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Nonlinear enhanced dissipation in viscous Burgers-type equations

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Abstract. We construct a class of infinite mass functions for which solutions of the viscous Burgers equation decay at a better rate than the solution of the heat equation for the same initial data in this class. In other words, we show an enhanced dissipation coming from a nonlinear transport term. We compute the asymptotic profile in this class for both equations. For the viscous Burgers equation, the main novelty is the construction and description of a time-dependent profile with a boundary layer, which enhances the dissipation. This profile will be stable up to a computable nonlinear correction depending on the perturbation. We also extend our results to other convection–diffusion equations.

1. Introduction and presentation of the results

We are interested in this paper in the long-time behavior of solutions to a generalized viscous Burgers equation on the real line,

$$\begin{cases} \partial_t u - \partial_x^2 u + \partial_x\Big(\dfrac{u^2}{2} + J(u)\Big) = 0, \\ u_{|t=0}(x) = \dfrac{\kappa_\pm}{|x|^\alpha}(1 + o_{x\to\pm\infty}(1)), \end{cases} \tag{1.1}$$

for $\alpha \in]0,1[$, $\kappa_+, \kappa_- > 0$ and J a smooth function satisfying $|J(u)| \leqslant C|u|^3$. Note that $u_{|t=0}$ is not integrable, and $J = 0$ corresponds to the classical viscous Burgers equation.

It is well known that for the heat equation $\partial_t u - \partial_x^2 u = 0$, for an initial datum $u_0 \in L^1(\mathbb{R})$ we have the asymptotic profile

$$\sqrt{t}u(z\sqrt{t}, t) \to \frac{\int_{\mathbb{R}} u_0}{\sqrt{4\pi}} e^{-\frac{z^2}{4}}$$

when $t \to +\infty$, uniformly in $z \in \mathbb{R}$. A similar result holds for the viscous Burgers equation $\partial_t u - \partial_x^2 u + u\partial_x u = 0$ for initial data $u_0 \in L^1(\mathbb{R})$ (see [12, 20, 22]), as we have

$$\sqrt{t}u(z\sqrt{t}, t) \to \frac{2(e^{\frac{M}{2}} - 1)e^{-z^2/4}}{e^{\frac{M}{2}}\sqrt{4\pi} + (1 - e^{\frac{M}{2}})\int_{-\infty}^z e^{-s^2/4}\,ds}$$

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when $t \to +\infty$, uniformly in $z \in \mathbb{R}$, where $M = \int_{\mathbb{R}} u_0$. The same result holds with the term $J(u)$ in equation (1.1).

Although the limit profile is changed by the Burgers term $u\partial_x u$, the decay rate and the scaling in time are still the same as for the heat equation. In both cases, the L^∞ norm of the solution decays like $t^{-\frac{1}{2}}$.

Other asymptotic behavior results have been established in other convection–diffusion equations for initial data in $L^1(\mathbb{R})$; we refer to [23] and references therein, as well as [7, 9, 10, 18, 21]. See also [14] for some results on nonintegrable initial data.

Going back to our problem (1.1), as a comparison, we look first at the heat equation for this type of infinite mass initial data. There, up to a rescaling, we can show that the solution converges to a global attractor.

Proposition 1.1. *For $\kappa > 0$, $\alpha \in]0, 1[$, consider f the solution of the heat equation $\partial_t f - \partial_x^2 f = 0$ for an initial condition $f_0 \in C^0(\mathbb{R})$ that satisfies*

$$f_0(x) = \frac{\kappa(1 + o_{|x|\to+\infty}(1))}{(1 + |x|)^\alpha}.$$

Then, uniformly in $z \in \mathbb{R}$, we have the convergence

$$t^{\frac{\alpha}{2}} f(\sqrt{t}z, t) \to \frac{\kappa}{\sqrt{4\pi}} \int_{\mathbb{R}} \frac{1}{|y|^\alpha} e^{-(z-y)^2/4}\, dy$$

when $t \to +\infty$.

This result is first proven in [16] and the proof is redone in Appendix A to make this paper self-contained. Note that the decay in time of f is slower than if f_0 were in $L^1(\mathbb{R})$; in fact, $t^{-\frac{\alpha}{2}}$ is the size of $t^{-\frac{1}{2}} \int_{-\sqrt{t}}^{\sqrt{t}} f_0$. Furthermore, the asymptotic profile is smooth, and behaves like $\kappa|z|^{-\alpha}$ at infinity, connecting back to the initial data.

In this paper, we will construct a stable solution of (1.1) that converges, up to a rescaling, to an asymptotic profile. However, it will have two main differences compared to the result of Proposition 1.1. First, the rescaling will not be the same, and surprisingly, the solution will decay in time like $t^{-\frac{\alpha}{1+\alpha}}$, which is faster than the heat equation for the same initial data. The scales of the rescaling are thus dictated by the nonlinear term, which happens also in the nonlinear heat equation with a pure power term; see for instance [16, 19]. Second, the asymptotic profile will be, in the rescaling where it is of size 1, discontinuous at one point. This discontinuity can be seen as a sort of boundary layer as in [8, 15] (although there are no boundaries in this problem) that helps the dissipation. Regarding this enhanced dissipation, we can state the following result.

Theorem 1.2. *Given $\alpha \in]\frac{1}{4}, 1[$, $\kappa_+, \kappa_- > 0$, there exists an initial datum $u_{|t=0}$ with $u_{|t=0}(x) = \frac{\kappa_\pm}{|x|^\alpha}(1 + o_{x\to\pm\infty}(1))$ such that the solution of $\partial_t u - \partial_x^2 u + u\partial_x u = 0$ for this initial datum satisfies*

$$t^{\frac{\alpha}{1+\alpha}} \|u(\cdot, t)\|_{L^\infty(\mathbb{R})} \leqslant c_0,$$

where $c_0 > 0$ is a constant independent of time. Furthermore, this solution is stable in some sense.

See Theorem 1.4 for a more precise statement and the shape of the asymptotic profile, and Proposition 1.5 for a statement in the case $J \neq 0$. By Proposition 1.1, for the initial data of Theorem 1.2, if it was instead evolving following the heat equation, we would have $t^{\frac{\alpha}{1+\alpha}}\|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \to +\infty$ when $t \to +\infty$. This means that the additional nonlinear transport term improved the dissipation. Enhanced dissipation results are well known for the heat equation with an additional linear transport term (see for instance [1, 2, 4, 5, 11] and references therein) or for Navier–Stokes on $\mathbb{T} \times \mathbb{R}$ (see [3, 6, 13, 17]). We require $\kappa_+, \kappa_- > 0$ in Theorem 1.2, and although we can require less, we do not know how to show that this enhancement is true in general for any $\kappa_+, \kappa_- \in \mathbb{R}^*$.

1.1. Profile for the viscous Burgers equation

We focus first on the case $J = 0$ of equation (1.1). There, the Hopf–Cole formula gives us an explicit formulation of the solution of the equation. However, since our goal is to be able to generalize it for any J, we will not use it here. The results we can obtain with the Hopf–Cole formula will be the subject of a companion paper.

Here, we want to construct an approximate solution of the viscous Burgers equation in the right scaling.

1.1.1. The rescaled problem and the underlying ODE. We consider the viscous Burgers equation

$$\partial_t u - \partial_x^2 u + u\partial_x u = 0$$

and $\alpha \in]0,1[$. We want to make a change of variable such that the terms $\partial_t u$ and $u\partial_x u$ are the dominant ones. We define $\varepsilon(t) := t^{\frac{\alpha-1}{\alpha+1}}$ and

$$h(z,\varepsilon(t)) = t^{\frac{\alpha}{1+\alpha}} u(zt^{\frac{1}{1+\alpha}}, t),$$

leading to the equivalent equation

$$\frac{1-\alpha}{\alpha+1}\varepsilon\partial_\varepsilon h + \frac{\alpha}{1+\alpha}h + \frac{z\partial_z h}{1+\alpha} - h\partial_z h + \varepsilon\partial_z^2 h = 0. \tag{1.2}$$

Note that the term coming from the Laplacian, $\varepsilon\partial_z^2 h$, is small when $\varepsilon \to 0$ (that is, $t \to +\infty$). This means that at this scale, the nonlinear effect dominates the dynamic. Interestingly, if we simply remove the Laplacian, we get the Burgers equation $\partial_t u + u\partial_x u = 0$, for which the L^∞ norm is conserved before the formation of shocks. Since we will show some decay stronger than the heat equation, this means that although the Laplacian is fading out, it still has a major effect on the dynamic.

We want to construct, for $\varepsilon > 0$ small, a solution to the ODE problem

$$\begin{cases} \dfrac{\alpha}{1+\alpha}h + \left(\dfrac{z}{1+\alpha} - h\right)\partial_z h + \varepsilon\partial_z^2 h = 0, \\ h(z) = \kappa_+|z|^{-\alpha}(1 + o_{z\to+\infty}(1)), \\ h(z) = \kappa_-|z|^{-\alpha}(1 + o_{z\to-\infty}(1)), \end{cases} \tag{1.3}$$

for $\kappa_+, \kappa_- \in \mathbb{R}^*$. This will give us an approximate solution of (1.1). Note that the problem (1.3) is doubly degenerate when $\varepsilon \to 0$: the coefficient in front of the term with two derivatives goes to 0, but also, the limit problem when $\varepsilon = 0$ is ill defined when $h(z) = \frac{z}{1+\alpha}$, since then the coefficient in front of the term with one derivative cancels out.

1.1.2. The case $\varepsilon = 0$. We consider in this section the ODE

$$\begin{cases} \dfrac{\alpha}{1+\alpha}h + \dfrac{z\partial_z h}{1+\alpha} - h\partial_z h = 0, \\ h(z_0) = b, \end{cases}$$

for some given $(z_0, b) \in \mathbb{R}^2$ and $\alpha \in \,]0,1[$. This is the equation of (1.3) with $\varepsilon = 0$. We summarize here the properties of the solutions.

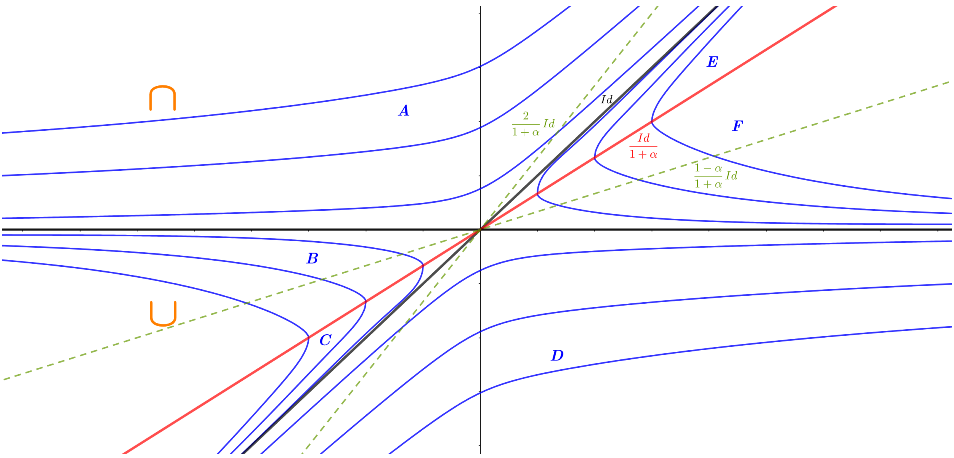

Figure 1

First, remark that $h(z) = z$ and $h(z) = 0$ are solutions of this equation (they are the two black lines in Figure 1). Furthermore, we can write the equation as

$$\partial_z h\Big(\frac{z}{1+\alpha} - h\Big) + \frac{\alpha}{1+\alpha}h = 0,$$

which is ill defined if $h(z) = \frac{z}{1+\alpha}$ at some point. This is the red line in Figure 1. The blue curves are the solution of the equation. In particular, we have to take $(z_0, b) \in \mathbb{R}^2$ with $b \neq \frac{z_0}{1+\alpha}$ for the equation to make sense.

Now we divide the set

$$\Big\{(z_0, b) \in \mathbb{R}^2,\ b \neq \frac{z_0}{1+\alpha}\Big\} = \mathbf{A} \cup \mathbf{B} \cup \mathbf{C} \cup \mathbf{D} \cup \mathbf{E} \cup \mathbf{F},$$

with

$$
\begin{aligned}
\mathbf{A} &:= \left\{(z_0, b) \in \mathbb{R}^2,\ b > \max(0, z_0)\right\},\\
\mathbf{B} &:= \left\{(z_0, b) \in \mathbb{R}^2,\ 0 > b > \frac{z_0}{1+\alpha}\right\},\\
\mathbf{C} &:= \left\{(z_0, b) \in \mathbb{R}^2,\ \frac{z_0}{1+\alpha} > b > z_0\right\},\\
\mathbf{D} &:= \left\{(z_0, b) \in \mathbb{R}^2,\ b < \min(0, z_0)\right\},\\
\mathbf{E} &:= \left\{(z_0, b) \in \mathbb{R}^2,\ z_0 > b > \frac{z_0}{1+\alpha}\right\},\\
\mathbf{F} &:= \left\{(z_0, b) \in \mathbb{R}^2,\ \frac{z_0}{1+\alpha} > b > 0\right\}.
\end{aligned}
$$

In Figure 1, the separations between these sets are the red and black lines (the role of the dotted green lines will be explained later).

The equation has a symmetry: if $z \to h(z)$ is a solution then so is $z \to -h(-z)$. Note also that since this is a first-order ODE, solutions cannot cross the axes 0, Id and $\frac{\mathrm{Id}}{1+\alpha}$. In particular, if a solution has a point in a bold set $\mathbf{J} \in \{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}, \mathbf{F}\}$, then it is fully included in $\mathbf{J}$.

If $(z_0, b) \in \mathbf{A}$, then the solution h is global, and

$$
\lim_{z\to+\infty} h(z) - z = 0, \quad h(z) \sim \kappa |z|^{-\alpha} \quad \text{when } z \to -\infty,
$$

for some $\kappa > 0$ determined by (z_0, b).

If $(z_0, b) \in \mathbf{B}$, then the solution is defined on $]-\infty, z_b[$ for some $z_b > z_0$ determined by (z_0, b), and

$$
\lim_{z\to z_b} h(z) = \frac{z_b}{1+\alpha}, \quad \lim_{z\to z_b} h'(z) = -\infty, \quad h(z) \sim -\kappa |z|^{-\alpha} \quad \text{when } z \to -\infty,
$$

for some $\kappa > 0$ determined by (z_0, b).

In both cases, $(z_0, b) \to \kappa$ is surjective in $\mathbb{R}_+^*$.

If $(z_0, b) \in \mathbf{C}$, then the solution is defined on $]-\infty, z_b[$ for some $z_b > z_0$ determined by (z_0, b), and

$$
\lim_{z\to z_b} h(z) = \frac{z_b}{1+\alpha}, \quad \lim_{z\to z_b} h'(z) = +\infty, \quad \lim_{z\to-\infty} h(z) - z = 0.
$$

By symmetry, we similarly describe the domains and limits if $(z_0, b) \in \mathbf{D} \cup \mathbf{E} \cup \mathbf{F}$.

In particular, remark that there are no continuous solutions to the problem

$$
\begin{cases}
\dfrac{\alpha}{1+\alpha} h + \dfrac{z\partial_z h}{1+\alpha} - h\partial_z h = 0,\\
h(z) = \kappa_+ |z|^{-\alpha}(1 + o_{z\to+\infty}(1)),\\
h(z) = \kappa_- |z|^{-\alpha}(1 + o_{z\to-\infty}(1)),
\end{cases}
$$

for $\kappa_+, \kappa_- \in \mathbb{R}^*$. Therefore, we expect the solution of (1.3) to have jumps in the limit $\varepsilon \to 0$.

In the next subsection, we will give some conditions to describe what jumps are possible in the limit $\varepsilon \to 0$. We will show that for $\kappa_+, \kappa_- > 0$, at most one is a viscosity solution.

1.1.3. Viscosity solutions. First, if h is a solution of (1.3) with $\varepsilon = 0$ in the distribution sense, then it must satisfy the Rankine–Hugoniot conditions. Here, it states that at any discontinuity $z_c \in \mathbb{R}$, we must have

$$\frac{1}{2}(h(z_c^+) + h(z_c^-)) = \frac{z_c}{1+\alpha}.$$

In Figure 1, this means that the middle point of any jump must be on the red line. This prevents, for instance, jumps from one bold set to itself, but also, for instance, from **F** to **D**.

The dotted green lines $\frac{2}{1+\alpha}\mathrm{Id}$ and $\frac{1-\alpha}{1+\alpha}\mathrm{Id}$ are those such that the red line is in the middle between 0 and $\frac{2}{1+\alpha}\mathrm{Id}$, and in the middle between Id and $\frac{1-\alpha}{1+\alpha}\mathrm{Id}$. Note that for any $\alpha \in \,]0,1[$, we have the order $0 < \frac{1-\alpha}{1+\alpha} < \frac{1}{1+\alpha} < 1 < \frac{2}{1+\alpha}$.

To continue, note that if

$$\frac{\alpha}{1+\alpha}h_\varepsilon + \frac{z\partial_z h_\varepsilon}{1+\alpha} - h_\varepsilon \partial_z h_\varepsilon + \varepsilon \partial_z^2 h_\varepsilon = 0$$

and $\partial_z h_\varepsilon(z) = 0$ for some $z \in \mathbb{R}$, then

$$\partial_z^2 h_\varepsilon(z) = \frac{-\alpha}{(1+\alpha)\varepsilon} h_\varepsilon(z).$$

This is represented by the two orange cups in Figure 1: if $h_\varepsilon(z) > 0, h_\varepsilon'(z) = 0$, then $h_\varepsilon''(z) < 0$, so near z the function h looks like the cup.

This means that, if we expect h, a solution of (1.3) with $\varepsilon = 0$ with some discontinuities, to be the limit when $\varepsilon \to 0$ of a sequence of functions h_ε solving (1.3), since the h_ε are smooth, then some jumps cannot happen. For instance, although the Rankine–Hugoniot condition allows jumps from **F** to **E**, they are not viscous (this would require the existence of z such that $h_\varepsilon(z) > 0$, $h_\varepsilon'(z) = 0$ and $h_\varepsilon''(z) \geqslant 0$).

Continuing, we infer that it is not possible to have two jumps that cross the axis $\{z = 0\}$. Indeed, otherwise we denote by $z_a < z_b$ two consecutive values such that $h_\varepsilon(z_a) = h_\varepsilon(z_b) = 0$, and integrating the equation between z_a and z_b leads to

$$\frac{\alpha-1}{\alpha+1}\int_{z_a}^{z_b} h_\varepsilon(s)\,ds + \varepsilon(h_\varepsilon'(z_b) - h_\varepsilon'(z_a)) = 0,$$

but this is impossible since either $h_\varepsilon'(z_b) - h_\varepsilon'(z_a) > 0$ and $h_\varepsilon > 0$ on $[z_a, z_b]$, or $h_\varepsilon'(z_b) - h_\varepsilon'(z_a) < 0$ and $h_\varepsilon < 0$ on $[z_a, z_b]$.

Finally, suppose that a solution has a point $z_a < 0$, where $h(z_a) = \frac{2}{1+\alpha}z_a$, $h'(z_a) > 0$ and a point $z_b > z_a$ such that $h(z_b) = 0$. Then, integrating the equation between z_a and z_b leads to

$$\frac{\alpha-1}{\alpha+1}\int_{z_a}^{z_b} h_\varepsilon(s)\,ds + \varepsilon(h_\varepsilon'(z_b) - h_\varepsilon'(z_a)) = 0.$$

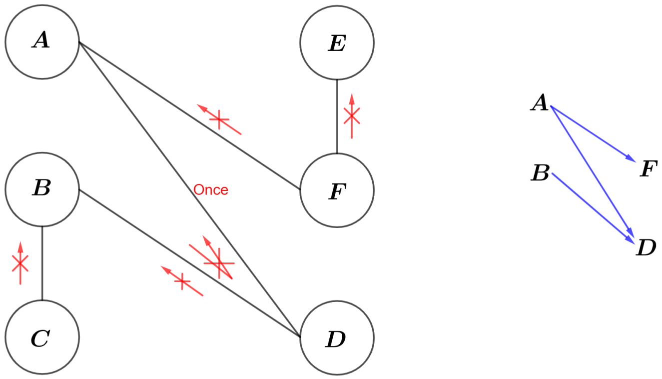

Figure 2

When $\varepsilon \to 0$, this also leads to a contradiction. This prevents the possibility of solutions having jumps between **B** and **D** followed by a jump from **D** to **A**.

We summarize these conditions in Figure 2.

Jumps are not possible from a bold set to itself. Two bold sets are connected by a black line if there is a possible jump between them satisfying the Rankine–Hugoniot condition. Crossed red arrows are jumps that are forbidden by the viscosity conditions. The jump between **A** and **D** can only be done once.

These viscosity conditions severely limit what jumps are allowed. We are looking for solutions starting from **A** or **B** and ending at **D** or **F**. These conditions impose that, for the cases **A** $\to$ **F**, **A** $\to$ **D** and **B** $\to$ **D**, only a single jump is possible. Furthermore, in these cases (that is, κ_+, κ_- having the same sign or $\kappa_- > 0$, $\kappa_+ < 0$), the position of the jump is fully determined by κ_+ and κ_-. For instance if $\kappa_+, \kappa_- > 0$, these two values determine on which blue curves in **A** and **F** the solution is, and we can check that there is only one value z_c for which the red line is the middle point of these two blue curves.

