Elemente der Mathematik 79 (2024), 80–83
DOI 10.4171/EM/502

Short note

The generalized Binet formula for k-bonacci numbers

Harold R. Parks and Dean C. Wills

Abstract. Using Vandermonde determinants, we give a simple proof of the generalization of the Binet formula to the k-bonacci numbers.

For $k \geq 2$, define the k-bonacci numbers $F_n^{(k)}$ by the initial values and recursion

$$F_n^{(k)} = \begin{cases} 0, & n \leq k-2, \\ 1, & n = k-1, \\ \sum_{i=1}^{k} F_{n-i}^{(k)}, & n \geq k. \end{cases} \tag{1}$$

For any linear recurrence, it is always possible to express the nth term as a linear combination of the nth powers of the roots of the characteristic equation of the recurrence. For the particular case of the k-bonacci numbers, the formula is the following:

$$\begin{aligned} F_n^{(k)} &= \sum_{i=1}^{k} \frac{\phi_i^n}{(\phi_i - \phi_1)\cdots(\phi_i - \phi_{i-1})(\phi_i - \phi_{i+1})\cdots(\phi_i - \phi_k)} \\ &= \sum_{i=1}^{k} \frac{\phi_i^n}{\prod_{j \neq i}(\phi_i - \phi_j)}, \end{aligned} \tag{2}$$

where the ϕ_i are the roots of $x^k - x^{k-1} - \cdots - 1 = 0$, the characteristic equation of the k-bonacci recurrence.[1] Note that, when $k = 2$, the k-bonacci numbers are the usual Fibonacci numbers and (2) reduces to the well-known Binet formula for the Fibonacci numbers.

[1]The roots ϕ_i are all distinct because, if the characteristic polynomial had a multiple root, then that same multiple root would be a multiple root of $(x-1)(x^k - x^{k-1} - \cdots - 1) = x^{k+1} - 2x^k + 1$ and thus a root of $d(x^{k+1} - 2x^k + 1)/dx = (k+1)x^k - 2kx^{k-1}$. This last polynomial has only rational roots, and the Rational Root Theorem tells us that they are not roots of $x^k - x^{k-1} - \cdots - 1$.

Our goal in this paper is to give a clear and simple proof of (2) that relies solely on Vandermonde determinants. Indeed, we hope our proof is as clear and simple as is possible, considering the attempts by many other papers in this area and environs, e.g. [1–11].

1 Vandermonde determinants

The Vandermonde determinant is the $k \times k$ determinant appearing in the next equation. The important elementary fact is that the determinant has the value given on the right-hand side of the equation:

$$\mathcal{V}(\phi,k) = \begin{vmatrix} 1 & \phi_1 & \phi_1^2 & \cdots & \phi_1^{k-2} & \phi_1^{k-1} \\ 1 & \phi_2 & \phi_2^2 & \cdots & \phi_2^{k-2} & \phi_2^{k-1} \\ 1 & \phi_3 & \phi_3^2 & \cdots & \phi_3^{k-2} & \phi_3^{k-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \phi_k & \phi_k^2 & \cdots & \phi_k^{k-2} & \phi_k^{k-1} \end{vmatrix} = \prod_{1 \le m < n \le k} (\phi_n - \phi_m).$$

The preceding equation holds for all choices of the ϕ_i in any field, not only the roots of the characteristic equation.

Similarly, if one forms a minor by omitting the rightmost column and the ith row of the above Vandermonde determinant, one obtains another Vandermonde determinant for which the next equation holds:

$$\mathcal{V}_i(\phi,k) = \begin{vmatrix} 1 & \phi_1 & \phi_1^2 & \cdots & \phi_1^{k-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \phi_{i-1} & \phi_{i-1}^2 & \cdots & \phi_{i-1}^{k-2} \\ 1 & \phi_{i+1} & \phi_{i+1}^2 & \cdots & \phi_{i+1}^{k-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \phi_k & \phi_k^2 & \cdots & \phi_k^{k-2} \end{vmatrix} = \prod_{\substack{1 \le m < n \le k \\ m,n \ne i}} (\phi_n - \phi_m).$$

Again, the preceding equation holds for all choices of the ϕ_i in any field, not only the roots of the characteristic equation.

Lemma 1. *Let the ϕ_i be elements of any field and let f_i, for $1 \le i \le k$, be elements in the same field. If the ϕ_i are all distinct, then*

$$\sum_{i=1}^{k} \frac{f_i}{\prod_{j \ne i} (\phi_i - \phi_j)} = \frac{1}{\mathcal{V}(\phi,k)} \begin{vmatrix} 1 & \phi_1 & \phi_1^2 & \cdots & \phi_1^{k-2} & f_1 \\ 1 & \phi_2 & \phi_2^2 & \cdots & \phi_2^{k-2} & f_2 \\ 1 & \phi_3 & \phi_3^2 & \cdots & \phi_3^{k-2} & f_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \phi_k & \phi_k^2 & \cdots & \phi_k^{k-2} & f_k \end{vmatrix} \tag{3}$$

holds.

