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Short note The circle as generator of Pythagorean triangles

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1 Introduction

A right-angled triangle whose side lengths are natural numbers is termed a *Pythagorean triangle*. The central idea in this paper is that circles generate Pythagorean triangles and they do so in a systematic way. Every circle of integral radius generates a set number of such triangles and every Pythagorean triangle is generated by some circle. Thus, circles provide a simple way of enumerating Pythagorean triangles.

2 Generating right-angled triangles

Starting with any circle, we construct a right-angled triangle whose incircle is the given circle as follows (Figure 1).

Let r be the length of the radius and let $d = 2r$ be the length of the diameter.

(i) Construct a tangent at any point T_1 of the circle.

(ii) From T_1, mark off a length r along the tangent giving point C.

(iii) With C as centre, draw an arc of radius length r to intersect the circle at T_2. Draw the ray CT_2 to give the second tangent from C. These two tangents are perpendicular to each other and form the right angle of the triangle.

(iv) From T_1 mark off a length, greater than r, along the tangent in a direction opposite to C, to give the point B. Let the length T_1B be $r + x$, $x > 0$, $x \in \mathbb{R}$.

(v) From B, construct a tangent to the circle that intersects the circle at T_3 and intersects the tangent CT_2 at A. Let the length $T_2A = r + y$, $y > 0$, $y \in \mathbb{R}$.

Then $\triangle ABC$ is the required triangle and the circle is its incircle. By varying x, an infinite number of triangles arise. From Figure 1, we see that if x is lengthened, y is shortened, and if x is shortened, y lengthens. We can read off the lengths of the three sides from Figure 1 as

$$a = CB = d + x, \tag{1}$$

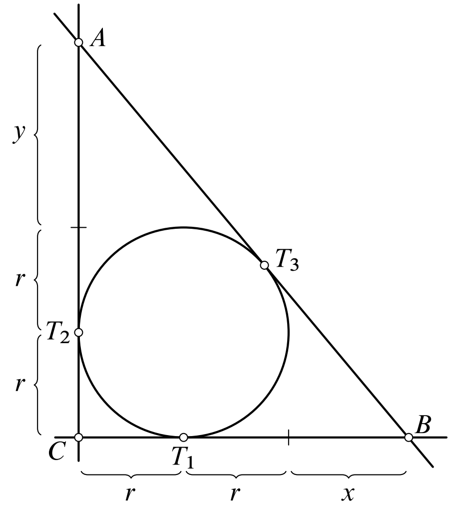

Figure 1. Constructing a right triangle from its incircle

$$b = CA = d + y, \tag{2}$$
$$c = BA = d + x + y \tag{3}$$

because

$$AT_3 = AT_2 = r + y \quad \text{and} \quad BT_3 = BT_1 = r + x.$$

By adding (1) and (2) and subtracting (3), we can obtain the diameter and hence the radius in terms of the three sides a, b and c as

$$a + b = 2d + x + y.$$

Therefore,

$$a + b - c = d. \tag{4}$$

Hence,

$$r = \frac{a + b - c}{2}. \tag{5}$$

By subtracting (2) from (3) and (1) from (3), we obtain

$$x = c - b, \tag{6}$$
$$y = c - a \tag{7}$$

in terms of the sides. It follows that $c = b + x = a + y$. We find the relationship between x and y by using the theorem of Pythagoras.

From $a^2 + b^2 = c^2$, we get $(d + x)^2 + (d + y)^2 = (d + x + y)^2$, whence $d^2 = 2xy$. Therefore,

$$xy = \frac{d^2}{2} = 2r^2. \tag{8}$$

We have now proved the following theorem.

Theorem 1. *For $d, x, y \in \mathbb{R}^+$, where d is the length of the diameter of a circle, the triangle with side lengths*

$$(d + x, d + y, d + x + y), \quad \text{with } xy = \frac{d^2}{2} = 2r^2,$$

is a right-angled triangle whose incircle has diameter length d.

