

H^2 solutions for a Degasperis–Procesi-type equation

GIUSEPPE MARIA COCLITE (*) – LORENZO DI RUVO (**)

ABSTRACT – The Degasperis–Procesi equation can be regarded as a model for shallow water dynamics and its asymptotic accuracy is the same as for the Camassa–Holm equation. Here we consider a generalization of the Degasperis–Procesi equation and prove the existence and uniqueness of H^2 solutions for the Cauchy problem.

MATHEMATICS SUBJECT CLASSIFICATION 2020 – 35K55 (primary); 35G25 (secondary).

KEYWORDS – existence, uniqueness, stability, Degasperis–Procesi equation, Cauchy problem.

1. Introduction

In this study, we investigate the existence and uniqueness of the following Cauchy problem:

$$(1.1) \quad \begin{cases} \partial_t u + \delta \partial_x u - \frac{4\gamma}{\beta^2} u \partial_x u + \kappa \partial_x^3 u \\ \quad - \beta^2 \partial_t \partial_x^2 u + \gamma \partial_x u \partial_x^2 u + \gamma u \partial_x^3 u = 0, & t > 0, x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

with

$$(1.2) \quad \delta, \gamma, \beta, \kappa \in \mathbb{R}, \quad \gamma, \beta \neq 0.$$

(*) *Indirizzo dell'A.*: Dipartimento di Meccanica, Matematica e Management, Politecnico di Bari, via E. Orabona 4, 70125 Bari, Italy; giuseppemaria.coclite@poliba.it

(**) *Indirizzo dell'A.*: Dipartimento di Matematica, Università di Bari, via E. Orabona 4, 70125 Bari, Italy; lorenzo.diruvo@uniba.it, lorenzo.diruvo77@gmail.com

On the initial datum, we assume

$$(1.3) \quad u_0 \in H^2(\mathbb{R}), \quad u_0 \neq 0.$$

Equation (1.1) is a particular case of

$$(1.4) \quad \partial_t u + \delta \partial_x u + au \partial_x u + \kappa \partial_x^3 u - \beta^2 \partial_t \partial_x^2 u + \alpha \partial_x u \partial_x^2 u + \gamma u \partial_x^3 u = 0,$$

which is analyzed by Degasperis and Procesi [21]. In particular, they found that there are only three equations that satisfy the asymptotic integrability.

The first family is given by taking $\delta = \beta = \alpha = \gamma = 0$ in (1.4), that is,

$$(1.5) \quad \partial_t u + au \partial_x u + \kappa \partial_x^3 u = 0.$$

Equation (1.5), known as the Korteweg–de Vries equation, models weakly nonlinear unidirectional long waves, and arises in various physical contexts. For example, it models surface waves of small amplitude and long wavelength in shallow water. In this context, $u(t, x)$ represents the wave height above a flat bottom, with x being proportional to distance in the propagation direction and t being proportional to the elapsed time. Equation (1.5) is completely integrable and possesses solitary wave solutions that are solitons. The Cauchy problem for (1.5) is studied in [6, 35] and the references cited therein.

The second family is given by taking $\alpha = 2\gamma$ in (1.4), that is,

$$(1.6) \quad \partial_t u + \delta \partial_x u + au \partial_x u + \kappa \partial_x^3 u - \beta^2 \partial_t \partial_x^2 u + 2\gamma \partial_x u \partial_x^2 u + \gamma u \partial_x^3 u = 0.$$

Equation (1.6), known as the Camassa–Holm equation, models the unidirectional propagation of shallow water waves over a flat bottom. In this case $u(t, x)$ represents the fluid velocity at time t in the spatial x direction, while δ is a parameter related to the critical shallow water speed [4, 24, 33]. Equation (1.6) is also a model for the propagation of axially symmetric waves in hyperelastic rods [19, 20]. It has a bi-Hamiltonian structure [25, 37] and is completely integrable [4, 15].

From a mathematical point of view, local existence and uniqueness results for (1.6) are proven in [16, 27, 39, 42, 44], while the existence of global solutions for a certain class of initial data and solutions that blow up in finite time for a large class of initial data are proven in [14, 16, 17]. Existence and uniqueness results for global weak solutions for (1.6) are proven in [2, 3, 17, 18, 28–30, 34, 46, 47]. The existence of the traveling wave solutions can be found in [38], and the existence and uniqueness of periodic solutions for the Cauchy problem in [13, 43]. Finally, in [5], using a compensated compactness argument in the L^p setting [36, 45], the convergence of the solution of (1.1) to the unique entropy solution of the Burgers equation is proven, and in [31], using a kinetic approach [32].

The third family is given by taking $a = -4\gamma/\beta^2$ and $\alpha = \gamma$ in (1.4), which is precisely equation (1.1) and it is known as the Degasperis–Procesi equation.

After an appropriate rescaling, equation (1.1) can be written in the form

$$(1.7) \quad \partial_t u + \partial_x u + 3\partial_x u^2 + \partial_x^3 u - \beta^2 \partial_t \partial_x^2 u - \frac{9\beta^2}{2} \partial_x u \partial_x^2 u - \frac{3\beta^2}{2} u \partial_x^3 u = 0.$$

By rescaling, shifting the dependent variable, and finally applying a Galilean boost, equation (1.7), can be transformed into the famous form (see [22, 23])

$$(1.8) \quad \partial_t u - \partial_t \partial_x^2 u + 4u \partial_x u = 3\partial_x u \partial_x^2 u + u \partial_x^3 u.$$

From a mathematical point of view, the local and global existence and uniqueness of (1.8) in energy spaces is proven in [26, 48–50] and the references cited therein. In [9], the existence and uniqueness of the entropy solution is proven, while in [11] the well-posedness of the homogeneous initial boundary value problem is studied. In [10] the existence and uniqueness of the periodic solution of (1.8) is analyzed, while in [12] the convergence of some numerical schemes is proven. Possible estimates on the blow-up time T for (1.8) are given in [39–41] and the references cited therein.

We use the following definition of a solution.

DEFINITION 1.1. We say that a function $u: [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ is a weak solution of (1.1) if

$$u \in H^1((0, T) \times \mathbb{R}) \cap L^\infty(0, T; H^2(\mathbb{R})), \quad \partial_t \partial_x u \in L^\infty(0, T; L^2(\mathbb{R})), \quad T > 0, \\ u(0, \cdot) = u_0 \text{ a.e. in } (0, \infty) \times \mathbb{R}$$

and for every test function $\varphi \in C^\infty(\mathbb{R}^2)$ with compact support,

$$\int_0^\infty \int_{\mathbb{R}} \left(\partial_t u \varphi + \delta \partial_x u \varphi - \frac{4\gamma}{\beta^2} u \partial_x u \varphi - \kappa \partial_x^2 u \partial_x \varphi \right. \\ \left. + \beta^2 \partial_t \partial_x u \partial_x \varphi + \gamma u \partial_x^2 u \partial_x \varphi \right) dt dx = 0.$$

The main result of this paper is the following theorem.

THEOREM 1.1. Assume (1.2) and (1.3). Given $\delta, \gamma, \kappa, \beta$, there exists a unique weak solution u of (1.1) in the sense of Definition 1.1, such that for every $T > 0$,

$$(1.9) \quad u \in H^1((0, T) \times \mathbb{R}) \cap L^\infty(0, T; H^2(\mathbb{R})) \cap W^{1,\infty}((0, T) \times \mathbb{R}), \\ \partial_t \partial_x u \in L^\infty(0, T; L^2(\mathbb{R})).$$

Moreover, if u_1 and u_2 are two weak solutions of (1.1) in correspondence with the initial data $u_{1,0}$ and $u_{2,0}$, we have that, for every $T > 0$,

$$(1.10) \quad \|u_1(t, \cdot) - u_2(t, \cdot)\|_{H^1(\mathbb{R})}^2 \leq \frac{\tau_2^2 e^{C(T)t}}{\tau_3^2} \|u_{1,0} - u_{2,0}\|_{H^1(\mathbb{R})}^2,$$

for some suitable $C(T) > 0$, and every $t \in [0, T]$, where

$$(1.11) \quad \tau_1^2 = \min\{1, \beta^2\}, \quad \tau_2^2 = \max\{1, \beta^2\}.$$

We note that Theorem 1.1 affirms that equation (1.1) admits a unique global-in-time weak solution. Unfortunately, some of the estimates blow up as $T \rightarrow \infty$, and indeed we prove them in every stripe $(0, T) \times \mathbb{R}$, $T > 0$. Finally, we cannot use the arguments of [9] because the L^2 norm is not a conserved quantity of (1.1); we can only prove that it stays bounded in any finite time interval $(0, T)$ (see (2.3) below).