For the last case **B** $\to$ **F**, where it seems that no connection is possible, we omitted the case where the jump does not finish in a bold set, but finishes on the identity line. It therefore may be possible to go from **B** to **F** with two jumps, both connecting to $\{(z, z), z \in \mathbb{R}\}$. It is however difficult to prove or disprove that such a thing might happen. It might also be possible that the solution of the viscous Burgers equation with such an initial datum simply does not converge with this rescaling.

The results of the three subsections above are shown in Sections 2.1 to 2.3.

1.1.4. Construction of the profile for small $\varepsilon > 0$. In this section, given $\kappa_+, \kappa_- > 0$, $\alpha \in]0, 1[$ and $\varepsilon > 0$ small enough, we want to construct a solution of the ODE problem

$$\begin{cases} \dfrac{\alpha}{1+\alpha}\mathbb{h}_\varepsilon + \dfrac{z\partial_z \mathbb{h}_\varepsilon}{1+\alpha} - \mathbb{h}_\varepsilon \partial_z \mathbb{h}_\varepsilon + \varepsilon \partial_z^2 \mathbb{h}_\varepsilon = 0, \\ \mathbb{h}_\varepsilon(z) = \kappa_\pm |z|^{-\alpha}(1 + o_{z\to\pm\infty}(1)). \end{cases} \tag{1.4}$$

But first, we define the function h_0 by the unique solution to the problem

$$\begin{cases} \frac{\alpha}{1+\alpha}h_0 + \frac{z\partial_z h_0}{1+\alpha} - h_0\partial_z h_0 = 0, \\ h_0(z) = \kappa_+|z|^{-\alpha}(1+o_{z\to+\infty}(1)), \end{cases} \tag{1.5}$$

if $z > z_c$, and the unique viscous solution to

$$\begin{cases} \frac{\alpha}{1+\alpha}h_0 + \frac{z\partial_z h_0}{1+\alpha} - h_0\partial_z h_0 = 0, \\ h_0(z) = \kappa_-|z|^{-\alpha}(1+o_{z\to-\infty}(1)), \end{cases} \tag{1.6}$$

if $z < z_c$, where $z_c \in \mathbb{R}$ is the position of the jump given by the conditions described above, uniquely determined by $\kappa_+, \kappa_-, \alpha$. We will show in Section 2.3 the existence and uniqueness of the solution of these problems.

We define

$$h_0(z_c^\pm) := \lim_{\nu\to 0} h_0(z_c \pm \nu),$$

as the function h_0 is discontinuous at z_c.

We construct a solution of (1.4) using a shooting method, and this solution will be close to h_0 far from z_c; see the following result.

Proposition 1.3. *For any $\kappa > 0$, $\alpha \in]0,1[$, there exist $z_c \in \mathbb{R}$, $\varepsilon_0 > 0$ such that, for $\varepsilon_0 > \varepsilon > 0$, there exist two C^1 functions $\varepsilon \to z_c(\varepsilon), a(\varepsilon)$ with*

$$z_c(\varepsilon) \to z_c, \quad a(\varepsilon) \to \frac{h_0(z_c^-) - h_0(z_c^+)}{2}$$

when $\varepsilon \to 0$, such that the solution of the ODE problem

$$\begin{cases} \frac{\alpha}{1+\alpha}\mathbb{h}_\varepsilon + \left(\frac{z}{1+\alpha} - \mathbb{h}_\varepsilon\right)\partial_z \mathbb{h}_\varepsilon + \varepsilon\partial_z^2 \mathbb{h}_\varepsilon = 0, \\ \mathbb{h}_\varepsilon(z_c(\varepsilon)) = \frac{z_c(\varepsilon)}{1+\alpha}, \quad \mathbb{h}_\varepsilon'(z_c(\varepsilon)) = \frac{-a(\varepsilon)^2}{2\varepsilon}, \end{cases}$$

satisfies

$$\mathbb{h}_\varepsilon(z) = \kappa_\pm|z|^{-\alpha}(1+o_{z\to\pm\infty}(1)).$$

Furthermore, there exists $w_0 > 0$ depending only on α and κ, such that

$$\left\| \mathbb{h}_\varepsilon(z) - h_0(z_c^+) - \frac{2a(\varepsilon)e^{-a(\varepsilon)(\frac{z-z_c(\varepsilon)}{\varepsilon})}}{1+e^{-a(\varepsilon)(\frac{z-z_c(\varepsilon)}{\varepsilon})}} \right\|_{L^\infty([z_c(\varepsilon)-w_0\varepsilon\ln\frac{1}{\varepsilon}, z_c(\varepsilon)+w_0\varepsilon\ln\frac{1}{\varepsilon}])} \to 0$$

and

$$\|(1+|z|)^\alpha(\mathbb{h}_\varepsilon - h_0)(z)\|_{L^\infty(\mathbb{R}\setminus[z_c(\varepsilon)-w_0\varepsilon\ln\frac{1}{\varepsilon}, z_c(\varepsilon)+w_0\varepsilon\ln\frac{1}{\varepsilon}])} \to 0$$

when $\varepsilon \to 0$. Finally, we have $|\partial_\varepsilon z_c(\varepsilon)| + |\partial_\varepsilon a(\varepsilon)| \leqslant K(\ln\frac{1}{\varepsilon})^2$.

Section 2 is devoted to the proof of this result. For $\varepsilon \neq 0$ and $a = \frac{h_0(z_c^-)-h_0(z_c^+)}{2}$, the solution to

$$\begin{cases} \frac{\alpha}{1+\alpha} h_\varepsilon + \Big(\frac{z}{1+\alpha} - h_\varepsilon\Big)\partial_z h_\varepsilon + \varepsilon \partial_z^2 h_\varepsilon = 0, \\ h_\varepsilon(z_c) = \frac{z_c}{1+\alpha}, \quad h_\varepsilon'(z_c(\varepsilon)) = \frac{-a^2}{2\varepsilon}, \end{cases}$$

does not satisfy $h_\varepsilon(z) = \kappa_\pm |z|^{-\alpha}(1 + o_{z\to\pm\infty}(1))$, but

$$h_\varepsilon(z) = (\kappa_\pm + o_{\varepsilon\to 0}(1))|z|^{-\alpha}(1 + o_{z\to\pm\infty}(1)).$$

This is why, to get the exact same equivalent at $+\infty$, we need to slightly change $z_c(\varepsilon)$ and $a(\varepsilon)$. We use the notation $\mathbb{h}_\varepsilon$ for solutions of problem (1.4) (that is, depending on the behavior at $\pm\infty$) and h_ε for solutions depending on its value at z_c.

The function $\mathbb{h}_\varepsilon$ in Proposition 1.3 is close to the discontinuous function h_0 except in the vicinity of z_c, the discontinuity point. The solution $\mathbb{h}_\varepsilon$ solves (1.3) but not (1.2) because of the term $\frac{1-\alpha}{1+\alpha}\varepsilon\partial_\varepsilon \mathbb{h}_\varepsilon$; however, this term is small compared to the others. We will show the stability of $\mathbb{h}_\varepsilon$ in a space that in particular contains this error term in the next subsection.

To construct $\mathbb{h}_\varepsilon$, we found the right scale around z_c to now have a continuous function (it is $\frac{z-z_c(\varepsilon)}{\varepsilon} \simeq 1$ rather than $z \simeq 1$). The proof of Proposition 1.3 is done in two parts. First, we compute the first order in ε of the solution in $[z_c(\varepsilon) - w_0\varepsilon \ln\frac{1}{\varepsilon}, z_c(\varepsilon) + w_0\varepsilon\ln\frac{1}{\varepsilon}]$ for some $w_0 > 0$ large but independent of ε, and we show that at the boundaries of this interval, it becomes close to the value of h_0 at the same point. Then, outside this interval, h_0 and $\mathbb{h}_\varepsilon$ verify a similar equation for small ε, and start with similar values. We thus show that they stay close.

1.1.5. Stability of the profile. We recall that $\varepsilon(t) = t^{\frac{\alpha-1}{\alpha+1}}$ and $\mathbb{h}_\varepsilon$ is the solution described in Proposition 1.3. We want to show that if at a time $T > 0$ large, we solve the viscous Burgers equation with the initial data $\mathbb{h}_{\varepsilon(T)} + f_0$ at time T in the rescaled variables, then for all times $t \geqslant T$ we stay close to $\mathbb{h}_{\varepsilon(t)}$. Interestingly, $\mathbb{h}_{\varepsilon(t)}$ will not be first order; we need to modify it nonlinearly, depending on f_0. It turns out that the mass of f_0 will change the profile near z_c, in a nonnegligible way. The stability result is as follows.

Theorem 1.4. *Given $\alpha \in]\frac{1}{4}, 1[$, $\kappa_+, \kappa_- > 0$, there exists $T_0 > 0$ such that, for any $T \geqslant T_0$, there exists $\nu > 0$ depending on T such that, considering $\mathbb{h}_{\varepsilon(t)}$, $z_c(t)$, $a(t)$ defined in Proposition 1.3, the solution u to the problem*

$$\begin{cases} \partial_t u - \partial_x^2 u + u\partial_x u = 0, \\ u_{|t=T}(x) = T^{-\frac{\alpha}{1+\alpha}} \mathbb{h}_{\varepsilon(T)}(T^{-\frac{1}{1+\alpha}} x) + f_0(x), \end{cases}$$

with $f_0 \in H^2(\mathbb{R})$ and

$$\|(1+|x|)^3 f_0(x)\|_{L^\infty(\mathbb{R})} + \|\partial_x f_0\|_{H^1(\mathbb{R})} + \left|\int_{\mathbb{R}} f_0\right| \leqslant \nu$$

satisfies for any $t \geqslant T$ that

$$\left\| t^{\frac{\alpha}{1+\alpha}} u(x,t) - \mathbb{h}_{\varepsilon(t)}(t^{-\frac{1}{1+\alpha}} x) - \mathfrak{u}\big(t^{\frac{1-\alpha}{1+\alpha}}(xt^{-\frac{1}{1+\alpha}} - z_c(t))\big) \right\|_{L^\infty(\mathbb{R})} = o_{t\to+\infty}(1),$$

where $\mathfrak{u}$ is the unique solution to the problem

$$\begin{cases} -\partial_x \mathfrak{u} + \Big(\dfrac{a(t)(e^{-a(t)x} - 1)}{1 + e^{-a(t)x}}\Big)\mathfrak{u} + \dfrac{\mathfrak{u}^2}{2} = 0, \\ \displaystyle\int_{\mathbb{R}} \mathfrak{u} = \int_{\mathbb{R}} f_0. \end{cases}$$

Section 3 is devoted to the proof of this result. Let us make some remarks about it:

- This result implies Theorem 1.2 and gives us the asymptotic profile

$$t^{\frac{\alpha}{1+\alpha}} u(zt^{\frac{1}{1+\alpha}}, t) \to h_0(z)$$

  when $t \to +\infty$ for any $z \neq z_c$. The convergence is uniform on $\mathbb{R}$ if we remove any open set containing z_c. In the vicinity of z_c we still have convergence to some limit, and there this limit depends on $\mathfrak{u}$, that is, f_0. Note that $\mathfrak{u}$ depends nonlinearly on f_0, and thus the correction coming from $\mathfrak{u}$ is not simply a modulation on the parameters of $\mathbb{h}_{\varepsilon(t)}$ (that is, $\varepsilon(t)$, $a(t)$ or $z_c(t)$), even if for small values of $\int_{\mathbb{R}} f_0$, we have $\mathfrak{u} \simeq \partial_{z_c} \mathbb{h}_{\varepsilon(t)}$.
- With the conditions on f_0, we check that our initial data

$$T^{-\frac{\alpha}{1+\alpha}} \mathbb{h}_{\varepsilon(T)}(T^{-\frac{1}{1+\alpha}} x) + f_0(x)$$

  decays like $\kappa_\pm |x|^{-\alpha}$ when $x \to \pm\infty$, and f_0 is small when compared to the main profile, since ν depends on T. Also, the condition $\alpha > \frac{1}{4}$ is a technical one; we expect the result to hold for any $\alpha \in]0,1[$. This condition will be used to show that $\partial_\varepsilon \mathbb{h}_\varepsilon$ has enough decay at $\pm\infty$ to estimate it in $H^1(\mathbb{R})$; see Section 3.4.

The core idea of the proof is to write the solution for $t \geqslant T$ as

$$u(x,t) = t^{-\frac{\alpha}{1+\alpha}} \Big(\mathbb{h}_{\varepsilon(t)}(t^{-\frac{1}{1+\alpha}} x) + \mathfrak{u}\big(t^{\frac{1-\alpha}{1+\alpha}}(xt^{-\frac{1}{1+\alpha}} - z_c(t))\big) + f(x,t) \Big),$$

and now the error f is massless (that is, $\int_{\mathbb{R}} f = 0$). We write it as $f = \partial_x g$, and it turns out that we can integrate the equation to have a new equation on g. We show there some coercivity on the linear part on g in $H^2(\mathbb{R})$, and we control the nonlinear part, from which we deduce that $\|g\|_{H^2(\mathbb{R})} \to 0$ when $t \to +\infty$.

1.2. Generalization to equation (1.1)

Our approach also works for the equation $\partial_t u - \partial_x^2 u + \partial_x(\frac{u^2}{2} + J(u)) = 0$, if J satisfies

$$|J(x)| + |xJ'(x)| + |x^2 J''(x)| \leqslant C_0 |x|^3$$

for some $C_0 > 0$.

Proposition 1.5. *For $\alpha \in]\frac{2}{3}, 1[$ and T_0, ν depending on C_0, the result of Theorem 1.4 also holds for the problem*

$$\begin{cases} \partial_t u - \partial_x^2 u + \partial_x\Big(\dfrac{u^2}{2} + J(u)\Big) = 0, \\ u_{|t=T}(x) = T^{-\frac{\alpha}{1+\alpha}} \mathbb{h}_{\varepsilon(T)}(T^{-\frac{1}{1+\alpha}} x) + f_0(x). \end{cases}$$

Section 3.6 is devoted to the proof of this result. It is done simply by checking that the term $\partial_x(J(u))$ can be considered as an error term in the proof of the stability of Theorem 1.4. This is true because at the scale where we see the profile $\mathbb{h}_\varepsilon$, this term is small compared to the others. As before, the condition $\alpha > \frac{2}{3}$ is a technical one, and is here to make sure that $J(u)$ has enough decay at $\pm\infty$ to estimate it in $H^1(\mathbb{R})$.

1.3. Some related open problems

Our results should extend easily for values of $\kappa_+, \kappa_- \in \mathbb{R}^*$ except in the case $\kappa_+ < 0$, $\kappa_+ > 0$. There, it may be possible to construct a specific solution, but it is likely that it is an unstable one. If we generalize to the equation $\partial_t u - \partial_x^2 u + u^k \partial_x u = 0$ for $k \in \mathbb{N}^*$, it is likely that a similar result can be shown with some improvements in the proofs.

For now, it seems difficult to generalize this result to higher dimensions, but it would be of interest, in particular if similar profiles can be constructed for the two-dimensional Euler or Navier–Stokes equation.

2. Construction of the profile $\mathbb{h}_\varepsilon$

This section is devoted to the proofs of Proposition 1.3 and the viscosity properties described in the introduction. First, in Section 2.1 we set the change of scaling and compute the Rankine–Hugoniot condition for the viscous Burgers equation. Section 2.2 is devoted to the case $\varepsilon = 0$. Section 2.3 is about the construction of h_0 (which will be the limit of $\mathbb{h}_\varepsilon$ when $\varepsilon \to 0$), as well as the study of its properties. Sections 2.4 and 2.5 are the study of the shooting problem at the heart of Proposition 1.3, respectively close and far from the shooting point z_c. Section 2.6 regroups all these elements and concludes the proof of Proposition 1.3.

2.1. Change of variable and viscosity conditions

In this subsection, our goal is to prove some results of Sections 1.1.1 to 1.1.3.

2.1.1. Computation of the underlying ODE problem. We consider here the equation

$$\partial_t u - \partial_x^2 u + u\partial_x u = 0.$$

We define

$$g(z,t) = t^{\frac{\alpha}{1+\alpha}} u(z t^{\frac{1}{1+\alpha}}, t),$$

and we have

$$t\partial_t g = \frac{\alpha}{1+\alpha} g + \frac{z}{1+\alpha} t^{\frac{\alpha}{1+\alpha}+\frac{1}{1+\alpha}} \partial_x u + t^{\frac{\alpha}{1+\alpha}+1} \partial_t u$$

and

$$\partial_z g = t\partial_x u, \quad \partial_z^2 g = t^{1+\frac{1}{1+\alpha}} \partial_x^2 u.$$

Therefore,

$$t\partial_t g = \frac{\alpha}{1+\alpha} g + \frac{z}{1+\alpha} t^{\frac{\alpha}{1+\alpha}+\frac{1}{1+\alpha}-1} \partial_z g + t^{\frac{\alpha}{1+\alpha}+1} (t^{-1-\frac{1}{1+\alpha}} \partial_z^2 g - t^{-1-\frac{\alpha}{1+\alpha}} g \partial_z g),$$

that is,

$$t\partial_t g = \frac{\alpha}{1+\alpha} g + \frac{z\partial_z g}{1+\alpha} - g\partial_z g + t^{\frac{\alpha-1}{1+\alpha}} \partial_z^2 g.$$

We define $\varepsilon(t) = t^{\frac{\alpha-1}{\alpha+1}}$ and we make the change of variable

$$h(z,\varepsilon) = g(z,t).$$

Since

$$t\partial_t \varepsilon = \frac{\alpha-1}{\alpha+1}\varepsilon$$

we have

$$\frac{1-\alpha}{1+\alpha}\varepsilon\partial_\varepsilon h + \frac{\alpha}{1+\alpha} h + \frac{z\partial_z h}{1+\alpha} - h\partial_z h + \varepsilon\partial_z^2 h = 0. \tag{2.1}$$

By Proposition 1.1, this scaling is not adapted to the heat equation with the same initial condition. Note that if we tried to use this scale anyway, we would get the same equation (2.1) but without the term $-h\partial_z h$. When $\varepsilon \to 0$, the limit problem will be $\frac{\alpha}{1+\alpha} h + \frac{z\partial_z h}{1+\alpha} = 0$, which only has the solution $h = Cz^{-\alpha}$ for some $C > 0$, which is unbounded at $z = 0$.

2.1.2. Rankine–Hugoniot condition. For the equation $\frac{\alpha}{1+\alpha} h + \frac{z\partial_z h}{1+\alpha} - h\partial_z h = 0$, integrating it between $z_c - \nu$ and $z_c + \nu$ leads, after some computations, to

$$\begin{aligned} \frac{\alpha-1}{1+\alpha} \int_{z_c-\nu}^{z_c+\nu} h &+ \frac{1}{1+\alpha}((z_c+\nu)h(z_c+\nu) - (z_c-\nu)h(z_c-\nu)) \\ &- \frac{1}{2}(h^2(z_c+\nu) - h^2(z_c-\nu)) \\ &= 0. \end{aligned}$$

Therefore, letting $\nu \to 0$ leads to

$$\frac{z_c}{1+\alpha}(h(z_c^+) - h(z_c^-)) - \frac{1}{2}(h^2(z_c^+) - h^2(z_c^-)) = 0,$$

which we factorize as

$$(h(z_c^+) - h(z_c^-))\Big(\frac{z_c}{1+\alpha} - \frac{1}{2}(h(z_c^+) + h(z_c^-))\Big) = 0,$$

which is the Rankine–Hugoniot condition stated in the introduction.

2.2. Some properties of solutions of $\frac{\alpha}{1+\alpha}h+(\frac{\mathrm{Id}}{1+\alpha}-h)h'=0$

Take $(z_0,b)\in\mathbb{R}^2$ with $b\neq\frac{z_0}{1+\alpha}$, and we consider here the problem

$$\begin{cases}\dfrac{\alpha}{1+\alpha}h+\Big(\dfrac{z}{1+\alpha}-h\Big)\partial_z h=0,\\ h(z_0)=b,\end{cases}\tag{2.2}$$

which is the problem described in Section 1.1.2. The fact that if $(z_0,b)\in\mathbf{J}\in\{\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D},\mathbf{E},\mathbf{F}\}$ implies that $(z,h(z))\in\mathbf{J}$ for all values of z on which the solution h of (2.2) is well defined is a consequence of standard Cauchy theory arguments, since the boundary between two bold sets is either a solution of (2.2), or the set $\{(z_0,b)\in\mathbb{R}^2,\ b=\frac{z_0}{1+\alpha}\}$, on which $\partial_z h$ explode.

2.2.1. The case $(z_0,b)\in\mathbf{A}$.

Lemma 2.1. *The solution h of (2.2) with $(z_0,b)\in\mathbf{A}$ is defined on $\mathbb{R}$ and satisfies*

$$\lim_{z\to+\infty}h(z)-z=0.$$

We leave the study of the behavior when $z\to-\infty$ for Section 2.3.

Proof. We consider $(z_0,b)\in\mathbf{A}=\{(z_0,b)\in\mathbb{R}^2,\ b>\max(0,z_0)\}$. As long as the solution h of (2.2) for this initial condition exists, we have $(z,h(z))\in\mathbf{A}$, therefore $h(z)>\max(0,z)$. We denote by $]z_-,z_+[$ the maximum domain of existence of h with $z_-,z_+\in\mathbb{R}\cup\{\pm\infty\}$ (by definition we have $z_0\in]z_-,z_+[$).

