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Aspects of Aperiodic Order

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Abstract. The theory of aperiodic order expanded and developed significantly since the discovery of quasicrystals, and continues to bring many mathematical disciplines together. The focus of this workshop was on harmonic analysis and spectral theory, dynamical systems and group actions, Schrödinger operators, and their roles in aperiodic order – with links into a full range of problems from number theory to operator theory.

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Introduction by the Organizers

In the last 10 years, the theory of aperiodic order has seen a rapid development, and expanded significantly, both in depth and breadth. Simultaneously, the field has continued to attract mathematicians from various directions, and to provide a stimulating environment for cross-discipline cooperations.

Nowadays, a rather central role is taken by dynamical systems theory, which features in tiling theory, spectral theory of Schrödinger operators, group actions on Cantor spaces, harmonic analysis, and number theory – to mention some of the key topics of our meeting.

The workshop was organised within the standard scheduling preferences of the MFO. Monday saw a collection of introductory and survey talks on the main research topics. This helped to get the discussions and collaborations going, which all the participants longed for and enjoyed after several years without enough personal contacts. We kept the mix of themes also on the other days to support our perspective and the strong unifying potential of the field, as we now summarise.

Boris Solomyak discussed substitution tilings, and provided a sufficient condition for the spectrum of a primitive substitution to be purely singular via the spectral cocycle. Together with the parallel development for singular continuous diffraction measures by other groups, this gives a promising entry point for a future classification of this important measure class.

Jake Fillman gave an introduction to the general theory of ergodic Schrödinger operators, with a special emphasis on how models that are relevant in the study of aperiodic order arise in this framework. He then surveyed many of the known results and discussed some of the central open problems in this area. It is noteworthy that the gap between physical expectations and mathematical results is slowly narrowing, but is still substantial for dimensions two and higher.

There was an introductory talk by Christoph Richard on the Fourier analysis of translation-bounded measures via the theory of mild distribution. This is an alternative approach to diffraction theory, which simplifies the Fourier-analytic aspects of the theory considerably. In particular, it has many aspects that are similar to the theory of tempered distributions, but has the significant advantage of working in the general framework of locally compact Abelian groups.

Neil Mañibo introduced the audience to substitutions on compact alphabets. He showed that, under mild assumptions, many properties of finite alphabet substitutions (such as minimality, unique ergodicity, natural tile length) can be extended to this general setting. He showed constructively that every $\lambda > 2$ is the inflation multiplier of some substitution on a compact alphabet. In particular, there exist substitution tilings over compact alphabets with transcendental inflation factor, which is impossible for finite alphabets.

A general talk by Michael Coons covered scaling properties of the diffraction measure around the origin for some models such as period doubling, Rudin–Shapiro and Thue–Morse. The notion of a spectral measure (as in the Thue–Morse case) has an analogue for the Stern sequence, and was later generalised to regular sequences, opening a road to a spectral classification that complements existing approaches.

Philipp Gohlke discussed in detail the scaling properties of the Thue–Morse measure and the odd behaviour of its decay at certain locations. The key to the progress emerges from an extension of g-measures with a bounded potential to potentials with a singularity, and a careful analysis of the consequences.

The connection to number theory, which already appeared in Michael Coons' talk, featured predominantly in the talks by Sebastián Donoso, Alan Haynes and Olga Lukina. Donoso discussed ergodic averages for multiplicative actions, in particular for the multiplicative action of $\mathbb{Z}[\mathrm{i}]$, with applications to number theory. Alan Haynes discussed the accumulation points of normalised integer translates of rotations in Euclidean space, and their odd behaviour for some particular choices of the fixed point $x \in \mathbb{R}^d$. Lukina showed particular properties of groups of automorphisms of trees that appear as the arboreal representations of polynomials.

Johannes Kellendonk delved into the relationship between the non-tameness property of a dynamical system and the complexity of its Ellis semigroup, identifying a characteristic dichotomy that is practically approachable. Later, Reem Yassawi showed a connection between some algebraic properties of the Ellis semigroup of constant-length substitutions and their Toeplitz-like factors. Here, Yassawi exploited the structure of the semigroup generated by the columns and known factorisations using collaring.

Motivated by the work of Furstenberg, Bryna Kra introduced the notion of chaotic almost minimal systems (CAM). A dynamical system (X, G) given by a continuous action of G on the compact metric space X is CAM if it is transitive, faithful, X has a dense set of G-periodic points, and every proper closed G-invariant subset of X is invariant. Mixing $\mathbb{Z}$-subshifts of finite type are examples of CAM systems. Kra has shown that there are restrictions on phase spaces and groups which admit a CAM system. For example, if X is locally connected, X does not support expansive CAM $\mathbb{Z}$-systems. On the other side, if G admits a CAM action, G is residually finite, and it is still an open question whether any residually finite group admits actions of this type.

Samuel Petite precisely described the self-isomorphisms of $\mathbb{Z}^n$-odometers and some substitution $\mathbb{Z}^n$-subshifts. Formulated in terms of centralisers and normalisers of the group action within the homeomorphism group of the space, the discovery of new cases with large normaliser showed that this recently observed phenomenon deserves further study.

Felipe García-Ramos explained that it is impossible to decide, with countably many arguments, whether two minimal Cantor systems are conjugate. For that, he studied the notion of Borel reduction for equivalence relations of homeomorphisms on the Cantor sets. Using these tools, García-Ramos has shown that the conjugacy equivalence classes of minimal homeomorphism on the Cantor set are not Borel, which can be interpreted in terms of decidability.

Tobias Jäger discussed some characterisations of tame systems, showing that tame minimal group actions, which are almost 1-1 extensions of their maximal equicontinuous factors, are necessarily regular systems. His approach is constructive, and relies on linking irregularity to the existence of an independence pair.

Felix Pogorzelski exhibited examples of linearly repetitive Delone sets on the Heisenberg group. He also showed that such (coloured) Delone sets provide suitable models for Schrödinger potentials on the Heisenberg group, whose spectra can be studied using periodic approximants.

Paulina Cecchi-Bernales discussed the realisation of simplices of invariant measures for minimal subshifts given by the actions of general amenable groups that are not necessarily residually finite. Cecchi-Bernales also showed an analogous result for Toeplitz subshifts with actions of residually finite groups that are not necessarily amenable.

When studying Schrödinger operators with aperiodically ordered potentials, one is interested in the structure of the spectrum and the type of the spectral measures. A good amount of information is available in one space dimension, but some important and interesting questions still remain open.

One of them was recently solved in work presented by Siegfried Beckus, joint with Ram Band and Raphael Loewy, on the gap structure of the spectra of Sturmian Hamiltonians. Extending work of Laurent Raymond, who had proved that all gaps allowed by the gap labelling theorem are open whenever the coupling constant obeys $\lambda > 4$, they showed that this result holds on the full range $\lambda > 0$.

Quantum graphs based on Sturmian sequences were discussed in Ram Band's lecture. As one can employ trace map methods, originally developed for discrete Schrödinger operators in $\ell^2(\mathbb{Z})$, in this context as well, they provide examples of quantum graphs for which quite a detailed spectral analysis is possible and whose properties are novel phenomena in the quantum graph setting.

Anton Gorodetski discussed the following natural question: how do the spectral properties of a Schrödinger operator with aperiodically ordered potential change when a (small) random perturbation is added? The general expectation is that the randomness should dominate any deterministic background, and as consequences, the spectral gaps should no longer be dense and the spectral measures should now be pure point (with exponentially decaying eigenfunctions). Such results have indeed been established recently under suitable assumptions; the first statement is due to Avila–Damanik–Gorodetski and the second is due to Gorodetski–Kleptsyn.

Since the proposal was submitted, various new developments took place that we did not anticipate. Among them was the mathematically rigorous analysis of (localised) eigenfunctions in aperiodic Schrödinger operators on Penrose and Ammann–Beenker tilings, and the discovery of new families of aperiodic monotiles for the Euclidean plane. Each of these two topics was covered by an informal evening session, with several shorter contributions and ample time for discussions.

The evening session on Tuesday featured contributions by May Mei, Mark Embree, and Jan Mazáč. These presentations discussed the presence of finitely supported eigenfunctions of the Laplacian on Penrose and Ammann–Beenker tilings. Due to the linear repetitivity of these tilings, such finitely supported eigenfunctions will cause discontinuities of the integrated density of states, occurring at the eigenvalues corresponding to these eigenfunctions. In order to understand the sizes of these jumps, one needs to understand the frequencies of the patches that support the eigenfunctions. Mei and Embree explained the computational methods that allowed for an identification of finitely supported eigenfunctions with rather large supports, while Mazáč explained how the frequencies of their supports can be determined effectively and precisely.

The monotile session on Thursday evening began with a summary of the problem up to the present state of the art, by Jamie Walton, followed by an in-depth analysis of the Hat and the Spectre tilings (Franz Gähler), which also proved that these tilings have pure point spectrum and can be obtained by the projection method. The new feature is the quasiperiodicity of the Hat and the Spectre, in contrast to the limit-periodicity of previous examples such as the Taylor–Socolar monotile. An interpretation via translation surfaces (by Pierre Arnoux) and an independent aperiodicity proof (by Shigeki Akiyama) concluded the presentations, which were followed by a lively discussion.

The enthusiastic closing lecture on Friday afternoon, by Tobias Hartnick, developed a robust framework for models of aperiodic order, inspired by insight from point process theory. It showed both the potential and the maturity of the field, and gave all participants a wonderful perspective to consider on their way home.

In summary, all participants enjoyed the inspiring in-person atmosphere of the workshop. Those who returned from previous visits underlined the importance and irreplaceability of such meetings in the perfect working environment of the MFO, while the newcomers, of which there were many, were immediately hooked by the place and its special flair. All agreed that the MFO might have been copied many times, but it retains its role as one of the best places for an interactive research workshop – a point of view all organisers thankfully adopt as well.

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Workshop: Aspects of Aperiodic Order

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Abstracts

Which substitutions have pure singular spectrum?

Boris Solomyak
(joint work with Alexander Bufetov)

The talk is based on [5], with some additions from the work by Yaari [7], which was a part of his Master's Thesis at the Bar-Ilan University. The emphasis is on substitutions of non-constant length, since the constant length case, although still mysterious, is somewhat better understood from the spectral viewpoint. Our main tool is the *spectral cocycle*, introduced in [4].

Given a primitive aperiodic substitution ζ on the alphabet $\mathcal{A} = \{1, \ldots, d\}$, let $\zeta(b) = u_1^{(b)} \ldots u_{k_b}^{(b)}$, with $b \in \mathcal{A}$. The *Fourier matrix function* $\mathscr{M}_\zeta : \mathbb{R}^d \to M_d(\mathbb{C})$ (the space of complex $d \times d$ matrices), is defined by the formula, for $\boldsymbol{x} = (x_1 \ldots, x_d)$:

$$[\mathscr{M}_\zeta(\boldsymbol{x})]_{(b,c)} := \Big(\sum_{j \leqslant k_b,\ u_j^{(b)} = c} \exp\big(-2\pi i \sum_{k=1}^{j-1} x_{u_k^{(b)}}\big)\Big)_{(b,c) \in \mathcal{A}^2}, \quad \boldsymbol{x} \in \mathbb{R}^d.$$

Since $\mathscr{M}_\zeta$ is $\mathbb{Z}^d$-periodic, we obtain a continuous matrix function on the torus $\mathbb{T}^d$.

Note that $\mathscr{M}_\zeta(0) = \mathsf{S}_\zeta^\mathsf{T}$, where S_ζ is the substitution matrix. The entries of the matrix $\mathscr{M}_\zeta(\boldsymbol{x})$ are trigonometric polynomials with coefficients 0 and 1, and they are less than or equal to the corresponding entries of $\mathsf{S}_\zeta^\mathsf{T}$ in absolute value for every $\boldsymbol{x} \in \mathbb{T}^d$. Crucially, the following *cocycle condition* holds: for any substitutions ζ_1, ζ_2 on $\mathcal{A}$,

$$\mathscr{M}_{\zeta_1 \circ \zeta_2}(\boldsymbol{x}) = \mathscr{M}_{\zeta_2}(\mathsf{S}_{\zeta_1}^\mathsf{T} \boldsymbol{x}) \mathscr{M}_{\zeta_1}(\boldsymbol{x}),$$

which is verified by a direct computation. Suppose that $\det(\mathsf{S}_\zeta) \neq 0$ and consider the endomorphism of the torus $\mathbb{T}^d$ defined by

$$\boldsymbol{x} \mapsto \mathsf{S}_\zeta^\mathsf{T} \boldsymbol{x} \pmod{\mathbb{Z}^d}, \tag{1}$$

which preserves the Haar measure m_d. Then,

$$\mathscr{M}_\zeta(\boldsymbol{x}, n) := \mathscr{M}_\zeta\big((\mathsf{S}_\zeta^\mathsf{T})^{n-1} \boldsymbol{x}\big) \cdots \mathscr{M}_\zeta(\boldsymbol{x}),$$

is a complex matrix cocycle over the endomorphism (1), which we call the *spectral cocycle*, associated to ζ. Assuming that this endomorphism is ergodic, which is equivalent to S_ζ having no eigenvalues that are roots of unity, the following limit exists and is constant m_d-a.e. by the Furstenberg–Kesten theorem:

$$\chi_\zeta = \lim_{n \to \infty} \frac{1}{n} \log \|\mathscr{M}_\zeta(\boldsymbol{x}, n)\|.$$

It is called the *top Lyapunov exponent* of the spectral cocycle and denoted by χ_ζ. On the other hand, for any point $\boldsymbol{x} \in \mathbb{T}^d$, there exists the local upper Lyapunov exponent $\chi_{\zeta,\boldsymbol{x}}^+$, defined in the same way, just with lim sup. Let θ be the Perron–Frobenius eigenvalue of S_ζ. Then $\chi_{\zeta,\boldsymbol{x}}^+(0) = \log \theta$, by the definition and the Perron–Frobenius theorem. Interestingly, one always has $\chi_\zeta \leqslant \frac{1}{2} \log \theta$. There is a more direct formula for the Lyapunov exponent χ_ζ, which follows from Kingman's theorem and makes it possible to obtain rigorous numerical estimates,

$$\chi_\zeta = \inf_k \frac{1}{k} \int_{\mathbb{T}^d} \log \|\mathscr{M}_{\zeta^k}(\boldsymbol{x})\| \, \mathrm{d}m_d(\xi).$$

Now consider the tiling $\mathbb{R}$-action, associated with ζ, determined by a positive vector of tile lengths $\vec{s} \in \mathbb{R}^d$. We showed in [4] that if $\chi^+_{\zeta,\omega\vec{s}} < \frac{1}{2}\log\theta$ for Lebesgue-a.e. $\omega \in \mathbb{R}$, the tiling flow has pure singular spectrum. However, this condition is not easy to check for specific systems. The following was proved in [5]:

Theorem 1. *Let ζ be a primitive aperiodic substitution on $\mathcal{A} = \{0, \dots, d-1\}$, with $d \geqslant 2$, such that the substitution matrix S_ζ has a characteristic polynomial that is irreducible over $\mathbb{Q}$. Let θ be the Perron–Frobenius eigenvalue of S_ζ. If*

$$\chi_\zeta < \tfrac{1}{2}\log\theta, \tag{2}$$

the substitution $\mathbb{Z}$-action has pure singular spectrum.

The idea is to apply a theorem of Host [6], which implies that under our conditions, for Lebesgue-a.e. ω, the orbit of the point on the diagonal of the torus $\omega\vec{1}$ under the endomorphism $\xi \mapsto \mathsf{S}_\zeta^\mathsf{T}\xi \pmod{\mathbb{Z}^d}$ equidistributes, and then one can show that (2) implies $\chi^+_{\zeta,\omega\vec{1}} < \frac{1}{2}\log\theta$ for a.e. ω. It is well known that the suspension flow over a $\mathbb{Z}$-action, with a constant height 1 (this corresponds to $\vec{s} = \vec{1}$), has the same spectral properties as the $\mathbb{Z}$-action.

Earlier, Baake and collaborators [1–3] obtained somewhat similar results for the (closely related) diffraction spectrum of the self-similar tiling flow, which one gets by taking $\vec{s} = \boldsymbol{e}_1$, the PF eigenvalue of the matrix $\mathsf{S}_\zeta^\mathsf{T}$. Then,

$$\mathscr{M}_\zeta(\omega\mathbf{e}_1, n) := \mathscr{M}_\zeta\big(\theta^{n-1}\omega\mathbf{e}_1\big) \cdot \ldots \cdot \mathscr{M}_\zeta(\theta\,\omega\mathbf{e}_1) \cdot \mathscr{M}_\zeta(\omega\mathbf{e}_1) \tag{3}$$

is a cocycle on $\mathbb{R}$ over the infinite-measure preserving action $\omega \mapsto \theta\omega$. In particular, Baake, Gähler, and Mañibo [3, Thm. 3.28] proved that if the upper Lyapunov exponent satisfies the condition $\chi^+_{\zeta,\omega\mathbf{e}_1} < \frac{1}{2}\log\theta - \epsilon$, for some $\epsilon > 0$, for Lebesgue-a.e. $\omega \in \mathbb{R}$, then the absolutely continuous component of the diffraction measure must be trivial, hence it is purely singular. (Their method is different, although there are some common features with our approach.) This allowed the authors of [1–3] to prove singularity of the spectrum for many examples of non-Pisot self-similar tiling flows. Now, we obtain singularity of the corresponding $\mathbb{Z}$-actions as well.

Yaari [7] found an alternative proof of Theorem 1 and extended it to the case of reducible and singular matrices, with appropriate modifications, using uniform distribution theory. This allowed him to simultaneously obtain results on singularity of substitution $\mathbb{Z}$-actions and $\mathbb{R}$-actions for any $\vec{s}$ under the assumptions of Theorem 1.

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Aspects of of Schrödinger operators whose potentials exhibit aperiodic order

Jake Fillman

The spectral theory of Schrödinger operators with ergodic dynamically defined potentials is a vast subject with connections to many fields of mathematics, including group theory, topology, harmonic analysis, fractal geometry, and others. The talk surveys some important background information and results for these operators in order to facilitate discussions among the workshop participants. The content of the talk is based on the recent book [2]; see also the books [1, 3].

The basic setup is given by the following data: given an ergodic topological dynamical system (Ω, T, μ) and a bounded Borel measurable function $v : \Omega \to \mathbb{R}$, one defines the *ergodic family* $\{H_\omega\}_{\omega\in\Omega}$ by $H_\omega = \Delta + V_\omega$, where

$$V_\omega(n) = v(T^n\omega). \tag{4}$$

More precisely, for each point $\omega \in \Omega$, one obtains a corresponding linear operator $H_\omega : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ via

$$[H_\omega\psi](n) = \psi(n-1) + V_\omega(n)\psi(n) + \psi(n+1). \tag{5}$$

Since v was assumed to be bounded and real-valued, each H_ω is a bounded self-adjoint operator on the Hilbert space $\ell^2(\mathbb{Z})$. Studying the operators $\{H_\omega\}_{\omega\in\Omega}$ as a family allows one to leverage ideas and techniques from ergodic theory and dynamical systems. For instance, there is a fixed compact set $\Sigma \subset \mathbb{R}$ with the property that

$$\mathrm{spec}(H_\omega) = \Sigma, \quad \mu\text{-a.e. } \omega \in \Omega. \tag{6}$$

This talk surveys some important special cases of the framework above that produce potentials exhibiting aperiodic order. Significant examples include random sequences, Sturmian sequences, irrational circle rotations, and subshifts generated by primitive aperiodic substitutions.

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On non-tame Ellis semigroups

Johannes Kellendonk

We are interested in dynamical systems which are close to equicontinuous but whose Ellis semigroup has a cardinality which is bigger than that of the continuum (it is not tame). By close to equicontinuous we mean that the factor map onto the maximal equicontinuous factor has at least one finite fibre. We investigate the question which algebraic component of the Ellis semigroup has such a big cardinality. Indeed, like any semigroup, $E(X,T)$ has an ideal structure and can be decomposed into the equivalence classes of Green's relations and so it is interesting to know which of these parts are especially big.

Let (X,d) be a compact metric space with an action of a group T by homeomorphisms. Let $F(X)$ be the set of functions from X to X, equipped with the topology of point-wise convergence. Then, $F(X)$ is a compact right-topological semigroup. The *Ellis* (or enveloping) *semigroup* $E(X,T)$ of (X,T) is the closure in $F(X)$ of the group of homeomorphisms $\{\alpha^t : t \in T\}$ coming from the action α of T. The Ellis semigroup has rich algebraic and topological properties, and these can be used to characterise the dynamical system. Its elements capture the notion of proximality. Indeed, two points $x,y \in X$ are called *proximal* if $\inf_{t\in T} d(\alpha^t(x),\alpha^t(y)) = 0$ and this is the case if and only if $f(x) = f(y)$ for some $f \in E(X,T)$.

