varepsilon^2 = \pi d^2$), as well as the modulus part of the gradient: $\int |\nabla |u_{\varepsilon,d}||^2 < +\infty$. Notice that $u_{\varepsilon,d}$ has winding number d at infinity, in the sense that for each radius $r > 0$ large enough (actually for any radius $r > 0$ in the case considered here) the map $\psi_r : \partial B_r \simeq S^1 \to S^1$ given by

$$\partial B_r \ni z \mapsto \frac{u_{\varepsilon,d}(z)}{|u_{\varepsilon,d}(z)|}$$

has topological degree d. Actually it can easily be proved that any continuous field which does not vanish outside a compact set and has a nonzero degree at infinity has infinite energy.

In this paper, we wish to study multi-vortex configurations, and the dynamics near such configurations. More precisely, for given points $a_1, \ldots, a_l$ in $\mathbb{R}^2$, and integers $d_1, \ldots, d_l$ in $\mathbb{Z}^*$, we have in mind initial data of the form

$$u^\star_\varepsilon(a_i, d_i)(z) \equiv \prod_{i=1}^{l} u_{\varepsilon,d_i}(z - a_i) = \prod_{i=1}^{l} f_{1,d_i}\Big(\frac{|z - a_i|}{\varepsilon}\Big)\Big(\frac{z - a_i}{|z - a_i|}\Big)^{d_i}, \tag{1.2}$$

as well as small perturbations of the maps defined above.

Notice that if $\Sigma d_i \neq 0$, then

$$\int_{\mathbb{R}^2} |\nabla u_\varepsilon^\star(a_i, d_i)|^2 = +\infty,$$

and that

$$J(u_\varepsilon^\star(a_i, d_i)) \rightharpoonup \pi \sum_{i=1}^{l} d_i \delta_{a_i} \text{ in } \mathcal{D}'(\mathbb{R}^2),$$

where for a map $v : \mathbb{R}^2 \to \mathbb{C}$, Jv denotes its Jacobian

$$Jv = \frac{1}{2} v_{x_1} \times v_{x_2}.$$

We will often use the notation

$$jv = v \times \nabla v.$$

Physically, jv represents the momentum density associated with a wave function v. Note that $Jv = \frac{1}{2}\nabla \times jv$, so that Jv is naturally interpreted as vorticity.

This program has already been successfully carried out, in the case $|d_i| = 1$ for all i, on bounded domains with periodic, Dirichlet or Neumann boundary conditions by Colliander and Jerrard [8], Lin and Xin [13] and Jerrard and Spirn [11], for suitable modification $\tilde{u}_\varepsilon^\star(a_i, d_i)$ of $u_\varepsilon^\star(a_i, d_i)$, according to the boundary condition. These papers show that the vortex dynamics is governed in the limit $\varepsilon \to 0$ by exactly the same ordinary differential equations that describe the motion of vortices in an ideal incompressible fluid, with suitable boundary condition.

In the bounded case, a crucial observation is the fact that the total energy is bounded and that $\tilde{u}_\varepsilon^\star(a_i, d_i)$ is almost energy minimizing for the given vortex configuration. More precisely, it is proved in [8] that if

$$J(u_\varepsilon) \rightharpoonup \pi \sum_{i=1}^{l} d_i \delta_{a_i}$$

then

$$\liminf_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon) - E_\varepsilon(\tilde{u}_\varepsilon^\star(a_i, d_i)) \geq 0.$$

Moreover the last inequality is coercive in the sense that, if the left-hand side is small, then u_ε is close to $\tilde{u}_\varepsilon^\star(a_i, d_i)$ in various norms. This last property makes it possible to compare the dynamics of $(GP)_\varepsilon$ with that of the finite dimensional Hamiltonian system whose Hamiltonian is essentially given by $E_\varepsilon(\tilde{u}_\varepsilon^\star(a_i, d_i))$.

Our main aim here is to extend the result to the whole plane and to initial data of the form $u^\star_\varepsilon(a_i, d_i)$ and perturbations thereof. One of the main additional difficulties we have to face is the divergence of the total energy and various losses of control at infinity. A first issue is to solve the Cauchy problem. This is done in [6]. It is proved there that the Cauchy problem is globally well-posed in $\{U\} + H^1(\mathbb{R}^2)$, for any $U \in \mathcal{V}$, where

$$\mathcal{V} = \left\{U \in L^\infty(\mathbb{R}^2, \mathbb{C}),\ \nabla^k U \in L^2, \forall k \geq 2,\ \nabla |U| \in L^2,\ (1 - |U|^2) \in L^2\right\}.$$

In particular, for any configuration (a_i, d_i), since $u^\star_\varepsilon(a_i, d_i)$ is in $\mathcal{V}$, the Cauchy problem is globally well-posed in $\{u^\star_\varepsilon(a_i, d_i)\} + H^1(\mathbb{R}^2)$. It turns out that, for any $U \in \mathcal{V}$, one may define a renormalized energy in $\{U\} + H^1(\mathbb{R}^2)$, denoted $\mathcal{E}_{\varepsilon,U}$, whose definition depends on U, and that this renormalized energy remains constant in time, for $(GP)_\varepsilon$ and intial data in $\{U\} + H^1(\mathbb{R}^2)$, i.e

$$\forall t \in \mathbb{R}, \qquad \mathcal{E}_{\varepsilon,U}(u_\varepsilon(., t)) = \mathcal{E}_{\varepsilon,U}(u_\varepsilon(., 0)). \tag{1.3}$$

More precisely, $\mathcal{E}_{\varepsilon,U}$ is given by

$$\mathcal{E}_{\varepsilon,U}(U + v) = \int_{\mathbb{R}^2} \frac{|\nabla v|^2}{2} - \int_{\mathbb{R}^2} (\Delta U) \cdot v + \int_{\mathbb{R}^2} \frac{(|U + v|^2 - 1)^2}{4\varepsilon^2}. \tag{1.4}$$

If moreover the map U verifies the additional condition $|\nabla U(z)| \leq \frac{C}{\sqrt{|z|}}$ then the renormalized energy $\mathcal{E}_{\varepsilon,U}$ may be defined as follows (see Section 3)

$$\mathcal{E}_{\varepsilon,U}(u) = \lim_{R\to\infty} \int_{B(R)} [e_\varepsilon(u) - \frac{|\nabla U|^2}{2}]. \tag{1.5}$$

We therefore restrict ourselves to the class

$$\mathcal{V}_* = \left\{U \in \mathcal{V},\ |\nabla U(z)| \leq \frac{C}{\sqrt{|z|}}\right\}.$$

In contrast with the classical energy, we will show in Section 4 that the renormalized energy is unbounded from below when the degree at infinity of U is greater or equal to 2 in absolute value.

Working in $\{U\} + H^1(\mathbb{R}^2)$ for a single reference field u is in some places too restrictive. In this direction, we introduce an equivalence relation on the set $\mathcal{V}_*$. First, observe that if $U \in \mathcal{V}$, its zero set is bounded so that its topological degree at infinity $\deg(U, \infty)$ is well defined. We write

$$U \sim U' \quad \text{iff} \quad \deg(U, \infty) = \deg(U', \infty) \quad \text{and} \quad |\nabla U|^2 - |\nabla U'|^2 \in L^1(\mathbb{R}^2)$$

and denote by $[U]$ the corresponding equivalence class. As a consequence of the second condition, if $U' \in [U]$,

$$\lim_{R\to+\infty} \int_{B(R)} e_\varepsilon(u) - \frac{|\nabla U|^2}{2} \quad \text{exists}$$

for any $u \in \{U'\} + H^1(\mathbb{R}^2)$. This allows us to extend the definition of $\mathcal{E}_{\varepsilon,U}$ to $[U] + H^1(\mathbb{R}^2)$.

For every $d \in \mathbb{Z}$, we choose a smooth reference field U_d such that

$$U_d = \left(\frac{z}{|z|}\right)^d = \exp id\theta, \qquad \forall z \in \mathbb{R}^2 \backslash B(1). \tag{1.6}$$

Notice that for any configuration of vortices (a_i, d_i) with $\sum d_i = D$, one has $u_\varepsilon^*(a_i, d_i) \in [U_d]$, whereas $u_\varepsilon^*(a_i, d_i) \in \{U_d\} + H^1(\mathbb{R}^2)$ if and only if (see Lemma 4.4) $\sum d_i a_i = 0$.

In the sequel, we decompose, for suitable choices of integer $n_0 \in \mathbb{N}^*$ the plane $\mathbb{R}^2$ as $\mathbb{R}^2 = B(2^{n_0}) \cup \left(\cup_{n=n_0}^{+\infty} A_n\right)$, where $A_n = B(2^{n+1}) \backslash B(2^n)$.

Definition 1. *Let $a_1, \ldots, a_l$ be l points in $\mathbb{R}^2$, let $d_i = \pm 1$, for $i = 1, \ldots, l$, and set $d = \Sigma d_i$. We say that a family $\{u_\varepsilon\}_{0<\varepsilon<1}$ of maps in $[U_d] + H^1(\mathbb{R}^2)$ is well-prepared with respect to the configuration (a_i, d_i) if and only if there exists $R = 2^{n_0} > \max\{|a_i|\}$ and $K_0 > 0$ such that*

$$\lim_{\varepsilon\to 0} \|Ju_\varepsilon - \pi \Sigma d_i \delta_{a_i}\|_{[C_c^{0,1}(B(R))]^*} = 0, \tag{1.7}$$

$$\sup_{0<\varepsilon<1} E_\varepsilon(u_\varepsilon, A_n) \le K_0 \quad \forall n \ge n_0, \tag{1.8}$$

and

$$\lim_{\varepsilon\to 0} [\mathcal{E}_{\varepsilon,U_d}(u_\varepsilon) - \mathcal{E}_{\varepsilon,U_d}(u_\varepsilon^\star(a_i, d_i))] = 0. \tag{1.9}$$

Our main theorem then can be stated as

Theorem 1. *Assume that $\{u_\varepsilon^0\}_{0<\varepsilon<1}$ is well-prepared with respect to the configuration (a_i^0, d_i). Let $\{a_i(t)\}_{i=1,\ldots,l}$ denote the solution of the point-vortex system with initial data $(a_i^0)_{i=1,\ldots,l}$, that is*

$$\begin{cases} \dfrac{d}{dt} a_i(t) = \displaystyle\sum_{j\neq i} d_j \frac{(a_i(t) - a_j(t))^\perp}{|a_i(t) - a_j(t)|^2}, & \text{for } i = 1, \ldots, l \\ a_i(0) = a_i^0, & \text{for } i = 1, \ldots, l \end{cases} \tag{1.10}$$

and let (T_*, T^*) denote its maximal interval of existence. Then, for every $T_* \le t < T^*$, the sequence $\{u_\varepsilon(.,t)\}_{0<\varepsilon<1}$ is well-prepared with respect to the configuration $(a_i(t), d_i)$.

