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## An interesting theorem on circumscribed hexagon

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### 1 Introduction

In [3], an interesting theorem of Dao Thanh Oai about six circumcenters associated with a cyclic hexagon was introduced.

**Theorem 1** (Dao). *Let  $A_i$ ,  $i = 1, 2, \dots, 6$ , be six points on a circle. Taking subscripts modulo 6, we denote, for  $i = 1, 2, \dots, 6$ , the intersection of the lines  $A_i A_{i+1}$  and  $A_{i+2} A_{i+3}$  by  $B_{i+3}$ , and the circumcenter of the triangle  $A_i A_{i+1} B_{i+2}$  by  $C_{i+3}$ . The lines  $C_1 C_4$ ,  $C_2 C_5$ , and  $C_3 C_6$  are concurrent. (See Figure 1.)*

With a dual way of thinking, when we replace a cyclic hexagon with a circumscribed hexagon and the circumscribed circle centers with the inscribed circle centers, we get

Gelegentlich gelingt es, durch analoges Denken aus bekannten Theoremen neue Sätze zu gewinnen. Diese Methode ist besonders fruchtbar in der ebenen Euklidischen Geometrie. In der vorliegenden Arbeit gehen die Autoren von einem erst vor wenigen Jahren gefundenen Satz von Dao aus. Indem sie in diesem Satz statt eines einbeschriebenen ein umbeschriebenes Sechseck betrachten und Umkreise durch Inkreise ersetzen, gelangen sie zu einem Theorem, in dem statt kopunktalen Geraden kollineare Punkte auftreten. Der Beweis erfolgt durch Rechnung in komplexen Koordinaten.

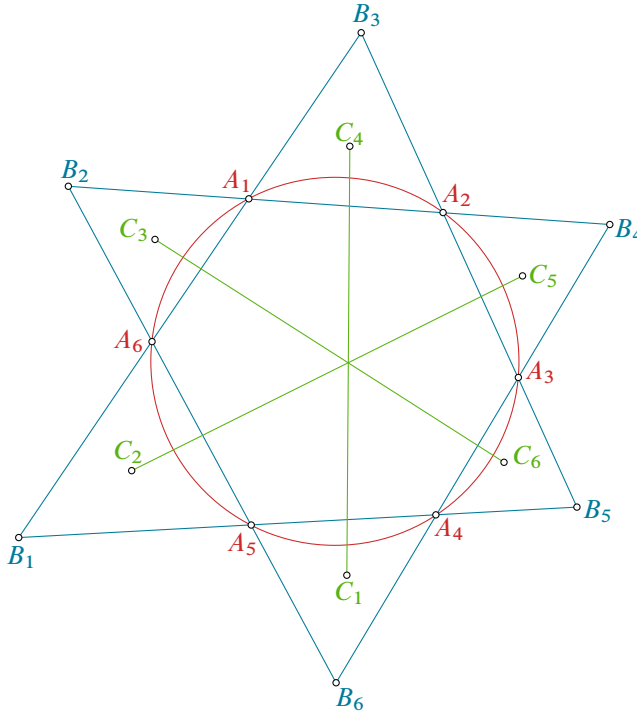


Figure 1. Illustration of Dao's theorem.

a similar theorem on three collinear points (instead of three concurrent lines). Our theorem is as follows.

**Theorem 2** (Main theorem). *Let  $A_1 A_2 A_3 A_4 A_5 A_6$  be a circumscribed hexagon on a plane. Taking subscripts modulo 6, we denote, for  $i = 1, 2, \dots, 6$ , the intersection of the lines  $A_i A_{i+1}$  and  $A_{i+2} A_{i+3}$  by  $B_{i+1}$ , and the incenter of the triangle  $A_i B_i A_{i+1}$  by  $I_i$ , and the intersection of the lines  $I_i I_{i+3}$  and  $B_i B_{i+3}$  by  $X_i$ . Then the three points  $X_1, X_2$ , and  $X_3$  are collinear. (See Figure 2.)*

We shall give a solution to Theorem 2 using complex coordinates in the next section.

