Mathematisches Forschungsinstitut Oberwolfach

Report No. 40/2023

DOI: 10.4171/OWR/2023/40

Tensor-Triangular Geometry and Interactions

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17 September – 22 September 2023

Abstract. The workshop brought together experts in a rapidly growing field of tensor triangular geometry highlighting applications to and techniques coming from homotopy theory, algebraic geometry, modular representation theory, motivic homotopy theory and noncommutative algebra.

Mathematics Subject Classification (2020): 13D, 14C, 16D, 18F, 18G, 20C, 55N, 55P, 55U.

Introduction by the Organizers

Spectra are ubiquitous throughout modern mathematics: The Zariski spectrum of a commutative ring, the topological spectrum representing a generalized cohomology theory, and the Balmer spectrum of a tensor-triangulated category are important instances of the same underlying concept. In each case, the spectral representation of a more familiar object reveals its hidden geometry and symmetries. Amplified by modern homotopical and representation-theoretic techniques, recent years have seen a whirlwind of activity and groundbreaking progress in the development and application of spectral techniques, which may be loosely organized in the following themes.

(1) Global classification problems: Classification of thick tensor ideals and localizing tensor ideals as the key to capturing the global structure of categories; construction of novel support theories.

(2) Local-to-global principles: Assembly and disassembly in homotopy theory and modular representation theory; adelic techniques in rational equivariant homotopy theory; reconstruction theorems in (non-)commutative algebraic geometry.

(3) Invariants, duality, and descent: The computation of Picard groups and higher invariants like Brauer groups via ∞-categorical descent techniques; local and global dualities.

Structurally the workshop consisted of five integral components: hour-long research talks reporting on the most recent developments in tt-geometry; shorter talks on Thursday morning given by all the graduate student participants; a *motivic master class* on Monday evening; a *problem session* on Wednesday evening; and wide ranging and unrestricted discussions and collaborations during all the remaining unscheduled time.

Motivic master class. On Monday evening Sasha Vishik and Tom Bachmann gave a series of introductory lectures on the derived category of motives introducing the other participants to the foundations of Voevodsky's theory. The lectures were given in a very accessible and interactive way and were much appreciated by the non-geometers in the audience equipping them with the background necessary to appreciate the next day's talks on current motivic developments.

Problem session. On Wednesday night John Greenlees led an open problem session which turned out to be very lively due to a special auction based format. The potential contributors were mildly but firmly encouraged to bid for time to describe their problems. The key principle was that those asking for the least time got priority. This led to admirably succinct descriptions of the problems, and left time for some discussion after each problem was stated. In several cases the discussion led to a real prospect of progress. Keeping in character with the format of our problem session we provide a very cursory list of the topics discussed:

(1) *Paul Balmer* enquired whether there was a uniform bound for nilpotence degrees in tt-categories;
(2) *Maxime Ramzi and Drew Heard* posed questions on Tambara spectra which led to a very good discussion revealing that Markus Hausmann thought about the topic quite a bit;
(3) *Dave Benson* formulated questions around Greenlees's conjecture for generating the category of finitely generated modules over C^*BG, and the role of the nucleus in understanding singularity categories;
(4) *Greg Stevenson* asked a question about the relationship between dimension of Balmer spectrum of a tt-category K and the Rouquier dimension of K; followed by a finiteness question about "primes" for big tt-categories;
(5) *Jan Stovicek* suggested further explorations towards classification of compactly generated t-structures;
(6) *Sasha Vishik* posed a problem about "points" of the spectrum of $DM(C, F_p)$: Are there only two and what is the relationship? Subsequent discussion led to tangible progress;
(7) *Dan Nakano and Kent Vashaw* formulated questions about extending tt-geometry to the non-symmetric setting; in particular about the relationship of the (non-commutative) Balmer spectrum vs the homological spectrum;

(8) *Josh Pollitz* asked about compact objects in the homotopy category of injective dg-modules for a differential graded algebra; this was followed by constructive discussions with Jan Stovicek;
(9) *Ivo Dell'Ambrogio* discussed the Bootstrap category and its connection with KU_G-modules; there was some tangible progress through Morita theory.

Research presentations.

Towards spectral calculations in representation theory. Dave Benson reported on joint work with Iyengar, Krause and Pevtsova on locally dualisable modules in minimal cellularisations of the stable module category, giving numerous equivalent definitions, including some concrete interpretations in terms of representation theory. This led on to a classification of thick tensor ideals of dualisables in terms of specialization closed subsets of the completed localization. Dan Nakano gave an account of joint work with Matthew Hamil of a nilpotence theorem for Lie superalgebras, including a complete description of the Balmer spectrum in many cases. Eike Lau and Henning Krause (joint with Barthel, Benson, Iyengar and Pevtsova) described in two independent talks the Balmer spectrum and stratification for stable categories of finite group schemes defined over commutative Noetherian rings.

tt-geometry and stable homotopy theory. Natàlia Castellana gave an account of joint work with Barthel, Heard, Naumann and Pol giving analogues of a strong Quillen stratification for a rather general equivariant cohomology theory. Clover May described a number of results with Drew Heard on Balmer spectra of categories of modules over equivariant Eilenberg–MacLane spectra for constant Mackey functors. Scott Balchin reported on a joint work with Barnes and Barthel on the tt-geometry of the category of genuine equivariant G-spectra for G a profinite group.

Other spectral calculations. Drew Heard described joint work with Arone, Barthel and Sanders on the calculation of the Balmer spectrum of n-excisive functors from spectra to spectra in Goodwillie calculus. He presented a complete answer including topology and exhibited a close parallel with what happens for G-spectra for a finite group G.

Structural questions in triangulated categories. Kent Vashaw described general constructions in "non-commutative" tt-geometry, including the functorial map from the Balmer spectrum to the categorical center. Jan Stovicek gave an account of work generalizing the classification of compactly generated t-structures due to Alonso, Jeremias and Saorin, going beyond the Noetherian case and into certain non-commutative contexts. Janina Letz described joint work with Marc Stephan giving significantly improved bounds on the generation time by using suitable exact bifunctors. Ivo Dell'Ambrogio described how to apply the methods of tt-geometry in C^*-algebras by considering cellular algebras and applying countable versions of various constructions, which he showed to be particularly effective for finite groups. Josh Pollitz described joint work with Ballard, Iyengar, Lank and Mukhophadyay understanding singularities in prime characteristic through the lens of the Frobenius endomorphism. Their results show that a Frobenius pushforward of a module of full support gives a strong generator and leads to illuminating proofs of many Kunz-like regularity theorems. In Beren Sanders's talk the question of the tt-geometry properties of the "monogenisation" of a tt-category was posed and some theorems and examples, coming from motivic and equivariant settings, were stated. The choice of terminology proved controversial with the audience and a lively discussion ensued.

tt-geometry and the category of motives. Sasha Vishik reported on the progress towards Balmer spectrum for the Voevodsky and Morel–Voevodsky category of motives. Vishik's talk was intimately connected with the talk by Martin Gallauer on the tt-geometry for the category of permutation modules for a profinite group which was motivated by the tt-questions in the category of Artin motives. Tom Bachmann gave an account of progress on the C_2-equivariant motivic stable homotopy theory, highlighting analogues of the geometric fixed points and also a complete t-structure with an explicitly describable heart.

Graduate student research presentations. Charalampos Verasdanis discussed the "local-to-global" approach to costratification of tt-vategories, and applications to hypersurface singularities.

Changhan Zou explored tt-geometry in non-noetherian settings, introducing the notion of small support.

Maxime Ramzi talked about work with Naumann and Pol giving a complete classification of separable algebras in G-spectra for G a finite p-group.

Anish Chedalavada presented a generalization of the classical reconstruction theorem for schemes from the Balmer spectrum of the perfect category of sheaves to the setting of symmetric monoidal stable infinity categories, building on joint work with Aoki, Barthel, Schlank and Stevenson.

Acknowledgement: We are grateful to the Mathematisches Forschungsinstitut Oberwolfach for providing such an excellent environment for our workshop. The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-2230648, "US Junior Oberwolfach Fellows".

Workshop: Tensor-Triangular Geometry and Interactions

Table of Contents

Abstracts

C_2-equivariant motivic stable homotopy theory

Tom Bachmann

Fix a finite group G and a field k in which $|G|$ is invertible. The definition of the G-equivariant motivic stable category was contemplated by various authors, until it was put into definite form in the work of Hoyois [4]. The definition can be made to look very familiar to the non-equivariant motivic practitioner. Write Sm_k^G for the category of smooth k-schemes with a G-action. We start with the category

$$PSh(Sm_k^G)$$

of presheaves of spaces on Sm_k^G. We then perform a Bousfield localization in order to arrive at *motivic spaces*

$$Spc^G(k) = L_{\mathbb{A}^1,Nis} PSh(Sm_k^G).$$

The pointed version $Spc^G(k)_*$ acquires a symmetric monoidal structure called the *smash product*, and the G-equivariant motivic stable category is obtained as

$$SH^G(k) = Spc^G(k)_*[((\mathbb{P}^1)^{\wedge G})^{-1}].$$

The first result that I presented is a motivic variant of the presentation of classical equivariant stable homotopy theory using *spectral Mackey functors*. To explain this, denote by $Span(Sm_k^G, fet, all)$ the $(2,1)$-category with the same objects as Sm_k^G, but morphisms from X to Y given by *groupoids* of spans of the form

$$X \xleftarrow{p} Z \xrightarrow{f} Y,$$

where p is a finite étale morphism (and f is an arbitrary morphism). The composition of morphisms is via pullback. The category

$$PSh(Span(Sm_k^G, fet, all))$$

is thus the category of presheaves of spaces on Sm_k^G together with "wrong-way" or "transfer" maps along finite étale morphisms. Out of this we can build the category of motivic spaces with finite étale transfers

$$Spc^{fet,G}(k) = L_{\mathbb{A}^1,Nis} PSh(Span(Sm_k^G, fet, all)).$$

Theorem 1. *There is a canonical equivalence of categories*

$$SH^G(k) \simeq Spc^{fet,G}(k)[((\mathbb{P}^1)^{\wedge G})^{-1}].$$

This is the main result of [1]. It is essentially a categorified form of the motivic tom Dieck splitting Theorem of Gepner–Heller [3].

One curious fact that we can read off from this theorem is that the category $SH^G(k)$ is compactly generated by objects of the form $\Sigma^\infty_+ X \wedge \mathbb{G}_m^{\wedge n}$ for $X \in Sm_k^G$ and $n \in \mathbb{Z}$. Here $\mathbb{G}_m$ denotes the *trivial* representation (and the trivial action is the curious aspect).

Definition 2. The homotopy t-structure on $SH^G(k)$ is the one with non-negative part generated under colimits and extensions by the above objects.

It holds essentially by construction that the homotopy t-structure is left complete, and one may also show that it is right complete. Moreover, there are certain "(generalized) fixed point functors" which are t-exact for the homotopy t-structure. For simplicity, we explain what these are in the case $G = C_2$. The fixed point functors come in three forms: there is the usual C_2-fixed points functor

$$(-)^{C_2} : SH^{C_2}(k) \to SH(k),$$

there is the "forgetful" functor

$$(-)^e : SH^{C_2}(k) \to SH(k),$$

and for every field l/k with a non-trivial C_2-action, there is

$$(-)^l : SH^{C_2}(k) \to SH(l^{C_2}).$$

We also have the following result which might be of particular interest for this workshop.

Theorem 3. *The generalized fixed point functors form a conservative family.*

The final result that I sketched was that in the case $G = C_2$, the heart of the homotopy t-structure on $SH^{C_2}(k)$ can be described explicitly. This is done as follows. A sheaf of abelian groups

$$F \in Shv_{Nis}(Sm_k^{C_2})$$

is called *strictly homotopy invariant* if for every $X \in Sm_k^{C_2}$ the canonical map

$$H^*_{Nis}(X, F) \to H^*_{Nis}(\mathbb{A}^1 \times X, F)$$

is an isomorphism. Given such a sheaf F, one may prove that the presheaf $F_{-1}(X) = F(X \times \mathbb{G}_m)/F(X)$ is again a sheaf. Now by a *homotopy module (over k, with C_2-action)* we mean a sequence of sheaves with finite étale transfers

$$F_i \in Shv_{Nis}(Span(Sm_k^G, fet, all))$$

together with isomorphisms $(F_{i+1})_{-1} \simeq F_i$, such that each F_i is strictly homotopy invariant.

Theorem 4. *Let k be infinite. The heart $SH^{C_2}(k)^\heartsuit$ of the homotopy t-structure is equivalent to the category of homotopy modules over k with a C_2-action.*

The result is obtained by mimicking the non-equivariant proof. The key input is a C_2-equivariant generalization of Gabber's presentation lemma from [2]. This is a rather technical geometric statement, which is used to deduce a crucial connectivity estimate:

Lemma 5 (C_2-equivariant stable connectivity)**.** *Let $E : (Sm_k^G)^{op} \to Sp$ be a spectral presheaf, which is Nisnevich locally connected. Then $L_{\mathbb{A}^1, Nis}E$ is also connected.*

Open question. It seems fairly clear that the results can be generalized from the case of $G = C_2$ to other groups. This seems particularly likely if all irreducible representations of G are one dimensional. *But what if G affords irreducible representations of dimension > 1. Does stable connectivity still hold?*

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The tensor-triangular geometry of G-spectra for profinite G

Scott Balchin

(joint work with David Barnes, Tobias Barthel)

We study the tensor-triangular geometry of the category of genuine equivariant G-spectra where G is a profinite group. Taking a leaf from the structure theorem of profinite groups — which says that any profinite group G is the cofiltered limit of finite quotient groups G_i [7] — we define the category of G-spectra for a profinite group as the filtered colimit of the G_i-spectra. This *continuous* construction of the category Sp_G allows us to lift structural results form the finite group case to the profinite case. For example, a result of Gallauer [4] allows us to compute the Balmer spectrum as the limit of the Balmer spectra of the Sp_{G_i}. All in all we prove a generalization (and strengthening) of the nilpotence theorem [1, 8] and the thick subcategory theorem [1, 2] for finite groups:

Theorem 1. *Let G be a profinite group. Then:*

- *The nilpotence theorem holds for Sp_G unconditionally. That is, the geometric functors $(K_p(n)_*\Phi^H_G)_{H,p,n}$ jointly detect $\otimes$-nilpotence of morphisms in Sp_G with compact source.*
- *If G is moreover abelian, then there is a full classification of thick ideals of Sp^ω_G.*

We then focus our attention onto the rationalized category $\mathsf{Sp}_{G,\mathbb{Q}}$, with a particular focus on the question of stratification, that is, when the localizing ideals are in bijection with the subsets of the Balmer spectrum [3]. We prove the following characterization of stratification using work of Gartside–Smith [5, 6]:

Theorem 2. *Let G be a profinite group. The category of rational G-spectra $\mathsf{Sp}_{G,\mathbb{Q}}$ is a tensor-triangulated category with Balmer spectrum*

$$\mathsf{Spc}(\mathsf{Sp}^\omega_{G,\mathbb{Q}}) \cong \mathrm{Sub}(G)/G,$$

and the following conditions are equivalent:

(1) $\mathsf{Sp}_{G,\mathbb{Q}}$ *is stratified;*
(2) $\mathrm{Sub}(G)/G$ *is countable;*
(3) $A_{\mathbb{Q}}(G)$*, the rational Burnside ring of* G*, is semi-Artinian.*

Moreover, if G *is abelian, then these conditions hold if and only if* G *is topologically isomorphic to* $A \times \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_r}$ *for pairwise distinct primes* $p_1, \ldots, p_r$ *and* A *a finite abelian group.*

The conditions appearing in this characterization are rather delicate as it relies on the cardinality of conjugacy classes of subgroups. For example, one is able to show that while $\mathrm{SL}_2(\mathbb{Z}_p)$ contains uncountably many subgroups, it only contains countably many up to conjugacy, and hence $\mathsf{Sp}_{\mathrm{SL}_2(\mathbb{Z}_p),\mathbb{Q}}$is stratified.