One property which has recently attracted a lot of attention is tameness. The semigroup $E(X,T)$ (and (X,T)) is called *tame* if all its elements are Baire class 1 functions, that is, can be obtained as a limit of a sequence of continuous functions. An equivalent characterisation is that $E(X,T)$ is tame if its cardinality is at most c, that of the reals. A third characterisation is that $E(X,T)$ is not tame if X contains an independence sequence. A good review on enveloping semigroups and tameness is [1].

Recall that (X,T) is equicontinuous if the family of homeomorphisms coming from the action of T is equicontinuous. If T is abelian, this implies that X carries a group structure such that the action of T on it is given by left multiplication with elements of a subgroup of it. If (X,T) is also minimal, it carries a unique T-invariant probability measure, namely the Haar measure. Coming back to general dynamical systems (X,T), they have always a maximal equicontinuous factor $\pi : (X,T) \to (X_{max},T)$. A point $z \in X_{max}$ is called *singular* if the fibre $\pi^{-1}(z)$

contains two proximal points. Otherwise, it is called *regular*. A system is called *almost automorphic* if it is minimal and there is $z \in X_{max}$ such that $\pi^{-1}(z)$ contains a single point. This implies that the regular points are exactly the points which have a unique pre-image under π.

A recent result in [2] states that if a minimal system with abelian T is tame then it is almost automorphic and the set of regular points has full Haar measure in X_{max}. The converse need not be true. Toeplitz shifts are almost automorphic but not always tame, although their set of regular points has full measure [3]. For instance, of the shift dynamical systems associated to the two substitutions

$$\begin{array}{rcl} a & \mapsto & aabaa \\ b & \mapsto & abbaa \end{array} \qquad \begin{array}{rcl} a & \mapsto & aabaa \\ b & \mapsto & ababa \end{array}$$

the first is tame, whereas the second is not tame. Note that the substitutions differ only in the order of two letters, in particular their associated dynamical systems are strong orbit equivalent.

In the light of the question which parts of the Ellis semigroup of non-tame system are especially big, we investigate here when the so-called Ellis group $eE(X,T)e$ has cardinality $2^{\mathfrak{c}}$, that of the power set of the continuum. Here, e is a minimal idempotent of $E(X,T)$ and $eE(X,T)e = \{efe : f \in E(X,T)\}$, the Ellis group depending on e only up to isomorphism. This group is actually isomorphic to the Rees structure group of the kernel of $E(X,T)$. Below, we denote the group structure on X_{max} additively and the set of singular points of X_{max} by X_{max}^{sing}.

Theorem 1. *Let (X,T) be a minimal dynamical system with abelian group T. Suppose that there exists a regular point $z \in X_{max} \backslash X_{max}^{sing}$ with finite fibre $\pi^{-1}(z)$ and an uncountable set $A \subset X_{max}$ such that, for all $a \neq b \in A$, we have that $X_{max}^{sing} + a \cap X_{max}^{sing} + b = \varnothing$.*

(1) If (X,T) is almost automorphic, the Ellis group is isomorphic to X_{max}.

(2) If (X,T) is not almost automorphic, the Ellis group has cardinality $2^{\mathfrak{c}}$.

Note that the second condition of the theorem is always satisfied if there are only finitely many orbits of singular points in X_{max} while X_{max} is uncountable. If there is only one singular orbit, more can be said about $eE(X,T)e$; see [4]. For instance, if $(X_\theta, \mathbb{Z})$ is the shift dynamical system associated to a bijective substitution θ of length ℓ with trivial generalised height, then the maximal equicontinuous factor is the ℓ-adic odometer $\mathbb{Z}_\ell$, and $eE(X_\theta, \mathbb{Z})e$ is algebraically isomorphic to a semi-direct product of its fibre preserving part

$$\{f \in eE(X_\theta, \mathbb{Z})e : f(\pi^{-1}(z)) \subset \pi^{-1}(z), z \in \mathbb{Z}_\ell\}$$

with $\mathbb{Z}_\ell$. Furthermore, the fibre preserving part is topologically and algebraically isomorphic to the group of all functions $g : \mathbb{Z}_\ell/\mathbb{Z} \to G_\theta$ from the space of orbits to the group G_θ generated by the column maps of the substitution. Here, the group multiplication on the group of functions is point-wise and the topology is that of point-wise convergence. This tells us that, for any function $g : \mathbb{Z}_\ell/\mathbb{Z} \to G_\theta$, there is an element in the Ellis group which preserves the fibres of π and acts on $\pi^{-1}(z)$

by the permutation of the letters given by $g(z \bmod \mathbb{Z})$. The fact that any function is allowed leads to the large cardinality.

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Ergodic averages of multiplicative actions and their applications

Sebastián Donoso

(joint work with A.N. Le, J. Moreira, W. Sun)

The study of multiplicative functions plays a central role in number theory. A function $f\colon \mathbb{N} \to \mathbb{C}$ is *completely multiplicative* if $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{N}$. It is *multiplicative* if the previous relation holds for all $m, n \in \mathbb{N}$ that are coprime. One of the reasons why multiplicative functions have been widely studied is that their statistics have deep connections with properties of the prime numbers. For instance, the prime number theorem can be restated as

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \lambda(n) = 0\,, \tag{7}$$

where $\lambda : \mathbb{N} \to \{-1, 1\}$ is the Liouville function. It is defined as $\lambda(n) = (-1)^{\Omega(n)}$, where $\Omega(n)$ is the number of prime factors of n (with multiplicity). So, (7) states that the prime number theorem is equivalent to the Liouville function having zero mean.

A dynamical approach. Recall that a topological dynamical system (TDS) is a tuple (X, T), where X is a compact metric space and $T\colon X \to X$ is a homeomorphism. The TDS (X, T) is uniquely ergodic if it has only one invariant measure.

Theorem 1 (Bergelson–Richter [1])**.** *If (X, T) is a uniquely ergodic topological dynamical system, with unique invariant measure μ, then, for any $x \in X$ and any continuous function $f\colon X \to \mathbb{C}$,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^{\Omega(n)}x) = \int_X f d\mu\,.$$

Remark 2. *The condition of unique ergodicity of (X, T) and continuity of f are essential for this result to hold; compare* [6].

Theorem 1 is simple to state and has deep consequences. For instance, if (X, T) is a (cyclic) rotation on 2 points, Theorem 1 allows us to recover (7), that is, the prime number theorem. One could consider more complicated uniquely ergodic systems and obtain other number-theoretic results. For instance, for a finite (cyclic) group rotation, Theorem 1 allows us to recover a theorem by Pillai and Selberg, which states that $\Omega(n)$ is equally distributed over all residue classes mod q for all $q \in \mathbb{N}$. If we consider a system $(\mathbb{T}, T)$, where $T(x) = x + \alpha$ for an irrational α (that is, an irrational rotation on the circle), by taking $x = 0$ and $f(x) = e^{2\pi i k x}$, for $k \in \mathbb{Z}\setminus\{0\}$, Theorem 1 allows one to deduce that

$$\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k\Omega(n)\alpha} \to 0$$

This recovers a theorem by Erdős and Delange, which establishes that the sequence $(\Omega(n)\alpha)_{n\in\mathbb{N}}$ is uniformly distributed mod 1.

In our work [3], we are mainly interested in the multi-parameter case of Theorem 1. That is, what can we say if we replace $\Omega(n)$ by $\Omega(P(n,m))$ for some polynomial P in two variables? It is worth highlighting that averages like

$$\frac{1}{N}\sum_{n=1}^{N} \lambda(P(n)) \tag{8}$$

are very hard to deal with. For instance, it is still open (this is a conjecture by Chowla) whether for a polynomial $P \in \mathbb{Z}[x]$ such that $P \neq cQ^2$, for every $c \in \mathbb{Z}, Q \in \mathbb{Z}[x]$, the average (8) converge to 0 as N goes to infinity (note that the condition on P is to discard obvious cases). However, if one is allowed to introduce more variables, the corresponding averages seem to be more tractable. For instance, the corresponding multi-parameter version of (8),

$$\frac{1}{N^2}\sum_{m,n=1}^{N} \lambda(P(m,n)) \tag{9}$$

is known to have a zero limit for various P (see [3] and references therein). In [3], we obtain the following variant of Theorem 1.

Theorem 3. *Let (X, T) be a uniquely ergodic system with the unique invariant measure μ. Then, for any $x \in X$ and any $f \in C(X)$,*

$$\lim_{N\to\infty} \mathop{\mathbb{E}}_{1\leqslant m,n\leqslant N} f(T^{\Omega(m^2+n^2)}x) = \int_X f \, d\mu.$$

Of course, from Theorem 3, one can derive various natural conclusions. The following one is an example.

Theorem 4. *The multi-parameter sequence $\Omega(m^2 + n^2)_{n,m}$ is equally distributed over all residue classes mod q for all q. Moreover, if $Q \in \mathbb{R}[x]$ has at least one irrational coefficient, then $Q\big(\Omega(m^2 + n^2)\big)_{n,m}$ is uniformly distributed mod 1.*

Main ideas. Theorem 3 follows from a more general theorem, and uses the idea of multiplicative actions of Gaussian integers (such actions have been studied in the past in, for instance, [2, 4, 7]). Recall that $\mathbb{G} := \{m + in : m, n \in \mathbb{N}\}$ and $G^* = \mathbb{G}\backslash\{0\}$. Denote by $\mathbb{P}$ the set of primes in $\mathbb{G}$.

Theorem 5. *Let X be a compact metric space and T be an action of $(G^*, \cdot)$, where $\cdot$ denotes complex multiplication. Assume that $\{T_p : p \in \mathbb{P}\}$ is finite and that there is a unique invariant measure μ that is invariant under all these transformations. Then, for any continuous function f and $x \in X$,*

$$\frac{1}{N^2}\sum_{n,m=1}^{N} f(T_{(m,n)}x) \to \int f d\mu\,.$$

Going back to the average (7), one could ask what happens for multiplicative functions other than Liouville. For real-valued multiplicative functions, a celebrated theorem of Wirsing in the 1960s established that $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(n)$ exists, answering a long-standing conjecture by Erdős and Wintner.

As corollaries in number theory, Theorem 5 allows us to obtain the following versions of Wirsing's theorem.

Corollary 6. *Let $f\colon \mathbb{N} \to \mathbb{R}$ be a bounded multiplicative function.*
Then, $\lim_{N\to\infty}\frac{1}{N^2}\sum_{n,m=1}^{N} f(m^2+n^2)$ exists.

Corollary 7. *Let $f\colon \mathbb{G}^* \to \mathbb{R}$ be a bounded multiplicative function.*
Then, $\lim_{N\to\infty}\frac{1}{N^2}\sum_{n,m=1}^{N} f(m+in)$ exists.

We remark that, in Theorem 5, we can replace the average over $[1, N]^2$ by an average over a *dilated Følner sequence*. This concept includes squares and disks and many others (see [3] for details). Of course, all our corollaries can be strengthened by allowing averages to be taken along dilated Følner sequences. The following question by Frantzikinakis and Host that is still out of reach.

Question 8 ([5, p. 91])**.** *Let $f : \mathbb{N} \to \mathbb{R}$ be a real-valued and bounded, completely multiplicative function, and let $P \in \mathbb{Z}[x, y]$ be a homogeneous polynomial with values on the positive integers. Does the limit $\lim_{N\to\infty}\frac{1}{N^2}\sum_{m,n=1}^{N} f(P(m,n))$ exist?*

We believe that some new ideas will be needed for this problem.

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A complete characterisation of length-ℓ substitution shifts with an almost automorphic factor

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(joint work with Álvaro Bustos-Gajardo, Johannes Kellendonk)

Recall that a minimal topological dynamical system (X, T) is almost automorphic over (Y, S) if there exists a factor map $\pi : (X, T) \to (Y, S)$ such that $\pi^{-1}(y)$ is a singleton for some $y \in Y$. Based on work in [1], we give a complete answer to the question of when a minimal length-ℓ substitution shift (X_θ, σ) factors onto a shift which is an almost automorphic extension of the maximal equicontinuous factor of (X_θ, σ).

A length-ℓ substitution is a map $\theta : \mathcal{A} \to \mathcal{A}^\ell$, where $\mathcal{A}$ is a finite alphabet. By concatenation, we extend θ to act on words on $\mathcal{A}$. In this way, we define a language $\mathcal{L}_\theta := \{w \in \mathcal{A}^+ : w \text{ is a subword of } \theta^n(a) \text{ for some } n \in \mathbb{N} \text{ and } a \in \mathcal{A}\}$. This language is left- and right-extendable, and closed under taking subwords. It thus defines a shift space $X_\theta := \{x \in \mathcal{A}^\mathbb{Z} : x_m \dots x_n \in \mathcal{L}_\theta \text{ for each } m, n \in \mathbb{Z}\}$, and the resulting shift (X_θ, σ) is called a length-ℓ substitution shift.

By looking at θ as an ordered collection of maps, i.e., writing

$$\theta(a) = \theta_0(a)\theta_1(a), \dots \theta_{\ell-1}(a),$$

we have that $\theta_i \in \mathcal{A}^\mathcal{A}$ for each i, and with the operation of function composition we define S_θ, the semigroup associated to θ, as

$$S_\theta := \langle \theta_0, \theta_1, \dots, \theta_{\ell-1} \rangle .$$

Note that $S_\theta = G_\theta$ is a group if and only if each θ is a bijection.

Martin [5] partly characterised the existence of an almost automorphic factor in certain cases when the substitution θ is bijective. He was a student of Veech [6], who gave a structure theorem for shifts with a residual set of distal points, characterising them as having an almost automorphic extension which is an inverse limit of alternating almost automorphic and isometric extensions of a one-point system. Substitution shifts are a natural class of systems to which one can apply Veech's structure theorem, and this was the rationale for Martin's work. Later, Herning [3] characterised the existence of an almost automorphic factor in certain cases when the substitution θ is bijective and of prime length.

By a result in [4], any shift factor of (X_θ, σ) is topologically conjugate to another length-ℓ shift (X_η, σ). It is also known that any almost automorphic extension of an odometer is a shift. Thus, as (X_θ, σ) has an odometer as a maximal equicontinuous factor, we deduce that we need only characterise when an almost automorphic

(X_η, σ) exists as a factor of (X_θ, σ). Now, using known results, we can limit the radius of the factor map, to having left- and right-radius at most one. In other words, we can limit our study to radius zero factor maps of *collarings* $\theta^{(-l,r)}$ of θ, with $0 \leqslant l, r \leqslant 1$.

At this point, we can check the existence, or not, of η by brute force calculation, but here we give an elegant construction which links the existence of η to Green's $\mathcal{R}$-relation. The rest of the proof is entirely algebraic.

The kernel $\ker S_\theta$ of S_θ consists of elements of S_θ of minimal rank; it is a two-sided ideal. Two elements $f, g \in \ker S_\theta$ are R-related if they generate the same right ideal, i.e., if $f \ker S_\theta = g \ker S_\theta$. As $\ker S_\theta$ is completely simple, it follows that $f, g \in \ker S_\theta$ are R-related if and only if they have the same image. We can define the substitution η_θ defined by $\mathcal{R}$ and θ. Namely, for the substitutions we consider, the images of elements in $\ker S_\theta$ form a cover $\mathcal{U}_\theta$ of $\mathcal{A}$. We take the partition $\mathcal{P}_\theta$ defined to be the transitive closure of $\mathcal{U}_\theta$. On this partition, we can define η_θ in a natural way. Namely, if $\pi : \mathcal{A} \to \mathcal{P}_\theta$ is the projection map assigning to $a \in \mathcal{A}$ the partition element in $\mathcal{P}_\theta$ to which it belongs, it can be verified that $\eta_\theta := \pi\theta\pi^{-1}$ is well defined as a substitution on $\mathcal{P}_\theta$. We can assume that θ has trivial height, as from there, in our work, we can easily extend the following to substitutions of nontrivial height.

We show that X_{η_θ} must be almost automorphic over $(\mathbb{Z}_\ell, +1)$, which is known [2] to be the maximal equicontinuous factor of (X_θ, σ). Furthermore, we show that any other almost automorphic factor of X_θ with a zero-radius factor map *must* factor through X_{η_θ}.

Note that X_{η_θ} may be finite. One concludes that X_θ has an almost automorphic factor if and only if one of $\eta_{\theta^{(-l,r)}}$ is aperiodic, for some l, r with $0 \leqslant l, r \leqslant 1$.

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A problem at the interface of number theory and aperiodic order

Michael Coons

Sequences arising from binary constant-length substitutions are important in many areas, most notably for the current audience are the interest in aperiodic order and number theory. In what follows, we will consider certain asymptotic behaviours of two objects associated to such a sequence, the diffraction measure (aperiodic order) and the generating power series (number theory). Further, we assume that the underlying alphabet is balanced—the sequence takes the values 1 and -1.

On the aperiodic order side, given a sequence f arising from a constant-length substitution, one can form the diffraction measure by means of the following Wiener diagram

$$\begin{array}{ccc} \omega := \sum_{n\in\mathbb{Z}} f(n)\,\delta_n & \xrightarrow{\circledast} & \gamma := \omega \circledast \widetilde{\omega} = \sum_{m\in\mathbb{Z}} \eta(m)\,\delta_m \\ \mathcal{F}\big\downarrow & & \big\downarrow\mathcal{F} \\ \widehat{\omega} & \xrightarrow{\;|\cdot|^2\;} & \widehat{\gamma} = \widehat{\omega \circledast \widetilde{\omega}} \end{array}$$

where ω is the (weighted) Dirac comb with weights w, $\circledast$ represents Eberlein convolution, $\mathcal{F}$ is Fourier transformation, and the values

$$\eta(m) := \lim_{N\to\infty} \frac{1}{2N+1} \sum_{i=-N}^{N} f(i) f(i+m)$$

are the autocorrelation coefficients. Note that the lower route in the diagram needs some careful justification that we supress here. Here, we will be interested in the scaling of the diffraction measure near the origin, that is, with the asymptotics of

$$Z(x) := \widehat{\gamma}\big((0,x]\big)$$

as $x \to 0^+$. In particular, we note two examples, the binary Rudin–Shapiro sequence and the Thue–Morse sequence. For details concerning these examples, see Baake and Grimm [2].

The binary Rudin–Shapiro sequence RS is the fixed point of the substitution

$$\varrho_{\mathrm{RS}}: \quad a \mapsto ab, \quad b \mapsto ad, \quad c \mapsto cd, \quad d \mapsto cb,$$

where we use the values $a = c = 1$ and $b = d = -1$. The associated diffraction measure $\widehat{\gamma}_{\mathrm{RS}}$ is purely absolutely continuous and is equal to Lebesgue measure, so that

$$Z_{\mathrm{RS}}(x) = x^1\,, \tag{10}$$

where the exponent 1 has been emphasised for later use. For a more interesting example, we turn to the Thue–Morse sequence TM, which is the fixed point of the substitution

$$\varrho_{\mathrm{TM}}: \quad a \mapsto ab, \quad b \mapsto ba,$$

where we use the values $a = 1$ and $b = -1$. The associated diffraction measure $\widehat{\gamma}_{\mathrm{TM}}$ is purely singular continuous and the exact behaviour of $Z_{\mathrm{TM}}(x)$ near the origin is not known. Our current best understanding is that there are positive constants c_1 and c_2, such that, as $x \to 0^+$,

$$c_1\, x^{2+\alpha}\, 2^{-\log_2^2(x)} \leqslant Z_{\mathrm{TM}}(x) \leqslant c_2\, x^{\alpha}\, 2^{-\log_2^2(x)}\,, \tag{11}$$

where $\alpha = -\log_2(\pi^2/2)$ and $\log_2(\cdot)$ denotes the binary logarithm.

As someone coming into aperiodic order from the number theory community—specifically from the community interested in automatic sequences and their generating power series—the asymptotics in (10) and (11) give a sense of *déjà vu*, which I will elaborate in what follows.