Suppose that $z_+\neq+\infty$. There exists $C_0>0$ depending on z_0 and b such that, for $z\in[z_0,z_+[$ we have $|\frac{z}{1+\alpha}-h(z)|\geqslant C_0(1+|z|)$. Indeed, we have $h(z)>\max(0,z)$ and $\frac{1}{1+\alpha}<1$. In particular, $\frac{z}{1+\alpha}-h(z)\neq 0$ on $[z_0,z_+[$ and since $h(z)>0$ we have

$$\Big|\frac{\partial_z h}{h}\Big|\leqslant\frac{K}{(1+|z|)}$$

on $[z_0,z_+[$. With $z_+<+\infty$, we deduce that h and $\partial_z h$ are bounded near z_+, which is a contradiction, therefore $z_+=+\infty$. We define for $z\geqslant z_0$ the function $u(z)=h(z)-z>0$. It satisfies the equation

$$\Big(\frac{\alpha}{1+\alpha}-\partial_z h(z)\Big)u+\Big(\frac{z}{1+\alpha}-h(z)\Big)\partial_z u=0\tag{2.3}$$

on $[z_0,+\infty[$. Now we compute, using the equation satisfied by h, that

$$\Big(\frac{\alpha}{1+\alpha}-\partial_z h(z)\Big)\Big(\frac{z}{1+\alpha}-h(z)\Big)=\frac{\alpha}{1+\alpha}\Big(\frac{z}{1+\alpha}-h(z)\Big)+\frac{\alpha}{1+\alpha}h(z)=\frac{\alpha z}{(1+\alpha)^2},$$

hence

$$\frac{\alpha}{1+\alpha}-\partial_z h(z)=\frac{\alpha z}{(1+\alpha)^2(\frac{z}{1+\alpha}-h(z))},$$

and we can write equation (2.3) as

$$\frac{\alpha z}{(1+\alpha)^2}u + \left(\frac{z}{1+\alpha} - h(z)\right)^2 \partial_z u = 0.$$

Using $|\frac{z}{1+\alpha} - h(z)| \geqslant C_0(1+|z|)$, we deduce that for $z \geqslant \max(1, z_0)$ we have

$$\frac{\partial_z u}{u} \leqslant \frac{-C_1}{z}$$

for some $C_1 > 0$, therefore $u(z) \leqslant K z^{-C_1}$ for $z \geqslant \max(1, z_0)$ for some constant $K > 0$, hence $u(z) \to 0$ when $z \to +\infty$, leading to $h(z) - z \to 0$ when $z \to +\infty$.

On $]z_-, z_0]$, by similar arguments to previously, we have $h > 0$ and $\frac{z}{1+\alpha} - h(z) < 0$. Therefore, on $]z_-, z_0]$ we have $\partial_z h(z) > 0$, $h(z) > 0$, hence $z_- = -\infty$. ■

2.2.2. The case $(z_0, b) \in \mathbf{F}$.

Lemma 2.2. *The solution h of (2.2) with $(z_0, b) \in \mathbf{F}$ is defined on $]z_-, +\infty[$ for some $z_- > 0$.*

We also leave the study of the behavior when $z \to +\infty$ for Section 2.3.

Proof. We consider here $(z_0, b) \in \mathbf{F} = \{(z_0, b) \in \mathbb{R}^2, \frac{z_0}{1+\alpha} > b > 0\}$ in problem (2.2). As in the previous subsection, we consider the largest interval on which the solution is defined, which we write as $]z_-, z_+[$. We have $z_0 \in]z_-, z_+[$ and for $z \in]z_-, z_+[$, we have

$$\frac{z}{1+\alpha} > h(z) > 0.$$

A consequence of this and the equation $\frac{\alpha}{1+\alpha}h + (\frac{z}{1+\alpha} - h)\partial_z h = 0$ is that $\partial_z h < 0$ on $]z_-, z_+[$, and with $h > 0$, we deduce that $z_+ = +\infty$. We also see that $z_- > 0$ because the condition $\frac{z}{1+\alpha} > h(z) > 0$ can no longer hold at $z = 0$. ■

2.2.3. The remaining cases. For $(z_0, b) \in \mathbf{E}$, we can deal with the limit for large z as in the case of $\mathbf{A}$, and we can show that $z_- > 0$ as in the case of $\mathbf{F}$. By symmetry, we show similar properties in $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$.

2.3. Definition and properties of the profile h_0

The goal of this subsection is to show that, given $\alpha \in]0, 1[$ and $\kappa_+, \kappa_- > 0$, there exist a unique value of z_c and a unique viscous solution of (1.5)–(1.6) in the sense of the introduction. We will also study its properties.

2.3.1. A connected implicit problem. We look for an implicit solution of $\frac{\alpha}{1+\alpha}h + (\frac{z}{1+\alpha} - h)\partial_z h = 0$ of the form $z = g(h)$. Differentiating with respect to z, we have $1 = \partial_z h g'(h)$ and replacing, we deduce that

$$g'(h) = \frac{-g(h)}{\alpha h} + \frac{1+\alpha}{\alpha}.$$

The solution of this equation is of the form $g(h) = h + \frac{\kappa}{|h|^\alpha}$.

This is why we define for $\alpha \in]0,1[$ and $\kappa > 0$ the function

$$g_\kappa(y) := y + \frac{\kappa}{|y|^\alpha}.$$

We are interested in the solutions of the implicit problem $z = g_\kappa(y(z))$. First, remark that $g_\kappa(y) \to +\infty$ when $y \to 0^\pm$, $g_\kappa(y) \to \pm\infty$ when $y \to \pm\infty$ and $g'_\kappa(y) \to 1$ when $y \to \pm\infty$. We compute for $y \neq 0$ that

$$g'_\kappa(y) = 1 - \frac{\kappa\alpha}{y|y|^\alpha}.$$

In particular, $g'_\kappa > 0$ on $]-\infty, 0[$. We have $g'_\kappa(y) = 0$ if and only if $y = y_\kappa = (\kappa\alpha)^{\frac{1}{1+\alpha}}$. This implies that on $[y_0, +\infty[$, we have $g'_\kappa(y) > 0$. We compute easily that

$$g_\kappa(y_\kappa) = \kappa^{\frac{1}{1+\alpha}}(\alpha^{\frac{1}{1+\alpha}} + \alpha^{-\frac{\alpha}{1+\alpha}}) > 0.$$

By the implicit function theorem, given $\kappa_+, \kappa_- > 0$ we construct two particular branches of functions. First, a smooth function $y^*_-\colon \mathbb{R} \to]-\infty, 0[$, a solution of $z = g_{\kappa_-}(y^*_-(z))$ for any $z \in \mathbb{R}$, defined as the inverse of the invertible function $g_{\kappa_-}\colon]-\infty, 0[\to \mathbb{R}$, and another smooth function

$$y^*_+\colon]g_{\kappa_+}(y_{\kappa_+}), +\infty[\to]y_{\kappa_+}, +\infty[,$$

a solution of $z = g_{\kappa_+}(y^*_+(z))$, defined as the inverse of

$$g_{\kappa_+}\colon]y_{\kappa_+}, +\infty[\to]g_{\kappa_+}(y_{\kappa_+}), +\infty[.$$

We define here $h_\pm(z) := \frac{\kappa_\pm}{|y^*_\pm(z)|^\alpha} = z - y^*_\pm(z)$. Since $g'_\kappa(y) \to 1$ when $|y| \to +\infty$, we have that $y^*_\pm(z) \to z$ when $z \to \pm\infty$ and therefore $h_\pm(z) \sim \frac{\kappa_\pm}{|z|^\alpha}$ when $|z| \to +\infty$. Let us show that these functions are solutions of

$$\frac{\alpha}{1+\alpha}h + \Big(\frac{z}{1+\alpha} - h\Big)\partial_z h = 0.$$

Lemma 2.3. *The functions $h_\pm$ satisfy, on their domains of definition, the equation*

$$\frac{\alpha}{1+\alpha}h_\pm + \Big(\frac{z}{1+\alpha} - h_\pm\Big)\partial_z h_\pm = 0.$$

Proof. We first check that $g'_\kappa(y) = 1 - \frac{\kappa\alpha}{y|y|^\alpha} = (1+\alpha)y - \alpha g_\kappa(y)$, and since $g_{\kappa_\pm}(y^*_\pm(z)) = z$, we have

$$y^*_\pm(z)g'_{\kappa_\pm}(y^*_\pm(z)) = (1+\alpha)y^*_\pm(z) - \alpha z.$$

Furthermore, differentiating the equation $z = g_\kappa(y(z))$ with respect to z leads to

$$\partial_z y^*_\pm(z)g'_{\kappa_\pm}(y^*_\pm(z)) = 1,$$

therefore

$$\partial_z y^*_\pm(z)\Big(y^*_\pm(z) - \frac{\alpha}{1+\alpha}z\Big) = \frac{y^*_\pm(z)}{1+\alpha}.$$

Since $h_\pm(z) = z - y^*_\pm(z)$, we have

$$\begin{aligned}\Big(\frac{z}{1+\alpha} - h_\pm\Big)\partial_z h_\pm &= \Big(\frac{z}{1+\alpha} - h_\pm\Big)(1 - \partial_z y^*_\pm) \\ &= \frac{z}{1+\alpha} - h_\pm - \partial_z y^*_\pm\Big(y^*_\pm(z) - \frac{\alpha}{1+\alpha}z\Big)\end{aligned}$$

and thus

$$\Big(\frac{z}{1+\alpha} - h_\pm\Big)\partial_z h_\pm = \frac{z}{1+\alpha} - h_\pm - \frac{y^*_\pm(z)}{1+\alpha} = \frac{-\alpha}{1+\alpha}h_\pm. \qquad \blacksquare$$

This result allows us to complete the study of problem (2.2) for $(z_0, b) \in \mathbf{A}$ and $(z_0, b) \in \mathbf{F}$.

Lemma 2.4. *Given $(z_0, b) \in \mathbf{A}$, the solution of problem (2.2) is h_- defined above for the value $\kappa_- = b|b - z_0|^\alpha$. It satisfies in particular*

$$\lim_{z\to-\infty} |z|^\alpha h_-(z) = b|b - z_0|^\alpha.$$

Furthermore, for $z \leqslant z_0$, $n \in \mathbb{N}$, there exists $C_n > 0$ depending on n, z_0, α such that

$$|\partial_z^n h_-(z)| \leqslant \frac{C_n}{(1 + |z|)^{\alpha+n}}.$$

Similarly, for $(z_0, b) \in \mathbf{F}$, the solution of problem (2.2) is h_+ defined above for the value $\kappa_+ = b|b - z_0|^\alpha$. It satisfies

$$\lim_{z\to+\infty} |z|^\alpha h_+(z) = b|b - z_0|^\alpha$$

and for $z \geqslant z_0$,

$$|\partial_z^n h_+(z)| \leqslant \frac{C_n}{(1 + |z|)^{\alpha+n}}.$$

Proof. Given $(z_0, b) \in \mathbf{A}$, we look for a value κ such that

$$z_0 = g_\kappa(z_0 - b)$$

with $g_\kappa(z) = z + \frac{\kappa}{|z|^\alpha}$. We check that then $\kappa = b|b - z_0|^\alpha$. Defining h_- as above, remark that $h_-(z_0) = b$ and h_- satisfies

$$\frac{\alpha}{1+\alpha}h_- + (\frac{z}{1+\alpha} - h_-)\partial_z h_- = 0.$$

It is therefore the solution of (2.2) for this choice of $(z_0, b) \in \mathbf{A}$. The proof is identical if $(z_0, b) \in \mathbf{F}$ with $\kappa_+ = b|b - z_0|^\alpha$, the only difference being that we have to check that $z_0 - b = y^*(z_0) \in]y_{\kappa_+}, +\infty[$, that is,

$$z_0 - b \geqslant (\kappa\alpha)^{\frac{1}{1+\alpha}} = |z_0 - b|^{\frac{\alpha}{1+\alpha}} b^{\frac{1}{1+\alpha}} \alpha^{\frac{1}{1+\alpha}}.$$

This inequality is a consequence of the fact that $(z_0, b) \in \mathbf{F}$, which implies that $\frac{z_0}{1+\alpha} > b > 0$, and thus $z_0 - b \geqslant \alpha b$.

Now, concerning the computations of $\partial_z^n h_\pm$, we have

$$\partial_z h_\pm = \frac{\frac{-\alpha}{1+\alpha} h_\pm}{\frac{z}{1+\alpha} - h_\pm},$$

hence

$$|\partial_z h_\pm(z)| \leqslant \frac{C_1}{(1+|z|)^{\alpha+1}}$$

and we can conclude by induction. ∎

We complete this subsection with a technical lemma on the dependency on b and z_0 of $h_\pm$.

Lemma 2.5. *The function $(b, z_0) \to h_\pm$ is differentiable, and there exists $K > 0$ depending on b, z_0 such that*

$$\left| h_\pm(z) - \frac{b|b - z_0|^\alpha}{|z|^\alpha} \right| \leqslant \frac{K}{(1+|z|)^{1+2\alpha}},$$

as well as

$$\left| \partial_b h_\pm(z) - \frac{\partial_b (b|b - z_0|^\alpha)}{|z|^\alpha} \right| \leqslant \frac{K}{(1+|z|)^{1+2\alpha}}$$

on the domain of definition of $h_\pm$.

Proof. Take $y(z)$ a function solution of the implicit problem

$$y(z) + \frac{\kappa}{|y(z)|^\alpha} = z$$

defined on some interval $[z_0, +\infty[$. By the remarks above Lemma 2.3 we have $y(z) \sim z$ when $z \to +\infty$. Writing $y = z + \bar{y}(z)$, we check that

$$|z|^\alpha \bar{y}(z) \left| 1 + \frac{\bar{y}(z)}{z} \right|^\alpha = -\kappa,$$

hence

$$\bar{y}(z) \sim \frac{-\kappa}{|z|^\alpha}$$

when $z \to +\infty$. We deduce that $h(z) = \frac{\kappa}{|y(z)|^\alpha}$ satisfies

$$h(z) - \frac{\kappa}{|z|^\alpha} \sim \frac{\alpha\kappa^2}{|z|^{1+2\alpha}}$$

when $z \to +\infty$. We then check easily that we have, similarly,

$$\partial_\kappa h(z) - \frac{1}{|z|^\alpha} \sim \frac{2\alpha\kappa}{|z|^{1+2\alpha}}$$

when $z \to +\infty$, which implies the result of the lemma for h_+, and a similar proof works for h_-. ∎

2.3.2. Connection between the jump and the limits at $\pm\infty$. We recall the notation

$$h(z_c^{\pm}) = \lim_{\nu \to 0} h(z_c \pm \nu).$$

Lemma 2.6. *Take $\alpha \in]0,1[$, $z_c > 0$ and $\frac{z_c}{1+\alpha} > a > \frac{\alpha}{1+\alpha} z_c$. Then there exists a unique function $h \in C^\infty(\mathbb{R}\backslash\{z_c\},]0,+\infty[)$ a solution to the problem*

$$\begin{cases} \frac{\alpha}{1+\alpha} h + \Big(\frac{z}{1+\alpha} - h\Big)\partial_z h = 0, \\ h(z_c^+) = \frac{z_c}{1+\alpha} - a, \\ h(z_c^-) = \frac{z_c}{1+\alpha} + a. \end{cases}$$

Furthermore, there exists $\kappa_{\pm}(z_c, a) > 0$ such that

$$h(z) = \frac{\kappa_{\pm}(z_c, a)}{|z|^\alpha}(1 + o_{z\to\pm\infty}(1)),$$

and

$$(z_c, a) \to (\kappa_+(z_c,a), \kappa_-(z_c,a))$$

is a smooth function and a bijection from $\{(z_c, a) \in]0,+\infty[^2, \frac{z_c}{1+\alpha} > a > \frac{\alpha}{1+\alpha} z_c\}$ to $]0,+\infty[^2$.

Proof. By Lemma 2.1, the solution of

$$\begin{cases} \frac{\alpha}{1+\alpha} h + \Big(\frac{z}{1+\alpha} - h\Big)\partial_z h = 0, \\ h(z_c) = \frac{z_c}{1+\alpha} - a, \end{cases}$$

with $z_c > 0$ and $0 < h(z_c) < \frac{z_c}{1+\alpha}$ (that is, $\frac{z_c}{1+\alpha} > a > 0$) is well defined for all $z \geqslant z_c$, and we have $g_{\kappa_+(z_c,a)}(z - h(z)) = z$ for

$$\kappa_+(z_c, a) := h(z_c)|z_c - h(z_c)|^\alpha = \Big(\frac{z_c}{1+\alpha} - a\Big)\Big|\frac{\alpha}{1+\alpha} z_c + a\Big|^\alpha.$$

Similarly, the solution of

$$\begin{cases} \frac{\alpha}{1+\alpha} h + \Big(\frac{z}{1+\alpha} - h\Big)\partial_z h = 0, \\ h(z_c) = \frac{z_c}{1+\alpha} + a, \end{cases}$$

with $h(z_c) > z_c$ (that is, $a > \frac{\alpha}{1+\alpha} z_c$) is well defined for all $z \leqslant z_c$, and we have $g_{\kappa_-(z_c,a)}(z - h(z)) = z$ for

$$\kappa_-(z_c, a) := h(z_c)|z_c - h(z_c)|^\alpha = \Big(\frac{z_c}{1+\alpha} + a\Big)\Big|\frac{\alpha}{1+\alpha} z_c - a\Big|^\alpha.$$

We deduce that

$$(z_c, a) \to (\kappa_+(z_c, a), \kappa_-(z_c, a))$$

is a smooth function from $\{(z_c, a) \in \mathbb{R},\ z_c > 0,\ a \in]\frac{\alpha}{1+\alpha} z_c, \frac{1}{1+\alpha} z_c[\}$ to $]0, +\infty[^2$. Let us show that it is a bijection. Writing $a = z_c b$ with $b \in]\frac{\alpha}{1+\alpha}, \frac{1}{1+\alpha}[$, we have

$$\kappa_+ = z_c^{1+\alpha}\Big(\frac{1}{1+\alpha} - b\Big)\Big|\frac{\alpha}{1+\alpha} + b\Big|^\alpha$$

and

$$\kappa_- = z_c^{1+\alpha}\Big(\frac{1}{1+\alpha} + b\Big)\Big|b - \frac{\alpha}{1+\alpha}\Big|^\alpha,$$

hence

$$\frac{\kappa_+}{\kappa_-} = \zeta(b) := \frac{(\frac{1}{1+\alpha} - b)|b + \frac{\alpha}{1+\alpha}|^\alpha}{(\frac{1}{1+\alpha} + b)|b - \frac{\alpha}{1+\alpha}|^\alpha}.$$

The variable ζ is a smooth function of b in $]\frac{\alpha}{1+\alpha}, \frac{1}{1+\alpha}[$, and

$$\lim_{b \to (\frac{\alpha}{1+\alpha})^+} \zeta(b) = +\infty, \qquad \lim_{b \to (\frac{1}{1+\alpha})^-} \zeta(b) = 0.$$

We check that

$$\zeta'(b) = \frac{-2b|b + \frac{\alpha}{1+\alpha}|^{\alpha-1}|b - \frac{\alpha}{1+\alpha}|^{\alpha-1}}{((\frac{1}{1+\alpha} + b)|b - \frac{\alpha}{1+\alpha}|^\alpha)^2} < 0,$$

hence ζ is a bijection from $]\frac{\alpha}{1+\alpha}, \frac{1}{1+\alpha}[$ to $\mathbb{R}^{+*}$. This completes the proof of the lemma. ∎

We now can construct the function h_0: for $\kappa_+, \kappa_- > 0$, take $(z_c(\kappa_+, \kappa_-), a(\kappa_+, \kappa_-)) \in]0, +\infty[^2$ such that $\kappa_+(z_c(\kappa_+, \kappa_-), a(\kappa_+, \kappa_-)) = \kappa_+$, $\kappa_-(z_c(\kappa_+, \kappa_-), a(\kappa_+, \kappa_-)) = \kappa_-$, then h_0 is the solution of Lemma 2.6 for these values. It is almost a solution of (1.3), but it is discontinuous at z_c. It satisfies the Rankine–Hugoniot condition, and by Lemma 2.6 it is the only solution among the ones behaving like $\kappa_\pm |z|^{-\alpha}$ at $\pm\infty$ doing so.

Our goal in the next subsection is to construct a better approximation $\mathbb{h}_\varepsilon$, that will be continuous at z_c, and be close to h_0 away from it when ε is small.

2.4. Shooting from z_c and shape of the profile near it

We want to understand the solution to the problem

$$\begin{cases} \dfrac{\alpha}{1+\alpha} h_\varepsilon + \Big(\dfrac{z}{1+\alpha} - h_\varepsilon\Big)\partial_z h_\varepsilon + \varepsilon \partial_z^2 h_\varepsilon = 0, \\ h_\varepsilon(z_c) = \dfrac{z_c}{1+\alpha}, \quad h_\varepsilon'(z_c) = \dfrac{-a^2}{2\varepsilon}, \end{cases} \tag{2.4}$$

for some given parameters $z_c, a > 0$ with $\frac{z_c}{1+\alpha} > a > \frac{\alpha}{1+\alpha} z_c$ that for now are independent of ε. In this subsection we take h to be the solution of Lemma 2.6 associated to the values of z_c and a. The function h is discontinuous at z_c. We want to show that for the right choice of z_c and a, h_ε is close to h far from z_c, and we want to compute the shape of h_ε near z_c.

