Elemente der Mathematik (Online first)
DOI 10.4171/EM/527

Short note

A synthetic proof of the Morley trisector theorem using congruent and similar triangles

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Abstract. This note presents a synthetic proof of Morley's trisector theorem, using only congruent and similar triangles.

The Morley trisector theorem stands as an intriguing proposition in plane geometry. Since Frank Morley's discovery in 1899, the historical journey of this theorem has been marked by various innovative proofs. Approaches have ranged from purely geometric methods to trigonometric considerations and even a proof rooted in algebraic group theory. A lot of proofs have recently been published, emphasizing the enduring relevance and intrigue of the theorem [1–5, 7–10, 13]. Furthermore, an extension of this theorem to three and more space dimensions appears in [11, 12]. In this article, we present a novel proof, notable for its exclusion of circle geometry (as found in Euclid's work [6, Book IV]), relying exclusively on the principles of congruent triangles (as detailed in [6, Book I]) and similar triangles (as outlined in [6, Book VI]).

The construction entails starting with the construction of an equilateral triangle XYZ and subsequently establishing points A, B, and C to illustrate that XYZ serves as a Morley triangle for ABC. It is worth noting that this construction, originally attributed to Child and Coxeter [3, 4], along with contributions from other mathematicians and a recent exploration by Hashimoto in [8], significantly enriches the array of solutions available for this captivating theorem.

Theorem (Morley, 1899). *In any triangle, the three points of intersection of the adjacent angle trisectors form an equilateral triangle.*

Proof. (See Figure 1). Consider arbitrary positive angles α, β, γ, where $\alpha + \beta + \gamma = 60°$. Let XYZ be an equilateral triangle. Construct point A opposite to X with respect to YZ such that $\angle AZY = 60° + \beta$ and $\angle AYZ = 60° + \gamma$. Similarly, define points B and C. In order to prove that XYZ is Morley triangle of ABC, it suffices to demonstrate that $\angle XBC = \angle XBZ$.

Draw a parallel line from X to YZ, intersecting YC and ZB at M and N, respectively. This results in $\angle ZXN = \angle YXM = 60°$. From here, we can easily see that $\angle ZXN <$

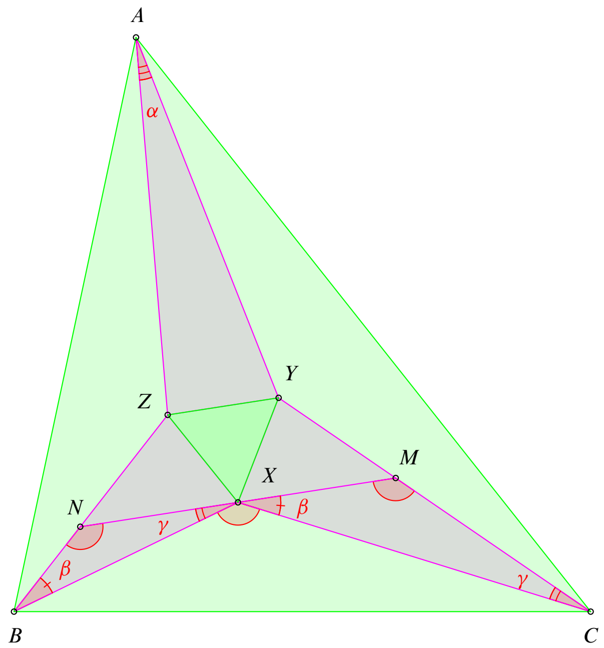

Figure 1. A synthetic proof of Morley's theorem using similar triangles.

$\angle ZXB$ and $\angle YXM < \angle YXC$; consequently, N lies on the line segment BZ, while M lies on the line segment CY.

Utilizing the given constructions $XZ = XY$ and $\angle XZB = \angle XYC$ $(= 60^\circ + \alpha)$, we conclude that the triangles ZXN and YXM are congruent (a.s.a.). Consequently, we have $XN = XM$. Furthermore,

$$\angle BXN = \angle BXZ - \angle ZXN = (60^\circ + \gamma) - 60^\circ = \gamma = \angle XCM.$$

Similarly, $\angle CXM = \angle XBN$. These two angle conditions imply that the triangles BNX and XMC are similar (a.a.). This leads to

$$\frac{XB}{XC} = \frac{BN}{XM} = \frac{BN}{XN}. \tag{1}$$

Simple angle chasing yields

$$\begin{aligned} \angle BXC &= 180^\circ - \angle NXB - \angle MXC \\ &= 180^\circ - \gamma - \beta \\ &= 180^\circ - \angle NXB - \angle NBX \\ &= \angle BNX. \end{aligned} \tag{2}$$

From (1) and (2), we conclude that the triangles BNX and BXC are similar (s.a.s.). This implies $\angle XBC = \angle XBZ$, completing the proof. ∎

Acknowledgements. The author is grateful to the referee for reading and reviewing the manuscript very carefully throughout the process.

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