The discussions in number theory for sequences arising from constant-length substitutions, which therein are called automatic sequences, often consider transcendence and algebraic independence properties of special values of their generating power series. This turns out to be approachable, since these generating power series are Mahler functions. In particular, if f is an automatic sequence and $F(z) := \sum_{n\geqslant 0} f(n)z^n \in \mathbb{Z}[[z]]$, then there is a positive integer d and polynomials $p_0(z), p_1(z), \ldots, p_d(z) \in \mathbb{Z}[z]$ such that

$$p_0(z)F(z) + p_1(z)F(z^k) + \cdots + p_d(z)F(z^{k^d}) = 0\,,$$

where the positive integer k is the constant length of the underlying substitution. A functional equation of this form allows one to understand the radial asymptotics of Mahler functions (functions that satisfy a Mahler functional equation as above). In joint work with Jason Bell [3], we showed that, under certain conditions, there is a real number $\lambda_F > 0$ such that

$$F(z) \asymp \frac{1}{(1-z)^{\log_k(\lambda_F)}}, \qquad \text{as } z \to 1^-,$$

where $\log_k(\cdot)$ denotes the base-k logarithm. This relationship can be combined with the Mahler functional equation to describe the radial asymptotics towards all roots of unity of order k^j for all j. But any set of roots of unity is only countable. We have yet to understand the almost everywhere radial asymptotics of Mahler functions. As a toy example, one can consider a Mahler function where $d = 1$, which is an infinite product. To this end, let β be a generic complex number with $|\,\beta\,| = 1$. We can compare the partial products of $F(z) = \prod_{j\geqslant 0} p(z^{k^j})$, say the first N factors, with the number of factors. In this case, using both Birkhoff's ergodic theorem and Jensen's formula, we have

$$\lim_{N\to\infty} \log_k \prod_{j\leqslant N} \left| p\left(e^{2\pi \mathrm{i}\beta k^j}\right)\right|^{1/N} = \lim_{N\to\infty} \frac{1}{N} \sum_{j\leqslant N} \log_k \left| p\left(e^{2\pi \mathrm{i}\beta k^j}\right)\right| = \log_k \mathfrak{M}(p),$$

where for $\mathfrak{M}(p)$ is the *Mahler measure* of the polynomial $p(z)$. Thus, for almost all β on the unit circle, as z radially approaches β, we expect, for any $\varepsilon > 0$,

$$|F(z)| < \frac{1}{(1-|z|)^{\log_k \mathfrak{M}(p)-\varepsilon}}.$$

Can such a suggested asymptotic be realised? One should compare our work [1] with Baake and Mañibo, where given a primitive, binary constant-length substitution, we showed that the maximal Lyapunov exponent is equal to the logarithmic Mahler measure of an associated (Peter) Borwein polynomial. But I digress.

We return to our two examples, the Rudin–Shapiro and Thue–Morse sequences. To understand the asymptotics of the generating power series of the Rudin–Shapiro sequence, we note that the partial sums satisfy

$$\sum_{k=0}^{n} \mathrm{RS}(n) \asymp \sqrt{n},$$

which implies that

$$\left| \sum_{n\geqslant 0} \mathrm{RS}(n) z^n \right|^2 \asymp \frac{1}{(1-z)^1}, \qquad \text{as } z \to 1^-,$$

where the squaring impulse was provided directly from the bottom of the Wiener diagram. This exponent (1 as highlighted in the denominator on the right-hand side) is precisely the power of the scaling of $Z_{\mathrm{RS}}(x) = x^1$ as $x \to 0^+$. While it is nice that $1 = 1$, the Thue–Morse example is much more interesting. Note that the generating function of TM is given by

$$\sum_{n\geqslant 0} \mathrm{TM}(n) z^n = \prod_{j\geqslant 0} \left(1 - z^{2^j}\right).$$

Unfortunately, the asymptotics for this function cannot be provided by my above-mentioned result with Bell. But fortunately, a result of de Bruijn [4] implies that

$$\left| \sum_{n\geqslant 0} \mathrm{TM}(n) z^n \right|^2 \asymp (1-z) \cdot 2^{-\log_2^2(1-z)}, \qquad \text{as } z \to 1^-.$$

This behaviour is certainly reflective of that of $Z_{\mathrm{TM}}(x)$ as $x \to 0^+$. The big question is whether one really can relate the order of the scaling of the diffraction measure near zero with the radial asymptotics of the associated Mahler function. Are these coincidences coincidental? Or, is something deeper going on?

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Almost chaotic systems

Bryna Kra

(joint work with Van Cyr and Scott Schmieding)

In his seminal paper on disjointness, Furstenberg [3] derives a Diophantine corollary showing that, if $p, q \geqslant 2$ are multiplicatively independent integers and $\alpha \in \mathbb{R}$ is irrational, then

$$\{p^n q^m \alpha \mod p : n, m \geqslant 1\}$$

is dense in the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. This result has been generalised in numerous directions, studying other sequences for which the same result holds, studying measurable analogues, and studying generalisations in other groups.

We study the question of what abstract properties lead to such a rigidity result. Namely, Furstenberg's theorem says that orbits under the semigroup of transformations $\times p, \times q$ for multiplicatively independent $p, q \geqslant 2$ on the circle have one of two behaviours: they are either finite (for rational points) or dense (for irrational points). To make the appropriate definition precise, we introduce some notation.

Let X be a compact metric space and assume that a group G acts on X by homeomorphisms,

$$G \to \mathrm{Homeo}(X)$$

(we can also consider a semigroup G). Given $g \in G$, write $T_g \colon X \to X$ for the action of left multiplication by g on X, and we call the pair (X, G) a *system*.

The system (X, G) is *transitive* if for all nonempty open sets $U, V \subset X$, there exists $g \in G$ such that $T_g U \cap V \neq \varnothing$ and the action of G on X is *faithful* if for every non-identity element $g \in G$, there exists $x \in X$ such that $gx \neq x$. Define the system (X, G) to be *chaotic almost minimal* if it satisfies the following properties:

(1) The system (X, G) is transitive.

(2) The action of G on X is faithful.

(3) The space X contains a dense set of G-periodic points.

(4) Every proper, closed G-invariant subset of X is finite.

If there exists a system (X, G) that is chaotic almost minimal, we refer to the group G as being *chaotic almost minimal*. The motivation for the terminology is that such systems are chaotic in the sense of Devaney [2] and are almost minimal in the sense defined by Schmidt [4] for a $\mathbb{Z}^d$-action by automorphisms on a compact group.

In joint work with Cyr and Schmieding, we have studied the properties of chaotic almost minimal systems. Examples include:

- The $\mathbb{N}^2$-system used by Furstenberg in his disjointness paper.
- The $\mathbb{Z}^2$-system that is an invertible model for this system in the full shift on six symbols.
- If (X, σ) is a mixing shift of finite type and $\mathrm{Aut}(X, \sigma)$ denotes its automorphism group, then the action of $\mathrm{Aut}(X, \sigma)$ on X (this follows from results in [1]).

There are many other systems scattered throughout the literature that are chaotic almost minimal.

Some of the results we have proved include:

- If (X, G) is an infinite chaotic almost minimal system, then X is perfect (and so uncountable).
- Any chaotic almost minimal system admits a Borel probability invariant measure with full support.
- If X is a locally connected compact metric space, then X does not support an expansive $\mathbb{Z}$-chaotic almost minimal action.

- If $\alpha\colon X \to X$ is an automorphism of a compact group X, then (X,α) is not a chaotic almost minimal system.
- if (X,G) is a chaotic almost minimal system, then G is residually finite.

In novel constructions, we have given an example of a $\mathbb{Z}$-action such that (X,T) is a chaotic almost minimal system that supports multiple nonatomic ergodic invariant measures.

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The Dry Ten Martini problem for Sturmian dynamical systems

Siegfried Beckus

(joint work with Ram Band, Raphael Loewy)

Are all possible spectral gaps, predicted by the gap labeling theorem, there? This is the so called *Dry Ten Martini problem (Dry TMP)* motivated by the *Ten Martini Problem (TMP)*. The names were coined by Simon [9] after Kac offered in 1981 ten Martinis to anyone who solves it. Originally, the almost Mathieu operator (AMO) was considered, whereas we treat Sturmian Hamiltonians.

The potential of Sturmian Hamiltonians reflects the aperiodicity coming from a solid and not from a magnetic field like for the AMO. Specifically, we study the one-dimensional Schrödinger operator $H_{\alpha,V} \in \mathcal{L}\big(\ell^2(\mathbb{Z})\big)$,

$$(H_{\alpha,V}\psi)(n) := \psi(n-1) + \psi(n+1) + V\chi_{[1-\alpha,1)}(\{n\alpha\})\psi(n), \qquad \psi \in \ell^2(\mathbb{Z}),\ n \in \mathbb{Z},$$

where $\{n\alpha\} := n\alpha - \lfloor n\alpha \rfloor$ is the fractional part of $n\alpha$. We note that the potential is characterised in terms of two parameters: the frequency $\alpha \in [0,1]$ and the potential strength, also known as the coupling constant $V \in \mathbb{R}$. The family of Schrödinger operators $H_{\alpha,V}$ for $\alpha \in [0,1]\backslash\mathbb{Q}$ and $V \neq 0$ is called *Sturmian Hamiltonians*. The *Kohmoto butterfly* is the corresponding plot of the spectra of $H_{\alpha,V}$ as it varies with α, see Figure 1 (a).

The integrated density of states (IDS) $N_{\alpha,V} : \mathbb{R} \to [0,1]$ is defined by

$$N_{\alpha,V}(E) := \lim_{n\to\infty} \frac{\#\big\{\lambda \in \sigma\big(H_{\alpha,V}|_{[1,n]}\big) \,\big|\, \lambda \leqslant E\big\}}{n}, \qquad E \in \mathbb{R},$$

where $H_{\alpha,V}|_{[1,n]}$ is the $n \times n$ matrix obtained by restricting $H_{\alpha,V}$ to $\ell^2(1,\dots,n)$. We call $g := (a,b)$ a *spectral gap* if $a,b \in \sigma(H_{\alpha,V})$ and $(a,b) \cap \sigma(H_{\alpha,V}) = \varnothing$. A spectral gap is characterised by the plateaus of the IDS. The corresponding value $N_{\alpha,V}(E)$ for some (any) $E \in g$ is called the *gap label* of g. The gap labeling theorem [4] prescribes all the possible labels that the spectral gaps may have:

$$\{N_{\alpha,V}(E) \,|\, E \in \mathbb{R}\backslash\sigma(H_{\alpha,V})\} \subseteq \big\{\{n\alpha\} \,|\, n \in \mathbb{Z}\big\} \cup \{1\}.$$

One asks if this inclusion is an equality – this is the Sturmian Dry TMP. A complete solution of this problem appears in [2] (see also [6] for first announcement).

Theorem 1 (Sturmian Dry TMP)**.** *For all $\alpha \in [0,1]\backslash\mathbb{Q}$ and all $V \neq 0$, all spectral gaps of $H_{\alpha,V}$ are open, i.e.,*

$$\{N_{\alpha,V}(E) \,|\, E \in \mathbb{R}\backslash\sigma(H_{\alpha,V})\} = \big\{\{n\alpha\} \,|\, n \in \mathbb{Z}\big\} \cup \{1\}.$$

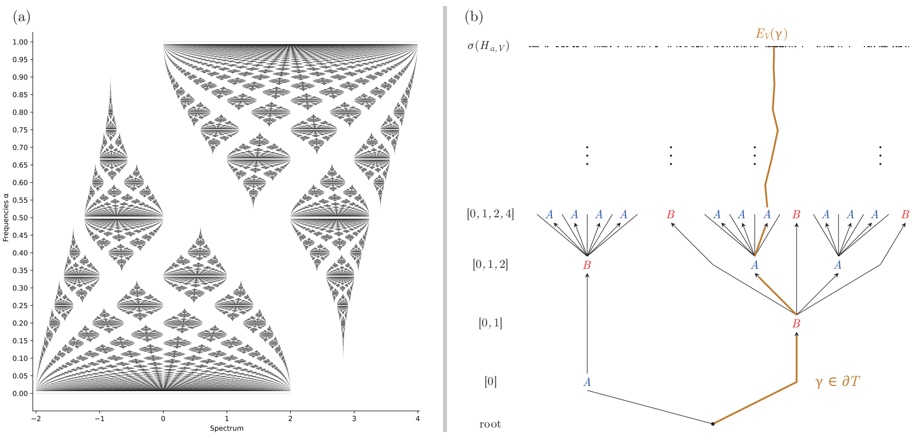

Figure 1. (a) Kohmoto butterfly [3] for $V = 2$: The vertical axis represents the frequency values α. For each α, the spectrum of $H_{\alpha,V}$ is plotted horizontally. (b) An example of the tree T for $\alpha = [0, 1, 2, 4, \ldots]$ is sketched. A specific infinite path starting at the root and approaching $E_V(\gamma)$ in the spectrum is indicated in brown, see Theorem 4 (a).

For $V > 4$, the previous statement was proved in [8] (see also [3] for a review). For the Fibonacci Hamiltonian (α equals to the golden mean $\varphi := \frac{\sqrt{5}-1}{2}$), a complete solution (i.e., for all $V \neq 0$) has been provided in [5]. If $\alpha \in [0,1]\backslash\mathbb{Q}$ has eventually periodic continued fraction expansion, then all gaps are open for sufficiently small values of V, see [7].

Let us shortly outline the strategy of the general proof, where we refer the reader to the extended version of this MFO report [1] for more details.

Each $\alpha \in [0,1]\backslash\mathbb{Q}$ is uniquely determined by its continued fraction expansion

$$\alpha = c_0 + \cfrac{1}{c_1 + \frac{1}{c_2 + \ldots}} =: [c_0, c_1, c_2, \ldots], \qquad c_k \in \mathbb{N},\ k \geqslant 0.$$

For $k \in \mathbb{N}$, $\alpha_k := [0, c_1, \ldots, c_k] \in \mathbb{Q}$ defines the periodic operator $H_{\alpha_k,V}$ and $\sigma(H_{\alpha_k,V})$ is a finite union of intervals – so-called spectral bands.

Theorem 2. *Each spectral band $I(V)$ in $\sigma(H_{\alpha_k,V})$ is either*

- *of* type A, *i.e., $I(V)$ is strictly contained in a spectral band of $\sigma(H_{\alpha_{k-1},V})$, or*
- *of* type B, *i.e., $I(V)$ is not of type A and it is strictly contained in a spectral band of $\sigma(H_{\alpha_{k-2},V})$.*

Remark 3. *For $V > 4$, Theorem 2 was proved in* [8]*, while the general case $V \neq 0$ is treated in [2]. The extra difficulty for small couplings $V \neq 0$ comes from possible overlaps of spectral bands. This issue is resolved in [2] by combining trace maps together with a new viewpoint – applying an interlacing theorem to matrix eigenvalues of the periodic approximations. Another crucial ingredient is changing the perspective to consider the whole space of all finite continued fraction expansions using a two-level induction instead of a single approximation $(\alpha_k)_{k\in\mathbb{N}}$.*

The type of $I(V)$ and c_{k+1} uniquely determine how many spectral bands of type A it contains from $\sigma(H_{\alpha_{k+1},V})$ and of type B it contains from $\sigma(H_{\alpha_{k+2},V})$, as well as how these spectral bands interlace. The details are sketched in Figure 2 (b).

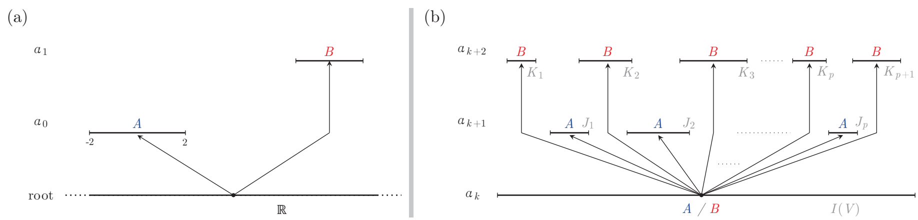

FIGURE 2. (a) The root of the tree T and two adjacent vertices are plotted. (b) The spectral band $I(V)$ and the spectral bands of type A and B it contains in the subsequent levels where $p = c_{k+1} - 1$ if $I(V)$ is of type A and $p = c_{k+1}$ if $I(V)$ is of type B.

This defines a V-independent tree graph T (see e.g. Figure 1 (b)) with the basic rules sketched in Figure 2, where each level k represents all the spectral bands of $\sigma(H_{\alpha_k,V})$. It encodes the local relations and order of these spectral bands. Then, the boundary of the tree ∂T is the set of all infinite paths starting at the root.

This leads to the following connection of ∂T with $\sigma(H_{\alpha,V})$ and a V-independent representation of the IDS, which is proven in [8] for $V > 4$ and in [2] for all $V \neq 0$.

Theorem 4. *Let $V > 0$ and $\alpha \in [0,1]\backslash\mathbb{Q}$.*

(a) There exists a bijective map

$$E_V : \partial T \to \sigma(H_{\alpha,V}), \quad \gamma \mapsto E_V(\gamma).$$

(b) For $\gamma \in \partial T$ and $k \geqslant -1$, there exists $\pi_k(\gamma) \in \{0, \dots, c_{k+1}\}$ such that

$$N_{\alpha,V}\big(E_V(\gamma)\big) = -\alpha + \sum_{k=-1}^{\infty} (-1)^k \pi_k(\gamma)(q_k\alpha - p_k),$$

We highlight that a major difficulty lies in proving the injectivity of the map E_V for small coupling constants V. This injectivity is the crucial ingredient to prove the complete solution to the Sturmian Dry TMP stated in Theorem 1.

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Mild distributions in diffraction theory

Christoph Richard
(joint work with Hans G. Feichtinger, Christoph Schumacher, Nicolae Strungaru)

We consider locally compact abelian (LCA) groups G together with their dual groups $\widehat{G}$, and fix Haar measures on G and $\widehat{G}$ such that Plancherel's formula holds. Take any non-constant $g \in L^1(G) \cap C_0(G)$ and define $S_0(G) = \{f \in L^1(G) \cap C_0(G) : \|f\|_{S_0} < \infty\}$, where $\|f\|_{S_0} = \int_{\widehat{G}} \|(\chi f) * g\|_1 \mathrm{d}\chi$. Here, $*$ denotes convolution, and $(\chi f)(x) = \chi(x)f(x)$ for $\chi \in \widehat{G}$ and $f \in L^1(G)$. Then $(S_0(G), \|\cdot\|_{S_0})$ is a Banach algebra with respect to both pointwise multiplication and convolution, and different choices of g lead to equivalent norms. The space $S_0(G)$ has been introduced by Feichtinger [5]; see [7] for a recent review. The Bruhat–Schwartz functions on G are contained in $S_0(G)$ as a dense subspace, and the Fourier transform provides a bijection between $S_0(G)$ and $S_0(\widehat{G})$. This makes distribution theory based on Feichtinger's algebra a powerful tool for harmonic analysis on general LCA groups. Elements of the dual space $S_0'(G)$ of bounded linear functionals on $S_0(G)$ are called *mild distributions.*

Diffraction analysis on LCA groups [9, 14] uses Fourier analysis of translation bounded Radon measures, as developed by Argabright and Gil de Lamadrid [1, 6].

Recall that a Radon measure μ on G is translation-bounded if

$$\sup\{|\mu|(x+K) : x \in G\} < \infty \,,$$

for all compact $K \subseteq G$, where $|\mu|$ denotes the total variation measure of μ. As any such measure can be identified with a mild distribution [4, Thm. B1], their Fourier theory is subsumed by that of mild distributions. Somewhat surprisingly, this connection has not systematically been used so far.

As an initial step, we re-analyse model set diffraction [2, Sec. 7.2]. Since model sets are projections of certain lattice subsets, one might expect that the Poisson summation formula of the underlying lattice determines their diffraction. This point of view has been put forward by Lagarias [8, Thm. 2.9], and summation formulae have been given for Euclidean space [2, Lem. 9.3] and for general LCA groups [13]. Observing that the Poisson summation formula continues to hold for test functions from S_0, see e.g. [7, Ex. 5.12], it is straightforward to obtain a summation formula for weighted model sets with weight functions from Feichtinger's algebra. This subsumes the results mentioned above and extends a recent result by Matusiak [10, Thm. 4.2] from Euclidean space to the general setting.

Fix LCA groups G, H and a lattice $\mathcal{L} \subseteq G \times H$, i.e., a discrete co-compact subgroup of $G \times H$. Using the notation of [13], we consider for $h \in S_0(H)$ the weighted Dirac comb $\omega_h = \sum_{(x,y)\in\mathcal{L}} h(y)\delta_x$ on G, which is a translation-bounded measure on G. There is a corresponding weighted Dirac comb on $\widehat{G}$ constructed from the lattice $\mathcal{L}^\circ \subseteq \widehat{G} \times \widehat{H}$ dual to $\mathcal{L}$, i.e., from the annihilator of $\mathcal{L}$ in $\widehat{G} \times \widehat{H}$. We write $\omega_\psi = \sum_{(\chi,\eta)\in\mathcal{L}^\circ} \psi(\eta)\delta_\chi$ for $\psi \in S_0(\widehat{H})$. The autocorrelation measure γ_{ω_h} of ω_h is a translation-bounded measure on G defined by a certain averaging procedure. Take any van Hove net $(A_\iota)_{\iota\in\mathbb{I}}$ in G, for example $A_\iota = U + F_\iota$ with U any compact zero neighbourhood and with $(F_\iota)_{\iota\in\mathbb{I}}$ any Følner net in G; see [12, Prop. 5.10]. In Euclidean space, the sequence of centred closed balls of radius $n \in \mathbb{N}$ constitutes a van Hove net. Consider the so-called Eberlein convolution

$$\gamma_{\omega_h} = \omega_h \circledast \widetilde{\omega_h} = \lim_{\iota\in\mathbb{I}} \frac{\omega_h|_{A_\iota} * \widetilde{\omega_h|_{A_\iota}}}{m_G(A_\iota)} \,.$$

Here, m_G denotes the Haar measure, $|_A$ denotes restriction to A and, for a Radon measure ω, its reflected version $\widetilde{\omega}$ is defined by $\widetilde{\omega}(f) = \overline{\omega(\widetilde{f})}$ where $\widetilde{f}(x) = \overline{f(-x)}$. We assume that $\mathcal{L}$ projects densely into H. Then, the above limit indeed exists and is independent of the chosen van Hove net. In fact, $\gamma_{\omega_h} = \operatorname{dens}(\mathcal{L}) \cdot \omega_{h*\widetilde{h}}$, where $\operatorname{dens}(\mathcal{L})$ is the density of the lattice $\mathcal{L}$, which equals the reciprocal Haar measure of a fundamental domain of $\mathcal{L}$ in $G \times H$. The summation formula can now be cast in terms of the following commutative Wiener diagram, which resembles the physicist's recipe for computing diffraction intensities of a finite sample from ω_h.

$$\begin{array}{ccc}
\omega_h & \xrightarrow{\circledast} & \mathrm{dens}(\mathcal{L}) \cdot \omega_{h * \widetilde{h}} \\
\widehat{}\downarrow & & \downarrow\widehat{} \\
\mathrm{dens}(\mathcal{L}) \cdot \omega_{\widecheck{h}} & \xrightarrow{|\cdot|^2} & \mathrm{dens}(\mathcal{L})^2 \cdot \omega_{|\widecheck{h}|^2}
\end{array}$$

Here, all objects are discrete translation-bounded measures, whose sums on any function from S_0 are absolutely convergent. The measure Fourier transform is denoted by $\widehat{}$, and $|\cdot|^2$ means squaring the weights of the weighted Dirac comb.

