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On quasi-Heronian equable triangles

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This article is dedicated to the memory of Prof. John Steinig who passed away on the 25th of March 2023.

Triangles having integer area and side lengths are said to be *Heronian*. Moreover, if their perimeter and area have the same value, they are called *equable*. There are exactly five equable Heronian triangles whose side lengths are $(5, 12, 13)$, $(6, 8, 10)$, $(6, 25, 29)$, $(7, 15, 20)$ and $(9, 10, 17)$. Proofs of the preceding result figure in [1–3]. Our goal in this note is to study the set of equable triangles having side lengths $b + \sqrt{a}$, $b - \sqrt{a}$, c, where $a, b, c \in \mathbb{N}$ and c is a non-square, which we define as *quasi-Heronian*. For example, by

Wie die babylonische Keilschrifttafel Plimpton 322 zu zeigen scheint, geht eine mögliche Parametrisierung von rechtwinkligen Dreiecken mit ganzzahligen Seiten bereits auf das Jahr 1750 vor Christus zurück. Etwa 3500 Jahre später erweiterte Leonhard Euler das Ergebnis, indem er alle Dreiecke mit ganzzahligen Seiten und Flächen parametrisierte. Inspiriert durch die Form der Lösung von rechtwinkligen Dreiecken mit rationalem Umfang und rationaler Fläche ersetzt der Autor der vorliegenden Arbeit in der Voraussetzung den rechten Winkels durch die Eigenschaft *equabel* (d. h. Fläche = Umfang) und eine rationale Seite. Dies führt ihn auf natürliche Weise zur Definition von *quasi-heronischen equablen Dreiecken*. Die Parametrisierung der obigen Familie ist eines der Ziele dieser Arbeit. Insbesondere wird damit ein Problem gelöst, das der Autor seinen Studenten vorschlägt: *Finde eine unendliche Menge von rechtwinkligen equablen Dreiecken mit ganzzahliger Hypotenuse und ganzzahligem Umfang und zwei irrationalen Schenkeln* (siehe Theorem 1 (I)). Der Artikel ist Teil eines aktuellen Forschungsgebiets, das darin besteht, allgemeine Eigenschaften von equablen Polygonen zu erforschen, deren Scheitelpunkte auf einem Gauß- oder Eisenstein-Gitter liegen.

applying Heron's formula [4, p. 172],

$$16K^2 = (r+s+t)(-r+s+t)(r-s+t)(r+s-t),$$

to the triangle $(r,s,t) = (11+\sqrt{2}, 11-\sqrt{2}, 6)$, we get an area $K = 28$ which corresponds also to its perimeter.

As a motivation to the above definition, we recall a result concerning right triangles.

Lemma 1. *If ABC is a right triangle of rational area and perimeter of legs x, y and hypotenuse z, then $x = b - \sqrt{a}$, $y = b + \sqrt{a}$ for $a, b \in \mathbb{Q}$ and hence $z \in \mathbb{Q}$.*

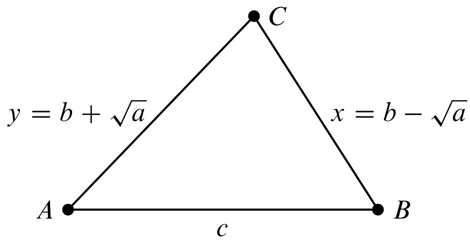
Proof. By a rational normalisation of the perimeter, we may suppose

$$x + y + \sqrt{x^2 + y^2} = 1.$$

Isolating the square root, squaring and reordering gives $x + y - xy = \frac{1}{2}$. Consequently, the legs are either rational or conjugate roots of a second-degree equation with rational coefficients, and hence so are the original side lengths. ∎

In order to obtain a similar conclusion for oblique triangles, we impose the following.

Proposition. *Let ABC be a triangle with rational perimeter, $\overline{AB} = c \in \mathbb{Q}$, $\overline{BC} = x$, $\overline{CA} = y$, with $x < y$. If ABC is equable, then $x = b - \sqrt{a}$, $y = b + \sqrt{a}$, where $a, b \in \mathbb{Q}$ and $a = \frac{c^2}{4} - 4 - \frac{8c}{2b-c}$.*



Proof. Let p denote the perimeter of ABC and so $x + y = p - c \in \mathbb{Q}$. Equability and Heron's formula as above give

$$16p^2 = p(p-2x)(p-2y)(p-2c) \iff \frac{16p}{p-2c} = p^2 - 2p(x+y) + 4xy$$
$$= p^2 - 2p(p-c) + 4xy. \quad (1)$$