Notice that the system (1.10) is Hamiltonian, with Hamiltonian given by the Kirchhoff-Onsager functional

$$W((a_i,d_i)) = -\pi \sum_{i\neq j} d_i d_j \log|a_i - a_j|$$

divided by π. As a matter of fact, this quantity appears in the computation of the expansion of the energy of $u_\varepsilon^\star(a_i,d_i)$. We will show in Section 4 that

$$\begin{aligned}\int_{B(R)} e_\varepsilon(u_\varepsilon^*(a_i,d_i)) = \pi\sum_{i=1}^{l} d_i^2|\log\varepsilon| + \pi d^2\log R + W((a_i,d_i))\\ + \sum_{i=1}^{l}\gamma(|d_i|) + o_{\varepsilon,R}(1),\end{aligned}$$

where $o_{\varepsilon,R}(1)\to 0$ as $\varepsilon\to 0$ and $R\to+\infty$, and where the constant $\gamma(|d_i|)$ is given by

$$\begin{aligned}\gamma(d) = \pi\Big(\int_0^1 |f_{1,d}'|^2 r\,dr - \int_1^\infty (1-f_{1,d}^2)\frac{d^2}{r}\,dr \\ + \int_0^1 f_{1,d}^2\frac{d^2}{r}\,dr + \int_0^\infty \tfrac{(1-f_{1,d}^2)^2}{2} r\,dr\Big).\end{aligned} \tag{1.11}$$

Therefore,

$$\begin{aligned}\mathcal{E}_{\varepsilon,U_d}(u_\varepsilon^\star(a_i,d_i)) = \pi\sum_{i=1}^{l} d_i^2|\log\varepsilon| + W((a_i,d_i)) + \sum_{i=1}^{l}\gamma(|d_i|)\\ + \int_{B(1)}\frac{|\nabla U_d|^2}{2} + o_\varepsilon(1).\end{aligned}$$

The proof of Theorem 1 borrows many ideas from [8] and [11]. The starting point in [8] is the remarkable identity for the evolution of the Jacobian, valid for any solution u of $(GP)_\varepsilon$,

$$\frac{d}{dt}Ju = (|u_{x_2}|^2 - |u_{x_1}|^2)_{x_1x_2} + (u_{x_1}\cdot u_{x_2})_{x_1x_1} - (u_{x_1}\cdot u_{x_2})_{x_2x_2}. \tag{1.12}$$

Integrating against a test function $\chi\in\mathcal{D}(\mathbb{R}^2)$, this yields

$$\frac{d}{dt}\int_{\mathbb{R}^2}\chi Ju = \int_{\mathbb{R}^2}(|u_{x_2}|^2 - |u_{x_1}|^2)\chi_{x_1x_2} + (u_{x_1}\cdot u_{x_2})(\chi_{x_1x_1} - \chi_{x_2x_2}), \tag{1.13}$$

which may be reformulated in complex notations as

$$\frac{d}{dt}\int_{\mathbb{R}^2}\chi Ju = -2\int_{\mathbb{R}^2}\mathrm{Im}\Big(\omega(u)\frac{\partial^2\chi}{\partial\bar z^2}\Big), \tag{1.14}$$

where ω denotes the Hopf differential given by

$$\omega(u) = |u_{x_1}|^2 - |u_{x_2}|^2 - 2iu_{x_1} \cdot u_{x_2} = 4\partial_z u \overline{\partial_{\bar z} u}. \tag{1.15}$$

To derive the motion law, one specifies (1.14) for test functions $\chi = \chi_{\rm aff}$ which are affine near a point a_i and vanish near all the other ones, and one takes advantage of the special form of the integral in (1.14) when u is close to a map $u^*_\varepsilon(a_i, d_i)$. Indeed, for the map

$$u^*(a_i, d_i) = \lim_{\varepsilon\to 0} u^*_\varepsilon(a_i, d_i) = \prod_{i=1}^{l} \left(\frac{z - a_i}{|z - a_i|}\right)^{d_i},$$

one computes

$$J((u^*(a_i, d_i)) = \pi \sum_{i=1}^{l} d_i \delta_{a_i} \quad \text{and} \quad \omega(u^*(a_i, d_i)) = -\left(\sum_{i=1}^{l} \frac{d_i}{z - a_i}\right)^2, \tag{1.16}$$

so that

$$\int_{\mathbb{R}^2} \chi J u^*(a_i, d_i) = \pi d_i \chi(a_i), \tag{1.17}$$

and

$$-2\int_{\mathbb{R}^2} {\rm Im}\Big(\omega(u^*(a_i, d_i))\frac{\partial^2 \chi}{\partial \bar z^2}\Big) = \pi \sum_{j\neq i} d_i d_j \frac{(a_i - a_j)^\perp}{|a_i - a_j|^2} \cdot \nabla\chi(a_i). \tag{1.18}$$

Replacing $u(\cdot, t)$ by $u^*(a_i(t), d_i)$ one obtains formally from (1.12)

$$\frac{d}{dt} a_i(t) \cdot \nabla\chi(a_i(t)) = \sum_{j\neq i} d_j \frac{(a_i(t) - a_j(t))^\perp}{|a_i(t) - a_j(t)|^2} \cdot \nabla\chi(a_i(t)).$$

By varying $\nabla\chi(a_i)$ one is therefore led to (1.10). A rigorous justification of the previous limiting procedure requires precise control of the distance between $u(\cdot, t)$ and $u^*_\varepsilon(a_i(t), d_i)$. For bounded domains (using $\tilde u^*_\varepsilon$ instead of u^*_ε), this control was provided combining conservation of energy with the already mentioned coercivity property near the $\tilde u^*_\varepsilon(a_i, d_i)$. In our context, the conservation of energy is replaced by the conservation of renormalized energy. The important new point is to establish a kind of coercivity of the renormalized energy about the reference map $u^*_\varepsilon(a_i, d_i)$, with respect to perturbations at infinity. For that purpose, we use almost minimizing properties of the map $\left(\frac{z}{|z|}\right)^d$ on annuli. This property is strongly connected to topological properties of Ginzburg-Landau maps on annuli, which we expose in the next section. Thanks to the coercivity properties on annuli, we are able to adapt the stoppiing time argument of [8] to our setting. This adaptation which leads to the proof of Theorem 1 is carried out in Section 7 and 8.

2. Topological sectors and almost minimizing properties

Let $A = B(2) \setminus B(1)$ be a reference annulus. Although the zero set of Ginzburg-Landau maps on A may be nonempty, a restriction on the Ginzburg-Landau energy allows us to define a notion of degree with suitable continuity properties. First, notice that by Sobolev embedding, for $u \in H^1(A)$, the restriction $u_{|\partial B(r)}$ is continuous for almost every $r \in [1,2]$. In particular, if it does not vanish, we may define the degree of $\frac{u}{|u|}_{|\partial B(r)}$. We therefore define, for $u \in H^1(A)$, the set $B(u)$ as the subset of of radii r of $[1,2]$ for which the restriction of u to $\partial B(r)$ is continuous and does not vanish. We set

$$T_d = \Big\{u \in H^1(A) \text{ s.t. } \exists B \subset B(u),\ \text{meas}(B) \geq \frac{3}{4}, \quad \text{and } \forall r \in B \ \text{deg(u}, \partial\text{B(r))} = \text{d}\Big\}.$$

It is clear from the definition that $\left(\frac{z}{|z|}\right)^d \in T_d$ and that $T_d \cap T_{d'} = \emptyset$ if $d \neq d'$. Of course, $\cup_{d\in\mathbb{Z}} T_d \neq H^1(A)$. Next, we restrict ourselves to the sublevel sets E_ε^Λ of $H^1(A)$ defined by

$$E_\varepsilon^\Lambda = \left\{u \in H^1(A),\ \text{s.t. } E_\varepsilon(u,A) \equiv \int_A e_\varepsilon(u) < \Lambda\right\}$$

and set

$$S_{d,\varepsilon}^\Lambda = E_\varepsilon^\Lambda \cap T_d.$$

The following result was proved by Almeida.

Theorem 2.1 ([1])**.** *Let $\Lambda > 0$ be given. There exists a constant $\varepsilon_\Lambda > 0$, such that for every $0 < \varepsilon < \varepsilon_\Lambda$, we have the partition*

$$E_\varepsilon^\Lambda = \bigcup_{d\in\mathbb{Z}} S_{d,\varepsilon}^\Lambda. \tag{2.1}$$

Moreover, the map

$$\deg\ : E_\varepsilon^\Lambda \to \mathbb{Z},\ u \in S_{d,\varepsilon}^\Lambda \mapsto d$$

is continuous.

It is also proved in [1] that, for $d \neq d'$, the threshold energy between $S_{d,\varepsilon}^\Lambda$ and $S_{d',\varepsilon}^\Lambda$ satisfies the lower bound, for every $0 < \varepsilon < \varepsilon_\Lambda$,

(2.2)
$$\inf\left\{\sup_{s\in[0,1]} E_\varepsilon(p(s),A),\ p \in \mathcal{C}([0,1],H^1(A)),\ p(0) \in S_{d,\varepsilon}^\Lambda,\ p(1) \in S_{d',\varepsilon}^\Lambda\right\} \geq \sigma|\log\varepsilon| \geq 2\Lambda,$$

where $\sigma > 0$ is some universal constant. The set $S^\Lambda_{d,\varepsilon}$ is referred to in [1] as the topological sector of degree d. An elementary computation shows that $E_\varepsilon(\exp id\theta, A) = \pi d^2 \log 2$, so that the condition $\exp id\theta \in E^\Lambda_\varepsilon$ is equivalent to $\Lambda > \pi d^2 \log 2$.

Our next results stresses the almost minimizing properties of the map $\exp id\theta$.[1] Note that the proof shows that every energy-minimizer in a topological sector $S^\Lambda_{d,\varepsilon}$ is equivariant.

Lemma 2.1. *Let $d \in \mathbb{Z}$. There exists a constant $C > 0$ depending only on d such that, for every $0 < \varepsilon < 1$, and for every Λ such that $S^\Lambda_{d,\varepsilon}$ is nonempty,*

$$\pi d^2 \log 2 = E_\varepsilon(\exp id\theta, A) \le \inf_{v \in S^\Lambda_{d,\varepsilon}} E_\varepsilon(v, A) + C\varepsilon^2. \tag{2.3}$$

Proof. It suffices of course to prove that (2.3) is satisfied for ε sufficiently small, the other cases being treated by considering a sufficiently large constant C. It is proved in [1] that E_ε satisfies the Palais-Smale condition in E^Λ_ε and also (therefore) that the infimum appearing in (2.3) is achieved in each topological sector $S^\Lambda_{d,\varepsilon}$. We denote by V_d one such minimizer. Since $E_\varepsilon(V_d) \le \pi d^2 \log 2$ and V_d is a solution of the Euler-Lagrange equation with Neumann boundary conditions on ∂A, it follows from the η-ellipticity results proved in [4] (for the interior) and [7] (Theorem 3, for the boundary) that $|V_d| \ge \frac{2}{3}$ on A. We may hence write

$$V_d = \rho \exp(i\varphi) \qquad \text{on } A$$

where $\rho \ge \frac{2}{3}$ and $\varphi : A \mapsto \mathbb{T}^1$ are smooth and satisfy

$$\int_0^{2\pi} \frac{\partial \varphi}{\partial \theta}(r \exp(i\theta))\, d\theta = 2\pi d \qquad \forall\, r \in [1, 2]. \tag{2.4}$$

We claim that V_d is equivariant, in the sense that

$$\exists\, \alpha \in \mathbb{T}^1 \text{ s.t. } \varphi(r \exp(i\theta)) = \alpha + d\theta. \tag{2.5}$$

Proof of the claim (2.5). Let W_d denote a minimizer among equivariant maps in $S^\Lambda_{d,\varepsilon}$. Up to a constant phase shift, we may assume that $W_d(r \exp(i\theta)) = M(r) \exp(id\theta)$, and we have

$$E_\varepsilon(W_d) = \pi \int_1^2 \Big[M'(r)^2 + M(r)^2 \frac{d^2}{r^2} + \frac{(1 - M(r)^2)^2}{2\varepsilon^2} \Big]\, r dr.$$

[1] See [2] for a somewhat related result.