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On the lack of equidistribution on fat Cantor sets

Gabriel Fuhrmann

Given an irrational rotation number $\omega \in \mathbb{R}\backslash\mathbb{Q}$, a set $W \subseteq \mathbb{T}^1 (= \mathbb{R}/\mathbb{Z})$ and a point $x \in \mathbb{T}^1$, we define

$$S_W^n(x) = 1/n \cdot \sum_{\ell=0}^{n-1} \mathbf{1}_W(x + \ell\omega) \qquad (n \in \mathbb{N}),$$

where $\mathbf{1}_W$ denotes the characteristic function of W.

It is well known and straightforward to see that, if W is a Cantor set, there is a dense (in fact, residual) set of $x \in \mathbb{T}^1$ with $\lim_{n\to\infty} S_W^n(x) = 0$. This is a consequence of Baire's category theorem. On the other hand, if W is a *fat* Cantor set (that is, $\mathrm{Leb}_{\mathbb{T}^1}(W) > 0$), we have $\lim_{n\to\infty} S_W^n(x) = \mathrm{Leb}_{\mathbb{T}^1}(W) > 0$ for $\mathrm{Leb}_{\mathbb{T}^1}$-a.e. x. This is a consequence of Birkhoff's ergodic theorem.

But what other frequencies of visits to W may occur? In the words of a recent MathOverflow post [1], what is the set

$$S_W = \bigcup_{x\in\mathbb{T}^1} \bigcap_{N\in\mathbb{N}} \overline{\{S_W^n(x)\colon n \geqslant N\}}?$$

Due to the above dichotomy between the topological and the measure-theoretical perspective, purely topological tools are just as useless in answering this question as purely measure-theoretical ones. Instead, we utilise almost automorphic symbolic extensions and their representation via ordered Bratteli diagrams to show that every irrational rotation admits certain fat Cantor sets C (which, interestingly, include those constructed in [2]) such that S_C is maximal, that is, $S_C = [0, \mathrm{Leb}_{\mathbb{T}^1}(C)]$. This observation is, in fact, a corollary of a more general theorem about at most 2-to-1 almost 1-to-1 factor maps between minimal homeomorphisms on the Cantor set.

The overall strategy of the proof of this theorem is to reduce the computation of the asymptotic frequencies (where $n \to \infty$) to finite-time frequencies in an associated *extended Bratteli diagram*—a notion introduced in [4] for Toeplitz shifts. The corollary is then obtained by applying the theorem to 2-to-1 extensions of Sturmian shifts.

We also show—by an explicit construction—that every irrational rotation admits for certain fat Cantor sets C such that S_C is *not* maximal. Again, this observation is a corollary of a more general theorem—this time about at most 3-to-1 almost 1-to-1 factor maps between minimal homeomorphisms on the Cantor set.

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Counting patches and discontinuities of Penrose integrated densities of states

May Mei

(joint work with David Damanik, Mark Embree, Jake Fillman)

In [2], the authors show that, if (Ω, T) is a Delone dynamical system of finite type, then there exists a Delone dynamical system of finite type (Ω^b, T) and a random operator of finite range (A^b_ω) on (Ω^b, T) such that (Ω, T) and (Ω^b, T) are mutually locally derivable and (A^b_ω) has locally supported eigenfunctions with the same eigenvalues for every $\omega \in \Omega^b$. Moreover, (A^b_ω) can be chosen to be the nearest neighbour Laplacian of a suitable graph. Further, if (Ω, T) is strictly ergodic and A is a self-adjoint random operator of finite range, then E is a point of discontinuity of the spectral measure ρ^A if and only if there exists a locally supported eigenfunction of A_ω corresponding to E for some $\omega \in \Omega$.

In [1], we compute lower bounds for the heights of jumps in the integrated density of states for the nearest neighbour Laplacian on graphs associated to members of the MLD class of Penrose tilings. We consider the dual graph of a tiling, namely vertices are associated to tiles and two vertices are adjacent via an edge just in case two tiles are adjacent in the tiling. Consider the Robinson triangle tiling. We show that the proportion of eigenvalues $E = 2$ is bounded below by $\frac{65-29\sqrt{5}}{20}$. The proportion of eigenvalues $E = 4$ is given by the same number. This is obtained through a combinatorial argument and recursion based on the proportion of acute and obtuse triangles. Now consider the Penrose pentagon boat star tiling and the associated substitution. The support of an eigenfunction for $E = 4$ appears after the two-fold application of the substitution to a pentagon. Thus, an explicit calculation of the number of pentagon tiles yields a lower bound for the proportion of eigenvalues.

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Spectral computations for aperiodic models

Mark Embree

(joint work with David Damanik, Jake Fillman, May Mei)

With a finite section of the Penrose tiling we can associate a graph, identifying tiles as nodes and placing edges between tiles that share a common boundary segment. We study spectral properties of the graph Laplacian, the matrix constructed by subtracting the graph's adjacency matrix from its (diagonal) degree matrix.

For four different Penrose constructions (Robinson triangles; kites and darts; rhombuses; boats and stars), one can construct eigenvectors of the graph Laplacian that are supported on finitely many tiles away from the boundary, as illustrated in [1]. The existence of such finitely-supported modes (identified in the physics literature in the 1980s, see, e.g., [2], and for the vertex-based variant of the Penrose Laplacian [3]) implies that the integrated density of states associated with the infinite tiling will exhibit a jump at the corresponding energy.

This talk describes some computational tools we use to investigate these locally supported modes, and the numerical results they produce. Some questions (such as computation of the integrated density of states) require computation of the entire spectrum; others require more local information about the spectrum and thus permit calculations with much larger matrices.

To count the multiplicity of a specific energy E, we utilise a technique called *spectrum slicing* [4]. This technique works on the following premise: The symmetric matrix A can be factored as the product $A = LDL^T$, where L is an invertible lower-triangular matrix, and the D is block diagonal, having 1-by-1 or 2-by-2 diagonal blocks. Since A is a congruence transformation of D, Sylvester's law of inertia ensures that A and D have the same inertia (number of positive, zero, and negative eigenvalues); the structure of D makes this inertia easy to compute. By comparing the inertia of $A-(E+\varepsilon)I$ and $A-(E-\varepsilon)I$ for, say, $\varepsilon = 10^{-10}$, one can estimate the multiplicity of E as an eigenvalue of A. This knowledge can support the search for novel finitely supported mode shapes on large sections of the tiling.

We utilise the same technique to investigate the existence of gaps in the spectrum for the Robinson triangle construction. If the inertia of $A - \alpha I$ and $A - \beta I$ agree, we have no eigenvalues in the interval $[\alpha, \beta]$. We produce numerical evidence for the existence of gaps that persist on larger and larger sections of the Robinson triangle tiling.

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How to compute exact patch frequencies in certain projection tilings

Jan Mazáč

The exact computation of frequencies of large patches of various tilings is a tricky task. We present an efficient algorithm for obtaining them for tilings that can be described via the dualisation method. This method, due to Kramer and

Schlottmann [4], directly applies to the rhombic Penrose tiling [2] or the Ammann–Beenker tiling [1]. It provides a one-to-one correspondence between a tile and a dual tile, which is a convex polytope in the internal space. Due to equidistribution results, its area is proportional to the frequency of the given object.

In the case of rhombic Penrose tilings, a dual triangle is assigned to each rhombus. One can list all possible tiles and their relative positions with respect to some chosen origin. By applying the duality, one gets a list of triangles in internal space. The vertex coordinates are elements of $\mathbb{Q}(\sqrt{5})$, and the intersection of all triangles can be exactly computed using a clipping algorithm. The resulting convex polygon still has its coordinates within the same number field. The area of such a polygon can be calculated via the shoelace formula, and, again, the result remains in $\mathbb{Q}(\sqrt{5})$. After dividing by the total area of the window for the vertex points, this yields the exact relative patch frequency.

In [5], we present the frequencies of large (possibly disconnected) patches of the Penrose rhombic tiling and the Ammann–Beenker tiling, which appear when studying Schrödinger operators on graph structures arising from those tilings [3].

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Large normalizer of odometers and automatic $\mathbb{Z}^d$-sequences

Samuel Petite

(joint work with Christopher Cabezas)

For a $\mathbb{Z}^d$ topological dynamical system $(Z, T, \mathbb{Z}^d)$, an *isomorphism* is a self-homeomorphism $\phi : X \to X$ such that, for some matrix $M \in \mathrm{GL}(d, \mathbb{Z})$ and any $\boldsymbol{n} \in \mathbb{Z}^d$, $\phi \circ T^{\boldsymbol{n}} = T^{M\boldsymbol{n}} \circ \phi$, where $T^{\boldsymbol{n}}$ denotes the self-homeomorphism of X given by the action of $\boldsymbol{n}$. The collection of all such isomorphisms is a group that is the normalizer of the set of transformations $T^{\boldsymbol{n}}$. In the one-dimensional case, isomorphisms corresponds to the notion of *flip conjugacy* of dynamical systems and by this fact are also called *reversing symmetries* (see [1, 3]).

These isomorphisms are not well understood even for classical systems. In [2], we present a description of them for odometers and, more precisely, for $\mathbb{Z}^2$-constant base odometers, which, surprisingly, is not simple. This is a classification where we give computable arithmetical conditions to determine the elements of the group.

We deduce a complete description of the isomorphisms of some $\mathbb{Z}^d$, $d > 1$, minimal substitution subshifts. We give the first example known of a minimal zero-entropy subshift with the largest possible normalizer group. In particular, we show that any matrix $M \in \mathrm{GL}(d, \mathbb{Z})$ occurs as a matrix associated to an isomorphism of the half-hex tiling system.

Surprisingly, even if our examples have the smallest complexity among aperiodic subshifts, their normalizers are not amenable. This is in contrast with what happens in dimension $d = 1$, where the group of isomorphisms (which is a finite extension of the automorphism group) is known to be amenable for a large class of zero-entropy subshifts.

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Scaling properties of Thue–Morse measure

Philipp Gohlke

(joint work with Michael Baake, Marc Kesseböhmer, Georgios Lamprinakis, Tanja Schindler, Jörg Schmeling)

The Thue–Morse measure μ is a paradigmatic example of a singular continuous measure on the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, initially studied by Mahler [6]. It arises as a diffraction measure of the Thue–Morse substitution $\varrho\colon a \mapsto ab, b \mapsto ba$ with balanced weights and can be represented as the Riesz product

$$\mu = \prod_{n=0}^{\infty} \big(1 - \cos(2\pi 2^n x)\big),$$

to be understood as a weak limit of probability densities. We are interested in the local scaling properties of the measure μ, that is, we ask: How does the measure $\mu(B_r(x))$ of the ball $B_r(x) = \{y \in \mathbb{T} : |x - y| < r\}$ decay as $r \to 0$? Since the answer to this question depends heavily on the point x, we in fact aim to quantify the Hausdorff dimension $\dim_H$ of all points x with a given local scaling behaviour.

As a first step, we quantify this question in terms of the local dimension of μ at x, given by

$$d_\mu(x) := \lim_{r \downarrow 0} \frac{\log \mu(B_r(x))}{\log r},$$

provided that the limit exists. To gain sufficient control over the local decay behaviour, it is useful to note that μ falls into the class of g-measures (studied by

Keane [4]) over the doubling map $(\mathbb{T},T)$, with $T\colon x\mapsto 2x \mod 1$. More precisely, μ is invariant under the dual of the transfer operator $\varphi_g\colon C(\mathbb{T})\to C(\mathbb{T})$, with

$$(\varphi_g f)(x)=\sum_{y\in T^{-1}x} g(y)f(y),\quad g\colon y\mapsto \frac{1}{2}\big(1-\cos(2\pi y)\big).$$

The relation $\int \varphi_g f\,\mathrm{d}\mu=\int f\,\mathrm{d}\mu$ can be used to estimate the measure on dyadic intervals of the form $I=[k2^{-n},(k+1)2^{-n}]$, with $0\leqslant k<2^n$, by sampling the potential function $\psi=\log(g)$ along the doubling map. More precisely,

$$\mu(I)=\int_{\mathbb{T}}\exp\big(S_n\psi(2^{-n}(k+x))\big)\,\mathrm{d}\mu(x),\quad S_n\psi(x)=\sum_{k=0}^{n-1}\psi(T^k x).$$

Hence, the value of μ on small intervals is intimately connected to appropriately defined Birkhoff sums. The description of ψ as a *potential function* goes back to the observation of Ledrappier [5] that μ is an *equilibrium measure* for ψ, that is, it achieves the maximum in

$$\mathcal{P}(\psi)=\sup_{\nu\in\mathcal{M}_T} h_\nu+\int\psi\,\mathrm{d}\mu,$$

where $\mathcal{M}_T$ is the set of T-invariant probability measures on $\mathbb{T}$, $h_\nu(T)$ denotes the entropy of ν, and $\mathcal{P}(\psi)$ is called the pressure of ψ. Introducing a real parameter t leads to the pressure function

$$p\colon\mathbb{R}\to\mathbb{R},\quad t\mapsto\mathcal{P}(t\psi).$$

For Hölder-continuous potential functions, there is a well-understood relation between the Legendre transform of the pressure function and the local dimensions of the corresponding equilibrium measure [7]. This relation extends to the case at hand, despite the technical obstacle that ψ has a singularity at the origin.

Theorem 1 ([1]). *The dimension spectrum*

$$f(\alpha)=\dim_H\{x\in\mathbb{T}: d_\mu(x)=\alpha\}$$

is related to $p^*(\alpha)=\inf_{t\in\mathbb{R}}(\alpha t-p(t))$ *via*

$$f(\alpha)=\max\left\{\frac{-p^*(-\alpha\log 2)}{\log 2},0\right\},$$

for all $\alpha\in\mathbb{R}$.

For an illustration of the dimension spectrum, we refer the reader to [1]. In contrast to equilibrium measures that are related to Hölder-continuous potential functions, the function f is strictly positive on a half-line; in fact, $f(\alpha)=1$ for all $\alpha\geqslant 2$. That is, points with arbitrarily large local dimension occur with full Hausdorff dimension. The fastest decay of μ occurs precisely at dyadic points y, where it satisfies

$$\lim_{r\downarrow 0}\frac{\log_2\mu(B_r(y))}{-(\log_2 r)^2}=1,$$

compare [2] for a more refined estimate at $y = 0$. The dyadic points form a (countable) set of vanishing Hausdorff dimension, but demanding a slightly slower decay already produces sets of full Hausdorff dimension.

Theorem 2 ([3])**.** *For all* $\gamma \in (1,2)$ *and* $c \geqslant 0$*, we have*

$$\dim_H \left\{ x \in \mathbb{T} : \lim_{r \downarrow 0} \frac{\log_2 \mu(B_r(x))}{-(\log_2 r)^\gamma} = c \right\} = 1.$$

In this sense, $\gamma = 2$ is the critical exponent. However, except at dyadic points, the family $(h_r(x))_{r>0}$ with

$$h_r(x) = \frac{\log_2 \mu(B_r(y))}{-(\log_2 r)^2}$$

does not possess a limit as $r \downarrow 0$. In the following result, we quantify the spreading of accumulation points explicitly.

Theorem 3 ([3])**.** *Let* $\underline{h}(x) = \liminf_{r\downarrow 0} h_r(x)$ *and* $\overline{h}(x) = \limsup_{r\downarrow 0} h_r(x)$ *for all* $x \in \mathbb{T}$*. Then, for all non-dyadic* $x \in \mathbb{T}$*, we have* $\underline{h}(x) \leqslant 1/2$ *and*

$$\overline{h}(x) \geqslant \frac{\underline{h}(x)}{1 - \underline{h}(x)}.$$

The bound provided by this theorem is sharp in the sense that, for all $\beta \in [0, 1/2]$ and $\alpha \geqslant \beta/(1-\beta)$, there is a point $x \in \mathbb{T}$ with $\overline{h}(x) = \alpha$ and $\underline{h}(x) = \beta$. In [3], we give an explicit analytic expression for the joint dimension spectrum

$$F(\alpha,\beta) = \dim_H\{x \in \mathbb{T} : \overline{h}(x) = \alpha,\, \underline{h}(x) = \beta\},$$

for all possible values $\alpha, \beta \in \mathbb{R}$.

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On spectrum of aperiodic Schrödinger operators with random noise

Anton Gorodetski
(joint work with A. Avila, D. Damanik, V. Kleptsyn)

The spectrum of a discrete Schrödinger operator with a periodic potential is known to be a finite union of intervals. The same is true for Anderson's model, i.e., for a Schrödinger operator where the potential is defined to be a sequence of i.i.d. random variables. The "intermediate" case of deterministic aperiodic potentials, or "one dimensional quasicrystals" (Fibonacci Hamiltonian, Sturmian, Almost Mathieu, limit periodic, substitution potentials, thus etc.), tends to present a Cantor set as its spectrum, even if it is not easy to prove this in many cases.

What happens if one adds some random noise on top of an aperiodic potential, or, more generally, a given ergodic potential? It turns out that, in many cases, "randomness" wins, both in terms of spectral type, and in terms of topological structure of the spectrum. More specifically, one can prove Anderson localization for such models, i.e., to show that the spectrum must be pure point almost surely. And, under the additional assumption that the phase space of the dynamical system that defines the background potential is connected, one can show that the almost-sure spectrum must be a finite union of intervals, exactly as in Anderson's model.

Let us now provide a formal setting and statements of some of these results. Given a compact metric space X, a homeomorphism $T : X \to X$, an ergodic Borel probability measure μ with full topological support, $\operatorname{supp}\mu = X$, and a sampling function $f \in C(X, \mathbb{R})$, we generate potentials

$$V_x(n) = f(T^n x),\ x \in X,\ n \in \mathbb{Z}$$

and Schrödinger operators

$$[H_x\psi](n) = \psi(n+1) + \psi(n-1) + V_x(n)\psi(n)$$

on $\ell^2(\mathbb{Z})$. By the general theory of ergodic Schrödinger operators on $\ell^2(\mathbb{Z})$, the spectrum of H_x, denoted by $\sigma(H_x)$, is almost surely independent of x. That is, there is a compact set Σ_0 such that

$$\Sigma_0 = \sigma(H_x) \quad \text{for } \mu\text{-almost every } x \in X.$$

The specific cases one might want to consider here are periodic potentials (in this setting, X is a finite set, and T is a cyclic permutation), the almost Mathieu potential (X is the circle, T is an irrational rotation, and $f : S^1 \to \mathbb{R}$ is given by $f(x) = 2\lambda\cos(2\pi x)$), the Fibonacci substitution sequence, Sturmian potentials (X is a Cantor set), etc. Any one-dimensional aperiodic structure can be modeled by an ergodic potential by considering the hull of a sequence, and hence can be included into this setting.

The random perturbation is given by

$$W_\omega(n) = \omega_n, \quad \omega \in \Omega,\ n \in \mathbb{Z},$$

where $\Omega = (\operatorname{supp}\nu)^{\mathbb{Z}}$ and ν is a compactly-supported probability measure on $\mathbb{R}$ with topological support $S := \operatorname{supp}\nu$ satisfying $\#S \geqslant 2$. It turns out that

Anderson localisation can be proved for the operator H_x+W_ω in this case. Namely, in our work with V. Kleptsyn [3] based on previously obtained non-stationary version of Furstenberg theorem on random matrix products [2], we show that the following holds.