The proof of Theorem 1.1 is given in the next section.

2. Proof of the Theorem 1.1

This section is devoted to the proof of Theorem 1.1.

Our existence argument is based on passing to the limit in a vanishing viscosity approximation of (1.1); see e.g. [1].

Fix a small number $0 < \varepsilon < 1$, and let $u_\varepsilon = u_\varepsilon(t, x)$ be the unique classical solution of the following problem [8]:

$$(2.1) \quad \begin{cases} \partial_t u_\varepsilon + \delta \partial_x u_\varepsilon - \frac{4\gamma}{\beta^2} u_\varepsilon \partial_x u_\varepsilon + \kappa \partial_x^3 u_\varepsilon - \beta^2 \partial_t \partial_x^2 u_\varepsilon \\ \quad + \gamma \partial_x u_\varepsilon \partial_x^2 u_\varepsilon + \gamma u_\varepsilon \partial_x^3 u_\varepsilon = -\varepsilon \partial_x^4 u_\varepsilon, & t > 0, x \in \mathbb{R}, \\ u_\varepsilon(0, x) = u_{\varepsilon,0}(x), & x \in \mathbb{R}, \end{cases}$$

where $u_{\varepsilon,0}$ is a C^∞ approximation of u_0 , such that

$$(2.2) \quad \|u_{\varepsilon,0}\|_{H^2(\mathbb{R})} \leq \|u_0\|_{H^2(\mathbb{R})}, \quad \sqrt{\varepsilon} \|\partial_x^3 u_{\varepsilon,0}\|_{L^2(\mathbb{R})} \leq C_0, \quad u_0 \neq 0,$$

where C_0 is a positive constant independent of ε .

Let us prove some a priori estimates on u_ε , denoting by C_0 the constants which depend only on the initial data, and by $C(T)$ the constants which depend also on T .

Inspired by the arguments in [7] we prove the following lemma.

LEMMA 2.1 (H^2 estimate). *Given $\delta, \gamma, \kappa, \beta$, the following estimates hold:*

$$(2.3) \quad \|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}, \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}, \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} \leq C(T),$$

$$(2.4) \quad \|\partial_x u_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})} \leq C(T),$$

$$(2.5) \quad \|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})} \leq C(T),$$

$$(2.6) \quad \varepsilon \int_0^t \|\partial_x^3 u_\varepsilon(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \leq C(T),$$

for every $0 \leq t \leq T$.

PROOF. Consider F a positive constant. Multiplying (2.1) by $2u_\varepsilon - 2F\partial_x^2 u_\varepsilon$, an integration on $\mathbb{R}$ gives

$$\begin{aligned} & \frac{d}{dt} (\|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + (\beta^2 + F)\|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + F\beta^2\|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2) \\ &= 2 \int_{\mathbb{R}} u \partial_t u_\varepsilon dx - 2\beta^2 \int_{\mathbb{R}} u_\varepsilon \partial_t \partial_x^2 u_\varepsilon dx - 2F \int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_t u_\varepsilon dx \\ & \quad + 2F\beta^2 \int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_t \partial_x^2 u_\varepsilon dx \\ &= -2\delta \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon dx + 2F \int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dx + \frac{8\gamma}{\beta^2} \int_{\mathbb{R}} u_\varepsilon^2 \partial_x u_\varepsilon dx \\ & \quad - \frac{4F\gamma}{\beta^2} \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dx - 2\kappa \int_{\mathbb{R}} u_\varepsilon \partial_x^3 u_\varepsilon dx + 2F\kappa \int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_x^3 u_\varepsilon dx \\ & \quad - 2\gamma \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dx + 2F\gamma \int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x^2 u_\varepsilon)^2 dx - 2\gamma \int_{\mathbb{R}} u_\varepsilon^2 \partial_x^3 u_\varepsilon dx \\ & \quad + 2F\gamma \int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_\varepsilon \partial_x^3 u_\varepsilon dx - 2\varepsilon \int_{\mathbb{R}} u_\varepsilon \partial_x^4 u_\varepsilon dx + 2F\varepsilon \int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_x^4 u_\varepsilon dx \\ &= \frac{2F\gamma}{\beta^2} \int_{\mathbb{R}} (\partial_x u_\varepsilon)^3 dx + 2\kappa \int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dx + \gamma \int_{\mathbb{R}} (\partial_x u_\varepsilon)^3 dx \\ & \quad + F\gamma \int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x^2 u_\varepsilon)^2 dx + 4\gamma \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dx + 2\varepsilon \int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^3 u_\varepsilon dx \\ & \quad - 2F\varepsilon \|\partial_x^3 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ &= \frac{2F\gamma}{\beta^2} \int_{\mathbb{R}} (\partial_x u_\varepsilon)^3 dx - \gamma \int_{\mathbb{R}} (\partial_x u_\varepsilon)^3 dx + F\gamma \int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x^2 u_\varepsilon)^2 dx \\ & \quad - 2\varepsilon \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 - 2F\varepsilon \|\partial_x^3 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

Therefore, we have

$$\begin{aligned} & \frac{d}{dt} (\|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + (\beta^2 + F)\|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + F\beta^2\|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2) \\ & \quad + 2\varepsilon \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2F\varepsilon \|\partial_x^3 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ (2.7) \quad &= \frac{2F\gamma}{\beta^2} \int_{\mathbb{R}} (\partial_x u_\varepsilon)^3 dx - \gamma \int_{\mathbb{R}} (\partial_x u_\varepsilon)^3 dx + F\gamma \int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x^2 u_\varepsilon)^2 dx. \end{aligned}$$

Introducing the function

$$(2.8) \quad \begin{aligned} X_\varepsilon(t) &:= \|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + (\beta^2 + F) \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ &\quad + F\beta^2 \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2, \end{aligned}$$

equation (2.7) reads

$$(2.9) \quad \begin{aligned} \frac{dX_\varepsilon(t)}{dt} &+ 2\varepsilon \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2F\varepsilon \|\partial_x^3 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ &= \frac{2F\gamma}{\beta^2} \int_{\mathbb{R}} (\partial_x u_\varepsilon)^3 dx - \gamma \int_{\mathbb{R}} (\partial_x u_\varepsilon)^3 dx + F\gamma \int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x^2 u_\varepsilon)^2 dx. \end{aligned}$$