The minimality of W_d amounts to

$$E_\varepsilon(W_d)=\min\left\{\pi\int_1^2\Big[m'(r)^2+m(r)^2\frac{d^2}{r^2}+\frac{(1-m(r)^2)^2}{2\varepsilon^2}\Big]\,rdr,\ \ m\in H^1([1,2])\right\}.$$

For $r\in[1,2]$, we set

$$m(r):=\fint_{\partial B(r)}\rho. \tag{2.6}$$

On the one hand, by Jensen's inequality we obtain

$$\begin{aligned}\int_A\frac{|\nabla\rho|^2}{2}=\pi\int_1^2 r\Big(\fint_{\partial B(r)}|\nabla\rho|^2\Big)\,dr &\geq \pi\int_1^2 r\Big(\fint_{\partial B(r)}|\nabla\rho|\Big)^2 dr\\ &\geq \pi\int_1^2(m'(r))^2 r\,dr.\end{aligned} \tag{2.7}$$

On the other hand,

$$\int_A\frac{\rho^2|\nabla\varphi|^2}{2}+\frac{(1-\rho^2)^2}{4\varepsilon^2}\geq\int_1^2 r\left[2\pi^2\frac{d^2}{r^2}\Big(\int_0^{2\pi}\rho^{-2}\Big)^{-1}+\int_0^{2\pi}\frac{(1-\rho^2)^2}{4\varepsilon^2}\right]dr. \tag{2.8}$$

Indeed, for each $r\in[1,2]$, minimization over $\varphi\in H^1([0,2\pi])$ with the constraint (2.4) leads to $\partial_\theta(\rho^2\partial_\theta\varphi)=0$ and therefore to

$$\frac{\partial\varphi}{\partial\theta}=\frac{C}{\rho^2}\qquad\text{where}\qquad C=2\pi d\Big(\int_0^{2\pi}\rho^{-2}\Big)^{-1},$$

from which (2.8) follows. Notice that since $\rho\geq\frac12$, for each $r\in[1,2]$

$$2\pi^2\frac{d^2}{r^2}\Big(\int_0^{2\pi}\rho^{-2}\Big)^{-1}+\int_0^{2\pi}\frac{(1-\rho^2)^2}{4\varepsilon^2}=2\pi^2\frac{d^2}{r^2}\Big(\int_0^{2\pi}S(\rho)\Big)^{-1}+\int_0^{2\pi}\frac{B(\rho)}{4\varepsilon^2}$$

where S is a smooth function with uniformly bounded derivatives at all orders which coincide with ρ^{-2} on the interval $[2/3,+\infty)$, and B is a strictly convex smooth function which coincides with $(1-\rho^2)^2$ on $[2/3,+\infty)$ and which satisfies the growth condition $B(\rho)\leq C(1+\rho^4)$. For ε sufficiently small, the functional

$$I(\rho)=2\pi^2\frac{d^2}{r^2}\Big(\int_0^{2\pi}S(\rho)\Big)^{-1}+\int_0^{2\pi}\frac{B(\rho)}{4\varepsilon^2}$$

is well-defined, smooth and strictly convex on $L^4([0,2\pi])$. It possesses therefore a unique critical point over the affine space defined by the constraint (2.6). Since the constant function $\rho(r\exp(i\theta)) \equiv m(r)$ is clearly such a critical point, it is also the unique minimizer and we are lead to improve (2.8) by

$$\int_A \frac{\rho^2|\nabla\varphi|^2}{2} + \frac{(1-\rho^2)^2}{4\varepsilon^2} \geq \int_1^2 \pi\Big[m(r)^2\frac{d^2}{r^2} + \frac{(1-m(r)^2)^2}{2\varepsilon^2}\Big]rdr. \tag{2.9}$$

Combining (2.7) and (2.9) we obtain

$$E_\varepsilon(V_d) \geq E_\varepsilon(W_d)$$

from which the claim follows since all the previous inequalities are strict unless V_d is equivariant.

Proof of Lemma 2.1 completed. Since $V_d = W_d$ up to a constant phase shift, we have $\rho(r\exp(i\theta)) = m(r)$. On the other hand, $m(r)$ solves the ordinary differential equation

$$-m''(r) - \frac{1}{r}m'(r) + \frac{d^2}{r^2}m(r) + \frac{1}{\varepsilon^2}(m(r)^2-1)m(r) = 0 \qquad \text{on } [1,2]$$

with Neumann boundary condition. Since the constant functions $r \mapsto 1$ and $r \mapsto \sqrt{1-\varepsilon^2 d^2}$ are respectively upper and lower solutions of the same equation, we infer from the maximum principle that

$$1 - \varepsilon^2 d^2 \leq m(r)^2 \leq 1 \qquad \forall\, r \in [1,2],$$

from which (2.3) follows, noticing that $|e_\varepsilon(U_d) - e_\varepsilon(W_d)| \leq C[(1-m(r)^2) + (1-m(r)^2)^2/\varepsilon^2]$ pointwise on A. ∎

3. Energy at infinity and topological sectors

We use Theorem 2.1 next to define the smallest radius from which the degree (at infinity) is well defined and constant, even for functions whose zero set is unbounded. For that purpose, given $\Lambda > \Lambda_d := 2\pi d^2 \log 2$ we set

$$S_d = S^\Lambda_{d,\varepsilon_\Lambda}.$$

An easy consequence of the definition of $[U_d]$ and (2.2) is

Corollary 3.1. *Let $d \in \mathbb{Z}$, $\Lambda > \Lambda_d$ and $u \in [U_d] + H^1(\mathbb{R}^2)$. There exists an integer $n \in \mathbb{N}^*$, such that for any $k \geq n$, the function defined on the reference annulus A by $z \mapsto u(2^k z)$, belongs to the topological sector S_d.*

This leads us to

Definition 2. *For $u \in [U_d] + H^1(\mathbb{R}^2)$, $n(u)$ is the smallest integer such that for $k \geq n(u)$ the function defined on the reference annulus A by $z \mapsto u(2^k z)$, belongs to the topological sector S_d. This integer is finite in view of Corollary* 3.1.

We would like to emphasize that our definition of $n(u)$ does not depend on ε. By scaling and summation over the annuli $A_n = B(2^{n+1}) \setminus B(2^n)$ we infer from Lemma 2.1

Lemma 3.1. *Let $d \in \mathbb{Z}$, and $u \in [U_d] + H^1(\mathbb{R}^2)$. Then, for any $k \geq n(u)$, we have, for every $0 < \varepsilon < \varepsilon_\Lambda$*

$$\int_{A^k} \left[e_\varepsilon(u) - \frac{|\nabla U_d|^2}{2} \right] \geq -C 2^{-2k} \varepsilon^2. \tag{3.1}$$

It follows in particular that

$$\lim_{R \to +\infty} \int_{B(R) \setminus B(2^k)} \left[e_\varepsilon(u) - \frac{|\nabla U_d|^2}{2} \right] \geq -C 2^{-2k} \varepsilon^2 \tag{3.2}$$

and therefore

$$\mathcal{E}_{\varepsilon, U_d}(u) \geq -C 2^{-2n(u)} \varepsilon^2 - \int_{B(2^{n(u)})} \frac{|\nabla U_d|^2}{2}. \tag{3.3}$$

Proof. By definition of $n(u)$, for $k \geq n(u)$ the map $T_k(u) : A \to \mathbb{C}$, $z \mapsto u(2^k z)$ belongs to S_d, so that in view of Lemma 2.1 we have for every $0 < \tilde{\varepsilon} < 1$

$$\pi d^2 \log 2 \leq E_{\tilde{\varepsilon}}(T_k u, A) + c \tilde{\varepsilon}^2. \tag{3.4}$$

On the other hand, by scaling we derive the identity

$$E_\varepsilon(u, A) = E_{2^{-k}\varepsilon}(T_k u, A),$$

whereas a direct compution shows that

$$\int_{A^k} \frac{|\nabla U_d|^2}{2} = \pi d^2 \log 2.$$

Choosing $\tilde{\varepsilon} = 2^{-k} \varepsilon$ in (3.4) inequality (3.1) follows. Inequality (3.2) is obtained by summation of (3.1), for $k \geq n$. Finally, inequality (3.3) follows from (3.2) and (1.5). ∎

Inequality (3.2) expresses the almost minimizing properties of U_d at infinity. Taking into account the fact that $u^*_\varepsilon(a_i, d_i)$ is very close to U_d at infinity, we infer

Lemma 3.2. *Let* $d \in \mathbb{Z}$, $u \in [U_d] + H^1(\mathbb{R}^2)$, $a_1, \dots, a_l \in \mathbb{R}^2$ *and* $d_1, \dots, d_l \in \mathbb{Z}^*$ *such that* $\sum d_i = d$. *Then, for* $k \geq 1 + \max\{\log_2|a_1|, \dots, \log_2|a_l|, n(u)\}$ *and* $R = 2^k$ *we have*

$$\int_{B(R)} e_\varepsilon(u) - e_\varepsilon(u_\varepsilon^*(a_i, d_i)) \leq \mathcal{E}_{\varepsilon,U_d}(u) - \mathcal{E}_{\varepsilon,U_d}(u_\varepsilon^*(a_i, d_i)) + \frac{C}{R}, \tag{3.5}$$

where C *depends only on* l *and* d.

Proof. In view of (1.5), we have

$$\begin{aligned} &\int_{B(R)} e_\varepsilon(u_\varepsilon) - e_\varepsilon(u_\varepsilon^*(a_i, d_i)) - [\mathcal{E}_{\varepsilon,U_d}(u_\varepsilon) - \mathcal{E}_{\varepsilon,U_d}(u_\varepsilon^*(a_i, d_i))] \\ &\qquad = \lim_{R' \to +\infty} \int_{B(R') \setminus B(R)} \Big[e_\varepsilon(u_\varepsilon) - \frac{|\nabla U_d|^2}{2} \Big] \\ &\qquad\qquad - \lim_{R' \to +\infty} \int_{B(R') \setminus B(R)} \Big[e_\varepsilon(u_\varepsilon^*(a_i, d_i)) - \frac{|\nabla U_d|^2}{2} \Big]. \end{aligned} \tag{3.6}$$

It follows from (3.2) that the first limit in (3.6) is bounded from below by $-CR^{-2}\varepsilon^2$. For the second limit, we first infer from the explicit form of u_ε^* and the know facts (see e.g. [12])

$$f_{\varepsilon,d} = 1 + O\Big(\Big(\frac{\varepsilon}{|z|}\Big)^2\Big) \qquad \text{and} \qquad |\nabla f_{\varepsilon,d}| = O\Big(\Big(\frac{\varepsilon}{|z|}\Big)^3\Big)$$

that for $|z| \geq R$,

$$|e_\varepsilon(u_\varepsilon^*(a_i, d_i))(z) - |\nabla U_d|^2(z)| \leq \frac{C}{|z|^3}, \tag{3.7}$$

where C depends only on d. Integrating (3.7) on $B(R') \setminus B(R)$ yields the conclusion (3.5). ■

Notice in particular that, if the family $\{u_\varepsilon\}_{0<\varepsilon<1}$ is well-prepared with respect to the configuration (a_i, d_i), and if the sequence $\{n(u_\varepsilon)\}_{0<\varepsilon<1}$ is bounded by a constant k (which we will prove always holds) then for $R \geq 2^{k+1}$

$$\int_{B(R)} e_\varepsilon(u_\varepsilon) \leq \int_{B(R)} e_\varepsilon(u_\varepsilon^*(a_i, d_i)) + \frac{C}{R} + o(1) \qquad \text{as } \varepsilon \to 0$$

and we may then rely on the coercivity results on $B(R)$ proved in [8, 11] to show that u_ε is sufficiently close to $u_\varepsilon^*(a_i, d_i)$.