2.4.1. Estimates in $]z_c, z_c + w_0\varepsilon \ln \frac{1}{\varepsilon}]$.

Lemma 2.7. *There exists $w_0 > 0$ depending only on α, z_c, a such that h_ε, the solution of (2.4), is well defined on $]z_c, z_c + w_0\varepsilon \ln \frac{1}{\varepsilon}]$ and satisfies*

$$\left\| h_\varepsilon(z) - h(z) - \frac{2ae^{-a(\frac{z-z_c}{\varepsilon})}}{1 + e^{-a(\frac{z-z_c}{\varepsilon})}} \right\|_{L^\infty(]z_c, z_c + w_0\varepsilon \ln \frac{1}{\varepsilon}])} \to 0,$$

as well as

$$\left\| h'_\varepsilon(z) + \frac{2a^2 e^{-a(\frac{z-z_c}{\varepsilon})}}{\varepsilon(1 + e^{-a(\frac{z-z_c}{\varepsilon})})^2} \right\|_{L^\infty(]z_c, z_c + w_0\varepsilon \ln \frac{1}{\varepsilon}])} \leqslant C$$

for some constant $C > 0$ depending only on α, z_c, a when $\varepsilon \to 0$, and also

$$\left| h_\varepsilon\Big(z_c + w_0\varepsilon \ln \frac{1}{\varepsilon}\Big) - h(z_c^+) \right| + \left| h'_\varepsilon\Big(z_c + w_0\varepsilon \ln \frac{1}{\varepsilon}\Big) - h'(z_c^+) \right| \leqslant K\varepsilon \ln \frac{1}{\varepsilon}$$

This lemma implies that, when we are at distance $w_0\varepsilon \ln \frac{1}{\varepsilon}$ to the right of z_c for some constant $w_0 > 0$ independent of ε, the functions h_ε and h and their derivatives are close. In particular, $h'_\varepsilon(z_c) = \frac{-a^2}{2\varepsilon}$ is large when ε is small, but $h'_\varepsilon(z_c + w_0\varepsilon \ln \frac{1}{\varepsilon})$ is of size 1. In other words, at $z_c + w_0\varepsilon \ln \frac{1}{\varepsilon}$ the jump has ended, and h_ε, h'_ε from now on will be bounded uniformly in ε. The choice of $w_0\varepsilon \ln \frac{1}{\varepsilon}$ is not necessarily optimal – it might be improved – but it is enough here. We also compute the first-order correction in C^1 between h_ε and h to the right of z_c.

Proof of Lemma 2.7. We decompose, for $z > z_c$, $Z = \frac{z-z_c}{\varepsilon} > 0$, the solution of (2.4) as

$$h_\varepsilon(z) = h(z) + (F(Z) + a) + \varepsilon G(Z),$$

and we recall that $\frac{\alpha}{1+\alpha} h(z) + (\frac{z}{1+\alpha} - h(z))h'(z) = 0$. The function h is discontinuous at $z = z_c$, but we focus here only on $z \in]z_c, z_c + w_0\varepsilon \ln \frac{1}{\varepsilon}]$. We choose F the solution of

$$\begin{cases} F''(Z) - F(Z)F'(Z) = 0, \\ F(0) = 0, \quad F'(0) = \dfrac{-a^2}{2}, \end{cases}$$

that is,

$$F(Z) = \frac{a(e^{-aZ} - 1)}{1 + e^{-aZ}}.$$

Note that

$$F(+\infty) = -a, \quad a = \frac{h(z_c^-) - h(z_c^+)}{2} \quad \text{and} \quad \frac{z_c}{1+\alpha} = \frac{h(z_c^-) + h(z_c^+)}{2}$$

by Lemma 2.6. We also check that $G(0) = 0$, $G'(0) = -h'(z_c^+)$. Indeed,

$$h_\varepsilon(z_c^+) = h(z_c^+) + (F(0) + a) = \frac{z_c}{1+\alpha}$$

and

$$h'_\varepsilon(z_c^+) = h'(z_c^+) + \frac{1}{\varepsilon}F'(0) + G'(0) = \frac{-a^2}{2\varepsilon}.$$

Let us compute the equation satisfied by G. We have

$$\begin{aligned}
\frac{\alpha}{1+\alpha}h_\varepsilon &= \frac{\alpha}{1+\alpha}(h(z) + (F(Z)+a) + \varepsilon G(Z)),\\
\Big(\frac{z}{1+\alpha} - h_\varepsilon\Big) &= \Big(\frac{z}{1+\alpha} - h(z)\Big) - (F(Z)+a) - \varepsilon G(Z),\\
\partial_z h_\varepsilon &= h'(z) + \frac{1}{\varepsilon}F'(Z) + G'(Z),
\end{aligned}$$

so

$$\begin{aligned}
\Big(\frac{z}{1+\alpha} - h_\varepsilon\Big)\partial_z h_\varepsilon = &-\frac{1}{\varepsilon}F'(Z)F(Z) + \Big(\frac{z}{1+\alpha} - h(z)\Big)h'(z)\\
&+ \frac{1}{\varepsilon}F'(Z)\Big(\frac{z}{1+\alpha} - h(z) - a\Big) + \Big(\frac{z}{1+\alpha} - h(z)\Big)G'(Z)\\
&- (F(Z)+a)(h'(z) + G'(Z)) - G(Z)F'(Z)\\
&- \varepsilon G(Z)(h'(z) + G'(Z)).
\end{aligned}$$

Finally,

$$\varepsilon\partial_z^2 h_\varepsilon = \varepsilon h''(z) + \frac{1}{\varepsilon}F''(Z) + G''(Z).$$

Using $F''(Z) - F(Z)F'(Z) = 0$ and $\frac{\alpha}{1+\alpha}h(z) + (\frac{z}{1+\alpha} - h(z))h'(z) = 0$ we infer that on $Z > 0$, G satisfies

$$\begin{aligned}
G''(Z) &+ \Big(\frac{z}{1+\alpha} - h(z) - (F(Z)+a)\Big)G'(Z)\\
&+ \Big(-F'(Z) + \varepsilon\Big(\frac{\alpha}{1+\alpha} - h'(z)\Big)\Big)G(Z) - \varepsilon G(Z)G'(Z)\\
&+ \frac{1}{\varepsilon}F'(Z)\Big(\frac{z}{1+\alpha} - h(z) - a\Big) + \Big(\frac{\alpha}{1+\alpha} - h'(z)\Big)(F(Z)+a)\\
&+ \varepsilon h''(z)\\
= 0.&
\end{aligned}$$

We define the source part as

$$\begin{aligned}
S := &\frac{1}{\varepsilon}F'(Z)\Big(\frac{z}{1+\alpha} - h(z) - a\Big) + \Big(\frac{\alpha}{1+\alpha} - h'(z)\Big)(F(Z)+a)\\
&+ \varepsilon h''(z)
\end{aligned}$$

and the operator on G as

$$\begin{aligned}
\mathcal{O}(G) := G''(Z) &+ \Big(\frac{z}{1+\alpha} - h(z) - (F(Z)+a)\Big)G'(Z)\\
&+ \Big(-F'(Z) + \varepsilon\Big(\frac{\alpha}{1+\alpha} - h'(z)\Big)\Big)G(Z) - \varepsilon G(Z)G'(Z),
\end{aligned}$$

leading to the equation

$$\mathcal{O}(G) + S = 0.$$

Let us estimate $S(Z)$ for $Z > 0$. We have

$$\frac{z_c}{1+\alpha} - h(z_c^+) - a = 0,$$

therefore

$$\begin{aligned} S(Z) = {} & F'(Z)\Big(\frac{Z}{1+\alpha} - \frac{h(z_c + \varepsilon Z) - h(z_c)}{\varepsilon}\Big) \\ & + \Big(\frac{\alpha}{1+\alpha} - h'(z_c + \varepsilon Z)\Big)(F(Z) + a) + \varepsilon h''(z_c + \varepsilon Z). \end{aligned}$$

For now, take any $w_0 > 0$, independent of ε. Then, since

$$|F(Z) + a| + |F'(Z)| + |F''(Z)| \leqslant K e^{-aZ},$$

we deduce that

$$|S(Z)| \leqslant K(\varepsilon + e^{-\frac{a}{2}Z})$$

for $Z \in]0, w_0 \varepsilon \ln \frac{1}{\varepsilon}]$ for a constant $K > 0$ depending only on w_0, α, a.

Let us now look at the coefficient in the operator $\mathcal{O}(G)$. We write it

$$\mathcal{O}(G) = G''(Z) + A_1(Z)G'(Z) + A_2(Z)G(Z) - \varepsilon G(Z)G'(Z)$$

with

$$A_1(Z) := \frac{z_c + \varepsilon Z}{1+\alpha} - h(z_c + \varepsilon Z) - (F(Z) + a)$$

and

$$A_2(Z) := -F'(Z) + \varepsilon\Big(\frac{\alpha}{1+\alpha} - h'(z_c + \varepsilon Z)\Big).$$

In particular, A_1 and A_2 are bounded by constants independent of ε if $\varepsilon < 1$. By the estimates on S, for any $Z_0 > 0$, if $\varepsilon > 0$ is small enough depending on Z_0 (so that the nonlinear term $\varepsilon G(Z)G'(Z)$ can be neglected), there exists $K(Z_0) > 0$ such that

$$|G(Z)| + |G'(Z)| \leqslant K(Z_0) \tag{2.5}$$

for $Z \in [0, Z_0]$. This is because the equation satisfied by G is, except for the term $-\varepsilon GG'$, linear with a bounded source term. Without this nonlinear term the solution would then be global, and taking $\varepsilon > 0$ small enough depending on Z_0, since $G(0)$, $G'(0)$, A_1 and A_2 are bounded uniformly in ε, the solution exists at least on $[0, Z_0]$ with a uniform estimate depending on Z_0.

Now, remark that $A_1(Z) \to \frac{z_c}{1+\alpha} - h(z_c^+)$ and

$$\frac{A_2(Z)}{\varepsilon} \to \Big(\frac{\alpha}{1+\alpha} - h'(z_c^+)\Big)$$

if $Z \geqslant w_0 \ln \frac{1}{\varepsilon}$ with w_0 large (such that $F'(w_0 \ln \frac{1}{\varepsilon}) \leqslant \varepsilon^2$ for instance) when $\varepsilon \to 0$. We therefore write the equation on G as

$$G''(Z) + \Big(\frac{z_c}{1+\alpha} - h(z_c^+)\Big)G'(Z) + \varepsilon\Big(\frac{\alpha}{1+\alpha} - h'(z_c^+)\Big)G(Z) = S + \mathcal{R}(G)$$

with

$$\begin{aligned}\mathcal{R}(G) := G'(Z)\Big(&\frac{z_c}{1+\alpha} - h(z_c^+) - \Big(\frac{z}{1+\alpha} - h(z^+) - (F(Z)+a)\Big)\Big)\\ &+ G(Z)\Big(\varepsilon\Big(\frac{\alpha}{1+\alpha} - h'(z_c^+)\Big) - \Big(-F'(Z) + \varepsilon\Big(\frac{\alpha}{1+\alpha} - h'(z)\Big)\Big)\Big)\\ &+ \varepsilon G(Z)G'(Z).\end{aligned}$$

We simplify:

$$\begin{aligned}\mathcal{R}(G) = G'(Z)\Big(&\frac{-\varepsilon Z}{1+\alpha} + h(z_c + \varepsilon Z) - h(z_c^+) + F(Z) + a\Big)\\ &+ G(Z)\big(F'(Z) + \varepsilon(h'(z) - h'(z_c^+))\big)\\ &+ \varepsilon G(Z)G'(Z).\end{aligned}$$

To simplify the notation we define $\lambda := \frac{z_c}{1+\alpha} - h(z_c^+) > 0$, $\mu := \frac{\alpha}{1+\alpha} - h'(z_c^+) > 0$ (since $h'(z_c^+) < 0$) so that the equation on G can be written as

$$G''(Z) + \lambda G'(Z) + \varepsilon\mu G(Z) = S + \mathcal{R}(G).$$

For $\varepsilon > 0$ small enough we have $\lambda^2 - 4\varepsilon\mu > 0$, and then we can write, with

$$\lambda_\pm := \frac{-\lambda \pm \sqrt{\lambda^2 - 4\varepsilon\mu}}{2} < 0,$$

satisfying $\lambda_+ + \lambda_- = -\lambda$, $\lambda_+ \sim -\frac{\varepsilon\mu}{\lambda}$, $\lambda_- \sim -\lambda$ when $\varepsilon \to 0$, that (we recall that $G(0) = 0$, $G'(0) = -h'(z_c^+)$)

$$\begin{aligned}G(Z) = &\frac{-h'(z_c^+)}{\sqrt{\lambda^2 - 4\varepsilon\mu}}(e^{\lambda_+ Z} - e^{\lambda_- Z})\\ &+ e^{\lambda_+ Z}\int_0^Z e^{\lambda_- u}\Big(\int_0^u e^{\lambda v}(S(v) + \mathcal{R}(G)(v))\,dv\Big)\,du.\end{aligned} \tag{2.6}$$

Let us show that for $C_0 > 0$ large enough (independently of ε) and ε small enough, we have

$$|G(Z)| + |G'(Z)| \leqslant C_0\Big(\varepsilon \ln\frac{1}{\varepsilon} + e^{-\frac{a}{2}Z}\Big) \tag{2.7}$$

for $Z \in [0, w_0 \ln \frac{1}{\varepsilon}]$. This is true on $[0, Z_0]$ for some $Z_0 > 0$ by (2.5). Now, if the result is not true, we denote by $w_0 \ln \frac{1}{\varepsilon} \geqslant Z_c \geqslant Z_0$ the first value such that this estimate becomes an equality. Then, on $[0, Z_c]$ we have

$$|S + \mathcal{R}(G)| \leqslant K(1 + C_0)\Big(\varepsilon \ln\frac{1}{\varepsilon} + e^{-\frac{a}{2}Z}\Big),$$

and plotting this estimate in (2.6) leads to

$$|G(Z_c)| \leqslant K(1+C_0 e^{-C\varepsilon Z_c})\Big(\varepsilon \ln \frac{1}{\varepsilon} + e^{-\frac{a}{2}Z_c}\Big),$$

for some constants $K, C > 0$ independent of ε and C_0. We can easily show a similar estimate on $G'(Z_c)$, up to an increase on K_0. We deduce that if Z_0, C_0 are large enough and ε small enough, then

$$\frac{C_0}{2}\Big(\varepsilon \ln \frac{1}{\varepsilon} + e^{-\frac{a}{2}Z_c}\Big) > |G(Z_c)| + |G'(Z_c)| = C_0\Big(\varepsilon \ln \frac{1}{\varepsilon} + e^{-\frac{a}{2}Z_c}\Big),$$

which is a contradiction.

This completes the proof of (2.7). Going back to $h_\varepsilon(z) = h(z) + (F(Z)+a) + \varepsilon G(Z)$, we deduce that

$$\|h_\varepsilon(z) - h(z) - (F(Z)+a)\|_{L^\infty(]z_c, z_c + w_0 \varepsilon \ln \frac{1}{\varepsilon}])} \to 0$$

when $\varepsilon \to 0$, and taking w_0 large enough, by Lemma 2.4,

$$\Big|h_\varepsilon\Big(z_c + w_0\varepsilon \ln \frac{1}{\varepsilon}\Big) - h(z_c^+)\Big| \leqslant \Big|h\Big(z_c + w_0\varepsilon \ln \frac{1}{\varepsilon}\Big) - h(z_c^+)\Big| + K\varepsilon^2 \ln \frac{1}{\varepsilon} \leqslant K\varepsilon \ln \frac{1}{\varepsilon}.$$

Finally,

$$h_\varepsilon'(z) = h'(z) + \frac{1}{\varepsilon}F'(Z) + G'(Z),$$

leading to $|h_\varepsilon'(z) - \frac{1}{\varepsilon}F'(Z)| \leqslant C$ a constant independent of ε, and since if w_0 is large enough $|G'(w_0 \ln \frac{1}{\varepsilon})| \leqslant K\varepsilon^{1/2}$ and $\frac{1}{\varepsilon}|F'(w_0 \ln \frac{1}{\varepsilon})| \leqslant \frac{K}{\varepsilon}\varepsilon^{aw_0}$, we conclude the proof of this lemma by

$$\Big|h_\varepsilon'\Big(z_c + w_0\varepsilon \ln \frac{1}{\varepsilon}\Big) - h'(z_c^+)\Big| \leqslant K\varepsilon \ln \frac{1}{\varepsilon}. \qquad \blacksquare$$

By standard Cauchy theory, at fixed z_c and a, $\varepsilon \to h_\varepsilon$ is a smooth function. We conclude this subsection with some estimates on $\partial_\varepsilon h_\varepsilon$.

Lemma 2.8. *For $\alpha \in]0,1[$, $z_c > 0$, $\frac{z_c}{1+\alpha} > a > \frac{\alpha}{1+\alpha}z_c$, there exist $\varepsilon_0, C > 0$ depending only on α, z_c, w_0 such that, if $\varepsilon_0 > \varepsilon > 0$ and h_ε is the solution of (2.4) for these parameters, then*

$$\varepsilon \to \Big(h_\varepsilon\Big(z_c + w_0\varepsilon \ln \frac{1}{\varepsilon}\Big), \partial_z h_\varepsilon\Big(z_c + w_0\varepsilon \ln \frac{1}{\varepsilon}\Big)\Big) \in C^1(]0,\varepsilon_0[, \mathbb{R}^2),$$

with

$$\Big|\partial_\varepsilon\Big(h_\varepsilon\Big(z_c + w_0\varepsilon \ln \frac{1}{\varepsilon}\Big)\Big)\Big| + \Big|\partial_\varepsilon\Big(\partial_z h_\varepsilon\Big(z_c + w_0\varepsilon \ln \frac{1}{\varepsilon}\Big)\Big)\Big| \leqslant C\Big(\ln \frac{1}{\varepsilon}\Big)^2.$$

Furthermore, for $z \in]z_c, z_c + w_0\varepsilon \ln \frac{1}{\varepsilon}]$ we have

$$|\varepsilon \partial_\varepsilon h_\varepsilon(z)| + \frac{|\varepsilon \partial_z \partial_\varepsilon h_\varepsilon(z)|}{\ln \frac{1}{\varepsilon}} \leqslant C e^{-\frac{a}{2}|\frac{z-z_c}{\varepsilon}|}.$$

Proof. We recall that with $Z = \frac{z-z_c}{\varepsilon}$, we have

$$h_\varepsilon(z) = h(z) + F(Z) + a + \varepsilon G(Z,\varepsilon).$$

In the previous lemma, we did not write the dependency of G in ε since we did not differentiate with respect to it, but we do so here. We deduce that

$$\varepsilon\partial_\varepsilon h_\varepsilon = -ZF'(Z) + \varepsilon(G - Z\partial_Z G) + \varepsilon^2\partial_\varepsilon G.$$

With the explicit formula for F and (2.7) we check that for $z \in]z_c, z_c + w_0\varepsilon\ln\frac{1}{\varepsilon}]$ we have

$$|-ZF'(Z) + \varepsilon(G - Z\partial_Z G)| \leqslant Ke^{-\frac{a}{2}|Z|},$$

where $K > 0$ depends only on α, z_c, w_0. Furthermore, from the proof of Lemma 2.7 we know that G satisfies the equation

$$\partial_Z^2 G + \lambda\partial_Z G + \varepsilon\mu G = S + \mathcal{R}(G),$$

with $\lambda = \frac{z_c}{1+\alpha} - h(z_c^+) > 0$, $\mu = \frac{\alpha}{1+\alpha} - h'(z_c^+) > 0$, hence

$$\partial_Z^2\partial_\varepsilon G + \lambda\partial_Z\partial_\varepsilon G + \varepsilon\mu\partial_\varepsilon G = \partial_\varepsilon S + \partial_\varepsilon(\mathcal{R}(G)) - \mu G.$$

We check that

$$|\varepsilon\partial_\varepsilon S(Z) - \mu G| \leqslant K(\varepsilon + e^{-\frac{a}{2}Z})$$

and by similar arguments to the proof of Lemma 2.7, we conclude that

$$|\varepsilon\partial_\varepsilon G(Z)| \leqslant K(\varepsilon + e^{-\frac{a}{2}Z}),$$

for some constant $C > 0$ depending only on α, z_c, w_0. Finally, we have

$$\partial_\varepsilon\Big(h_\varepsilon\Big(z_c + w_0\varepsilon\ln\frac{1}{\varepsilon}\Big)\Big) = \partial_\varepsilon h_\varepsilon\Big(z_c + w_0\varepsilon\ln\frac{1}{\varepsilon}\Big) + w_0\partial_\varepsilon\Big(\varepsilon\ln\frac{1}{\varepsilon}\Big)\partial_z h_\varepsilon\Big(z_c + w_0\varepsilon\ln\frac{1}{\varepsilon}\Big),$$

leading to

$$\Big|\partial_\varepsilon\Big(h_\varepsilon\Big(z_c + w_0\varepsilon\ln\frac{1}{\varepsilon}\Big)\Big)\Big| \leqslant K\ln\frac{1}{\varepsilon}.$$