## 2 Proof of Theorem 2 using complex numbers

In this section, we shall use complex numbers to approach the theorem as in [3]. An interesting observation is that, when we replace the secant line (of cyclic quadrilateral) with the tangents (of circumscribed quadrilateral), we get a similar formula for complex numbers. This formula will also lead to another complex number solution to our new theorem on the circumscribed hexagon.

We first introduce and prove a lemma for tangents (for a similar lemma, see [3]).

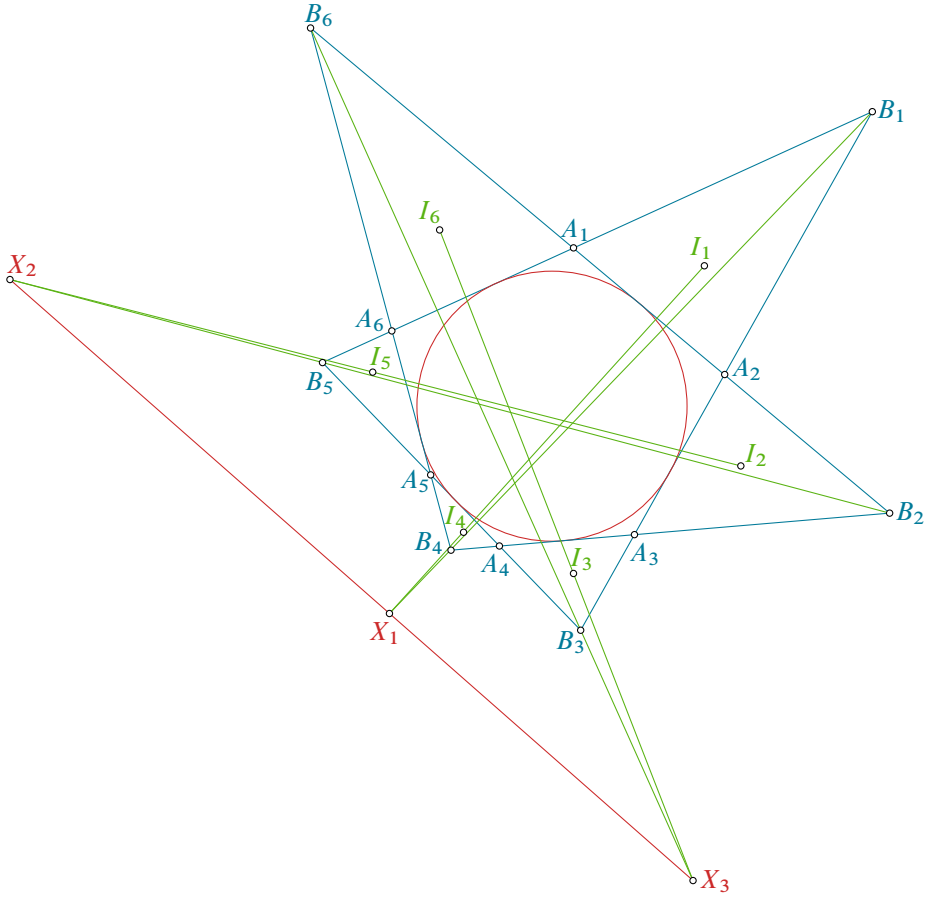


Figure 2. Illustration of the main theorem.