To finish this section, we provide a proof of (1.5)

Lemma 3.3. *Let $U \in \mathcal{V}_*$ and $u \in \{U\} + H^1(\mathbb{R}^2)$. Then*

$$\mathcal{E}_{\varepsilon,U}(u) = \lim_{R\to\infty} \int_{B(R)} \Big[e_\varepsilon(u) - \frac{|\nabla U|^2}{2}\Big].$$

Proof. Writing $u = U + v$ where $v \in H^1(\mathbb{R}^2)$, we have by definition (1.4) and the continuity of integration

$$\mathcal{E}_{\varepsilon,U}(u) = \lim_{R\to+\infty} \int_{B(R)} \frac{|\nabla v|^2}{2} - (\Delta U)\cdot v + \int_{\mathbb{R}^2} \frac{(|u|^2-1)^2}{4\varepsilon^2}.$$

By integration by parts

$$\int_{B(R)} \frac{|\nabla v|^2}{2} - (\Delta U)\cdot v + \int_{\mathbb{R}^2} \frac{(|u|^2-1)^2}{4\varepsilon^2} = \int_{B(R)} e_\varepsilon(u) - \frac{|\nabla U|^2}{2} - \int_{\partial B(R)} \frac{\partial U}{\partial r} v.$$

We claim that

$$\lim_{R\to+\infty} \int_{\partial B(R)} \frac{\partial U}{\partial r} v = 0. \tag{3.8}$$

To establish the claim (3.8) it suffices in view of the assumption $|\nabla U(z)| \le C/\sqrt{|z|}$ and Cauchy-Schwarz inequality to establish that

$$\lim_{R\to+\infty} \int_{\partial B(R)} |v|^2 = 0. \tag{3.9}$$

This follows from the next lemma and the inclusion $W^{1,1}(\mathbb{R})$ into $\mathcal{C}_0(\mathbb{R})$. ■

Lemma 3.4. *For $v \in H^1(\mathbb{R}^2)$, the function $f : [1,+\infty) \to \mathbb{R}$ defined by $f(r) = \int_{\partial B(r)} |v|^2$ belongs to $W^{1,1}([1,+\infty))$.*

Proof. We write

$$f(r) = r \int_{\partial B(1)} |v|^2(ry)\, dy$$

so that

$$\begin{aligned} f'(r) &= \int_{\partial B(1)} |v|^2(ry)\, dy + 2r \int_{\partial B(1)} v(ry) \frac{\partial v}{\partial r}(ry)\, dy \\ &= \frac{1}{r} \int_{\partial B(r)} |v|^2(y)\, dy + 2 \int_{\partial B(r)} v(y) \frac{\partial v}{\partial r}(y)\, dy. \end{aligned}$$

Hence

$$|f'(r)| \le \Big(1 + \frac{1}{r}\Big) \int_{\partial B(r)} |v|^2(y)\, dy + \int_{\partial B(r)} |\nabla v|^2(y)\, dy$$

and the conclusion follows by integration. ■

4. Some properties of u^*_ε

In this Section we present some properties of the reference maps $u^*_\varepsilon(a_i, d_i)$, where $a_1, \dots, a_l$ are l distinct points in $\mathbb{R}^2$, and $d_1, .., d_l$ are l integers in $\mathbb{Z}^\star$. We set $d = \sum_{i=1}^l d_i$. We consider also the limiting map $u^*_0 \equiv u^*$ given by

$$u^*(a_i, d_i) = \prod_{i=1}^{l} \left(\frac{z - a_i}{|z - a_i|} \right)^{d_i}.$$

We recall that (see e.g. [3])

$$|\nabla u^*| = |\nabla \psi^*|, \quad \text{where} \quad \psi^* = \psi^*(a_i, d_i) \equiv \sum_{i=i}^{l} \log |z - a_i|, \tag{4.1}$$

in particular, the real-valued function $\psi^*(a_i, d_i)$ satisfies the equation

$$\Delta \psi^*(a_i, d_i) = 2\pi \sum_{i=i}^{l} d_i \delta_{a_i} \qquad \text{on } \mathbb{R}^2. \tag{4.2}$$

For $R \geq R_a := 2 \max_i \{|a_i|\}$ and $r \leq r_a := \frac{1}{8} \min_{i \neq j} \{|a_i - a_j|\}$, we introduce the domain

$$\Omega_{R,r} = B(R) \setminus \bigcup_{i=1}^{l} B(a_i, r).$$

We first have

Lemma 4.1. *As $R \to +\infty$,*

$$\int_{\Omega_{R,r}} \frac{|\nabla u^*(a_i, d_i)|^2}{2} = -\pi \sum_{i=1}^{l} d_i^2 \log r - \pi \sum_{i \neq j} d_i d_j \log |a_i - a_j| + \pi d^2 \log R + O(\tfrac{R_a}{R}), \tag{4.3}$$

whereas for $0 < \varepsilon < 1$,

$$\int_{\mathbb{R}^2 \setminus \cup_{i=1}^l B(a_i, r)} |e_\varepsilon(u^*_\varepsilon(a_i, d_i)) - \frac{|\nabla u^*(a_i, d_i)|^2}{2}| \leq C \left(\frac{\varepsilon}{r} \right)^2, \tag{4.4}$$

the constant C depending only on l and $\max_i(|d_i|)$.

Proof. The proof of identity (4.3) is classical (see e.g. [3]). It relies first on the identity (4.1), so that we may replace the integrand on the r.h.s of (4.3) by $|\nabla \psi^*(a_i, d_i)|^2$. To derive the result, one then uses equation (4.2) and integrations by parts, with suitable estimates for the boundary terms. For the proof of (4.4) one uses the aforementioned estimates for $f_{\varepsilon,d}$. ∎

Concerning the energy near the core, we have

Lemma 4.2. *Let* $0 < \varepsilon < 1$. *The following expansion holds for* $j = 1, \ldots, l$,

$$\int_{B(a_j,r)} e_\varepsilon(u_\varepsilon^*(a_i,d_i)) = \pi d_j^2 \log\frac{\varepsilon}{r} + \gamma(|d_j|) + O\Big(\frac{r}{r_a}\Big)^2 + O\Big(\frac{\varepsilon}{r}\Big)^2. \tag{4.5}$$

We omit the proof. Here γ denotes a fixed function $\gamma : \mathbb{N} \to \mathbb{R}^+$ whose values are given by (1.11). We are now in position to assert

Proposition 4.1. *Let* $0 < \alpha < 1$ *be given. Let* $0 < \varepsilon < 1$ *be such that*

$$r_a = \frac{1}{8}\min_{i\neq j}\big\{|a_i - a_j|\big\} \geq \varepsilon^\alpha. \tag{4.6}$$

Then, for $R > R_a + 1$, *we have*

$$\begin{aligned}\int_{B(R)} e_\varepsilon(u_\varepsilon^*(a_i,d_i)) = \pi\sum_{i=1}^{l} d_i^2|\log\varepsilon| - \pi\sum_{i\neq j} d_i d_j \log|a_i - a_j| \\ + \sum_{i=1}^{l}\gamma(|d_i|) + \pi d^2 \log R + O\Big(\frac{R_a}{R}\Big) + r_\varepsilon,\end{aligned} \tag{4.7}$$

where the remainder term r_ε *satisfies, for some constant* C_α *depending only on* α, l *and* $\max_i |d_i|$,

$$r_\varepsilon \leq C_\alpha \varepsilon^{1-\alpha}. \tag{4.8}$$

Proof. We write

$$\int_{B(R)} e_\varepsilon(u_\varepsilon^*(a_i,d_i)) = \int_{\Omega_{R,r}} e_\varepsilon(u_\varepsilon^*(a_i,d_i)) + \sum_{j=1}^{l}\int_{B(a_j,r)} e_\varepsilon(u_\varepsilon^*(a_i,d_i)).$$

It follows therefore from (4.3), (4.4) and (4.5) that

$$\begin{aligned}\int_{B(R)} e_\varepsilon(u_\varepsilon^*(a_i,d_i)) = \pi\sum_{i=1}^{l} d_i^2|\log\varepsilon| + W(a_i,d_i) + \pi d^2\log R + \\ O\Big(\frac{R_a}{R}\Big) + O\Big(\frac{\varepsilon}{r}\Big)^2 + O\Big(\frac{r}{r_a}\Big)^2.\end{aligned}$$

Choosing r such that $\frac{\varepsilon}{r} = \frac{r}{r_a}$, we obtain the conclusion (4.7) with the estimate (4.8), taking into account (4.6). ∎

Proposition 4.1 has the following consequence for the renormalized energy

Corollary 4.1. *Let $0 < \alpha < 1$ be given and let $0 < \varepsilon < 1$ be such that the condition* (4.6) *holds. Then, we have*

$$\mathcal{E}_{\varepsilon,U_d}(u_\varepsilon^\star(a_i,d_i)) = \pi\sum_{i=1}^{l} d_i^2|\log\varepsilon| + W(a_i,d_i) + \sum_{i=1}^{l}\gamma(|d_i|) + \int_{B(1)}\frac{|\nabla U_d|^2}{2} + r_\varepsilon, \tag{4.9}$$

where $d = \sum_{i=1}^{l} d_i$ and the remainder term r_ε satisfies the bound (4.8).

On exterior domains, similar computations yield

Lemma 4.3. *For $R > R_a$ we have*

$$\left|\int_{B(2R)\setminus B(R)} e_\varepsilon(u_\varepsilon^*(a_i,d_i)) - \pi d^2\log 2\right| \le C\frac{R_a}{R}.$$

We finish this section with the following elementary observations. First we have

Lemma 4.4. *Let $\{(a_i,d_i)\}_{i=1,\dots,l}$ and $\{(a_i',d_i')\}_{i=1,\dots l'}$ be two vortex configurations. For every $0 \le \varepsilon < 1$ the difference $u_\varepsilon^*(a_i,d_i) - u_\varepsilon^*(a_i',d_i')$ belongs to $L^2(\mathbb{R}^2)$ if and only if*

$$\sum_{i=1}^{l} d_i = \sum_{i=1}^{l'} d_i' \qquad \text{and} \qquad \sum_{i=1}^{l} d_i a_i = \sum_{i=1}^{l'} d_i' a_i'. \tag{4.10}$$

In particular, for $0 < \varepsilon < 1$ the map $u_\varepsilon^*(a_i,d_i)$ belongs to $\{u_\varepsilon^*(a_i',d_i')\} + H^1(\mathbb{R}^2)$ if and only if condition (4.10) *is satisfied.*

Proof. For a given vortex configuration $\{(a_i,d_i)\}_{i=1,\dots,l}$ we write for $i = 1,\dots,l$,

$$\left(\frac{z-a_i}{|z-a_i|}\right)^{d_i} = \left(\frac{z}{|z|}\right)^{d_i}\left(\frac{1-\frac{a_i}{z}}{|1-\frac{a_i}{z}|}\right)^{d_i}.$$

Expanding for $|z|\to\infty$, we have

$$\begin{aligned}\left(\frac{1-\frac{a_i}{z}}{|1-\frac{a_i}{z}|}\right)^{d_i} &= \left(1 - \frac{d_i a_i}{z} + O\Big(\frac{1}{z^2}\Big)\right)\left(1 + \mathrm{Re}\frac{d_i a_i}{z} + O\Big(\frac{1}{z^2}\Big)\right)\\ &= 1 - \mathrm{Im}\frac{d_i a_i}{z} + O\Big(\frac{1}{z^2}\Big),\end{aligned}$$

so that

$$u^*(a_i,d_i) = \left(\frac{z}{|z|}\right)^d - \left(\frac{z}{|z|}\right)^d\left(\mathrm{Im}\sum_{i=1,\dots,l}\frac{d_i a_i}{z} + O\Big(\frac{1}{z^2}\Big)\right).$$