Theorem 1. *Almost surely (with respect to $\nu^{\mathbb{Z}}$), the spectrum of the Schrödinger operator H_x+W_ω is pure point, with exponentially decreasing eigenfunctions. Also, dynamical localisation holds almost surely for the operator $H_x + W_\omega$.*

Let us now address the question about the topological structure of the spectrum of $H_x + W_\omega$. Since the product of μ and $\tilde{\mu} := \nu^{\mathbb{Z}}$ is ergodic with respect to the product of T and the left shift, there is, again by the general theory of ergodic Schrödinger operators in $\ell^2(\mathbb{Z})$, a compact set Σ_1 such that

$$\Sigma_1 = \sigma(H_x + W_\omega) \quad \text{for } \mu \times \tilde{\mu}\text{-almost every } (x,\omega) \in X \times \Omega.$$

Since $\operatorname{supp}(\mu \times \tilde{\mu}) = X \times S^{\mathbb{Z}}$, we also have

$$\Sigma_1 = \bigcup_{(x,\omega)\in X\times S^{\mathbb{Z}}} \sigma(H_x + W_\omega),$$

that is, the spectra corresponding to exceptional points can only be smaller than the almost sure spectrum. In particular,

$$\sigma(H_x + W_\omega) \subseteq \Sigma_1 \quad \text{for every } (x,\omega) \in X \times S^{\mathbb{Z}}.$$

Before stating the result, we introduce the following operation on pairs of compact subsets of $\mathbb{R}$. Suppose A and B are compact subsets of $\mathbb{R}$. We define the compact set $A \star B$ as follows. If $\operatorname{diam}(A) \geqslant \operatorname{diam}(B)$, then $A \star B = A + \operatorname{ch}(B)$, and if $\operatorname{diam}(A) < \operatorname{diam}(B)$, then $A \star B = \operatorname{ch}(A) + B$. Here, $\operatorname{diam}(S)$ denotes the diameter, $\operatorname{ch}(S)$ denotes the convex hull of a compact $S \subset \mathbb{R}$, and $S_1 + S_2$ denotes the Minkowski sum $\{s_1 + s_2 : s_1 \in S_1,\, s_2 \in S_2\}$.

In [1], jointly with A. Avila and D. Damanik, we prove the following.

Theorem 2. *Consider the setting described above and assume that X is connected. Then, we have $\Sigma_1 = \Sigma_0 \star S$.*

This theorem provides an affirmative answer to a question by Bellissard.

Corollary 3. *If X is connected, the almost sure spectrum Σ_1 is given by a finite union of non-degenerate compact intervals.*

Notice that the condition that X is connected cannot be removed, and, while it holds in many important cases (e.g., almost Mathieu potential as background), it also excludes many interesting models. We will formulate a specific open question.

Question 4. *Let H be the Fibonacci Hamiltonian (the discrete Schrödinger operator with the potential given by the Fibonacci substitution sequence). What can one say about topological structure of the spectrum of its random perturbation $H+W_\omega$?*

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Transversal Hölder regularity in tiling spaces and applications

Rodrigo Treviño

Let Ω be the tiling space of a primitive substitution tiling of finite local complexity. Anderson and Putnam [1] showed us how this is homeomorphic to the inverse limit

$$\Omega \cong \lim_{\leftarrow}(\Gamma,\gamma) := \{(x_0,x_1,\dots)\in\Gamma^\infty : x_i = \gamma(x_{i+1})\} \tag{12}$$

where Γ is a compact flat branched manifold of dimension $d \geqslant 1$ and $\gamma:\Gamma\to\Gamma$ is a locally expanding affine surjective map, which I will assume here to be uniformly expanding and has derivative $D\gamma = \lambda\cdot\mathrm{Id}$ outside the branches of Γ, for some $\lambda > 1$. For each k, there is a projection map $\pi_k:\Omega\to\Gamma$ to the k^{th} coordinate. The space Ω is a compact metric space and has the local product structure of a Euclidean ball of dimension d and a Cantor set. More precisely, the local k^{th} transversal around a point $x\in\Omega$ is the set $C_k^\perp(x) := \{y\in\Omega : y_i = x_i \text{ for all } 0\leqslant i\leqslant k\}$, which is a Cantor set. The tiling space Ω is foliated by orbits of a minimal $\mathbb{R}^d$ action which can be described in the coordinates (12)

$$\varphi_t(x) = \big(x_0 - t, x_1 - (D\gamma)^{-1}t, x_2 - (D\gamma)^{-2}t, \dots\big)\ , \tag{13}$$

for $x = (x_0,x_1,x_2,\dots)\in\Omega$ and $t\in\mathbb{R}^d$. Let μ be the unique $\mathbb{R}^d$-invariant probability measure, with local product structure $\mu = \mathrm{Leb}\times\nu$.

My talk focuses on trying to understand spaces of functions of varying regularity. The lowest amount of regularity one can ask for is continuity, and the algebra $C(\Omega)$ of continuous functions is well known. At the highest extreme of regularity are the C^∞ *transversally locally constant (tlc) functions* defined as

$$C^\infty_{tlc}(\Omega) = \bigcup_{k\geqslant 0}\pi_k^* C^\infty(\Gamma)$$

where $C^\infty(\Gamma)$ is the space of C^∞-smooth functions on Γ. This set captures the maximum regularity in both directions when seen locally: C^∞ in the Euclidean direction and locally constant in the Cantor direction.

Are there good spaces of functions of intermediate regularity? It is easy to imagine the space C^r_{tlc} of functions which are C^r in the Euclidean direction and locally constant in the Cantor direction, but is there a way to relax the regularity in the Cantor direction?

One of the motivations comes from cohomology. Recall that the leafwise or tangential de Rham cohomology of a tiling space $H^*_{tlc}(\Omega;\mathbb{R})$ is the cohomology of the complex of forms with coefficients in C^∞_{tlc} (this is also known as the strong pattern-equivariant cohomology). By [2], $H^*_{tlc}(\Omega;\mathbb{R})$ is isomorphic to the real Čech cohomology $\check{H}^*(\Omega;\mathbb{R})$. Here, every function $f\in C^\infty_{tlc}$ has a cohomology class $[f]\in H^d_{tlc}(\Omega;\mathbb{R})$, and every class in the finite dimensional vector space $H^d_{tlc}(\Omega;\mathbb{R})$ is represented by a function in C^∞_{tlc}. Pick a norm $\|\cdot\|$ on $H^d_{tlc}(\Omega;\mathbb{R})$ and, for each k, a function $f_k\in\pi_k^*C^\infty(\Gamma)\subset C^\infty_{tlc}$ such that $f_k\neq\gamma^*f_{k-1}$. Now consider the sequence of functions

$$F_k=\sum_{i=0}^{k}\varepsilon_i f_i, \tag{14}$$

where $\varepsilon_i=2^{-i}\|[f_i]\|^{-1}$. Then, $F_k\in C^\infty_{tlc}$ for all $k\geqslant 0$ and so $[F_k]\in H^d_{tlc}(\Omega;\mathbb{R})$ for all $k\geqslant 0$. Moreover, by construction, $[F_k]$ converges in $H^d_{tlc}(\Omega;\mathbb{R})$ but $F_\infty\notin C^\infty_{tlc}$ so F_∞ cannot be assigned a cohomology class even though it is really natural to assign it the limit of $[F_k]$. How can this conundrum be resolved? The reason $[F_k]$ converges has to do with the fast decay with k and so it would be nice to see that this is related to the regularity of F_∞ in the (transverse) Cantor direction.

In order to do this, consider the σ sub-algebra $\mathcal{A}_k$ of the Borel σ-algebra $\mathcal{A}$ of Ω which is generated by sets of the form $\pi_k^{-1}(A)$, where $A\subset\Gamma$ is an open set, and denote by $E(\cdot|\mathcal{A}_k):L^1(\mathcal{A},\mu)\to L^1(\mathcal{A}_k,\mu)$ the conditional expectation, which has the explicit form

$$E(f|\mathcal{A}_k)(x)=\nu(C_k^\perp(x))^{-1}\int_{C_k^\perp(x)}f(z)\,d\nu(z).$$

Denote $\Pi_k f:=E(f|\mathcal{A}_k)$ and let me make two remarks. First, $\Pi_k f$ is a transversally locally constant function. Second, by the increasing martingale theorem, $\Pi_k f\to f$ in L^1. Thus we can approximate arbitrary measurable functions by functions which are transversally locally constant. One way to think about $\Pi_k f$ is the amount of f one has access to if one only knows the coordinates $x_0,\dots,x_k$ of a point in Ω.

We can now rewrite $\Pi_k f$. Let $\Pi_{-1}f=0$ and, for $k\geqslant 0$, let $\delta_k:=\Pi_k f-\Pi_{k-1}f$. Then, $\Pi_k f=\sum_{i=0}^k\delta_k f$ with $\delta_k f=\pi_k^*g_k$, for some measurable function g_k on Γ and $\delta_k f\neq\gamma^*\delta_{k-1}f$. Thus, we see how any $f\in L^1$ can be expressed as the sum of functions which are pullbacks of functions on Γ: $f=\sum_{k\geqslant 0}\delta_k f$. Given the way one can think about $\Pi_k f$, one way to think about $\delta_k f=\Pi_k f-\Pi_{k-1}f$ is the amount of information we have about $f(x)$ if we only know the k^{th} coordinate x_k.

Now that we know how to consider functions both as infinite sums and as limits of transversally locally constant functions, we can talk about the decay rates. For $r,\alpha\geqslant 0$, let

$$\mathcal{S}^r_\alpha(\Omega):=\left\{f=\sum_{k\geqslant 0}f_k:\begin{array}{c}f_k=\pi_k^*(f^{(k)})\text{ for some }f^{(k)}\in C^r(\Gamma)\text{ and there}\\ \text{exists a }C_f\text{ such that }\|f^{(k)}\|_{C^r(\Gamma)}\leqslant C_f\lambda^{-k\alpha}\end{array}\right\}.$$

Two comments are in order about these spaces of functions. First, denoting by C^r_α the space of functions on Ω whose first r derivatives are continuous and transversally α-Hölder, we have the inclusion $\mathcal{S}^r_\alpha\subset C^r_\alpha$. Note that any notion of Hölder continuity has an implicit choice of metric, and so there is an implicit choice in the containment $\mathcal{S}^r_\alpha\subset C^r_\alpha$. Secondly, if $f\in\mathcal{S}^r_\alpha$, then $\partial_v f\in\mathcal{S}^{r-1}_{\alpha+1}$ for any $v\in\mathbb{R}^d$, where

∂_v denotes the leafwise directional derivative. This last property follows from the definition of $\mathcal{S}^r_\alpha$ and (13) by direct computation.

These spaces of functions separately capture regularity in the leaf direction as well as regularity in the transverse (Cantor) direction through the parameters $r \in \mathbb{N}$, $\alpha \in (0, \infty]$. The main reported result is the following de Rham regularization-like result.

Theorem 1. *Let $H^*_{r,\alpha}(\Omega; \mathbb{R})$ be the leafwise cohomology with coefficients in $\mathcal{S}^r_\alpha$. If $r \in \mathbb{N}$ and $\alpha > 1$, then $H^*_{r,\alpha}(\Omega; \mathbb{R})$ is isomorphic to $H^*_{tlc}(\Omega; \mathbb{R})$.*

This solves the conundrum of giving F_∞ in (14) a cohomology class. But it does more: the spaces $\mathcal{S}^r_\alpha$ can be completed to define so-called anisotropic Banach spaces which are crucial in understanding the rates of mixing of the hyperbolic map $\Phi : \Omega \to \Omega$ given by $\gamma(x) = (\gamma(x_0), x_0, x_1, \dots)$ through its induced action on cohomology. In addition, they help understand solutions to the cohomological equation $f = u \circ \phi - u$ for functions f in the Hölder class whenever ϕ is a minimal substitution subshift. More details and proofs of all of the mentioned results are found in the preprint [3].

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Substitutions on compact alphabets

Neil Mañibo

(joint work with Dirk Frettlöh, Alexey Garber, Dan Rust, Jamie Walton)

Let $\mathcal{A}$ be a compact Hausdorff topological space. A *substitution* ϱ on $\mathcal{A}$ is a continuous map from $\mathcal{A}$ to the set $\mathcal{A}^+$ of (non-empty) finite words over $\mathcal{A}$. From ϱ, one can build a subshift $X_\varrho \subset \mathcal{A}^{\mathbb{Z}}$ using the language $\mathcal{L}_\varrho = \overline{\{v \in \mathcal{A}^+ : v \lhd \varrho^n(a)\}}$, where $\lhd$ denotes the subword relation and where the closure is taken with respect to the topology of $\mathcal{A}^+$. As in the finite alphabet case, one is interested in the properties of the topological dynamical system (X_ϱ, σ); see [5] for details and [1,2] for related works.

In this setting, one calls ϱ *primitive* if, for every non-empty open set $U \subset \mathcal{A}$, there exists an $n := n(U)$ such that $\varrho^n(a)$ contains a letter in U, for all $a \in \mathcal{A}$. This implies the minimality of (X_ϱ, σ). A weaker notion is that of *irreducibility*, which means that ϱ cannot be restricted to a proper subalphabet of $\mathcal{A}$.

Requiring that ϱ is continuous has immediate consequences. In particular, this implies that ϱ must be of constant length if $\mathcal{A}$ is connected. Moreover, one obtains a canonical space to work with, which is the space $E = C(\mathcal{A})$ of real-valued continuous functions on $\mathcal{A}$. This is a *Banach lattice*, whose positive cone K is the set of functions with $f(a) \geqslant 0$ for all a. One can define the *substitution operator* $M := M_{\varrho} \colon E \to E$ via

$$(Mf)(a) = \sum_{b \lhd \varrho(a)} f(b),$$

where $\varrho(a)$ is seen as a multiset. This generalises the (transpose of the) substitution matrix for substitutions on finite alphabets. If ϱ is primitive, one has the bounds $\min_{a\in\mathcal{A}} |\varrho^n(a)| \leqslant r(M)^n \leqslant \max_{a\in\mathcal{A}} |\varrho^n(a)|$ involving the spectral radius $r(M)$. Since $MK \subset K$, M is a *positive* operator.

We call $0 \neq \ell \in K$ a *natural length function* if $M\ell = \lambda\ell$, for some $\ell > 0$. Two questions one can ask regarding substitutions on compact alphabets are

(Q1) When does ϱ admit a strictly positive natural length function with $\lambda > 1$?

(Q2) When is (X_{ϱ}, σ) uniquely ergodic?

A positive answer to **Q1** allows one to realise ϱ as a geometric inflation rule that generates tilings of $\mathbb{R}$ (which are typically of infinite local complexity). Here, λ is called the *inflation factor*.

To answer these questions, one needs a suitable version of Perron–Frobenius theory for positive operators on Banach lattices (particularly AM-spaces). The classical Kreĭn–Rutman theorem cannot be applied here since the operator M is never compact whenever ϱ is primitive; see [5].

An operator $M \colon E \to E$ is called *quasi-compact* if there exists a compact operator C and a power n such that $\|M^n - C\| < r(M)^n$. Equivalently, $r_{\text{ess}}(M) < r(M)$, where $r_{\text{ess}}(M)$ is the essential spectral radius of M. We briefly remark that if $\mathcal{A}$ is finite, M is trivially quasi-compact since it is of finite range (and hence compact). The next result shows that this property leads to affirmative answers to the two questions above.

Theorem 1 ([5]). *Let ϱ be a substitution on a compact Hausdorff alphabet $\mathcal{A}$. Suppose ϱ is primitive and M is quasi-compact. Then*

(1) ϱ admits a strictly positive natural length function with $\lambda = r(M)$.
(2) (X_{ϱ}, σ) is uniquely ergodic. □

To get (1), it suffices to satisfy the weaker assumption that ϱ is *irreducible* and M is *mean ergodic*. On the other hand, (2) is guaranteed if ϱ is irreducible and M is *strongly power convergent*; see [5] and references therein. When ϱ is irreducible, one has the following hierarchy of convergence properties

$$M \text{ quasi-compact} \implies M \text{ strongly power convergent} \implies M \text{ mean ergodic}.$$

In this talk, we chose to focus on the stronger assumptions of Theorem 1 as they are easier to check for examples. In particular, we have the following combinatorial condition which implies the quasi-compactness of M.

Theorem 2 ([5]). *Let ϱ be a substitution on a compact Hausdorff alphabet $\mathcal{A}$. Let $F \subset \mathcal{A}$ be finite. For $k \in \mathbb{N}$, consider $C_k(F) := \max_{a\in\mathcal{A}} \#\{b \lhd \varrho^k(a),\, b \notin F\}$. If, for some $k \in \mathbb{N}$,*

(i) $C_k(F) < r(M)^k$ *and* F *consists only of isolated points,* or

(ii) $2C_k(F) < r(M)^k$,

then M *is quasi-compact.* □

The proof uses the finite set F to construct the compact operator C in the definition of quasi-compactess. The lower bound for $r(M)^k$ involving $|\varrho^n(a)|$ mentioned above turns this into a checkable condition.

A family of examples parametrised by a sequence $\boldsymbol{m} = (m_i)_{i\geqslant 0}$ of non-negative integers was discussed during the talk. For $n \in \mathbb{N}_0$, one builds the rule

$$\varrho_{\boldsymbol{m}}\colon \begin{cases} 0 \mapsto & 0^{m_0}\,1 \\ 1 \mapsto & 0^{m_1}\,0\,1 \\ \vdots & \vdots \\ n \mapsto & 0^{m_n}\,(n{-}1)\,(n{+}1), \end{cases}$$

where 0^m refers to the concatenation of m copies of 0. This is the generalisation of the example with the constant sequence $\boldsymbol{m} = (1)_{i\geqslant 0}$ in [6].

This rule is then extended to a suitable compactification of the set $\mathbb{N}_0$, that is, $\mathcal{A} = \overline{\iota(\mathbb{N}_0)}$, where $\iota(\mathbb{N}_0)$ is an appropriate embedding of $\mathbb{N}_0$ in some full shift Y. Both the ambient full shift Y and the embedding ι depend on $\boldsymbol{m}$. The closure is taken with respect to the topology on Y. Depending on $\boldsymbol{m}$, the set $\mathcal{A}\backslash\iota(\mathbb{N}_0)$ of accumulation points can be finite, countably infinite, or uncountable; see [4] for a detailed account.

When $\boldsymbol{m} = (1)_{i\geqslant 0}$, the alphabet is $\mathcal{A} = \mathbb{N}_0 \cup \{\infty\}$, ϱ_m is primitive, and M is quasi-compact. The inflation factor is $\lambda = 5/2$, which is impossible to achieve with a substitution on a finite alphabet (where λ is always an algebraic integer). Moreover, the length function is given by $\ell(n) = 2 - \frac{1}{2^n}$ for $n \in \mathbb{N}_0$ and $\ell(\infty) = 2$. It turns out that this generalisation allows one to realise an even larger class of possible inflation factors; see [4].

Theorem 3 ([4]). *Let* $\lambda > 2$. *Then, there exists a sequence* $\boldsymbol{m} = (m_i)_{i\geqslant 0}$ *of non-negative integers (satisfying some mild assumptions) such that*

(1) $\varrho_{\boldsymbol{m}}$ *is primitive and* M *is quasi-compact and*

(2) $\varrho_{\boldsymbol{m}}$ *has inflation factor* λ. □

Closed forms for λ and $\ell(n)$ (for $n \in \mathbb{N}_0$) are available and are given by

$$\lambda = \mu + \frac{1}{\mu} \qquad \text{and} \qquad \ell(n) = \mu^n + \sum_{j=1}^{n}\sum_{i=j}^{\infty} m_i \mu^{i+n+1-2j},$$

where μ is the unique solution in $(0,1)$ of the equation $\frac{1}{\mu} = \sum_{i=0}^{\infty} m_i \mu^i$; see [3, 4].

An explicit example where λ is transcendental is also given in [4]. This is the case when $\boldsymbol{m} = (2,1,1,2,1,2,2,1,\ldots)$ is the Thue–Morse sequence on $\{1,2\}$. This provides the first example of a substitution tiling with transcendental inflation factor. Primitivity and quasi-compactness also lead to the existence of a *spectral* *gap*, which is useful in discrepancy estimates; see [3] for Catalan-like behaviour manifested by elements of this infinite family of substitutions.

Other interesting examples include the following respective generalisations of Thue–Morse and Rudin–Shapiro substitutions,

$$\varrho_{\mathrm{TM},\alpha} \colon \theta \mapsto (\theta)\,(\theta{+}\alpha) \qquad \text{and} \qquad \varrho_{\mathrm{RS},\alpha} \begin{cases} (\theta,0) & \mapsto (\theta,0)\,(\theta{+}\pi,1) \\ (\theta,1) & \mapsto (\theta{+}\alpha,0)\,(\theta{+}\alpha,1) \end{cases},$$

where $\alpha \in \mathbb{T} \simeq \mathbb{R}/\mathbb{Z}$ is irrational. The corresponding alphabets are $\mathcal{A} = \mathbb{T}$ for $\varrho_{\mathrm{TM},\alpha}$ and $\mathcal{A} = \mathbb{T} \times \{0,1\}$ for $\varrho_{\mathrm{RS},\alpha}$; see [6]. Both substitutions are primitive. The substitution operators are not quasi-compact, but are strongly power convergent. It was shown in [6] that $\varrho_{\mathrm{TM},\alpha}$ has a purely singular dynamical spectrum, while $\varrho_{\mathrm{RS},\alpha}$ has countably infinite Lebesgue spectrum, countably infinite singular continuous spectrum, and a pure point component coming from the dyadic odometer.