Similarly,

$$\begin{aligned}
&\partial_\varepsilon\Big(\partial_z h_\varepsilon\Big(z_c + w_0\varepsilon\ln\frac{1}{\varepsilon}\Big)\Big)\\
&\quad = \partial_\varepsilon\partial_z h_\varepsilon\Big(z_c + w_0\varepsilon\ln\frac{1}{\varepsilon}\Big) + w_0\partial_\varepsilon\Big(\varepsilon\ln\frac{1}{\varepsilon}\Big)\partial_z^2 h_\varepsilon\Big(z_c + w_0\varepsilon\ln\frac{1}{\varepsilon}\Big),
\end{aligned}$$

and since $\frac{\alpha}{1+\alpha}h_\varepsilon + (\frac{z}{1+\alpha} - h_\varepsilon)\partial_z h_\varepsilon + \varepsilon\partial_z^2 h_\varepsilon = 0$, we have

$$\begin{aligned}
&\partial_z^2 h_\varepsilon\Big(z_c + w_0\varepsilon\ln\frac{1}{\varepsilon}\Big)\\
&\quad = \frac{-1}{\varepsilon}\Big(\frac{\alpha}{1+\alpha}(h_\varepsilon - h)\Big(z_c + w_0\varepsilon\ln\frac{1}{\varepsilon}\Big)\Big)\\
&\qquad - \frac{1}{\varepsilon}\Big(\frac{z_c + w_0\varepsilon\ln\frac{1}{\varepsilon}}{1+\alpha} - h\Big(z_c + w_0\varepsilon\ln\frac{1}{\varepsilon}\Big)\Big)(\partial_z h_\varepsilon - \partial_z h)\Big(z_c + w_0\varepsilon\ln\frac{1}{\varepsilon}\Big)\\
&\qquad - \frac{1}{\varepsilon}\partial_z h_\varepsilon\Big(z_c + w_0\varepsilon\ln\frac{1}{\varepsilon}\Big)(h - h_\varepsilon)\Big(z_c + w_0\varepsilon\ln\frac{1}{\varepsilon}\Big),
\end{aligned}$$

hence, by Lemma 2.7,

$$\left|\partial_z^2 h_\varepsilon\Big(z_c + w_0\varepsilon \ln\frac{1}{\varepsilon}\Big)\right| \leqslant K \ln\frac{1}{\varepsilon}.$$

This concludes the proof of this lemma. ∎

2.4.2. Estimates in $[z_c - w_0\varepsilon\ln\frac{1}{\varepsilon}, z_c[$.

Lemma 2.9. *For $\alpha \in]0,1[$, $z_c > 0$, $\frac{z_c}{1+\alpha} > a > \frac{\alpha}{1+\alpha} z_c$, there exists $w_0 > 0$ depending on α, z_c, a such that h_ε, the solution of (2.4), is well defined on $[z_c - w_0\varepsilon\ln\frac{1}{\varepsilon}, z_c[$, satisfies*

$$\left\| h_\varepsilon(z) - h(z) + \frac{2a}{1+e^{-a(\frac{z-z_c}{\varepsilon})}} \right\|_{L^\infty([z_c - w_0\varepsilon\ln\frac{1}{\varepsilon}, z_c[)} \to 0$$

and

$$\left\| h'_\varepsilon(z) + \frac{2a^2 e^{-a(\frac{z-z_c}{\varepsilon})}}{\varepsilon(1+e^{-a(\frac{z-z_c}{\varepsilon})})^2} \right\|_{L^\infty([z_c - w_0\varepsilon\ln\frac{1}{\varepsilon}, z_c[)} \leqslant C$$

for some constant $C > 0$ depending only on α, z_c, a when $\varepsilon \to 0$, and also

$$\left|h_\varepsilon\Big(z_c - w_0\varepsilon\ln\frac{1}{\varepsilon}\Big) - h(z_c^-)\right| + \left|h'_\varepsilon\Big(z_c - w_0\varepsilon\ln\frac{1}{\varepsilon}\Big) - h'(z_c^-)\right| \leqslant C\varepsilon\ln\frac{1}{\varepsilon}.$$

Proof. For $z < z_c$, keeping the notation $Z = \frac{z-z_c}{\varepsilon} < 0$, we decompose h_ε, a solution of (2.4) as

$$h_\varepsilon(z) = h(z) + (F(Z) - a) + \varepsilon G(Z),$$

for the same function F as in the proof of Lemma 2.7, but another function G. We recall that the function h is not continuous at z_c, and since we consider here $z < z_c$, it will have a different limit for $z - z_c < 0$ close to 0. We take $G(0) = 0$ and $G'(0) = -h'(z_c^-)$ so that we match the conditions at z_c of h_ε: $h_\varepsilon(z_c) = h(z_c^-) + F(0) - a = \frac{z_c}{1+\alpha}$ and $h'_\varepsilon(z_c) = h'(z_c^-) + \frac{1}{\varepsilon}F'(0) + G'(0) = \frac{-a^2}{2\varepsilon}$. As in the proof of Lemma 2.7, we check that G satisfies the equation

$$\mathcal{O}(G) + S = 0,$$

with

$$S(Z) := \frac{1}{\varepsilon}F'(Z)\Big(\frac{z}{1+\alpha} - h(z) + a\Big) + \Big(\frac{\alpha}{1+\alpha} - h'(z)\Big)(F(Z) - a) + \varepsilon h''(z)$$

and

$$\mathcal{O}(G) = G''(Z) + A_1(Z)G'(Z) + A_2(Z)G(Z) - \varepsilon G(Z)G'(Z),$$

with

$$A_1(Z) := \frac{z_c + \varepsilon Z}{1+\alpha} - h(z_c + \varepsilon Z) - (F(Z) - a)$$

and

$$A_2(Z) := -F'(Z) + \varepsilon\Big(\frac{\alpha}{1+\alpha} - h'(z_c + \varepsilon Z)\Big).$$

We now define

$$\widetilde{G}(Z) = G(-Z)$$

satisfying the equation

$$\widetilde{G}''(Z) - A_1(-Z)\widetilde{G}'(Z) + A_2(-Z)\widetilde{G}(Z) + \varepsilon\widetilde{G}(Z)\widetilde{G}'(Z) = S(-Z).$$

We therefore consider $Z > 0$ in the rest of the proof. Now, remark that

$$-A_1(-Z) \to -\Big(\frac{z_c}{1+\alpha} - h(z_c^-)\Big) > 0$$

and

$$\frac{A_2(-Z)}{\varepsilon} \to \Big(\frac{\alpha}{1+\alpha} - h'(z_c^-)\Big) > 0$$

if $Z \geqslant w_0 \ln \frac{1}{\varepsilon}$ for w_0 large when $\varepsilon \to 0$. We therefore define $\lambda := -(\frac{z_c}{1+\alpha} - h(z_c^-)) > 0$ and $\mu = \frac{\alpha}{1+\alpha} - h'(z_c^-) > 0$, and we can complete the proof in a similar fashion to Lemma 2.7. ∎

Lemma 2.10. *For $\alpha \in]0,1[$, $z_c > 0$, $\frac{z_c}{1+\alpha} > a > \frac{\alpha}{1+\alpha} z_c$, there exist $\varepsilon_0, C > 0$ depending only on a, z_c, w_0 such that, if $\varepsilon_0 > \varepsilon > 0$ and h_ε is the solution of (2.4) for these parameters, then*

$$\varepsilon \to \Big(h_\varepsilon\Big(z_c - w_0\varepsilon \ln \frac{1}{\varepsilon}\Big), h'_\varepsilon\Big(z_c - w_0\varepsilon \ln \frac{1}{\varepsilon}\Big)\Big) \in C^1(]0,\varepsilon_0[, \mathbb{R}^2)$$

with

$$\Big|\partial_\varepsilon\Big(h_\varepsilon\Big(z_c - w_0\varepsilon \ln \frac{1}{\varepsilon}\Big)\Big)\Big| + \Big|\partial_\varepsilon\Big(h'_\varepsilon\Big(z_c - w_0\varepsilon \ln \frac{1}{\varepsilon}\Big)\Big)\Big| \leqslant C\Big(\ln \frac{1}{\varepsilon}\Big)^2$$

when $\varepsilon \to 0$. Furthermore, for $z \in [z_c - w_0\varepsilon \ln \frac{1}{\varepsilon}, z_c[$ we have

$$|\varepsilon\partial_\varepsilon h_\varepsilon(z)| + \frac{|\varepsilon\partial_\varepsilon h_\varepsilon(z)|}{\ln \frac{1}{\varepsilon}} \leqslant C e^{-\frac{a}{2}|\frac{z-z_c}{\varepsilon}|}.$$

The proof of this result is similar to the proof of Lemma 2.8 and we omit it.

2.5. Profile far from z_c

2.5.1. Profile on the right of z_c. We start with an a priori estimate on solutions to the ODE problem.

Lemma 2.11. *For any $z_d > 0$, there exists $K > 0$ such that the solution to the problem*

$$\begin{cases} \dfrac{\alpha}{1+\alpha}h_\varepsilon + \Big(\dfrac{z}{1+\alpha} - h_\varepsilon\Big)\partial_z h_\varepsilon + \varepsilon\partial_z^2 h_\varepsilon = 0, \\ \dfrac{z_d}{1+\alpha} > h_\varepsilon(z_d) > 0, \quad h'_\varepsilon(z_d) < 0, \end{cases}$$

for $\varepsilon > 0$ small enough (depending on $z_d, h_\varepsilon(z_d), h'_\varepsilon(z_d)$) is well defined on $[z_d, +\infty[$ and satisfies

$$|h_\varepsilon(z)| + z|h'_\varepsilon(z)| \leqslant \frac{K}{z^\alpha}$$

for any $z \geqslant z_d$.

Proof. Define

$$u = \frac{h'_\varepsilon}{h_\varepsilon}$$

so that $\frac{h''_\varepsilon}{h_\varepsilon} = u' + u^2$ and $h_\varepsilon(z) = h_\varepsilon(z_d)e^{\int_{z_d}^z u(s)\,ds}$. Then $u(z_d) = \frac{h'_\varepsilon(z_d)}{h_\varepsilon(z_d)} < 0$ and

$$\varepsilon(u' + u^2) + \Big(\frac{z}{1+\alpha} - h_\varepsilon\Big)u + \frac{\alpha}{1+\alpha} = 0.$$

We write it as

$$u' = -u^2 - \frac{1}{\varepsilon}\Big(\Big(\frac{z}{1+\alpha} - h_\varepsilon\Big)u + \frac{\alpha}{1+\alpha}\Big).$$

First, we have $u(z_d) < 0$, and we show that as long as u exists, we have

$$u(z) < 0.$$

Indeed, if $u(z) = 0$ for the first time at some point $z \geqslant z_d$, then $u'(z) = \frac{-\alpha}{(1+\alpha)\varepsilon} < 0$, which is impossible. Using $h_\varepsilon(z) = h_\varepsilon(z_d)e^{\int_{z_d}^z u(s)\,ds}$, this implies that h_ε is decreasing, and in particular $\frac{z}{1+\alpha} - h_\varepsilon(z) > 0$ for $z \geqslant z_d$.

Also, for $\varepsilon > 0$ small enough, if $u(z_d) > \frac{-1}{2\sqrt{\varepsilon}}$ say, then we always have $u(z) > \frac{-1}{2\sqrt{\varepsilon}}$. This is because if at some point $u(z) = \frac{-1}{2\sqrt{\varepsilon}}$, then $u'(z) > 0$, which is impossible. We deduce that u is bounded, and therefore global.

Using the same idea, we can show that $u(z) \leqslant \frac{-\lambda}{z}$ for some small (but independent of ε if ε is small enough) constant $\lambda > 0$. In particular, $h_\varepsilon(z)z^{\lambda/2} \to 0$ when $z \to +\infty$. Similarly, we can check that $u(z) \geqslant \frac{-1}{\lambda z}$ if λ is small enough.

Now define

$$v(z) = u(z) + \frac{\alpha}{z};$$

then

$$v'(z) + \frac{z}{(1+\alpha)\varepsilon}v(z) = \frac{\alpha}{z^2} - u^2 + \frac{1}{\varepsilon}h_\varepsilon(z)u(z).$$

We have that

$$\Big|\frac{\alpha}{z^2} - u^2 + \frac{1}{\varepsilon}h_\varepsilon(z)u(z)\Big| \leqslant \frac{K}{\varepsilon z^{1+\frac{\lambda}{2}}}$$

and therefore, by a comparison principle, on $z \geqslant z_d$ we have

$$|v(z)| \leqslant \frac{K}{z^{2+\frac{\lambda}{2}}}.$$

Using these estimates in the equation $h'_\varepsilon = uh_\varepsilon$ completes the proof of the lemma. ∎

We recall that h is a solution of

$$\begin{cases} \frac{\alpha}{1+\alpha}h + \Big(\frac{z}{1+\alpha} - h\Big)\partial_z h = 0, \\ h_0(z) = \kappa_\pm(z_c, a)|z|^{-\alpha}(1 + o_{z\to\pm\infty}(1)), \end{cases} \tag{2.8}$$

and is discontinuous at z_c.

Lemma 2.12. *The function h_ε, the solution of (2.4), satisfies*

$$\|(1+|z|^\alpha)(h_\varepsilon(z) - h(z))\|_{L^\infty([z_c + w_0\varepsilon\ln\frac{1}{\varepsilon}, +\infty[)} \to 0$$

when $\varepsilon \to 0$. Furthermore,

$$\lim_{z\to+\infty} z^\alpha h_\varepsilon(z) =: \kappa_{\varepsilon,+}(z_c, a) > 0$$

is well defined, and

$$|\kappa_{\varepsilon,+}(z_c, a) - \kappa_+(z_c, a)| \to 0$$

when $\varepsilon \to 0$. Finally, for fixed values of z_c and a, the function $\varepsilon \to \kappa_{\varepsilon,+}(z_c, a)$ is smooth, and

$$|\partial_\varepsilon(\kappa_{\varepsilon,+}(z_c, a))| \leqslant K\Big(\ln\frac{1}{\varepsilon}\Big)^2$$

for some constant K independent of ε if ε is small enough. Furthermore,

$$\Big|\partial_\varepsilon h_\varepsilon(z) - \frac{\partial_\varepsilon(\kappa_{\varepsilon,+}(z_c, a))}{|z|^\alpha}\Big| \leqslant \frac{K(\ln\frac{1}{\varepsilon})^2}{(1+|z|)^{1+2\alpha}}$$

for $|z|$ large enough (uniformly in ε).

Note that this does not imply that $\lim z^\alpha h_\varepsilon(z) = \kappa_+(z_c, a)$ when $z \to +\infty$, but simply that their difference goes to 0 when $\varepsilon \to 0$.

Proof of Lemma 2.12. We introduce first a generic problem that we will use to estimate both h_ε and $\partial_\varepsilon h_\varepsilon$.

We consider for now the problem

$$J_1 v + J_2\partial_z v + \varepsilon\partial_z^2 v = S \tag{2.9}$$

for given functions J_1, J_2, S of z, and initial values of v at some point z_d, and with J_2 that does not cancel. We introduce the function A defined by $A(z_d) = 1$ and

$$J_1 A + J_2 A' = 0,$$

that is,

$$A(z) = \exp\Big(-\int_{z_d}^z \frac{J_1}{J_2}\Big) > 0.$$

Then, writing $v = Au$, we have

$$\varepsilon\partial_z^2 u + J_2\partial_z u + \varepsilon\Big(2\frac{A'}{A}\partial_z u + \frac{A''}{A}u\Big) = \frac{S}{A}.$$

To continue, we introduce B with $B(z_d) = 1$ and

$$\varepsilon B' = J_2 B,$$

that is,

$$B(z) = \exp\Big(\frac{1}{\varepsilon}\int_{z_d}^{z} J_2\Big) > 0.$$

We introduce for $\gamma > 0$ the quantity

$$\mathbb{B}_\gamma(z) := \frac{1}{B(z)}\int_{z_d}^{z}\frac{B(s)}{s^\gamma}\,ds,$$

a solution to the equation $\mathbb{B}_\gamma'(z) + \frac{J_2}{\varepsilon}\mathbb{B}_\gamma(z) = \frac{1}{z^\gamma}$ with $\mathbb{B}_\gamma(z_d) = 0$. If there exists a constant $C_0 > 0$ independent of ε such that $\frac{1}{C_0} \geqslant \frac{J_2(z)}{z} \geqslant C_0$, then by comparison there exists $K > 0$ (depending only on C_0 and γ) such that

$$\mathbb{B}_\gamma(z) \leqslant \frac{K\varepsilon}{z^{\gamma+1}}. \tag{2.10}$$

Continuing, we have

$$\varepsilon\partial_z^2 u + J_2\partial_z u = \frac{\varepsilon\partial_z(\partial_z u B)}{B}$$

and therefore

$$\partial_z(\partial_z u B) = B\Big(\frac{S}{\varepsilon A} - 2\partial_z u\frac{A'}{A} + \frac{A''}{A}u\Big).$$

Integrating between z_d and z leads to

$$\partial_z u = \frac{\partial_z u(z_d)}{B(z)} + \frac{1}{B(z)}\int_{z_d}^{z} B\Big(\frac{S}{\varepsilon A} - 2\partial_z u\frac{A'}{A} + \frac{A''}{A}u\Big). \tag{2.11}$$

Step 1. *Existence and properties of* $\kappa_{\varepsilon,+}(z_c, a)$. We take $z_d = z_c + w_0\varepsilon\ln\frac{1}{\varepsilon}$, and we recall that, for $\varepsilon > 0$ small enough, h_ε satisfies the equation

$$\frac{\alpha}{1+\alpha}h_\varepsilon + \Big(\frac{z}{1+\alpha} - h_\varepsilon\Big)\partial_z h_\varepsilon + \varepsilon\partial_z^2 h_\varepsilon = 0$$

and we have $\frac{z_d}{1+\alpha} > h_\varepsilon(z_d) > 0$, $h_\varepsilon'(z_d) < 0$ since h_ε solved (2.4). We decompose $h_\varepsilon = h + g$ with h a solution of (2.8), so that the equation satisfied by g is

$$\varepsilon g'' + g'\Big(\frac{z}{1+\alpha} - h_\varepsilon\Big) + g\Big(\frac{\alpha}{1+\alpha} - h'\Big) + \varepsilon h'' = 0.$$

This is equation (2.9) with $J_1 = \frac{\alpha}{1+\alpha} - h'$, $J_2 = \frac{z}{1+\alpha} - h_\varepsilon$ and $\frac{S}{\varepsilon} = -h''$. Note that for $z \geqslant z_d$, we have $J_2(z) > 0$. Now we have

$$\frac{J_1}{J_2} = \frac{\alpha - (1+\alpha)h'}{z - (1+\alpha)h_\varepsilon} = \frac{\alpha}{z} - \frac{(1+\alpha)(zh' - h_\varepsilon)}{z(z-(1+\alpha)h_\varepsilon)},$$

hence

$$A(z) = \exp\Big(-\int_{z_d}^{z} \frac{J_1}{J_2}\Big) = \Big(\frac{z}{z_d}\Big)^{-\alpha} \exp\Big(-\int_{z_d}^{z} \frac{(1+\alpha)(sh' - h_\varepsilon)}{s(s-(1+\alpha)h_\varepsilon)}\,ds\Big),$$

and with

$$\Big|\frac{(1+\alpha)(sh' - h_\varepsilon)}{s(s-(1+\alpha)h_\varepsilon)}\Big| \leqslant \frac{K}{s^2}$$

for $s \geqslant z_d$, we deduce that there exists $K > 0$ depending on z_d, α such that

$$\frac{1}{K} \leqslant z^\alpha(|A(z)| + |zA'(z)| + |z^2 A''(z)|) \leqslant K$$

for $z \geqslant z_d$, and $z^\alpha A(z)$ converges when $z \to +\infty$ to a finite constant bounded uniformly in ε. With $g = Au$, we define $N(z) := \|u\|_{L^\infty([z_d,z])} + \|zu'\|_{L^\infty([z_d,z])}$. We have, for $z_d \leqslant s \leqslant z$,

$$\Big|B\Big(\frac{S}{\varepsilon A} - 2\partial_z u\frac{A'}{A} + \frac{A''}{A}u\Big)\Big|(s) \leqslant \frac{B(s)(1+N(z))}{s^2},$$

and by (2.10) we deduce that

$$\Big|\frac{1}{B(z)}\int_{z_d}^{z} B\Big(\frac{S}{\varepsilon A} - 2\partial_z u\frac{A'}{A} + \frac{A''}{A}u\Big)\Big| \leqslant \frac{K\varepsilon(1+N(z))}{z^3}.$$

Now we have

$$\frac{1}{B(z)} = \exp\Big(\frac{-1}{\varepsilon}\int_{z_d}^{z}\Big(\frac{s}{1+\alpha} - h_\varepsilon(s)\Big)\,ds\Big) \leqslant \exp\Big(\frac{-K(z-z_d)^2}{\varepsilon}\Big).$$

Combining these estimates in (2.11) and the integral of (2.11), we deduce that

$$N(z) \leqslant C(\varepsilon + N(z_d))$$

for some constant $C > 0$ independent of ε and for all $z \geqslant z_d$. Furthermore,

$$u(+\infty) = u(z_d) + \int_{z_d}^{+\infty} \frac{\partial_z u(z_d)}{B(s)}\,ds + \int_{z_d}^{+\infty} \frac{1}{B(s)}\int_{z_d}^{s} B\Big(\frac{S}{\varepsilon A} - 2\partial_z u\frac{A'}{A} + \frac{A''}{A}u\Big)$$

is a finite quantity that satisfies $|u(+\infty)| \leqslant K(|u(z_d)| + |\partial_z u(z_d)| + \varepsilon)$, and

$$|u(z) - u(+\infty)| \leqslant \frac{K\varepsilon}{z^2} \tag{2.12}$$

for $z \geqslant z_d$. By Lemma 2.7, we have $|g(z_d)| + |g'(z_d)| \leqslant K\varepsilon \ln \frac{1}{\varepsilon}$ and thus $|u(z_d)| + |\partial_z u(z_d)| \leqslant K\varepsilon \ln \frac{1}{\varepsilon}$. Defining

$$\kappa_{\varepsilon,+}(z_c, a) = \kappa_+(z_c, a) + u(+\infty) \lim_{z \to +\infty} (z^\alpha A(z)),$$

we deduce that

$$\lim_{z \to +\infty} z^\alpha h_\varepsilon(z) = \kappa_{\varepsilon,+}(z_c, a).$$

More precisely,

$$\begin{aligned} h_\varepsilon(z) &= h(z) + A(z)u(z) \\ &= \frac{\kappa_{\varepsilon,+}(z_c, a)}{z^\alpha} + \Big(\frac{z^\alpha A(z) - \lim_{x \to +\infty}(x^\alpha A(x))}{z^\alpha}\Big) u(+\infty) \\ &\quad + \Big(h(z) - \frac{\kappa_+(z_c, a)}{z^\alpha}\Big) + A(z)(u(z) - u(+\infty)), \end{aligned}$$

and with the explicit definition of A, we have

$$\Big|\frac{z^\alpha A(z) - \lim_{x \to +\infty}(x^\alpha A(x))}{z^\alpha}\Big| \leqslant \frac{K}{z^{1+2\alpha}},$$

and by Lemma 2.5 we have

$$\Big|h(z) - \frac{\kappa_+(z_c, a)}{z^\alpha}\Big| \leqslant \frac{K}{z^{1+2\alpha}}.$$

With (2.12), we conclude that for $z \geqslant z_d$,

$$|z^\alpha h_\varepsilon(z) - \kappa_{\varepsilon,+}(z_c, a)| \leqslant \frac{K}{z^{1+\alpha}}$$

and

$$|\kappa_{\varepsilon,+}(z_c, a) - \kappa_+(z_c, a)| \to 0$$

when $\varepsilon \to 0$. Lemmas 2.5, 2.7 and

$$N(z) \leqslant C(\varepsilon + N(z_d)) \leqslant K\varepsilon \ln \frac{1}{\varepsilon}$$

also imply that for $z \geqslant z_d$,

$$|\partial_z h_\varepsilon(z)| \leqslant \frac{K}{z^{1+\alpha}},$$

as well as

$$|\partial_z^2 h_\varepsilon(z)| \leqslant \frac{K \ln \frac{1}{\varepsilon}}{z^{2+\alpha}}.$$

This last estimate can be improved (we can remove the $\ln \frac{1}{\varepsilon}$) but it is not needed here; we will only need $\varepsilon \partial_\varepsilon h_\varepsilon$ to be small and not $\partial_\varepsilon h_\varepsilon$ itself.