Thus, both the sum and product of x and y are rational and hence are conjugate roots of a second-degree equation with rational coefficients. Let $x = b - \sqrt{a}$, $y = b + \sqrt{a}$, implying $p = 2b + c \in \mathbb{Q}$ and $xy = b^2 - a \in \mathbb{Q}$. Substituting in the right of (1) gives

$$16\frac{2b+c}{2b-c} = 2pc - p^2 + 4xy = 4(b^2 - a) - (p-c)^2 + c^2$$
$$= 4(b^2 - a) - (2b)^2 + c^2,$$

which can be written equivalently as

$$16 \cdot \frac{2b+c}{2b-c} = c^2 - 4a \quad \text{or} \quad a = \frac{c^2}{4} - 4 - \frac{8c}{2b-c}. \quad (2)$$

∎

1 Quasi-Heronian equable triangles

Heronian and quasi-Heronian equable triangles correspond to those for which a, b, c are strictly positive integers in the preceding proposition. Our aim is to study the three cases depending on whether the length of c is greater than, smaller than or in between the lengths of the two other sides. For the rest of this paper, ABC will always represent a Heronian or quasi-Heronian equable triangle with side lengths $b \pm \sqrt{a}, c$ and will be noted by $[a, b, c]$. For convenience, we indicate by (TI) the triangle inequalities

$$c < 2b \quad \text{and} \quad 4a < c^2. \tag{TI}$$

Lemma 2. *ABC verifies $2 \mid c$, and if $3 \mid a$, then $3 \mid b$ and $3 \mid c$.*

Proof. Multiplying the left of (2) by $2b - c$ and observing the product modulo 4 implies $2 \mid c$, if $3 \mid a$, then the same product is equivalent to $b - c \equiv (b + c)c^2 \pmod 3$, which gives a contradiction if $3 \nmid c$ and implies therefore $3 \mid b$. ∎

By letting $c = 2d$ for $d \in \mathbb{N}$, we transform the left of (2) and (TI) into

$$4 \cdot \frac{b+d}{b-d} = d^2 - a, \quad 0 < a < d^2 < b^2. \tag{3}$$

Theorem 1. *If ABC verifies $b + \sqrt{a} < c$, then $[a, b, c]$ is one of the following two forms:*

(I) $[(n-2)^2 - 8, n + 2, 2n]$ *for $n \geq 5$ or*

(II) $[(n-4)^2 - 20, n + 1, 2n]$ *for $n \geq 9$.*

Notice first before proving the above theorem that if $n = 5$ in (I), then $a = 1$, $b = 7$, $c = 10$ gives the equable Pythagorean triangle $(6, 8, 10)$, and if $n = 10$, then (II) gives the equable Heronian triangle with sides $(7, 15, 20)$.

Proof. We break the proof down into four steps.

- If $b - d \geq 4$, then $a \geq d^2 - d - b \geq d^2 - 2d - 4 > (d-2)^2$ for $d \geq 4$, which implies $c = 2d > b + \sqrt{a} > b + (d-2) = (b-d) + c - 2 > c + 2$, a contradiction. Concerning the cases $d = 1, 2$ or 3, we have
  - $d = 1$ contradicts the right of (3).
  - If $d = 2$, then $4\left(1 + \frac{4}{b-2}\right) = 4 - a$ gives $a < 0$ for each solution, which is absurd.
  - If $d = 3$, then $4\left(1 + \frac{6}{b-3}\right) = 9 - a$ implies again $a < 0$, except for $b = 9$, $a = 1$ and $c = 6$, which contradicts $b + \sqrt{a} < c$.
- If $b - d = 3$, then (3) implies $d = 3m$ for some $n \in \mathbb{N}$, which gives

$$a = 9m^2 - 8m - 4 \geq (3m-2)^2 \quad \text{for } m \geq 2.$$

  Hence $c = 6m > b + \sqrt{a} \geq 3m + 3 + (3m - 2) = 6m + 1$, a contradiction. In case $m = 1$, then $a = -3$, which is absurd.

Consequently, $b - d \leq 2$, and hence, by (TI), $b - d$ equals either 1 or 2.