Proof. Expanding the determinant on the right-hand side of (3) along the last column, we see that

$$\begin{vmatrix} 1 & \phi_1 & \phi_1^2 & \cdots & \phi_1^{k-2} & f_1 \\ 1 & \phi_2 & \phi_2^2 & \cdots & \phi_2^{k-2} & f_2 \\ 1 & \phi_3 & \phi_3^2 & \cdots & \phi_3^{k-2} & f_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \phi_k & \phi_k^2 & \cdots & \phi_k^{k-2} & f_k \end{vmatrix} = \sum_{i=1}^{k} (-1)^{k-i} f_i \cdot \mathcal{V}_i(\phi, k).$$

For each i, observe that

$$\prod_{1 \le m < n \le k} (\phi_n - \phi_m) = (-1)^{k-i} \prod_{\substack{1 \le m < n \le k \\ m,n \ne i}} (\phi_n - \phi_m) \prod_{j \ne i} (\phi_i - \phi_j),$$

so

$$\sum_{i=1}^{k} \frac{f_i}{\prod_{j \ne i} (\phi_i - \phi_j)} = \sum_{i=1}^{k} \frac{(-1)^{k-i} f_i \cdot \mathcal{V}_i(\phi, k)}{\mathcal{V}(\phi, k)}$$

holds, and the result follows. ∎

Corollary 2. *If the ϕ_i are distinct elements of any field, then*

$$\sum_{i=1}^{k} \frac{\phi_i^n}{\prod_{j \ne i} (\phi_i - \phi_j)} = \begin{cases} 0, & n < k, \\ 1, & n = k-1, \end{cases} \tag{4}$$

holds.

Proof. For $n = 0, 1, \ldots, k-2$, Lemma 1 tells us that the left-hand side of (4) equals $\mathcal{V}(\phi, k)^{-1}$ times the value of a determinant with a repeated column; thus the right-hand side of (4) equals 0. When $n = k-1$, Lemma 1 tells us that the left-hand side of (4) equals $\mathcal{V}(\phi, k)^{-1}$ times $\mathcal{V}(\phi, k)$. ∎

2 Completion of the proof of equation (2)

To prove that the recurrence in (1) holds, we must now require that the ϕ_i be the solutions of the characteristic equation of that recurrence. That is, we require that $\phi_i^k = \phi_i^{k-1} + \cdots + 1$ holds for each $i = 1, 2, \ldots, k$.

Supposing that $n \ge k$, we argue inductively, as we may, since (1) and Corollary 2 show that (2) holds for $n = 0, 1, \ldots, k-1$. We have

$$\sum_{i=1}^{k} \frac{\phi_i^n}{\prod_{j \ne i} (\phi_i - \phi_j)} = \sum_{i=1}^{k} \frac{\phi_i^{n-k} \phi_i^k}{\prod_{j \ne i} (\phi_i - \phi_j)} = \sum_{i=1}^{k} \frac{\phi_i^{n-k} (\phi_i^{k-1} + \cdots + 1)}{\prod_{j \ne i} (\phi_i - \phi_j)}$$

$$= \sum_{m=1}^{k} \sum_{i=1}^{k} \frac{\phi_i^{n-m}}{\prod_{j \ne i} (\phi_i - \phi_j)} = \sum_{m=1}^{k} F_{n-m}^{(k)} = F_n^{(k)}.$$

References

[1] G. P. B. Dresden and Z. Du, A simplified Binet formula for k-generalized Fibonacci numbers. *J. Integer Seq.* **17** (2014), no. 4, Article 14.4.7 Zbl 1360.11031 MR 3181762

[2] D. E. Ferguson, An expression for generalized Fibonacci numbers. *Fibonacci Quart.* **4** (1966), 270–273 Zbl 0139.26703 MR 207622

[3] I. Flores, Direct calculation of k-generalized Fibonacci numbers. *Fibonacci Quart.* **5** (1967), 259–266 Zbl 0163.03703 MR 222016

[4] H. Gabai, Generalized Fibonacci k-sequences. *Fibonacci Quart.* **8** (1970), no. 1, 31–38 Zbl 0211.07301 MR 263734

[5] G. S. Hathiwala and D. V. Shah, Binet-type formula for the sequence of Tetranacci numbers by alternate methods. *Math. J. Interdiscip. Sci.* **6** (2017), 37–48

[6] D. Kalman, Generalized Fibonacci numbers by matrix methods. *Fibonacci Quart.* **20** (1982), no. 1, 73–76 Zbl 0472.10016 MR 660765

[7] D. Kessler and J. Schiff, A combinatoric proof and generalization of Ferguson's formula for k-generalized Fibonacci numbers. *Fibonacci Quart.* **42** (2004), no. 3, 266–273 Zbl 1076.11008 MR 2093882

[8] G.-Y. Lee, S.-G. Lee, J.-S. Kim, and H.-K. Shin, The Binet formula and representations of k-generalized Fibonacci numbers. *Fibonacci Quart.* **39** (2001), no. 2, 158–164 Zbl 0989.11008 MR 1829526

[9] C. Levesque, On mth order linear recurrences. *Fibonacci Quart.* **23** (1985), no. 4, 290–293 Zbl 0574.10016 MR 811309

[10] E. P. Miles, Jr., Generalized Fibonacci numbers and associated matrices. *Amer. Math. Monthly* **67** (1960), 745–752 Zbl 0103.27203, MR 123521

[11] W. R. Spickerman, Binet's formula for the Tribonacci sequence. *Fibonacci Quart.* **20** (1982), no. 2, 118–120 Zbl 0486.10011 MR 673292

Harold R. Parks
Department of Mathematics
Oregon State University
Corvallis, Oregon 97331 USA
hal.parks@oregonstate.edu

Dean C. Wills
AppDynamics
500 Terry A Francois Blvd
San Francisco, California 94158 USA
dean@lifetime.oregonstate.edu