Step 2. *Differentiation with respect to ε at fixed z_d.* We consider here $\mathfrak{h}_\varepsilon$ the solution to the problem

$$\frac{\alpha}{1+\alpha}\mathfrak{h}_\varepsilon + \Big(\frac{z}{1+\alpha} - \mathfrak{h}_\varepsilon\Big)\partial_z \mathfrak{h}_\varepsilon + \varepsilon \partial_z^2 \mathfrak{h}_\varepsilon = 0$$

with $\mathfrak{h}_\varepsilon(z_d)$, $\partial_z \mathfrak{h}_\varepsilon(z_d)$ given satisfying $\frac{z_d}{1+\alpha} > \mathfrak{h}_\varepsilon(z_d) > 0$, $\mathfrak{h}_\varepsilon'(z_d) < 0$, and they are, with z_d, independent of ε. We have as previously that for $z \geqslant z_d$,

$$|\mathfrak{h}_\varepsilon(z)| + z|\partial_z \mathfrak{h}_\varepsilon(z)| \leqslant \frac{K}{z^\alpha}, \quad |\partial_z^2 \mathfrak{h}_\varepsilon(z)| \leqslant \frac{K \ln \frac{1}{\varepsilon}}{z^{2+\alpha}}. \tag{2.13}$$

We introduce this new notation since we want to differentiate h_ε with respect to ε, but its dependency on ε comes from the ε in the equation but also from z_c and the value of h_ε here. For $\mathfrak{h}_\varepsilon$, the dependency on ε comes only from the ε in front of $\partial_z^2 \mathfrak{h}_\varepsilon$ in the equation. By standard

Cauchy theory. , $\varepsilon \to \mathfrak{h}_\varepsilon$ is differentiable, and $v = \partial_\varepsilon \mathfrak{h}_\varepsilon$ satisfies the problem

$$\begin{cases} \Big(\frac{\alpha}{1+\alpha} - \partial_z \mathfrak{h}_\varepsilon\Big)v + \Big(\frac{z}{1+\alpha} - \mathfrak{h}_\varepsilon\Big)\partial_z v + \varepsilon \partial_z^2 v = -\partial_z^2 \mathfrak{h}_\varepsilon, \\ v(z_d) = v'(z_d) = 0. \end{cases}$$

This is equation (2.9) with $J_1 = \frac{\alpha}{1+\alpha} - \partial_z \mathfrak{h}_\varepsilon$, $J_2 = \frac{z}{1+\alpha} - \mathfrak{h}_\varepsilon$ and $\frac{S}{\varepsilon} = \frac{-1}{\varepsilon}\partial_z^2 \mathfrak{h}_\varepsilon$. Following a similar proof to Step 1, we check that, with $\mathfrak{h}_\varepsilon = Au$, we have that $u(z)$ converges to a finite limit $u(+\infty)$, with $|u(+\infty)| \leqslant K \ln \frac{1}{\varepsilon}$, and that

$$|u(z) - u(+\infty)| \leqslant \frac{K \ln \frac{1}{\varepsilon}}{z^2}$$

for $z \geqslant z_d$. We also check, as in Step 1, that

$$|z^\alpha v(z) - k_0| \leqslant \frac{K \ln \frac{1}{\varepsilon}}{z^{1+\alpha}}$$

and

$$|\partial_z v| \leqslant \frac{K \ln \frac{1}{\varepsilon}}{z^{1+\alpha}}$$

for some k_0 depending on ε and $K > 0$.

Step 3. *Differentiation with respect to z_d.* We consider here $\mathfrak{h}_\varepsilon$ the solution to the problem

$$\frac{\alpha}{1+\alpha}\mathfrak{h}_\varepsilon + \Big(\frac{z}{1+\alpha} - \mathfrak{h}_\varepsilon\Big)\partial_z \mathfrak{h}_\varepsilon + \varepsilon \partial_z^2 \mathfrak{h}_\varepsilon = 0,$$

with $\mathfrak{h}_\varepsilon(z_d) = a$, $\partial_z \mathfrak{h}_\varepsilon(z_d) = b$, where $\frac{z_d}{1+\alpha} > a > 0$, $b < 0$ are independent of z_d and verify $|\frac{\alpha}{1+\alpha}a + (\frac{z_d}{1+\alpha} - a)b| \leqslant K\varepsilon \ln \frac{1}{\varepsilon}$ for some $K > 0$ independent of ε. As previously, estimate (2.13) holds. We want to compute $v = \partial_{z_d} \mathfrak{h}_\varepsilon$. It is a solution of

$$\begin{cases} \Big(\frac{\alpha}{1+\alpha} - \partial_z \mathfrak{h}_\varepsilon\Big)v + \Big(\frac{z}{1+\alpha} - \mathfrak{h}_\varepsilon\Big)\partial_z v + \varepsilon \partial_z^2 v = 0, \\ v(z_d) = -\partial_z \mathfrak{h}_\varepsilon(z_d), \quad v'(z_d) = -\partial_z^2 \mathfrak{h}_\varepsilon(z_d). \end{cases}$$

We have $-\partial_z \mathfrak{h}_\varepsilon(z_d) = -b$ and since $(\frac{\alpha}{1+\alpha}\mathfrak{h}_\varepsilon + (\frac{z}{1+\alpha} - \mathfrak{h}_\varepsilon)\partial_z \mathfrak{h}_\varepsilon + \varepsilon\partial_z^2 \mathfrak{h}_\varepsilon)(z_d) = 0$, we have

$$-\partial_z^2 \mathfrak{h}_\varepsilon(z_d) = \frac{1}{\varepsilon}\Big(\frac{\alpha}{1+\alpha}a + \Big(\frac{z_d}{1+\alpha} - a\Big)b\Big),$$

which is bounded by $K \ln\frac{1}{\varepsilon}$ with $K > 0$ independent of ε. As in the previous case, we check that

$$|z^\alpha v(z) - k_0| \leqslant \frac{K\ln\frac{1}{\varepsilon}}{z^{1+\alpha}}$$

for some $k_0, K > 0$ and

$$|\partial_z v| \leqslant \frac{K\ln\frac{1}{\varepsilon}}{z^{1+\alpha}}.$$

Step 4. *Differentiation with respect to* $\mathfrak{h}_\varepsilon(z_d)$ *and* $\partial_z\mathfrak{h}_\varepsilon(z_d)$. We consider here $\mathfrak{h}_\varepsilon$ the solution to the problem

$$\frac{\alpha}{1+\alpha}\mathfrak{h}_\varepsilon + \Big(\frac{z}{1+\alpha} - \mathfrak{h}_\varepsilon\Big)\partial_z\mathfrak{h}_\varepsilon + \varepsilon\partial_z^2\mathfrak{h}_\varepsilon = 0,$$

with $\mathfrak{h}_\varepsilon(z_d) = a$, $\partial_z\mathfrak{h}_\varepsilon(z_d) = b$, where $\frac{z_d}{1+\alpha} > a > 0$, $b < 0$. Then, $v = \partial_a\mathfrak{h}_\varepsilon$ satisfies

$$\begin{cases} \Big(\frac{\alpha}{1+\alpha} - \partial_z\mathfrak{h}_\varepsilon\Big)v + \Big(\frac{z}{1+\alpha} - \mathfrak{h}_\varepsilon\Big)\partial_z v + \varepsilon\partial_z^2 v = 0,\\ v(z_d) = 1, \quad v'(z_d) = 0.\end{cases}$$

This is similar to the previous steps, and we also check that $\partial_b\mathfrak{h}_\varepsilon$ can be estimated similarly.

Step 5. *Conclusion.* The function h_ε is a solution to

$$\frac{\alpha}{1+\alpha}h_\varepsilon + \Big(\frac{z}{1+\alpha} - h_\varepsilon\Big)\partial_z h_\varepsilon + \varepsilon\partial_z^2 h_\varepsilon = 0,$$

with the initial condition at $z_d = z_c + w_0\varepsilon\ln\frac{1}{\varepsilon}$ satisfying (by Lemmas 2.7 and 2.8)

$$|h_\varepsilon(z_d) - h(z_c^+)| + |h_\varepsilon'(z_d) - h'(z_c^+)| \leqslant K\varepsilon\ln\frac{1}{\varepsilon}$$

and

$$|\partial_\varepsilon h_\varepsilon(z_d)| + |\partial_\varepsilon h_\varepsilon'(z_d)| \leqslant K\Big(\ln\frac{1}{\varepsilon}\Big)^2.$$

Therefore, $\partial_\varepsilon h_\varepsilon$ can be written as the sum of the functions v of Steps 2 to 4, and since $|\partial_\varepsilon z_d| \leqslant w_0\ln\frac{1}{\varepsilon}$, this concludes the proof of this lemma. ∎

2.5.2. Profile on the left of z_c.

Lemma 2.13. *The function* h_ε*, the solution of* (2.4)*, is well defined on* $]-\infty, z_c - w_0\varepsilon\ln\frac{1}{\varepsilon}]$ *and satisfies*

$$\|(1+|z|^\alpha)(h_\varepsilon(z) - h(z))\|_{L^\infty(]-\infty, z_c - w_0\varepsilon\ln\frac{1}{\varepsilon}])} \to 0$$

when $\varepsilon \to 0$. Furthermore,

$$\lim_{z\to-\infty} z^\alpha h_\varepsilon(z) =: \kappa_{\varepsilon,-}(z_c,a) > 0$$

is well defined, and

$$|\kappa_{\varepsilon,-}(z_c,a) - \kappa_-(z_c,a)| \to 0$$

when $\varepsilon \to 0$. Finally, for fixed values of z_c and a, the function $\varepsilon \to \kappa_{\varepsilon,-}(z_c,a)$ is smooth, and

$$|\partial_\varepsilon(\kappa_{\varepsilon,-}(z_c,a))| \leqslant K\Big(\ln\frac{1}{\varepsilon}\Big)^2$$

for a constant $K > 0$ independent of ε if ε is small enough. Furthermore,

$$\left|\partial_\varepsilon h_\varepsilon(z) - \frac{\partial_\varepsilon(\kappa_{\varepsilon,-}(z_c,a))}{|z|^\alpha}\right| \leqslant \frac{K(\ln\frac{1}{\varepsilon})^2}{(1+|z|)^{1+2\alpha}}$$

for $|z|$ large enough.

The proof is similar to that of Lemma 2.12 and we omit it.

2.6. End of the proof of Proposition 1.3

Take $\kappa_+,\kappa_- > 0$, $\alpha \in]0,1[$. By Lemma 2.6, we choose $z_c, a > 0$ such that

$$\kappa_+(z_c,a) = \kappa_+, \kappa_-(z_c,a) = \kappa_-.$$

We infer that for ε small enough, we can take $z_c(\varepsilon), a(\varepsilon) > 0$ such that

$$\kappa_{\varepsilon,+}(z_c(\varepsilon),a(\varepsilon)) = \kappa_+, \quad \kappa_{\varepsilon,-}(z_c(\varepsilon),a(\varepsilon)) = \kappa_-,$$

with

$$|z_c(\varepsilon) - z_\kappa| + |a(\varepsilon) - a_\kappa| \to 0$$

when $\varepsilon \to 0$, and that this choice is unique, and determines $\mathbb{h}_\varepsilon$. This is a consequence of the implicit function theorem on the function

$$\mathfrak{K}(\varepsilon, z_c, a) := (\kappa_{+,\varepsilon}(z_c,a) - \kappa_+, \kappa_{-,\varepsilon}(z_c,a) - \kappa_-).$$

Indeed, by Lemmas 2.6, 2.12 and 2.13, we have $\mathfrak{K}(0,z_c,a) = 0$ and the Jacobian at $\varepsilon = 0$ is invertible. By Lemma 2.12 and 2.13 we have the estimates

$$|\partial_\varepsilon z_c(\varepsilon)| + |\partial_\varepsilon a(\varepsilon)| \leqslant K\Big(\ln\frac{1}{\varepsilon}\Big)^2 \tag{2.14}$$

The other properties in Proposition 1.3 are a consequence of Lemmas 2.7, 2.9, 2.12 and 2.13.

2.7. Properties of $\partial_\varepsilon \mathbb{h}_\varepsilon$

We recall that the function $\mathbb{h}_\varepsilon$ is a solution h_ε of the previous subsection, with a particular choice of $z_c(\varepsilon), a(\varepsilon)$ such that the limits at $\pm\infty$ of $|z|^\alpha \mathbb{h}_\varepsilon(z)$ are $\kappa_\pm$ respectively, quantities independent of ε. The function $\mathbb{h}_\varepsilon$ depends on ε by h_ε as above, but also through $z_c(\varepsilon)$ and $a(\varepsilon)$.

Lemma 2.14. *The function $\partial_\varepsilon \mathbb{h}_\varepsilon$ satisfies*

$$|\partial_\varepsilon \mathbb{h}_\varepsilon(z)| \leqslant \frac{K(\ln\frac{1}{\varepsilon})^2}{(1+|z|)^{1+2\alpha}}$$

for $|z - z_c(\varepsilon)| \geqslant 1$ and a constant K independent of ε. For $z \in [z_c(\varepsilon)-1, z_c(\varepsilon)+1]$ we have

$$|\varepsilon \partial_\varepsilon \mathbb{h}_\varepsilon(z)| \leqslant K e^{-\frac{a}{2}|\frac{z-z_c(\varepsilon)}{\varepsilon}|}.$$

Finally, $\partial_\varepsilon \mathbb{h}_\varepsilon \in L^1(\mathbb{R})$ and

$$\int_{\mathbb{R}} \partial_\varepsilon \mathbb{h}_\varepsilon = 0.$$

Proof. By Lemma 2.12, we check that for $z \geqslant z_c(\varepsilon)+1$, we have

$$\left|\partial_\varepsilon \mathbb{h}_\varepsilon(z) - \frac{\partial_\varepsilon(\kappa_{+,\varepsilon}(z_c(\varepsilon), a(\varepsilon)))}{|z|^\alpha}\right| \leqslant \frac{K(\ln\frac{1}{\varepsilon})^2}{(1+|z|)^{1+2\alpha}}$$

and

$$|\partial_\varepsilon \partial_z \mathbb{h}_\varepsilon(z)| \leqslant \frac{K(\ln\frac{1}{\varepsilon})^2}{(1+|z|)^{1+\alpha}},$$

but $\kappa_{+,\varepsilon}(z_c(\varepsilon), a(\varepsilon)) = \kappa_+$, which is independent of ε, hence $\partial_\varepsilon(\kappa_{+,\varepsilon}(z_c(\varepsilon), a(\varepsilon))) = 0$. Now, on $[z_c(\varepsilon), z_c(\varepsilon)+1]$, the estimate is a consequence of Lemma 2.8. For $z \leqslant z_c(\varepsilon)$ the proof is similar. The decay at infinity of $\partial_\varepsilon \mathbb{h}_\varepsilon$ implies in particular that it is in $L^1(\mathbb{R})$.

Now, $\mathbb{h}_\varepsilon$ satisfies

$$\frac{\alpha}{1+\alpha}\mathbb{h}_\varepsilon + \frac{z}{1+\alpha}\partial_z \mathbb{h}_\varepsilon - \mathbb{h}_\varepsilon \partial_z \mathbb{h}_\varepsilon + \varepsilon \partial_z^2 \mathbb{h}_\varepsilon = 0,$$

and integrating between $-x$ and x for some large $x > 0$ leads to

$$\frac{\alpha-1}{1+\alpha}\int_{-x}^{x} \mathbb{h}_\varepsilon + \left[\frac{z}{1+\alpha}\mathbb{h}_\varepsilon - \frac{1}{2}\mathbb{h}_\varepsilon^2 + \partial_z \mathbb{h}_\varepsilon\right]_{-x}^{x} = 0.$$

Differentiating with respect to ε leads to

$$\frac{\alpha-1}{1+\alpha}\int_{-x}^{x} \partial_\varepsilon \mathbb{h}_\varepsilon + \left[\frac{z}{1+\alpha}\partial_\varepsilon \mathbb{h}_\varepsilon - \partial_\varepsilon \mathbb{h}_\varepsilon \mathbb{h}_\varepsilon + \partial_z \partial_\varepsilon \mathbb{h}_\varepsilon\right]_{-x}^{x} = 0,$$

and going to the limit $x \to +\infty$, we check with $|\partial_\varepsilon \mathbb{h}_\varepsilon(z)| \leqslant \frac{K(\ln\frac{1}{\varepsilon})^2}{(1+|z|)^{1+2\alpha}}$ that

$$\frac{\alpha-1}{1+\alpha}\int_{\mathbb{R}} \partial_\varepsilon \mathbb{h}_\varepsilon = 0. \qquad \blacksquare$$

3. Stability of $\mathbb{h}_\varepsilon$

This section is devoted to the proof of Theorem 1.4. For the viscous Burgers equation $\partial_t u - \partial_x^2 u + u\partial_x u = 0$ and $\varepsilon(t) = t^{\frac{\alpha-1}{\alpha+1}}$, we introduce now the rescaling

$$y = \frac{x - t^{\frac{1}{1+\alpha}} z_c}{\varepsilon} = t^{\frac{2-\alpha}{1+\alpha}}(z - z_c)$$

and the rescaled function

$$H(y,t) = t^{\frac{\alpha}{1+\alpha}} u\big((z_c + \varepsilon(t)y)t^{\frac{1}{1+\alpha}}, t\big).$$

This scale is the right one to understand the profile near the discontinuity point z_c ($y = 0$ corresponds to $z = z_c$). The previous scaling (in z), where the profile h_ε was constructed, was

$$h(z, \varepsilon(t)) = t^{\frac{\alpha}{1+\alpha}} u(zt^{\frac{1}{1+\alpha}}, t),$$

and they are connected by

$$H(y,t) = h(z_c(t) + \varepsilon(t)y, \varepsilon(t)).$$

In particular, we define

$$H_\varepsilon(y,t) := \mathbb{h}_\varepsilon(z_c(t) + \varepsilon(t)y).$$

We recall that $\mathbb{h}_\varepsilon$ satisfies

$$\frac{\alpha}{1+\alpha}\mathbb{h}_\varepsilon + \frac{z}{1+\alpha}\partial_z \mathbb{h}_\varepsilon - \mathbb{h}_\varepsilon \partial_z \mathbb{h}_\varepsilon + \varepsilon \partial_z^2 \mathbb{h}_\varepsilon = 0,$$

therefore H_ε satisfies

$$\frac{\alpha}{1+\alpha} H_\varepsilon + \frac{t^{\frac{1-\alpha}{1+\alpha}} z_c(t) + y}{1+\alpha}\partial_y H_\varepsilon - t^{\frac{1-\alpha}{1+\alpha}} H_\varepsilon \partial_y H_\varepsilon + t^{\frac{1-\alpha}{1+\alpha}} \partial_y^2 H_\varepsilon = 0,$$

that is,

$$t^{-\frac{2\alpha}{1+\alpha}}(-\partial_y^2 H_\varepsilon + H_\varepsilon \partial_y H_\varepsilon) - \frac{\alpha}{1+\alpha} t^{-1} H_\varepsilon - t^{-1}\frac{(t^{\frac{1-\alpha}{1+\alpha}} z_c(t) + y)}{1+\alpha}\partial_y H_\varepsilon = 0.$$

Now we have

$$\partial_y H = t^{\frac{2\alpha}{1+\alpha}}\partial_x u, \quad \partial_y^2 H = t^{\frac{3\alpha}{1+\alpha}}\partial_x^2 u$$

and

$$\partial_t H = t^{\frac{\alpha}{1+\alpha}}\partial_t u + \frac{\alpha}{1+\alpha} t^{-1} H + \frac{t^{-1}(\alpha y + t^{\frac{1-\alpha}{1+\alpha}} z_c)}{1+\alpha}\partial_y H + \partial_t z_c t^{\frac{1-\alpha}{1+\alpha}}\partial_y H;$$

therefore, the equation on H (that is, the rescaled viscous Burgers equation in this new scaling) is

$$\begin{aligned}\partial_t H - \frac{\alpha}{1+\alpha} t^{-1} \partial_y (yH) - \frac{t^{-1+\frac{1-\alpha}{1+\alpha}} z_c}{1+\alpha} \partial_y H \\ - \partial_t z_c t^{\frac{1-\alpha}{1+\alpha}} \partial_y H + t^{-\frac{2\alpha}{1+\alpha}} (-\partial_y^2 H + H \partial_y H) \\ = 0.\end{aligned}$$

We now decompose $H = H_\varepsilon + f$. Then f satisfies the equation

$$\begin{aligned}&\partial_t f - \frac{\alpha}{1+\alpha} t^{-1} \partial_y (yf) \\ &\quad + t^{-\frac{2\alpha}{1+\alpha}} \Big(-\partial_y^2 f + f \partial_y H_\varepsilon + \Big(H_\varepsilon - \frac{z_c}{1+\alpha} \Big) \partial_y f + f \partial_y f \Big) - \partial_t z_c t^{\frac{1-\alpha}{1+\alpha}} \partial_y f \\ &\quad + \partial_t H_\varepsilon - \partial_t z_c t^{\frac{1-\alpha}{1+\alpha}} \partial_y H_\varepsilon + \frac{1-\alpha}{1+\alpha} t^{-1} y \partial_y H_\varepsilon \\ &= 0.\end{aligned}$$

We check that

$$S := \partial_t H_\varepsilon - \partial_t z_c t^{\frac{1-\alpha}{1+\alpha}} \partial_y H_\varepsilon + \frac{1-\alpha}{1+\alpha} t^{-1} y \partial_y H_\varepsilon = \partial_t \varepsilon \partial_\varepsilon \mathbb{h}_\varepsilon (z_c + \varepsilon y).$$

Note that the problem takes the form

$$\begin{aligned}t^{\frac{2\alpha}{1+\alpha}} \partial_t f + \partial_y \Big(-\partial_y f + \frac{-\alpha\varepsilon}{1+\alpha} y f + \Big(H_\varepsilon - \frac{z_c}{1+\alpha} \Big) f + \frac{f^2}{2} - t \partial_t z_c f \Big) \\ + S = 0. \qquad (3.1)\end{aligned}$$

From Lemma 2.14, $S \in L^1(\mathbb{R})$ and $\int_{\mathbb{R}} S = 0$. We therefore write $S = \partial_y \widetilde{S}$ with

$$\widetilde{S} := \int_{-\infty}^{y} S.$$

However, the perturbation f can have a mass. To deal with it, we introduce an additional term $\mathfrak{u}_M$ in the profile, and we will decompose $f = \mathfrak{u}_M + \partial_x g$, where all the mass is in $\mathfrak{u}_M$.