Observe that, thanks to (2.8),

$$\begin{aligned} \frac{2F\gamma}{\beta^2} \int_{\mathbb{R}} (\partial_x u_\varepsilon)^3 dx &\leq \frac{2F|\gamma|}{\beta^2} \int_{\mathbb{R}} |\partial_x u_\varepsilon|^3 dx \\ &\leq \frac{2F|\gamma|}{\beta^2} \|\partial_x u_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})} \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ &\leq \frac{2(F + \beta^2)|\gamma|}{\beta^2} \|\partial_x u_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})} \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ &\leq \frac{2|\gamma|}{\beta^2} \|\partial_x u_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})} (\|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ &\quad + (F + \beta^2) \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2) \\ &\quad + \frac{2|\gamma|}{\beta^2} \|\partial_x u_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})} F\beta^2 \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ &= \frac{2|\gamma|}{\beta^2} \|\partial_x u_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})} X_\varepsilon(t), \\ -\gamma \int_{\mathbb{R}} (\partial_x u_\varepsilon)^3 dx &\leq |\gamma| \int_{\mathbb{R}} |\partial_x u_\varepsilon|^3 dx \\ &\leq \frac{|\gamma|}{\beta^2} \beta^2 \|\partial_x u_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})} \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ &\leq \frac{|\gamma|}{\beta^2} (F + \beta^2) \|\partial_x u_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})} \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ &\leq \frac{|\gamma|}{\beta^2} \|\partial_x u_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})} (\|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ &\quad + (F + \beta^2) \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2) \\ &\quad + \frac{|\gamma|}{\beta^2} \|\partial_x u_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})} F\beta^2 \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ &= \frac{|\gamma|}{\beta^2} \|\partial_x u_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})} X_\varepsilon(t), \end{aligned}$$

$$\begin{aligned}
 F\gamma \int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x^2 u_\varepsilon)^2 dx &\leq F|\gamma| \int_{\mathbb{R}} |\partial_x u_\varepsilon| (\partial_x^2 u_\varepsilon)^2 dx \\
 &\leq \frac{|\gamma|}{\beta^2} F\beta^2 \|\partial_x u_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})} \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
 &\leq \frac{|\gamma|}{\beta^2} (\|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + F\beta^2 \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2) \\
 &\quad + \frac{|\gamma|}{\beta^2} (F + \beta^2) \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
 &= \frac{|\gamma|}{\beta^2} \|\partial_x u_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})} X_\varepsilon(t).
 \end{aligned}$$

Consequently, by (2.9), we have

$$(2.10) \quad \frac{dX_\varepsilon(t)}{dt} \leq \frac{4|\gamma|}{\beta^2} \|\partial_x u_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})} X_\varepsilon(t).$$

Observe that, by (2.8),

$$(2.11) \quad (\beta^2 + F) \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq X_\varepsilon(t), \quad F\beta^2 \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq X_\varepsilon(t).$$

Therefore, thanks to (2.11) and the Hölder inequality,

$$\begin{aligned}
 (\partial_x u_\varepsilon(t, x))^2 &= 2 \int_{-\infty}^x \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dy \leq 2 \int_{\mathbb{R}} |\partial_x u_\varepsilon| |\partial_x^2 u_\varepsilon| dx \\
 &\leq 2 \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} \\
 &= \frac{2}{\sqrt{F}\beta^2} \sqrt{\beta^2 \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2} \sqrt{F\beta^2 \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2} \\
 &\leq \frac{2}{\sqrt{F}\beta^2} \sqrt{(F + \beta^2) \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2} \sqrt{F\beta^2 \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2} \\
 &\leq \frac{2}{\sqrt{F}\beta^2} X_\varepsilon(t).
 \end{aligned}$$

Hence,

$$(2.12) \quad \|\partial_x u_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})}^2 \leq \frac{2}{\sqrt{F}\beta^2} X_\varepsilon(t).$$

It follows from (2.10) and (2.12) that

$$\frac{dX_\varepsilon(t)}{dt} \leq \frac{4\sqrt{2}|\gamma|}{\sqrt[4]{F}|\beta|^3} X_\varepsilon^{\frac{3}{2}}(t),$$

which gives

$$X_\varepsilon^{-\frac{3}{2}}(t) \frac{dX_\varepsilon(t)}{dt} \leq \frac{4\sqrt{2}|\gamma|}{\sqrt[4]{F}|\beta|^3}.$$

Integrating on $(0, t)$, we obtain

$$(2.13) \quad \frac{1}{\sqrt{X_\varepsilon(t)}} \geq \frac{1}{\sqrt{X_\varepsilon(0)}} - \frac{8\sqrt{2}|\gamma|t}{\sqrt[4]{F}|\beta|^3}.$$

Using (2.8) in (2.13), we have

$$(2.14) \quad \begin{aligned} & \frac{1}{\sqrt{A_\varepsilon(t) + (\beta^2 + F)B_\varepsilon(t) + F\beta^2 E_\varepsilon(t)}} + \frac{8\sqrt{2}|\gamma|t}{\sqrt[4]{F}|\beta|^3} \\ & \geq \frac{1}{\sqrt{A_\varepsilon(0) + (\beta^2 + F)B_\varepsilon(0) + F\beta^2 E_\varepsilon(0)}} \end{aligned}$$

for every $0 \leq t \leq T$, where F is an arbitrary positive constant, and

$$(2.15) \quad \begin{aligned} A_\varepsilon(t) &:= \|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2, & B_\varepsilon(t) &:= \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2, \\ E_\varepsilon(t) &:= \|\partial_x^2 u(t, \cdot)\|_{L^2(\mathbb{R})}^2, & A_\varepsilon(0) &:= \|u_{\varepsilon,0}\|_{L^2(\mathbb{R})}^2, \\ B_\varepsilon(0) &:= \|\partial_x u_{\varepsilon,0}\|_{L^2(\mathbb{R})}^2, & E_\varepsilon(0) &:= \|\partial_x^2 u_{\varepsilon,0}\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

We prove (2.3). Observe that, by (2.2),

$$(2.16) \quad \begin{aligned} (F + \beta^2)\|\partial_x u_{\varepsilon,0}\|_{L^2(\mathbb{R})}^2 &\leq (F + \beta^2)\|\partial_x u_0\|_{L^2(\mathbb{R})}^2, \\ F\beta^2\|\partial_x^2 u_{\varepsilon,0}\|_{L^2(\mathbb{R})}^2 &\leq F\beta^2\|\partial_x^2 u_0\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

Consequently, (2.2), (2.15), (2.9), and (2.16) give

$$\begin{aligned} X_\varepsilon(0) &= A_\varepsilon(0) + (F + \beta^2)B_\varepsilon(0) + F\beta^2 E_\varepsilon(0) \\ &\leq Y_0 + (F + \beta^2)Y_1 + F\beta^2 Y_2, \end{aligned}$$

where

$$Y_0 := \|u_0\|_{L^2(\mathbb{R})}^2, \quad Y_1 := \|\partial_x u_0\|_{L^2(\mathbb{R})}^2, \quad Y_2 := \|\partial_x^2 u_0\|_{L^2(\mathbb{R})}^2.$$

Therefore, we have

$$(2.17) \quad \begin{aligned} & \frac{1}{\sqrt{A_\varepsilon(0) + (F + \beta^2)B_\varepsilon(0) + F\beta^2 E_\varepsilon(0)}} \\ & \geq \frac{1}{\sqrt{Y_0 + (F + \beta^2)Y_1 + F\beta^2 Y_2}}. \end{aligned}$$

Moreover,

$$(2.18) \quad \frac{8\sqrt{2}|\gamma|t}{\sqrt[4]{F}|\beta|^3} \leq \frac{8\sqrt{2}|\gamma|T}{\sqrt[4]{F}|\beta|^3} \Rightarrow -\frac{8\sqrt{2}|\gamma|t}{\sqrt[4]{F}|\beta|^3} \geq -\frac{8\sqrt{2}|\gamma|T}{\sqrt[4]{F}|\beta|^3}.$$

It follows from (2.14), (2.15), (2.8), (2.13), (2.17), and (2.18) that

$$(2.19) \quad \begin{aligned} \frac{1}{\sqrt{X_\varepsilon(t)}} &\geq \frac{1}{\sqrt{Y_0 + (F + \beta^2)Y_1 + F\beta^2Y_2}} - \frac{8\sqrt{2}|\gamma|T}{\sqrt[4]{F}|\beta|^3} \\ &= \frac{\sqrt[4]{F}|\beta|^3 - 8\sqrt{2}|\gamma|T\sqrt{Y_0 + (F + \beta^2)Y_1 + F\beta^2Y_2}}{\sqrt[4]{F}|\beta|^3\sqrt{Y_0 + (F + \beta^2)Y_1 + F\beta^2Y_2}}. \end{aligned}$$