- If $b - d = 2$, then (3) implies $a = d^2 - 4d - 4$, hence solution (I) for $d = n$.
- If $b - d = 1$, then (3) implies $a = d^2 - 8d - 4$, hence solution (II) for $d = n$. ∎

Remark 1. Observe that the above triangles of case (I) are right angled at C since

$$(b - \sqrt{a})^2 + (b + \sqrt{a})^2 = 2(b^2 + a) = 2((n + 2)^2 + (n - 2)^2 - 8)$$
$$= 4n^2 = c^2.$$

On the other hand, those of case (II) are obtuse at C since

$$2(b^2 + a) = 4n^2 - 12n - 6 < 4n^2 = c^2.$$

Theorem 2. *If ABC verifies $c < b - \sqrt{a}$, then $[a, b, c]$ is one of the following four forms:*

(I) $[9n^2 - 7, 11n, 6n]$ *for* $n \in \mathbb{N}$,

(II) $[n^2 - 8, 3n, 2n]$ *for* $n \geq 3$,

(III) $[n^2 - 6, 5n, 2n]$ *for* $n \geq 3$,

(IV) $[n^2 - 5, 9n, 2n]$ *for* $n \geq 3$.

For example, if $n = 3$ in case (IV), then $a = 4$, $b = 27$, $c = 6$ corresponds to the Heronian equable triangle with sides $(25, 29, 6)$.

Proof. Since $c < b - \sqrt{a} < b$, then $0 < \frac{c}{b} < 1$. Rewriting (2) gives

$$a = \frac{c^2}{4} - 4 - \frac{8c}{2b - c} = \frac{c^2}{4} - 4 - \frac{8\frac{c}{b}}{2 - \frac{c}{b}}.$$

Let $f(x) := \frac{8x}{2-x}$. Since f is increasing on $]0, 1]$, and $f(1) = 8$, the only solutions to test are $f(x) \in \{1, 2, \ldots, 8\}$:

- $\frac{8c}{2b-c} = 1$ implies $\frac{c}{b} = \frac{2n}{9n}$, $n \in \mathbb{N}$, and hence $a = n^2 - 5$, $b = 9n$, $c = 2n$ as in (IV).
- $\frac{8c}{2b-c} = 2$ implies $\frac{c}{b} = \frac{2n}{5n}$, and hence $a = n^2 - 6$, $b = 5n$, $c = 2n$ as in (III).
- $\frac{8c}{2b-c} = 3$ implies $\frac{c}{b} = \frac{6n}{11n}$, and so $a = 9n^2 - 7$, $b = 11n$, $c = 6n$ as in (I).
- $\frac{8c}{2b-c} = 4$ implies $\frac{c}{b} = \frac{2n}{3n}$, and so $a = n^2 - 8$, $b = 3n$, $c = 2n$ as in (II).
- $\frac{8c}{2b-c} = 5$ implies $\frac{c}{b} = \frac{10n}{13n}$, which gives $a = 25n^2 - 9$, $b = 13n$, $c = 10n$. If $n = 1$, then we get the Heronian equable triangle $(9, 17, 10)$ which however does not verify $c < b - \sqrt{a}$. More generally, the inequality $25n^2 - 9 < (13n - 10n)^2$ holds only for $n = 0$.

Being all quite similar, we treat summarily the cases for $f(x) = 6, 7$ and 8.

- $\frac{8c}{2b-c} = 6 \Rightarrow \frac{c}{b} = \frac{6n}{7n} \Rightarrow a = 9n^2 - 10$. However, $a < (b - c)^2 \Leftrightarrow 9n^2 - 10 < n^2$, which holds only for $n = 1$ and implies the contradiction $a = -1$.
- $\frac{8c}{2b-c} = 7 \Rightarrow \frac{c}{b} = \frac{14n}{15n} \Rightarrow a = 49n^2 - 11 \Rightarrow 49n^2 - 11 < n^2$, a contradiction.
- $\frac{8c}{2b-c} = 8 \Rightarrow b = c \Rightarrow a < (b - c)^2 = 0$, a contradiction. ∎

Remark 2. The angle at B is obtuse if $(b+\sqrt{a})^2 > (b-\sqrt{a})^2 + c^2$, i.e., $16b^2a > c^4$. Replacing all four parametrisations gives the respective quadratic inequalities

$$9n^2 - 7 > \Big(\frac{9n}{11}\Big)^2, \quad n^2 - 8 > \Big(\frac{n}{9}\Big)^2, \quad n^2 - 6 > \Big(\frac{n}{5}\Big)^2, \quad n^2 - 5 > \Big(\frac{n}{9}\Big)^2.$$

Easy calculations guarantee the above condition is verified, except for $n = 3$ in case (II), where we obtain once again, up to a permutation, the right triangle $(8, 10, 6)$.

