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Gap distributions of Fourier quasicrystals with integer weights via Lee–Yang polynomials

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Abstract. Recent work of Kurasov and Sarnak provides a method for constructing one-dimensional Fourier quasicrystals (FQ) from the torus zero sets of a special class of multivariate polynomials called Lee–Yang polynomials. In particular, they provided a non-periodic FQ with unit coefficients and uniformly discrete support, answering an open question posed by Meyer. Their method was later shown to generate all one-dimensional Fourier quasicrystals with $\mathbb{N}$-valued coefficients ($\mathbb{N}$-FQ).

In this paper, we characterize which Lee–Yang polynomials give rise to non-periodic $\mathbb{N}$-FQs with unit coefficients and uniformly discrete support, and show that this property is generic among Lee–Yang polynomials. We also show that the infinite sequence of gaps between consecutive atoms of any $\mathbb{N}$-FQ has a well-defined distribution, which, under mild conditions, is absolutely continuous. This generalizes previously known results for the spectra of quantum graphs to arbitrary $\mathbb{N}$-FQs. Two extreme examples are presented: first, a sequence of $\mathbb{N}$-FQs whose gap distributions converge to a Poisson distribution. Second, a sequence of random Lee–Yang polynomials that results in random $\mathbb{N}$-FQs whose empirical gap distributions converge to that of a random unitary matrix (CUE).

1. Introduction

An $\mathbb{N}$-FQ is an $\mathbb{N}$-valued measure supported on a discrete set whose Fourier transform is also supported on a discrete set and has moderate growth (see Definition 2.1). A recent sequence of works [3, 14, 19] established that all one-dimensional $\mathbb{N}$-FQs arise from the torus zero sets of a special class of multivariate polynomials, called Lee–Yang polynomials.

Given a discrete periodic set $\Lambda \subset \mathbb{R}$ with period $\Delta > 0$, the Poisson summation formula states that

$$\sum_{x\in\Lambda} f(x) = \frac{2\pi}{\Delta} \sum_{k\in\Lambda^*} \hat{f}(k), \quad \text{for all } f \in \mathcal{S}(\mathbb{R}),$$

where $\Lambda^* = \{k \in \mathbb{R} : \forall x \in \Lambda, e^{ikx} = 1\}$ and $\mathcal{S}(\mathbb{R})$ is the space of Schwartz functions: smooth functions on $\mathbb{R}$ that rapidly decay to zero at $\pm\infty$ (properly defined in Section 2).

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Fourier quasicrystals are generalizations of the Poisson summation formula to sets which are not periodic but exhibit similar features.

1.1. Fourier quasicrystals

Elements in the dual space $\mathcal{S}'(\mathbb{R})$ are called *tempered distributions*, and the Fourier transform of $\mu \in \mathcal{S}'(\mathbb{R})$ is the tempered distribution $\widehat{\mu}$ defined by duality, $\int f\, d\widehat{\mu} := \int \widehat{f}\, d\mu$. For example, if Λ is periodic as above, then $\mu = \sum_{x\in\Lambda} \delta_x$ is tempered and its Fourier transform is $\widehat{\mu} = \frac{2\pi}{\Delta} \sum_{k\in\Lambda} \delta_k$, by the Poisson summation formula. However, in general, if a tempered distribution μ is supported on a non-periodic discrete set Λ, i.e., $\mu = \sum_{x\in\Lambda} a_x \delta_x$ for some complex coefficients $(a_x)_{x\in\Lambda}$, it is unlikely that $\widehat{\mu}$ will also be supported on a discrete set.

If $\mu \in \mathcal{S}'(\mathbb{R})$ satisfies the condition that both μ and $\widehat{\mu}$ are supported on discrete (locally finite) sets, then μ is called a *crystalline measure* [17]. A crystalline measure $\mu = \sum_{x\in\Lambda} a_x \delta_x$ with Fourier transform $\widehat{\mu} = \sum_{k\in S} c_k \delta_k$ (so that $S \subset \mathbb{R}$ is some discrete set) is called a *Fourier quasicrystal* if

$$|\mu| = \sum_{x\in\Lambda} |a_x|\,\delta_x \quad \text{and} \quad |\widehat{\mu}| = \sum_{k\in S} |c_k|\,\delta_k$$

are tempered as well, [15]. We say that μ is $\mathbb{N}$-valued if $a_x \in \mathbb{N}$ for all $x \in \Lambda$, and we abbreviate $\mathbb{N}$-valued Fourier quasicrystals as $\mathbb{N}$-FQs.

1.2. Lee–Yang polynomials

Following [21], we call a polynomial $p \in \mathbb{C}[z_1, \ldots, z_n]$ a *Lee–Yang polynomial* if it has no zeros in the product $\mathbb{D}^n$ of the open unit disk, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and it has no zeros in the product of the outer disk $(\mathbb{C} \setminus \overline{\mathbb{D}})^n$. One fundamental example is a determinant:

$$p(z_1, z_2, \ldots, z_n) = \det(\operatorname{diag}(z_1, \ldots, z_n) + U),$$

where U is an $n \times n$ unitary matrix. The name Lee–Yang polynomials refers to the elegant proof of the Lee–Yang circle theorem [9, 10] by Brändén and Borcea. Lee–Yang polynomials are intimately related, by Möbius transformations, to the class of real stable polynomials, i.e., $p \in \mathbb{C}[z_1, z_2, \ldots, z_n]$ with the property that $p(\mathbf{a})$ is nonzero whenever $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{C}^n$ has imaginary part $\operatorname{Im}(a_j) > 0$ for all $j = 1, \ldots, n$ or $\operatorname{Im}(a_j) < 0$ for all $j = 1, \ldots, n$. Brändén and Borcea developed a classification of linear operations preserving stability and used this to prove the Lee–Yang circle theorem [9, 10], among many other things. See [22] for a survey of these techniques. Many properties of determinants, especially those involving eigenvalues, also hold and have elegant proofs for general real stable polynomials. See, for example, [5].

1.3. $\mathbb{N}$-FQs and Lee–Yang polynomials

Meyer posed an intriguing question: Are there any non-periodic crystalline measures $\mu = \sum_{x\in\Lambda} \delta_x$, with unit coefficients ($a_x \equiv 1$) and uniformly discrete[1] support Λ?

[1] A set $\Lambda \subset \mathbb{R}$ is said to be *uniformly discrete* if $\exists\, r > 0$ such that $|x - x'| \geq r > 0$ for any distinct $x, x' \in \Lambda$.

In their notable work [14], Kurasov and Sarnak presented a general construction of $\mathbb{N}$-FQs. Using this construction, they answered Meyer's question by providing an explicit example of a non-periodic FQ μ with unit coefficients and a uniformly discrete support. A question addressed in this paper is whether these properties are common among all the $\mathbb{N}$-FQs.

To describe the Kurasov–Sarnak construction, suppose $p(z_1, z_2, \ldots, z_n) = \sum_{\boldsymbol{\alpha}} c_{\boldsymbol{\alpha}} \mathbf{z}^{\boldsymbol{\alpha}}$ is a Lee–Yang polynomial (we use here the multi-index notation $\mathbf{z}^{\boldsymbol{\alpha}} = \prod_{j=1}^{n} z_j^{\alpha_j}$), and let $\ell = (\ell_1, \ldots, \ell_n) \in \mathbb{R}_+^n$. Then, the univariate exponential polynomial

$$f(x) = p(\exp(ix\ell)) = p(e^{ix\ell_1}, \ldots, e^{ix\ell_n}) = \sum_{\boldsymbol{\alpha}} c_{\boldsymbol{\alpha}}\, e^{ix\langle \boldsymbol{\alpha}, \ell\rangle}$$

is *real-rooted*, namely $f(x) = 0 \Rightarrow \mathrm{Im}(x) = 0$, since $\exp(ix\ell) \in \mathbb{D}^n \cup (\mathbb{C} \setminus \overline{\mathbb{D}})^n$ when $\mathrm{Im}(x) \neq 0$. If $f(x) = 0$, let $\mathrm{m}(x)$ denote the multiplicity[2] of x as a zero of f.

Theorem 1.1 (Kurasov–Sarnak construction [14]). *Given a positive vector $\ell \in \mathbb{R}_+^n$ and a Lee–Yang polynomial $p(z_1, z_2, \ldots, z_n)$, let Λ denote the zero set of $f(x) = p(\exp(ix\ell))$ and let $\mathrm{m}(x)$ be the multiplicity of $x \in \Lambda$. Then, the measure*

$$\mu_{p,\ell} := \sum_{x \in \Lambda} \mathrm{m}(x)\, \delta_x,$$

is an $\mathbb{N}$-FQ.

Example 1.2. The polynomial

$$p(z_1, z_2) = 16(1 + z_1^2 z_2^2) - 8(z_1 + z_2 + z_1^2 z_2 + z_1 z_2^2) + (z_1 - z_2)^2$$

is Lee–Yang, and the vector $\ell = (5\pi/22, 1)$ has $\mathbb{Q}$-linearly independent entries. Let $\Lambda = \{t \in \mathbb{R} : p(\exp(it\ell)) = 0\}$ be the support of $\mu_{p,\ell}$. Figure 1 (top) shows the points of Λ in the interval $[0, 10\pi]$. The bottom left picture shows the zero set of $p(e^{ix}, e^{iy})$ and line $(x, y) = t\ell$ for $0 \le t \le 10\pi$; the bottom right image represents these sets in $\mathbb{R}^2/(2\pi\mathbb{Z})^2$.