We search for F such that

$$(2.20) \quad \sqrt[4]{F}|\beta|^3 - 8\sqrt{2}|\gamma|T\sqrt{Y_0 + (F + \beta^2)Y_1 + F\beta^2Y_2} > 0.$$

Taking $|\beta|$ as

$$(2.21) \quad |\beta| = F^n, \quad n > \frac{1}{8},$$

(2.20) reads

$$F^{\frac{12n+1}{4}} - 8\sqrt{2}|\gamma|T\sqrt{Y_0 + (F + F^{2n})Y_1 + F^{2n+1}Y_2} > 0,$$

which gives

$$(2.22) \quad F^{\frac{12n+1}{2}} > 128\gamma^2T^2(Y_0 + (F + F^{2n})Y_1 + F^{2n+1}Y_2).$$

Consequently,

$$(2.23) \quad \begin{aligned} &F^{\frac{12n+1}{2}} - 128\gamma^2T^2Y_2F^{2n+1} - 128\gamma^2T^2Y_1F^{2n} \\ &- 128\gamma^2T^2Y_1F - 128\gamma^2T^2Y_0 > 0. \end{aligned}$$

Taking n as in (2.21), we have

$$(2.24) \quad \begin{aligned} &\lim_{F \rightarrow \infty} F^{\frac{12n+1}{2}} - 128\gamma^2T^2Y_2F^{2n+1} - 128\gamma^2T^2Y_1F^{2n} \\ &- 128\gamma^2T^2Y_1F - 128\gamma^2T^2Y_0 = \infty. \end{aligned}$$

In fact, we have (2.24) if only if

$$\frac{12n+1}{2} > 2n+1,$$

that is

$$n > \frac{1}{8}.$$

Therefore, thanks to (2.24), we have that (2.20) (which is equivalent to (2.22)) holds. Taking F very large and up to rescaling, we can have $|\beta| = F^n$, with n defined in (2.21).

Consequently, by (2.19), (2.20), (2.22), (2.23), and (2.24),

$$(2.25) \quad \frac{1}{\sqrt{X_\varepsilon(t)}} \geq C(T).$$

Thanks to (2.8) and (2.25), we get

$$\|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + (\beta^2 + F) \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + F\beta^2 \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq C(T),$$

which gives (2.3).

We prove (2.4). By (2.8) and (2.12) with $F = 1$, we have

$$(2.26) \quad \begin{aligned} \|\partial_x u_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})}^2 &\leq \frac{2}{\beta^2} (\|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + (\beta^2 + 1) \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ &\quad + \beta^2 \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2). \end{aligned}$$

Inequality (2.4) follows from (2.3) and (2.26).

We prove (2.5). Thanks to (2.3) and the Hölder inequality,

$$\begin{aligned} u_\varepsilon^2(t, x) &= 2 \int_{-\infty}^x u_\varepsilon \partial_x u_\varepsilon dy \leq 2 \int_{\mathbb{R}} |u_\varepsilon| |\partial_x u_\varepsilon| dx \\ &\leq 2 \|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} \leq C(T). \end{aligned}$$

Hence,

$$\|u_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})}^2 \leq C(T),$$

which gives (2.5).

Finally, we prove (2.6). Observe that (2.7) with $F = 1$ is written

$$(2.27) \quad \begin{aligned} &\frac{d}{dt} (\|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + (\beta^2 + 1) \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \beta^2 \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2) \\ &\quad + 2\varepsilon \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon \|\partial_x^3 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ &= \frac{\gamma(2 - \beta^2)}{\beta^2} \int_{\mathbb{R}} (\partial_x u_\varepsilon)^3 dx + \gamma \int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x^2 u_\varepsilon)^2 dx. \end{aligned}$$

Thanks to (2.3) and (2.4),

$$(2.28) \quad \begin{aligned} &\frac{|\gamma(2 - \beta^2)|}{\beta^2} \int_{\mathbb{R}} |\partial_x u_\varepsilon|^3 dx \\ &\leq \frac{|\gamma(2 - \beta^2)|}{\beta^2} \|\partial_x u_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})} \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq C(T), \\ &|\gamma| \int_{\mathbb{R}} |\partial_x u_\varepsilon| (\partial_x^2 u_\varepsilon)^2 dx \\ &\leq |\gamma| \|\partial_x u_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})} \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq C(T). \end{aligned}$$

It follows from (2.27) and (2.28) that

$$\begin{aligned} \frac{d}{dt} (\|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + (\beta^2 + 1) \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \beta^2 \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2) \\ + 2\varepsilon \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon \|\partial_x^3 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq C(T). \end{aligned}$$

Integrating on $(0, t)$, by (2.2) we get

$$\begin{aligned} \|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + (\beta^2 + 1) \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \beta^2 \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ + 2\varepsilon \int_0^t \|\partial_x^2 u_\varepsilon(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds + 2\varepsilon \int_0^t \|\partial_x^3 u_\varepsilon(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \\ \leq C_0 + C(T)t \leq C(T), \end{aligned}$$

which gives (2.6). ■

LEMMA 2.2. *Given $\delta, \gamma, \kappa, \beta$, we have*

$$\begin{aligned} (2.29) \quad \varepsilon \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \beta^2 \varepsilon \|\partial_x^3 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ + \varepsilon^2 \int_0^t \|\partial_x^4 u_\varepsilon(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \leq C(T), \end{aligned}$$

for every $0 \leq t \leq T$.

PROOF. Let $0 \leq t \leq T$. Multiplying (2.1) by $2\varepsilon \partial_x^4 u_\varepsilon$, an integration on $\mathbb{R}$ gives

$$\begin{aligned} \frac{d}{dt} (\varepsilon \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \beta^2 \varepsilon \|\partial_x^3 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2) \\ = 2\varepsilon \int_{\mathbb{R}} \partial_x^4 u_\varepsilon \partial_t u_\varepsilon dx - 2\beta^2 \varepsilon \int_{\mathbb{R}} \partial_x^4 u_\varepsilon \partial_t \partial_x^2 u_\varepsilon dx \\ = -2\delta \varepsilon \int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^4 u_\varepsilon dx + \frac{8\gamma\varepsilon}{\beta^2} \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_x^4 u_\varepsilon dx \\ - 2\kappa \varepsilon \int_{\mathbb{R}} \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon dx - 2\gamma \varepsilon \int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^2 u_\varepsilon \partial_x^4 u_\varepsilon dx \\ - 2\gamma \varepsilon \int_{\mathbb{R}} u_\varepsilon \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon dx - 2\varepsilon^2 \|\partial_x^4 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ = 2\delta \varepsilon \int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_x^3 u_\varepsilon dx + \frac{8\gamma\varepsilon}{\beta^2} \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_x^4 u_\varepsilon dx \\ + 3\gamma \varepsilon \int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x^3 u_\varepsilon)^2 dx - 2\varepsilon^2 \|\partial_x^4 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ = \frac{8\gamma\varepsilon}{\beta^2} \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_x^4 u_\varepsilon dx + 3\gamma \varepsilon \int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x^3 u_\varepsilon)^2 dx \\ - 2\varepsilon^2 \|\partial_x^4 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

Therefore, we have

$$(2.30) \quad \begin{aligned} & \frac{d}{dt} (\varepsilon \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \beta^2 \varepsilon \|\partial_x^3 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2) + 2\varepsilon^2 \|\partial_x^4 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ &= \frac{8\gamma\varepsilon}{\beta^2} \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_x^4 u_\varepsilon dx + 3\gamma\varepsilon \int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x^3 u_\varepsilon)^2 dx. \end{aligned}$$