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Galois groups and Cantor dynamics

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(joint work with María Isabel Cortez)

This is a brief summary of results on the properties of arboreal representations of absolute Galois groups of number fields in the joint work with Cortez [2]. The dynamical system we discuss is an action of a profinite group on the boundary of a rooted tree T, constructed as follows.

Let $f(x)$ be a polynomial of degree $d \geqslant 2$ over K, where K is a finite field extension of $\mathbb{Q}$. Fix $\alpha \in K$ and, for $n \geqslant 1$ and $f^n = f \circ f^{n-1}$, consider the equation $f^n(x) = \alpha$. We assume that the polynomial $f^n(x) - \alpha$ is irreducible over K for all $n \geqslant 1$, hence the equation has d^n distinct solutions. We define $V_0 = \{\alpha\}$ and $V_n = f^{-n}(\alpha)$, with $n \geqslant 1$, to be the *vertex level sets* of the tree T. We join two vertices $v \in V_{n+1}$ and $w \in V_n$ by an edge if and only if $f(v) = w$.

Since the polynomial $f^n(x) - \alpha$ is irreducible for all $n \geqslant 1$, the finite field extensions $K[f^{-n}(\alpha)]/K$ are Galois, and the Galois group H_n is finite and permutes the roots in V_n transitively. The action of H_{n+1} on V_{n+1} is compatible with the action of H_n, in the sense that it preserves the relation of connectedness by an edge on vertices. Thus, this construction defines a homomorphism

$$\rho_{f,\alpha} : \mathrm{Gal}(\overline{K}/K) \to \mathrm{Gal}_{f,\alpha} \subset \mathrm{Aut}(T)\,, \tag{15}$$

called the *arboreal representation* of $f(x)$ over K, where $\mathrm{Aut}(T)$ is the group of automorphisms of T, and

$$\mathrm{Gal}_{f,\alpha} = \varprojlim\{H_n \to H_{n-1} \mid n \geqslant 1\} \subset \mathrm{Aut}(T)\,.$$

The study of arboreal representations goes back to Odoni [7], who used the dynamics of arboreal representations to obtain an estimate on the density of prime divisors in the set $\{f^n(\alpha) \mid n \geqslant 1\}$, for polynomials over $\mathbb{Q}$. The study of arboreal representations is an active research area in arithmetic dynamics. Since the image of an arboreal representation is a group acting on the boundary of a rooted tree, it is natural to use topological dynamics and geometric group theory to study its properties.

In [5], I showed how to study the action of the profinite group $\mathrm{Gal}_{f,\alpha}$ on the boundary of the tree T, using the methods developed for actions of countable groups on Cantor sets. In [5], I also gave the first examples of computations of algebraic invariants for the actions of arboreal representations, based on invariants developed for actions of countable groups on Cantor sets in [3, 4]. More examples were computed in [6].

One of the open questions about arboreal representations of Galois groups is the properties of the images of Frobenius elements in $\mathrm{Gal}(\overline{K}/K)$ under such representations. A conjecture by Boston and Jones [1] states that the images of Frobenius elements in $\mathrm{Aut}(T)$ are *settled*, which means that they have a specific cycle structure. By the Chebotarev density theorem, Frobenius elements are dense in the absolute Galois group of $\mathbb{Q}$, which motivates the following conjecture, due to Boston and Jones [1].

Conjecture 1. *Let $f(x)$ be a polynomial of degree $d \geqslant 2$ over a number field K, let $\alpha \in K$, and let $\mathrm{Gal}_{f,\alpha} \subset \mathrm{Aut}(T_d)$ be the image of the corresponding arboreal representation. Then, the images of Frobenius elements in $\mathrm{Aut}(T)$ are* settled*, and so the set of settled elements is dense in $\mathrm{Gal}_{f,\alpha}$.*

To define the notion of a settled element, note that each $g \in \mathrm{Aut}(T)$ induces a permutation $\pi_{g,n}$ of V_n, $n \geqslant 1$. Each cycle τ of $\pi_{g,n+1}$ projects onto a cycle τ' of $\pi_{g,n}$. Such a projection restricted to τ is an m-to-1 map, for an integer $1 \leqslant m \leqslant d$.

Definition 2. *A vertex $v_n \in V_n$ is in a* stable cycle *$\tau_{g,n}$ if, for any $m \geqslant n$ and any cycle $\tau_{g,m}$ in $\pi_{g,m}$ which projects onto $\tau_{g,n}$, the projection $\tau_{g,m} \to \tau_{g,n}$ is a d^{m-n}-to-1 map. We say that $g \in \mathrm{Aut}(T)$ is* settled *if*

$$\lim_{n\to\infty} \frac{|\{v \in V_n \mid v \text{ is in a stable cycle}\}|}{|V_n|} = 1\,.$$

Conjecture 1 holds for arboreal representations whose image has finite index in $\mathrm{Aut}(T)$ [1], for post-critically finite quadratic polynomials over $\mathbb{Q}$, for which the orbit of the critical point is strictly periodic of length 1, or pre-periodic of length

2 [1], and for certain infinite wreath products of finite groups [2]. Further results were obtained in [2].

Definition 3. *A quadratic polynomial $f(x)$ is* post-critically finite*, or* PCF*, if the orbit*

$$P_c = \{f(c), f^2(c), \ldots\}$$

of its critical point c under forward iterations of f is finite.

The orbit P_c of c is pre-periodic *if there exists $k \geqslant 1$, and $m_c > 1$ such that, for all $s \geqslant 1$ and all $m \geqslant m_c$, we have $f^{m+sk}(c) = f^m(x)$.*

Let K be a finite extension of $\mathbb{Q}$. We consider the generic case which is handled as follows. Let t be a transcendental element. Adjoining to $K(t)$ the roots of the equation $f^n(x) = t$, we obtain a Galois extension of K, for $n \geqslant 1$. Implementing the construction similar to the one used to define (15), we obtain a profinite group $\mathrm{Gal}_{\mathrm{arith}}(f)$, called the *profinite arithmetic iterated monodromy group.* The profinite group $\mathrm{Gal}_{\mathrm{arith}}(f)$ contains a closed subgroup $\mathrm{Gal}_{\mathrm{geom}}(f)$, called the *profinite geometric iterated monodromy group.* Pink observed that $\mathrm{Gal}_{\mathrm{geom}}(f)$ is isomorphic to the closure of the discrete iterated monodromy group studied in geometric group theory, and so the methods of the latter can be used to study the properties of this group. One can specialize t to α, thus obtaining the group $\mathrm{Gal}_{f,\alpha}$ which is not larger, and in most cases equal to $\mathrm{Gal}_{\mathrm{arith}}(f)$.

Let $r = |P_c|$, the length of the orbit of c. In [2], we obtained the following results.

Theorem 4. *Let $f(x)$ be a quadratic PCF polynomial over a number field K with pre-periodic post-critical orbit of length $r = 2$. Then,* $\mathrm{Gal}_{\mathrm{geom}}(f)$ *is not densely settled, and* $\mathrm{Gal}_{\mathrm{arith}}(f)$ *is densely settled.*

Thus, Conjecture 1 holds for arboreal representations associated to quadratic polynomials over number fields with pre-periodic post-critical orbit of length 2. For $r \geqslant 3$, the situation is as follows.

Theorem 5. *Let $f(x)$ be a quadratic PCF polynomial over a number field K with pre-periodic post-critical orbit of length $r \geqslant 3$. Then we have:*

(1) Substantial evidence *that* $\mathrm{Gal}_{\mathrm{geom}}(f)$ *is densely settled;*

(2) Substantial to good *(depending on K)* evidence *that* $\mathrm{Gal}_{\mathrm{arith}}(f)$ *is densely settled.*

The proof uses the analysis of the *Weyl groups of maximal tori.* Weyl groups and maximal tori for actions of profinite groups were introduced in [2], where also the notions of good and substantial evidence for the conjecture are made precise. We refer to [2] for details.

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The conjugacy relation of Cantor minimal systems

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(joint work with Konrad Deka, Kosma Kasprzak, Philipp Kunde, Dominik Kwietniak)

The existence of an isomorphism between two objects within a certain context tells us that these objects cannot be distinguished from the point of view of that theory. The objective of the classification problem is to have a method to determine all possible (non-isomorphic) objects of the theory. The best scenario is a *classification* result, which identifies all possible classes of isomorphic objects and a *concrete* technique to determine the isomorphism class of each object. For example, a result of Halmos and von Neumann states that every minimal equicontinuous dynamical system is conjugated to a rotation on a compact abelian group, and two of such systems are conjugated if and only if the set of eigenvalues coincide. On the other hand an *anti-classification* result tells us when a classification result of a certain type is not possible [2]. The first step towards anti-classification requires fixing a representation for our problem by providing an appropriate model (or universal space). Here, by a *model* we understand a topological space such that every isomorphism class corresponds to at least one point in the space. Furthermore, we obtain an equivalence relation on the model space by identifying points that represent the same isomorphism class within our model space.

Here, we are interested in explaining an anti-classification result for Cantor minimal dynamical systems that rules out characterisations based on countable arguments.

We say (X, T) is a *topological dynamical system (TDS)* if X is a compact metrisable space and $T : X \to X$ is a homeomorphism. Let K be a Cantor space. A TDS (K, T) is called a *Cantor system*.

The space of dynamical systems on a given set (our model) will be denoted by $\mathrm{Homeo}(X) = \{T : (X, T) \text{ is a TDS}\}$. We equip $\mathrm{Homeo}(X)$ with the following metric

$$d_s(T_1, T_2) = \sup\{d(T_1^{\ell} x, T_2^{\ell} x) : x \in X, \ell \in \{-1, 1\}\}.$$

This makes $\mathrm{Homeo}(X)$ a Polish space.

Two TDS (X_1, T_1) and (X_2, T_2) are *conjugated* if there exists a homeomorphism $f : X_1 \to X_2$ such that $f \circ T_1 = T_2 \circ f$. In this case, we write $(X_1, T_1) \approx (X_2, T_2)$.

The equivalence relation generated by conjugacy is denoted by

$$\mathcal{R}_{\approx}(X) = \{(T_1, T_2) : (X, T_1) \approx (X, T_2)\} \subset \mathrm{Homeo}(X) \times \mathrm{Homeo}(X).$$

The following result indicates that $\mathcal{R}_{\approx}(K)$ is a complicated set. Informally, it means that Cantor systems cannot be classified using inherently countable techniques.

Theorem 1 (Camerlo and Gao [1]). *$\mathcal{R}_{\approx}(K)$ is not a Borel subset of the product Homeo$(K) \times$ Homeo(K).*

A TDS is *minimal* if for every closed subset $A \subset X$ such that $T(A) \subset A$ we have that $A = \varnothing$ or $A = X$. We define

$$\mathcal{R}^{min}_{\approx}(K) = \{(T_1, T_2) \in \mathcal{R}_{\approx}(K) : (X, T_1) \text{ is minimal}\}.$$

The following question was studied for pointed Cantor minimal systems.

Question 2 (Gao). *Is $\mathcal{R}^{min}_{\approx}(K)$ a Borel subset?*

The triplet (X, T, x) is a pointed TDS if (X, T) is a TDS and $x \in X$. We say (X_1, T_1, x_1) and (X_2, T_2, x_2) are *conjugated* if there exists a homeomorphism $f : X_1 \to X_2$ such that $f \circ T_1 = T_2 \circ f$ and $f(x_1) = x_2$.

Theorem 3 (Kaya [3]). *The equivalence relation generated by conjugacy of pointed Cantor minimal systems is a Borel subset.*

We prove that the situation for classical Cantor minimal systems is different.

Theorem 4. *$\mathcal{R}^{\mathrm{min}}_{\approx}(K)$ is not a Borel subset.*

Similar to the work of Foreman, Rudolph and Weiss [2], we will use ill-founded trees to obtain our result.

A *tree* over an at most countable alphabet B is a prefix-closed collection of words over B, that is $T \subset B^{<\mathbb{N}}$ is a tree if for each $w \in T$ every prefix of w also belongs to T, see [4, Ch. 2]. One can endow the set of all trees Tr over B with the topology given by the metric D obtained in the following way: first enumerate all words in the countable set $B^{<\mathbb{N}}$ to form a sequence $w_0, w_1, \ldots$. Then, for trees $S, T \in \mathrm{Tr}$, set

$$D(S, T) = \begin{cases} 0, & \text{if } S = T, \\ 2^{-\min\{j \geqslant 0 : w_j \in S \div T\}}, & \text{if } S \neq T. \end{cases}$$

We say that a sequence $\omega \in B^{\mathbb{N}}$ is an infinite branch of the tree T if, for every $n \in \mathbb{N}$, the word $\omega_1 \ldots \omega_n$ is in T. We say that a tree with at least one infinite branch is *ill-founded*.

We denote the set of rooted trees (every word starts with the same letter) with arbitrarily long branches on $\mathbb{N}$ with Tr and the subset of ill-founded trees with **IF**. The space Tr can be viewed as a Polish space using the Hausdorff metric on subsets of $B^{<\mathbb{N}}$. The set **IF** is complete analytic [4, Thm. 26.1] and hence it is not a Borel subset.

We define

$$\mathrm{Homeo}^{min}(K) = \{T \in \mathrm{Homeo}(K) : (K, T) \text{ is minimal}\}.$$

The following proposition is the main step to prove Theorem 4, and it is proved using extensions of odometers.

Proposition 5. *There exists a Borel function*

$$f\colon \mathrm{Tr} \to \mathrm{Homeo}^{min}(K)$$

such that $t \in \boldsymbol{IF}$ *if and only if* $(K, f(t))$ *is conjugated to* $(K, (f(t))^{-1})$.

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Tame implies regular

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(joint work with Gabriel Fuhrmann, Eli Glasner, Christian Oertel)

We call (X, G) a *topological dynamical system* (TDS) if X is a compact metric space and G is a topological group acting on X by homeomorphisms $\varphi_g : X \to X$. When G is non-discrete, we assume in addition that the joint action $(g, x) \mapsto \varphi_g(x)$ is continuous. We say (X, G) is *minimal* if there exists no non-empty G-invariant compact proper subset of X. An independence pair for (X, G) is a pair of closed and disjoint subsets $U_0, U_1 \subseteq X$ such that there exists an infinite set $S \subseteq G$ such that

$$\text{for all } a \in \{0,1\}^S \text{ there exists } x \in X \text{ with } \varphi_g(x) \in U_{a_g}\,. \tag{16}$$

If G is amenable and S has positive asymptotic density in G, we call (U_0, U_1) a positive density independence pair.

A TDS is called *tame* if the cardinality of its Ellis semigroup is of cardinality at most $2^{\aleph}$. The following result of Kerr and Li allows to define tameness via independence pairs, and thus in a more tangible way from a dynamical systems perspective.

Theorem 1 ([1]). *A TDS* (X, G) *is tame if and only if it has no independence pair.*

In order to put this characterisation of tameness into perspective, it is insightful to compare it to the following equivalent characterisation of of positive entropy systems.

Theorem 2 ([1]). *A* $\mathbb{Z}$*-action* $(X, \mathbb{Z})$ *has positive topological entropy if and only if it has a positive entropy independence pair.*

Hence, one may view non-tameness of a dynamical system as a weakening of positive entropy, and therefore as a mild form of chaoticity.

Our aim is to obtain a better understanding of the structural implications of tameness. To this end, recall that any TDS (X, G) has a *maximal equicontinuous factor* (MEF), which we denote by (Y, G, π), where (Y, G) is an equicontinuous TDS and $\pi : X \to Y$ is the factor map. *'Maximal'* here refers to the fact that any other equicontinous factor $(\tilde{Y}, G, \tilde{\pi})$ can be obtained as a subfactor of (Y, G, π), that is, there exists a factor map $h : Y \to \tilde{Y}$ such that $\tilde{\pi} = h \circ \pi$.

When (X, G) is minimal, then (Y, G) is minimal and hence (as an equicontinuous system) uniquely ergodic. If (X, G) and (Y, G) are uniquely ergodic, we denote by μ and ν the respective unique invariant measures. We say that

- (X, G) is an *isomorphic extension* (of its MEF) if it is uniquely ergodic and π is a measure-theoretic isomorphism between (X, G, μ) and (Y, G, ν);
- (X, G) is an *almost one-to-one extension* if $\#\pi^{-1}(y) = 1$ for some $y \in Y$;
- (X, G) is a *regular* extension if $\#\pi^{-1}(y) = 1$ holds for ν-almost every $y \in Y$.

In [2], Glasner established a comprehensive structure theory for minimal tame group actions. One important result is the following.

Theorem 3 ([2]). *Suppose (X, G) is a minimal and tame TDS which has an invariant probability measure. Then, (X, G) is an almost one-to-one extension of its maximal equicontinuous factor.*

It was left open in [2], however, whether tame systems are actually regular extensions. Our main result is an affirmative answer to this question.

Theorem 4 ([3]). *Suppose that (X, G) is a minimal TDS that is an almost one-to-one extension of its MEF. If (X, G) is tame, it is a regular extension of its MEF.*

Remark 5. *We note that previously García-Ramos established regularity of $\mathbb{Z}$-action $(X, \mathbb{Z})$ that are null (have zero topological sequence entropy) [4]. As nullness can be characterised in terms of independence pairs with arbitrarily large finite (but not necessarily infinite) index sets S, it is a natural strengthening of tameness.*

In the proof of Theorem 4, a crucial step is to reduce the problem for the original system (X, G) to one in the MEF (Y, G). Key to this is an application of Lusin's theorem to the set-valued mapping $Y \to \mathcal{K}(X),\ y \mapsto \pi^{-1}(y)$, where $\mathcal{K}(X)$ denotes the space of compact subsets of X (equipped with the Hausdorff metric). This allows to construct a pair of closed sets $V_0, V_1 \subseteq Y$, obtained as projections of disjoint closed sets $U_0, U_1 \subseteq X$, such that property (16) for V_0, V_1, with φ replaced by ψ, implies property (16) also for U_0, U_1. Instead of being disjoint like U_0, U_1, however, the common boundary $\partial V_0 \cap \partial V_1$ has positive measure, while the interiors are still disjoint. These particular properties of the sets V_0, V_1, together with the group structure of Y, eventually allow to define an infinite S for which (16) is satisfied, in a careful inductive construction.

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Symbolic substitutions in the Heisenberg group and spectral approximation

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(joint work with Ram Band, Siegfried Beckus, Tobias Hartnick, Lior Tenenbaum)

Prelude. We consider the discrete Heisenberg group $\Gamma = H_3(2\mathbb{Z})$ over the even integers with group multiplication

$$(x, y, z)(a, b, c) = \big(x + a, y + b, z + c - 1/2(xb - ya)\big)\,.$$

Most of the presented assertions can be extended to a large class of lattices in rationally homogeneous Lie groups with rational spectrum, cf. [2], but for the sake of a simple exposition, we stick to the Heisenberg group. The restriction to even integers is a technicality that has some illustrative advantages, but one could consider $H_3(\mathbb{Z})$ as well. Apparently, Γ is generated by the set $S = \{\pm(2, 0, 0), \pm(0, 2, 0)\}$. The (left) Cayley graph with respect to (Γ, S) consists of Γ as the vertex set, and $x, y \in \Gamma$ are connected by an edge if and only if $x^{-1}y \in S$. We write d_S for the underlying path metric, and denote by $B_R^S(x)$ the closed ball of radius $R > 0$ around $x \in \Gamma$ with respect to d_S. Throughout the talk, we fix a finite set $\mathcal{A}$. Then, each $\omega \in \mathcal{A}^\Gamma$ can be interpreted as a colouring of Γ with the finitely many colours in $\mathcal{A}$. Note further that Γ acts on $\mathcal{A}^\Gamma$ by translations as $\gamma.\omega(x) = \omega(\gamma^{-1}x)$ for $\gamma, x \in \Gamma$.

For each colouring ω, we will be concerned with the linear Schrödinger-type operators $H_\omega : \ell^2(\Gamma) \to \ell^2(\Gamma)$ acting as

$$H_\omega u(x) = \sum_{y\in\Gamma} b_\omega(x, y)\big(u(x) - u(y)\big) + c_\omega(x)u(x), \quad u \in \ell^2(\Gamma).$$

Here, we call such an operator *nice* if the following properties are satisfied:

- $b_\omega(x, y) \geqslant 0$ and $c_\omega(x) \geqslant 0$ (non-negativity);
- $b_\omega(x, y) = b_\omega(y, x)$ (symmetry);
- $b_{\gamma.\omega}(\gamma x, \gamma y) = b_\omega(x, y)$ and $c_{\gamma.\omega}(\gamma x) = c_\omega(x)$ (translation equivariance);
- There exists $P > 0$ such that $\omega_{|B_P^S(x)\cup B_P^S(y)} = \omega'_{|B_P^S(x)\cup B_P^S(y)}$ implies $b_\omega(x, y) = b_{\omega'}(x, y)$ and $c_\omega(x) = c_{\omega'}(x)$ (pattern equivariance);
- There exists $R > 0$ such that $d_S(x, y) > R$ implies $b_\omega(x, y) = 0$ (finite hopping range).