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Locally dualisable modules for finite groups

Dave Benson

(joint work with Srikanth Iyengar, Henning Krause, Julia Pevtsova)

Let G be a finite group and k a field of characteristic p. The cohomology ring $H^*(G,k) = \mathsf{Ext}^*_{kG}(k,k)$ is a graded commutative Noetherian k-algebra. The stable module category $\mathsf{StMod}(kG)$ has as objects the kG-modules, and the arrows $\underline{\mathsf{Hom}}_{kG}(M,N)$ are given by the quotient of the kG-module homomorphisms by those that factor through a projective module. Then $\mathsf{StMod}(kG)$ is a tensor triangulated category that comes with a map from cohomology to the graded centre

$$H^*(G,k) \to Z^*\mathsf{StMod}(kG)$$

If $\mathfrak{p}$ is a (homogeneous) prime ideal in $H^*(G,k)$ then there are functors

$$\Gamma_{\mathfrak{p}},\, \Lambda^{\mathfrak{p}} \colon \mathsf{StMod}(kG) \to \mathsf{StMod}(kG)$$

that are idempotent, and pick out the minimal localising tensor ideals, respectively colocalising tensor coideals of $\mathsf{StMod}(kG)$. These induce bijections

$$\{\text{localising tensor ideals}\} \longleftrightarrow \mathsf{Proj}\, H^*(G,k) \longleftrightarrow \{\text{colocalising tensor coideals}\}$$

and we have $\Gamma_{\mathfrak{p}}(M) \cong \Gamma_{\mathfrak{p}}k \otimes M$, $\Lambda^{\mathfrak{p}}(M) \cong \mathsf{Hom}_k(\Gamma_{\mathfrak{p}}k, M)$. The functors $\Gamma_{\mathfrak{p}}$, $\Lambda^{\mathfrak{p}}$ induce equivalences $\Gamma_{\mathfrak{p}}\mathsf{StMod}(kG) \longleftrightarrow \Lambda^{\mathfrak{p}}\mathsf{StMod}(kG)$. The cohomology is given by $\hat{H}^*(G,\Gamma_{\mathfrak{p}}k) \cong H^*(G,k)_{\mathfrak{p}}$, the localisation of the cohomology ring at $\mathfrak{p}$, and we have $\widehat{\mathrm{Ext}}^*_{kG}(\Gamma_{\mathfrak{p}}k,\Gamma_{\mathfrak{p}}k) \cong H^*(G,k)^{\wedge}_{\mathfrak{p}}$, the completion of the localisation at $\mathfrak{p}$. For details, see [1, 2].

In general, let $\mathcal{T}$ be a tensor triangulated category with tensor identity $\mathbb{1}$, and assume that $\mathcal{T}$ has Brown representability: contravariant exact functors $\mathcal{T} \to \mathcal{A}\mathrm{b}$ are representable. So there exists Hom objects:

$$[X \otimes Y, Z] \cong [X, \mathcal{H}om(Y,Z)]$$

where $[X,Y]$ denotes the morphisms in $\mathcal{T}$ from X to Y. We define the *Spanier–Whitehead dual* of X to be $D^{\mathrm{sw}}(X) = \mathcal{H}om(X,\mathbb{1})$. Then there is a natural map $D^{\mathrm{sw}}(X) \otimes Y \to \mathcal{H}om(X,Y)$, and we say that an object X is *dualisable* if this is an isomorphism for all Y. We say that X is *compact* if the natural map $\bigoplus [X, Y_\alpha] \to [X, \bigoplus Y_\alpha]$ is an isomorphism, and *functionally compact* if the natural map $\bigoplus \mathcal{H}om(X, Y_\alpha) \to \mathcal{H}om(X, \bigoplus Y_\alpha)$ is an isomorphism. Then

(1) dualisable $\Rightarrow$ functionally compact,
(2) X dualisable $\Rightarrow$ $D^{\mathrm{sw}}X$ dualisable, and $X \to D^{\mathrm{sw}}D^{\mathrm{sw}}X$ is iso,
(3) if $\mathcal{T}$ is generated by a set of dualisables then functionally compact $\Rightarrow$ dualisable,
(4) if $\mathbb{1}$ is compact then functionally compact $\Rightarrow$ compact.

In $\Gamma_{\mathfrak{p}}\mathsf{StMod}(kG)$, we have $\mathcal{H}om(M,N) = \Gamma_{\mathfrak{p}}\mathsf{Hom}_k(M,N)$, and so $D^{\mathrm{sw}}M = \Gamma_{\mathfrak{p}}\mathsf{Hom}_k(M,\Gamma_{\mathfrak{p}}k)$. But beware that the tensor identity $\Gamma_{\mathfrak{p}}k$ in this category is usually not compact, so compact and dualisable are different conditions. Compact objects are dualisable, but there are usually more dualisables than compacts.

Recall from [5] that a π-point α of kG consists of a field extension $K \supseteq k$ and a *flat* homomorphism of algebras $\alpha\colon K[t]/(t^p) \to KG$ that factors through KE for some elementary abelian p-subgroup E of G. We say that $\alpha\colon K[t]/(t^p) \to KG$ and $\beta\colon L[t]/(t^p) \to LG$ are *equivalent* (written $\alpha \sim \beta$) if for all finitely generated kG-modules M, $\alpha^*(M_K)$ is projective if and only if $\beta^*(M_L)$ is projective.

Given a π-point $\alpha\colon K[t]/(t^p) \to KG$, we look at the composite

$$H^*(G,k) \to H^*(G,K) \xrightarrow{\alpha^*} H^*(K[t]/(t^p),K).$$

Since $H^*(K[t]/(t^p),K)$ is of the form $K[x]$ ($p=2$, $|x|=1$) or $K[x]\otimes\Lambda(y)$ (p odd, $|x|=2$, $|y|=1$), the inverse image in $H^*(G,k)$ of the radical is a prime ideal $\mathfrak{p}_\alpha$ in $\mathsf{Proj}\, H^*(G,k)$. In [5], it is proved that $\alpha \sim \beta$ if and only if $\mathfrak{p}_\alpha = \mathfrak{p}_\beta$, and that this gives a bijection between the set of equivalence classes of π-points and the set $\mathsf{Proj}\, H^*(G,k)$. Furthermore, given $\mathfrak{p}$, there exists a finite extension K of $k(\mathfrak{p})$ and a π-point $\alpha\colon K[t]/(t^p) \to KG$ such that $\mathfrak{p} = \mathfrak{p}_\alpha$. We say that such a π-point is *good*. So every equivalence class contains good π-points.

Now let $\mathfrak{p}$ be a point in $\mathsf{Proj}\, H^*(G,k)$. Let $\alpha\colon K[t]/(t^p) \to KG$ be a good π-point with $\mathfrak{p}_\alpha = \mathfrak{p}$. We define a kG-module

$$\Delta_\alpha = (K_{K[t]/(t^p)}{\uparrow}^{KG}){\downarrow}_{kG}.$$

Let C be a compact generator for $\mathsf{StMod}(kG)$. For example, we can take for C the permutation module on the cosets of a Sylow p-subgroup, or the direct sum of the simple kG-modules. Then we have the following characterisations of compacts and dualisables.

Theorem 1. *For a module M in $\Gamma_\mathfrak{p}\mathsf{StMod}(kG)$, the following are equivalent:*

(1) M is compact,
(2) $\hat{H}^*(G, M\otimes C)$ has finite length over $H^*(G,k)_\mathfrak{p}$,
(3) M is in the thick subcategory generated by Δ_α.

Theorem 2. *For a module M in $\Gamma_\mathfrak{p}\mathsf{StMod}(kG)$, the following are equivalent:*

(1) M is dualisable,
(2) M is functionally compact,
(3) M is in the thick subcategory generated by $\Gamma_\mathfrak{p}C$,
(4) $\alpha^*(M_K)$ is a direct sum of a finite dimensional and a projective $K[t]/(t^p)$-module,
(5) $\widehat{\mathrm{Ext}}^*_{kG}(M,\Delta_\alpha)$ has finite length over $H^*(G,k)_\mathfrak{p}$,
(6) $\widehat{\mathrm{Ext}}^*_{kG}(\Delta_\alpha, M)$ has finite length over $H^*(G,k)_\mathfrak{p}$,
(7) $\hat{H}^*(G, M\otimes C)$ is Artinian over $H^*(G,k)_\mathfrak{p}$,
(8) $\widehat{\mathrm{Ext}}^*_{kG}(M\otimes C, \Gamma_\mathfrak{p}k)$ is Noetherian over $\widehat{\mathrm{Ext}}^*_{kG}(\Gamma_\mathfrak{p}k,\Gamma_\mathfrak{p}k) \cong H^*(G,k)^\wedge_\mathfrak{p}$,
(9) $\mathsf{Hom}_k(M,\Gamma_\mathfrak{p}k)$ is in the thick subcategory of $\Lambda^\mathfrak{p}\mathsf{StMod}(kG)$ generated by $\Lambda^\mathfrak{p}C$.

The Balmer spectrum of dualisable objects in $\Gamma_\mathfrak{p}\mathsf{StMod}(kG)$ is $\mathsf{Proj}\, H^*(G,k)^\wedge_\mathfrak{p}$. The reader should beware here that completion can change the spectrum. For example, if k has odd characteristic and we take the ring $R = k[x,y]$, with $\mathfrak{p} = (x,y)$, then the polynomial $x^2 - y^2(1-y)$ is prime in $R_\mathfrak{p}$, but factorises in $R^\wedge_\mathfrak{p}$ as $(x + y\sqrt{1-y})(x - y\sqrt{1-y})$. This can be used to give a similar example in the stable module category of $(\mathbb{Z}/p)^4$ by first homogenising, then adding another variable so that $\mathfrak{p}$ is not maximal, and then interpreting in the polynomial part of the cohomology ring.

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Quillen stratification in equivariant homotopy theory

Natàlia Castellana Vila
(joint work with Tobias Barthel, Drew Heard, Niko Naumann, Luca Pol)

In this project (see [5]) we prove a generalization of strong Quillen's stratification theorem in the context of equivariant homotopy theory, formulated in the language of tensor-triangular geometry.

Quillen [9] published a celebrated theorem known as strong Quillen stratification theorem. The strong stratification theorem provides a decomposition of the Zariski spectrum of the cohomology of any finite group G with coefficients in a field k

$$\operatorname{Spec} H^\bullet(G,k) = \bigsqcup_{(E)\subseteq G} \mathcal{V}^+_{G,E}.$$

in terms of locally closed subsets indexed on the conjugacy classes of elementary abelian subgroups and the strata $\mathcal{V}^+_{G,E}$ are orbits of the Weyl group action on an open subset of the Zariski spectrum $\operatorname{Spec} H^\bullet(E,k)$ of the cohomology of E, which is well-known.

The weak version of this result in Quillen's paper describes the spectrum as a colimit of $\operatorname{Spec} H^\bullet(E,k)$ on the orbit category on elementary abelian subgroups. Generalizations of this result had been obtained previously. On the one hand, Mathew–Naumann–Noel [8] proved a generalization for coefficients in an arbitrary commutative equivariant ring spectrum; on the other hand, this statement had also found a tt-geometric version for the spectrum of $\mathrm{D}^b(\mathbb{F}_pG)$ in Balmer's [1].

The goal is to conceptualize these results in a uniform way getting together equivariant tensor-triangular geometry, Quillen's stratification of group cohomology, and stratifications in modular representation theory. In particular, we establish a Quillen-type decomposition of the Balmer spectrum of equivariant tensor-triangulated category and study the extent to which it is reflected in a stratification of the category defined over it.

Suppose that $\mathcal{T}$ is a rigidly-compactly generated tt-category whose Balmer spectrum of compact objects is Noetherian. One would like to understand when the Balmer spectrum of compact objects also parameterizes the localizing $\otimes$-ideals. Balmer–Favi [4] and Stevenson [10] have extended the notion of Balmer support from compact objects,$\mathcal{T}^c$, to all of $\mathcal{T}$.

$$\{\text{Localizing } \otimes\text{-ideals of } \mathcal{T}\} \xrightarrow{\text{Supp}} \{\text{Subsets of } \operatorname{Spc}(\mathcal{T}^c)\} \tag{1}$$

If the map Supp from (1) is a bijection, then we say that $\mathcal{T}$ *is stratified.* So, the first question is to decide when this happens. Techniques to approach this question are developed in [6]. Another question then is to identify $\operatorname{Spc}(\mathcal{T}^c)$ as a set or as a topological space. If R is the graded endomorphism ring of the unit, Balmer [3] defines a natural continuous comparison map $\rho\colon \operatorname{Spc}(\mathcal{K}) \to \operatorname{Spec}^h(R)$ which combined with $\operatorname{Spec}^h(R) \to \operatorname{Spec}(R_0)$ (that sends a homogeneous prime ideal $\mathfrak{p}$ to $\mathfrak{p}\cap R_0$), gives rise to an *ungraded comparison map*

$$\rho_0\colon\ \operatorname{Spc}(\mathcal{K}) \to \operatorname{Spec}(R_0).$$

We specialize to the following context. Let G be a finite group. We let Sp_G denote the stable ∞-category of G-spectra. Given a commutative equivariant ring spectrum R, let $\mathrm{Mod}_G(R)$ denote the ∞-category of R-modules internal to Sp_G, and write $\mathrm{Perf}_G(R)$ for its full subcategory of compact (or perfect) modules. Given a subgroup $H \subseteq G$, the geometric fixed point functor is denoted by Φ^H.

In the following main result we establish an analogue of Quillen stratification for an arbitrary commutative equivariant ring spectrum R and we show that the category is stratified in terms of geometric fixed points.

Theorem 1. *Let R be a commutative equivariant ring spectrum and write $\mathrm{Mod}_G(R)$ for the category of G-equivariant modules over R. Then:*

(1) The spectrum of perfect R-modules admits a locally-closed decomposition

$$\mathrm{Spc}(\mathrm{Perf}_G(R)) \simeq \bigsqcup_{(H)\subseteq G} \mathrm{Spc}(\mathrm{Perf}(\Phi^H R))/W_G(H),$$

with the set-theoretic disjoint union being indexed on conjugacy classes of subgroups of G;

(2) $\mathrm{Mod}_G(R)$ is stratified if the categories $\mathrm{Mod}(\Phi^H R)$ are stratified with Noetherian spectrum for all subgroups H in G.

In both (a) and (b) it suffices to index on a family $\mathcal{F}$ of subgroups $H \subseteq G$ such that R is $\mathcal{F}$-nilpotent.

Specializing to the Borel-equivariant theory for $H\mathbb{F}_p$, the Eilenberg-MacLane spectrum for $\mathbb{F}_p$, one recovers a version of Quillen's theorem (restricted to homogeneous prime ideals). We apply our methods to the relevant case of the Borel-equivariant Lubin-Tate E-theory.

Theorem 2. *Let $\underline{E} = \underline{E_n}$ be a G-Borel-equivariant Lubin–Tate E-theory of height n and at the prime p. The category $\mathrm{Mod}_G(\underline{E})$ is cohomologically stratified, and there is a decomposition into locally-closed subsets.*

$$\mathrm{Spc}(\mathrm{Perf}_G(\underline{E})) \cong \mathrm{Spec}(E^0(BG)) \simeq \bigsqcup_{A} \mathrm{Spec}(\pi_0 \Phi^A \underline{E})/W_G^Q(A),$$

where the disjoint union is indexed on abelian p-subgroups A of G generated by at most n elements. In particular, the generalized telescope conjecture holds for $\mathrm{Mod}_G(\underline{E})$ and there are explicit bijections

$$\left\{\begin{matrix}\textit{Thick } \otimes\textit{-ideals of} \\ \mathrm{Perf}_G(\underline{E})\end{matrix}\right\} \xrightarrow{\sim} \left\{\begin{matrix}\textit{Specialization closed} \\ \textit{subsets of } \mathrm{Spec}(E^0(BG))\end{matrix}\right\}$$

and

$$\left\{\begin{matrix}\textit{Localizing } \otimes\textit{-ideals of} \\ \mathrm{Mod}_G(\underline{E})\end{matrix}\right\} \xrightarrow{\sim} \left\{\begin{matrix}\textit{Subsets of} \\ \mathrm{Spec}(E^0(BG))\end{matrix}\right\}.$$

A key input for the proof of the previous theorem is the following result which allows to understand stratification for modules over the geometric points following [7].

Theorem 3. *The commutative ring $\pi_0\Phi^A\underline{E}$ is regular Noetherian for any finite abelian p-group A.*

Other examples obtained by applying the main result are the following:

(1) The integral constant Green functor $R = H\underline{\mathbb{Z}}$ for any cyclic p-group G.
(2) Equivariant K-theory $R = KU_G$ for any finite group G. In this case, $\mathrm{Spc}(\mathrm{Perf}_G(KU_G)) \cong \mathrm{Spec}(\pi_0 KU_G)$, where $\pi_0 KU_G \cong R(G)$ is the complex representation ring of G.

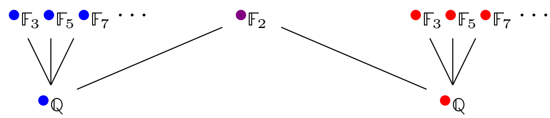

FIGURE 1. Here $\mathrm{Spc}(\mathrm{Perf}_{C_2}(KU_{C_2})) \cong \mathrm{Spec}(R(C_2))$. The closure goes upwards, and the primes are labeled by their residue fields.