Due to (2.3), (2.4), (2.5), and the Young inequality,

$$\begin{aligned} \frac{8|\gamma|\varepsilon}{\beta^2} \int_{\mathbb{R}} |u_\varepsilon| |\partial_x u_\varepsilon| |\partial_x^4 u_\varepsilon| dx &\leq \frac{8|\gamma|\varepsilon}{\beta^2} \|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})} \int_{\mathbb{R}} |\partial_x u_\varepsilon| |\partial_x^4 u_\varepsilon| dx \\ &\leq 2C(T)\varepsilon \int_{\mathbb{R}} |\partial_x u_\varepsilon| |\partial_x^4 u_\varepsilon| dx \\ &= 2 \int_{\mathbb{R}} |C(T)\partial_x u_\varepsilon| |\varepsilon \partial_x^4 u_\varepsilon| dx \\ &\leq C(T) \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \varepsilon^2 \|\partial_x^4 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ &\leq C(T) + \varepsilon^2 \|\partial_x^4 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2, \\ 3|\gamma|\varepsilon \int_{\mathbb{R}} |\partial_x u_\varepsilon| (\partial_x^3 u_\varepsilon)^2 dx &\leq 3|\gamma|\varepsilon \|\partial_x u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})} \|\partial_x^3 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ &\leq C(T)\varepsilon \|\partial_x^3 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

It follows from (2.30) that

$$\begin{aligned} & \frac{d}{dt} (\varepsilon \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \beta^2 \varepsilon \|\partial_x^3 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2) + \varepsilon^2 \|\partial_x^4 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ &\leq C(T)\varepsilon \|\partial_x^3 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + C(T). \end{aligned}$$

Integrating on $(0, t)$, by (2.2) and (2.6), we get

$$\begin{aligned} & \varepsilon \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \beta^2 \varepsilon \|\partial_x^3 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \varepsilon^2 \int_0^t \|\partial_x^4 u_\varepsilon(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \\ &\leq C_0 + C(T)\varepsilon \int_0^t \|\partial_x^3 u_\varepsilon(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds + C(T)t \leq C(T), \end{aligned}$$

which gives (2.29). ■

LEMMA 2.3. *Given $\delta, \gamma, \kappa, \beta$, the following estimates hold:*

$$(2.31) \quad 2\|\partial_t u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \beta^2 \|\partial_t \partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq C(T),$$

$$(2.32) \quad \|\partial_t u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})} \leq C(T),$$

for every $0 \leq t \leq T$.

PROOF. Let $0 \leq t \leq T$. Multiplying (2.1) by $2\partial_t u_\varepsilon$, an integration on $\mathbb{R}$ gives

$$\begin{aligned}
 & 2\|\partial_t u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\beta^2\|\partial_t u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
 &= 2 \int_{\mathbb{R}} (\partial_t u_\varepsilon)^2 dx - 2\beta^2 \int_{\mathbb{R}} \partial_t u_\varepsilon \partial_t \partial_x^2 u_\varepsilon dx \\
 &= -2\delta \int_{\mathbb{R}} \partial_x u_\varepsilon \partial_t u_\varepsilon dx + \frac{8\gamma}{\beta^2} \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_t u_\varepsilon dx \\
 &\quad - 2\kappa \int_{\mathbb{R}} \partial_x^3 u_\varepsilon \partial_t u_\varepsilon dx - 2\gamma \int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^2 u_\varepsilon \partial_t u_\varepsilon dx \\
 &\quad - 2\gamma \int_{\mathbb{R}} u_\varepsilon \partial_x^3 u_\varepsilon \partial_t u_\varepsilon dx - 2\varepsilon \int_{\mathbb{R}} \partial_x^4 u_\varepsilon \partial_t u_\varepsilon dx \\
 &= -2\delta \int_{\mathbb{R}} \partial_x u_\varepsilon \partial_t u_\varepsilon dx + \frac{8\gamma}{\beta^2} \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_t u_\varepsilon dx \\
 &\quad + 2\kappa \int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_t \partial_x u_\varepsilon dx + 2\gamma \int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_\varepsilon \partial_t \partial_x u_\varepsilon dx \\
 &\quad + 2\varepsilon \int_{\mathbb{R}} \partial_x^3 u_\varepsilon \partial_t \partial_x u_\varepsilon dx.
 \end{aligned}$$

Therefore, we have

$$\begin{aligned}
 & 2\|\partial_t u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\beta^2\|\partial_t u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
 &= -2\delta \int_{\mathbb{R}} \partial_x u_\varepsilon \partial_t u_\varepsilon dx + \frac{8\gamma}{\beta^2} \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_t u_\varepsilon dx \\
 &\quad + 2\kappa \int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_t \partial_x u_\varepsilon dx + 2\gamma \int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_\varepsilon \partial_t \partial_x u_\varepsilon dx \\
 (2.33) \quad &+ 2\varepsilon \int_{\mathbb{R}} \partial_x^3 u_\varepsilon \partial_t \partial_x u_\varepsilon dx.
 \end{aligned}$$

Since $0 < \varepsilon < 1$, thanks to (2.3), (2.5), (2.29), and the Young inequality,

$$\begin{aligned}
 2|\delta| \int_{\mathbb{R}} |\partial_x u_\varepsilon| |\partial_t u_\varepsilon| dx &= \int_{\mathbb{R}} |2\delta \partial_x u_\varepsilon| |\partial_t u_\varepsilon| dx \\
 &\leq 2\delta^2 \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{1}{2} \|\partial_t u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
 &\leq C(T) + \frac{1}{2} \|\partial_t u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2, \\
 \frac{8|\gamma|}{\beta^2} \int_{\mathbb{R}} |u_\varepsilon| |\partial_x u_\varepsilon| |\partial_t u_\varepsilon| dx &\leq \frac{8|\gamma|}{\beta^2} \|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})} \int_{\mathbb{R}} |\partial_x u_\varepsilon| |\partial_t u_\varepsilon| dx \\
 &\leq C(T) \int_{\mathbb{R}} |\partial_x u_\varepsilon| |\partial_t u_\varepsilon| dx
 \end{aligned}$$

$$\begin{aligned}
&\leq C(T) \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{1}{2} \|\partial_t u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
&\leq C(T) + \frac{1}{2} \|\partial_t u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2, \\
2|\kappa| \int_{\mathbb{R}} |\partial_x^2 u_\varepsilon| |\partial_t \partial_x u_\varepsilon| dx &= \int_{\mathbb{R}} \left| \frac{2\kappa \partial_x^2 u_\varepsilon}{\beta} \right| |\beta \partial_t \partial_x u_\varepsilon| dx \\
&\leq \frac{2\kappa^2}{\beta^2} \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{\beta^2}{2} \|\partial_t \partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
&\leq C(T) + \frac{\beta^2}{2} \|\partial_t \partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2, \\
2|\gamma| \int_{\mathbb{R}} |u_\varepsilon| |\partial_x^2 u_\varepsilon| |\partial_t \partial_x u_\varepsilon| dx &\leq 2|\gamma| \|u_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})} \int_{\mathbb{R}} |\partial_x^2 u_\varepsilon| |\partial_t \partial_x u_\varepsilon| dx \\
&\leq C(T) \int_{\mathbb{R}} |\partial_x^2 u_\varepsilon| |\partial_t \partial_x u_\varepsilon| dx \\
&\leq \int_{\mathbb{R}} \left| \frac{C(T) \partial_x^2 u_\varepsilon}{\beta} \right| |\beta \partial_t \partial_x u_\varepsilon| dx \\
&\leq C(T) \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{\beta^2}{2} \|\partial_t \partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
&\leq C(T) + \frac{\beta^2}{2} \|\partial_t \partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2, \\
2\varepsilon \int_{\mathbb{R}} |\partial_x^3 u_\varepsilon| |\partial_t \partial_x u_\varepsilon| dx &= \int_{\mathbb{R}} \left| \frac{2\varepsilon \partial_x^3 u_\varepsilon}{\beta} \right| |\beta \partial_t \partial_x u_\varepsilon| dx \\
&\leq \frac{2\varepsilon^2}{\beta^2} \|\partial_x^3 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{\beta^2}{2} \|\partial_t \partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
&\leq \frac{2\varepsilon}{\beta^2} \|\partial_x^3 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{\beta^2}{2} \|\partial_t \partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
&\leq C(T) + \frac{\beta^2}{2} \|\partial_t \partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2.
\end{aligned}$$

It follows from (2.33) that

$$\|\partial_t u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{\beta^2}{2} \|\partial_t u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq C(T),$$

which gives (2.31).