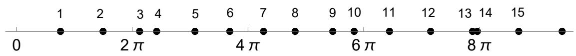
Olevskyii and Ulanovskii [19] proved that any one-dimensional $\mathbb{N}$-FQ has the form $\mu = \sum_{x \in \Lambda} \mathrm{m}(x)\delta_x$, where Λ and $(\mathrm{m}(x))_{x \in \Lambda}$ are the zero set and multiplicities for some real-rooted exponential polynomial f. Together with Cohen [3], the authors showed that every real-rooted exponential polynomial f is of the form[3] $f(x) = p(\exp(ix\ell))$, for some Lee–Yang polynomial p and some positive vector $\ell \in \mathbb{R}_+^n$ that has $\mathbb{Q}$-linearly independent entries. All together, this gives the following.

Theorem 1.3 (Inverse result, [3,19]). *Let $\mu \in \mathcal{S}'(\mathbb{R})$ be an $\mathbb{N}$-FQ. Then, μ is equal to $\mu_{p,\ell}$ as in the Kurasov–Sarnak construction, for some $n \in \mathbb{N}$, a Lee–Yang polynomial $p \in \mathbb{C}[z_1, z_2, \ldots, z_n]$ and a positive vector $\ell \in \mathbb{R}_+^n$ whose entries are $\mathbb{Q}$-linearly independent.*

Given a set $A \subset \mathbb{R}$, let $\dim_{\mathbb{Q}}(A)$ denote the dimension (as a $\mathbb{Q}$-vector space) of the $\mathbb{Q}$-linear span of the elements of A. For a vector $\ell \in \mathbb{R}^n$, $\dim_{\mathbb{Q}}(\ell) = n$ means that its entries are $\mathbb{Q}$-linearly independent.

[2] The multiplicity of a zero x of an analytic function f is the minimal $n \in \mathbb{N}$ for which the n-th derivative is non-zero, $f^{(n)}(x) \neq 0$.

[3] Up to a non-vanishing factor.



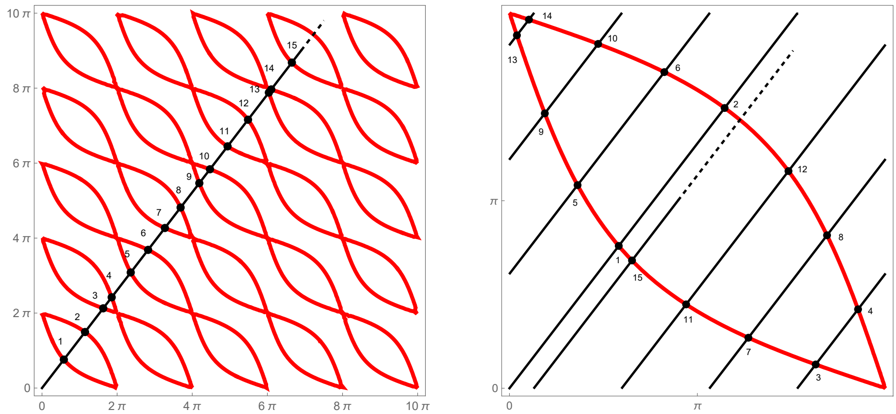


Figure 1. The Kurasov–Sarnak construction of an $\mathbb{N}$-FQ from the zero-set of a Lee–Yang polynomial in the torus $\mathbb{T}^2$. See Example 1.2.

Theorem 1.4 (Theorem 3 in [14]). *Any $\mathbb{N}$-FQ, say $\mu = \sum_{x\in\Lambda} \mathrm{m}(x)\,\delta_x$, has uniformly bounded weights $\mathrm{m}(x)$ and has two integers $r, c \geq 0$ such that its support $\Lambda = L_1 \cup L_2 \cup \cdots \cup L_r \cup N$ is the union of r infinite arithmetic progressions and a set N which, if not empty, has $\dim_{\mathbb{Q}}(N) = \infty$ and $|N \cap L| \leq c$ for any arithmetic progression L.*

We elaborate on Theorem 3 in [14] and the relation between the decomposition of the measure and the decomposition of the polynomial into irreducible factors (a proof provided in Section 7). A polynomial is said to be *binomial* if it has only two monomials.

Theorem 1.5 (Decomposition and non-periodicity). *Given an $\mathbb{N}$-FQ μ, there are an $n\in\mathbb{N}$, a Lee–Yang polynomial p in n variables, and a $\mathbb{Q}$-linearly independent vector $\ell \in \mathbb{R}^n_+$ such that $\mu = \mu_{p,\ell}$. The polynomial p decomposes into distinct irreducible Lee–Yang polynomials $p = \prod_{j=1}^{N} q_j^{c_j}$, where each factor q_j appears with a power $c_j \in \mathbb{N}$. Let Λ be the support of μ, and let Λ_j be the support of $\mu_{q_j,\ell}$ for each q_j. Then,*

$$\mu_{p,\ell} = \sum_{j=1}^{N} c_j\, \mu_{q_j,\ell} \quad \text{and} \quad \Lambda = \bigcup_{j=1}^{N} \Lambda_j .$$

If q_j is binomial, then $\mu_{q_j,\ell}$ has unit coefficients and Λ_j is an infinite arithmetic progression.

If q_j is non-binomial, let D denote its total degree and let $\mu_{q_j,\ell} = \sum_{x\in\Lambda_j} m_j(x)\,\delta_x$. Then we have the following:

(1) (*Almost all unit coefficients*) *The coefficients are bounded by* $m_j(x) \le D$, *and* $m_j(x) = 1$ *for almost every* $x \in \Lambda_j$:

$$\lim_{R\to\infty} \frac{|\{|x| < R : x \in \Lambda_j,\ m_j(x) = 1\}|}{|\{|x| < R : x \in \Lambda_j\}|} = 1.$$

(2) (*Dimension over* $\mathbb{Q}$) *The support has* $\dim_{\mathbb{Q}}(\Lambda_j) = \infty$, *with uniform bounds* $|\Lambda_j \cap A| \le c = c(m, D)$ *for any set* $A \subset \mathbb{R}$ *with* $\dim_{\mathbb{Q}}(A) = m$.

Remark 1.6 (Quasicrystals and cut-and-project sets). The mathematical definition of a *quasicrystal* (not to be confused with a Fourier quasicrystal) is a set $\Lambda \subset \mathbb{R}^n$ which is uniformly discrete, relatively dense[4], and its set of differences $\Lambda - \Lambda = \{x - y : x, y \in \Lambda\}$ is contained in finitely many translates of Λ, see Definition 6 in [16]. A *model set* (also known as cut-and-project set) $\Lambda \subset \mathbb{R}^n$ is the projection of a set $(B \times \mathbb{R}^n) \cap L$, where $L \subset \mathbb{R}^m \times \mathbb{R}^n$ is a lattice in generic location[5] and $B \subset \mathbb{R}^m$ is bounded with non-empty interior. Meyer showed that any model set is a quasicrystal, and any quasicrystal lies in finitely many translates of model sets, see Theorem 1 in [16]. In particular, in such case, $\dim_{\mathbb{Q}}(\Lambda) \le n + m$.

Corollary 1.7. *If* p *is an irreducible non-binomial Lee–Yang polynomial, then the support of* $\mu_{p,\ell}$, *for any* $\mathbb{Q}$*-linearly independent* $\ell \in \mathbb{R}^n_+$, *intersects any quasicrystal and any model set in at most finitely many points.*

Proof. According to Remark 1.6, if A is the support of a quasicrystal or a model set, then $\dim_{\mathbb{Q}}(A) < \infty$, and now the corollary follows from Theorem 1.5 (2). ∎

Remark 1.8 (Non-uniqueness of the decomposition). Even though multivariate polynomials p factor uniquely into irreducibles, the measure $\mu_{p,\ell}$ depends only on the exponential polynomial $f(x) = p(\exp(ix\ell))$. The ring of exponential polynomials is not a unique factorization domain and, as a result, the decomposition of the measures in Theorem 1.5 is not unique. As a simple example, consider

$$1 - \exp(ix) = (1 - \exp(ix/2))(1 + \exp(ix/2)) = (1 - \exp(ix/2^k)) \prod_{j=1}^{k-1} (1 + \exp(ix/2^j))$$

for any $k \ge 1$. The corresponding measure is $\sum_{x\in\mathbb{Z}} \delta_x$. The first factorization gives this measure as $\left(\sum_{x\in 2\mathbb{Z}} \delta_x\right) + \left(\sum_{x\in 2\mathbb{Z}+1} \delta_x\right)$. The subsequent factorizations decompose the measure further. For $n > 1$, this decomposition can also fail to be unique in non-trivial ways. The Lee–Yang polynomial $p(z_1, z_2)$ in Example 1.2 is irreducible, but $p(z_1^2, z_2^2)$ factors as the product of four Lee–Yang polynomials $q_\sigma = 2 + \sigma_1 z_1 + \sigma_2 z_2 + 2\sigma_1\sigma_2 z_1 z_2$ for $(\sigma_1, \sigma_2) \in \{\pm 1\}^2$. Therefore, for any $\ell \in \mathbb{R}^2_+$,

$$\mu_{p,\ell} = \sum_{\sigma\in\{\pm 1\}^2} \mu_{q_\sigma,\ell/2}.$$

[4] Relatively dense means that there exists $R > 0$ such that Λ intersects any ball of radius R.