(3) Atiyah's K-theory with reality $R = K\mathbb{R}$ for $G = C_2$. In this case, $\mathrm{Spc}(\mathrm{Perf}_{C_2}(K\mathbb{R})) \cong \mathrm{Spec}(\mathbb{Z})$.

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A derived refinement of a classical theorem in tt-geometry

Anish Chedalavada

We let $2\mathrm{CAlg} := \mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{perf}})_{\mathrm{rig}}$ the ∞-category of *2-rings* be the underlying $(\infty, 1)$-category of *symmetric monoidal, stable, idempotent complete* ∞-categories with a biexact tensor product that are *rigid*, meaning every object is dualizable (here the symmetric monoidal structure on $\mathrm{Cat}_\infty^{\mathrm{perf}}$ is that of [BGT13, 3.1]). In particular, for any object $\mathcal{K} \in 2\mathrm{CAlg}$, the homotopy category $\mathrm{ho}(\mathcal{K})$ is canonically tensor-triangulated and rigid [Bar+23, 5.12]. To any small tensor-triangulated category $\mathcal{K}_0$, one may equip the basic open sets of its Balmer spectrum $\mathrm{Spc}(\mathcal{K}_0)$ with a "structure presheaf" of triangulated categories [Bal02, §5], specializing to the following assignment:

$$U(a) \mapsto \mathcal{K}_0(U(a)) := (\mathcal{K}_0/a)^\natural$$

where $U(a) \subseteq \mathrm{Spc}(\mathcal{K}_0)$ is the basic open set corresponding to the primes which contain $a \in \mathcal{K}_0$, $\mathcal{K}_0/a$ denotes the Verdier quotient of $\mathcal{K}_0$ by the thick tensor-ideal generated by a, and $(-)^\natural$ denotes idempotent completion.

Our first result demonstrates that for any $\mathcal{K} \in 2\mathrm{CAlg}$, the "structure presheaf" on $\mathrm{Spc}(\mathrm{ho}\,\mathcal{K})$ upgrades to a full structure sheaf valued in 2CAlg, with an appropriate locality condition. We recall the following definition.

Definition 1. [Bal10, 4.1] A tensor-triangulated category $\mathcal{K}_0$ is called *local* if the thick tensor ideal $\{0\} \subseteq \mathcal{K}_0$ is prime.

We now have the following:

Theorem 2. *For $\mathcal{K} \in 2\mathrm{CAlg}$, there is a natural sheaf $\mathcal{O}_\mathcal{K} \in \mathrm{Shv}_{2\mathrm{CAlg}}(\mathrm{Spc}(\mathrm{ho}\,\mathcal{K}))$ such that for any $a \in \mathcal{K}$, $\mathrm{ho}(\mathcal{O}_\mathcal{K}(U(a))) = (\mathrm{ho}\,\mathcal{K}/a)^\natural$. Furthermore, for every $x \in \mathrm{Spc}(\mathrm{ho}\,\mathcal{K})$, the homotopy category of its stalk $\mathcal{O}_{\mathcal{K},x}$ is a local tt-category.*

Remark 3. Any $x \in \mathrm{Spc}(\mathrm{ho}\,\mathcal{K})$ corresponds to a prime tt-ideal $\mathcal{P} \subseteq \mathcal{K}$, and the homotopy category of the stalk $\mathcal{O}_{\mathcal{K},x}$ is exactly $(\mathrm{ho}\,\mathcal{K}/\mathcal{P})^\natural$.

This motivates the following definition, which we write informally for the purpose of exposition.

Definition 4. The ∞-category $\mathrm{Top}^{\mathrm{loc}}_{2\mathrm{CAlg}}$ of *locally 2-ringed spaces* is the $(\infty, 1)$-category whose objects are pairs $(X, \mathcal{O}_X)$ where $X \in \mathrm{Top}$ and $\mathcal{O}_X \in \mathrm{Shv}_{2\mathrm{CAlg}}(X)$ has *local* homotopy categories of stalks, and morphisms $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ are given by pairs

$$[f_\# : X \to Y] \in \mathrm{Top}^{\Delta^1},\ [f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X] \in \mathrm{Shv}_{2\mathrm{CAlg}}(Y)^{\Delta^1}$$

where $f^\#$ induces *conservative* functors on homotopy categories of stalks.

We will henceforth utilise the notation $\mathrm{Spc}(\mathcal{K})$ to refer to the locally 2-ringed space $(\mathrm{Spc}(\mathrm{ho}\,\mathcal{K}), \mathcal{O}_\mathcal{K})$. Our next result provides both functoriality and a universal property for $\mathrm{Spc}(\mathcal{K})$ among all locally 2-ringed spaces.

Theorem 5. *The assignment* $\mathcal{K} \mapsto \mathrm{Spc}(\mathcal{K})$ *promotes to a fully faithful functor* $\mathrm{Spc}(-) : \mathrm{2CAlg}^{op} \to \mathrm{Top}^{\mathrm{loc}}_{\mathrm{2CAlg}}$. *Furthermore, for any* $\mathcal{X} \in \mathrm{Top}^{\mathrm{loc}}_{\mathrm{2CAlg}}$, *one has the following equivalence*

$$\mathrm{Map}_{\mathrm{Top}^{\mathrm{loc}}_{\mathrm{2CAlg}}}(\mathcal{X}, \mathrm{Spc}(\mathcal{K})) \simeq \mathrm{Map}_{\mathrm{2CAlg}}(\mathcal{K}, \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}))$$

via the map that takes a morphism $f : \mathcal{X} \to \mathrm{Spc}(\mathcal{K})$ *to the induced map on global sections of structure sheaves.*

We note that the results above have been obtained independently by joint work of Ko Aoki, Tobias Barthel, Tomer Schlank, and Greg Stevenson.

We go on to apply the machinery above in proving a derived-geometric extension of a classical result of Balmer-Thomason on the reconstruction of coherent schemes from their categories of perfect complexes (stated in full generality as [KP17, 4.2.5]). To formulate the same, we need the following definition.

Definition 6. We write $\mathrm{Spc}^{\mathrm{LRS}}(\mathcal{K}) \in \mathrm{Top}^{\mathrm{loc}}_{\mathrm{CAlg}}$ to denote the locally spectrally ringed space given by $(\mathrm{Spc}(\mathcal{K}), \mathbf{End}_{\mathbb{1}}(\mathcal{O}_{\mathcal{K}}))$, where $\mathbf{End}_{\mathbb{1}} : \mathrm{2CAlg} \to \mathrm{CAlg}$ denotes the functor sending a 2-ring to the *endomorphism ring spectrum* of its unit object.

Remark 7. The fact that the spectrally ringed space above is *locally* spectrally ringed is an observation originally made in [Bal10, 6.6]. Furthermore, the proposition preceding this observation implies that one has a functor $\mathbf{LRS} : \mathrm{Top}^{\mathrm{loc}}_{\mathrm{2CAlg}} \to \mathrm{Top}^{\mathrm{loc}}_{\mathrm{CAlg}}$ by sending $(X, \mathcal{O}_X) \mapsto (X, \mathbf{End}_{\mathbb{1}}(\mathcal{O}_{\mathcal{X}}))$.

Our main result is the following version of reconstruction for a certain class of spectral schemes.

Theorem 8 (Reconstruction). *Let* $\mathcal{X} \in \mathrm{SpSch}^{\mathrm{nc}}$ *be a nonconnective spectral scheme whose underlying classical scheme* $\mathcal{X}^{\heartsuit}$ *is coherent and has affine diagonal. Then there is a canonical map of locally spectrally ringed spaces* $\gamma_{\mathcal{X}} : \mathrm{Spc}^{\mathrm{LRS}}(\mathrm{Perf}_{\mathcal{X}}) \to \mathcal{X}$, *satisfying the following:*

(1) *Any open immersion of an affine spectral subscheme* $\iota : \mathrm{Spec}(R) \hookrightarrow \mathcal{X}$, *induces an open inclusion* $U := \mathrm{Spc}^{\mathrm{LRS}}(\mathrm{Perf}_R) \hookrightarrow \mathrm{Spc}^{\mathrm{LRS}}(\mathrm{Perf}_{\mathcal{X}})$, *and the restriction* $\gamma_{\mathcal{X}}|_U : \mathrm{Spc}^{\mathrm{LRS}}(\mathrm{Perf}_R) \to \mathcal{X}$ *is given by a composition* $\mathrm{Spc}^{\mathrm{LRS}}(\mathrm{Perf}_R) \xrightarrow{\rho_R} \mathrm{Spec}(R) \xrightarrow{\iota} \mathcal{X}$.

   The map ρ_R *is the affinization map associated to the locally spectrally ringed space* $\mathrm{Spc}^{\mathrm{LRS}}(\mathrm{Perf}_R)$, *and in particular on underlying classically ringed spaces it recovers the comparison map* $\mathrm{Spc}(\mathrm{ho}\,\mathrm{Perf}_R) \to \mathrm{Spec}(\pi_0 R)$ *of* [Bal10].

(2) *One has a natural equivalence*

$$\mathrm{Map}_{\mathrm{2CAlg}}(\mathrm{Perf}_{\mathcal{X}}, \mathcal{K}) \simeq \mathrm{Map}_{\mathrm{Top}^{\mathrm{loc}}_{\mathrm{CAlg}}}(\mathrm{Spc}^{\mathrm{LRS}}(\mathcal{K}), \mathcal{X})$$

   for any $\mathcal{K} \in \mathrm{2CAlg}$, *where the map is induced by the functor composite* $\mathbf{LRS} \circ \mathrm{Spc}(-)$ *following by pushing forward along* $\gamma_{\mathcal{X}}$.

We remark that part (1) of the above theorem can be interpreted as saying that Balmer spectra of spectral schemes are governed by a "geometric direction" corresponding to the underlying classical scheme of $\mathcal{X}$, and a "homotopy theoretic" direction governed by the comparison maps on an affine chart.

Remark 9. We indicate a few examples of (1) below.

(1) From (1) one can immediately deduce that for any *connective* spectral scheme satisfying the conditions of the theorem, the comparison map γ of the theorem is surjective using the results of [Bal10, §7].
(2) The full reconstruction for classical schemes (with affine diagonal) as stated in [KP17, 4.2.5] is a direct consequence of the reconstruction theorem above, combined with [Bal10, 8.1].
(3) Given any locally even periodic spectral scheme whose underlying classical scheme is regular noetherian and satisfies the conditions of the theorem, γ is an equivalence. This promotes (and extends) the scheme case of [Mat15, 1.7] to a statement on ringed Balmer spectra, essentially by reformulating [Mat15, 1.4] in terms of Balmer's comparison map ρ_R. We intend to discuss the reconstruction theorem for spectral DM-stacks in future work.

We end by indicating an application of part (2) of the theorem above.

Corollary 10. *Given any locally monogenic 2-ring $\mathcal{K}$ such that $\mathrm{Spc}^{\mathrm{LRS}}(\mathcal{K})$ is itself a coherent nonconnective spectral scheme whose underlying classical scheme has affine diagonal, one has an equivalence of 2-rings*

$$\mathcal{K} \simeq \mathrm{Perf}_{\mathrm{Spc}^{\mathrm{LRS}}(\mathcal{K})}$$

In particular, this yields an equivalence of tensor-triangulated categories upon passage to homotopy categories.

A key example of categories satisfying the above include the principal blocks of any ∞-categorical enhancement of the stable module categories of a finite flat group scheme G over a field k of characteristic $p > 0$, by results of [FP07]. Equivalences of this form enable the computation of invariants in these categories using descent-theoretic techniques based on the associated spectral scheme: these have been utilised in chromatic homotopy theory to great effect, for example in the computation of the Picard group of TMF via an étale descent spectral sequence as in [MS16], or in the classification of certain Azumaya algebras for TMF as in [BMS22]. We end our discussion with two questions:

Question 11. Can the Picard and Brauer groups of the principal blocks for stable module categories be completely computed by a descent spectral sequence based on their associated spectral schemes?

Question 12. Do the spectral schemes appearing in the equivalences above admit natural spectral moduli-theoretic interpretations? I am presently able to provide an affirmative answer only for elementary abelian groups.

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Stratification in Kasparov theory

Ivo Dell'Ambrogio

(joint work with Rubén Martos)

Kasparov's KK-theory is an important tool for operator algebraists and noncommutative geometers. It features prominently e.g. in the proof of the Baum-Connes conjecture for large classes of groups [Kas88]. For every (say) countable discrete group G, it defines an essentially small tt-category KK^G and therefore a spectrum $\mathrm{Spc}(\mathsf{KK}^G)$. Long ago I proved [Del10] that if the natural map

$$\sqcup_{H\leq G \text{ finite}}\mathrm{Spc}(\mathsf{KK}^H) \to \mathrm{Spc}(\mathsf{KK}^G)$$

is surjective, then a strong form of the Baum-Connes conjecture holds for G. This criterion, alas, appears to be useless, since the computation of such triangular spectra has prove elusive, indeed KK^G is a rather badly behaved tt-category to which the usual techniques do not apply.

Recently, a little hope has been restored by replacing KK^G with the more reasonable subcategory $\mathsf{Cell}^G \subset \mathrm{KK}^G$ of *cellular G-algebras*, and the Balmer spectrum with a countable version, Spc_ω:

Theorem 1 (D.–Martos 2022)**.** If the map $\sqcup_{H\leq G \text{ finite}}\mathrm{Spc}_\omega(\mathsf{Cell}^H) \to \mathrm{Spc}_\omega(\mathsf{Cell}^G)$ is surjective, G satisfies a strong form of the Baum-Connes conjecture.

The difference is that, this time, we know how to calculate $\mathrm{Spc}_\omega(\mathsf{Cell}^G)$ at least in the case of some small finite groups (and possibly also *rationally* for all finite groups; work in progress!). The way this works for G finite is that we observe that if Cell^G is stratified by the Balmer-Favi support (in a countable version of the usual meaning, as these categories only have countable coproducts), then $\mathrm{Spc}_\omega(\mathsf{Cell}^G)$ is just $\mathrm{Spc}(\mathsf{Cell}^G_c)$, the usual spectrum of the tt-subcategory of compact-rigid objects.

The latter space is known for $G = 1$ and $G = \mathbb{Z}/p\mathbb{Z}$ with p prime [DM21]. Even better, in these cases we can apply the proposition because:

Theorem 2 (D.–Martos 2023)**.** *Stratification holds for* Cell^1 *and* $\mathsf{Cell}^{\mathbb{Z}/p\mathbb{Z}}$.

The case of Cell^1 (= the Rosenberg-Schochet bootstrap category) is easy, but for $\mathsf{Cell}^{\mathbb{Z}/p\mathbb{Z}}$ we first use ∞-categorical enhancements to construct (for any finite G) a rigidly-compactly generated tt-category Cell^G_{big} which is nicely functorial in G and which contains Cell^G in a way which preserves the existing countable coproducts. We then apply the stratification theory of Barthel-Heard-Sanders [BHS21] to Cell^G_{big}, and manage to prove stratification when $G = \mathbb{Z}/n\mathbb{Z}$. Finally, tricks from Neeman's theory of well-generated categories let us deduce 'countable' stratification for the subcategory Cell^G when (usual) stratification holds for Cell^G_{big}.

The above ideas may well apply to more general finite groups. For G infinite, however, all the usual techniques again break down, essentially because the natural generating objects of Cell^G are neither dualizable nor compact…New ideas are welcome!

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Going pro in tt-geometry

Martin Gallauer

(joint work with Paul Balmer)

For any field $\mathbb{F}$ there is a canonical equivalence of tt-categories between Artin motives over $\mathbb{F}$ (in the sense of Voevodsky) and the derived category of permutation modules over the pro-finite absolute Galois group G of $\mathbb{F}$:

$$\mathrm{DAM}(\mathbb{F}) \simeq \mathrm{DPerm}(G)$$

The latter can also be understood as the derived category of cohomological Mackey functors, or as modules over the constant Green functor, see [1]. Motivated by these equivalences, in recent work [2, 3, 4] we have been studying in detail the tt-geometry of $\mathrm{DPerm}(G; k)$ for arbitrary pro-finite groups G with coefficients in a field k.

I will start by recalling our results in the case of a finite group. After that the focus will be on the passage to pro-finite groups, leading us to the following general task. Frequently, a tt-category T can be expressed as a filtered colimit of tt-categories T_i that are more accessible. Ideally, one would like to reduce questions about the tt-geometry of T to that of the T_i's. The formula $\mathrm{Spc}(T) = \varprojlim \mathrm{Spc}(T_i)$, while useful for theoretical purposes, doesn't entirely solve this problem. Instead, I will discuss a *density theorem* that, in good cases, identifies a subspace of $\mathrm{Spc}(T)$ that is accessible and yet determines the entire space. Finally, this result will be applied to the derived category of permutation modules (and therefore to Artin motives).