Finally, we prove (2.32). Due to (2.31) and the Hölder inequality,

$$\begin{aligned}
(\partial_t u_\varepsilon(t, x))^2 &= 2 \int_{-\infty}^x \partial_t u_\varepsilon \partial_t \partial_x u_\varepsilon dy \leq 2 \int_{\mathbb{R}} |\partial_t u_\varepsilon| |\partial_t \partial_x u_\varepsilon| dx \\
&\leq 2 \|\partial_t u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} \|\partial_t \partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} \leq C(T).
\end{aligned}$$

Hence,

$$\|\partial_t u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}^2 \leq C(T),$$

which gives (2.32). ■

Using the Sobolev immersion theorem, we prove the following result:

LEMMA 2.4. *Assume (1.2) and (1.3). There exist a subsequence $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ of $\{u_\varepsilon\}_{\varepsilon>0}$ and a limit function u which satisfies (1.9) such that*

$$(2.34) \quad u_{\varepsilon_k} \rightarrow u \text{ a.e. and in } L_{\text{loc}}^p((0, \infty) \times \mathbb{R}), \quad 1 \leq p < \infty.$$

Moreover, u is a solution of (1.1).

PROOF. Thanks to Lemmas 2.1 and 2.3,

$$\{u_\varepsilon\}_{\varepsilon>0} \text{ is uniformly bounded in } H^1((0, T) \times \mathbb{R}), \quad T \geq 0,$$

which gives (2.34).

Observe that, thanks to Lemma 2.1,

$$u \in L^\infty(0, T; H^2(\mathbb{R})), \quad T \geq 0,$$

while, by Lemma 2.3,

$$u \in W^{1,\infty}((0, T) \times \mathbb{R}), \quad T \geq 0.$$

Moreover, by Lemma 2.3, we have

$$\partial_t \partial_x u \in L^2((0, T) \times \mathbb{R}), \quad T \geq 0.$$

Therefore, (1.9) holds and u is a solution of (1.1). ■

We are finally ready for the proof of Theorem 1.1.

PROOF OF THEOREM 1.1. Lemma 2.4 gives the existence of a solution u of (1.1) such that (1.9) holds.

We prove (1.10). Let u_1, u_2 be two solutions of (1.1), which satisfy (1.9), that is,

$$\begin{cases} \partial_t u_i + \delta \partial_x u_i - \frac{2\gamma}{\beta^2} \partial_x u_i^2 + \kappa \partial_x^3 u_i \\ \quad - \beta^2 \partial_t \partial_x^2 u_i + \gamma \partial_x (u_i \partial_x^2 u_i) = 0, & t > 0, x \in \mathbb{R}, \\ u_i(0, x) = u_{i,0}(x), & x \in \mathbb{R}, \end{cases}$$

for $i = 1, 2$. Then the function

$$(2.35) \quad \omega = u_1 - u_2$$

is the solution of the following Cauchy problem:

$$(2.36) \quad \begin{cases} \partial_t \omega + \delta \partial_x \omega - \frac{2\gamma}{\beta^2} \partial_x (u_i^2 - u_2^2) + \kappa \partial_x^3 \omega \\ - \beta^2 \partial_t \partial_x^2 \omega + \gamma \partial_x (u_i \partial_x^2 u_i - u_2 \partial_x^2 u_2) = 0, & t > 0, x \in \mathbb{R}, \\ \omega_0(x) = u_{1,0}(x) - u_{2,0}(x), & x \in \mathbb{R}. \end{cases}$$

Observe that, thanks to (2.35),

$$(2.37) \quad \begin{aligned} \partial_x (u_1^2 - u_2^2) &= \partial_x ((u_1 + u_2)(u_1 - u_2)) = \partial_x ((u_1 + u_2)\omega), \\ \partial_x (u_1 \partial_x^2 u_1 - u_2 \partial_x^2 u_2) &= \partial_x (u_1 \partial_x^2 u_1 - u_1 \partial_x^2 u_2 + u_1 \partial_x^2 u_2 - u_2 \partial_x^2 u_2) \\ &= \partial_x (u_1 \partial_x^2 \omega - \partial_x^2 u_2 \omega) + \partial_x (\partial_x^2 u_2 \omega). \end{aligned}$$

Consequently, by (2.37), equation (2.36) reads

$$(2.38) \quad \begin{aligned} \partial_t \omega + \delta \partial_x \omega - \beta^2 \partial_t \partial_x^2 \omega - \frac{2\gamma}{\beta^2} \partial_x ((u_1 + u_2)\omega) \\ + \kappa \partial_x^3 \omega + \gamma \partial_x (u_1 \partial_x^2 \omega - \partial_x^2 u_2 \omega) + \gamma \partial_x (\partial_x^2 u_2 \omega) = 0. \end{aligned}$$

Assume $T \geq 0$ is given. Since $u_1, u_2 \in L^\infty(0, T; H^2(\mathbb{R}))$,

$$(2.39) \quad \|\partial_x u_1\|_{L^\infty((0,T) \times \mathbb{R})}, \|\partial_x u_2\|_{L^\infty((0,T) \times \mathbb{R})} \leq C(T).$$

Moreover, by (2.39), we have

$$(2.40) \quad |\partial_x u_1 + \partial_x u_2| \leq |\partial_x u_1| + |\partial_x u_2| \leq C(T).$$

Multiplying (2.38) by 2ω , an integration on $\mathbb{R}$ gives

$$\begin{aligned} & \frac{d}{dt} (\|\omega(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \beta^2 \|\partial_x \omega(t, \cdot)\|_{L^2(\mathbb{R})}^2) \\ &= 2 \int_{\mathbb{R}} \omega \partial_t \omega - 2\beta^2 \int_{\mathbb{R}} \omega \partial_t \partial_x^2 \omega \, dx \\ &= \frac{4\gamma}{\beta^2} \int_{\mathbb{R}} \omega \partial_x ((u_1 + u_2)\omega) \, dx - 2\delta \int_{\mathbb{R}} \omega \partial_x \omega \, dx - 2\kappa \int_{\mathbb{R}} \omega \partial_x^3 \omega \, dx \\ &\quad - 2\gamma \int_{\mathbb{R}} \omega \partial_x (u_1 \partial_x^2 \omega - \partial_x^2 u_2 \omega) - 2\gamma \int_{\mathbb{R}} \omega \partial_x (\partial_x^2 u_2 \omega) \, dx \\ &= -\frac{4\gamma}{\beta^2} \int_{\mathbb{R}} (u_1 + u_2) \omega \partial_x \omega \, dx + 2\kappa \int_{\mathbb{R}} \partial_x \omega \partial_x^2 \omega \, dx + 2\gamma \int_{\mathbb{R}} u_1 \partial_x \omega \partial_x^2 \omega \, dx \\ &= \frac{2\gamma}{\beta^2} \int_{\mathbb{R}} (\partial_x u_1 + \partial_x u_2) \omega^2 \, dx - \gamma \int_{\mathbb{R}} \partial_x u_1 (\partial_x \omega)^2 \, dx. \end{aligned}$$

Therefore, we have

$$(2.41) \quad \begin{aligned} & \frac{d}{dt} (\|\omega(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \beta^2 \|\partial_x \omega(t, \cdot)\|_{L^2(\mathbb{R})}^2) \\ &= \frac{2\gamma}{\beta^2} \int_{\mathbb{R}} (\partial_x u_1 + \partial_x u_2) \omega^2 dx - \gamma \int_{\mathbb{R}} \partial_x u_1 (\partial_x \omega)^2 dx. \end{aligned}$$