[5] Lattice in $\mathbb{R}^m \times \mathbb{R}^n$ such that the projection to $\mathbb{R}^m$ is dense and the projection to $\mathbb{R}^n$ is injective

1.4. Main results

To set up notations, let $\mathrm{LY}_{\mathbf{d}}(n)$ denote the set of Lee–Yang polynomials $p \in \mathbb{C}[z_1, \ldots, z_n]$ of degrees $\mathbf{d} = (d_1, \ldots, d_n)$, i.e., p that has degree d_j in every z_j. Let $|\mathbf{d}| = d_1 + d_2 + \cdots + d_n$ denote the total degree. Let

$$\mathbb{T}^n = \{\mathbf{z} \in \mathbb{C}^n : |z_j| = 1, \forall j\}.$$

Theorem 1.9 (Density and maximal gap). *Let $p \in \mathrm{LY}_{\mathbf{d}}(n)$ and $\ell \in \mathbb{R}^n_+$. Then,*

(1) *$\mu_{p,\ell}$ has density $\langle \mathbf{d}, \ell \rangle / 2\pi$ with uniformly bounded error term*:

$$\mu_{p,\ell}([x, x+T]) = \frac{\langle \mathbf{d}, \ell \rangle}{2\pi} T + \mathrm{err}(x, T), \quad \textit{with } |\mathrm{err}(x, T)| \leq |\mathbf{d}|,$$

for all $x \in \mathbb{R}, T > 0$.

(2) *The gap between any pair of consecutive atoms in $\mu_{p,\ell}$ is at most $2\pi|\mathbf{d}|/\langle \mathbf{d}, \ell \rangle$.*

Remark 1.10. The bounds in Theorem 1.9 are tight: for any choice of $n \in \mathbb{N}$ and $\mathbf{d} \in \mathbb{N}^n$, we can construct $\mu_{p,\ell}$ with error term that gets arbitrarily close to $|\mathbf{d}|$ and gaps that get arbitrarily close to $2\pi|\mathbf{d}|/\langle \mathbf{d}, \ell \rangle$. Let $p(\mathbf{z}) = \prod_{j=1}^n (1 - z_j)^{d_j} \in \mathrm{LY}_{\mathbf{d}}(n)$, and let μ_j be the sum of delta masses at $\frac{2\pi}{\ell_j}\mathbb{Z}$, so that $\mu_{p,\ell} = \sum_{j=1}^n d_j \mu_j$. In particular, there is an atom at 0 with coefficient $|\mathbf{d}|$, so $\mu_{p,\ell}([-\varepsilon, \varepsilon]) = |\mathbf{d}|$ for sufficiently small $\varepsilon > 0$, and therefore $\mathrm{err}(-\varepsilon, 2\varepsilon) = |\mathbf{d}| - \langle \mathbf{d}, \ell \rangle \varepsilon / \pi \to 0$ as $\varepsilon \to 0$. Moreover, the gap to the next atom is $\min 2\pi/\ell_j \leq 2\pi|\mathbf{d}|/\langle \mathbf{d}, \ell \rangle$, and if ℓ is arbitrary close to $(2\pi, 2\pi, \ldots, 2\pi)$, then $2\pi|\mathbf{d}|/\langle \mathbf{d}, \ell \rangle - \min 2\pi/\ell_j$ is arbitrary close to zero.

The next theorem shows that, generically, an $\mathbb{N}$-FQ enjoys the desired properties of having uniformly discrete support and having all unit coefficients. For this end, we define the following.

Definition 1.11. We define $\mathrm{mingap}(p) \in [0, 2\pi)$ for $p \in \mathrm{LY}_{\mathbf{d}}(n)$ as follows. When $n = 1$, if p has multiple root, we set $\mathrm{mingap}(p) = 0$; otherwise, we let $\mathrm{mingap}(p)$ be the minimal angle between different roots of p.[6] When $n > 1$, we set $\mathrm{mingap}(p)$ to be the minimum of $\mathrm{mingap}(p_{\mathbf{z}})$ over all $\mathbf{z} \in \mathbb{T}^n$, where $p_{\mathbf{z}}(s) := p(sz_1, sz_2, \ldots, sz_n)$ is a univariate Lee–Yang polynomial for any fixed $\mathbf{z} \in \mathbb{T}^n$.

Theorem 1.12 (Minimal gap for generic FQ). *Let $n \geq 2$ be an integer and let $\mathbf{d} \in \mathbb{Z}^n_{>0}$.*

(1) (Characterization) *For any $\mathbb{Q}$-linearly independent $\ell \in \mathbb{R}^n_+$ and $p \in \mathrm{LY}_{\mathbf{d}}(n)$, the measure $\mu_{p,\ell}$ is non-periodic with unit coefficients and uniformly discrete support if and only if p satisfies*
   - (i) $\nabla p(\mathbf{z}) \neq 0$ *whenever $\mathbf{z} \in \mathbb{T}^n$ such that $p(\mathbf{z}) = 0$, and*
   - (ii) *p has a non-binomial factor.*

(2) (Explicit lower estimate) *The polynomial p has* $\mathrm{mingap}(p) > 0$ *if and only if p satisfies* (i). *Denote the ordered atoms of $\mu_{p,\ell}$ by $(x_j)_{j \in \mathbb{Z}}$. Then*

$$\frac{\mathrm{mingap}(p)}{\ell_{\max}} \leq \inf_{j \in \mathbb{Z}} (x_{j+1} - x_j) \leq \frac{\mathrm{mingap}(p)}{\ell_{\min}}.$$

[6] The roots of a univariate Lee–Yang polynomial lie on the unit circle.

The lower bound holds for any $\ell \in \mathbb{R}^n_+$, the upper bound holds for any $\mathbb{Q}$-linearly independent $\ell \in \mathbb{R}^n_+$, and $\ell_{\max}$ (respectively, $\ell_{\min}$) stands for the largest (respectively, smallest) entry of ℓ.

(3) (Genericity) *The set of Lee–Yang polynomials in $\mathrm{LY}_{\mathbf{d}}(n)$ that satisfy both* (i) *and* (ii) *is a semialgebraic open dense subset of $\mathrm{LY}_{\mathbf{d}}(n)$.*

Furthermore, we provide explicit perturbation taking any $p \in \mathrm{LY}_{\mathbf{d}}(n)$ to a one parameter family polynomials $p_\lambda = p + \sum_{j=1}^{|\mathbf{d}|-1} \lambda^j q_j$, such that $p_0 = p$ and p_λ satisfies (i) *for any $\lambda > 0$.*

The polynomial p from Example 1.2 has $\operatorname{mingap}(p) = 0$, because $p(1,1) = 0$ and $\nabla p(1,1) = 0$. The measure $\mu_{p,\ell}$ then fails to have uniformly discrete support. Figure 2 shows the effect of the perturbation $p \mapsto p_\lambda$. For $\lambda > 0$, $\operatorname{mingap}(p_\lambda) > 0$ and $\mu_{p_\lambda,\ell}$ is uniformly discrete.

Remark 1.13. There is no loss of generality by considering only $\mathbb{Q}$-linearly independent ℓ's, due to [3]. Nevertheless, we point out that if p satisfies (i), then $\mu_{p,\ell}$ will have unit coefficients and uniformly discrete support for any $\ell \in \mathbb{R}^n_+$.

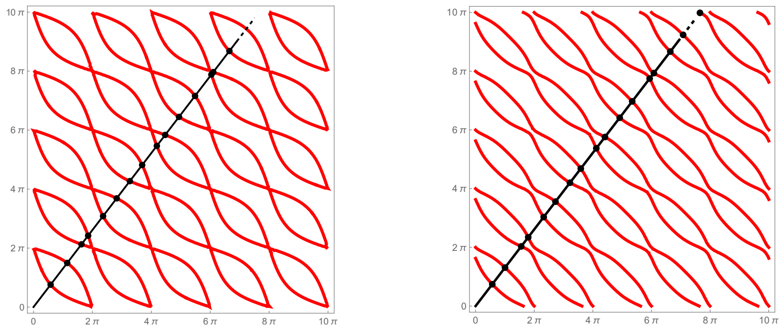


Figure 2. (Left) The singular zero set of p and the line in direction ℓ, as in Figure 1. (Right) The regular zero set of the perturbed polynomial p_λ for $\lambda = 0.2$ and the same line in direction ℓ.

Given $p \in \mathrm{LY}_{\mathbf{d}}(n)$ and $\ell \in \mathbb{R}^n_+$, let $(x_j)_{j\in\mathbb{Z}}$ be the zeros of $p(\exp(ix\ell))$, numbered increasingly with multiplicities (so that a zero of order m appears m times). Then $\mu_{p,\ell} = \sum_{j\in\mathbb{Z}} \delta_{x_j}$. A random measure of the form $\sum_{j\in\mathbb{Z}} \delta_{x_j}$, for random x_j's, is called a *point process*, and it can be defined in terms of the gaps $\Delta_j = x_{j+1} - x_j$, which are often taken to be i.i.d. Δ_j samples from some probability distribution. Next theorem shows that the gaps between atoms in $\mu_{p,\ell}$ obey a well-defined "gap-distribution" $\rho_{p,\ell}$, by which we mean that

$$\frac{1}{N}\sum_{j=1}^{N} \delta_{(x_{j+1}-x_j)} \xrightarrow{\mathfrak{D}} \rho_{p,\ell},$$

where $\xrightarrow{\mathfrak{D}}$ stands for convergence in distribution. Equivalently, for any continuous f,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} f(x_{j+1}-x_j) = \int f \, d\rho_{p,\ell}. \tag{1.1}$$

Theorem 1.14 (Existence of gap distribution). *Every $\mathbb{N}$-valued FQ μ has a well-defined gap distribution ρ with the following properties*:

(1) *It has finitely many atoms, say $(r_j)_{j=1}^M$, such that $\rho = \rho_{\mathrm{ac}} + \sum_{j=1}^M \rho(\{r_j\})\delta_{r_j}$, and ρ_{ac} is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$.*

(2) *$\rho_{\mathrm{ac}} = 0$ if and only if μ is periodic.*

(3) *If $\Delta \geq 0$ is any gap between consecutive atoms of μ, then $\rho(I) > 0$ for any open neighborhood $I \subset \mathbb{R}$ of Δ.*

(4) *The average gap $\mathbb{E}(\rho)$ is the reciprocal of its density.*

When $\mu = \mu_{p,\ell}$ we denote the resulting gap distribution $\rho_{p,\ell}$. Together with Theorem 1.9, part (4) implies that the average gap of $\mu_{p,\ell}$ is given by $\mathbb{E}(\mu_{p,\ell}) = 2\pi/\langle \mathbf{d}, \ell\rangle$.