In particular $u^*(a_i,d_i) - u^*(0,d')$ belongs to $L^2(\mathbb{R}^2)$ if and only if $d = d'$ and $\sum_{i=1}^l d_i a_i = 0$. The conclusion follows for $\varepsilon = 0$ translating the origin. For the general case $0 < \varepsilon < 1$, one observes that, since $1 - f_{\varepsilon,d}$ belongs to $L^2(\mathbb{R}^2)$, the difference

$$u^*(a_i,d_i) - u^*_\varepsilon(a_i,d_i) = \Big(1 - \prod_{i=1}^l f_{\varepsilon,d_i}(|z-a_i|)\Big) u^*(a_i,d_i)$$

belongs to $L^2(\mathbb{R}^2)$. ∎

Next, we have

Lemma 4.5. *The renormalized energy $\mathcal{E}_{d,U_d}$ is bounded from below if and only if $|d| \leq 1$.*

Proof. Assume $d \geq 2$. We consider the configuration of two vortices $\{a_i^n, d_i\}$ given by $a_1^n = (0,0)$, $d_1 = 1$, $a_2^n = (n,0)$, $d_2 = d - 1 > 0$. We deduce from (4.9) that $\mathcal{E}_{\varepsilon,U_d}(u^\star_\varepsilon(a_i,d_i))$ behaves like $-\log n$ as $n \to +\infty$, and hence the conclusion.

When $d = 1$, the conclusion follows from the locally minimizing properties of u_1^* established by Mironescu [14].

When $d = 0$ the renormalized energy is defined by integration of a pointwise non negative function. ∎

5. Kirchhoff-Onsager functional and the renormalized energy

Proposition 4.1 shows that, removing the diverging and constant part of the energy, the Kirchhoff-Onsager functional is the next important part of the expansion in (4.7). In order to bridge our work with coercivity properties derived on bounded domains in [8, 11] we need to compare the Kirchhoff-Onsager functional with the renormalized energy considered there. For that purpose, let Ω be a bounded simply connected domain in $\mathbb{R}^2$ with $\mathcal{C}^1$ boundary, and let $G(a_i,d_i)$ be the function defined on Ω by

$$\Delta G(a_i,d_i) = 2\pi \sum_{i=1}^l d_i \delta_{a_i} \text{ in } \Omega, \qquad G \equiv 0 \text{ on } \partial\Omega.$$

Let also $H(\cdot,y)$, for $y \in \Omega$, denote the solution of

$$\Delta_x H(\cdot,y) = 0 \text{ in } \Omega, \qquad H(x,y) = -\log|x-y| \text{ for } x \in \partial\Omega.$$

Then, for $x \in \Omega$,

$$G(a_i, d_i)(x) = \sum_{i=1}^{l} d_i \left[\log|x - a_i| + H(x, a_i)\right].$$

The renormalized energy $W_\Omega(a_i, d_i)$ is defined to be

$$W_\Omega(a_i, d_i) = -\pi\bigg(\sum_{i\neq j} d_i d_j \log|a_i - a_j| + \sum_{i,j} d_i d_j H(a_i, a_j)\bigg). \tag{5.1}$$

We next specify the domain Ω to the case $\Omega = B(R)$ and set $W_R \equiv W_{B(0,R)}$.

Proposition 5.1. *Let $\{a_i, d_i\}_{i=1,\dots,l}$ be a configuration of vortices and set $R_a = 2\max\{|a_i|\}$. Then, for $R > R_a + 1$, we have*

$$W_R(a_i, d_i) = W(a_i, d_i) + \pi d^2 \log R + O\Big(\frac{R_a}{R}\Big), \tag{5.2}$$

where $d = \sum_{i=1}^{l} d_i$.

Proof. In view of (5.1) we have

$$W_R(a_i, d_i) - W(a_i, d_i) = -\pi \sum_{i,j} d_i d_j H(a_i, a_j).$$

Since H is harmonic in each of its variables and for each j

$$H(\cdot, a_j) = -\log R + O\Big(\frac{R_a}{R}\Big)$$

on the boundary of Ω, we obtain

$$\sum_{i,j} d_i d_j H(a_i, a_j) = -\sum_{i,j} d_i d_j \log R + O\Big(\frac{R_a}{R}\Big)$$

and the conclusion follows from the identity $\sum_{i,j} d_i d_j = d^2$. (In fact we give an explicit formula for $H(x, a_i)$ below.) ■

Finally, we also recall the canonical harmonic map u^*_Ω on a bounded domain Ω, with vortices (a_i, d_i) and Neumann boundary conditions. This is characterized (up to a constant phase) by the fact that

$$ju^*_\Omega = \sum_{i=1}^{\ell} d_i \nabla \times \left[\log|x - a_i| + H(x, a_i)\right]$$

for G as defined above, depending on Ω. When $\Omega = B(R)$ we wil write u^*_R. Similar to the previous proposition, we have

Proposition 5.2. *Let* $\{a_i, d_i\}, R_a, R$ *be as in Proposition* 5.1. *Then*

$$|ju^*_R(x; a_i, d_i) - ju^*(x; a_i, d_i)| \ = O(\frac{R_a}{R^2}). \tag{5.3}$$

for all $x \in B(R)$.

Proof. It is a classical fact that $H(x, y) = -\log\left(\frac{|y|}{R}|x - \frac{yR^2}{|y|^2}|\right)$ as long as $y \neq 0$, and that $H(x, 0) = -\log R$. Thus

$$|ju^*_R(x; a_i, d_i) - ju^*(x; a_i, d_i)| \ \leq \ \sum_{i=1}^{\ell} |\nabla \times H(x, a_i)| = \sum_{i=1}^{\ell} |x - \frac{a_i R^2}{|a_i|^2}|^{-1}.$$

If $a_i = 0$ for some i, the corresponding term in the sum is of course replaced by 0. It is now easy to deduce the conclusion, since $|x| < R^2/R_a$ in B_R, and $|\frac{a_i R^2}{|a_i|^2}| \geq 2R^2/R_a$ for all i. ■

6. Coercivity for $\mathcal{E}_{\varepsilon,U_d}$

In this Section, we adapt to our setting coercivity results established in [8, 11]. To that purpose, for a given configuration of vortices $\{a_i, d_i\}_{i=1,\dots,l}$ and $u \in [U_d] + H^1(\mathbb{R}^2)$ where $d = \sum d_i$, the excess energy is defined as

$$\Sigma_\varepsilon \equiv \Sigma_\varepsilon(u, a_i, d_i) = \mathcal{E}_{\varepsilon,U_d}(u) - \mathcal{E}_{\varepsilon,U_d}(u^*_\varepsilon(a_i, d_i)).$$

We also set

$$r_a = \frac{1}{8}\min\{|a_i - a_j|, \ i \neq i\} \quad \text{and} \quad R_a = \max\{|a_i|\}.$$

We have

Theorem 6.1. *Assume that* $d_i \in \{-1, +1\}$ *for all* i, *and let* $r \leq r_a$ *and* $R > R_a$ *be given. There exist constants* $\varepsilon_0 > 0$ *and* $\eta_0 > 0$ *(depending only on* l, r, r_a, R_a, R*) such that if* $\varepsilon \leq \varepsilon_0$,

$$\eta \equiv \left\| Ju - \pi \sum d_i \delta_{a_i} \right\|_{[W_0^{1,\infty}(B(R))]^*} \leq \eta_0 \tag{6.1}$$

and

$$2^{n(u)} \leq R, \tag{6.2}$$

then

$$\int_{B(R)\setminus \cup B(a_i,r)} e_\varepsilon(|u|) + \frac{1}{8}\left|\frac{j(u)}{|u|} - j(u^*(a_i, d_i))\right|^2 \leq \Sigma_\varepsilon + C\big(\eta, \varepsilon, \frac{1}{R}\big) \tag{6.3}$$

where C is a continuous function which vanishes at the origin. Moreover, there exist points $b_i \in B(a_i, r/2)$ such that

$$\left\| Ju - \pi \sum d_i \delta_{b_i} \right\|_{[W_0^{1,\infty}(B(R))]^*} \leq D(R, \Sigma_\varepsilon)\varepsilon|\log\varepsilon| \tag{6.4}$$

where D is a continuous function on $\mathbb{R}^2$.

Condition (6.1) suggests that the vortex structure of u inside $B(R)$ is sufficiently well approximated by the configuration (a_i, d_i), and condition (6.2) ensures that no vortex of u is hidden far away. Under those assumptions, the conclusion (6.3) asserts that the deviation of u from the canonical map $u^*(a_i, d_i)$ is controlled by the excess Σ_ε away from the vortices.

For sake of conciseness we will not present a self-contained proof Theorem 6.1 but instead rely on Theorem 2 in [11], which we use as a black box.

Proof of Theorem 6.1. First notice that there exists $\varepsilon_2 > 0$ such that if $\varepsilon < \varepsilon_2$ and $\varepsilon E_\varepsilon(u, B(R)) > \sqrt{\varepsilon}$ then (6.3) trivially holds. Indeed, since $E_\varepsilon(u_\varepsilon^*(a_i, d_i)) \leq C|\log\varepsilon| + \log R$, for ε small enough one even has $\frac{1}{2}E_\varepsilon(u) \leq \Sigma_\varepsilon$ so that (6.3) holds with $C = 0$. In the sequel, we assume that $\varepsilon \leq \varepsilon_2$ and $\varepsilon E_\varepsilon(u, B(R)) \leq \sqrt{\varepsilon}$.

Let K_1 be the constant given by Theorem 2 in [11]. We choose $0 < \varepsilon_1 \leq \varepsilon_2$ and η_0 sufficiently small so that

$$4x \leq \frac{r_a}{l^3}\sqrt{x} \quad \forall x \in [0, \eta_0] \quad \text{and} \quad \frac{r_a}{l^3}\sqrt{\eta_0 + \sqrt{\varepsilon_1}} \leq \min\left(r, \frac{r_a}{lK_1}\right). \tag{6.5}$$

Finally, we choose $0 < \varepsilon_0 \leq \varepsilon_1$ sufficiently small so that

$$\varepsilon_0 \sqrt{\log\left(\frac{r_a}{lK_1}\right)} \leq \eta_0.$$

Assume that u is such that (6.1) is satisfied. We distinguish two cases.

Case 1: $\varepsilon\sqrt{\log(\frac{r_a}{\varepsilon})} \leq \eta$. In that case, we will apply Theorem 2 in [11] with the choice $s_\varepsilon = \eta$. From (6.5) and the fact that $\varepsilon \leq \varepsilon_2$ we infer that $4s_\varepsilon \leq \sigma^* \equiv \sqrt{\frac{r_a}{l^3}(s_\varepsilon + \varepsilon E_\varepsilon(u, B(R)))} \leq \frac{r_a}{lK_1}$ and therefore the conditions of Theorem 2 in [11] are satisfied and we get

$$\begin{aligned} &\int_{B(R)\setminus \cup B(a_i,\sigma^*)} e_\varepsilon(|u|) + \frac{1}{4}\left|\frac{j(u)}{|u|} - j(u_R^*(a_i, d_i))\right|^2 \\ &\qquad \leq E_\varepsilon(u, B(R)) - \pi l|\log\varepsilon| - l\gamma(1) - W_R(a_i, d_i) + C\sqrt{\frac{l^5}{r_a}(\eta + \sqrt{\varepsilon})}, \end{aligned} \tag{6.6}$$

where C is universal.