This theorem can be easily verified by using Pythagoras' theorem and equation (4). Equation (8) represents a hyperbola whose branches are in the first and third quadrants and the line $y = x$ is an axis of symmetry. Since $x, y > 0$, we need only be concerned with the branch in the first quadrant. For a fixed value of r, each point (x, y) on that branch generates a right-angled triangle whose side lengths are $a = d + x$, $b = d + y$, $c = d + x + y$. Because the hyperbola is symmetric about the line $y = x$, we can restrict the point (x, y) to lie only on that portion of the hyperbola which lies in the upper half of the first quadrant, i.e., above the axis of symmetry. This is done by making $x < y$, which confines x to the interval $0 < x < \sqrt{2}r$.

3 Generating Pythagorean triangles

A Pythagorean triangle with side lengths (a, b, c) is such that a, b, c are natural numbers and $a^2 + b^2 = c^2$. A *Pythagorean triple* is an ordered triple (a, b, c) of natural numbers such that $a^2 + b^2 = c^2$. Two Pythagorean triples (a, b, c) and (b, a, c) identify the same Pythagorean triangle. For this reason, we make the arbitrary choice that a Pythagorean triangle is identified by the single triple (a, b, c) with $a < b$. This is consistent with restricting (x, y) to lie on the upper half of that portion of the hyperbola which lies in the first quadrant.

The results established in Section 2, that is, equations (1)–(8) and Theorem 1 carry over to here. Because a, b, c are now natural numbers, equations (5), (6) and (7) show that r, x and y are natural numbers too. Since all our variables are natural numbers, we shall make a slight modification; for r, we shall write n, for x, i and for y, j. Hence, equations (1), (2) and (3) become

$$\begin{aligned} a &= d + i = 2n + i, \\ b &= d + j = 2n + j, \\ c &= d + i + j = 2n + i + j. \end{aligned}$$

Equations (6) and (7) become

$$i = c - b, \tag{9}$$

$$j = c - a. \tag{10}$$

Because $a < b$, it follows that $i < j$. Equation (8) becomes

$$ij = \frac{d^2}{2} = 2n^2. \tag{11}$$

To ensure $i < j$, i must be in the range $0 < i < \sqrt{2}n$ (see [1]).

Theorem 1 now becomes the following theorem.

Theorem 2. *For $n, i, j \in \mathbb{N}$, $d = 2n$, where d is the length of the diameter of a circle, and $i < j$, the triangle with side lengths $(d + i, d + j, d + i + j)$, with $ij = \frac{d^2}{2} = 2n^2$, is a right-angled triangle whose incircle has diameter length d. Furthermore, every Pythagorean triangle (a, b, c) is of this form.*

Proof. The first two claims are easy to verify, so we prove the third one only.

Take any Pythagorean triangle (a, b, c). Equations (4), (9) and (10) enable us to calculate d, i and j; hence, we get

$$\begin{aligned} a &= (a + b - c) + (c - b) = d + i, \\ b &= (a + b - c) + (c - a) = d + j, \\ c &= (a + b - c) + (c - b) + (c - a) = d + i + j, \end{aligned}$$

which ends the proof of Theorem 2. ■

Example 1. We take a Pythagorean triangle that was known more than three thousand eight hundred years ago. It is one of the triangles listed on the Mesopotamian clay tablet, known nowadays by its catalogue number in the George A. Plimpton Collection at Columbia University, New York, as Plimpton 322. The tablet comes from the ancient city of Larsa [3] and in two of its columns lists the shorter leg and hypotenuse of a Pythagorean triangle. The triangle, from the fourth row of the tablet, is (12709, 13500, 18541) written in modern notation. From $a = 12709$, $b = 13500$, $c = 18541$, we obtain

$$d = a + b - c = 7668, \quad n = 3834, \quad i = c - b = 5041, \quad j = c - a = 5832.$$

Hence,

$$\begin{aligned} a &= d + i = 7668 + 5041, \\ b &= d + j = 7668 + 5832, \\ c &= d + i + j = 7668 + 5041 + 5832. \end{aligned}$$

4 Counting the Pythagorean triangles generated by a circle

From equation (11) ($ij = 2n^2$), it is clear that the number of triangles generated by the circle of radius length n is given by the number of divisor pairs (i, j) of $2n^2$. To count the number of divisor pairs of $2n^2$, we make use of the arithmetical function $d(m)$ which counts the number of divisors of m. When m is written as a product of its prime divisors as $m = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, where each p_i, $i = 1, \dots, k$, is prime, then

$$d(m) = (\alpha_1 + 1)(\alpha_2 + 1) \dots (\alpha_k + 1).$$

Theorem 3. *The circle with radius length $n = 2^m p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$, where the indices are non-negative integers and each p_i, $i = 1, 2, \dots, k$, is an odd prime number, generates*

$$(m + 1)(2a_1 + 1)(2a_2 + 1) \dots (2a_k + 1)$$

Pythagorean triangles.