**Lemma 1.** *On the unit complex circle (centered at  $O$ ), consider the consecutive points  $b$ ,  $a$ , and  $c$ . Let the tangents at these points meet at the points  $A_1$ ,  $B_1$ , and  $C_1$  such that  $O$  is the  $A_1$ -excenter of triangle  $A_1B_1C_1$ . Then the incenter  $K$  of the triangle  $A_1B_1C_1$  has the complex coordinate*

$$k = \frac{4abc}{(a+b)(a+c)}.$$

*Proof.* (See Figure 3.) The tangents at the points  $b$ ,  $c$  of the unit circle have the equations  $z + b^2\bar{z} - 2b = 0$  and  $z + c^2\bar{z} - 2c = 0$ . Therefore, the complex coordinate of the intersection  $A_1$  is the solution of the system

$$\begin{cases} z + b^2\bar{z} - 2b = 0, \\ z + c^2\bar{z} - 2c = 0. \end{cases} \quad (1)$$

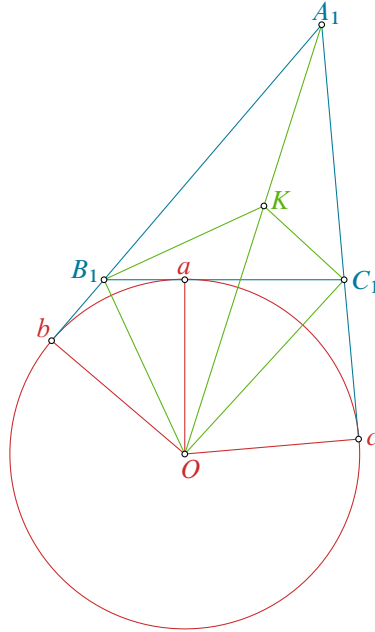


Figure 3. Proof of Lemma 1.

Solving system (1) yields

$$a_1 = \frac{2bc}{b+c}. \tag{2}$$

Similarly, we have

$$b_1 = \frac{2ca}{c+a} \tag{3}$$

and

$$c_1 = \frac{2ab}{a+b}.$$

Since  $O$  is the  $A_1$ -excenter of triangle  $A_1B_1C_1$ , the perpendicular to  $OB_1$  at  $B_1$  is the angle bisector of  $\angle A_1B_1C_1$ . From (3), this line has the equation

$$(a+b)z + ab(a+b)\bar{z} - 4ab = 0.$$

The angle bisector of  $\angle B_1A_1C_1$  is the line  $OA_1$ . From (2), line  $OA_1$  has the equation  $z - bc\bar{z} = 0$ . Hence the complex coordinate of the incenter  $K$  of the triangle  $A_1B_1C_1$  is the solution of the system

$$\begin{cases} (a+b)z + ab(a+b)\bar{z} - 4ab = 0, \\ z - bc\bar{z} = 0. \end{cases} \tag{4}$$

Solving system (4), we get  $k = \frac{4abc}{(a+b)(a+c)}$ . This completes the proof of Lemma 1. ■