By (6.5), $\sigma^* \le r$ and we may therefore replace σ^* by r in (6.6). By (6.2) we may apply Lemma 3.2, and combining (3.5), (4.7) and (5.2) we obtain

$$E_\varepsilon(u, B(R)) - \pi l|\log\varepsilon| - l\gamma(1) - W_R(a_i, d_i) \le \Sigma_\varepsilon + C\Big(\frac{R_a}{R} + \sqrt{\varepsilon}\Big).$$

Also, it follows from Proposition 5.2 that

$$\|ju_R(a_i, d_i) - ju(a_i, d_i)\|^2_{L^2(B(R)} \le CR_a^2/R^2 \le CR^a/R.$$

Combining these estimates we find that

$$\int_{B(R)\setminus\cup B(a_i,r)} e_\varepsilon(|u|) + \frac{1}{8}\left|\frac{j(u)}{|u|} - j(u^*(a_i, d_i))\right|^2 \le \Sigma_\varepsilon + C\left(\sqrt{\frac{l^5}{r_a}(\eta + \sqrt{\varepsilon})} + \frac{R_a}{R} + \sqrt{\varepsilon}\right).$$

Case 2: $\varepsilon\sqrt{\log(\frac{r_a}{\varepsilon})} > \eta$. In that case, we apply Theorem 2 in [11] with the choice $s_\varepsilon = \varepsilon\sqrt{\log(\frac{r_a}{\varepsilon})}$. This similarly leads to

$$\int_{B(R)\setminus\cup B(a_i,r)} e_\varepsilon(|u|) + \frac{1}{8}\left|\frac{j(u)}{|u|} - j(u^*(a_i, d_i))\right|^2 \le \Sigma_\varepsilon + C\left(\sqrt{\frac{l^5}{r_a}\varepsilon\log\left(\frac{r_a}{\varepsilon}\right) + \sqrt{\varepsilon}} + \frac{R_a}{R} + \sqrt{\varepsilon}\right).$$

The maximum between those two error terms may serve as a definition of the function C which appears in (6.3). Without loss of generality, we may assume that $\eta_0 \le r_a/(8K_2 l^5)$, where K_2 is the constant appearing in [11, Theorem 3], so that the existence of the points b_i and the estimate (6.4) follow by [11, Theorem 3]. ■

7. Lipschitz continuity of vortex paths

The results in this section apply to initial data slightly more general than the one in Definition 1. More precisely, we keep assumptions (1.7) and (1.8), and we replace (1.9) by

$$\text{(7.1)} \qquad \sup_{0<\varepsilon<1}\,[\mathcal{E}_{\varepsilon,U_d}(u_\varepsilon) - \mathcal{E}_{\varepsilon,U_d}(u_\varepsilon^\star(a_i, d_i))] \le K_1$$

for some constant $K_1 < +\infty$. Without loss of generality, we may assume, increasing possibly K_0, that $K_0 > K_1 + \pi d^2 \log 2$.

The main result in this section is

Theorem 7.1. *Let* (a_i^0, d_i) *be a configuration of vortices such that* $d_i \in \{-1, +1\}$. *Let* $(u_\varepsilon^0)_{0<\varepsilon<1}$ *satisfy* (1.7),(1.8) *and* (7.1) *with* $u_\varepsilon^0 \in [U_d] + H^1(\mathbb{R}^2)$ *for all* $0 < \varepsilon < 1$. *There exist a time* $T > 0$, *depending only on* K_0, K_1, r_a *and* R, *a sequence* $\varepsilon_k \to 0$, *and Lipschitz paths* $t \mapsto b_i(t)$ *defined on* $[0, T]$ *such that* $b_i(0) = a_i^0$ *and*

$$\sup_{t\in[0,T]} \|Ju_{\varepsilon_k}(\cdot, t) - \pi\Sigma d_i \delta_{b_i(t)}\|_{[W_0^{1,\infty}(B(R))]^*} \to 0, \quad \text{as } k \to +\infty. \tag{7.2}$$

Moreover, there exist constants $C_0 > 0$ *and* $C_1 > 0$, *depending only on* K_0, K_1, r_a *and* R, *such that for all* $t \in [0, T]$, $n \geq n_0$ *and* $k \in \mathbb{N}$,

$$E_{\varepsilon_k}(u_{\varepsilon_k}(\cdot, t), A_n) \leq C_0, \tag{7.3}$$

and

$$\mathcal{E}_{\varepsilon_k,U_d}(u_{\varepsilon_k}(\cdot, t)) - \mathcal{E}_{\varepsilon_k,U_d}(u^*_{\varepsilon_k}(b_i(t), d_i)) \leq C_1. \tag{7.4}$$

The proof relies on several arguments which we present separately.

Lemma 7.1. *Assume* $\Lambda > K_0$ *and let* $(u_\varepsilon^0)_{0<\varepsilon<1}$ *be as in Theorem* 7.1. *Then, for* $0 < \varepsilon < \varepsilon_\Lambda$ *and for* $n \geq n_0$ *we have* $u_\varepsilon^0(2^n\cdot) \in S_d = S^\Lambda_{d,\varepsilon_\Lambda}$.

Proof. By Corollary 3.1, $u_\varepsilon^0(2^n\cdot) \in S_d$ for n sufficiently large, say $n \geq n_1(\varepsilon)$, depending possibly on ε. Since by assumption $\Lambda > K_0$, for each $n \geq n_0$ the map $u_\varepsilon^0(2^n\cdot)$ belongs to some $S_{d(n)}$. It remains to prove that $d(n) \equiv d$ for all $n \geq n_0$. Assume by contradiction that this does not hold, and let N be the largest integer such that $d(N) \neq d$. Consider the mapping $p : [0, 1] \to H^1(A)$ defined by $p(s) = u_\varepsilon^0(2^{N+s}\cdot)$. We have $p(0) \in S_{d(N)}$ whereas $p(1) \in S_d$. It follows therefore by (2.2) that there exists $s \in [0, 1]$ such that $E_\varepsilon(p(s)) > 2\Lambda$. In particular, this would imply

$$E_\varepsilon(u_\varepsilon^0, A_N \cup A_{N+1}) > 2\Lambda > 2K_0,$$

a contradiction with assumption (1.8). ■

Throughout the rest of this paper, we assume

$$\Lambda > K_0. \tag{7.5}$$

The core argument (as in [8]) relies on the evolution equation (1.12) for the Jacobians.

Lemma 7.2. *Let* η_0 *be given by Theorem* 6.1 *for the choice* $r = r_a$, *and let* T_ε *be the largest time for which for all* $0 \leq s \leq T_\varepsilon$

$$\|Ju_\varepsilon(\cdot, s) - \pi\Sigma d_i \delta_{a_i^0}\|_{[\mathcal{W}_0^{1,\infty}(B(R))]^*} \leq \eta_0 \quad \text{and} \quad u_\varepsilon(\cdot, s) \in S_d \ \forall n \geq n_0.$$

Then we have

$$\liminf_{\varepsilon\to 0} T_\varepsilon \geq T(r_a, R, K_0, K_1) > 0. \tag{7.6}$$

Proof. First notice that since $u_\varepsilon(\cdot,t) - u_\varepsilon^0$ belongs to $\mathcal{C}(\mathbb{R}, H^1(\mathbb{R}^2))$, the map $t \mapsto Ju_\varepsilon(\cdot,t)$ is continuous with values in $L^1(B(R))$ and the maps $t \mapsto E_\varepsilon(u_\varepsilon(\cdot,t), A_n)$ are uniformly continuous with respect to $n \geq n_0$. This implies that

$$T_\varepsilon > 0 \qquad \forall\, 0 < \varepsilon < \varepsilon_\Lambda.$$

Step 1. We have, for $s \in [0, T_\varepsilon]$

$$E_\varepsilon(u_\varepsilon(\cdot,s), B(R)) \geq \pi l |\log \varepsilon| - C$$

for some constant $C > 0$ depending only on r_a and R.

Proof. This directly follows from the positivity of e_ε and the inequality

$$E_\varepsilon(u_\varepsilon(\cdot,s), B(a_i^0, r_a)) \geq \pi |\log \varepsilon| - C \tag{7.7}$$

valid for any $i = 1, \dots, l$. This last inequality is itself a consequence of the bound $\|Ju_\varepsilon(\cdot,s) - \pi d_i \delta_{a_i^0}\|_{[W_0^{1,\infty}(B(a_i^0, r_a))]^*} \leq \eta_0$ and Theorem 3 in [10].

Step 2. There exists a constant $D > 0$ such that for all $s \in [0, T_\varepsilon]$

$$E_\varepsilon(u_\varepsilon(\cdot,s), A_n) \leq D, \qquad \forall n \geq n_0.$$

Proof. We write the difference $E_\varepsilon(u_\varepsilon(\cdot,t), A_n) - E_\varepsilon(u_\varepsilon^*(a_i^0, d_i), A_n)$ as

$$\begin{aligned}\sum_{k=n_0,\, k\neq n}^{+\infty} &\big(E_\varepsilon(u_\varepsilon^*(a_i^0, d_i), A_k) - E_\varepsilon(u_\varepsilon(\cdot,t), A_k)\big) \\ &+ E_\varepsilon(u_\varepsilon^*(a_i^0, d_i), B(R)) - E_\varepsilon(u_\varepsilon(\cdot,t), B(R)) \\ &\qquad + \mathcal{E}_{\varepsilon,U_d}(u_\varepsilon(\cdot,t)) - \mathcal{E}_{\varepsilon,U_d}(u_\varepsilon^*(a_i^0, d_i)).\end{aligned} \tag{7.8}$$

For the first term on the r.h.s of (7.8) we invoke Lemma 3.1 and Lemma 4.3 to assert that

$$\sum_{k=n_0,\, k\neq n}^{+\infty} E_\varepsilon(u_\varepsilon^*(a_i^0, d_i), A_k) - E_\varepsilon(u_\varepsilon(\cdot,t), A_k) \leq C_1. \tag{7.9}$$

The second term is bounded thanks to Step 1, whereas the last one is bounded by conservation of $\mathcal{E}_{\varepsilon,U_d}$ and hypothesis (7.1). The conclusion follows.

Notice that the computation in Step 2 leading to the definition of the constant D does not depend on the precise choice of the constant Λ entering in the definition of the topological sectors S_d. Therefore, we may assume that

$$\Lambda > D.$$

Step 3. We claim that

$$n(u_\varepsilon(\cdot, T_\varepsilon)) \le n_0 \qquad \forall\, t \in [0, T_\varepsilon]$$

and that

$$\|Ju_\varepsilon(\cdot, T_\varepsilon) - \pi \Sigma d_i \delta_{a_i^0}\|_{[\mathcal{W}_0^{1,\infty}(B(R))]^*} = \eta_0.$$

This is an immediate consequence of Step 2, (2.2) for the first assertion, and the definition of T_ε for the second assertion.

At this stage, we invoke Theorem 6.1 with the choice $r = r_a$. Assumptions (6.1) and (6.2) are satisfied by definition of T_ε so that we may assert that for all $s \in [0, T_\varepsilon]$

$$\int_{B(R)\setminus \cup B(a_i^0, r_a)} e_\varepsilon(|u_\varepsilon(\cdot, s)|) + \frac{1}{8}\left|\frac{j(u_\varepsilon(\cdot, s))}{|u_\varepsilon(\cdot, s)|} - j(u^*(a_i^0, d_i))\right|^2 \le C$$

where C is independent of ε and s.