As discussed above, every $\mathbb{N}$-valued FQ μ can be written as $\mu_{p,\ell}$ for some Lee–Yang polynomial p and some vector $\ell \in \mathbb{R}^n_+$, where ℓ has linearly independent entries over $\mathbb{Q}$. We explore the dependence of the gap distribution $\rho_{p,\ell}$ on both the polynomial p and the vector ℓ. First we note that the gap distribution is independent of torus actions on p, and give conditions on the factorization of p under which the $\rho_{p,\ell}$ has atoms.

Theorem 1.15 (p-dependence of the gap distribution). *Suppose $p \in \mathrm{LY}_{\mathbf{d}}(n)$ and let $\ell \in \mathbb{R}^n_+$ with $\mathbb{Q}$-linearly independent entries. Then*

(1) *for any fixed $\mathbf{x} \in \mathbb{R}^n$, the polynomial $q(\mathbf{z}) = p(\exp(i\mathbf{x})\mathbf{z}) = p(e^{ix_1}z_1, \ldots, e^{ix_n}z_n)$ is in $\mathrm{LY}_{\mathbf{d}}(n)$, and $\rho_{q,\ell} = \rho_{p,\ell}$.*

(2) *The distribution $\rho_{p,\ell}$ has an atom at $\Delta \geq 0$ if and only if there are two irreducible factors of p, say q_i and q_j, such that $q_j(\mathbf{z}) = q_i(\exp(i\Delta\ell)\mathbf{z})$. Moreover,*

(3) *if $\Delta > 0$ and $q_i = q_j$, namely $q_i(\mathbf{z}) = q_i(\exp(i\Delta\ell)\mathbf{z})$, then q_i is binomial.*

Corollary 1.16. *Suppose $p \in \mathrm{LY}_{\mathbf{d}}(n)$ and let $\ell \in \mathbb{R}^n_+$ with $\mathbb{Q}$-linearly independent entries.*

(1) *If p is irreducible and not binomial, then $\rho_{p,\ell}$ is absolutely continuous.*

(2) *If p is binomial, then $\rho_{p,\ell}$ is the atomic measure at $2\pi/\langle \mathbf{d}, \ell\rangle$.*

(3) *$\rho_{p,\ell}$ has an atom at 0 if and only if p has a square factor.*

(4) *Suppose that p has $N + M$ distinct irreducible factors, M which are binomial and N non-binomial. Then, $\rho_{p,\ell}$ has at most $\binom{N}{2} + M + 1$ atoms.*

Next we show that the gap distribution $\rho_{p,\ell}$ varies continuously in ℓ when we restrict to vectors ℓ with $\mathbb{Q}$-linearly independent entries. For arbitrary $\ell \in \mathbb{R}^n_+$, this gives rise to a well-defined limiting distribution $\nu_{p,\ell}$ that agrees when $\rho_{p,\ell}$ when ℓ has $\mathbb{Q}$-linearly independent entries. The limiting measure $\nu_{p,\ell}$ is defined explicitly in Definition 9.1.

Theorem 1.17 (ℓ-dependence of the gap distribution). *Let $p \in \mathrm{LY}_{\mathbf{d}}(n)$ and $\ell \in \mathbb{R}^n_+$. Then the gap distribution $\rho_{p,\ell}$ is supported inside $[0, 2\pi|\mathbf{d}|/\langle \mathbf{d}, \ell\rangle]$. There is a distribution $\nu_{p,\ell}$ such that, for any converging sequence $\ell^{(j)} \to \ell$ in which each $\ell^{(j)}$ has $\mathbb{Q}$-linearly independent entries,*

$$\rho_{p,\ell^{(j)}} \xrightarrow{\mathcal{D}} \nu_{p,\ell}.$$

In particular, $\nu_{p,\ell} = \rho_{p,\ell}$ whenever ℓ has $\mathbb{Q}$-linearly independent entries.

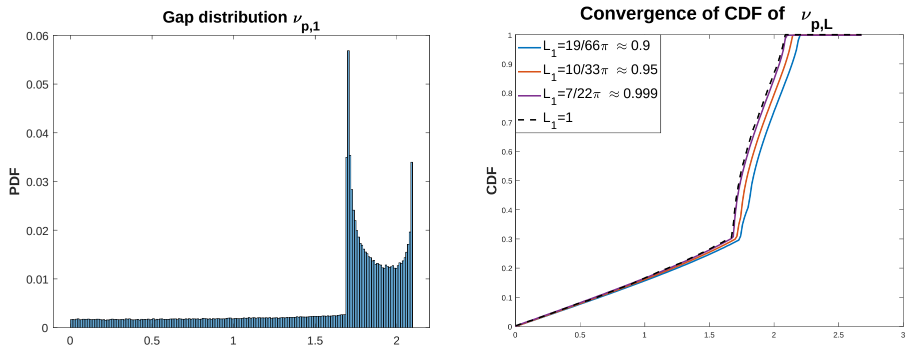


Figure 3. Example of $\rho_{p,\ell} \xrightarrow{\mathcal{D}} \nu_{p,\mathbf{1}}$. The gap distributions for p as in Figure 1, and $\ell = (L_1, 1)$ with L_1 converging to 1. (Left) The probability distribution function of $\nu_{p,\mathbf{1}}$. (Right) The cumulative distribution functions of $\nu_{p,\mathbf{1}}$ (dashed) and of $\rho_{p,\ell}$ for three different values of L_1. Each $\rho_{p,\ell}$ was computed from the gaps in the interval $[0, 10^4]$, while $\nu_{p,\ell}$ was computed as in Theorem 1.19, by sampling 10^4 random points on the torus.

A particularly interesting case is the limit $\nu_{p,\mathbf{1}}$ for $\ell = \mathbf{1} := (1, 1, \ldots, 1)$, which can be calculated explicitly, as follows. Figure 4 displays the distributions $\nu_{p,\mathbf{1}}$ for two important examples of Lee–Yang polynomials p.

Definition 1.18. Let $p \in \mathrm{LY}_N(1)$ be a univariate Lee–Yang polynomial of degree N, and denote its roots by $\{e^{i\theta_j}\}_{j=1}^N$, with $0 \le \theta_1 \le \cdots \le \theta_N < 2\pi$. By convention, $\theta_{N+1} = \theta_1 + 2\pi$. Then the *gap distribution* of p is a probability measure on $[0, 2\pi]$ given by

$$\mathrm{gaps}(p) = \frac{1}{N} \sum_{j=1}^{N} \delta_{\theta_{j+1}-\theta_j}.$$

If U is a unitary matrix, then $p(s) = \det(s - U)$ and $q(s) = \det(1 - sU)$ have the same gap distribution, and we denote it by $\mathrm{gaps}(U)$.

For a fixed $p \in \mathrm{LY}_{\mathbf{d}}(n)$ and a fixed point $\mathbf{x} \in [0, 2\pi]^n$, define the univariate polynomial $p_{\mathbf{x}}(s) := p(se^{ix_1}, se^{ix_2}, \ldots, se^{ix_n})$ so that $p_{\mathbf{x}} \in \mathrm{LY}_N(1)$ with $N = |\mathbf{d}|$. We may then take $\mathbf{x}$ uniformly at random.

Theorem 1.19 ($\ell \to \mathbf{1}$). *Let $p \in \mathrm{LY}_{\mathbf{d}}(n)$. Let $\mathbf{x}$ be a uniformly random point in $[0, 2\pi]^n$. Then $\nu_{p,\mathbf{1}}$, for $\ell = \mathbf{1}$, is given by*

$$\nu_{p,\mathbf{1}} = \mathbb{E}[\mathrm{gaps}(p_{\mathbf{x}})].$$

Namely, for any sequence $\ell^{(j)} \to \mathbf{1}$ such that each $\ell^{(j)}$ has $\mathbb{Q}$-linearly independent entries,

$$\lim_{j\to\infty} \int f \, d\rho_{p,\ell^{(j)}} = \frac{1}{(2\pi)^n} \int_{\mathbf{x}\in[0,2\pi]^n} \left[\frac{1}{|\mathbf{d}|} \sum_{j=1}^{|\mathbf{d}|} f(\theta_{j+1}(\mathbf{x}) - \theta_j(\mathbf{x})) \right] d\mathbf{x}, \quad \forall f \in C(\mathbb{R}),$$

where $\{e^{i\theta_j(\mathbf{x})}\}_{j=1}^{|\mathbf{d}|}$ are the ordered roots of $p_{\mathbf{x}}$ for every $\mathbf{x}$, and $\theta_{|\mathbf{d}|+1}(\mathbf{x}) := \theta_1(\mathbf{x}) + 2\pi$.