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The geometry of functor calculus

Drew Heard

(joint work with Greg Arone, Tobias Barthel, Beren Sanders)

1. Introduction

To any (essentially small) tensor-triangulated category $\mathcal{C}$, Balmer has associated a topological space $\mathrm{Spc}(\mathcal{C})$, an analog of the Zariski spectrum of a commutative ring [2]. The points of this space are the prime thick-$\otimes$ ideals of $\mathcal{C}$, and the Thomason subsets classify the thick $\otimes$-ideals of $\mathcal{C}$. In the case where $\mathcal{C} = \mathrm{D}(R)^c$, the compact objects in the unbounded derived category of a commutative ring, then $\mathrm{Spc}(\mathrm{D}(R)^c) \cong \mathrm{Spec}(R)$, the usual Zariski spectrum of R.

The purpose of this talk was to explain a new computation in tensor-triangulated geometry, namely a determination of the spectrum of Goodwillie's category of d-excisive functors from finite spectra to spectra.

2. The tt-category of d-excisive functors

We recall the following definitions. For more details, see Goodwillie's original paper [6].

Definition 1. Let $\underline{d} = \{1, \ldots, d\}$. A d-cube in finite spectra is a functor $\mathcal{P}(\underline{d}) \to \mathrm{Sp}^c$, where $\mathcal{P}(\underline{d})$ denotes the poset of subsets of $\underline{d}$. A d-cube $\mathcal{X}$ is said to be

(1) cartesian, if the canonical map

$$\mathcal{X}(\varnothing) \to \lim_{\varnothing \neq S \subseteq \underline{d}} \mathcal{X}(S)$$

is an equivalence, and

(2) cocartesian, if the canonical map

$$\operatorname{colim}_{S \subsetneq \underline{d}} \mathcal{X}(S) \to \mathcal{X}(\underline{d})$$

is an equivalence.

Finally, a d-cube $\mathcal{X}$ is *strongly cocartesian* if it is left Kan extended from subsets of cardinality at most 1 (equivalently, any 2-face in the cube is a pushout).

Definition 2. A functor $\mathrm{Sp}^c \to \mathrm{Sp}$ from finite spectra to spectra is said to be d-excisive if it takes strongly cocartesian $(d+1)$-cubes to cartesian $(d+1)$-cubes. We let

$$\mathrm{Exc}_d(\mathrm{Sp}^c, \mathrm{Sp}) \subseteq \mathrm{Fun}(\mathrm{Sp}^c, \mathrm{Sp})$$

denote the full subcategory of functors which are d-excisive and reduced, i.e., $F(*) \simeq *$.

For example, a reduced functor $F\colon \mathrm{Sp}^c \to \mathrm{Sp}$ is 1-excisive if and only if F carries pushout squares to pullback squares (i.e., F is exact). In fact, this sets up an equivalence of categories $\mathrm{Exc}_1(\mathrm{Sp}^c, \mathrm{Sp}) \simeq \mathrm{Sp}$, given by evaluation at the sphere spectrum.

In order to apply methods from tensor-triangulated geometry to the category of d-excisive functors, we prove the following result.

Theorem 3. *The category* $\mathrm{Exc}_d(\mathrm{Sp}^c, \mathrm{Sp})$ *is a rigidly-compactly generated tensor-triangulated category, that is, it is compactly generated and the compact and dualizable objects coincide.*

Here the symmetric monoidal structure on $\mathrm{Exc}_d(\mathrm{Sp}^c, \mathrm{Sp})$ is given as the localization of the Day convolution monoidal structure on $\mathrm{Fun}(\mathrm{Sp}^c, \mathrm{Sp})$.

3. The spectrum of d-excisive functors

The strategy for constructing prime ideals in $\mathrm{Exc}_d(\mathrm{Sp}^c, \mathrm{Sp})^c$ follows the methods of Balmer and Sanders in equivariant homotopy theory, namely pulling back primes from the stable homotopy category [4]. In equivariant homotopy the functor used is geometric fixed points, while in the current work we use the Goodwillie derivatives ∂_i which are symmetric monoidal functors

$$\partial_i\colon \mathrm{Exc}_d(\mathrm{Sp}^c, \mathrm{Sp}) \to \mathrm{Sp}$$

for $1 \le i \le d$. We recall from [3, 7] that the spectrum of compact objects in the stable homotopy category Sp^c has points $\mathcal{C}_{p,h}$ for p a prime number or 0, and $1 \le h \le \infty$. Inclusions are essentially determined p-locally: we have

$$\mathcal{C}_{p,\infty} \subsetneq \ldots \subsetneq \mathcal{C}_{p,h} \subsetneq \mathcal{C}_{p,h-1} \subsetneq \mathcal{C}_{p,2} \subsetneq \mathcal{C}_{0,1}.$$

Here $\mathcal{C}_{p,1} := \mathcal{C}_{0,1}$ is the subcategory of torsion finite spectra, independently of p. For a complete description, see [3, Section 9]. This allows us to make the following definition.

Definition 4. Let $\mathcal{P}_d(\underline{i}, p, h)$ be the image of $\mathcal{C}_{p,h}$ under the map

$$\mathrm{Spc}(\partial_i)\colon\ \mathrm{Spc}(\mathrm{Sp}^c) \to \mathrm{Spc}(\mathrm{Exc}_d(\mathrm{Sp}^c, \mathrm{Sp})^c).$$

These give all the primes in $\mathrm{Spc}(\mathrm{Exc}_d(\mathrm{Sp}^c, \mathrm{Sp})^c)$ by the following theorem, which describes the spectrum completely as a set.

Theorem 5. *Every prime in* $\mathrm{Spc}(\mathrm{Exc}_d(\mathrm{Sp}^c, \mathrm{Sp})^c)$ *is of the form* $\mathcal{P}_d(\underline{i}, p, h)$ *for some triple* (i, p, h) *consisting of an integer* $1 \le i \le d$, *p a prime number or* $p = 0$, *and a chromatic height* $1 \le h \le \infty$. *Moreover, we have* $P_d(\underline{i}, p, h) = P_d(\underline{j}, q, l)$ *if and only if* $\underline{i} = \underline{j}$ *and* $\mathcal{C}_{p,h} = \mathcal{C}_{q,l}$ *in* Sp^c *(i.e.,* $h = l$ *and if* $h = l > 1$, *then also* $p = q$*).*

To determine the topology on $\mathrm{Spc}(\mathrm{Exc}_d(\mathrm{Sp}^c, \mathrm{Sp})^c)$ we show that it suffices to describe all the inclusions among prime ideals. Using the comparison map to the Zariski spectrum of the endomorphism ring of $\mathrm{Exc}_n(\mathrm{Sp}^c, \mathrm{Sp})^c$ (which we also compute), we are able to determine the following basic restrictions:

Proposition 6. *Suppose that* $\mathcal{P}_d(\underline{k}, p, h) \subseteq \mathcal{P}_d(\underline{l}, p, m)$ *for* $1 \le k, l \le d$, *a prime p and chromatic integers* $1 \le h, m \le \infty$. *Then the following hold:*

(1) $p - 1 \mid k - l \ge 0$;
(2) if $h = 1$ *then* $m = 1$ *and* $k = l$, *so the two primes are equal.*

To determine the topology, we translate the problem into determining the blue-shift for a certain 'Tate construction' on the category of d-excisive functors. We are able to solve this by using work of Arone and Ching [1] to translate this to a problem in equivariant homotopy for a product of symmetric groups. Using techniques from equivariant homotopy, we show that we can then reduce to studying the case of cyclic groups, allowing us to apply the main theorems of [5] and [8]. The result can be stated as follows, where $s_p(k)$ denotes the weight of the p-adic expansion of k, i.e., the sum of the coefficients of the p-adic expansion of k.

Theorem 7. *Let p, q be prime numbers,* $1 \le k, l \le d$ *integers, and suppose* $1 \le h, h' \le \infty$. *There is an inclusion* $\mathcal{P}_d(\underline{k}, p, h') \subseteq \mathcal{P}_d(\underline{l}, q, h)$ *if and only if the following conditions hold:*

(1) $p - 1 \mid k - l \ge 0$;
(2) $h' \ge h + \delta_p(k, l)$; *and*
(3) if $h > 1$, *then* $p = q$.

Here

$$\delta_p(k,l) = \begin{cases} 0 & \text{if } k = l; \\ 1 & \text{if } p-1 \mid k-l > 0 \text{ and } l \ge s_p(k); \\ 2 & \text{if } p-1 \mid k-l > 0 \text{ and } l < s_p(k); \\ \infty & \text{otherwise.} \end{cases}$$

We note the perhaps surprising result that the maximum shift is 2.

As applications, we give a classification of all the thick $\otimes$-ideals of $\mathrm{Exc}_d(\mathrm{Sp}^c, \mathrm{Sp})$ and give a d-excisive version of Kuhn and Lloyd's chromatic Floyd and chromatic Smith theorems [9].

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Lattices over finite group schemes and stratification

Henning Krause

My talk was devoted to explaining the following recent result from joint work with Tobias Barthel, Dave Benson, Srikanth Iyengar, and Julia Pevtsova [1].

Theorem 1. *For G a finite flat group scheme over a commutative noetherian ring R, the tensor triangulated category $\mathrm{Rep}(G, R)$ is stratified and costratified by the canonical action of the cohomology ring $H^*(G, R)$.*

The theorem unifies results from commutative algebra and representation theory of finite groups, because one may specialise either by taking for G the trival group or by taking for R a field and for G a discrete group. In the first case we obtain a theorem of Neeman [4], while the second case recovers the main result from [2]. A consequence of stratification is a classification of all tensor ideal localising subcategories of $\mathrm{Rep}(G, R)$ via subsets of the homogeneous prime ideal spectrum of $H^*(G, R)$, while the costratification implies a classification of all Hom closed colocalising subcategories of $\mathrm{Rep}(G, R)$. An essential ingredient is a recent theorem of van der Kallen which states that the cohomology ring $H^*(G, R)$ is noetherian [5]. The category $\mathrm{Rep}(G, R)$ arises as a suitable ind-completion of the bounded derived category $\mathrm{rep}(G, R)$ of lattices, that is, representations of G that are finitely generated and projective over R. In particular we are able to identify the Balmer spectrum of the tensor-triangulated category $\mathrm{rep}(G, R)$ with the Zariski spectrum of $H^*(G, R)$. This is in line with recent work of Lau which discusses the case of a discrete group [3].

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The Balmer spectrum of the category of perfect complexes on a finite flat groupoid of affine schemes

Eike Lau

Let k be a noetherian ring. All rings, schemes, and stacks are assumed to be of finite type over k. For a rigid tt category $\mathcal{T}$ one can ask if the comparison map

$$\rho : \mathrm{Spc}(\mathcal{T}) \to \mathrm{Spec}^h(R)$$

from the prime ideal spectrum of $\mathcal{T}$ to the homogeneous prime ideal spectrum of the graded ring $R = \mathrm{End}^*_{\mathcal{T}}(1)$ is a homeomorphism.

Here we consider the case $\mathcal{T} = \mathrm{Perf}(X)$, the category of perfect complexes on an algebraic stack X. When X is a scheme, $\mathrm{Spc}(\mathcal{T})$ is homeomorphic to X by a result of Thomason [8]. It follows that ρ is a homeomorphism when X is an affine scheme. Our main result is that this continues to hold if X is an algebraic stack which is sufficiently close to an affine scheme.

Theorem 1. *Let X be an algebraic stack such that there is a surjective finite flat morphism $Y \to X$ where Y is an affine scheme. Then the comparison map ρ for $\mathcal{T} = \mathrm{Perf}(X)$ is a homeomorphism.*

Example 1. Let k be a field and $X = BG = [\mathrm{Spec}(k)/G]$ where G is a finite group or a finite group scheme over k. Then $\mathcal{T} \cong D^b(kG)$ and $R = H^*(G, k)$, and the theorem is a reformulation of classical results of [3, 5].

Example 2. The case $X = [Y/G]$ where G is a finite group that acts on an affine scheme $Y = \mathrm{Spec}(A)$ is treated in [7]. Here $\mathcal{T} \cong D(AG)_{A\text{-perf}}$, the derived category of A-perfect complexes of AG-modules, and $R = H^*(G, A)$.

Example 3. The case $X = BG = [\mathrm{Spec}(k)/G]$ where G is a finite flat group scheme over the noetherian ring k is proved in [2] as a consequence of a stratification of a rigidly-compactly generated tt category with category of compact objects $\mathcal{T}$.

Example 4. The previous examples are subsumed by the case $X = [Y/G]$ where G is a finite flat group scheme over k that acts on an affine k-scheme Y.

Remark 2. The proof [7] for the case $X = [Y/G]$ as in Example 2 carries over to the general case of the theorem with some modifications; this will appear soon.

Remark 3. The category $\mathcal{T} = \mathrm{Perf}(X)$ can be made explicit as follows. The presentation $Y \to X$ gives rise to a finite flat groupoid $\mathrm{Spec}(B) \rightrightarrows \mathrm{Spec}(A)$ with $\mathrm{Spec}(A) = Y$ and $\mathrm{Spec}(B) = Y \times_X Y$, which in turn corresponds to a finite flat Hopf algebroid $A \rightrightarrows B$. Then $\mathcal{T}$ is equivalent to the category of A-perfect complexes in the derived category of comodules for this Hopf algebroid.

Finite generation of cohomology. The following technical condition seems to be crucial. A rigid tt category $\mathcal{T}$ is called noetherian if $\mathrm{End}^*_{\mathcal{T}}(M)$ is a noetherian module over $R = \mathrm{End}^*_{\mathcal{T}}(1)$ for each $M \in \mathcal{T}$. If this holds, the comparison map ρ is a homeomorphism iff it is bijective.

Proposition 4. *For X as in the theorem, the category $\mathcal{T} = \mathrm{Perf}(X)$ is noetherian.*

In the situation of Example 2 this is classical [6]. The general case is a consequence of results of van der Kallen on finite generation of cohomology of reductive groups [9], as was recently explained in [10] for the situation of Example 4.

Sketch of proof. It suffices to show that R is a noetherian ring. Assume that the morphism $\pi : Y \to X$ is finite flat of degree d. The stack X allows a representation $X \cong [Z/\mathrm{GL}_d]$ where $Z = \mathrm{Spec}(C)$ is an affine scheme. Explicitly, Z is the scheme that classifies trivializations of the locally free $\mathcal{O}_X$-module $\pi_*\mathcal{O}_Y$. It follows that $R \cong H^*(\mathrm{GL}_d, C)$. By [9, Th. 10.5], this graded ring is finitely generated over k iff it has bounded torsion. In our case, the cohomology in positive degrees is annihilated by d because the composition $\mathcal{O}_X \to \pi_*\mathcal{O}_Y \to \mathcal{O}_X$ (where the second map is the trace) is equal to d, and $R = H^*(X, \mathcal{O}_X)$, while $H^{>0}(X, \pi_*\mathcal{O}_Y)$ vanishes. □

Decomposition into fibers. The theorem is proved by a reduction to the known cases where X is an affine scheme or where $X = BG$ for a finite group scheme over a field. For simplicity, we explain this when $X = [\mathrm{Spec}(A)/G]$ for a finite group G as in Example 2. Let A^G be the ring of G-invariants, i.e. the degre zero part of $R = H^*(G, A)$. We consider fibers over $\mathfrak{q} \in \mathrm{Spec}(A^G)$ in two ways.

Spectral fibers. There is a natural map $\mathrm{Spec}^h(R) \to \mathrm{Spec}(A^G)$, and we can take the fibers over $\mathfrak{q}$ in the source and target of ρ with respect to this map. This gives the lower line of the following diagram.

Geometric fiber. There is a natural morphism $X \to \mathrm{Spec}(A^G)$. The reduced fiber over $\mathfrak{q}$ under this morphism, denoted $X(\mathfrak{q})$, gives rise to another instance of the comparison map; this is the upper line of the following diagram. Explicitly, we have $X(\mathfrak{q}) = [\mathrm{Spec}(A(\mathfrak{q}))/G]$ where $A(\mathfrak{q})$ is the product of the residue fields of the primes of A over $\mathfrak{q}$, and consequently $R(\mathfrak{q}) = H^*(G, A(\mathfrak{q}))$.

$$\begin{array}{ccc} \mathrm{Spec}(T(\mathfrak{q})) & \xrightarrow{\rho(\mathfrak{q})} & \mathrm{Spec}^h(R(\mathfrak{q})) \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\alpha} \\ \mathrm{Spc}(T)_{\mathfrak{q}} & \xrightarrow{\rho_{\mathfrak{q}}} & \mathrm{Spec}^h(R)_{\mathfrak{q}} \end{array}$$

The vertical arrows of the diagram arise by functoriality. Now the comparison map ρ is bijective (hence a homeomorphism) iff $\rho_{\mathfrak{q}}$ is bijective for each $\mathfrak{q}$. This will be a consequence of the following facts.