Next, for $i = 1, \ldots, l$, we consider real-valued functions $\chi_i \in \mathcal{D}(B(a_i^0, \frac{3r_a}{2}))$ such that χ_i is affine on $B(a_i^0, r_a)$ and $|\nabla \chi_i(a_i^0)| = 1$.

Step 4. We have, for $0 \le s, t \le T_\varepsilon$

$$\left|\int \langle Ju_\varepsilon(\cdot, t) - Ju_\varepsilon(\cdot, s), \chi_i\rangle\right| \le C|D^2\chi_i\|_{L^\infty}|t - s|. \tag{7.10}$$

Proof. Since χ_i is affine on $B(a_i^0, r_a)$, it follows that $\mathrm{supp}(\frac{\partial^2 \chi_i}{\partial \bar{z}^2}) \subset B(a_i^0, \frac{3r_a}{2}) \setminus B(a_i^0, r_a)$. The conclusion then follows from (1.12), (1.15) and the bound provided in Step 2, which yields, for all $s \in [0, T_\varepsilon]$,

$$\|\omega(u_\varepsilon(\cdot, s))\|_{L^1(B(a_i^0, \frac{3r_a}{2})\setminus B(a_i^0, r_a))} \le C.$$

Step 5. Proof of Lemma 7.2 completed. For $0 \le t \le T_\varepsilon$, let $b_i(t) \in B(a_i^0, \frac{r_a}{2})$ be the points provided by Theorem 6.1 for $u_\varepsilon(\cdot, t)$ and satisfying (6.4), that is

$$\|Ju_\varepsilon(\cdot, t) - \pi \sum d_i \delta_{b_i(t)}\|_{[W_0^{1,\infty}(B(R))]^*} \le C\varepsilon|\log \varepsilon|. \tag{7.11}$$

We write (in this Step $\|\cdot\|$ stands for $\|\cdot\|_{[W_0^{1,\infty}(B(R))]^*}$)

$$\begin{aligned}\eta_0 &= \|Ju_\varepsilon(\cdot, T_\varepsilon) - \pi\Sigma d_i \delta_{a_i^0}\| \\ &\le \|Ju_\varepsilon(\cdot, T_\varepsilon) - \pi\Sigma d_i\delta_{b_i(T_\varepsilon)}\| + \|\pi\Sigma d_i \delta_{a_i^0} - \pi \Sigma d_i \delta_{b_i(T_\varepsilon)}\|.\end{aligned}$$

From the definition of r_a and the fact that $b_i(T_\varepsilon) \in B(a_i^0, \frac{r_a}{2})$ is follows that there exist real-valued functions $\chi_{i,\varepsilon}$ with the same properties as in Step 4, such that

$$\|\pi\Sigma d_i\delta_{a_i^0} - \pi\Sigma d_i\delta_{b_i(T_\varepsilon)}\| = \left|\langle \pi\Sigma d_i \delta_{a_i^0} - \pi \Sigma d_i \delta_{b_i(T_\varepsilon)}, \Sigma\chi_{i,\varepsilon}\rangle\right|.$$

Moreover, the functions $\chi_{i,\varepsilon}$ may be choosed in such a way that $\|D^2\chi_{i,\varepsilon}\|_{L^\infty} \leq C$ independently of i and ε. Therefore, we obtain using Step 4 and (7.11)

$$\begin{aligned}\eta_0 &\leq \|Ju_\varepsilon(\cdot,T_\varepsilon) - \pi\Sigma d_i\delta_{b_i(T_\varepsilon)}\| + \left|\langle \pi\Sigma d_i\delta_{a_i^0} - \pi\Sigma d_i\delta_{b_i(T_\varepsilon)}, \Sigma\chi_{i,\varepsilon}\rangle\right| \\ &\leq C\left(\|Ju_\varepsilon(\cdot,T_\varepsilon) - \pi\Sigma d_i\delta_{b_i(T_\varepsilon)}\| + \|Ju_\varepsilon^0 - \pi\Sigma d_i\delta_{a_i^0}\|\right) \\ &\quad + \left|\int \langle Ju_\varepsilon(\cdot,T_\varepsilon) - Ju_\varepsilon^0, \Sigma\chi_{i,\varepsilon}\rangle\right| \\ &\leq C\varepsilon|\log\varepsilon| + CT_\varepsilon.\end{aligned}$$

The conclusion (7.6) follows. ■

We are now in position to present the

Proof of Theorem 7.1. Using Lemma 7.2, (7.11) and a diagonal argument, we obtain the existence of a sequence $\varepsilon_n \to 0$ and paths $b_i(t)$ defined for $t \in \mathcal{Q} \equiv \mathbb{Q} \cap [0, T(r_a, R, K_0, K_1)]$ such that $b_i(0) = a_i^0$ and the norm in (7.2) converges to zero for each $t \in \mathcal{Q}$. Passing to the limit $\varepsilon_k \to 0$ in (7.10) we infer that the paths $b_i(\cdot)$ are lipschitz on $\mathcal{Q}$. We still denote by $b_i(\cdot)$ their unique lipschitz extension on $[0, T(r_a, R, K_0, K_1)]$. By compactness and (7.10) once more, it follows that the norm in (7.2) converges to zero uniformly for every $t \in [0, T(r_a, R, K_0, K_1)]$.

By Lemma 7.2 Step 2, the bound (7.3) holds for the whole family of maps $(u_\varepsilon)_{0<\varepsilon<\varepsilon_\Lambda}$ and for $C_0 = D$.

Finally, the bound (7.4) is a direct consequence of (7.1), the conservation of $\mathcal{E}_{\varepsilon,U_d}$, the continuity of the $b_i(\cdot)$ and the continuous dependence of $\mathcal{E}_{\varepsilon,U_d}(u_\varepsilon^*(a_i,d_i))$ on a_i. ■

The convergence of the Jacobians in (7.2) actually holds on larger balls passing possibly to a further subsequence. We have

Lemma 7.3. *There exits a subsequence (still denoted by ε_k) such that for all for all $L \geq 2^{n_0}$,*

$$\sup_{t\in[0,T]} \|Ju_{\varepsilon_k}(\cdot,t) - \pi\Sigma d_i\delta_{b_i(t)}\|_{[W_0^{1,\infty}(B(L))]^*} \to 0 \qquad \text{as } k \to +\infty.$$

Proof. From (7.8) and (7.9) we infer that for each $n \geq n_0$ and for $L \equiv L_n = 2^n$ there exists $C > 0$ (depending on n) such that

$$E_{\varepsilon_k}(u^0_{\varepsilon^k}, B(L)) \leq \pi l|\log\varepsilon_k| + C,$$

and therefore by compactness of the Jacobians (see e.g. [9]) there exists a subsequence (still denoted by ε_k) and a configuration (a_i', d_i') such that

$$\|Ju^0_{\varepsilon_k} - \pi\Sigma d_i'\delta_{a_i'}\|_{[W_0^{1,\infty}(B(L_n))]^*} \to 0 \qquad \text{as } k \to +\infty.$$

By (1.8) and e.g. the lower energy bounds of Theorem 5 in [10] it follows that none of the a_i' lies in $B(2^n) \setminus B(2^{n_0})$. By (1.7) we therefore obtain that the configurations (a_i^0, d_i) and (a_i', d_i') are identical. We may then replace $R = 2^{n_0}$ by $R = L_n = 2^n$ when using Theorem 7.1. A diagonal argument finally allows us to construct a fix subsequence which works for all n. ∎

Let $\Sigma_{\mathfrak{v}}$ denote the trajectory set in $[0, T]$:

$$\Sigma_{\mathfrak{v}} := \{(b_i(t), t),\ t \in [0, T],\ i = 1, \ldots, l\},$$

and $\mathcal{G}$ denote its complement in $\mathbb{R}^2 \times [0, T]$. It follows from Lemma 7.3 and Theorem 6.1 that the sequence $(ju_{\varepsilon_k}/|u_{\varepsilon_k}|)$ is uniformly bounded in $L^2_{\mathrm{loc}}(\mathcal{G})$. The next lemma characterizes its weak limit.

Lemma 7.4. *There exists a subsequence (still denoted ε_k) such that*

$$\frac{j(u_{\varepsilon_k}(\cdot,\cdot))}{|u_{\varepsilon_k}(\cdot,\cdot)|} \rightharpoonup ju^*(b_i(\cdot), d_i))$$

weakly in $L^2_{\mathrm{loc}}(\mathcal{G})$ as $k \to +\infty$.

Proof. We already know from Lemma 7.3 that in $\mathcal{D}'(B \times [0, T])$

$$\mathrm{curl}\,(ju_{\varepsilon_k}) = 2Ju_{\varepsilon_k} \to 2\pi \sum d_i \delta_{b_i(\cdot)} = \mathrm{curl}\,(ju^*(b_i, d_i)).$$

On the other hand, since u_{ε_k} is a solution of $(GP)_{\varepsilon_k}$, we have in $\mathcal{D}'(B \times [0, T])$

$$\mathrm{div}\,(ju_{\varepsilon_k}) = -\varepsilon_k \frac{d}{dt} \frac{|u_{\varepsilon_k}|^2 - 1}{\varepsilon_k} \to 0 = \mathrm{div}\,(ju^*(b_i, d_i)).$$

Since by (6.3) and Lemma 7.3, $|u_{\varepsilon_k}| \to 1$ in $L^p_{\mathrm{loc}}(\mathcal{G})$ for every $p < +\infty$, we first infer that ju_{ε_k} is uniformly bounded in $L^q_{\mathrm{loc}}(\mathcal{G})$ for every $q < 2$ and then, taking possibly subsequences, that

$$ju_{\varepsilon_k} - ju^*(b_i, d_i) \rightharpoonup \nabla_x^{\perp} H$$

in $L^q_{\mathrm{loc}}(\mathcal{G})$ where H is harmonic on $\mathcal{G}$. It also follows from Theorem 6.1 that

$$\|\nabla_x H\|^2_{L^2(\mathcal{G})} \leq K_1 T.$$

Standard singularity removal theory yields then that H is harmonic on the whole $\mathbb{R}^2 \times [0, T]$, and then that it is constant (in x only). Using once more the fact that $|u_{\varepsilon_k}| \to 1$ in $L^p_{\mathrm{loc}}(\mathcal{G})$ for every $p < +\infty$, we thus obtain that $ju_{\varepsilon_k}/|u_{\varepsilon_k}|$ converges weakly to $ju^*(b_i, d_i)$ in $L^q_{\mathrm{loc}}(\mathcal{G})$ for every $q < 2$. The weak convergence in $L^2_{\mathrm{loc}}(\mathcal{G})$ then follows from the already mentioned uniform bound of $ju_{\varepsilon_k}/|u_{\varepsilon_k}|$ in that space. ∎

8. Dynamical law for the vortices

The purpose of this section is to present the proof of Theorem 1. To that aim, we consider a family $(u^0_\varepsilon)_{0<\varepsilon<1}$ well-prepared with respect to a configuration (a^0_i, d_i), i.e. such that $d_i \in \{-1,+1\}$ and such that the conditions (1.7), (1.8) and (1.9) are satisfied.