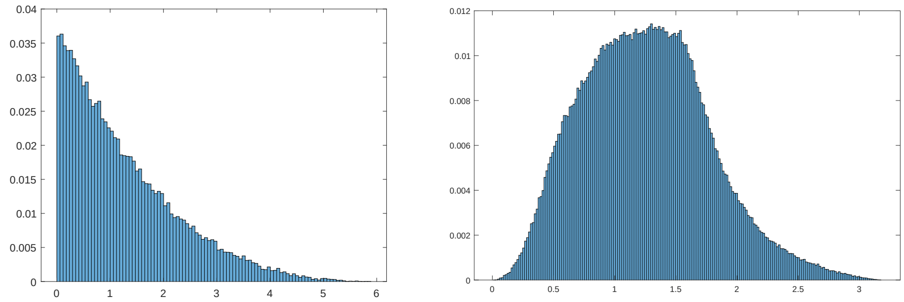


Figure 4. (Left) $\nu_{p,\mathbf{1}}$ for $p(\mathbf{z}) = \prod_{j=1}^{5}(z_j - 1)$. (Right) $\nu_{p,\mathbf{1}}$ for $p(\mathbf{z}) = \det(1 - \operatorname{diag}(\mathbf{z})U)$ for a fixed 5×5 unitary matrix U, chosen at random (Haar uniformly). Both calculated with 10^4 random points from the torus.

Using Theorem 1.19, we can provide examples of limiting gap distributions that correspond to the following special distributions.

Example 1.20 (Poisson). If

$$p(z_1, \dots, z_n) = \prod_{j=1}^{n}(1 - z_j) \quad \text{and} \quad \ell = \frac{2\pi}{n}\mathbf{1},$$

then $\nu_{p,\ell}$ is the distribution of gaps between n random points in a circle of circumference $n/(2\pi)$, chosen uniformly and independently. It is well known that this distribution converges to the gap distribution of a Poisson process, in the limit $n \to \infty$.

Example 1.21 (CUE). Given a fixed unitary $n \times n$ matrix u, let $p_u(z_1, \dots, z_n) := \det(1 - \operatorname{diag}(z_1, \dots, z_n)u)$. Then,

$$\nu_{p_u,\mathbf{1}} = \mathbb{E}\left[\operatorname{gaps}(\operatorname{diag}(\exp(i\mathbf{x}))u)\right], \quad \mathbf{x} \sim U([0, 2\pi]^n).$$

For a random u, Haar uniformly from $U(n)$, the *empirical gap distribution* is

$$\mathbb{E}(\nu_{p_u,\mathbf{1}}) = \mathbb{E}\left[\operatorname{gaps}(\operatorname{diag}(\exp(i\mathbf{x}))u)\right] = \mathbb{E}\left[\operatorname{gaps}(u)\right], \quad u \sim \operatorname{Haar}(U(n)).$$

The distribution $\mathbb{E}\left[\operatorname{gaps}(u)\right]$ for $u \sim \operatorname{Haar}(U(n))$ is well known, and when scaled to have average 1, by taking $\mathbb{E}(\nu_{p_u,\ell})$ with $\ell = \frac{2\pi}{n}\mathbf{1}$, it converges to the CUE (circular unitary ensemble) gap distribution as $n \to \infty$.

The paper is organized as follows. The first two sections provide background and preliminary results, Section 2 on crystalline measures and FQ's, and Section 3 on Lee–Yang polynomials and real stable polynomials. The torus zero sets of Lee–Yang polynomials are analyzed in Section 4. Theorem 1.9, the growth rate and upper bound on the gaps, is proved in Section 5. In Section 6, an ergodic dynamical system is defined on the torus zero set, which is being used in the subsequent sections. Theorem 1.5, decomposition and non-periodicity, is proven in Section 7. Theorem 1.12, minimal gap and genericity, is proved in Section 8. Section 9 focus on gap distributions, in which Theorems 1.14, 1.15, 1.17, and 1.19 are proved.

2. Preliminaries on crystalline measures and FQ

A *Schwartz function* on $\mathbb{R}$ is a smooth function $f \in C^\infty(\mathbb{R}, \mathbb{C})$ that decays, as $|x| \to \infty$, faster than any polynomial in $|x|$, and so does any of its derivatives. The *Schwartz space* $\mathcal{S}(\mathbb{R})$ is the infinite dimensional vector space of Schwartz functions. It can be defined in terms of the seminorms $\|f\|_{n,m} := \sup_{x\in\mathbb{R}} |x^n (d/dx)^m f(x)|$:

$$\mathcal{S}(\mathbb{R}) := \{f \in C^\infty(\mathbb{R}, \mathbb{C}) : \|f\|_{n,m} < \infty \text{ for all } m, n \in \mathbb{Z}_{\geq 0}\},$$

and it is a complete metric space with respect to the metric

$$d(f, g) := \sum_{n,m=0}^{\infty} \frac{\|f\|_{n,m}}{2^{n+m}(1 + \|f\|_{n,m})}.$$

A ($\mathbb{C}$-valued) Borel measure μ on $\mathbb{R}$ is *tempered* if $\langle f, \mu\rangle := \int f \, d\mu$ is finite for all $f \in \mathcal{S}(\mathbb{R})$. The vector space of tempered measures is the dual of $\mathcal{S}(\mathbb{R})$ and is denoted by $\mathcal{S}'(\mathbb{R})$. The *Fourier transform*

$$\mathcal{F}(f) := \widehat{f}, \quad \text{with} \quad \widehat{f}(k) := \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx,$$

is a linear automorphism of $\mathcal{S}(\mathbb{R})$, and it defines an automorphism on the dual space. Given a measure $\mu \in \mathcal{S}'(\mathbb{R})$, its Fourier transform is the measure $\widehat{\mu} \in \mathcal{S}'(\mathbb{R})$ defined by $\langle f, \widehat{\mu}\rangle := \langle \widehat{f}, \mu\rangle$ for all $f \in \mathcal{S}(\mathbb{R})$. Let $\delta_x \in \mathcal{S}'(\mathbb{R})$ denote the atom at $x \in \mathbb{R}$ (also known as a Dirac delta at x), which is defined by $\langle f, \delta_x\rangle := f(x)$. We say that a measure μ is *discrete* if it is supported on a discrete (locally finite) set, in which case it can be written as

$$\mu = \sum_{x\in\Lambda} a_x \delta_x := \lim_{T\to\infty} \sum_{x\in\Lambda\cap[-T,T]} a_x \delta_x, \tag{2.1}$$

with complex coefficients $a_x \in \mathbb{C}$ and discrete support $\Lambda \subset \mathbb{R}$. Whenever we write an infinite sum as in (2.1), it should be understood as the $T \to \infty$ limit of the $[-T, T]$ truncated sum. One can check that a discrete measure μ is tempered, i.e., $\mu \in \mathcal{S}'(\mathbb{R})$, if and only if $\mu([-T, T])$ is bounded by some polynomial in T, namely if there exist $C > 0$ and $m \in \mathbb{N}$ such that

$$\Bigg|\sum_{x\in\Lambda\cap[-T,T]} a_x\Bigg| \leq C(1 + T^m), \quad \forall T > 0.$$

If μ is a complex valued measure given by (2.1), then $|\mu| := \sum_{x\in\Lambda} |a_x|\delta_x$.

Definition 2.1 (FQ and $\mathbb{N}$-FQ, [15, 17]). A *crystalline measure* is a discrete measure that is a tempered distribution and whose Fourier transform is also discrete[7]. A *Fourier quasicrystal* (FQ) is a crystalline measure μ with the further restriction that $|\mu|$ and $|\widehat{\mu}|$ are also tempered. To write it explicitly, μ is an FQ if there exist discrete (locally finite) sets Λ

[7]The Fourier transform is tempered by definition.

and S, and complex coefficients $(a_x)_{x\in\Lambda}$ and $(c_k)_{k\in S}$, such that

$$\mu = \sum_{x\in\Lambda} a_x\,\delta_x, \quad \hat{\mu} = \sum_{k\in S} c_k\delta_k, \quad \text{and} \tag{2.2}$$
$$\sum_{x\in\Lambda\cap[-T,T]} |a_x| + \sum_{k\in S\cap[-T,T]} |c_k| \le C(1+T^m),$$

for some $C > 0, m \in \mathbb{N}$, for all $T > 0$. When $a_x \in \mathbb{N}$ for all $x \in \Lambda$, we call μ an $\mathbb{N}$-FQ.

For example, the measure $\mu = \sum_{x\in\Lambda}\delta_x$ for any periodic Λ is an FQ due to the Poisson summation formula.

3. Preliminaries on Lee–Yang polynomials

Let $\mathbb{C}[\mathbf{z}]$ denote the space $\mathbb{C}[z_1,\dots,z_n]$ of polynomials in indeterminates $\mathbf{z} = (z_1,\dots,z_n)$. For a nonnegative integer vector $\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_n) \in \mathbb{Z}^n_{\ge}$, we use $\mathbf{z}^{\boldsymbol{\alpha}}$ to denote the monomial $\prod_{j=1}^n z_j^{\alpha_j}$. The degree of a polynomial $p = \sum_{\boldsymbol{\alpha}} a_{\boldsymbol{\alpha}}\mathbf{z}^{\boldsymbol{\alpha}}$ in $\mathbb{C}[\mathbf{z}]$ in the variable z_j, denoted $\deg_j(p)$, is the maximum value of α_j appearing in a monomial with nonzero coefficient $a_{\boldsymbol{\alpha}} \neq 0$. For $\mathbf{d} = (d_1,\dots,d_n) \in \mathbb{N}^n$, let $\mathbb{C}[\mathbf{z}]_{\le\mathbf{d}}$ denote the $\mathbb{C}$-vector space of polynomials with $\deg_j(p) \le d_j$ in each variable z_j, i.e.,

$$\mathbb{C}[\mathbf{z}]_{\le\mathbf{d}} = \Big\{ \sum_{0\le\boldsymbol{\alpha}\le\mathbf{d}} a_{\boldsymbol{\alpha}}\mathbf{z}^{\boldsymbol{\alpha}} : a_{\boldsymbol{\alpha}} \in \mathbb{C}\Big\},$$

where $\boldsymbol{\alpha} \le \mathbf{d}$ is taken coordinate-wise.