(1) The map $\rho(\mathfrak{q})$ is a homeomorphism, in particular bijective,
(2) the map α is a homeomorphism, in particular bijective,
(3) the map β is surjective.

(1) can be deduced from the case of BG over a field (Example 1). (2) is based on the following two lemmas. A ring homomorphism is called a universal homeomorphism if it induces a homeomorphism on Spec after arbitrary base change.

Lemma 5. *For a G-invariant nilpotent ideal $I \subseteq A$, the ring homomorphism $H^*(G, A) \to H^*(G, A/I)$ is a universal homeomorphism.*

Lemma 6. *If $t \in A^G$ is an A-regular element, then $H^*(G, A)/t \to H^*(G, A/t)$ is a universal homeomorphism.*

The proof of Lemma 6 is related to an argument of [4]. It may be interesting to see how one uses that $R = H^*(G, A)$ is noetherian:

Sketch of proof. We can assume that a prime power p^r annihilates $H^{>0}(G, A)$. The key point is to show that for a homogeneous element $a \in H^*(G, A/t)$ of even degree, the connecting homomorphism $\delta : H^*(G, A/t) \to H^*(G, A)$ associated to $A \to A \to A/t$ sends a^{p^r} to zero. Since R is noetherian, after replacing t by a power t^m, we can assume that the annihilators in R of t and of t^2 coincide. Using that the connecting homomorphism $\bar{\delta} : H^*(G, A/t) \to H^*(G, A/t)$ associated to $A/t \to A/t^2 \to A/t$ is a graded derivation, we get $\bar{\delta}(a^{p^r}) = 0$. Hence $\delta(a^{p^r}) = tb$ for some $b \in R$. But then $t^2 b = t\delta(a^{p^r}) = 0$ and thus $tb = 0$ as desired. □

Finally, (3) is deduced from a surjectivity criterion of [1]: If a tt functor $\mathcal{T} \to \mathcal{T}'$ detects tensor nilpotence of morphisms, then $\mathrm{Spc}(\mathcal{T}') \to \mathrm{Spc}(\mathcal{T})$ is surjective. This applies due to the following counterparts of Lemmas 5 and 6.

Lemma 7. *If $A' = A/I$ for a G-invariant nilpotent ideal $I \subseteq A$, then the functor $D(AG)_{A\text{-perf}} \to D(A'G)_{A'\text{-perf}}$ detects tensor nilpotence of morphisms.*

Lemma 8. *If $t \in A^G$ is an A-regular element and $B = A/t \times A[t^{-1}]$, then the functor $D(AG)_{A\text{-perf}} \to D(BG)_{B\text{-perf}}$ detects tensor nilpotence of morphisms.*

The factor $A[t^{-1}]$ in Lemma 8 is of auxiliary nature; it has the effect that $A \to B$ and hence $A^G \to B^G$ are universally bijective on spectra. There is a version of Lemma 6 with $B = A/t \times A[t^{-1}]$ in place of A/t, which is not very useful in that context, but which would further streamline the structure of the argument.

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Generation time in triangulated categories under biexact functors and for Koszul objects

Janina C. Letz

(joint work with Marc Stephan)

In a triangulated category $\mathcal{T}$ an object X generates an object Y, if Y can be obtain from X by taking finite coproducts, rectracts, suspensions, desuspensions, and cones. The generation time of Y from X is the minimal number of cones necessary in this process. Explicitly, given X we define

- $\operatorname{thick}^0_{\mathcal{T}}(X)$ as the smallest strictly full subcategory containing the zero object;
- $\operatorname{thick}^1_{\mathcal{T}}(X)$ as the smallest strictly full subcategory containing X that is closed under finite coproducts, rectracts, suspensions and desuspensions;
- $\operatorname{thick}^n_{\mathcal{T}}(X)$ as the smallest strictly full subcategory of objects Y for which there exists an exact triangle

$$Y' \to Y \oplus \tilde{Y} \to Y'' \to \Sigma Y \quad \text{with } Y' \in \operatorname{thick}^1(X) \text{ and } Y'' \in \operatorname{thick}^{n-1}(X)\,.$$

These subcategories give an exhaustive filtration of the smallest thick subcategory of $\mathcal{T}$ containing X; they were first constructed by [BvdB03, Section 2.2]. The generation time is defined as

$$\operatorname{level}^X_{\mathcal{T}}(Y) := \inf\{n \geq 0 \mid Y \in \operatorname{thick}^n_{\mathcal{T}}(X)\}\,;$$

this invariant was introduced by [ABIM10, 2.3].

For various X the invariant level coincides with some algebraic invariants for modules, like the projective dimension. Level is also connected to the Rouquier dimension of a triangulated category; cf. [Rou08]. In general, it is hard to compute the Rouquier dimension, and it is not known for most triangulated categories. We expect, that understanding level will help computing Rouquier dimension.

It is straightforward to see that the value of level decreases along an exact functor of triangulated category. Along a bifunctor, for example a tensor product, the behavior is more complicated. In fact, we require additional conditions for the biexact functor to get a reasonable bound:

Theorem 1. *Let* $\mathsf{F}\colon \mathcal{S}\times\mathcal{T}\to\mathcal{U}$ *be a biexact bifunctor that admits a strong Verdier structure. Then*

$$\operatorname{level}_{\mathcal{U}}^{\mathsf{F}(X,X')}(\mathsf{F}(Y,Y')) \le \operatorname{level}_{\mathcal{S}}^{X}(Y) + \operatorname{level}_{\mathcal{T}}^{X'}(Y') - 1$$

for $X, Y \in \mathcal{S}$ *and* $X', Y' \in \mathcal{T}'$.

The notion of a strong Verdier structure is motivated by [May01, KN02]. It is satisfied in many examples that appear in nature. In fact, if the triangulated categories are the homotopy category of a model category, then any bifunctor that is induced by a biexact functor on the model categories, admits a strong Verdier structure. By [May01, GPS14a], the tensor product of many tensor triangulated categories admits a strong Verdier structure, for example for the derived category of modules over a commutative ring or the stable module category of a finite dimensional group algebra. Another class of bifunctors admitting a strong Verdier structure is the tensor product of dg bimodules

$$- \otimes_B^L - \colon \mathrm{D}(\mathrm{Bimod}(A,B)) \times \mathrm{D}(\mathrm{Bimod}(B,C)) \to \mathrm{D}(\mathrm{Bimod}(A,C))$$

for dg algebras A and B and C over a commutative ring. This includes in particular, the tensor action $\otimes_A^L$ of $\mathrm{D}(\mathrm{Bimod}(A,A))$ on $\mathrm{D}(\mathrm{Mod}(A))$.

Any action $\mathsf{F}\colon \mathcal{S}\times\mathcal{T}\to\mathcal{T}$ of a tensor triangulated category $(\mathcal{S},\otimes,\mathbb{1})$ on a triangulated category $\mathcal{T}$ induces a graded ring homomorphism

$$\operatorname{End}_{\mathcal{S}}^*(\mathbb{1}) \to \mathrm{Z}^*(\mathcal{T}) := \bigoplus_{d\in\mathbb{Z}} \{\alpha\colon \operatorname{id}_{\mathcal{T}} \to \Sigma^d \mid \alpha\Sigma = (-1)^d \Sigma\alpha\}$$

of the endomorphism ring to the center of $\mathcal{T}$. This homomorphism is given by

$$f \mapsto \alpha_f\,, \quad \alpha_f(X) := (X \cong \mathsf{F}(\mathbb{1},X) \to \mathsf{F}(\Sigma^{|f|}\mathbb{1},X) \cong \Sigma^{|f|}X)\,.$$

The elements in the center induced by an action, as well as their Koszul objects, have particularly nice properties.

The *Koszul object of* $\alpha_1,\dots,\alpha_c \in \mathrm{Z}^*(\mathcal{T})$ *on* $X \in \mathcal{T}$ is

$$X/\!\!/(\alpha_1,\dots,\alpha_c) := \begin{cases} X & c=0 \\ \operatorname{cone}(\alpha_1(X)) & c=1 \\ (X/\!\!/(\alpha_1,\dots,\alpha_{c-1}))/\!\!/\alpha_c & c>1\,. \end{cases}$$

Theorem 2. *Let* $\mathsf{F}_i\colon \mathcal{S}_i\times\mathcal{T}\to\mathcal{T}$ *be an action of a tensor triangulated category on* $\mathcal{T}$ *and* $f_i \in \operatorname{End}_{\mathcal{S}_i}(\mathbb{1})$. *If* F_i *admits a strong Verdier structure for all* $1\le i\le c$, *then*

$$\operatorname{level}_{\mathcal{T}}^{X}(X/\!\!/(\alpha_{f_1},\dots,\alpha_{f_c})) \le c+1\,.$$

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The Balmer spectrum for equivariant Eilenberg–MacLane spectra

Clover May

(joint work with Drew Heard)

For the cyclic group $G = C_p$, we compute the Balmer spectrum of the compact objects in equivariant HR-modules, where R is a (fairly nice) constant Mackey functor. Many of these Balmer spectra are known. For example, in the case $p = 2$ and $R = \mathbb{F}_2$, this was described in previous work joint with Dugger and Hazel, and computed independently by Balmer–Gallauer. Recent work of Balmer–Gallauer describes the spectrum of HR-modules where R is a field of characteristic p (and more generally for G any finite group). Our computations recover these cases for $G = C_p$, and generalize to include rings such as $R = \mathbb{Z}$ and $\mathbb{Z}_{(p)}$. We also compute the Balmer spectrum for HR_G-modules, where HR_G is inflated from the trivial group and now G is any finite group, generalizing a computation of Patchkori–Sanders–Wimmer. This is joint work in progress with Drew Heard.

The talk began in the classical (nonequivariant) setting, recalling that cohomology is represented by stable objects or spectra. More precisely, for R a commutative ring, singular cohomology with R coefficients $H^*(-;R)$ is represented by a commutative ring spectrum HR. The homotopy category of HR-modules is completely algebraic. Schwede–Shipley showed in [9] there is a Quillen equivalence

$$HR\text{–Mod} \simeq_Q \mathcal{C}h(R)$$

and thus an equivalence of homotopy categories

$$Ho(HR\text{–Mod}) \simeq \mathcal{D}(R).$$

Lurie showed there is a symmetric-monoidal equivalence of ∞-categories

$$HR\text{–Mod} \simeq \mathcal{C}h(R)$$

in Theorem 7.1.2.13 of *Higher Algebra*, so

$$Ho(HR\text{–Mod}) \simeq \mathcal{D}(R).$$

as tt-categories. For the Balmer spectrum of compact objects, we write

$$\mathrm{Spc}(\mathrm{Perf}(HR)) = \mathrm{Spc}(HR\text{–}\mathrm{Mod}^c) \cong \mathrm{Spc}(\mathcal{D}_{\mathrm{perf}}(R)) \cong \mathrm{Spec}(R)$$

where the last isomorphism is due to Hopkins–Neeman [6, 7] in the case where R is Noetherian and Thomason [10] for general R.

Example 1. *The commutative ring spectrum $H\mathbb{F}_p$ is a "field" in the following sense. Any $X \in H\mathbb{F}_p$–Mod splits as a wedge of suspensions of $H\mathbb{F}_p$, so*

$$X \simeq \bigvee_{i\in I} \Sigma^{n_i} H\mathbb{F}_p$$

and $\mathrm{Spc}(\mathrm{Perf}(H\mathbb{F}_p)) = \bullet$.

Now we turn to the equivariant setting. For G a finite group and V a real representation of G, the one-point compactification is a representation sphere $S^V = \widehat{V}$. Let $H^*_G(-;M)$ denote $RO(G)$-graded equivariant cohomology with coefficients in a commutative Mackey ring (or commutative Green functor) M. This is an equivariant cohomology theory graded on the real representation ring with a suspension isomorphism with respect to representation spheres. Like the nonequivariant case, this cohomology theory is represented by a stable object. Let $HM \in \mathcal{S}p_G$ denote the genuine equivariant Eilenberg–MacLane spectrum representing equivariant cohomology with coefficients in M. We are motivated by the following question.

Question 2. Can we describe the Balmer spectrum $\mathrm{Spc}(\mathrm{Perf}_G(HM))$?

One might also be motivated to ask this question from the algebraic side. Building on work of Patchkoria–Sanders–Wimmer, we show the following.

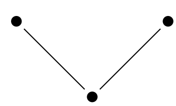
Theorem 3 (Patchkoria–Sanders–Wimmer [8], Heard–M. in progress)**.** *There is a symmetric-monoidal equivalence of ∞-categories*

$$\mathrm{Perf}_G(HM) \simeq \mathcal{D}_{\mathrm{perf}}(M).$$

The Balmer spectrum $\mathrm{Spc}(\mathrm{Perf}_G(HM))$ is known in several cases. For example, take $G = C_2$, the cyclic group of order 2, and $M = \underline{\mathbb{F}}_2$ the constant Mackey functor. The $RO(C_2)$-graded equivariant cohomology theory $H^*_{C_2}(-;\underline{\mathbb{F}}_2)$ is represented by the equivariant Eilenberg–MacLane spectrum $H\underline{\mathbb{F}}_2$. The Balmer spectrum was computed independently by Balmer and Gallauer, and by Dugger, Hazel and myself.

Theorem 4 (Balmer–Gallauer [2], Dugger–Hazel–M. [5])**.** *The Balmer spectrum*

$$\mathrm{Spc}(\mathrm{Perf}_{C_2}(H\underline{\mathbb{F}}_2)) =$$



In a sense, we took the wrong approach to computing the Balmer spectrum because we classified everything.

Theorem 5 (Dugger–Hazel–M. [5])**.** *If $X \in \mathrm{Perf}_{C_2}(H\underline{\mathbb{F}}_2)$ then X splits as a wedge of $RO(G)$-suspensions of*

$$H\underline{\mathbb{F}}_2, \qquad (S^n_a)_+ \wedge H\underline{\mathbb{F}}_2, \qquad \textit{and} \qquad \mathrm{cof}\,(\tau^m)\,,$$

where $n \geq 0$ and $m \geq 1$.

With this complete classification of compact objects, we were able to tensor everything together. Notice this classification and the computation of the Balmer spectrum show that $H\underline{\mathbb{F}}_2$ is not a field.

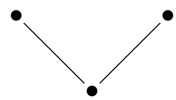
A similar approach will not work in more general settings. In work in progress with Grevstad, we show there is no complete classification for odd primes.

Theorem 6 (Grevstad–M. in progress)**.** *For $G = C_p$ with p an odd prime, the classification of $\mathrm{Perf}_{C_p}(H\underline{\mathbb{F}}_p)$ is wild.*

So we need another approach. Balmer and Gallauer have used different tactics to compute the Balmer spectrum for $\mathrm{Perf}_{C_p}(H\underline{\mathbb{F}}_p)$.

Theorem 7 (Balmer–Gallauer [3])**.** *The Balmer spectrum*

$$\mathrm{Spc}(\mathrm{Perf}_{C_p}(H\underline{\mathbb{F}}_p)) =$$



In fact, Balmer and Gallauer do much more. For G a finite group and k a field of characteristic $p > 0$, they describe $\mathrm{Spc}(\mathrm{Perf}_G(H\underline{k}))$. In their previous work on Artin–Tate motives [2] they also describe $\mathrm{Spc}(\mathrm{Perf}_{C_2}(H\underline{\mathbb{Z}}))$ using C_2 as the Galois group.

So we consider the case of a more general ring.

Question 8. For $G = C_p$ (with p either 2 or and odd prime) and $\underline{R}$ the constant Mackey functor for a commutative ring R, can we describe the Balmer spectrum $\mathrm{Spc}(\mathrm{Perf}_{C_p}(H\underline{R}))$?

Let us first deal with a fairly trivial case.

Remark 9. If p is invertible in R then $\mathrm{Spc}(\mathrm{Perf}_{C_p}(H\underline{R})) \cong \mathrm{Spec}(R)$.

So assume p is not invertible in R. We can compute $\mathrm{Spc}(\mathrm{Perf}_{C_p}(H\underline{R}))$ as a set using work we heard about earlier this week. Applied to $H\underline{R}$ we have the following.