Thanks to (2.39) and (2.40),

$$\begin{aligned} & \frac{2|\gamma|}{\beta^2} \int_{\mathbb{R}} |\partial_x u_1 + \partial_x u_2| \omega^2 dx \leq C(T) \|\omega(t, \cdot)\|_{L^2(\mathbb{R})}^2, \\ & |\gamma| \int_{\mathbb{R}} |\partial_x u_1| (\partial_x \omega)^2 dx \leq C(T) \|\partial_x \omega(t, \cdot)\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

It follows from (2.41) that

$$\begin{aligned} & \frac{d}{dt} (\|\omega(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \beta^2 \|\partial_x \omega(t, \cdot)\|_{L^2(\mathbb{R})}^2) \\ & \leq C(T) \|\omega(t, \cdot)\|_{L^2(\mathbb{R})}^2 + C(T) \|\partial_x \omega(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ & \leq C(T) (\|\omega(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \beta^2 \|\partial_x \omega(t, \cdot)\|_{L^2(\mathbb{R})}^2). \end{aligned}$$

The Grönwall lemma and (2.36) give

$$\|\omega(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \beta^2 \|\partial_x \omega(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq e^{C(T)t} (\|\omega_0\|_{L^2(\mathbb{R})}^2 + \beta^2 \|\partial_x \omega_0\|_{L^2(\mathbb{R})}^2).$$

By (1.11), we have

$$(2.42) \quad \tau_1^2 \|\omega\|_{H^1(\mathbb{R})}^2 \leq \tau_2^2 e^{C(T)t} \|\omega_0\|_{H^1(\mathbb{R})}^2.$$

Therefore, (1.10) follows from (2.36) and (2.42). ■

ACKNOWLEDGMENTS – GMC expresses his gratitude to the HIAS – Hamburg Institute for Advanced Study for their warm hospitality.

FUNDING – GMC is a member of the Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). GMC has been partially supported by the project funded under the National Recovery and Resilience Plan (NRRP), Mission 4 Component 2 Investment 1.4 – Call for tender No. 3138 of 16/12/2021 of the Italian Ministry of University and Research funded by the European Union – NextGenerationEUoAward Number: CN000023, Concession Decree No. 1033 of 17/06/2022 adopted by the Italian Ministry of University and Research, CUP: D93C22000410001, Centro Nazionale per la Mobilità Sostenibile and the Italian Ministry of Education, University and Research

under the Programme Department of Excellence Legge 232/2016 (Grant No. CUP – D93C23000100001), by the Research Project of National Relevance “Evolution problems involving interacting scales” granted by the Italian Ministry of Education, University and Research (MIUR Prin 2022, project code 2022M9BKBC, Grant No. CUP D53D23005880006).

REFERENCES

- [1] S. BIANCHINI – A. BRESSAN, [Vanishing viscosity solutions of nonlinear hyperbolic systems](#). *Ann. of Math. (2)* **161** (2005), no. 1, 223–342. Zbl [1082.35095](#) MR [2150387](#)
- [2] A. BRESSAN – A. CONSTANTIN, [Global conservative solutions of the Camassa–Holm equation](#). *Arch. Ration. Mech. Anal.* **183** (2007), no. 2, 215–239. Zbl [1105.76013](#) MR [2278406](#)
- [3] A. BRESSAN – A. CONSTANTIN, [Global dissipative solutions of the Camassa–Holm equation](#). *Anal. Appl. (Singap.)* **5** (2007), no. 1, 1–27. Zbl [1139.35378](#) MR [2288533](#)
- [4] R. CAMASSA – D. D. HOLM, [An integrable shallow water equation with peaked solitons](#). *Phys. Rev. Lett.* **71** (1993), no. 11, 1661–1664. Zbl [0972.35521](#) MR [1234453](#)
- [5] G. M. COCLITE – L. DI RUVO, [A note on the convergence of the solutions of the Camassa–Holm equation to the entropy ones of a scalar conservation law](#). *Discrete Contin. Dyn. Syst.* **36** (2016), no. 6, 2981–2990. Zbl [1336.35056](#) MR [3485428](#)
- [6] G. M. COCLITE – L. DI RUVO, [On the solutions for an Ostrovsky type equation](#). *Nonlinear Anal. Real World Appl.* **55** (2020), article no. 103141. Zbl [1451.35061](#) MR [4083153](#)
- [7] G. M. COCLITE – L. DI RUVO, [\$H^1\$ solutions for a Kuramoto–Velarde type equation](#). *Mediterr. J. Math.* **20** (2023), no. 3, article no. 110. Zbl [1509.35111](#) MR [4549888](#)
- [8] G. M. COCLITE – H. HOLDEN – K. H. KARLSEN, [Wellposedness for a parabolic-elliptic system](#). *Discrete Contin. Dyn. Syst.* **13** (2005), no. 3, 659–682. Zbl [1082.35056](#) MR [2152336](#)
- [9] G. M. COCLITE – K. H. KARLSEN, [On the well-posedness of the Degasperis–Procesi equation](#). *J. Funct. Anal.* **233** (2006), no. 1, 60–91. Zbl [1090.35142](#) MR [2204675](#)
- [10] G. M. COCLITE – K. H. KARLSEN, [Periodic solutions of the Degasperis–Procesi equation: Well-posedness and asymptotics](#). *J. Funct. Anal.* **268** (2015), no. 5, 1053–1077. Zbl [1316.35081](#) MR [3304593](#)
- [11] G. M. COCLITE – K. H. KARLSEN – Y.-S. KWON, [Initial-boundary value problems for conservation laws with source terms and the Degasperis–Procesi equation](#). *J. Funct. Anal.* **257** (2009), no. 12, 3823–3857. Zbl [1180.35348](#) MR [2557726](#)
- [12] G. M. COCLITE – K. H. KARLSEN – N. H. RISEBRO, [Numerical schemes for computing discontinuous solutions of the Degasperis–Procesi equation](#). *IMA J. Numer. Anal.* **28** (2008), no. 1, 80–105. Zbl [1246.76114](#) MR [2387906](#)