Given a circular region in the complex plane $C \subseteq \mathbb{C}$, we say that p is *stable* with respect to C if p has no zeros in C^n. For us, the circular regions of interest will be the upper half plane $\mathcal{H}_+ = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$, the lower half plane $\mathcal{H}_- = \{z \in \mathbb{C} : \operatorname{Im}(z) < 0\}$, and the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Stability with respect to $\mathbb{D}$ is often known as *Schur stability*. We use $\mathbb{T}$ to denote the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ in $\mathbb{C}$, and $\overline{\mathbb{D}}$ for the closed unit disk $\mathbb{D} \cup \mathbb{T}$. Of particular interest are polynomials stable with respect $\mathbb{D}$ and its inverse $\mathbb{C} \setminus \overline{\mathbb{D}}$.

Definition 3.1. We say that $p \in \mathbb{C}[\mathbf{z}]$ is a *Lee–Yang polynomial* if it is stable with respect to both $\mathbb{D}$ and $\mathbb{C} \setminus \overline{\mathbb{D}}$, and use $\mathrm{LY}_{\mathbf{d}}$ to denote the set of Lee–Yang polynomials in $\mathbb{C}[\mathbf{z}]_{\le\mathbf{d}}$ of multidegree equal to $\mathbf{d}$. That is, $\mathrm{LY}_{\mathbf{d}}$ is the set of polynomials $p = \sum_{0\le\boldsymbol{\alpha}\le\mathbf{d}} a_{\boldsymbol{\alpha}}\mathbf{z}^{\boldsymbol{\alpha}}$ so that $\deg_j(p) = d_j$ for all j with the property that $p(z_1,\dots,z_n) \neq 0$ whenever $|z_j| < 1$ for all j or $|z_j| > 1$ for all j. When n is not clear from the context, we will write $\mathrm{LY}_{\mathbf{d}}(n)$.

One property of stability that we will often use is that the set of multivariate polynomials that is stable with respect to either an open disk or halfplane is closed in the Euclidean topology on $\mathbb{C}[\mathbf{z}]_{\le\mathbf{d}}$. This follows immediately from Hurwitz's theorem.

Theorem (Hurwitz's theorem, see Theorem 1.3.8 of [20]). *Let $\Omega \subseteq \mathbb{C}^m$ be a connected open set, and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions, each analytic and nonvanishing on Ω, that converges to a limit f uniformly on compact subsets of Ω. Then f is either nonvanishing on Ω or identically zero.*

Möbius transformations map between circular regions in $\mathbb{C}$. Given a tuple of Möbius transformations $\boldsymbol{\phi} = (\phi_j(z_j))_j$, where $\phi_j(z) = \frac{a_j z + b_j}{c_j z + d_j}$, and a polynomial $p \in \mathbb{C}[\mathbf{z}]_{\leq \mathbf{d}}$, define

$$\boldsymbol{\phi} \cdot p = \prod_{j=1}^{n} (c_j z_j + d_j)^{\deg_j(p)} \cdot p(\phi_1(z_1), \ldots, \phi_n(z_n)) \in \mathbb{C}[\mathbf{z}]_{\leq \mathbf{d}}.$$

We will sometimes abuse notation and, for a single Möbius transformations $\phi(z) = \frac{az+b}{cz+d}$, use $\phi \cdot p$ to denote $(\phi, \ldots, \phi) \cdot p$. Then p is stable with respect to a region C if and only if $\phi \cdot p$ is stable with respect to $\phi^{-1}(C)$. See Lemma 1.8 in [10]. We will often fix ϕ to be a Möbius transformation taking $\mathcal{H}_+$ to $\mathbb{D}$. Explicitly, for fixed $\theta \in [0, 2\pi)$, consider the pair

$$\phi(z) = \frac{e^{i\theta}(z - i)}{z + i} \quad \text{and} \quad \phi^{-1}(z) = \frac{-i(z + e^{i\theta})}{z - e^{i\theta}},$$
$$\text{with } \rho(x) = \phi^{-1}(e^{ix}) = \cot\left(\frac{\theta - x}{2}\right). \tag{3.1}$$

The derivative of ρ, $\rho'(x) = \frac{1}{2}\csc^2((\theta - x)/2)$, is strictly positive everywhere it is defined, which is for $x \notin \theta + 2\pi\mathbb{Z}$. In particular, we can always choose θ so that ρ and its derivative are defined at any finite set $a_1, \ldots, a_n \in \mathbb{R}$.

The following are straightforward from the definitions of stability.

Proposition 3.2. *For $p \in \mathbb{C}[\mathbf{z}]_{\leq \mathbf{d}}$, the following are equivalent:*

(a) *p is a Lee–Yang polynomial,*

(b) *for every $\ell = (\ell_1, \ldots, \ell_n) \in \mathbb{R}_+^n$ and $x \in \mathbb{C}$, $p(\exp(ix\ell)) = 0$ implies $x \in \mathbb{R}$,*

(c) *for ϕ as in (3.1), $\phi \cdot p$ is stable with respect to $\mathcal{H}_+$ and $\mathcal{H}_-$.*

In order to understand polynomials stable with respect to $\mathbb{D}$ and $\mathbb{C} \setminus \overline{\mathbb{D}}$, we first recall some useful facts about real polynomials stable with respect to $\mathcal{H}_+$.

We define the support of a polynomial $q = \sum_{\boldsymbol{\alpha}} a_{\boldsymbol{\alpha}} \mathbf{z}^{\boldsymbol{\alpha}}$ to be the collection of exponents of monomials appearing in q, i.e., $\operatorname{supp}(q) = \{\boldsymbol{\alpha} \in \mathbb{Z}_{\geq 0}^n : a_{\boldsymbol{\alpha}} \neq 0\}$. For any vector $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{R}^n$, define the $\mathbf{w}$-initial form of q to be the sum over all terms in q maximizing $\langle \mathbf{w}, \boldsymbol{\alpha} \rangle$. That is, we can define

$$\deg_{\mathbf{w}}(q) = \max_{\boldsymbol{\alpha} \in \operatorname{supp}(q)} \langle \mathbf{w}, \boldsymbol{\alpha} \rangle,$$
$$\operatorname{in}_{\mathbf{w}}(q) = (t^{\deg_{\mathbf{w}}(q)} q(t^{-w_1} z_1, \ldots, t^{-w_n} z_n))|_{t=0} = \sum_{\boldsymbol{\alpha} \in A} a_{\boldsymbol{\alpha}} \mathbf{z}^{\boldsymbol{\alpha}},$$

where A is the subset of $\boldsymbol{\alpha} \in \operatorname{supp}(p)$ maximizing $\langle \mathbf{w}, \boldsymbol{\alpha} \rangle$.

Proposition 3.3. *Let $q = \sum_{\boldsymbol{\alpha}} a_{\boldsymbol{\alpha}} \mathbf{z}^{\boldsymbol{\alpha}} \in \mathbb{C}[z_1, \ldots, z_n]$ be stable with respect to $\mathcal{H}_+$. Then*

(a) *for any $\mathbf{w} \in \mathbb{R}^n$, $\operatorname{in}_{\mathbf{w}}(q)$ is stable with respect to $\mathcal{H}_+$,*

(b) *for any $\mathbf{a}_1, \ldots, \mathbf{a}_m \in \mathbb{R}_{\geq 0}^n$ and $\mathbf{b} \in \mathbb{R}^n$, the polynomial $q(\mathbf{b} + y_1 \mathbf{a}_1 + \cdots + y_m \mathbf{a}_m) \in \mathbb{C}[y_1, \ldots, y_m]$ is stable with respect to $\mathcal{H}_+$,*

(c) *if q is homogeneous, then all its coefficients have the same phase, and*

(d) *if $\mathbf{b} \in \mathbb{R}^n$ is a real zero of q of multiplicity m, namely $q(\mathbf{b}) = 0$ and $\partial^{\boldsymbol{\alpha}} q(\mathbf{b}) = 0$ for all $|\boldsymbol{\alpha}| < m$, then the nonzero entries of $\{\partial^{\boldsymbol{\alpha}} q(\mathbf{b}) : |\boldsymbol{\alpha}| = m\}$ all have the same phase.*

Proof. (a) Note that for any $t \in \mathbb{R}_{>0}$, the polynomial $t^{\deg_{\mathbf{w}}(q)} q(t^{-w_1} z_1, \ldots, t^{-w_n} z_n)$ is stable with respect to $\mathcal{H}_+$. By Hurwitz's theorem, the set of stable polynomials is closed in the Euclidean topology on $\mathbb{C}[\mathbf{z}]_{\leq \mathbf{d}}$, and taking the limit as $t \to 0$ shows that $\mathrm{in}_{\mathbf{w}}(q)$ is stable with respect to $\mathcal{H}_+$.

(b) First, suppose $\mathbf{a}_1, \ldots, \mathbf{a}_m \in \mathbb{R}^n_+$. If $\mathrm{Im}(y_j) > 0$ for all j, then the imaginary part of $\mathbf{b} + \sum_{i=1}^m y_i \mathbf{a}_i$ belongs to $\mathbb{R}^n_+$, and so, $q(\mathbf{b} + \sum_{i=1}^m y_i \mathbf{a}_i) \neq 0$. Hurwitz's theorem then shows that the polynomial $q(\mathbf{b} + \sum_{i=1}^m y_i \mathbf{a}_i) \neq 0$ is stable for any $\mathbf{a}_1, \ldots, \mathbf{a}_m \in \mathbb{R}^n_{\geq 0}$.

(c) This is the content of Theorem 6.1 in [11].