Notice that the results of Section 7, in particular Theorem 7.1, apply in the present situation, providing a sequence $\varepsilon_k \to 0$, Lipschitz paths $b_i(t)$ for $t \in [0,T]$ such that $b_i(0) = a^0_i$ and (7.2) (7.3) (7.4) are satisfied. The main point in order to prove Theorem 1 is to show that $b_i(t) = a_i(t)$ for all $i = 1, \dots, l$, where $a_i(\cdot)$ denote the unique solution of the Cauchy problem (1.10) on its maximal interval of existence.[2]

We set

$$\sigma(t) = \sum_{i=1}^{l} |a_i(t) - b_i(t)|$$

for $t \in [0,T]$. Note that $\sigma(0) = 0$ and that decreasing possibly the time T provided by Theorem 7.1, we may assume that $\sigma(t) \leq r_a$ for $t \in [0,T]$. We will show that σ is identically zero by a Gronwall type argument adapted from [8].

We set

$$\Sigma_{\varepsilon_k}(t) = \mathcal{E}_{\varepsilon_k,U_d}(u_{\varepsilon_k}(\cdot,t)) - \mathcal{E}_{\varepsilon,U_d}(u^*_\varepsilon(b_i(t),d_i)).$$

Lemma 8.1. *There exists $C > 0$ such that*

$$\limsup_{k\to+\infty} \sup_{t\in[0,T]} \Big(\Sigma_{\varepsilon_k}(t) - C\sigma(t)\Big) \leq 0.$$

Proof. By conservation of renormalized energy we have

$$\mathcal{E}_{\varepsilon_k,U_d}(u_{\varepsilon_k}(\cdot,t)) = \mathcal{E}_{\varepsilon_k,U_d}(u^0_{\varepsilon_k}).$$

By Corollary 4.1 we have

$$\limsup_{k\to+\infty} \big|\mathcal{E}_{\varepsilon,U_d}(u^*_\varepsilon(b_i(t),d_i)) - \mathcal{E}_{\varepsilon,U_d}(u^*_\varepsilon(a^0_i,d_i))\big| \leq |W(b_i(t),d_i) - W(a^0_i,d_i)|.$$

By conservation of W under (1.10) we have

$$W(a_i(t),d_i) = W(a^0_i,d_i).$$

[2] In order to prove that the full family $(u_\varepsilon(\cdot,t))_{0<\varepsilon<1}$ is well-prepared with respect to the configuration $(a_i(t),d_i)$, first notice that it suffices to show that for any sequence $\varepsilon_k \to 0$ the conditions corresponding to (1.7), (1.8) and (1.9) hold for a subsequence of ε_k. Indeed, the general case then follows from the uniqueness of the limits $(a_i(t),d_i)$.

The conclusion then follows from the inequality

$$|W(b_i(t), d_i) - W(a_i(t), d_i)| \le C\sigma(t)$$

which holds for $C > 0$ depending only on r_a and R_a and for $t \in [0, T]$. ■

From Section 6 we infer

Lemma 8.2. *There exists $C > 0$ such that for every $L > 0$ we have*

$$\limsup_{k\to+\infty} \sup_{t\in[0,T]} \int_U \left[e_{\varepsilon_k}(|u_{\varepsilon_k}(\cdot,t)|) + \left| \frac{j(u_{\varepsilon_k}(\cdot,t))}{|u_{\varepsilon_k}(\cdot,t)|} - j(u^*(b_i(t),d_i)) \right|^2 \right] - C\sigma(t) \le 0,$$

where $U = B(L) \setminus \cup_{i=1}^l B(a_i^0, \frac{3r_a}{2})$.

Proof. Let

$$\eta_k := \sup_{t\in[0,T]} \|Ju_{\varepsilon_k}(\cdot,t) - \pi\Sigma d_i \delta_{b_i(t)}\|_{[W_0^{1,\infty}(B(L))]^*},$$

and note that Lemma 7.3 implies that $\eta_k \to 0$ as $k \to \infty$. For each $n \ge n_0$, let $\varepsilon_0(n)$ and $\eta_0(n)$ denote the constants provided in the statement of Theorem 6.1 for the choice $R = 2^n$. Then exists $n_k \to \infty$ and nondecreasing (but not necessarily strictly increasing) in k such that $\varepsilon_k \le \varepsilon_0(n_k)$ and $\eta_k \le \eta_0(n_k)$ for every large enough k. We can then use Theorem 6.1 to conclude that for

$$\int_U e_{\varepsilon_k}(|u_{\varepsilon_k}(\cdot,t)|) + \left| \frac{j(u_{\varepsilon_k}(\cdot,t))}{|u_{\varepsilon_k}(\cdot,t)|} - j(u^*(b_i(t),d_i)) \right|^2 \le C\Sigma_{\varepsilon_k}(t) + C(\eta_k, \varepsilon_k, 2^{-n_k}).$$

The conclusion follows letting $k \to +\infty$ and using Lemma 8.1. ■

The control of the difference between u_ε and u^* obtained in Lemma 8.2 allows us to control the difference of their Hopf differential, which was defined in (1.15) as

$$\omega(u) = |u_{x_1}|^2 - |u_{x_2}|^2 - 2iu_{x_1} \cdot u_{x_2}.$$

In the sequel, we set $A = \cup_{i=1}^l B(a_i^0, 2r_a) \setminus B(a_i^0, \frac{3r_a}{2})$. Notice that pointwise on $A \times [0, T]$ we have

$$|\omega(u^*(a_i(t), d_i)) - \omega(u^*(b_i(t), d_i)| \le C\sigma(t). \tag{8.1}$$

Lemma 8.3. *There exists $C > 0$ such that for $0 \le t_1 \le t_2 \le T$ and $\varphi \in \mathcal{D}(A)$ we have*

$$\limsup_{k\to+\infty} \left| \int_{t_1}^{t_2} \int_A \left[\omega(u_{\varepsilon_k}) - \omega(u^*(b_i(\cdot), d_i))\right] \varphi \right| \le C\|\varphi\|_{L^\infty} \int_{t_1}^{t_2} \sigma(t)\, dt.$$

Proof. We start with the pointwise equality

$$u_{x_p} \cdot u_{x_q} = \frac{u \times u_{x_p}}{|u|} \frac{u \times u_{x_q}}{|u|} + |u|_{x_p} |u|_{x_q}$$

which applies to $u_{\varepsilon_k}(\cdot, t)$ and $u^*(b_i(t), d_i)$ almost everywhere on $U \times [0, T]$. Note that $u \times u_{x_p}$ is just the pth component of ju, which we will write $(ju)_p$. Hence, since $|u^*(b_i, d_i)| \equiv 1$, $\omega(u_{\varepsilon_k}) - \omega(u^*(b_i, d_i))$ has the form

$$\text{(8.2)} \quad \sum_{p,q=1}^{2} c_{pq} \Big[\frac{(ju_{\varepsilon_k})_p}{|u_{\varepsilon_k}|} \frac{(ju_{\varepsilon_k})_q}{|u_{\varepsilon_k}|} - (ju^*(b_i, d_i))_p (ju^*(b_i, d_i))_q \Big] + d_{pq} |u_{\varepsilon_k, x_p}| \, |u_{\varepsilon_k, x_q}|$$

for certain numbers $c_{pq}, d_{pq} \in \mathbb{C}$. For real numbers a_p, a_q and a_p^*, a_q^* be have the equality

$$\text{(8.3)} \qquad a_p a_q - a_p^* a_q^* = (a_p - a_p^*)(a_q - a_q^*) + a_p^*(a_q - a_q^*) + a_q^*(a_q - a_q^*).$$

We multiply (8.2) by φ and rewrite, using (8.3) with $a_q = j(u_{\varepsilon_k})_p / |u_{\varepsilon_k}|$ and $a_p^* = j(u^*(b_i, d_i))_p$, and similarly a_q, a_q^*. Integrating over $A \times [t_1, t_2]$, letting $k \to +\infty$, and using Lemma 7.4, we see that the terms that are linear in $(ju_{\varepsilon_k})_p - (ju^*(b_i, d_i))_p$ vanish, and the remaining terms can be easily estimated to obtain

$$\begin{aligned} &\limsup_{k \to +\infty} \int_{t_1}^{t_2} \int_A [\omega(u_{\varepsilon_k}(\cdot, t)) - \omega(u^*(b_i, d_i))] \, \varphi \\ &\qquad \le C \|\varphi\|_{L^\infty} \limsup_{k \to +\infty} \int_{t_1}^{t_2} \int_A \left[\left| \frac{j(u_{\varepsilon_k})}{|u_{\varepsilon_k}|} - j(u^*(b_i, d_i)) \right|^2 + |\nabla |u_{\varepsilon_k}||^2 \right]. \end{aligned}$$

The conclusion then follows from Lemma 8.2. ■

We are now in position to present the

Proof of Theorem 1 completed. We first consider the interval $[0, T]$ where T is as above. Let $t \in [0, T]$ be a point of differentiability of all the Lipschitz functions a_i and b_i (almost all points t have this property). Since all the points $a_i(t)$ and $b_i(t)$ belong to $B(a_i^0, \frac{r_a}{2})$, there exists $\chi \in \mathcal{D}(\mathbb{R}^2)$ (depending on t) such that χ is affine on each $B(a_i^0, r_a)$, $|\nabla \chi(a_i^0)| = 1$, $\operatorname{supp} \frac{\partial^2 \chi}{\partial \bar{z}^2} \subset A$ and

$$\text{(8.4)} \qquad \sigma(t) \equiv \sum_{i=1}^{l} |a_i(t) - b_i(t)| = \Big\langle \sum_{i=1}^{l} \delta_{a_i(t)} - \delta_{b_i(t)}, \chi \Big\rangle.$$

On the other hand, for every $0 \le \delta \le t$,

$$\text{(8.5)} \quad \sigma(t - \delta) \equiv \sum_{i=1}^{l} |a_i(t - \delta) - b_i(t - \delta)| \ge \Big\langle \sum_{i=1}^{l} \delta_{a_i(t-\delta)} - \delta_{b_i(t-\delta)}, \chi \Big\rangle.$$

Integrating (1.14) on $[t-\delta, t]$ and taking into account (1.17) and (1.18) we obtain

$$(8.6)\quad \Big\langle \big(Ju_{\varepsilon_k}(\cdot,t) - \pi\Sigma d_i\delta_{a_i(t)}\big) - \big(Ju_{\varepsilon_k}(\cdot,t-\delta) - \pi\Sigma d_i\delta_{a_i(t-\delta)}\big)\,,\chi\Big\rangle = -2\int_{t-\delta}^{t}\int_A \operatorname{Im}\left(\left[\omega(u_{\varepsilon_k}) - \omega(u^*(a_i,d_i))\right]\frac{\partial^2\chi}{\partial\bar z^2}\right).$$

Passing to the limsup $k\to+\infty$ we are led, taking into account Lemma 7.3, Lemma 8.3, (8.5) and (8.1), to

$$\sigma(t) - \sigma(t-\delta) \le C\int_{t-\delta}^{t}\sigma(s)\,ds.$$

Passing to the limit $\delta\to 0$ we finally obtain

$$\frac{d}{dt}\sigma(t) \le C\sigma(t),$$

and since $\sigma(0)=0$ Gronwall's lemma yields

$$\sigma(t) = 0 \quad \text{for } t\in[0,T].$$

The conditions (1.7) (1.8) and (1.9) are therefore satisfied for $a_i = a_i(t)$ and $t\in[0,T]$. Indeed, (1.7) is a consequence of Lemma 7.3 and the equality $a_i(t) = b_i(t)$, (1.8) was already proved in (7.3), and (1.9) follows from Lemma 8.1 and the fact that $\sigma\equiv 0$.

To conclude, it suffices to pass from the interval $[0,T]$ to $[T_*, T^*]$, i.e. the maximal interval of existence of (1.10). Since we obtained a lower bound on T which depends only on r_a and R_a, this is readily achieved considering translations in time and reversing time. ■

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