3.1. Definition and properties of $\mathfrak{u}_M$

This subsection is devoted to the proof of the following result.

Lemma 3.1. *Given $M \in \mathbb{R}$, $a > 0$, the problem*

$$\begin{cases} -\partial_x \mathfrak{u} + F \mathfrak{u} + \dfrac{\mathfrak{u}^2}{2} = 0, \\ \displaystyle\int_{\mathbb{R}} \mathfrak{u} = M, \end{cases}$$

with $F(Z) = \frac{a(e^{-aZ}-1)}{1+e^{-aZ}}$ *admits a unique solution denoted* $\mathfrak{u}_M$*, which satisfies for* $|M|$ *small enough (depending on* a*),*

$$\partial_x(F + \mathfrak{u}_M) < 0.$$

Proof. We look at the equation

$$\begin{cases} -\partial_x \mathfrak{u} + F\mathfrak{u} + \dfrac{\mathfrak{u}^2}{2} = 0, \\ \mathfrak{u}(0) \in \mathbb{R}, \end{cases} \tag{3.2}$$

where $F = \frac{a(e^{-aZ}-1)}{1+e^{-aZ}}$ was introduced in the proof of Lemma 2.7 (recall that a depends on time). A classical computation shows that

$$\mathfrak{u}(x) = \frac{\exp(\int_0^x F)}{\frac{1}{\mathfrak{u}(0)} - \frac{1}{2}\int_0^x \exp(\int_0^y F)\,dy}$$

(with $\mathfrak{u}(x) = 0$ if $\mathfrak{u}(0) = 0$). If $\mathfrak{u}(0) \neq 0$, then $\mathfrak{u}$ has the same sign as $\mathfrak{u}(0)$. Denoting $\mathcal{F} = \exp(\int_0^x F)$, remark that $\mathcal{F}(0) = 1$, $\mathcal{F}$ is positive, even, because F is odd, and

$$\int_{\mathbb{R}} \mathcal{F} < +\infty.$$

In particular,

$$\begin{aligned} \int_{\mathbb{R}} \mathfrak{u} &= -2\int_{\mathbb{R}} \partial_x\Big(\ln\Big(\frac{1}{\mathfrak{u}(0)} - \frac{1}{2}\int_0^x \mathcal{F}(y)\,dy\Big)\Big) \\ &= 2\ln\left(\frac{1 + \frac{\mathfrak{u}(0)}{4}\int_{\mathbb{R}}\mathcal{F}}{1 - \frac{\mathfrak{u}(0)}{4}\int_{\mathbb{R}}\mathcal{F}}\right). \end{aligned}$$

This means that $\mathfrak{u}(0) \to \int_{\mathbb{R}} \mathfrak{u}$ is a bijection from $]\frac{-4}{\int_{\mathbb{R}}\mathcal{F}}, \frac{4}{\int_{\mathbb{R}}\mathcal{F}}[$ to $\mathbb{R}$. Given $M \in \mathbb{R}$, we then define $\mathfrak{u}_M$ to be the solution of (3.2) with $\int_{\mathbb{R}} \mathfrak{u}_M = M$.

We have $F' + \frac{a^2}{2} = \frac{1}{2}F^2$ and $F(0) = 0$, hence by (3.2) we check that $F + \mathfrak{u}_M$ satisfies the equation

$$\partial_x(F + \mathfrak{u}_M) = \frac{1}{2}(F + \mathfrak{u}_M)^2 - \frac{a^2}{2},$$

which implies that

$$(F + \mathfrak{u}_M)(x) = a\tanh(c_M - ax)$$

with c_M defined by $\mathfrak{u}_M(0) = a\tanh(c_M)$ (for $|M|$ small enough, $\mathfrak{u}_M(0)$ is small, hence c_M is well defined by this equation). We deduce that

$$\partial_x(F + \mathfrak{u}_M) = \frac{-a^2}{\cosh^2(c_M - ax)} < 0. \tag{3.3}$$

■

3.2. Decomposition of f and equation on the norm

We denote $M = \int_{\mathbb{R}} f$ and we decompose

$$f = \mathfrak{u}_M + \partial_y g,$$

with

$$g(y) = \int_{-\infty}^{y} f - \mathfrak{u}_M.$$

First, remark that M is independent of time. This is because $\partial_t \mathbb{h}_\varepsilon = \partial_t \varepsilon \partial_\varepsilon \mathbb{h}_\varepsilon$ and by Lemma 2.14, $\int_{\mathbb{R}} \partial_\varepsilon \mathbb{h}_\varepsilon = 0$. Furthermore, $\mathfrak{u}_M$ depends on time but only through $a(t)$. However, since $\int_{\mathbb{R}} \mathfrak{u}_M = M$, we deduce that $\int_{\mathbb{R}} \partial_t \mathfrak{u}_M = 0$. We therefore write

$$\partial_t \mathfrak{u}_M = \partial_t a \partial_a \mathfrak{u}_M = \partial_t a \partial_y \mathfrak{v}_M,$$

where

$$\mathfrak{v}_M = \int_{-\infty}^{y} \partial_a \mathfrak{u}_M.$$

We check easily, with the explicit dependency on a of $\mathfrak{u}_M$, that $\mathfrak{v}_M$ also decays exponentially fast at $\pm\infty$ with similar bounds to $\mathfrak{u}_M$.

To continue, take some $\nu_0 > 0$ small and assume that at time T,

$$\|(1+|y|)^3 f(y,T)\|_{L^\infty(\mathbb{R})} + \|\partial_y f(\cdot,T)\|_{H^1(\mathbb{R})} + \left| \int_{\mathbb{R}} f(y,T) \right| \leqslant \nu_0.$$

Then, using the results of Section 3.1 to estimate $\mathfrak{u}_M$, we have

$$|g(y)| \leqslant \frac{K\nu_0}{(1+|y|)^2},$$

hence $\|g\|_{L^2(\mathbb{R})}(T) \leqslant K\nu_0$. Furthermore, $\partial_y g = f - \mathfrak{u}_M$ and $\partial_y^2 g = \partial_y f - \partial_y \mathfrak{u}_M$, hence

$$\|g\|_{H^2(\mathbb{R})}(T) \leqslant K\nu_0.$$

That is, taking ν_0 small enough, we can make g as small as we want in $H^2(\mathbb{R})$ at the initial time T.

Now, replacing it in (3.1) and integrating the equation between $-\infty$ and y implies that

$$\begin{aligned} t^{\frac{2\alpha}{1+\alpha}} \partial_t g - \partial_y^2 g - \frac{\alpha\varepsilon}{1+\alpha} y \partial_y g + \Big(H_\varepsilon - \frac{z_c}{1+\alpha} + \mathfrak{u}_M \Big) \partial_y g \\ + \frac{(\partial_y g)^2}{2} - t \partial_t z_c \partial_y g \\ + \widetilde{S} - \frac{\alpha\varepsilon}{1+\alpha} y \mathfrak{u}_M - t \partial_t z_c \mathfrak{u}_M + t^{\frac{2\alpha}{1+\alpha}} \partial_t a \mathfrak{v}_M \\ = 0. \end{aligned} \tag{3.4}$$

We recall that $\varepsilon(t) = t^{-\frac{1-\alpha}{1+\alpha}}$. We now take a weight $W \in C^2(\mathbb{R}, \mathbb{R}^{+*})$ that we will make precise later on. Taking the scalar product of the equation with gW leads to

$$\begin{aligned} & t^{\frac{2\alpha}{1+\alpha}} \partial_t(\|g\|^2_{L^2(W)}) + 2\|\partial_y g\|^2_{L^2(W)} \\ & \quad + \int_{\mathbb{R}} g^2 W\Big(\frac{\alpha\varepsilon}{(1+\alpha)} - \partial_y(H_\varepsilon + \mathfrak{u}_M) - \frac{\partial_y W}{W}\Big(H_\varepsilon - \frac{z_c + \alpha\varepsilon y}{1+\alpha} + \mathfrak{u}_M\Big)\Big) \\ & \quad + \int_{\mathbb{R}} g^2 W\Big(-t^{\frac{2\alpha}{1+\alpha}}\frac{\partial_t W}{W} - \frac{\partial_y^2 W}{W} + t\partial_t z_c \frac{\partial_y W}{W}\Big) \\ & \quad + \int_{\mathbb{R}} gW\Big((\partial_y g)^2 + 2\Big(\tilde{S} - \frac{\alpha\varepsilon}{1+\alpha} y\mathfrak{u}_M - t\partial_t z_c \mathfrak{u}_M + t^{\frac{2\alpha}{1+\alpha}}\partial_t a\mathfrak{v}_M\Big)\Big) \\ & = 0. \end{aligned} \tag{3.5}$$

Differentiating equation (3.4) leads to

$$\begin{aligned} & t^{\frac{2\alpha}{1+\alpha}} \partial_t \partial_y g - \partial_y^3 g - \frac{\alpha\varepsilon}{1+\alpha}\partial_y g - \frac{\alpha\varepsilon}{1+\alpha} y \partial_y^2 g + \Big(H_\varepsilon - \frac{z_c}{1+\alpha} + \mathfrak{u}_M\Big)\partial_y^2 g \\ & \quad + \partial_y(H_\varepsilon + \mathfrak{u}_M)\partial_y g + \partial_y\Big(\frac{(\partial_y g)^2}{2}\Big) \\ & \quad + \partial_y\Big(\tilde{S} - \frac{\alpha\varepsilon}{1+\alpha} y\mathfrak{u}_M - t\partial_t z_c \mathfrak{u}_M + t^{\frac{2\alpha}{1+\alpha}}\partial_t a\mathfrak{v}_M\Big) \\ & = 0. \end{aligned}$$

Its scalar product with $\partial_y g W$ gives us the equality

$$\begin{aligned} & t^{\frac{2\alpha}{1+\alpha}} \partial_t(\|\partial_y g\|^2_{L^2(W)}) + 2\|\partial_y^2 g\|^2_{L^2(W)} \\ & = \int_{\mathbb{R}} (\partial_y g)^2 W\Big(\frac{-\alpha\varepsilon}{(1+\alpha)} + \partial_y(H_\varepsilon + \mathfrak{u}_M) - \frac{\partial_y W}{W}\Big(H_\varepsilon - \frac{z_c + \alpha\varepsilon y}{1+\alpha} + \mathfrak{u}_M\Big)\Big) \\ & \quad + \int_{\mathbb{R}} (\partial_y g)^2 W\Big(-t^{\frac{2\alpha}{1+\alpha}}\frac{\partial_t W}{W} - \frac{\partial_y^2 W}{W} + t\partial_t z_c \frac{\partial_y W}{W}\Big) + \int_{\mathbb{R}} W\partial_y((\partial_y g)^2)\partial_y g \\ & \quad + \int_{\mathbb{R}} 2\partial_y g W \partial_y\Big(\tilde{S} - \frac{\alpha\varepsilon}{1+\alpha} y\mathfrak{u}_M - t\partial_t z_c \mathfrak{u}_M + t^{\frac{2\alpha}{1+\alpha}}\partial_t a\mathfrak{v}_M\Big). \end{aligned} \tag{3.6}$$

We compute, supposing that W is constant outside a compact set, that

$$\int_{\mathbb{R}} W\partial_y((\partial_y g)^2)\partial_y g = \frac{-1}{3}\int_{\mathbb{R}} \partial_y W(\partial_y g)^3.$$

Summing (3.5) and λ times (3.6) for some $\lambda > 0$ to be determined later reads

$$\begin{aligned} & t^{\frac{2\alpha}{1+\alpha}} \partial_t(\|g\|^2_{L^2(W)} + \lambda\|\partial_y g\|^2_{L^2(W)}) \\ & \quad + \int_{\mathbb{R}} g^2 W D_1 + \lambda\int_{\mathbb{R}} (\partial_y g)^2 W D_2 \\ & \quad - \|\partial_y g\|^2_{L^2(W)}\|g\|_{L^\infty} \end{aligned}$$

$$
\begin{aligned}
&+2\int_{\mathbb{R}} gW\Big(\widetilde{S}-\frac{\alpha\varepsilon}{1+\alpha}y\mathfrak{u}_M-t\partial_t z_c\mathfrak{u}_M+t^{\frac{2\alpha}{1+\alpha}}\partial_t a\mathfrak{v}_M\Big)\\
&+2\lambda\int_{\mathbb{R}}\partial_y gW\partial_y\Big(\widetilde{S}-\frac{\alpha\varepsilon}{1+\alpha}y\mathfrak{u}_M-t\partial_t z_c\mathfrak{u}_M+t^{\frac{2\alpha}{1+\alpha}}\partial_t a\mathfrak{v}_M\Big)\\
&+2\lambda\|\partial_y^2 g\|^2_{L^2(W)}-\frac{1}{3}\int_{\mathbb{R}}\partial_y W(\partial_y g)^3\\
\leqslant{}& 0, \qquad (3.7)
\end{aligned}
$$

with

$$
\begin{aligned}
D_1 :={}& \frac{\alpha\varepsilon}{(1+\alpha)}-\partial_y(H_\varepsilon+\mathfrak{u}_M)-\frac{\partial_y W}{W}\Big(H_\varepsilon-\frac{z_c+\alpha\varepsilon y}{1+\alpha}+\mathfrak{u}_M\Big)\\
&-t^{\frac{2\alpha}{1+\alpha}}\frac{\partial_t W}{W}-\frac{\partial_y^2 W}{W}+t\partial_t z_c\frac{\partial_y W}{W}
\end{aligned}
$$

and

$$
\begin{aligned}
D_2 :={}& \frac{2}{\lambda}+\frac{-\alpha\varepsilon}{(1+\alpha)}+\partial_y(H_\varepsilon+\mathfrak{u}_M)-\frac{\partial_y W}{W}\Big(H_\varepsilon-\frac{z_c+\alpha\varepsilon y}{1+\alpha}+\mathfrak{u}_M\Big)\\
&-t^{\frac{2\alpha}{1+\alpha}}\frac{\partial_t W}{W}-\frac{\partial_y^2 W}{W}+t\partial_t z_c\frac{\partial_y W}{W}.
\end{aligned}
$$

3.3. Estimates of D_1 and D_2 and choice of W

In this subsection we choose the values of λ and W so that D_1 and D_2 are strictly positive, and satisfy some good estimates.

3.3.1. Estimates for $y > 0$. The goal of this subsection is to show that for $y > 0$ we have

$$
D_1(y)\geqslant\frac{\varepsilon\alpha}{1+\alpha}+Ce^{-\frac{a}{2}|y|}\quad\text{and}\quad D_2(y)\geqslant 1
$$

for some constant $C > 0$ independent of ε. We recall that for $y > 0$,

$$
H_\varepsilon(y)=h_\varepsilon(z_c+\varepsilon y)=h_0(z_c+\varepsilon y)+F(y)+a+\varepsilon G(y),
$$

hence

$$
\begin{aligned}
D_1 ={}& \varepsilon\Big(\frac{\alpha}{(1+\alpha)}-h_0'(z_c+\varepsilon y)\Big)-\partial_y(F+\mathfrak{u}_M)-\varepsilon\partial_y G\\
&-\frac{\partial_y W}{W}\Big(h_0(z_c+\varepsilon y)-\frac{z_c}{1+\alpha}+F(y)+a-\frac{\alpha\varepsilon y}{1+\alpha}+\mathfrak{u}_M\Big)\\
&-t^{\frac{2\alpha}{1+\alpha}}\frac{\partial_t W}{W}-\frac{\partial_y^2 W}{W}+t\partial_t z_c\frac{\partial_y W}{W}.
\end{aligned}
$$

In this region, we choose $W = 1$. Then

$$
D_1=\varepsilon\Big(\frac{\alpha}{(1+\alpha)}-h_0'(z_c+\varepsilon y)\Big)-\partial_y(F+\mathfrak{u}_M)-\varepsilon\partial_y G.
$$

We recall that $-h_0'(z_c+\varepsilon y)\geqslant 0$, and from (3.3) we have

$$-\partial_y(F+\mathfrak{u}_M)=\frac{a^2}{\cosh^2(c_M-ax)}.$$

Finally, from (2.7) we have

$$\varepsilon|\partial_y G|\leqslant C_0\Big(\varepsilon^2\ln\frac{1}{\varepsilon}+\varepsilon e^{-\frac{a}{2}Z}\Big),$$

which implies that, for $t\geqslant T$ with $T>0$ large enough and $|M|$ small enough (so that c_M is close to 0), we have $-\partial_y(F+\mathfrak{u}_M)\geqslant Ce^{-\frac{a}{2}|y|}$ for some $C>0$ and thus

$$D_1(y)\geqslant\frac{\varepsilon\alpha}{1+\alpha}+Ce^{-\frac{a}{2}|y|}.$$

Now we have

$$D_2(y)=\frac{2}{\lambda}+\frac{-\alpha\varepsilon}{(1+\alpha)}+\partial_y(H_\varepsilon+\mathfrak{u}_M),$$

and since $|\partial_y(H_\varepsilon+\mathfrak{u}_M)|\leqslant a^2$, taking λ small enough (depending only on a) and $t\geqslant T$ with T large enough leads to

$$D_2(y)\geqslant 1.$$

3.3.2. Estimates for $y<0$. The goal of this subsection is to show that for $y<0$ and fixing a well-chosen weight W we have

$$D_1(y)\geqslant\frac{\alpha\varepsilon}{4(1+\alpha)}\quad\text{and}\quad D_2(y)\geqslant 1.$$

For $y<0$, we recall that

$$H_\varepsilon(y)=h_\varepsilon(z_c+\varepsilon y)=h_0(z_c+\varepsilon y)+F(y)-a+\varepsilon G(y),$$

hence

$$\begin{aligned}D_1&=\varepsilon\Big(\frac{\alpha}{(1+\alpha)}-h_0'(z_c+\varepsilon y)\Big)-\partial_y(F+\mathfrak{u}_M)-\varepsilon\partial_y G\\&\quad-\frac{\partial_y W}{W}\Big(h_0(z_c+\varepsilon y)-\frac{z_c}{1+\alpha}+F(y)-a-\frac{\alpha\varepsilon y}{1+\alpha}+\mathfrak{u}_M\Big)\\&\quad-t^{\frac{2\alpha}{1+\alpha}}\frac{\partial_t W}{W}-\frac{\partial_y^2 W}{W}+t\partial_t z_c\frac{\partial_y W}{W}.\end{aligned}$$