(d) Let m denote the multiplicity of q at $\mathbf{z} = \mathbf{b}$. Note that by replacing $q(\mathbf{z})$ with $q(\mathbf{z} + \mathbf{b})$, it suffices to address the case $\mathbf{b} = (0, \ldots, 0)$. The notation $\boldsymbol{\alpha}! = \prod_{j=1}^n \alpha_j!$ allows to write $\partial^{\boldsymbol{\alpha}} q(0) = \boldsymbol{\alpha}! \cdot a_{\boldsymbol{\alpha}}$, and so it is enough to prove that all nonzero $a_{\boldsymbol{\alpha}}$, with $|\boldsymbol{\alpha}| = m$, share the same phase. Fix $\mathbf{w} = (-1, \ldots, -1)$. Because $\boldsymbol{\alpha}! \cdot a_{\boldsymbol{\alpha}} = \partial^{\boldsymbol{\alpha}} q(0) = 0$ for all $|\boldsymbol{\alpha}| < m$, then the $\boldsymbol{\alpha} \in \mathrm{supp}(p)$ that maximize $\langle \mathbf{w}, \boldsymbol{\alpha} \rangle = -|\boldsymbol{\alpha}|$ are those with $|\boldsymbol{\alpha}| = m$, and in particular, $\mathrm{in}_{\mathbf{w}}(q) = \sum_{|\boldsymbol{\alpha}|=m} a_{\boldsymbol{\alpha}} \mathbf{z}^{\boldsymbol{\alpha}}$. By parts (a) and (b), this polynomial is stable and so all of its nonzero coefficients have the same phase, which proves the claim. ∎

We translate this statement for derivatives of trigonometric polynomials of the form $F(\mathbf{x}) = p(\exp(i\mathbf{x}))$, where $p \in \mathrm{LY}_{\mathbf{d}}$.

First, we need a technical lemma on derivatives of compositions.

Proposition 3.4 (Multivariate chain rule). *Let $\varphi: \mathbb{C} \to \mathbb{C}$ be a meromorphic function such that $\varphi'(x)$ is nonzero wherever defined. Consider $f(\mathbf{x}) = g(\varphi(\mathbf{x}))$, where φ is applied coordinate-wise. For any $\mathbf{a} \in \mathbb{C}^n$ at which φ is defined, the multiplicity m of f at $\mathbf{a}$ equals the multiplicity of g at $\mathbf{b} = \varphi(\mathbf{a})$, and for any $\boldsymbol{\alpha} \in \mathbb{N}^n$ with $|\boldsymbol{\alpha}| = m$,*

$$\partial^{\boldsymbol{\alpha}} f(\mathbf{a}) = \partial^{\boldsymbol{\alpha}} g(\mathbf{b}) \cdot \prod_{j=1}^n \varphi'(a_j)^{\alpha_j}.$$

Proof. The symbolic expansion of $\partial^{\boldsymbol{\alpha}} g(\varphi(\mathbf{a}))$ using the chain rule will be a sum of products of factors $\partial^{\boldsymbol{\beta}} g(\mathbf{b})$ and $\rho^{(k)}(a_j)$ for some $|\boldsymbol{\beta}| \leq |\boldsymbol{\alpha}|$ and $k \leq \alpha_j$. The unique such term involving $\partial^{\boldsymbol{\alpha}} g$ is $\partial^{\boldsymbol{\alpha}} g(\mathbf{b}) \prod_{j=1}^n \varphi'(a_j)^{\alpha_j}$, and all others have a factor of $\partial^{\boldsymbol{\beta}} g(\mathbf{b})$ with $|\boldsymbol{\beta}| < |\boldsymbol{\alpha}|$. If m is the multiplicity of g at $\mathbf{b}$, then $\partial^{\boldsymbol{\beta}} g(\mathbf{b}) = 0$ for all $|\boldsymbol{\beta}| < m$ and $\partial^{\boldsymbol{\alpha}} g(\mathbf{b}) \neq 0$ for some $|\boldsymbol{\alpha}| = m$. The calculation above shows that $\partial^{\boldsymbol{\beta}} f(\mathbf{a}) = 0$ for all $|\boldsymbol{\beta}| < m$ and $\partial^{\boldsymbol{\alpha}} f(\mathbf{a}) \neq 0$. ∎

Proposition 3.5. *Let $p \in \mathrm{LY}_{\mathbf{d}}$ and define $F: \mathbb{C}^n \to \mathbb{C}$ by $F(\mathbf{x}) = p(\exp(i\mathbf{x}))$. If $\mathbf{a} \in \mathbb{R}^n$ is a zero of F of multiplicity m, then nonzero elements of $\{\partial^{\boldsymbol{\alpha}} F(\mathbf{a}) : |\boldsymbol{\alpha}| = m\}$ have the same phase.*

Proof. Let ϕ be a Möbius transformation taking $\mathcal{H}_+$ to $\mathbb{D}$, as in (3.1), with θ such that $e^{i\theta} \neq e^{ia_j}$ for all the coordinates $e^{ia_1}, \ldots, e^{ia_n}$ of $\exp(i\mathbf{a})$. By Proposition 3.2, $q(\mathbf{z}) = \phi \cdot p(\mathbf{z})$ is stable with respect to $\mathcal{H}_+$ and $\mathcal{H}_-$. Then

$$p(\mathbf{z}) = \phi^{-1} \cdot q(\mathbf{z}) = r(\mathbf{z}) \cdot q(\phi^{-1}(\mathbf{z})) \quad \text{and} \quad F(\mathbf{x}) = p(\exp(i\mathbf{x})) = r(\exp(i\mathbf{x})) \cdot q(\rho(\mathbf{x})),$$

where $\rho(\mathbf{x}) = (\cot(\frac{a_1 - x_1}{2}), \ldots, \cot(\frac{a_n - x_n}{2}))$ and $r(\exp(i\mathbf{x})) = \prod_{j=1}^n (e^{ix_j} - e^{ia_j})^{d_j}$. In particular, $r(\exp(i\mathbf{a})) \neq 0$, so $q(\rho(\mathbf{a})) = 0$, hence $\rho(\mathbf{a})$ is a zero of q. An induction argument shows that $\mathbf{a}$ must be a zero of $q(\rho(\mathbf{x}))$ of multiplicity m. That is, $\partial^{\boldsymbol{\alpha}} q(\rho(\mathbf{x}))$ is zero at $\mathbf{x} = \mathbf{a}$ for all $|\boldsymbol{\alpha}| < m$ and nonzero for some $|\boldsymbol{\alpha}| = m$. To do so, suppose that $\partial^{\boldsymbol{\alpha}} q(\rho(\mathbf{x}))|_{\mathbf{x}=\mathbf{a}} = 0$ for all $|\boldsymbol{\alpha}| \le m' - 1$ for $m' < m$. Then, for any $\boldsymbol{\alpha} \in \mathbb{Z}^n_{\ge 0}$ with $|\boldsymbol{\alpha}| = m'$,

$$\partial^{\boldsymbol{\alpha}} F(\mathbf{x}) = \sum_{\boldsymbol{\beta}+\boldsymbol{\gamma}=\boldsymbol{\alpha}} \partial^{\boldsymbol{\beta}} r(\exp(i\mathbf{x}))\, \partial^{\boldsymbol{\gamma}} q(\rho(\mathbf{x})), \tag{3.2}$$

so

$$0 = \partial^{\boldsymbol{\alpha}} F(\mathbf{x})|_{\mathbf{x}=\mathbf{a}} = r(\exp(i\mathbf{a}))\partial^{\boldsymbol{\alpha}} q(\rho(\mathbf{x}))|_{\mathbf{x}=\mathbf{a}}.$$

Since $r(\exp(i\mathbf{a})) \neq 0$, then $\partial^{\boldsymbol{\alpha}} q(\rho(\mathbf{x}))|_{\mathbf{x}=\mathbf{a}} = 0$ for every $|\boldsymbol{\alpha}| = m'$, and by induction for any $|\boldsymbol{\alpha}| < m$. Together with Proposition 3.4, this gives that

$$\partial^{\boldsymbol{\alpha}} F(\mathbf{a}) = r(\exp(i\mathbf{a}))\, \partial^{\boldsymbol{\alpha}} q(\rho(\mathbf{x}))|_{\mathbf{x}=\mathbf{a}} = r(\exp(i\mathbf{a}))(\partial^{\boldsymbol{\alpha}} q)|_{\mathbf{z}=\rho(\mathbf{a})} \cdot \prod_{j=1}^{n} \rho'(a_j)^{\alpha_j}.$$

The phase of the non-zero factor $r(\exp(i\mathbf{a})) \prod_{j=1}^n \rho'(a_j)^{\alpha_j}$ is independent of $\boldsymbol{\alpha}$, since $\rho'(a_j)$ is positive for all j, so the nonzero elements of $\{\partial^{\boldsymbol{\alpha}} F(\mathbf{a}) : |\boldsymbol{\alpha}| = m\}$ have the same phase because the nonzero elements of $\{\partial^{\boldsymbol{\alpha}} q|_{\mathbf{z}=\rho(\mathbf{a})} : |\boldsymbol{\alpha}| = m\}$ have the same phase, by Proposition 3.3 (d). ■

Lemma 3.6. *For $t \in \mathbb{R}$, $\ell \in \mathbb{R}^n_{\ge 0}$ and $p \in \mathrm{LY}_{\mathbf{d}}(n)$, the following coincide:*

(a) *the multiplicity of $t \in \mathbb{R}$ as a zero of the function $f(t) = p(\exp(it\ell))$,*

(b) *the multiplicity of $\mathbf{x} = t\ell$ as a zero of $F(\mathbf{x}) = p(\exp(i\mathbf{x}))$,*

(c) *the multiplicity of $\mathbf{z} = \exp(it\ell) \in \mathbb{T}^n$ as a zero of $p(\mathbf{z})$, and*

(d) *the multiplicity of 1 as a root of the univariate polynomial $q(s) = p(s\exp(it\ell))$.*

Proof. Note that by replacing $p(\mathbf{z})$ with $p(e^{it\ell_1}z_1, \dots, e^{it\ell_n}z_n)$, it suffices to consider $t = 0$ for this equivalence.