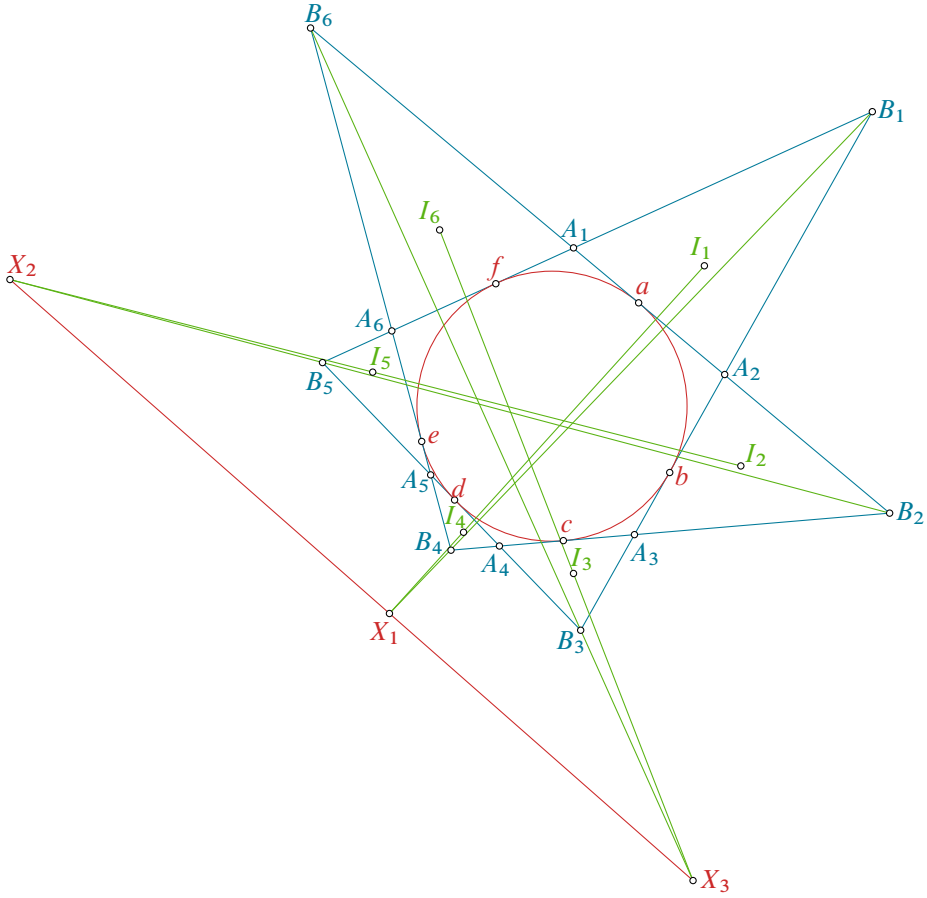


Figure 4. Illustration of the proof of the main theorem.

*Proof of Theorem 2.* (See Figure 2.) Assume that the incircle of the hexagon is the complex unit circle. Let  $a, b, c, d, e,$  and  $f$  be the complex coordinates of the contact points of the sides  $A_1A_2, A_2A_3, A_3A_4, A_4A_5, A_5A_6,$  and  $A_6A_1,$  respectively, with the unit circle (as shown in Figure 4). It follows from the proof of Lemma 1 that we have

$$b_1 = \frac{2fb}{f+b}, \quad b_2 = \frac{2ac}{a+c}, \quad b_3 = \frac{2bd}{b+d},$$

$$b_4 = \frac{2ce}{c+e}, \quad b_5 = \frac{2df}{d+f}, \quad b_6 = \frac{2ea}{e+a}.$$

On the other hand, we deduce from Lemma 1 that

$$i_1 = \frac{4fab}{(a+b)(a+f)}, \quad i_2 = \frac{4abc}{(b+c)(b+a)}, \quad i_3 = \frac{4bcd}{(b+c)(c+d)},$$

$$i_4 = \frac{4cde}{(c+d)(d+e)}, \quad i_5 = \frac{4def}{(d+e)(e+f)}, \quad i_6 = \frac{4efa}{(e+f)(f+a)},$$

where  $i_1, i_2, i_3, i_4, i_5,$  and  $i_6$  are the complex coordinates of the incenters of the triangles  $A_1A_2B_1, A_2A_3B_2, A_3A_4B_3, A_4A_5B_4, A_5A_6B_5,$  and  $A_6A_1B_6,$  respectively. From the coordinates of the points  $B_1$  and  $B_4,$  the line  $B_1B_4$  has the equation

$$\frac{z - b_1}{z - b_4} = \overline{\left(\frac{z - b_1}{z - b_4}\right)}$$

or

$$(c + e - f - b)z + (bce + cef - bcf - bef)\bar{z} + 2(bf - ce) = 0. \quad (5)$$

Similarly, from the coordinates of the points  $I_1$  and  $I_4,$  the line  $I_1I_4$  has the equation

$$\frac{z - i_1}{z - i_4} = \overline{\left(\frac{z - i_1}{z - i_4}\right)}$$

or

$$\begin{aligned} & (cda + cea + d^2a + dea - da^2 - dab - daf - dbf)z \\ & + (cdea^2 + cdeab + cdeaf + cdebf - cdabf - ceabf - d^2abf - deabf)\bar{z} \\ & + 4(dabf - cdea) = 0. \end{aligned} \quad (6)$$