If the two previous results depended merely on one parameter, unfortunately, the last case seems to rely heavily on two. Changing variables provides a more symmetrical layout: let $2b + c = 2n$ and $2b - c = 2m$ since c is even. Thus $c = n - m$ and $b = \frac{n+m}{2}$, and hence m, n have same parity and are both strictly positive by (3). Applying equation (2), we see $a = (\frac{n-m}{2})^2 - 4\frac{n}{m}$. Notice $n > m \geq 1$, and we obtain the following result.

Theorem 3. *The general solution of quasi-Heronian equable triangles may be written in the form*

$$\Big[\frac{(n-m)^2}{4} - 4\frac{n}{m}, \frac{n+m}{2}, n - m\Big].$$

Moreover, if the associated $[a, b, c]$ *verifies* $0 < b - \sqrt{a} < c < b + \sqrt{a} < b + c - \sqrt{a}$, *then we have the complement of the union of the two previous sets of Theorems 1 and 2, which we qualify of the third type.*

Remark 3. If $m = 1$ or 2, then substituting the expressions above in $c < b + \sqrt{a}$ gives second-degree inequalities with solutions $n < 1$, which is absurd. Hence $m \geq 3$, and if we let $n = 4m$, then the four inequalities above are equivalent to

$$\pm m + \sqrt{9m^2 - 64} > 0, \quad 4m \pm m - \sqrt{9m^2 - 64} > 0,$$

which one can readily verify are all true if $m \geq 3$. Hence, for every $m \geq 3$, there exists a sequence of quasi-Heronian equable triangles of the third type.

Theorem 4. *Acute equable Heronian or quasi-Heronian triangles do not exist.*

Proof. The only case to treat is when the condition $0 < b - \sqrt{a} < c < b + \sqrt{a}$ is verified. We shall prove that if the angle at B is acute, meaning $d^4 \geq ab^2$, then the angle at C is either obtuse or right. By (3), we have

$$\begin{aligned}
d^4 \geq b^2 a \iff\ & d^4 \geq b^2\Big(d^2 - 4\frac{b+d}{b-d}\Big) = \frac{b^3d^2 - d^3b^2 - 4b^3 - 4db^2}{b-d} \\
\iff\ & d^5 - d^4 b \leq d^3b^2 - b^3d^2 + 4b^3 + 4b^2d \\
\iff\ & d^2(d^3 + b^3) \leq d^3b^2 + d^4b + 4b^3 + 4b^2d \\
& \qquad = d^3b(d+b) + 4b^2(b+d) \\
\iff\ & d^2(d^2 - db + b^2) \leq d^3b + 4b^2 \\
\iff\ & d^2(d^2 - 2db + b^2) = d^2(b-d)^2 \leq 4b^2 = (2b)^2 \\
\iff\ & d(b-d) \leq 2b \iff 2 + \frac{4}{d-2} \geq b - d > 0.
\end{aligned}$$

Hence a finite number of cases are to be tested by applying polynomial division.

(a) If $d = 3$, then (3) gives $a = 5 - \frac{24}{b-3}$ and implies $\{[1, 9, 6], [2, 11, 6], [3, 15, 6], [4, 27, 6], [17, 1, 6], [29, 2, 6]\}$ as possible solutions. However, none of the preceding verify the above inequality condition.

(b) If $d = 4$, then by a similar method, we get the potential solutions $\{[4, 8, 8], [8, 12, 8], [10, 20, 8], [11, 36, 8], [28, 2, 8], [44, 3, 8]\}$, which once again do not verify the above inequality condition, except for the first one, which is merely the (6,10,8) triangle.

(c) Similarly, for $d = 5$, we obtain $\{[1, 7, 10], [11, 9, 10], [13, 10, 10], [16, 13, 10], [17, 15, 10], [19, 25, 10], [20, 45, 10], [31, 1, 10], [41, 3, 10], [61, 4, 10]\}$, which are either impossible or obtuse at B after verification.

(d) For $d = 6$, we get $\{[8, 8, 12], [16, 9, 12], [20, 10, 12], [24, 12, 12], [26, 14, 12], [28, 18, 12], [29, 22, 12], [30, 30, 12], [31, 54, 12], [44, 2, 12], [48, 3, 12], [56, 4, 12], [80, 5, 12]\}$, which we verify again are either impossible, right angled or obtuse.

Finally, the only two possibilities left are $b - d = 1$ or $b - d = 2$, which were already treated in Theorem 1, giving respectively an obtuse and a right triangle at C. ■

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