- [13] A. CONSTANTIN, [On the Cauchy problem for the periodic Camassa–Holm equation](#). *J. Differential Equations* **141** (1997), no. 2, 218–235. Zbl [0889.35022](#) MR [1488351](#)
- [14] A. CONSTANTIN, [Existence of permanent and breaking waves for a shallow water equation: A geometric approach](#). *Ann. Inst. Fourier (Grenoble)* **50** (2000), no. 2, 321–362. Zbl [0944.35062](#) MR [1775353](#)
- [15] A. CONSTANTIN, [On the scattering problem for the Camassa–Holm equation](#). *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* **457** (2001), no. 2008, 953–970. Zbl [0999.35065](#) MR [1875310](#)
- [16] A. CONSTANTIN – J. ESCHER, [Global existence and blow-up for a shallow water equation](#). *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **26** (1998), no. 2, 303–328. Zbl [0918.35005](#) MR [1631589](#)
- [17] A. CONSTANTIN – J. ESCHER, [Global weak solutions for a shallow water equation](#). *Indiana Univ. Math. J.* **47** (1998), no. 4, 1527–1545. Zbl [0930.35133](#) MR [1687106](#)
- [18] A. CONSTANTIN – L. MOLINET, [Global weak solutions for a shallow water equation](#). *Comm. Math. Phys.* **211** (2000), no. 1, 45–61. Zbl [1002.35101](#) MR [1757005](#)
- [19] A. CONSTANTIN – W. A. STRAUSS, [Stability of a class of solitary waves in compressible elastic rods](#). *Phys. Lett. A* **270** (2000), no. 3–4, 140–148. Zbl [1115.74339](#) MR [1763691](#)
- [20] H.-H. DAI, [Model equations for nonlinear dispersive waves in a compressible Mooney–Rivlin rod](#). *Acta Mech.* **127** (1998), no. 1–4, 193–207. Zbl [0910.73036](#) MR [1606738](#)
- [21] A. DEGASPERIS – G. GAETA (eds.), [Symmetry and perturbation theory](#), World Scientific, River Edge, NJ, 1999. Zbl [0944.00056](#) MR [1844102](#)
- [22] A. DEGASPERIS – D. D. HOLM – A. N. W. HONE, [Integrable and non-integrable equations with peakons](#). In *Nonlinear physics: Theory and experiment, II (Gallipoli, 2002)*, pp. 37–43, World Scientific, River Edge, NJ, 2003. Zbl [1053.37039](#) MR [2028761](#)
- [23] A. DEGASPERIS – D. D. HOLM – A. N. I. HONE, [A new integrable equation with peakon solutions](#). *Theoret. and Math. Phys.* **133** (2002), no. 2, 1463–1474. MR [2001531](#)
- [24] H. R. DULLIN – G. A. GOTTWALD – D. D. HOLM, [An integrable shallow water equation with linear and nonlinear dispersion](#). *Phys. Rev. Lett.* **87** (2001), no. 19, article no. 194501.
- [25] B. FUCHSSTEINER – A. S. FOKAS, [Symplectic structures, their Bäcklund transformations and hereditary symmetries](#). *Phys. D* **4** (1981/82), no. 1, 47–66. Zbl [1194.37114](#) MR [0636470](#)
- [26] G. GUI – Y. LIU, [On the Cauchy problem for the Degasperis–Procesi equation](#). *Quart. Appl. Math.* **69** (2011), no. 3, 445–464. Zbl [1229.35243](#) MR [2850740](#)
- [27] A. A. HIMONAS – G. MISIOLEK, [The Cauchy problem for an integrable shallow-water equation](#). *Differential Integral Equations* **14** (2001), no. 7, 821–831. Zbl [1009.35075](#) MR [1828326](#)
- [28] H. HOLDEN – X. RAYNAUD, [Global conservative solutions of the Camassa–Holm equation—A Lagrangian point of view](#). *Comm. Partial Differential Equations* **32** (2007), no. 10–12, 1511–1549. Zbl [1136.35080](#) MR [2372478](#)

- [29] H. HOLDEN – X. RAYNAUD, [Global conservative solutions of the generalized hyperelastic-rod wave equation](#). *J. Differential Equations* **233** (2007), no. 2, 448–484. Zbl [1116.35115](#) MR [2292515](#)
- [30] H. HOLDEN – X. RAYNAUD, [Dissipative solutions for the Camassa–Holm equation](#). *Discrete Contin. Dyn. Syst.* **24** (2009), no. 4, 1047–1112. Zbl [1178.65099](#) MR [2505693](#)
- [31] S. HWANG, [Singular limit problem of the Camassa–Holm type equation](#). *J. Differential Equations* **235** (2007), no. 1, 74–84. Zbl [1387.35526](#) MR [2309567](#)
- [32] S. HWANG – A. E. TZAVARAS, [Kinetic decomposition of approximate solutions to conservation laws: Application to relaxation and diffusion-dispersion approximations](#). *Comm. Partial Differential Equations* **27** (2002), no. 5-6, 1229–1254. Zbl [1020.35054](#) MR [1916562](#)
- [33] R. S. JOHNSON, [Camassa–Holm, Korteweg–de Vries and related models for water waves](#). *J. Fluid Mech.* **455** (2002), 63–82. Zbl [1037.76006](#) MR [1894796](#)
- [34] R. S. JOHNSON, [On solutions of the Camassa–Holm equation](#). *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* **459** (2003), no. 2035, 1687–1708. Zbl [1039.76006](#) MR [1997519](#)
- [35] C. E. KENIG – G. PONCE – L. VEGA, [Well-posedness and scattering results for the generalized Korteweg–de Vries equation via the contraction principle](#). *Comm. Pure Appl. Math.* **46** (1993), no. 4, 527–620. Zbl [0808.35128](#) MR [1211741](#)
- [36] P. G. LEFLOCH – R. NATALINI, [Conservation laws with vanishing nonlinear diffusion and dispersion](#). *Nonlinear Anal., Ser. A: Theory Methods* **36** (1999), no. 2, 213–230. Zbl [0923.35159](#) MR [1668856](#)
- [37] J. LENELL, [Conservation laws of the Camassa–Holm equation](#). *J. Phys. A* **38** (2005), no. 4, 869–880. Zbl [1076.35100](#) MR [2125239](#)
- [38] J. LENELL, [Traveling wave solutions of the Camassa–Holm equation](#). *J. Differential Equations* **217** (2005), no. 2, 393–430. Zbl [1082.35127](#) MR [2168830](#)
- [39] Y. A. LI – P. J. OLVER, [Well-posedness and blow-up solutions for an integrable nonlinearly dispersive model wave equation](#). *J. Differential Equations* **162** (2000), no. 1, 27–63. Zbl [0958.35119](#) MR [1741872](#)
- [40] Y. LIU – Z. YIN, [Global existence and blow-up phenomena for the Degasperis–Procesi equation](#). *Comm. Math. Phys.* **267** (2006), no. 3, 801–820. Zbl [1131.35074](#) MR [2249792](#)
- [41] Y. LIU – Z. YIN, [On the blow-up phenomena for the Degasperis–Procesi equation](#). *Int. Math. Res. Not. IMRN* (2007), no. 23, article no. rnm117. Zbl [1162.35067](#) MR [2380010](#)
- [42] D. LU – D. PENG – L. TIAN, [On the well-posedness problem for the generalized Dullin–Gottwald–Holm equation](#). *Int. J. Nonlinear Sci.* **1** (2006), no. 3, 178–186. Zbl [1394.35426](#) MR [2302020](#)
- [43] G. MISIOLEK, [Classical solutions of the periodic Camassa–Holm equation](#). *Geom. Funct. Anal.* **12** (2002), no. 5, 1080–1104. Zbl [1158.37311](#) MR [1937835](#)

- [44] G. RODRÍGUEZ-BLANCO, [On the Cauchy problem for the Camassa–Holm equation](#). *Nonlinear Anal., Ser. A: Theory Methods* **46** (2001), no. 3, 309–327. Zbl [0980.35150](#) MR [1851854](#)
- [45] M. E. SCHONBEK, [Convergence of solutions to nonlinear dispersive equations](#). *Comm. Partial Differential Equations* **7** (1982), no. 8, 959–1000. Zbl [0496.35058](#) MR [0668586](#)
- [46] Z. XIN – P. ZHANG, [On the weak solutions to a shallow water equation](#). *Comm. Pure Appl. Math.* **53** (2000), no. 11, 1411–1433. Zbl [1048.35092](#) MR [1773414](#)
- [47] Z. XIN – P. ZHANG, [On the uniqueness and large time behavior of the weak solutions to a shallow water equation](#). *Comm. Partial Differential Equations* **27** (2002), no. 9-10, 1815–1844. Zbl [1034.35115](#) MR [1941659](#)
- [48] Z. YIN, [Global existence for a new periodic integrable equation](#). *J. Math. Anal. Appl.* **283** (2003), no. 1, 129–139. Zbl [1033.35121](#) MR [1994177](#)
- [49] Z. YIN, [Global solutions to a new integrable equation with peakons](#). *Indiana Univ. Math. J.* **53** (2004), no. 4, 1189–1209. Zbl [1062.35094](#) MR [2095454](#)
- [50] Z. YIN, [Global weak solutions for a new periodic integrable equation with peakon solutions](#). *J. Funct. Anal.* **212** (2004), no. 1, 182–194. Zbl [1059.35149](#) MR [2065241](#)

Manoscritto pervenuto in redazione il 24 febbraio 2023.