Suppose now that $\omega \in \mathcal{A}^\Gamma$ is *non-periodic*, so $\mathrm{stab}_\Gamma(\omega) = 0$. Can one say something about the spectrum $\sigma(H_\omega)$ of H_ω, or its spectral distribution function? Of course, this is a very general question, and no straight-forward answers can be expected. However, one might attempt to approximate spectral quantities by finite volume of periodic analogues. We present two results in this direction.

Symbolic substitution systems. We must first determine the class of non-periodic ω that we would like to consider. One rich source of examples is given by symbolic substitution systems developed recently in [2]. We also refer to the Oberwolfach report [4]. Note that Γ is a lattice, i.e., a discrete co-compact subgroup of the continuous Heisenberg group $G = H_3(\mathbb{R})$. A relatively compact left-fundamental domain is given by $V = [-1,1)^3$. For $\lambda_0 \in \mathbb{N}$ with $\lambda_0 \geqslant 3$, we define the group automorphism $D = D_{\lambda_0} : G \to G$, $D(x,y,z) = (\lambda_0 x, \lambda_0 y, \lambda_0^2 z)$. Any map $S_0 : \mathcal{A} \to \mathcal{A}^{D(V)\cap\Gamma}$ is called a *substitution rule.* As was demonstrated in [2], any such map has a canonical extension $\widetilde{S}$ to all coloured patterns of Γ in such a way that the substitution map can be replicated by application to all letters in a pattern. In particular, one obtains a canonical continuous substitution map $S : \mathcal{A}^\Gamma \to \mathcal{A}^\Gamma$, cf. [2, Prop. 2.7]. For $\omega \in \mathcal{A}^\Gamma$, we define its *hull* as $\Omega_\omega := \overline{\{\gamma.\omega : \gamma \in \Gamma\}}$, where the closure is taken in the product topology. For the definition of primitivity and non-periodicity for substitution rules, we refer to [2,4].

Theorem 1 (Beckus–Hartnick–P.)**.** *Suppose that S_0 is a primitive and non-periodic substitution rule. Then, there is $\omega_0 \in \mathcal{A}^\Gamma$ such that $S^k(\omega_0) = \omega_0$ for some $k \in \mathbb{N}$, ω_0 is linearly repetitive with respect to the homogeneous metric inherited from G, and the action $\Gamma \curvearrowright \Omega_{\omega_0}$ is minimal, uniquely ergodic and free.*

The theorem has various interesting consequences, such as the existence of (explicitly constructable) strongly aperiodic, linearly repetitive Delone sets in the Heisenberg group $H_3(\mathbb{R})$. In particular, obtaining a free action from a non-periodic ω_0 is much harder than in the abelian setting, where it is an immediate consequence of minimality. For a systematic study of linear repetitivity for non-abelian groups and a proof of unique ergodicity in this context, we refer to [1]. In the following, we call ω_0 constructed by the above theorem *good.*

Spectral approximation. We return to the question on the spectra of Schrödinger-type operators as above, and suppose that ω_0 is good (in particular being a k-fixed point of a suitable substitution map $S : \mathcal{A}^\Gamma \to \mathcal{A}^\Gamma$), and H_{ω_0} is nice.

Integrated density of states. For $n \in \mathbb{N}$, we define the set $F_n := \big([-n,n]^2 \times [-n^2,n^2]\big) \cap \Gamma$, as well as the (matrix) operators $H^n_{\omega_0} = p_n H_{\omega_0} i_n$, where $i_n : \ell^2(F_n) \to \ell^2(\Gamma)$ and $p_n : \ell^2(\Gamma) \to \ell^2(F_n)$ are the canonical inclusion, respectively projection. Each $H^n_{\omega_0}$ is a self-adjoint matrix with a finite sequence (E_i) of real eigenvalues that can be ordered increasingly. Applying results of Lenz, Schwarzenberger and Veselić [5], we obtain a uniform approximation result for the spectral distribution function per unit volume, called the *integrated density of states*, via the empirical eigenvalue distributions.

Theorem 2. *Suppose ω_0 is good and H_{ω_0} is nice. Then,*

$$\lim_{n\to\infty} \frac{\sup\{i \leqslant \#F_n : E_i \leqslant E\}}{\#F_n} = \int_{\Omega_{\omega_0}} \langle \delta_0, 1_{(-\infty,E]}(H_\omega)\delta_0\rangle \, d\mu(\omega)$$

holds uniformly in $E \in \mathbb{R}$.

The proof of uniform convergence follows from a Banach space-valued ergodic theorem, cf. [5, Thm. 4.5], which is applicable due to unique ergodicity of $\Gamma \curvearrowright \Omega_{\omega_0}$. The limit is called the *Pastur–Shubin trace formula*. Here, $1_{(-\infty E]}$ is the spectral projector up to energy level E, and μ is the unique Γ-invariant probability measure on the hull.

Spectrum as a set. We fix one colour $a_0 \in \mathcal{A}$, and set $\omega_1 \equiv a_0$ as the constant configuration, i.e., $\omega_1(x) = a_0$ for every $x \in \Gamma$. Defining $\omega_{n,+} := S^n(\omega_1)$ for $n \in \mathbb{N}$ we obtain periodic configurations that are invariant under the group $\Gamma_n := D^n(\Gamma)$. Hence, the hull Ω_n of $\omega_{n,+}$ is finite. The following is from work in progress.

Theorem 3 (Band–Beckus–P.–Tenenbaum)**.** *Suppose that $\lambda_0 \geqslant 4$. Then, there is $T \subseteq D(V) \cap \Gamma$ such that, if the patch $\omega_{1|T}$ occurs in ω_0, there is $C > 0$ such that*

$$d_{\text{Hausdorff}}\big(\Omega_n, \Omega_{\omega_0}\big) \leqslant \frac{C}{\lambda_0^n} \quad \text{for all } n \in \mathbb{N}\,.$$

Applying a result from Beckus and Takase [3, Thm. 1.1 (c)] we obtain

Corollary 4. *Suppose ω_0 is good and H_{ω_0} is nice. In the situation of the previous theorem, there is $\widetilde{C} > 0$ such that*

$$d_{\text{Hausdorff}}\big(\sigma(H_{\omega_{n,+}}),\, \sigma(H_{\omega_0})\big) \leqslant \frac{\widetilde{C}}{\lambda_0^n} \quad \text{for all } n \in \mathbb{N}.$$

This shows that, as a set, the spectrum of the aperiodic operator H_{ω_0} can be approximated (exponentially fast) by the spectra of periodic operators. In particular, we have convergence for the corresponding bottoms of spectra. Via a study of positive harmonic functions, Richter [6] obtains certain formulae for the bottom of the spectrum of Schrödinger-type operators on graphs admitting a co-finite action by a nilpotent group. In particular, these results apply to $H_{\omega_{n,+}}$, and thus might open up a way to extract more explicit spectral information.

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The road to a geometric aperiodic monotile

Jamie Walton

The first aperiodic tile set was discovered by Berger in 1966, in his resolution of Wang's domino problem. I will survey the history of discoveries of aperiodic sets of few tiles following this. In the 1970s, Penrose discovered his remarkable aperiodic set of just two tiles [3], naturally leading to the monotile problem(s) of finding a single aperiodic tile. There are, in fact, many possible variants one may ask: should the tile be 'perfect' in the sense of local isomorphism, do we allow non-geometric matching rules, which isometries of the tile are allowed? etc. With examples such as the Schmitt–Conway–Danzer tile — a convex polyhedron in 3-space that tiles without translation symmetries but allows screw symmetries — one even has to be careful in what is meant by 'aperiodic'. I will explain the subtleties involved here. The first answer to one variant came in the early 2010s, with the (LI-perfect) Taylor–Socolar monotile [6], represented either as a simple hexagon but with next-nearest neighbour matching rules, or purely geometrically as a tile that is not a topological ball. The hexagonal grid and, in particular, the arrowed half-hex tiling, has been used several times as the scaffolding for interesting small aperiodic tile sets [1, 7], such as the Penrose $1 + \epsilon + \epsilon^2$ tilings. This year, the Hat monotile was discovered [4], receiving an unprecedented and welcome amount of public attention. Although it is in some sense commensurate with the hexagonal lattice, its global structure is completely different from the previous hexagonal examples. The Hat settles the monotile problem for a purely geometric tile that is a topological ball and only tiles nonperiodically (but is not translationally LI-perfect), and shortly after the Spectre [5], which settles the same problem but without requiring reflections of the tile. Within just a few months of discovery, lots is already known about the global dynamical and topological properties of these tilings [2], uncovered by an already advanced toolkit developed within the field of aperiodic order.

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The Hat tiling is topologically conjugate to a model set

Franz Gähler
(joint work with Michael Baake, Lorenzo Sadun)

In a recent preprint [1], it was shown that a certain non-convex polygon (the Hat), along with its mirror image, can tile the Euclidean plane, but only non-periodically. This tile was therefore called an aperiodic monotile. In any Hat tiling, all tiles have coordinates on a triangular lattice. Hats can be combined to certain clusters, called meta-tiles, and there is a combinatorial inflation symmetry present in these meta-tile tilings, leading to a hierarchical structure and aperiodicity. Moreover, it was shown that the meta-tiles can be deformed in such a way that the combinatorial inflation becomes a true geometric inflation, with a scaling factor ϕ^2, where ϕ is the golden mean.

In this talk, we show that this shape change of the meta-tiles, and other shape changes of the Hat tiles, are asymptotically negligible in the sense of [2], meaning that they do not mess up the long-range aperiodic order of the system. Hence, the original meta-tile tiling and the deformed one with true ϕ^2 inflation form topologically conjugate dynamical systems under the translation action. The same actually holds for all deformed Hat tilings, up to overall scale and orientation. As a result, their dynamical and diffraction spectra all coincide. Moreover, by the overlap algorithm [3, 4], we can show that these spectra are all pure point.

Inflation systems with pure-point spectra are expected to be cut-and-project sets, also known as model sets. This is the case also for the self-similar meta-tile tiling, for which we construct a cut-and-project scheme based on a 4d lattice with hexagonal symmetry and ϕ^2 inflation invariance. With a proper choice of control points of the meta-tiles, the set of control points can be described as a model set with fairly simple window. Its outer shape is a regular hexagon with a partly fractal subdivision for the different tile types. Other deformed Hat tilings are obtained from the same cut-and-project scheme and window, but with different projection direction. More details are given in [5].

Completely analogous results can also be obtained for a cousin of the Hat tiling, the Spectre tiling [6].

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Aperiodicity of Turtle

Shigeki Akiyama
(joint work with Yoshiaki Araki)

An aperiodic monotile is a tile which tiles a plane but only in a non-periodic way. We give a short self-contained alternative proof of this fact for a variant of Smith's Hat, called the Turtle.

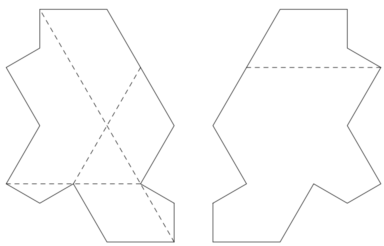

Figure 1. Turtle with one set of Ammann bars

The proof depends on interesting 1-dimensional stuructures (Sturmian edge and Ammann bar) which appear in the tilings by turtles.

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Invariant measures of Toeplitz subshifts on non-amenable groups

Paulina Cecchi-Bernales
(joint work with María Isabel Cortez, Jaime Gómez)

Let G be a countable discrete group, let X be a compact metric space and let $T\colon G \times X \to X$ be a continuous action of G on X. The associated topological dynamical system is denoted (X, T, G). The action of an element $g \in G$ on X is denoted by T^g. A classical problem in topological dynamics is the description of the set $\mathcal{M}(X, T, G)$ of all invariant measures of (X, T, G), i.e., the set of Borel probability measures μ on X such that $\mu(T^g(A)) = \mu(A)$ for all $g \in G$ and all Borel subset $A \subseteq X$. In particular, we are interested in the interplay between the properties of $\mathcal{M}(X, T, G)$ and certain group-theoretic properties of G.

It is well known [1] that a countable group G is *amenable* if and only if every continuous action of G on a compact metric space admits an invariant measure. Recall that a countable group is amenable if it admits a *Følner sequence*, which is a sequence $(F_n)_{n\in\mathbb{N}}$ of finite subsets of G satisfying $\lim_{n\to\infty}\frac{|F_n \triangle F_n g|}{|F_n|} = 0$ for all $g \in G$. This characterisation of amenability was strenghtened by Giordano and de la Harpe, who proved that a countable group G is amenable if and only if any continuous action of G on a Cantor space admits an invariant measure [8]. Furthermore, it is known that the family of *subshifts*, i.e., the symbolic continuous actions on the Cantor set, is a test family for amenability. More precisely, for every countable non-amenable group G, there exists a positive integer n and a subshift $X \subseteq \{0,1,\ldots,n-1\}^G$ such that $\mathcal{M}(X,\sigma,G) = \varnothing$, where σ denotes the *shift action* of G on $\{0,1,\ldots,n-1\}^G$, given by $\sigma^g((x_h)_{h\in G}) = (x_{g^{-1}h})_{h\in G}$ for all $g \in G$. This is a direct consequence of [10, Thm. 1.2]. More recently, the following general characterisation of amenability has appeared.

Theorem 1 ([7, Thm. 3.7.1])**.** *A countable group G is amenable if and only if, for any subshift $X \subseteq \{0,1\}^G$, one has $\mathcal{M}(X,T,G) \neq \varnothing$.*

All previous results are dynamical characterisations of amenability.

Considered as a subspace of the dual $C^\star(X,\mathbb{R})$ with the weak* topology, the set $\mathcal{M}(X,T,G)$ is a *Choquet simplex*, i.e., a convex compact metrizable subset K of a locally convex real vector space such that, for all $v \in K$, there exists a unique measure m_v supported on the extreme points of K such that $\int x dm_v = v$. The set of extreme points of $\mathcal{M}(X,T,G)$ corresponds to the *ergodic* invariant measures of (X,T,G), i.e., the invariant measures μ such that $\mu(A) \in \{0,1\}$ whenever $T^g(A) = A$ for all $g \in G$. Conversely, for any Choquet simplex K there exists a *Toeplitz* subshift $X \subseteq \{0,1\}^{\mathbb{Z}}$ such that $\mathcal{M}(X,\sigma,\mathbb{Z})$ is affine homeomorphic to K; see [6]. For a given finite alphabet Σ, a *Toeplitz* subshift is a subshift $X \subseteq \Sigma^{\mathbb{Z}}$ obtained as the shift orbit closure of a *Toeplitz sequence:* a sequence $x \in \Sigma^{\mathbb{Z}}$ such that, for all $k \in \mathbb{Z}$, there exists $p \in \mathbb{N}$ with $x_{k+p\ell} = x_k$ for all $\ell \in \mathbb{N}$. This extends to any group as follows: an element $x \in \Sigma^G$ is a Toeplitz configuration if for all $g \in G$ there exists a finite index subgroup $\Gamma \leqslant G$ such that $x_{\gamma g} = x_g$ for all $\gamma \in \Gamma$. A subshift $X \subseteq \Sigma^G$ is a *Toeplitz subshift* if it is the shift orbit closure of a Toeplitz configuration. The realisation of Choquet simplices in [6] was extended in [5], where the authors prove the following.

Theorem 2 ([5, Theorem A])**.** *For every Choquet simplex K and every residually finite amenable group G, there exists a Toeplitz subshift $X \subseteq \Sigma^G$ such that $\mathcal{M}(X,\sigma,G)$ is affine homeomorphic to K.*

Recall that a group G is *residually finite* if the intersection of all its finite index subgroups is trivial. Toeplitz subshifts are always minimal (see for instance [4,9]). On the other hand, if a Toeplitz subshift $X \subseteq \Sigma^G$ is *aperiodic*, i.e., if every point in X has trivial stabilizer under the shift action of G, then G must be residually finite. Indeed, a countable group G is residually finite if and only if there exists an aperiodic Toeplitz subshift $X \subseteq \{0,1\}^G$; see [4,9].

There are at least two possible ways to generalise the realisation result of Choquet simplices proved in [5]. One of them is to consider groups which are amenable but not necessarily residually finite (e.g., $\mathbb{Q}$ or any amenable divisible group). The second one is to consider residually finite groups which are not necessarily amenable (e.g., free groups). For the first type of generalisation, we introduce in [2] the notion of *congruent monotileable* amenable group. A countable amenable group G is *congruent monotileable* if it admits a Følner sequence $(F_n)_{n\in\mathbb{N}}$ such that each F_n is a *monotile* of G and each F_n is a disjoint union of translated copies of F_{n-1} (see [2] for precise definitions). Residually finite amenable groups are congruent monotileable, as well as any virtually nilpotent countable group [2]. We have the following realisation result for congruent monotileable amenable groups.

Theorem 3 ([2, Theorem 1]). *For any Choquet simplex K and any congruent monotileable amenable group G, there exists a minimal subshift $X \subseteq \Sigma^G$, which is aperiodic in a full measure set, such that $\mathcal{M}(X, \sigma, G)$ is affine homeomorphic to K.*

Regarding the second type of generalisation, let G be a countable residually finite group. Then, there exists a nested sequence $(\Gamma_n)_{n\in\mathbb{N}}$ of finite index normal subgroups of G with trivial intersection. Define the *G-odometer* $\overleftarrow{G}$ associated to $(\Gamma_n)_{n\in\mathbb{N}}$ as follows,

$$\overleftarrow{G} := \varprojlim_n (G/\Gamma_n, \tau_n) = \left\{ (x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} G/\Gamma_n \mid \tau_n(x_{n+1}) = x_n \text{ for all } n \in \mathbb{N} \right\},$$

where $\tau_n \colon G/\Gamma_{n+1} \to G/\Gamma_n$ is the canonical projection. Since the Γ_n's are finite index normal subgroups, $\overleftarrow{G}$ is a compact topological group with the induced topology from $\prod_{n\in\mathbb{N}} G/\Gamma_n$. Let φ denote the action of G on $\overleftarrow{G}$ by coordinate-wise product. The topological dynamical system $(\overleftarrow{G}, \varphi, G)$ is uniquely ergodic, the unique φ-invariant measure being the Haar measure on $\overleftarrow{G}$. We have the following realisation results for residually finite groups which are not necessarily amenable.

Theorem 4 ([3, Theorem 1.2]). *Let G be a countable residually finite group and let $\overleftarrow{G}$ be a G-odometer. Then, there exists a uniquely ergodic Toeplitz G-subshift (X, σ, G) and an almost 1-1 factor map $\pi : X \to \overleftarrow{G}$, such that, if ν is the unique probability measure of (X, σ, G), then π is a measure conjugacy between (X, σ, G, ν) and $(\overleftarrow{G}, \varphi, G)$ endowed with the Haar measure.*

Theorem 5 ([3, Theorem 1.3]). *Let G be a countable residually finite group and let $\overleftarrow{G}$ be a G-odometer. For every integer $r > 1$, there exists a Toeplitz G-subshift $X \subseteq \{1, \dots, r\}^G$ with at least r ergodic probability measures $\nu_1, \dots, \nu_r$, and whose maximal equicontinuous factor is $\overleftarrow{G}$. Furthermore, for every $1 \leqslant i \leqslant r$, the following holds,*

(1) (X, σ, G, ν_i) is measure conjugate to $(\overleftarrow{G}, \varphi, G)$ endowed with the Haar measure,

(2) $\nu_i\big(\{x \in X : x(1_G) = i\}\big) \geqslant \mu\big(\{x \in X : x(1_G) = i\}\big)$ for every invariant probability measure μ.

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Accumulation points of normalized integer translates of rotations in Euclidean space

Alan Haynes

(joint work with Kavita Dhanda)

Suppose that $d \in \mathbb{N}$ and that $\alpha \in \mathbb{R}^d$. A large portion of classical Diophantine approximation is concerned with understanding small values of the quantities

$$q\alpha - p \in \mathbb{R}^d,$$

where $q \in \mathbb{Z}$ and $p \in \mathbb{Z}^d$. A first version of this problem, which is sufficient for many applications, is to accurately determine, for $\eta \in \mathbb{R}$, the 'sizes' of the sets $\mathcal{W}(\eta)$ of *η-well approximable points*, and their complements $\mathcal{W}(\eta)^c$. For $\eta \in \mathbb{R}$, $\mathcal{W}(\eta)$ is defined to be the collection of $\alpha \in \mathbb{R}^d$ for which 0 is an accumulation point of the set

$$\mathrm{NA}_\eta(\alpha) = \{|q|^\eta\,(q\alpha - p) : q \in \mathbb{Z}\backslash\{0\}, p \in \mathbb{Z}^d\}.$$

We will call the elements of $\mathrm{NA}_\eta(\alpha)$ the *η-normalized approximations* to α.