Proof. From $n = 2^m p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$, we get $2n^2 = 2^{2m+1} p_1^{2a_1} p_2^{2a_2} \dots p_k^{2a_k}$. The number of divisors of $2n^2$ is

$$d(2n^2) = (2m+2)(2a_1+1)(2a_2+1)\dots(2a_k+1).$$

Therefore, the number of divisor pairs equals $(m+1)(2a_1+1)(2a_2+1)\dots(2a_k+1)$. Hence, the number of Pythagorean triangles is equal to

$$(m+1)(2a_1+1)(2a_2+1)\dots(2a_k+1). \qquad \blacksquare$$

Example 2. In Example 1, we found that the generating circle had a radius length $n = 3834$. From $n = 3834 = 54 \times 71 = 2 \times 3^3 \times 71$, we get $2n^2 = 2^3 \times 3^6 \times 71^2$. Therefore, we have $d(2n^2) = 4 \times 7 \times 3$. Consequently, the number of Pythagorean triangles equals $2 \times 7 \times 3 = 42$.

5 Counting primitive Pythagorean triangles generated by a circle

The Pythagorean triangle (a, b, c) is said to be *primitive* when a, b and c are relatively prime, i.e., there is no prime p that divides all three. For a, b, c to be relatively prime, it is sufficient that a and b are relatively prime. For it is evident from the formulae $c^2 = a^2 + b^2, a^2 = c^2 - b^2, b^2 = c^2 - a^2$ that if any pair taken from (a, b, c) shares a common divisor, $d > 1$, that the third member also does [2, p. 345].

Lemma 1. *Suppose that i, j and n are natural numbers such that $ij = 2n^2$ and $i < j$. Then $2n + i$ and $2n + j$ are relatively prime if and only if i and j are.*

Proof. The condition is necessary: if p is a prime that divides both i and j, then p divides $2n^2$ as well; hence, p divides $2n$ because all the prime divisors of $2n^2$ come from those of $2n$. Hence, p divides $2n + i$ and $2n + j$.

Conversely, if p is a prime that divides $2n + i$ and $2n + j$, then p^2 divides

$$(2n + i)^2 + (2n + j)^2 = (2n + i + j)^2.$$

Hence, p divides $2n + i + j$. Since p divides $2n + i$ and $2n + i + j$, p divides their difference j. Similarly, p divides i. $\blacksquare$

Theorem 4. *The number of primitive Pythagorean triangles generated by the circle of radius length n is 2^k, where k is the number of odd prime divisors of n.*

Proof. By Lemma 1, $2n + i$ and $2n + j$ are relatively prime whenever i and j are. Hence, we need to find the number of divisor pairs (i, j) of $2n^2$ for which i and j are relatively prime. Let $n = 2^m p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$ as before. Therefore, $2n^2 = 2^{2m+1} p_1^{2a_1} p_2^{2a_2} \dots p_k^{2a_k}$. Hence, $2n^2$ has $k + 1$ relatively prime divisors which are to be allocated to the divisor pair (i, j). Each of these relatively prime divisors is processed in two ways: allocated to i or to j. Hence, there are 2^{k+1} ways of allocating them. The number 2^{k+1} includes the duplicate pairs $i = q$, $j = r$ and $i = r$, $j = q$. Hence, there are 2^k distinct divisor pairs of relatively prime divisors. Consequently, there are 2^k primitive Pythagorean triangles generated. $\blacksquare$

Example 3. Using the same circle from Examples 1 and 2, the radius length is $n = 3834 = 2 \times 3^3 \times 71$. There are two odd prime divisors. Hence, there are $2^2 = 4$ primitive Pythagorean triangles generated by this circle. One of those is the triangle from Plimpton 322. It is generated by $i = 5041 = 71^2$ and $j = 5832 = 2^3 \times 3^6 = 2 \times 54^2$. Since i and j are relatively prime, so are the sides $(12709, 13500, 18541)$.

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