Theorem 10 (Barthel–Castellana–Heard–Naumann-Pol [1])**.** *As a set*

$$\mathrm{Spc}(\mathrm{Perf}_{C_p}(H\underline{R})) \simeq \coprod_{(H) \leq G} \mathrm{Spc}(\mathrm{Perf}(\Phi^H H\underline{R}))/W_G H.$$

In the case $G = C_p$ this reduces to

$$\mathrm{Spc}(\mathrm{Perf}_{C_p}(H\underline{R})) \simeq \mathrm{Spc}(\mathrm{Perf}(\Phi^{C_p} H\underline{R})) \sqcup \mathrm{Spc}(\mathrm{Perf}(\Phi^e H\underline{R}))/C_p.$$

Since $H\underline{R}$ is a global spectrum, the action is trivial and we have

$$\mathrm{Spc}(\mathrm{Perf}_{C_p}(H\underline{R})) \simeq \mathrm{Spc}(\mathrm{Perf}(\Phi^{C_p} H\underline{R})) \sqcup \mathrm{Spc}(\mathrm{Perf}(\Phi^e H\underline{R})).$$

In nice circumstances, we can also determine the topology.

Theorem 11 (Heard–M. in progress)**.** *A complete description of* $\mathrm{Spc}(\mathrm{Perf}_{C_p}(H\underline{R}))$ *as a space when* R *is commutative, Noetherian, regular, and* p*-torsion free, with* p *not invertible in* R*.*

The proof uses various comparison maps together with results of Dell'Ambrogio–Stanley [4].

Finally, for G an arbitrary finite group, one can inflate the Eilenberg–MacLane spectrum $H\mathbb{Z}$ to an equivariant spectrum by giving it the trivial action. This is denoted $H\mathbb{Z}_G = \mathrm{infl}_e^G H\mathbb{Z}$ and has the property $\Phi^H(H\mathbb{Z}_G) = H\mathbb{Z}$ for all subgroups $H \leq G$. In [8], Patchkoria, Sanders, and Wimmer computed $\mathrm{Spc}(\mathrm{Perf}_G(H\mathbb{Z}_G))$ as a space. Their proof involves reducing to the case $G = C_p$ and showing a particular inclusion. Our techniques recover this inclusion via $H\mathbb{Z}_{C_p} \to H\underline{\mathbb{Z}}$, the adjoint to the identity.

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A nilpotence theorem via homological residue fields for Lie superalgebra representations

Daniel K. Nakano

(joint work with Matthew H. Hamil)

For a small rigid symmetric tensor triangulated category (TTC), $\mathbf{K}$, Balmer introduced the concept of homological primes and homological residue fields [Bal20, BalC21]. For a TTC, the collection of homological primes, $\mathrm{Spc}^h(\mathbf{K})$, forms a topological space that can potentially realize the Balmer spectrum, $\mathrm{Spc}(\mathbf{K})$, and its supports in a concrete way. Such a realization enables one then to prove the tensor product property on support data. The central problem in the theory of homological primes is the following "Nerves of Steel" Conjecture (cf. [Bal20]).

Conjecture [NoS Conj] *Let* $\mathbf{K}$ *be a small rigid (symmetric) tensor triangulated category. Then the comparison map* $\phi : \mathrm{Spc}^h(\mathbf{K}) \to \mathrm{Spc}(\mathbf{K})$ *is a bijection.*

For the stable module category of finite group schemes, the [NoS Conj] can be verified by using a deep stratification result (see [Bal20, BIKP18]). The speaker (with Hamil) is interested in verifying the [NoS Conj] in the case of Lie superalgebra representations over $\mathbb{C}$.

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a classical Lie superalgebra. In this talk, we will consider the tensor triangular geometry for the stable category of finite-dimensional Lie superalgebra representations: $\mathrm{stab}(\mathcal{F}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})})$. The localizing subcategories for the detecting subalgebra $\mathfrak{f}$ are classified which answers a question of Boe, Kujawa and Nakano [BKN10]. As a consequence of these results, we prove a nilpotence theorem and determine the homological spectrum for the stable module category of $\mathcal{F}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}$.

The speaker (with Hamil) [HaN23] has verified [NoS Conj] for $\mathfrak{g} = \mathfrak{gl}(m|n)$ and has been working on the verification for other Type I classical simple Lie superalgebras. The method of proof involves using recent work of Serganova and Sherman [SS22] on the existence of "splitting subgroups" where one can find a copy of the trivial module in the induction of the trivial module from the splitting subgroup to the ambient supergroup, in addition to, results proved in [Bal20].

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Frobenius pushforwards and the bounded derived category

Josh Pollitz
(joint work with Matthew Ballard, Srikanth Iyengar, Pat Lank, Alapan Mukhophadyay)

Throughout R is a commutative noetherian ring having prime characteristic $p > 0$. In this setting one has access to the **Frobenius endomorphism** $F\colon R \to R$ given by $r \mapsto r^p$, and its iterates $F^e\colon R \to R$ given by $r \mapsto r^{p^e}$. As with any ring map one obtains functors

$$F^e_*\colon \operatorname{Mod} R \to \operatorname{Mod} R\,,$$

via restriction of scalars along F^e, called the $e^{\mathbf{th}}$ **Frobenius pushforward**. Explicitly, given an R-module M, the R-action on $F^e_*(M)$ is given by

$$r \cdot F^e_*(m) = F^e_*(r^{p^e} m) \quad \text{for} \quad r \in R,\ m \in M\,.$$

These are classical objects of study in prime characteristic commutative algebra and algebraic geometry as will be recalled below.

Looking forward, we will be interested with how these functors behave on the bounded derived category of finitely generated R-modules $\mathsf{D}^{\mathsf{b}}(\operatorname{mod} R)$, and so a mild (yet relevant) assumption will be that R is F-finite. That is, F is a finite ring map (i.e., F_*R is a finitely generated R-module). Such rings are exactly the rings where the pushforwards restrict as endofunctors on the category of finitely generated R-modules $\operatorname{mod} R$. Most natural examples in commutative algebra fit into this context: any ring that is essentially of finite type over a field k of characteristic p satisfying $[k : k^p] < \infty$.

The jumping off point in prime characteristic commutative algebra and algebraic geometry is the following classical theorem of Kunz [8]: *For an F-finite ring R, the following are equivalent*

(1) *R is regular*;
(2) *F^e_*R is projective over R for all $e > 0$;*
(3) *F^e_*R is projective over R for some $e > 0$.*

This theorem has motivated a tremendous amount of research in the study of "F-singularities", where one is interested in understanding singularities through the lens of the Frobenius; see, for example, [2, 5, 6, 11, 12]. The main result reported on gives a structural explanation of the theorem of Kunz and other theorems of this ilk, and provides a uniform way to deduce such results.

Theorem 1 (Ballard–Iyengar–Lank–Mukhophadyay–P. [3])**.** *When R is an F-finite commutative noetherian ring of prime characteristic p and M is an object of $\mathsf{D}^{\mathsf{b}}(\operatorname{mod} R)$ with $\operatorname{supp}_R M = \operatorname{Spec}(R)$, the R-complex F^e_*M is a strong generator for $\mathsf{D}^{\mathsf{b}}(\operatorname{mod} R)$ for any natural number $e > \log_p(\operatorname{codepth}(R))$. In particular F^e_*R is a strong generator for $\mathsf{D}^{\mathsf{b}}(\operatorname{mod} R)$ when $e > \log_p(\operatorname{codepth}(R))$.*

Recall a strong generator for a triangulated category T is an object that builds any other object of T using finitely many cones, suspensions, and retracts, where there is a uniform bound on the number of cones needed; cf. [4, 10]. The fact that

$\mathsf{D^b}(\operatorname{mod} R)$ admits a strong generator when R is F-finite (more generally, quasi-excellent) follows from [1, 7, 9]; however, these proofs do not identify explicit strong generators for $\mathsf{D^b}(\operatorname{mod} R)$. Hence part of the novelty of Theorem 1 is that over F-finite rings one can explicitly obtain many strong generators for $\mathsf{D^b}(\operatorname{mod} R)$ by applying enough Frobenius pushforwards to objects with full support (in particular to R itself).

The codepth is an invariant measuring the singularity of R: for example, R is regular if and only if $\operatorname{codepth}(R) = 0$. For an F-finite ring, this value is finite and is in fact at most the minimal number of generators for the R-module F_*R. An obvious question is whether, in Theorem 1, the specified bound on the number of pushforwards needed to obtain strong generators is always necessary.

Question 2. For an F-finite ring R, is F_*R a strong generator for $\mathsf{D^b}(\operatorname{mod} R)$?

The most decisive result we have concerning this question is the following.

Theorem 3 (Ballard–Iyengar–Lank–Mukhophadyay–P. [3])**.** *If R is an F-finite locally complete intersection ring and M in $\mathsf{D^b}(\operatorname{mod} R)$ has $\operatorname{supp}_R M = \operatorname{Spec}(R)$, then F_*M is a strong generator for $\mathsf{D^b}(\operatorname{mod} R)$.*

Outside of the locally complete intersection setting, we have verified that F_*R is a strong generator for $\mathsf{D^b}(\operatorname{mod} R)$ in a number of examples [3]. However, in full generality Question 2 remains open.

In the geometric context, we are also able to say something regarding strong generation via reduction to the affine case. For a noetherian scheme X, this reduction goes by regarding the category $\mathsf{D^b}(\operatorname{coh} X)$ as a module over the tensor-triangulated category $\mathsf{Perf}(X)$.

Theorem 4 (Ballard–Iyengar–Lank–Mukhophadyay–P. [3])**.** *Let X be a noetherian F-finite separated scheme of prime characteristic p, and G a generator for $\mathsf{Perf}(X)$. For any M in $\mathsf{D^b}(\operatorname{coh} X)$ with $\operatorname{supp}_X M = X$, the complex $F^e_*(G \otimes^{\mathsf{L}}_X M)$ is a strong generator for $\mathsf{D^b}(\operatorname{coh} X)$ for any natural number $e > \log_p(\operatorname{codepth} X)$.*

Here subtleties arise because $\mathsf{Perf}(X)$ is not typically generated by its tensor unit $\mathcal{O}_X$. For instance, there are examples where no pushforward $F^e_*(\mathcal{O}_X)$ is a strong generator for $\mathsf{D^b}(\operatorname{coh} X)$; examples include F-finite smooth curves of positive genus. Another delicate point is that a G, as in the theorem, is known to exist [4], however such a generator is only described explicitly in a handful of cases; one of the known cases is when X is a quasi-projective scheme in which case one can take G to be $\mathcal{O}_X \oplus L \oplus \cdots \oplus L^{\dim X}$ where L is a very ample line bundle. So getting your hands on explicit strong generators for $\mathsf{D^b}(\operatorname{coh} X)$, using Theorem 4, is less effective than in the affine case; nonetheless, it still provides a structural explanation for why pushforwards in the geometric setting are natural to study and detect singularities.

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Commutative separable algebras in genuine G-spectra

Maxime Ramzi

(joint work with Niko Naumann, Luca Pol)

Commutative separable algebras are a generalization of étale algebras, that one can define in an abitrary symmetric monoidal ∞-category C; namely a commutative algebra A is said to be separable if the multiplication map $\mu : A\otimes A \to A$ admits an $A\otimes A$-linear section. They were introduced in tt-geometry by Balmer in [1], where they were shown to have surprisingly nice features, for example a good theory of modules despite the lack of higher coherences. As étale algebras in classical algebraic geometry, they also encode some information about the geometry of a tt-category. The goal of this talk was to explain joint work with Niko Naumann and Luca Pol, where we compute separable commutative algebras in compact genuine G-spectra, for some finite p-group G. The restriction to compact objects is to avoid complications related to idempotent algebras (which are separable), i.e. smashing localizations - classifying them, even for trivial G, is a longstanding open problem in stable homotopy theory. I will say a few words about the restriction to p-groups at the end.

Similar classification results have been obtained previously: in algebraic geometry, Neeman classifies separable algebras in the tt-category of quasicoherent sheaves on a noetherian scheme X [5] - when restricting to the dualizable ones, he proves that they are all of the form $f_*\mathcal{O}_U$, where $f : U \to X$ is a finite étale morphism (if we keep the restriction to dualizable objects, Naumann and Pol [4] remove the noetherian hypothesis). In representation theory, Balmer and Carlson [2] classify the separable commutative algebras in the (small) stable module category of kC_p where k is a separably closed field of characteristic p and C_p is a cyclic group of order p (descent methods allow to more generally classify them for G any finite group of p-rank 1, see also [4, Prop 14.11, Theorem 14.14] for an alternative proof, using Galois theory under more restrictive assumptions): they are all of the form $(k[X])^\vee$ for X some finite G-set. Informally, in any "equivariant context" one has these "standard" separable algebras, given by duals of (linearizations of) finite G-sets. Since in Sp^ω, the category of finite spectra, the only separable algebras are "trivial", i.e. they are finite products of the unit, the following is not too surprising:

Theorem 1 (Naumann-Pol-R.)**.** *Let G be a finite p-group. The canonical functor $\mathrm{Fin}_G^{\mathrm{op}} \to \mathrm{CAlg}^{\mathrm{sep}}(\mathrm{Sp}_G^\omega)$ from the category of finite G-sets to the category of commutative separable algebras in compact genuine G-spectra*[1] *sending X to the dual of $\Sigma_+^\infty X$ is an equivalence of categories.*

More generally, given any family $\mathcal{F}$ of subgroups of G closed under conjugation and subgroups, letting $\mathrm{Sp}_G^\omega/\mathcal{F}$ denote the Verdier quotient of Sp_G^ω by the orbits $G/H, H \in \mathcal{F}$, and letting $\mathrm{Fin}_G^{\neg\mathcal{F}}$ denote the category of finite G-sets all of whose isotropy groups do not lie in $\mathcal{F}$, the canonical functor $(\mathrm{Fin}_G^{\neg\mathcal{F}})^{\mathrm{op}} \to \mathrm{CAlg}^{\mathrm{sep}}(\mathrm{Sp}_G^\omega/\mathcal{F})$ is an equivalence of categories.

The "More generally" part, more than being interesting in its own right, is used to prove the main result by descending induction on the family $\mathcal{F}$, using the following:

Proposition 2 (Krause, [3])**.** *Let G be a finite group, $\mathcal{F}$ a family of subgroups of G closed under conjugation and subgroups, and K a subgroup of G minimal among subgroups not in $\mathcal{F}$, there is a pullback square of stably symmetric monoidal ∞-categories:*

$$\begin{array}{ccc} \mathrm{Sp}_G^\omega/\mathcal{F} & \longrightarrow & \mathrm{Sp}_G^\omega/\mathcal{F}\vee\{K\} \\ \downarrow & & \downarrow \\ \mathrm{Sp}^{BW_G(K)} & \longrightarrow & \mathrm{Sp}^{BW_G(K)}/\langle\mathbb{S}[W_G(K)]\rangle \end{array}$$

Here, $W_G(K)$ is the Weyl group of K, i.e. $N_G(K)/K$, the quotient of the normalizer of K by K.

Using that the functor $\mathrm{CAlg}^{\mathrm{sep}}(-)$ preserves limits (a sort of descent property, see [4]), this allows for an inductive strategy (where we reduce the size of $\mathcal{F}$ at each stage) to prove Theorem 1. Indeed, using again the limit preservation property, we are able to compute the bottom left corner, using induction we are able to compute

[1] I have proved in [6] that this category does not depend on whether we view Sp_G^ω as a stably symmetric monoidal ∞-category, or as its associated homotopy tt-category : their (a priori ∞-)categories of separable commutative algebras are always equivalent, and they are both 1-categories.

the top right corner, so we are reduced to understanding the base case and the bottom right corner. The case where $\mathcal{F} = \mathcal{P}$ is the family of proper subgroups of G is easy to understand because $\mathrm{Sp}_G^\omega/\mathcal{P} \simeq \mathrm{Sp}^\omega$, so really the only difficulty lies in the bottom right corner, which is some form of stable module category over the sphere spectrum. Luckily, we do not need to classify all separable algebras therein, we only need to understand maps between "standard" ones (as all the separable algebras in $(\mathrm{Sp}^\omega)^{BW_G(K)}$ are standard, by descent). This is where the hypothesis that G is a p-group comes in. Using standard induction-coinduction techniques, this reduces the question to understanding idempotents in $\mathbb{S}^{tQ}$ for subquotients Q of G, and these are all trivial if and only if G is a p-group.

A topic which was not touched upon in the talk is the case of non-p-groups, which we are currently studying. In this case, we can give counterexamples to Theorem 1, but there is still some hope to describe explicitly the category $\mathrm{CAlg}^{\mathrm{sep}}(\mathrm{Sp}_G^\omega)$ using this inductive approach – the answer should be some more complicated combinatorial gadget associated to G.