For each η, the set $\mathcal{W}(\eta)$ satisfies a zero-full law, so that $\lambda(\mathcal{W}(\eta)) = 0$ or $\lambda(\mathcal{W}(\eta)^c) = 0$, where λ denotes Lebesgue measure on $\mathbb{R}^d$. In cases where the Lebesgue measure is zero, we seek finer information about Hausdorff dimensions of the corresponding sets. This is all well understood, primarily due to foundational work of Khintchine, Groshev, Jarník, Besicovitch, and Cassels, which is summarized in Table 1.

In line with the classical theory described above, it is natural and desirable to understand the collection of all accumulation points $\mathcal{A}_\eta(\alpha)$ of NA, both for generic choices of α and for specific choices with particular arithmetical properties. For

Table 1. Summary of results about $\mathcal{W}(\eta)$ and $\mathcal{W}(\eta)^c$, for $\eta \in \mathbb{R}$ and $d \in \mathbb{N}$.

$\eta < 0$	$\mathcal{W}(\eta) = \mathbb{R}^d$	$\mathcal{W}(\eta)^c = \varnothing$
$0 \leqslant \eta < 1/d$	$\mathcal{W}(\eta) = \mathbb{R}^d \backslash \mathbb{Q}^d$ (Dirichlet)	$\mathcal{W}(\eta)^c = \mathbb{Q}^d$
$\eta = 1/d$	$\lambda(\mathcal{W}(\eta)) = \infty,\ \lambda(\mathcal{W}(\eta)^c) = 0$ (Khintchine, Groshev)	$\dim_H(\mathcal{W}(\eta)^c) = d$ (Jarník, Schmidt)
$\eta > 1/d$	$\dim_H(\mathcal{W}(\eta)) = \dfrac{d+1}{1+\eta}$ (Jarník, Besicovitch)	$\lambda(\mathcal{W}(\eta)^c) = \infty,\ \lambda(\mathcal{W}(\eta)) = 0$ (Borel–Cantelli)

example, the density of $\mathbb{Q}^d$ in $\mathbb{R}^d$ guarantees that $\mathcal{A}_{-1}(\alpha) = \mathbb{R}^d$, for all $\alpha \in \mathbb{R}^d$. On the other hand, Kronecker's theorem, a much less trivial result, tells us that $\mathcal{A}_0(\alpha) = \mathbb{R}^d$ if and only if the numbers $1, \alpha_1, \ldots, \alpha_d$ are $\mathbb{Q}$-linearly independent. However, perhaps surprisingly, apart from these two fundamental examples, there do not appear to be any non-trivial cases of η (i.e., with $\eta > 0$) for which this problem is well understood.

In this talk, we explained new results about the sets $\mathcal{A}_\eta(\alpha)$, derived from two main points of view. From our first point of view, for $\eta \in \mathbb{R}$, we presented measure-theoretic and Hausdorff dimension results about the sets $\mathcal{D}(\eta)$ consisting of all $\alpha \in \mathbb{R}^d$ for which $\mathcal{A}_\eta(\alpha) = \mathbb{R}^d$, as well as results about the complementary sets $\mathcal{D}(\eta)^c$. These results are summarized in Table 2 (new results are labeled as Theorems 1-3). In analogy with the classical case, for $\eta \leqslant 1$, the property $\mathcal{A}_\eta(\alpha) = \mathbb{R}^d$ is satisfied for Lebesgue almost every $\alpha \in \mathbb{R}^d$. However, for $d \geqslant 2$ and $0 < \eta \leqslant 1$, the sets $\mathcal{D}(\eta)^c$ are related to what are called sets of singular points in Diophantine approximation, and are much more mysterious and interesting than the corresponding sets $\mathcal{W}(\eta)^c$ from the classical case.

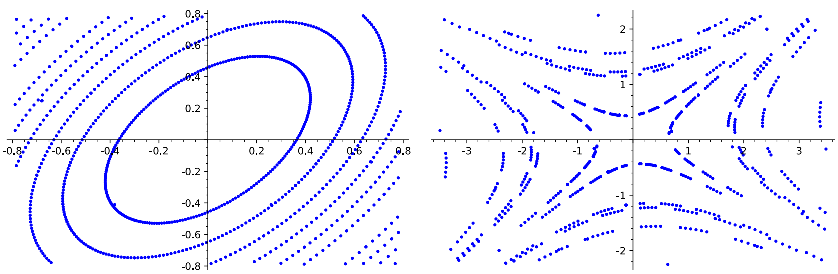
From our second point of view, we gave examples of particular choices of d, η, and α for which the sets $\mathcal{A}_\eta(\alpha)$ exhibit well ordered geometric structure. We focused primarily on the case when $d = 2$, $\eta = 1/2$, and the coordinates of α together with 1 form a $\mathbb{Q}$-basis for an algebraic number field. Using tools from algebraic number theory, we are able to show that, in this case, $\mathcal{A}_{1/2}(\alpha)$ is a union of countably many dilations (determined by norms of elements in an order in the corresponding cubic field) of either a single ellipse or a pair of hyperbolas, depending on whether or not the corresponding cubic field has a nontrivial complex embedding. Plots of some normalized approximations in this algebraic setting are displayed in Figure 1.

The results which we presented build on previous results due to many authors. A summary of key references is provided below.

Table 2. Summary of results about $\mathcal{D}(\eta)$ and $\mathcal{D}(\eta)^c$, for $\eta \in \mathbb{R}$ and $d \in \mathbb{N}$.

$\eta < 0$	$\mathcal{D}(\eta) = \mathbb{R}^d$	$\mathcal{D}(\eta)^c = \varnothing$
$\eta = 0$	$\mathcal{D}(\eta) = K_d$ (Kronecker)	$\mathcal{D}(\eta)^c = K_d^c$
$0 < \eta < 1/d$	$\lambda(\mathcal{D}(\eta)) = \infty$ (Theorem 1)	$0 < \dim_H(\mathcal{D}(\eta)^c \backslash K_d^c) < d$ ($d \geqslant 2$, Theorem 2)
$\eta = 1/d$	$\lambda(\mathcal{D}(\eta)) = \infty,\ \lambda(\mathcal{D}(\eta)^c) = 0$ (Theorem 1)	$\dim_H(\mathcal{D}(\eta)^c) = d$ ($\mathcal{W}(\eta)^c \subseteq \mathcal{D}(\eta)^c$)
$\eta > 1/d$	$\dim_H(\mathcal{D}(\eta)) = \dfrac{d+1}{1+\eta}$ (Theorem 3)	$\lambda(\mathcal{D}(\eta)^c) = \infty$, $\lambda(\mathcal{D}(\eta)) = 0$ ($\mathcal{D}(\eta) \subseteq \mathcal{W}(\eta)$)

Figure 1. (1/2)-normalized approximations to $(2^{1/3}, 2^{2/3})$ (left) and to (β, β^2), where β is the positive root of $x^3 + x^2 - 2x - 1$ (right).



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Spectral properties of Sturmian metric tree graphs

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(joint work with Gilad Sofer)

We study metric graph Laplacians defined on aperiodic tree graphs. The graph geometries are based on Sturmian sequences,

$$\forall n \in \mathbb{Z}, \quad s_\alpha(n) = \chi_{[1-\alpha,1)}(\{n\alpha\}),$$

where $\alpha \in [0,1]\backslash\mathbb{Q}$.
We consider two families of tree graphs:

(1) Infinite comb-like graphs. Specifically, consider the infinite line and two compact graphs which we denote by A and B. Decorate the real line graph by placing either the graph A or the graph B at the integer points, $\mathbb{Z}$. At each point, $n \in \mathbb{Z}$, the value of the Sturmian sequence $s_\alpha(n)$ determines whether to place graph A or B (see upper part of Figure 1). A particular simplification of this model is obtained when graph A is just the interval and graph B is the empty graph. This simplification yields the infinite comb graph, such that whether or not a tooth appears or not is determined by the Sturmian sequence (see lower part of Figure 1)
(2) Infinite radially symmetric tree graphs. Fix a sequence $(b_n)_{n=1}^{\infty}$ of branching numbers ($b_n \in \mathbb{N}$) and a sequence of edge lengths $(l_n)_{n=1}^{\infty}$, such that $l_n \in \mathbb{R}$ and $l_n > 0$. Construct a radially symmetric tree graph, such that at level n all the branching numbers are b_n (so that the corresponding vertex degrees are $b_n + 1$) and all edge lengths are l_n. Either of the sequences $(b_n)_{n=1}^{\infty}$, $(l_n)_{n=1}^{\infty}$, (or both) may be determined by a Sturmian sequence.

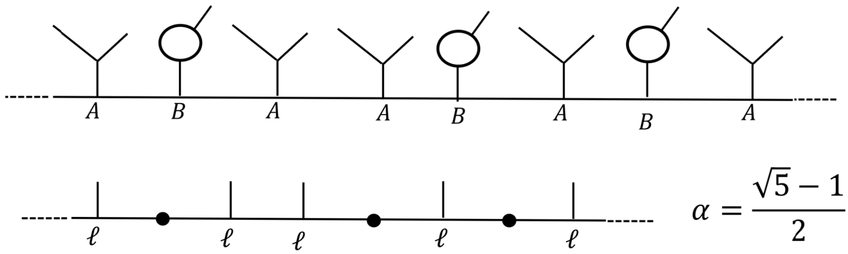

FIGURE 1. An example of an infinite comb-like graph generated by two particular compact decoration graphs A and B (upper part). The Sturmian sequence which determines the decoration locations is taken with the golden ratio, $\alpha = \frac{\sqrt{5}-1}{2}$ (lower part). The simplified graph, where A is taken to be the single interval and B is taken to be the empty graph.

Specifically, for the branching numbers fix two values $b_A, b_B \in \mathbb{N}$ and set

$$b_n = \begin{cases} b_A, & s_\alpha(n) = 0, \\ b_B, & s_\alpha(n) = 1, \end{cases}$$

and similarly for the edge lengths.

In each of these models, we take all vertex conditions to be of Neumann–Kirchhoff type. We study the spectral properties of these models, and their approximating periodic operators. Of specific interest are the following questions.

- What are the possible spectral types (and in particular whether eigenvalues exist)?
- Is the spectrum a (generalised) Cantor set?
- What is the Lebesgue measure of the spectrum?
- What is the Hausdorff dimension of the spectrum?
- Which values does the integrated density of states attain at the gaps?

In the spectral analysis for the graph families of the first type, we employ a classification of the spectral bands into types as in [1–3]. A few relevant and inspiring previous works on aperiodic metric graphs are [4–8].

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Transverse point processes – a robust framework for aperiodic order

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(joint work with Michael Björklund and Yakov Karasik)

Over the last decade, there has been a push to extend the theory of aperiodic order to include aperiodic subsets of non-abelian locally compact second countable (lcsc) groups and their homogeneous spaces (see e.g. [1]). In this wider setting, many classical tools such as van Hove sequences, ergodic theorems or distributions are not always available. This motivated us to systematically search the literature for formulations of basic concepts in aperiodic order (such as autocorrelation or diffraction) which do not require these tools and resulted in the theory of *transverse point process* [2, 3].

One key idea of our approach is to replace ergodic limits by more robust probabilistic concepts. To illustrate, the Hof diffraction of a uniquely ergodic FLC set in $\mathbb{R}^d$ coincides with the Bartlett spectrum of the point process defined by the invariant measure on the hull, hence we may think of the Bartlett spectrum as a probabilistic substitute for Hof diffraction. This substitute is more robust in the sense that is does not require van Hove sequences and hence can be applied in non-amenable situations. In the presence of a suitable ergodic theorem (e.g. in semisimple groups), the probabilistic definition can often be recast in analytic terms as a limit in the spirit of Hof; the geometry of the group then automatically determines the correct replacement of van Hove sequences. We emphasise that the additional robustness of the point process approach is beneficial even in the abelian case as it allows one to deal with very general classes of point sets.

We now formulate our general framework. Throughout this report, let G be a unimodular lcsc group with Haar measure m_G. Then, G admits a right-invariant proper metric d, and we fix such a metric once and for all. Given $r > 0$ we denote by $\mathrm{UD}_r(G)$ the space of r-uniformly discrete subsets of G, equipped with the Chabauty–Fell topology. This is a compact metrisable G-space under the action given by $g.P := Pg^{-1}$. If $G \curvearrowright (\Omega, \mathcal{A}, \mathbb{P})$ is an ergodic p.m.p. action on some auxiliary probability space, then we refer to a G-equivariant measurable map $\Lambda : \Omega \to \mathrm{UD}_{2r}(G)$, $\omega \mapsto \Lambda_\omega$ as a *hard-core point process* of *radius* r with *distribution* $\mu_\Lambda := \Lambda_* \mathbb{P}$ *realised* on $(\Omega, \mathcal{A}, \mathbb{P})$. Such processes can be seen as random packings of G by spheres of radius r, hence the name. We consider two processes as *equivalent* if they have the same distribution.

Up to equivalence, all hard-core point processes arise from a single dynamical construction: For this, let $G \curvearrowright (\Omega, \mathcal{A})$ be an arbitrary Borel action on an arbitrary standard Borel space. By a theorem of Conley, there then exists a *separated transversal* $\mathcal{T} \subset \Omega$, i.e. a Borel subset intersecting every orbit whose *return time set* $R_{\mathcal{T}} := \{g \in G \mid g.\mathcal{T} \cap \mathcal{T} = \emptyset\}$ intersects some identity neighbourhood in G only in the identity.

Proposition 1 (cf. [2]). *If $\mathcal{T} \subset \Omega$ is a separated transversal and $\mathbb{P}$ is a G-invariant probability measure on Ω, then $\Lambda_\omega := \{g \in G \mid g.\omega \in \mathcal{T}\}$ defines a hard-core process on $(\Omega, \mathcal{A}, \mathbb{P})$. Conversely, every hard-core process is equivalent to such a process.*

We refer to Λ as in the proposition as the *transverse process* associated with the *transverse triple* $(\Omega, \mathbb{P}, \mathcal{T})$. If Λ is an arbitrary hard-core process with distribution μ, then it is equivalent to the transverse process of the *canonical triple*

$$(\Omega^{\mathrm{can}} := \mathrm{supp}(\mu),\ \mathbb{P}^{\mathrm{can}} := \mu,\ \mathcal{T}^{\mathrm{can}} := \{Q \in \Omega^{\mathrm{can}} \mid e \in Q\}).$$

It is often more convenient to work with non-canonical realisations, though:

Example 2 (Cut-and-project process, cf. [3]). *Let H be an auxiliary lcsc group, $\Gamma < G \times H$ be a lattice and $\Omega = \Gamma\backslash(G \times H)$ with unique invariant probability measure $\mathbb{P}$. If $W \subset H$ is a relatively compact identity neighbourhood, then $\mathcal{T}_W := \{\Gamma(e,h) \mid h \in W\}$ is a separated transversal in Ω and the instances of the associated transverse process are the cut-and-project sets*

$$\Lambda_{\Gamma(g,h)} = \mathrm{proj}_G(\Gamma(g,h) \cap (G \times W)).$$

If Λ is a hard-core point process, then every Borel set $B \subset G$ defines an integer valued random variable $|\Lambda \cap B| : \omega \mapsto |\Lambda_\omega \cap B|$. There then exists a constant $i_\Lambda > 0$ (the *intensity*) and a Radon measure η_Λ on G such that

$$\mathbb{E}\left[|\Lambda \cap B|\right] = i_\Lambda \cdot m_G(B) \text{ and } \mathrm{Var}\left[|\Lambda \cap B|\right] = \int_G 1_B * 1_{B^{-1}} \, \mathrm{d}\eta_\Lambda - i_\Lambda^2 \cdot m_G(B)^2.$$

If $G = \mathbb{R}^n$, then η_Λ is the Hof autocorrelation of a generic instance of Λ, and hence we refer to η_Λ as the *autocorrelation* of Λ; it is our robust replacement for the Hof autocorrelation.

Both the intensity and the autocorrelation can be computed using transverse measure theory. Recall that if $(\Omega, \mathbb{P}, \mathcal{T})$ is a transverse triple, then in local flow box coordinates the measure $\mathbb{P}$ is of the form $m_G \otimes \nu$ for some *transverse measure* $\nu = {}_G\mathrm{Res}^{\Omega}_{\mathcal{T}}\mathbb{P}$ on $\mathcal{T}$ (cf. [2]). The total mass $\nu(\mathcal{T})$ is finite and given by the intensity of the associated transverse process Λ. Moreover, the autocorrelation of Λ satisfies

$$\eta_\Lambda(B) = \int_{\mathcal{T}} |\Lambda_\omega \cap B| \, \mathrm{d}\nu(\omega). \tag{17}$$

If one works with the canonical model, then the transverse measure is determined by its values on cylinder sets, since the Borel structure on $\mathcal{T}^{\mathrm{can}}$ is generated by these sets. If $G = \mathbb{R}^n$, then up to a factor of i_Λ these values turn out to be the patch frequencies of generic instances of Λ, and hence one should think of ν as a robust version of the collection of patch frequencies.

Example 3 (Patch frequencies of model sets)**.** *In the situation of Example* 2*, the pair* $(\mathcal{T}_W, \nu)$ *can be identified (up to null sets) with* $\left(W, \frac{m_H|_W}{\mathrm{covol}(\Gamma)}\right)$*, and under this identification cylinder sets correspond to the acceptance domains (up to null sets). This gives a robust proof of the fact that (up to normalisation) the patch frequencies of model sets are given by Haar volumes of acceptance domains and allows one to compute e.g., patch frequencies of model sets in hyperbolic space.*

Definition 4. *The* transverse groupoid *of a transverse triple* $(\Omega, \mathbb{P}, \mathcal{T})$ *is the étale groupoid* $\mathcal{G}$ *with* $\mathcal{G}^{(0)} = \mathcal{T}$*,* $\mathcal{G}^{(1)} = \{(Q_2, g, Q_1) \in \mathcal{T} \times G \times \mathcal{T} \mid Q_2 = g.Q_1\}$ *with partial composition* $(Q_3, h, Q_2) * (Q_2, g, Q_1) = (Q_3, hg, Q_1)$*.*

Transverse groupoids (and their cohomology) play a prominent role in the topology of tiling spaces; their relevance for us comes from the fact that transverse measures are invariant under transverse groupoids. A partial converse to this observation is given in [2]:

Proposition 5 (Transverse measure induction)**.** *If a finite measure* ν *on* $\mathcal{T}$ *is invariant under the transverse groupoid, then it is the transverse measure of a* σ*-finite (but in general not finite) measure* $\mu = {}_G\mathrm{Ind}_{\mathcal{T}}^{\Omega}\nu$ *on* Ω*.*

One of the main applications of Proposition 5 concerns the construction of *intersection measures*. For this, let $(\Omega_j, \mu_j, \mathcal{T}_j)$ be transverse triples with transverse measures ν_j. We then define the *intersection space*[1]

$$\Omega_1 \vee \Omega_2 := \{(g.Q_1, g.Q_2) \mid g \in G, Q_j \in \mathcal{T}_j\} \subset \Omega_1 \times \Omega_2.$$

Then, $\mathcal{T}_1 \times \mathcal{T}_2$ is a separated transversal for the G-action on $\Omega_1 \vee \Omega_2$, and $\nu_1 \otimes \nu_2$ is invariant under the transverse groupoid, hence induces a σ-finite measure $\mu_1 \vee \mu_2$ on $\Omega_1 \vee \Omega_2$. We say that the two systems are *commensurable* if this measure is finite; this is the case for instance if Ω_1 and Ω_2 are hulls of syndetic subsets of a strong uniform approximate lattice in the sense of [1]. If the systems are commensurable, then $\mu_1 \vee \mu_2$ can be normalised to a probability measure $\mathbb{P}$ which then defines a non-trivial joining of (Ω_1, μ_1) and (Ω_2, μ_2); here is a sample application from [2]:

Theorem 6. *For all i.i.d. random Meyer sets* $\Lambda_1, \Lambda_2 \subset G$*,*

$$\mathbb{P}\left[\, \underline{\mathrm{dens}}(\Lambda_1 \cap \Lambda_2) > 0 \mid \Lambda_1 \cap \Lambda_2 \neq \varnothing \,\right] = 1.$$

A very different class of intersection spaces arises by pairing aperiodic point processes in G with σ-finite invariant measures on homogeneous G-spaces. This leads to a notion of *Radon transform* for aperiodic point processes and allows one to define aperiodic versions of the Zak transform from time frequency analysis as well as the theta transform from the theory of automorphic forms. The former can be used to study the Schrödinger part of the spherical diffraction of Meyer sets in the Heisenberg group, whereas the latter can be used to study the continuous part of the spherical diffraction of non-uniform strong approximate lattices in hyperbolic spaces.

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[1] If the transverse triples are canonical, then $\Omega_1 \vee \Omega_2$ consists of those pairs in $\Omega_1 \times \Omega_2$ which intersect non-trivially, hence the name.

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