Thus, the intersection  $X_1$  of the two lines  $B_1B_4$  and  $I_1I_4$  is the solution of the system of equations (5) and (6). Solving this system, we get

$$x_1 = \frac{2(a^2cde - abcde + abcdf - abd^2f - abcef + abdef - acdef + bcdef)}{a^2cd - abd^2 - abce + a^2de + bcdf - ad^2f - acef + bdef}.$$

Similarly,

$$\begin{aligned} x_2 &= \frac{2(b^2def - bcdef + abcde - abce^2 - abcdf + abcef - abdef + acdef)}{b^2de - bce^2 - bcdf + b^2ef + acde - abe^2 - abdf + acef}, \\ x_3 &= \frac{2(ac^2ef - acdef + bcdef - bcdf^2 - bcdef + abcdf - abcef + abdef)}{c^2ef - cdf^2 - acde + ac^2f + bdef - bcf^2 - abce + abdf}. \end{aligned}$$

The points  $X_1, X_2,$  and  $X_3$  are collinear if and only if

$$\begin{vmatrix} x_1 & \bar{x}_1 & 1 \\ x_2 & \bar{x}_2 & 1 \\ x_3 & \bar{x}_3 & 1 \end{vmatrix} = 0 \quad (7)$$

(see [1,2]). Since

$$x_1 = \frac{2p_1}{r_1}, \quad \bar{x}_1 = \frac{2q_1}{r_1}, \quad x_2 = \frac{2p_2}{r_2}, \quad \bar{x}_2 = \frac{2q_2}{r_2}, \quad x_3 = \frac{2p_3}{r_3}, \quad \bar{x}_3 = \frac{2q_3}{r_3}$$

and after elimination of denominators, equation (7) is equivalent to

$$\begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix} = 0, \quad (8)$$

where

$$\begin{aligned} p_1 &= a^2cde - abcde + abcdf - abd^2f - abcef + abdef - acdef + bcdef, \\ p_2 &= b^2def - bcdef + abcde - abce^2 - abcdf + abcef - abdef + acdef, \\ p_3 &= ac^2ef - acdef + bcdef - bcdf^2 - bcdef + abcdf - abcef + abdef, \\ q_1 &= a^2d - abd + acd - ad^2 - ace + ade - adf + bdf, \\ q_2 &= b^2e - bce + bde - be^2 - bdf + bef - abe + ace, \\ q_3 &= c^2f - cdf + cef - cf^2 - ace + acf - bcf + bdf, \\ r_1 &= a^2cd - abd^2 - abce + a^2de + bcdf - ad^2f - acef + bdef, \\ r_2 &= b^2de - bce^2 - bcdf + b^2ef + acde - abe^2 - abdf + acef, \\ r_3 &= c^2ef - cdf^2 - acde + ac^2f + bdef - bcf^2 - abce + abdf. \end{aligned}$$

By determinant transformation, (8) is equivalent to

$$\begin{vmatrix} p_1 & q_1 & r_1 \\ p_1 + p_2 & q_1 + q_2 & r_1 + r_2 \\ p_3 - p_2 & q_3 - q_2 & r_3 - r_2 \end{vmatrix} = 0$$

or

$$\begin{vmatrix} p_1 & q_1 & r_1 \\ ace - bdf & a + c + e - b - d - f & ac + ae + ce - bd - bf - df \\ ace - bdf & a + c + e - b - d - f & ac + ae + ce - bd - bf - df \end{vmatrix} = 0,$$

which is true. This completes the proof. ■

### 3 Conclusion

We seem to see a duality in the statement of our new theorem on circumscribed hexagons with Dao's theorem on inscribed hexagons. Furthermore, the complex number tool once again shows its usefulness in both theorems. Both theorems are an interesting complement to each other and have a common view through complex numbers as we have seen.

### References

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