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Monogenization in tensor triangular geometry

Beren Sanders

Let $\mathcal{T}$ be a rigidly-compactly generated tensor-triangulated category; that is, a compactly generated tensor-triangulated category in which the compact objects coincide with the dualizable objects: $\mathcal{T}^c = \mathcal{T}^d$. In particular, the unit object $\mathbb{1}$ is a compact/dualizable object. We say that $\mathcal{T}$ is *monogenic* if $\mathrm{Loc}\langle\mathbb{1}\rangle = \mathcal{T}$; equivalently, if $\mathrm{thick}\langle\mathbb{1}\rangle = \mathcal{T}^c$. Basic examples of monogenic categories include the stable homotopy category SH and the derived category of a commutative ring $\mathrm{D}(R)$. On the other hand, the equivariant stable homotopy category $\mathrm{SH}(G)$ is not monogenic.

In general, $\mathrm{Loc}\langle\mathbb{1}\rangle$ is a rigidly-compactly generated tensor-triangulated subcategory of $\mathcal{T}$. More generally, if $\mathcal{G} \subset \mathcal{T}^c$ is a set of objects which contains $\mathbb{1}$ and is closed under the tensor-product and taking duals, then $\mathrm{Loc}\langle\mathcal{G}\rangle$ is a rigidly-compactly generated tensor-triangulated subcategory of $\mathcal{T}$.

A number of interesting examples fit into this framework. For example, the derived category of Tate motives $\mathrm{DTM}(k;R) := \mathrm{Loc}\langle R(n) \mid n \in \mathbb{Z}\rangle \subset \mathrm{DM}(k;R)$ sitting inside the derived category of all Voevodsky motives, or the category of cellular motivic spectra $\mathrm{SH}_{\mathrm{cell}}(k;R) \subset \mathrm{SH}(k;R)$ sitting inside the stable homotopy category of all motivic spectra. In these examples, k is a field and R is a commutative ring in which the characteristic of k (if positive) is invertible. (This latter assumption ensures that the categories are rigidly-compactly generated.)

We call $\mathcal{T}_{\langle \mathbb{1} \rangle} := \mathrm{Loc}\langle \mathbb{1} \rangle \subset \mathcal{T}$ the *monogenization* of $\mathcal{T}$. The goal of this project was to understand the tensor triangular geometry of this construction. This led to a general study of faithful and fully faithful functors in tensor triangular geometry.

To state the main theorem, let $F : \mathcal{T} \to \mathcal{S}$ be a geometric functor (that is, a tensor-triangulated functor which preserves coproducts) between rigidly-compactly generated tensor-triangulated categories. It preserves compact objects and hence induces a map

$$\varphi : \mathrm{Spc}(\mathcal{S}^c) \to \mathrm{Spc}(\mathcal{T}^c)$$

on Balmer spectra.

Theorem 1 (S.)**.** *Assume* $\mathrm{Spc}(\mathcal{S}^c)$ *is noetherian.*

(1) If F is faithful then φ is a topological quotient map.
(2) If F is fully faithful then the fibers of φ are connected.

Without assuming that $\mathrm{Spc}(\mathcal{S}^c)$ is noetherian, the theorem also holds, but in a slightly more technical form: in (1) "topological quotient map" is replaced by a suitably notion of "spectral quotient map" and (2) becomes "the fiber over each weakly visible point is connected".

We discussed how this theorem can be applied to numerous examples, including the motivic ones mentioned above, connecting with work of Gallauer, Balmer–Gallauer, Vishik, Deng–Vishik, Gheorge–Wang–Xu, and Burklund.

It is also fruitful to consider the relation with Balmer's comparison map

$$\rho : \mathrm{Spc}(\mathcal{T}^c) \to \mathrm{Spec}^h(\mathrm{End}^*(\mathbb{1})).$$

Since a category $\mathcal{T}$ and its monogenization $\mathcal{T}_{\langle \mathbb{1} \rangle}$ have the same graded endomorphism ring of the unit, we have a commutative diagram:

$$\begin{array}{ccc} \mathrm{Spc}(\mathcal{T}^c) & \xrightarrow{\ \rho\ } & \\ \varphi \downarrow & & \searrow \\ \mathrm{Spc}(\mathcal{T}^c_{\langle \mathbb{1} \rangle}) & \xrightarrow[\rho]{} & \mathrm{Spec}^h(\mathrm{End}^*(\mathbb{1})). \end{array}$$

It follows that φ is a homeomorphism for categories $\mathcal{T}$ for which the top ρ is a homeomorphism, such as $\mathcal{T} := \mathrm{KInj}(kG)$. That is, for such categories, monogenization induces a homeomorphism on Balmer spectra. On the other hand, it is easy to find examples in algebraic geometry in which the monogenization has a very different Balmer spectrum than the original category.

We also discussed how the above theorem provides a common strengthening of theorems due to Balmer and to Lau, namely:

Theorem 2 (Balmer, Lau, S.)**.** *The comparison map* $\rho : \mathrm{Spc}(\mathcal{T}^c) \to \mathrm{Spec}^h(\mathrm{End}^*(\mathbb{1}))$ *is a spectral quotient map if the graded ring* $\mathrm{End}^*(\mathbb{1})$ *is coherent.*

In the last part of the talk we discussed monogenization for the equivariant stable homotopy category $\mathrm{SH}(G)$:

Theorem 3 (S.). *Let G be a finite group.*

(1) Let $\mathcal{T} := \mathrm{SH}(G)_{\leq n}$ be the truncation below a finite height n. The monogenization $\mathcal{T}_{\langle \mathbb{1} \rangle} \hookrightarrow \mathcal{T}$ induces a homeomorphism on Balmer spectra.

(2) Let $\mathcal{T} := \mathrm{D}(\mathrm{H}\mathbb{Z}_G)$ be the category of derived Mackey functors. The spectrum of the monogenization coincides with the spectrum of the Burnside ring: $\mathrm{Spc}(\mathcal{T}^c_{\langle \mathbb{1} \rangle}) \cong \mathrm{Spec}(A(G))$.

(3) Let $\mathcal{T} = \mathrm{SH}(G)$. On Balmer spectra, monogenization $\mathcal{T}_{\langle \mathbb{1} \rangle} \hookrightarrow \mathcal{T}$ performs the gluing which occurs in the spectrum of the Burnside ring — but only at height ∞.

We illustrated part (3) of the theorem in the case $G = C_p$. Here the map on Balmer spectrum glues together two points (the points at height infinity at the prime p) and leaves everthing else untouched.

At the end of the talk, we opened up a discussion on the choice of terminology "monogenization". Many in the audience seemed to think this was an imperfect choice of terminology and some alternatives were suggested.

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Spectra and t-structures

Jan Šťovíček

The aim of the talk was to explain a known classification of t-structures in derived categories of commutative rings and ask about possible extensions to non-affine schemes and connective dg rings.

To start with, let R be a commutative noetherian ring and $\mathcal{D} = \mathrm{D}(\text{Mod-}R)$. Even in the case $R = \mathbb{Z}$, it is virtually impossible to classify all t-structures in $\mathcal{D}$. The reason is that there is a proper class of torsion pairs in the category of abelian groups and each such torsion pair induces a so-called Happel-Reiten-Smalø [4] t-structure in $\mathcal{D}$. So, unlike localizations [7], t-structures in $\mathcal{D}$ may form a proper class, as was noticed by Stanley [9].

However, it is possible to classify compactly generated t-structures $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$, i.e. t-structures with $\mathcal{D}^{\geq 0} = \operatorname{Ker}\operatorname{Hom}_{\mathcal{D}}(\mathcal{S}, -)$ for a set $\mathcal{S}$ of compact objects (= perfect complexes) in $\mathcal{D}$. This is a result of Alonso, Jeremías and Saorín [1], based on previous work of Deligne, Bezrukavnikov, Kashiwara and Stanley [9]: Such t-structures are in bijection with decreasing filtrations by supports, that is functions $f\colon \mathbb{Z} \to 2^{\mathrm{Spec}(R)}$ such that each $f(n) \subseteq \mathrm{Spec}(R)$ is closed under specialization and $f(n) \supseteq f(n+1)$ for each $n \in \mathbb{Z}$.

In one direction, given a compactly generated t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$, the corresponding filtration by supports is given by $f\colon n \mapsto \operatorname{Supp} H^n(\mathcal{D}^{\leq 0})$, where

$$H^n\colon \mathrm{D}(\text{Mod-}R) \to \text{Mod-}R$$

is the standard cohomology functor for complexes of R-modules. In the other direction, given a filtration by supports $f\colon \mathbb{Z} \to 2^{\mathrm{Spec}(R)}$, we have

$$\mathcal{D}^{\geq 0} = \operatorname{Ker}\operatorname{Hom}_{\mathcal{D}}(\mathcal{S}_f, -),$$

where $\mathcal{S}_f$ consists of those shifts of Koszul complexes $K_\bullet(\underline{x})[n]$ where $\underline{x} = (x_1, ..., x_r)$ runs over sequences of elements of R such that $\mathrm{V}(\underline{x}) \subseteq f(n)$ and $K_\bullet(\underline{x}) = \bigotimes_{i=1}^{n}(R \xrightarrow{x_i} R)$ lives in cohomological degrees $-r$ to 0.

In order to get a better intuition for the correspondence, here are a few examples:

(1) The canonical t-structure in $\mathrm{D}(\text{Mod-}R)$ corresponds to the filtration $f\colon \mathbb{Z} \to 2^{\mathrm{Spec}(R)}$ given by $f(n) = \mathrm{Spec}(R)$ for $n \leq 0$ and $f(n) = \emptyset$ for $n > 0$.
(2) More generally, if $V \subseteq \mathrm{Spec}(R)$ is a specialization closed set, there is a (hereditary) torsion pair $(\mathcal{T}_V, \mathcal{F}_V)$ with the torsion class $\mathcal{T}_V$ consisting of all modules supported in V, by a famous result by Gabriel [3]. The induced Happel-Reiten-Smalø t-structure corresponds to the filtration by supports given by $f(n) = \mathrm{Spec}(R)$ for $n < 0$, $f(0) = V$ and $f(n) = \emptyset$ for $n > 0$.

(3) Given a specialization closed set $V \subseteq \mathrm{Spec}(R)$ again, the constant filtration given by $f(n) = V$ for each $n \in \mathbb{Z}$ corresponds to the compactly generated localization of D(Mod-R) from Neeman's classification [7].

(4) If R has a dualizing complex, one can consider the height filtration $f(n) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathrm{ht}(\mathfrak{p}) \geq n\}$. The restriction of the corresponding compactly generated t-structure to $\mathrm{D^b}(\text{mod-}R)$ is a Grothendieck dual of the canonical t-structure on the bounded derived category. This example was considered by Yekutieli and Zhang [11].

Besides the classification, there are other aspects of the localization theory for D(Mod-R) with R noetherian, which extend to the realm of t-structures. For instance, a version of the Telescope Conjecture for t-structures, extending Neeman's results [7], was proved by Hrbek and Nakamura [6].

More remarkably, compactly generated t-structures have been recently classified in $\mathcal{D} = \mathrm{D}(\text{Mod-}R)$ for *all* commutative rings R by Hrbek [5]. This is completely analogous to the story for localizations [10], but the result is only available in the affine setting. Most of the above is still true if we drop noetherianness, the only difference is that we must consider decreasing functions $f\colon \mathbb{Z} \to 2^{\mathrm{Spec}(R)}$ where each $f(n) \subseteq \mathrm{Spec}(R)$ is a so-called *Thomason set*. That is, $f(n)$ is of the form $\bigcup_{I \in \mathcal{I}_n} \mathrm{V}(I)$, where $\mathcal{I}_n$ is some set of finitely generated ideals of R. Thomason sets form open sets of a topology on $\mathrm{Spec}(R)$ and they are none other than specialization closed sets if R is noetherian.

Our first, more down to earth, open question is whether such a classification extends to settings not entirely distant from commutative algebra.

Question 1. Can one classify (suitable) compactly generated t-structures in

(1) $\mathcal{D} = \mathrm{D}(\mathrm{Qcoh}(X))$, where X is a noetherian scheme?

(2) $\mathcal{D} = \mathrm{D}(R)$, where R is a cohomologically non-positive and cohomologically bounded commutative dg ring?

In the first case, results in [2] essentially settle the problem for Happel-Reiten-Smalø t-structures. Among others, [2, §5] establishes a bijection between specialization closed subsets $V \subseteq X$ and torsion pairs $(\mathcal{T}, \mathcal{F})$ of finite type (i.e. with $\mathcal{T}$ generated by a set of coherent sheaves) such that $\mathcal{T}$ is a $\otimes$-ideal in $\mathrm{Qcoh}(X)$. In general, there may be many more torsion classes of finite type in $\mathrm{Qcoh}(X)$ which are not $\otimes$-ideals, even for $X = \mathbb{P}^1$; see [2, Examples 5.5 and 6.14]. The condition of $\mathcal{T}$ being a $\otimes$-ideal has a geometric relevance as is explained in [2, Lemma 5.9], and it may not be very surprising from retrospective as the classification of compactly generated localizations of $\mathcal{D}$ due to Thomason [10] also treats only localizing classes which are (derived) $\otimes$-ideals. In view of this discussion, we can make Question 1(1) more precise:

Question 2. Let X be a noetherian scheme and $\mathcal{D} = \mathrm{D}(\mathrm{Qcoh}(X))$. Let us denote $(\mathcal{D}^{\leq 0}_{\mathrm{can}}, \mathcal{D}^{\geq 0}_{\mathrm{can}})$ the canonical t-structure and call a full subcategory $\mathcal{X} \subseteq \mathcal{D}$ a *t-$\otimes$-ideal* if $Y \otimes X \in \mathcal{X}$ whenever $Y \in \mathcal{D}^{\leq 0}_{\mathrm{can}}$ and $X \in \mathcal{X}$. Is there then a bijection between

(i) the compactly generated t-structures $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ for which $\mathcal{D}^{\leq 0}$ is a t-$\otimes$-ideal and
(ii) filtrations by supports $f\colon \mathbb{Z} \to 2^X$?

Let us conclude by quickly discussing Question 1(2). It is much less supported by evidence from existing results than Question 1(1), but we at least have a classification of localizations of $\mathrm{D}(R)$ if $H^*(R)$ is noetherian; see a recent preprint by Shaul and Williamson [8]. For experts, $\mathrm{D}(R)$ is then stratified by the canonical action of $H^0(R) = \mathrm{End}_{\mathrm{D}(R)}(R)$. As R is the $\otimes$-unit in $\mathrm{D}(R)$, this means that the canonical map $\rho\colon \mathrm{Spc}(\mathrm{Perf}(R)) \to \mathrm{Spec}(\mathrm{End}_{\mathrm{D}(R)}(R))$ from the Balmer spectrum of $\mathrm{Perf}(R)$ to the Zariski spectrum of $\mathrm{End}_{\mathrm{D}(R)}(R)$ is a homeomorphism. Based on discussions during the workshop, we conclude with a much more speculative question:

Question 3. Suppose that $(\mathcal{D}, \otimes, \mathbf{1})$ is a rigidly compactly generated $\otimes$-triangulated category such that

(i) the unit $\mathbf{1}$ is compact and generates a non-degenerate t-structure $(\mathcal{D}^{\leq 0}_{\mathrm{can}}, \mathcal{D}^{\geq 0}_{\mathrm{can}})$ in $\mathcal{D}$ and
(ii) the canonical map $\rho\colon \mathrm{Spc}(\mathbb{D}^c) \to \mathrm{Spec}(\mathrm{End}_{\mathcal{D}}(\mathbf{1}))$ is a homeomorphism.

Are then compactly generated t-structures classified by filtrations by supports of $\mathrm{Spec}(\mathrm{End}_{\mathcal{D}}(\mathbf{1}))$?

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Noncommutative algebra via tensor categories

Kent Vashaw

(joint work with Hongdi Huang, Daniel Nakano, Van Nguyen, Charlotte Ure, Padmini Veerapen, Xingting Wang, Milen Yakimov)

By a foundational theorem of Thomason, brought into the context of tensor-triangular geometry by Balmer, a scheme can be entirely reconstructed from tensor-triangular data. In this talk, we discuss recent approaches towards describing noncommutative algebra and noncommutative algebraic geometry via tensor and tensor-triangulated categories. We approach this problem by addressing two foundational goals:

(1) Associate a tensor category $\mathbf{C}$ to a given noncommutative algebra A such that fundamental homological and ring-theoretic properties of A are reflected in $\mathbf{C}$;
(2) Develop the tools for determining Balmer spectra of stable categories associated to abelian tensor categories.