Let us estimate the coefficient in the factor of $\frac{\partial_y W}{W}$ in the second line. For $y<0$, we have $-\frac{\alpha\varepsilon y}{1+\alpha}\geqslant 0$, $h_0(z_c+\varepsilon y)-\frac{z_c}{1+\alpha}>C_0$ a universal constant and

$$|F(y)-a+\mathfrak{u}_M|\leqslant Ke^{-\frac{a}{2}|y|}$$

if $|M|$ is small enough. Therefore, there exists y_0 independent of time such that

$$h_0(z_c+\varepsilon y)-\frac{z_c}{1+\alpha}+F(y)-a-\frac{\alpha\varepsilon y}{1+\alpha}+\mathfrak{u}_M\geqslant\frac{C_0}{2}$$

for $y \leqslant y_0$, and we choose y_0 to be the largest value in $\mathbb{R}^-$ such that this holds. We then define $W \in C^2$ by $W(y) = 1$ for $y \in [y_0, 0]$ and on $]-\infty, y_0]$,

$$\frac{\partial_y W}{W} = -w(y)\varepsilon,$$

where w is a C^2 function that satisfies $w(y) = C > 0$ if $y \in [-\gamma\varepsilon^{-1}, y_0 - 1]$ for some $C, \gamma > 0$ that will be determined later on, and $w(y) = 0$ if $y \leqslant -(\gamma+1)\varepsilon^{-1}$, with $|w| + |w'| \leqslant 2C$ everywhere, and $|w'| \leqslant 2C\varepsilon$ if $y \leqslant y_0 - 1$. Note in particular that W is then constant for $y \leqslant -(\gamma+1)\varepsilon^{-1}$, and that this constant is uniform in time (but depends on C and γ). Indeed, for $y \leqslant y_0$ we have

$$W(y) = \exp\left(-\varepsilon \int_{y_0}^{y} w\right).$$

This also shows that for $y \leqslant y_0 - 1$, we have

$$\left| -t^{\frac{2\alpha}{1+\alpha}} \frac{\partial_t W}{W} - \frac{\partial_y^2 W}{W} + t\partial_t z_c \frac{\partial_y W}{W} \right| \leqslant K\varepsilon^2.$$

We choose γ such that for $y \leqslant -\gamma\varepsilon^{-1}$ we have

$$\frac{\alpha}{(1+\alpha)} - h_0'(z_c + \varepsilon y) \geqslant \frac{\alpha}{2(1+\alpha)}.$$

This is possible thanks to Lemma 2.5. For $y \leqslant -\gamma\varepsilon^{-1}$, we have

$$-\frac{\partial_y W}{W}\left(h_0(z_c + \varepsilon y) - \frac{z_c}{1+\alpha} + F(y) - a - \frac{\alpha\varepsilon y}{1+\alpha} + \mathrm{u}_M\right) \geqslant 0,$$

and therefore

$$D_1 \geqslant \frac{\varepsilon\alpha}{2(1+\alpha)} - \partial_y(F + \mathrm{u}_M) - \varepsilon\partial_y G \geqslant \frac{\alpha\varepsilon}{4(1+\alpha)}$$

if γ is taken large enough (depending only on α, κ). Now, for $y \in [-\gamma\varepsilon^{-1}, y_0 - 1]$, we have

$$-\frac{\partial_y W}{W}\left(h_0(z_c + \varepsilon y) - \frac{z_c}{1+\alpha} + F(y) - a - \frac{\alpha\varepsilon y}{1+\alpha} + \mathrm{u}_M\right) \geqslant \frac{CC_0}{2}\varepsilon;$$

therefore, if we take C large enough we check that

$$D_1 \geqslant \frac{\alpha\varepsilon}{4(1+\alpha)}$$

there as well. Now, for $y \in [y_0 - 1, 0]$, we have

$$D_1 \geqslant -\partial_y(F + \mathrm{u}_M) - K\varepsilon \geqslant Ke^{-a|y|} > 0$$

if T is large enough.

For D_2, since $|\frac{\partial_y W}{W}| \leqslant 2C\varepsilon$, we check that taking λ large enough and $t \geqslant T$ with T large enough, we have $D_2(y) \geqslant 1$.

3.3.3. Summary. With the above choices for W and λ, we have

$$D_1 \geqslant \frac{\alpha\varepsilon}{4(1+\alpha)} + Ke^{-\frac{a}{2}|y|}$$

for some $K > 0$ independent of ε and

$$D_2 \geqslant 1.$$

Note that in the case $y > 0$, we could not have chosen a similar weight W, because D_1 contains a term $y\frac{\partial_y W}{W}$. For $y < 0$, $\frac{\partial_y W}{W} < 0$, this is a positive quantity, but for $y > 0$, this would pose an issue.

3.4. Estimates on the source terms

We focus here on estimates on

$$\widetilde{S} - \frac{\alpha\varepsilon}{1+\alpha} y \mathfrak{u}_M - t\partial_t z_c \mathfrak{u}_M + t^{\frac{2\alpha}{1+\alpha}} \partial_t a \mathfrak{v}_M$$

with $\widetilde{S} = \int_{-\infty}^{y} \partial_t \varepsilon \partial_\varepsilon \mathbb{h}_\varepsilon (z_c + \varepsilon x)\, dx$. We have

$$t\partial_t z_c = t\partial_t \varepsilon \partial_\varepsilon z_c = \frac{-1+\alpha}{1+\alpha} \varepsilon \partial_\varepsilon z_c$$

and $\partial_t a = \frac{-1+\alpha}{1+\alpha} t^{-1} \varepsilon \partial_\varepsilon a$, and we recall from (2.14) that $|\partial_\varepsilon z_c| + |\partial_\varepsilon a| \leqslant K(\ln \frac{1}{\varepsilon})^2$. By the estimates on $\mathfrak{u}_M$ from Section 3.1, we deduce easily that

$$\begin{aligned}
&\left| -\frac{\alpha\varepsilon}{1+\alpha} y \mathfrak{u}_M - t\partial_t z_c \mathfrak{u}_M + t^{\frac{2\alpha}{1+\alpha}} \partial_t a \mathfrak{v}_M \right| \\
&\qquad + \left| \partial_y \Big(-\frac{\alpha\varepsilon}{1+\alpha} y \mathfrak{u}_M - t\partial_t z_c \mathfrak{u}_M + t^{\frac{2\alpha}{1+\alpha}} \partial_t a \mathfrak{v}_M \Big) \right| \\
&\quad \leqslant K\varepsilon \Big(\ln \frac{1}{\varepsilon} \Big)^2 e^{-\frac{a}{2}|y|}
\end{aligned}$$

if $|M|$ is small enough. Concerning $\widetilde{S}$, we have $|\partial_t \varepsilon| \leqslant Kt^{-1}\varepsilon$, and using Lemma 2.14, we check that

$$\left| \int_{-\infty}^{y} \varepsilon \partial_\varepsilon \mathbb{h}_\varepsilon (z_c + \varepsilon x)\, dx \right| \leqslant \frac{K\varepsilon (\ln \frac{1}{\varepsilon})^2}{(1+|y|)^{2\alpha}} + Ke^{-2a|y|},$$

and if $\alpha > \frac{1}{4}$ (which is needed to get enough decay in y), $\widetilde{S} \in L^2(\mathbb{R})$ with

$$\|\widetilde{S}\|_{L^2(\mathbb{R})} \leqslant \frac{K}{t}.$$

We check also, with similar arguments, that $\partial_y \widetilde{S} \in L^2(\mathbb{R})$ and

$$\|\partial_y \widetilde{S}\|_{L^2(\mathbb{R})} \leqslant \frac{K}{t}.$$

3.5. End of the proof of Theorem 1.4

3.5.1. Estimates on $\|g\|^2_{L^2(W)} + \lambda\|\partial_y g\|^2_{L^2(W)}$. By Cauchy–Schwarz and the estimates on D_1 and D_2 from Section 3.3, for t large enough, equation (3.7) implies the inequality

$$
\begin{aligned}
&t^{\frac{2\alpha}{1+\alpha}}\partial_t(\|g\|^2_{L^2(W)} + \lambda\|\partial_y g\|^2_{L^2(W)}) \\
&\qquad + \int_{\mathbb{R}} g^2 W\Big(\frac{\alpha\varepsilon}{4(1+\alpha)} + Ke^{-2a|y|}\Big) + (\lambda - \|g\|_{L^\infty})\|\partial_y g\|^2_{L^2(W)} \\
&\qquad + 2\int_{\mathbb{R}} gW\Big(\widetilde{S} - \frac{\alpha\varepsilon}{1+\alpha}y\mathrm{u}_M - t\partial_t z_c \mathrm{u}_M\Big) \\
&\qquad - 2\lambda\|\partial_y g\|_{L^2(W)}\Big\|\partial_y\Big(\widetilde{S} - \frac{\alpha\varepsilon}{1+\alpha}y\mathrm{u}_M - t\partial_t z_c\mathrm{u}_M\Big)\Big\|_{L^2(W)} \\
&\qquad + 2\lambda\|\partial_y^2 g\|^2_{L^2(W)} - \frac{1}{3}\int_{\mathbb{R}} \partial_y W(\partial_y g)^3 \\
&\leqslant 0.
\end{aligned}
$$

By the computations of Section 3.4, we have by Cauchy–Schwarz that, for $\alpha > \frac{1}{4}$,

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}} gW\Big(\widetilde{S} - \frac{\alpha\varepsilon}{1+\alpha}y\mathrm{u}_M - t\partial_t z_c\mathrm{u}_M\Big)\Big| \\
&\qquad \leqslant K\int_{\mathbb{R}} |g|W\Big(\varepsilon\Big(\ln\frac{1}{\varepsilon}\Big)^2 e^{-\frac{a}{2}|y|} + \frac{t^{-1}}{(1+|y|)^{2\alpha}}\Big) \\
&\qquad \leqslant K\varepsilon\Big(\ln\frac{1}{\varepsilon}\Big)^2\sqrt{\int_{\mathbb{R}} g^2 W e^{-\frac{a}{2}|y|}} + t^{-1}\|g\|_{L^2(W)}.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\int_{\mathbb{R}} g^2 W\Big(\frac{\alpha\varepsilon}{4(1+\alpha)} + Ke^{-2a|y|}\Big) - \Big|\int_{\mathbb{R}} gW\Big(\widetilde{S} - \frac{\alpha\varepsilon}{1+\alpha}y\mathrm{u}_M - t\partial_t z_c\mathrm{u}_M\Big)\Big| \\
&\quad \geqslant \frac{\alpha\varepsilon}{8(1+\alpha)}\|g\|^2_{L^2(W)} - Kt^{-2}\varepsilon^{-1}
\end{aligned}
$$

if $\varepsilon(t)$ is small enough. We also check that

$$
\Big\|\partial_y\Big(\widetilde{S} - \frac{\alpha\varepsilon}{1+\alpha}y\mathrm{u}_M - t\partial_t z_c\mathrm{u}_M\Big)\Big\|_{L^2(W)} \leqslant K\varepsilon\Big(\ln\frac{1}{\varepsilon}\Big)^2,
$$

hence

$$
\begin{aligned}
&\lambda\|\partial_y g\|^2_{L^2(W)} - 2\lambda\|\partial_y g\|_{L^2(W)}\Big\|\partial_y\Big(\widetilde{S} - \frac{\alpha\varepsilon}{1+\alpha}y\mathrm{u}_M - t\partial_t z_c\mathrm{u}_M\Big)\Big\|_{L^2(W)} \\
&\quad \geqslant \frac{\lambda}{2}\|\partial_y g\|^2_{L^2(W)} - K\varepsilon^2\Big(\ln\frac{1}{\varepsilon}\Big)^4.
\end{aligned}
$$

Furthermore, by the Gagliardo-Nirenberg inequality, since W is bounded above and below by constants independent of time, we have

$$\|g\|_{L^\infty} \leqslant \widetilde{K}\|g\|_{L^2(\mathbb{R})}^{1/2}\|\partial_y g\|_{L^2(\mathbb{R})}^{1/2} \leqslant K\|g\|_{L^2(W)}^{1/2}\|\partial_y g\|_{L^2(W)}^{1/2}.$$

To continue,

$$\left|\int_{\mathbb{R}} \partial_y W(\partial_y g)^3\right| \leqslant 2C\varepsilon \int_{\mathbb{R}} W(\partial_y g)^3 \leqslant 2C\varepsilon\|\partial_y g\|_{L^2(W)}^2\|\partial_y g\|_{L^\infty(\mathbb{R})}$$

and

$$\|\partial_y g\|_{L^\infty(\mathbb{R})} \leqslant K\|\partial_y g\|_{L^2(W)}^{1/2}\|\partial_y^2 g\|_{L^2(W)}^{1/2}.$$

We deduce that

$$\begin{aligned}
&2\lambda\|\partial_y^2 g\|_{L^2(W)}^2 - \frac{1}{3}\left|\int_{\mathbb{R}} \partial_y W(\partial_y g)^3\right| \\
&\qquad \geqslant 2\lambda\|\partial_y^2 g\|_{L^2(W)}^2 - C\varepsilon\|\partial_y g\|_{L^2(W)}^{5/2}\|\partial_y^2 g\|_{L^2(W)}^{1/2} \\
&\qquad \geqslant -K\varepsilon^{4/3}\|\partial_y g\|_{L^2(W)}^{10/3}.
\end{aligned}$$

Combining these estimates leads to

$$\begin{aligned}
&t^{\frac{2\alpha}{1+\alpha}}\partial_t(\|g\|_{L^2(W)}^2 + \lambda\|\partial_y g\|_{L^2(W)}^2) \\
&\qquad + \frac{\alpha\varepsilon}{8(1+\alpha)}(\|g\|_{L^2(W)}^2 + \lambda\|\partial_y g\|_{L^2(W)}^2) - K\|\partial_y g\|_{L^2(W)}^{5/2}\|g\|_{L^2(W)}^{1/2} \\
&\qquad - K\varepsilon^2\Big(\ln\frac{1}{\varepsilon}\Big)^4 - K\varepsilon^{4/3}\|\partial_y g\|_{L^2(W)}^{10/3} \\
&\quad \leqslant 0.
\end{aligned}$$

With $\varepsilon(t) = t^{-\frac{1-\alpha}{1+\alpha}}$, dividing by $t^{\frac{2\alpha}{1+\alpha}}$ gives us the estimate

$$\begin{aligned}
&\partial_t(\|g\|_{L^2(W)}^2 + \lambda\|\partial_y g\|_{L^2(W)}^2) \\
&\qquad + \frac{\alpha t^{-1}}{8(1+\alpha)}(\|g\|_{L^2(W)}^2 + \lambda\|\partial_y g\|_{L^2(W)}^2) \\
&\qquad - Kt^{-\frac{2\alpha}{1+\alpha}}\|\partial_y g\|_{L^2(W)}^{5/2}\|g\|_{L^2(W)}^{1/2} - Kt^{\frac{-4-2\alpha}{3(1+\alpha)}}\|\partial_y g\|_{L^2(W)}^{10/3} \\
&\qquad - Kt^{-\frac{2}{1+\alpha}}(\ln t)^4 \\
&\quad \leqslant 0,
\end{aligned}$$

and since $\frac{-4-2\alpha}{3(1+\alpha)} < -1$ and $-\frac{2}{1+\alpha} < -1$, we deduce that

$$\partial_t\big(t^{-\frac{\alpha}{8(1+\alpha)}}(\|g\|_{L^2(W)}^2 + \lambda\|\partial_y g\|_{L^2(W)}^2)\big) \leqslant 0.$$

Therefore, if at time T large enough we take ν_0 small enough (depending on T), then for $t \geqslant T$,

$$\|g\|_{L^2(W)}^2 + \lambda\|\partial_y g\|_{L^2(W)}^2 \leqslant Kt^{-\frac{\alpha}{8(1+\alpha)}}$$

for some $K > 0$ large. We check also, with similar computations, that for some $\mu > 0$ we have

$$\|g\|^2_{L^2(W)} + \lambda \|\partial_y g\|^2_{L^2(W)} + \mu \|\partial^2_y g\|^2_{L^2(W)} \leqslant K t^{-\frac{\alpha}{8(1+\alpha)}}.$$

This can be done by differentiating equation (3.4) twice and taking its scalar product with $\partial^2_y g W$, and adding it to (3.7).

3.5.2. Returning to the original scaling. Since $g = f - \mathfrak{u}_M$, we have with $\delta = \frac{\alpha}{8(1+\alpha)}$ that

$$\|f - \mathfrak{u}_M\|^2_{L^2(\mathbb{R})} \leqslant K \|\partial_y g\|^2_{L^2(W)} \leqslant K t^{-\delta}$$

and

$$\|f - \mathfrak{u}_M\|^2_{L^\infty(\mathbb{R})} \leqslant K \|\partial_y g\|^2_{H^1(W)} \leqslant K t^{-\delta}.$$

We recall that with u solving the viscous Burgers equation, we wrote

$$H(y,t) = t^{\frac{\alpha}{1+\alpha}} u((z_c + \varepsilon(t) y) t^{\frac{1}{1+\alpha}}, t)$$

and

$$H(y,t) = H_{\varepsilon(t)}(y) + \mathfrak{u}_M(y) + (f - \mathfrak{u}_M)(y).$$

Therefore,

$$\left\| t^{\frac{\alpha}{1+\alpha}} u(x,t) - (H_{\varepsilon(t)} + \mathfrak{u}_M)\left(t^{\frac{1-\alpha}{1+\alpha}} (x t^{-\frac{1}{1+\alpha}} - z_c(t))\right)\right\|_{L^\infty(\mathbb{R})} = o_{t\to+\infty}(1),$$

and

$$H_{\varepsilon(t)}\left(t^{\frac{1-\alpha}{1+\alpha}} (x t^{-\frac{1}{1+\alpha}} - z_c(t))\right) = \mathbb{h}_{\varepsilon(t)}(t^{-\frac{1}{1+\alpha}} x).$$

This completes the proof of Theorem 1.4.

3.6. Proof of Proposition 1.5.

We consider here the equation

$$\partial_t u - \partial^2_x u + \partial_x \left(\frac{u^2}{2} + J(u)\right) = 0.$$

Making the same change of variable as for the proof of Theorem 1.4, the only change in equation (3.4) is that we add the term

$$E_J := t^{\frac{2\alpha}{1+\alpha}} J(t^{-\frac{\alpha}{1+\alpha}} (H_\varepsilon + \mathfrak{u}_M + \partial_y g)).$$

We recall that $|J(u)| \leqslant K|u|^3$. The scalar product of E_J with gW can be controlled by

$$\begin{aligned}\left|\int_{\mathbb{R}} E_J g W\right| &\leqslant K t^{-\frac{\alpha}{1+\alpha}} \int_{\mathbb{R}} |H_\varepsilon + \mathfrak{u}_M + \partial_y g|^3 |g| W \\ &\leqslant K t^{-\frac{\alpha}{1+\alpha}} \|g\|_{L^\infty} (1 + \|\partial_y g\|_{L^\infty}) \left(\|\partial_y g\|^2_{L^2(W)} + \int_{\mathbb{R}} W (H_\varepsilon + \mathfrak{u}_M)^2 \right).\end{aligned}$$

Since $H_\varepsilon(y) = h_\varepsilon(z_c + \varepsilon y)$, we check that if $\alpha > \frac{1}{2}$ we have

$$\int_{\mathbb{R}} W(H_\varepsilon + \mathfrak{u}_M)^2 \leqslant K\varepsilon(t)^{-1},$$

hence

$$\left|\int_{\mathbb{R}} E_J g W\right| \leqslant K t^{-\frac{\alpha}{1+\alpha}} \|g\|_{L^\infty}(1 + \|\partial_y g\|_{L^\infty})(\|\partial_y g\|^2_{L^2(W)} + K t^{\frac{1-\alpha}{1+\alpha}}),$$

and to consider it as an error term to conclude as in Section 3.5, we need $t^{-\frac{2\alpha}{1+\alpha}} \int_{\mathbb{R}} E_J g W$ to decay in time strictly faster than $t^{-1-\delta}$ for $\delta > 0$ small provided that $\|g\|^2_{H^2(\mathbb{R})} \leqslant K t^{-\delta}$. This is the case if $\frac{1-4\alpha}{1+\alpha} < -1$, that is, $\alpha > \frac{2}{3}$.

We can check similarly that we can treat $\int_{\mathbb{R}} \partial_y E_J \partial_y g W$ and $\int_{\mathbb{R}} \partial_y^2 E_J \partial_y^2 g W$ similarly. For the latter, we use the fact that we also control $\mu\|\partial_y^3 g\|^2_{L^2(W)}$.

A. Proof of Proposition 1.1

Proof. We recall that the solution of the heat equation is

$$f(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} f_0(y) e^{-\frac{(x-y)^2}{4t}}\, dy.$$

We compute

$$t^{\alpha/2} f(\sqrt{t}z, t) = \frac{t^{\alpha/2}}{\sqrt{4\pi}} \int_{\mathbb{R}} f_0(y\sqrt{t}) e^{-(y-z)^2/4}\, dy.$$

Take any $\beta \in \left]\frac{\alpha}{2}, \frac{1}{2}\right[$. Then

$$t^{\alpha/2}\left|\int_{|y|\leqslant t^{-\beta}} f_0(y\sqrt{t}) e^{-(y-z)^2/4}\, dy\right| \leqslant K t^{\frac{\alpha}{2}-\beta} \to 0$$

when $t \to +\infty$. Furthermore, for $|y| \geqslant t^{-\beta}$, we have $|y|\sqrt{t} \geqslant t^{\frac{1}{2}-\beta} \to +\infty$ when $t \to +\infty$, hence

$$t^{\alpha/2} f_0(y\sqrt{t}) \to \frac{-\kappa}{|y|^\alpha}$$

when $t \to +\infty$. We deduce that

$$\left|t^{\alpha/2} f(\sqrt{t}z, t) + \frac{\kappa}{\sqrt{4\pi}} \int_{|y|\geqslant t^{-\beta}} \frac{1}{|y|^\alpha} e^{-(y-z)^2/4}\, dy\right| = o_{t\to+\infty}(1)$$

and since $\alpha < 1$, $|y|^{-\alpha}$ is integrable near 0, which concludes the proof. ∎

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