In classical noncommutative projective algebraic geometry, the noncommutative projective scheme associated to a connected graded algebra is given by a quotient of its category of graded modules, playing the role of coherent sheaves on the (nonexistent) noncommutative scheme. Unlike in the commutative case, this category is not endowed with an intrinsically defined tensor product. To associate tensor categories to these module categories, we use the universal quantum groups defined by Manin [4]. If A is a connected graded algebra which is locally finite, then there is a Hopf algebra, denoted $\underline{\mathrm{aut}}(A)$, which coacts on A universally. We call two connected graded algebras A and B *quantum-symmetrically equivalent* if there is an equivalence of tensor categories $\mathrm{comod}(\underline{\mathrm{aut}}(A)) \cong \mathrm{comod}(\underline{\mathrm{aut}}(B))$, which sends A to B.

Some of the primary objects of study in noncommutative algebraic geometry are Artin–Schelter regular algebras, which are defined by homological and growth conditions resembling those of polynomial rings [1]. They play the role of noncommutative projective spaces. Classifying Artin–Schelter regular algebras has been a difficult problem over the past thirty years. We prove how the Artin–Schelter regular property behaves under quantum-symmetric equivalence.

Theorem 1 ([3])**.** *If A is a Koszul Noetherian Artin–Schelter regular algebra and B is quantum-symmetrically equivalent to A, then B is Artin–Schelter regular as well. If C is an Artin–Schelter regular algebra with the same Hilbert series as A, then A and C are quantum-symmetrically equivalent.*

The existence of quantum-symmetric equivalences in this theorem uses a foundational result of Radschaelders–Van den Bergh [6]. It remains an open question to fully determine the quantum-symmetric equivalence class of an Artin–Schelter regular algebra.

Given a finite tensor category $\mathbf{C}$, which is not necessarily symmetric or braided, we would like to determine the Balmer spectrum of its stable category $\underline{\mathbf{C}}$, thus recovering geometry from abstract tensor categories in the spirit of Balmer. To do this, we define a version of cohomological support varieties, which we call *central support*, relative to the *categorical center* of $\underline{\mathbf{C}}$, which consists of morphisms in the extended endomorphism ring of $\underline{\mathbf{C}}$ which tensor-commute with morphisms id_S for simple objects S of $\mathbf{C}$. Using these versions of support varieties, we are able to prove the following result.

Theorem 2 ([5]). *There is a continuous map of topological spaces $\rho : \mathrm{Spc}\,\underline{\mathbf{C}} \to \mathrm{Proj}\,C$, where C is the categorical center of $\underline{\mathbf{C}}$. If a weak finite generation condition holds, then ρ is surjective. If additionally the central support satisfies a tensor product property and extends to the big category $\underline{\mathrm{Ind}(\mathbf{C})}$, then ρ is a homeomorphism.*

The map ρ is a noncommutative analogue of the comparison map produced in [2] in the symmetric case. We conjecture that the map ρ is a homeomorphism for every finite tensor category.

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Costratification and actions of tensor-triangulated categories

Charalampos Verasdanis

Let T be a rigidly-compactly generated tensor-triangulated cateogry and let K be a comapctly generated triangulated category. An *action* of T on K is a coproduct-preserving triangulated functor $- * -\colon \mathsf{T} \times \mathsf{K} \to \mathsf{K}$ that satisfies a host of certain coherence conditions; see [4]. Let X be an object of T. Brown representability implies that the functor $X * -\colon \mathsf{K} \to \mathsf{K}$ has a right adjoint $[X, -]_*\colon \mathsf{K} \to \mathsf{K}$ called the *relative internal-hom*. We are interested in studying the *colocalizing* hom-*submodules* of K, i.e., those triangulated subcategories C of K that are closed under products and $[X, A]_* \in \mathsf{C}$, $\forall X \in \mathsf{T}$, $\forall A \in \mathsf{K}$. This leads us to the notions of cosupport and costratification, which we will explain after providing some motivation. It should be noted that for $\mathsf{K} = \mathsf{T}$ and $* = \otimes$, all concepts developed in this context specialize back to classical tensor-triangular geometry.

Motivation. (i) The main motivational example that leads one to develop the theory of cosupport and costratification in this more general setting, as described above, is the following: Let R be a commutative noetherian ring. The *singularity category* of R is $\mathrm{S}(R) = \mathrm{K}_{\mathrm{ac}}(\mathrm{Inj}R)$ the homotopy category of acyclic complexes of injective R-modules. By [2], $\mathrm{S}(R)$ is a compactly generated triangulated category. However, $\mathrm{S}(R)$ is not a tensor-trinagulated category and so it is not immediately clear how one can use support (resp. cosupport) theory to study the localizing (resp. colocalizing) subcategories of $\mathrm{S}(R)$. Nevertheless, there is an action of the derived category $\mathrm{D}(R)$ on $\mathrm{S}(R)$ defined as follows: if $X \in \mathrm{D}(R)$ and $A \in \mathrm{S}(R)$, then $X * A = \widetilde{X} \otimes_R A$, where $\widetilde{X}$ is a K-flat resolution of X. In the case where R is a locally hypersurface ring, it was proved in [5] that the localizing subcategories of $\mathrm{S}(R)$ stand in bijection with the subsets of $\mathrm{Sing}(R)$ the singular locus of R. This result was obtained by using a support for objects of $\mathrm{S}(R)$ induced by the action of $\mathrm{D}(R)$ on $\mathrm{S}(R)$. We are interested in a similar theorem concerning the colocalizing subcategories of $\mathrm{S}(R)$.

(ii) The second example that motivated this work is the following: Let R be a commutative noetherian ring. In [3] it was proved that the colocalizing subcategories of $\mathrm{D}(R)$ stand in bijection with the subsets of $\mathrm{Spec}(R)$. We generalize this result to derived categories of schemes. As we shall see, this is attained by reducing costratification to certain smashing localizations.

Cosupport and costratification. Assume that every point of $\mathrm{Spc}(\mathsf{T}^c)$ the Balmer spectrum of T is visible (so that the Balmer-Favi idempotents g_p are defined) and let A be an object of K. The *cosupport* of A is $\mathrm{Cosupp}(A) = \{p \in \mathrm{Spc}(\mathsf{T}^c) \mid [g_p, A]_* \neq 0\}$. We have maps:

$$\{\text{colocalizing hom-submodules of } \mathsf{K}\} \underset{\tau}{\overset{\sigma}{\rightleftarrows}} \{\text{subsets of } \mathrm{Spc}(\mathsf{T}^c)\},$$

where $\sigma(\mathsf{C}) = \bigcup_{A \in \mathsf{C}} \mathrm{Cosupp}(A)$ and $\tau(W) = \{A \in \mathsf{K} \mid \mathrm{Cosupp}(A) \subseteq W\}$.

Definition. We say that K is *costratified* if the maps σ and $\tau|_{\mathrm{Im}\sigma}$ are mutually inverse bijections.

In order to verify costratification in practise, it is easier to check the validity of two conditions that we introduce next.

Let A be a collection of objects of K. Then $\mathrm{coloc}^{\mathrm{hom}}(\mathsf{A})$ denotes the smallest colocalizing hom-submodule of K that contains A. Also, let I denote the cogenerator of K that is the product of the Brown–Comenetz duals of the compact objects of K.

Definition.

(a) K satisfies the *colocal-to-global principle* if

$$\mathrm{coloc}^{\mathrm{hom}}(A) = \mathrm{coloc}^{\mathrm{hom}}([g_p, A]_* \mid p \in \mathrm{Spc}(\mathsf{T}^c)),\ \forall A \in \mathsf{K}.$$

(b) K satisfies *cominimality* if $\mathrm{coloc}^{\mathrm{hom}}([g_p, I]_*)$ is minimal, for all $p \in \mathrm{Spc}(\mathsf{T}^c)$.

In the case $\mathsf{K} = \mathsf{T}$ and $* = \otimes$, the following theorems were also obtained independently in [1].

Theorem 1 ([6]). *The following are equivalent:*

(a) K *is costratified.*
(b) K *satsfies the colocal-to-global principle and cominimality.*

Theorem 2 ([6]). *Suppose that* $\mathrm{Spc}(\mathsf{T}^c) = \bigcup_{i \in I} U_i$ *is a cover of the Balmer spectrum by complements of Thomason subsets. Provided that* T *satisfies the colocal-to-global principle, the following are equivalent:*

(a) T *is costratified.*
(b) $\mathsf{T}(U_i)$ *is costratified, for all* $i \in I$.

Applications. Appealing to Theorem 1 and Theorem 2 and Neeman's classification of colocalizing subcategories of $\mathrm{D}(R)$ for a commutative noetherian ring R, one obtains the following result:

Theorem 3 ([6]). *Let* X *be a noetherian separated scheme. Then* $\mathrm{D}(X)$ *the derived category of quasi-coherent sheaves on* X *is costratified.*

Finally, we have the following result that will appear in a future paper.

Theorem 4. *Let* R *be a locally hypersurface ring. Then* $\mathrm{S}(R)$ *the singularity category of* R *is costratified; there is a bijection between the collection of colocalizing subcategories of* $\mathrm{S}(R)$ *and subsets of* $\mathrm{Sing}(R)$.

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On Balmer spectra of Voevodsky and Morel-Voevodsky categories

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(joint work with Peng Du)

The difference in complexity between algebraic geometry and topology is apparent from the comparison of "atomic objects" in both worlds. While in topology there is only one kind of a point, in algebraic geometry there are many types of them – spectra of various field extensions of the ground field k. This is reflected in the structure of the respective Balmer spectra.

Our approach is based on *isotropic realisations.* In the case of $DM(k)$, such realisations are parametrized by a choice of a prime number p and an extension E/k.

The isotropic motivic category $DM(k/k;\mathbb{F}_p)$ introduced in [2] is the localisation of $DM(k)$ by the subcategory generated by motives of p-anisotropic varieties, i.e varieties having no points of degree prime to p. If the field k is *flexible* (that is, a purely transcendental extension of infinite transcendence degree of some other field), then isotropic category will be really handy. Namely, as in the global case, on the compact part of this category there is a *weight structure* in the sense of Bondarko [1] whose heart is the category $Chow(k/k;\mathbb{F}_p)$ of *isotropic Chow motives*. Objects here are direct summands of isotropic motives of smooth projective varieties, while morphisms are given by *isotropic Chow groups* Ch^*_{iso}. The latter groups are obtained from the usual Chow groups by moding out *anisotropic classes*, that is, elements coming from p-anisotropic varieties. These naturally surject to numerical Chow groups Ch^*_{Num} (with $\mathbb{F}_p$-coefficients). It appears that over flexible fields both versions coincide.

Theorem 1. ([3, Theorem 1.2]) *Let k be flexible. Then* $\mathrm{Ch}^*_{iso} = \mathrm{Ch}^*_{Num}$.

This implies that the category of isotropic Chow motives over a flexible field is equivalent to the category of numerical Chow motives, which is semi-simple and where homs are finite groups. We get a family of *isotropic realisations* with values in flexible isotropic categories

$$\psi_{p,E} : DM(k) \to DM(\widetilde{E}/\widetilde{E};\mathbb{F}_p),$$

where E/k is a field extension and $\widetilde{E} = E(\mathbb{P}^\infty)$ is the *flexible closure* of E. Moreover, it is sufficient to consider extensions only up to $\overset{p}{\sim}$-equivalence relation, where $E/k \overset{p}{\sim} F/k$ iff the p-anisotropy of k-varieties is equivalent over E and F. Let $\mathfrak{a}_{p,E} := \ker(\psi^c_{p,E})$. With the help of Theorem 1 one obtains:

Theorem 2. ([3, Theorem 1.3])

(1) *The ideal $\mathfrak{a}_{p,E}$ of $DM^c(k)$ is prime and so, defines a point of the Balmer spectrum* $\mathrm{Spc}(DM^c(k))$.

(2) $\mathfrak{a}_{p,E} = \mathfrak{a}_{q,F} \quad\Leftrightarrow\quad p = q$ *and* $E/k \overset{p}{\sim} F/k$.

This provides a large supply of new points of the Balmer spectrum $\mathrm{Spc}(DM^c(k))$.

A similar technique may be applied to the study of the Balmer spectrum of $SH(k)$. Here we need to substantially generalize the notion of anisotropy, extending it to any oriented cohomology theory A^* on Sm_k. We say that a smooth projective variety $X \xrightarrow{\pi} \mathrm{Spec}(k)$ is *A-anisotropic*, if the push-forward map $\pi_* : A_*(X) \to A_*(\mathrm{Spec}(k)) = A$ is zero.

The theories we need are Morava K-theory $K(p,n)^*$ and $P(m)^*$-theory. Both of them are obtained from BP^*-theory (and so, also from the algebraic cobordism Ω^* of Levine-Morel), by change of coefficents: $K(p,m)^* = BP^* \otimes_{BP} \mathbb{F}_p[v_m, v_m^{-1}]$ and $P(m)^* = BP^*/I(m)$, where $I(m) = (p, v_1, \ldots, v_{m-1}) \subset BP$ is the invariant ideal of Landweber. Here $1 \leq m \leq \infty$ and $P(\infty)^* = K(p,\infty)^* = \mathrm{Ch}^* = \mathrm{CH}^*/p$. For any oriented cohomology theory A^* one may introduce the isotropic A^*_{iso} and numerical A^*_{Num} versions. The following vast generalization of Theorem 1 was proven in [3].

Theorem 3. ([3, Theorem 1.4]) *Let k be flexible. Then, for any $1 \leq m \leq \infty$, we have:*

(1) $P(m)^*_{iso} = P(m)^*_{Num}$;
(2) $K(p,m)^*_{iso} = K(p,m)^*_{Num}$.

This allows to construct isotropic variants of topological $\mathfrak{a}_{p,m}$-points. This was done in our joint paper with Peng Du [4]. For any prime p and any $1 \leq m \leq \infty$, we introduce the *Morava-isotropic stable homotopy category* $SH_{(p,m)}(k/k)$ as a localisation of the Morel-Voevodsky category $SH(k)$ by the subcategory generated by those compact objects U whose MGL-motive belongs to the subcategory of $MGL(k)-mod$ generated by the MGL-motives of $K(p,m)$-anisotropic varieties and $\mathbb{1}^{MGL}/v_m$. We get a family of *isotropic realisations*

$$\psi_{(p,m),E} : SH(k) \to SH_{(p,m)}(\widetilde{E}/\widetilde{E})$$

taking values in flexible Morava-isotropic stable homotopy categories, where as above, only the $K(p,m)$-equivalence class of an extension E/k matters. Here $E/k \stackrel{(p,m)}{\sim} F/k$ iff the $K(p,m)$-isotropy of k-varieties is equivalent over E and F. Denote as $\mathfrak{a}_{(p,m),E}$ (the compact part of) the kernel of $\psi_{(p,m),E}$. Theorem 3 permits to show that the zero ideals of the targets of these realisations are prime. Moreover, we obtain:

Theorem 4. ([4, Theorem 1.1])

(1) *The ideal $\mathfrak{a}_{(p,m),E}$ of $SH(k)$ is prime and so, defines a point of the Balmer spectrum* $\mathrm{Spc}(SH^c(k))$.
(2) $\mathfrak{a}_{(p,m),E} = \mathfrak{a}_{(q,n),F} \quad \Leftrightarrow \quad p = q,\ m = n$ *and* $E/k \stackrel{(p,m)}{\sim} F/k$.
(3) *The point $\mathfrak{a}_{(p,\infty),E}$ is the image of $\mathfrak{a}_{p,E}$ under the natural map* $\mathrm{Spc}(DM^c(k)) \to \mathrm{Spc}(SH^c(k))$ *of Balmer spectra.*

We obtain many new points of the Balmer spectrum. For example, for the field of real numbers $\mathbb{R}$, we get $2^{2^{\aleph_0}}$ new isotropic points $\mathfrak{a}_{(2,m),E}$, for every topological point $\mathfrak{a}_{(2,m),Top}$, $1 \leq m \leq \infty$, of characteristic two of the spectrum $\mathrm{Spc}(SH(\mathbb{R}))$. In particular, the cardinality of the Balmer spectrum is equal to the cardinality of the set of all subsets of $SH^c(\mathbb{R})$. The interesting feature is that the specialisation relations among isotropic points don't follow the topological pattern. The obstructions are provided (in particular) by the norm-varieties of Rost.

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Support theories for non-Noetherian tensor triangulated categories

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We will explain how the support theory developed by Benson, Iyengar, and Krause can be constructed without the Noetherian hypothesis. A key ingredient is a new notion of small support for modules over commutative rings that are not necessarily Noetherian, which depends on the notion of weakly associated prime. For a tensor triangulated category, we establish relations between the canonical BIK support and the tensor triangular support defined by Sanders, which generalizes the Balmer–Favi support. With these notions of support, we develop the associated stratification theories and show that the stable module category of a finite group over a field is canonically stratified by the Tate cohomology ring. With the support defined by Sanders we show that the p-local stable homotopy category satisfies the detection property. As a consequence, the non-zero dissonant spectra are exactly the spectra whose support is the single point at infinity in the Balmer spectrum.

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