(a) = (b) Let D_ℓ denote the differential operator $\sum_{j=1}^n \ell_j \frac{\partial}{\partial x_j}$. Then for any $m \in \mathbb{N}$,

$$D_\ell^m = \sum_{|\boldsymbol{\alpha}|=m} \binom{m}{\boldsymbol{\alpha}} \ell^{\boldsymbol{\alpha}} \partial^{\boldsymbol{\alpha}}, \quad \text{where} \quad \binom{m}{\boldsymbol{\alpha}} = \frac{m!}{\alpha_1! \cdots \alpha_n!},$$

and

$$f^{(m)}(0) = D_\ell^m F|_{\mathbf{x}=(0,\dots,0)} = \sum_{|\boldsymbol{\alpha}|=m} \binom{m}{\boldsymbol{\alpha}} \ell^{\boldsymbol{\alpha}} \partial^{\boldsymbol{\alpha}} F(0).$$

We see that the multiplicity of $(0,\dots,0)$ as a zero of $F(\mathbf{x}) = p(\exp(i\mathbf{x}))$ lower bounds on the multiplicity of 0 as a zero of the function f. Moreover, by Proposition 3.5, when m is the multiplicity of $(0,\dots,0)$ as a zero of $F(\mathbf{x})$, the nonzero values of $\{\partial^{\boldsymbol{\alpha}} F(0) : |\boldsymbol{\alpha}| = m\}$ have the same phase. By assumption, at least one of these is nonzero, ensuring that their sum, $f^{(m)}(0)$, is non-zero and that f has multiplicity m at $t = 0$.

(b) = (c) Follows from Proposition 3.4 with $\varphi(x) = e^{ix}$.

(b) = (d) Consider

$$q(s) = p(s, s, \dots, s) \quad \text{and} \quad h(t) = q(e^{it}) = p(e^{it}, \dots, e^{it}) = F(t, t, \dots, t).$$

By Proposition 3.4, the multiplicity of q at $s = 1$ equals the multiplicity of h at $t = 0$. By the equivalence (a) = (b) with $\ell = \mathbf{1}$, this equals the multiplicity of $F(\mathbf{x})$ at $\mathbf{x} = \mathbf{0}$. ■

3.1. Connectivity of $\mathrm{LY}_{\mathbf{d}}$ and perturbations

One of the early results on real hyperbolic polynomials is Nuij's result that the space of hyperbolic polynomials of a given degree is simply connected [18]. Here we adapt these techniques to better understand $\mathrm{LY}_{\mathbf{d}}$.

Nuij's proof relies on the following operators on univariate polynomial that preserve real rootedness. For $\lambda \in \mathbb{R}_{>0}$, define $\mathcal{D}_\lambda : \mathbb{C}[z] \to \mathbb{C}[z]$ by $\mathcal{D}_\lambda(q) = q + \lambda q'$. Nuij shows that if q is real rooted, then $\mathcal{D}_\lambda(q)$ is real rooted, $\mathcal{D}_\lambda$ decreases the multiplicity of roots of q by 1 for $\lambda \neq 0$, and all new roots of $\mathcal{D}_\lambda(q)$ are simple. In particular, for any real rooted polynomial $q \in \mathbb{R}[z]$ of degree d, the multiplicity of any root of q is at most d, and so applying $\mathcal{D}_\lambda$ d times to q results in a real rooted univariate polynomial with d simple roots. The roots of $\mathcal{D}_\lambda(q)$ interlace those of q in the following sense: if $a_1, \ldots, a_d$ are the roots of q and $b_1, \ldots, b_d$ are the roots of $\mathcal{D}_\lambda(q)$, then $b_j \leq a_j \leq b_{j+1}$ for all j.

Let $\mathbb{C}[\mathbf{y}, \mathbf{z}]_{\mathbf{d}}$ denote the set of polynomials in $y_1, \ldots, y_n$ and $z_1, \ldots, z_n$ that are homogeneous of degree d_j in each set of variables (y_j, z_j). The zero-set of such polynomials are well-defined subsets of $(\mathbb{P}^1(\mathbb{C}))^n$. Here we use $\mathbb{P}^1(K)$ to denote the projective line over a field K, which is $K^2 \setminus \{(0,0)\}$ modulo the equivalence $(a, b) \sim (\lambda a, \lambda b)$ for $\lambda \neq 0$. For any polynomial $p \in \mathbb{C}[\mathbf{y}, \mathbf{z}]_{\mathbf{d}}$ and $\lambda \in (\mathbb{C}^*)^n$, $p(\lambda_1 y_1, \ldots, \lambda_n y_n, \lambda_1 z_1, \ldots, \lambda_n z_n) = \lambda^{\mathbf{d}} p(y_1, \ldots, y_n, z_1, \ldots, z_n)$. In a slight abuse of notation, we will use $[\mathbf{a} : \mathbf{b}]$ to denote a point $([a_i : b_i])_{i \in [n]} \in (\mathbb{P}^1(K))^n$, where $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$. Similarly, for a subset $I \subseteq [n]$, we use $[\mathbf{a}_I : \mathbf{b}_I]$ to denote the point $([a_i : b_i])_{i \in I} \in (\mathbb{P}^1(K))^I$.

To understand the zero set of p on $(\mathbb{P}^1(K))^n$, we restrict to various affine charts. We can partition points $[\mathbf{a} : \mathbf{b}] \in (\mathbb{P}^1(K))^n$ by the set $I = \{i \in [n] : a_i \neq 0\}$. The affine chart of points $[\mathbf{a} : \mathbf{b}] \in (\mathbb{P}^1(K))^n$ with $a_i \neq 0$ for all i is isomorphic to K^n via the coordinate-wise correspondence $[a_i : b_i] \leftrightarrow b_i / a_i$. For $i \in I$, $b_i \neq 0$ and for $j \notin I$, $a_j \neq 0$, and so after rescaling we may take $b_i = 1$ and $a_j = 1$.

On this vector space of polynomials, define the linear operator

$$\mathcal{D}_\lambda : \mathbb{C}[\mathbf{y}, \mathbf{z}]_{\mathbf{d}} \to \mathbb{C}[\mathbf{y}, \mathbf{z}]_{\mathbf{d}} \quad \text{by} \quad \mathcal{D}_\lambda(q) = q + \lambda \sum_{j=1}^{n} y_j \partial_{z_j} q.$$

Let $\mathcal{D}_\lambda^{|\mathbf{d}|}$ denote the operator obtained from $\mathcal{D}_\lambda$ by applying it $|\mathbf{d}| = \sum_{j=1}^n d_j$ times.

For each $\mathbf{d} \in \mathbb{Z}_{\geq 0}^n$, consider the following sets of polynomials:

$$\mathfrak{S}_{\mathbf{d}} = \{q \in \mathbb{C}[\mathbf{y}, \mathbf{z}]_{\mathbf{d}} : \mathrm{coeff}(q, \mathbf{z}^{\mathbf{d}}) = 1 \text{ and } q(\mathbf{1}, \mathbf{z}) \text{ is stable with respect to } \mathcal{H}_+ \text{ and } \mathcal{H}_-\},$$
$$\mathfrak{S}_{\mathbf{d}}^{\circ} = \{q \in \mathfrak{S}_{\mathbf{d}} : q \text{ and } \nabla q \text{ have no common zeros in } (\mathbb{P}^1(\mathbb{R}))^n\}.$$

Proposition 3.7. *We have $\mathfrak{S}_{\mathbf{d}} \subseteq \mathbb{R}[\mathbf{y}, \mathbf{z}]_{\mathbf{d}}$.*

Proof. One can check directly from the definition that the stability of $q(\mathbf{1}, \mathbf{z})$ implies that for all $\mathbf{a} \in \mathbb{R}^n$, the polynomial $q(\mathbf{1}, \mathbf{a} + t\mathbf{1}) \in \mathbb{C}[t]$ has real roots, say $r_j \in \mathbb{R}$, for $j = 1, \ldots, |\mathbf{d}|$. By assumption, the coefficient of $\mathbf{z}^{\mathbf{d}}$ in q is 1, which implies that the coefficient of $t^{|\mathbf{d}|}$ in $q(\mathbf{1}, \mathbf{a} + t\mathbf{1})$ is 1, so $q(\mathbf{1}, \mathbf{a} + t\mathbf{1}) = \prod_{j=1}^{|\mathbf{d}|}(t - r_j)$, and therefore all of its coefficients must be real. If $q = g + ih$, where $g, h \in \mathbb{R}[\mathbf{y}, \mathbf{z}]_{\mathbf{d}}$, then we have shown that $h(\mathbf{1}, \mathbf{a} + t\mathbf{1}) \in \mathbb{R}[t]$ is the zero polynomial for all $\mathbf{a} \in \mathbb{R}^n$. In particular, $h(\mathbf{1}, \mathbf{a}) = 0$ for all $\mathbf{a} \in \mathbb{R}^n$, which implies that h is identically zero. ∎

Proposition 3.8. *If $q \in \mathbb{C}[z_1, \dots, z_n]_{\leq \mathbf{d}}$ is stable with respect to $\mathcal{H}_+$ and $\mathcal{H}_-$, then so is the polynomial $q + z_{n+1} \sum_{j=1}^n \partial_{z_j} q$ in $\mathbb{C}[z_1, \dots, z_{n+1}]$. Moreover, for any $\mathbf{a} \in \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^n_{\geq 0}$, the roots of $q + \sum_{j=1}^n \partial_{z_j} q$ interlace those of q when restricted to $\mathbf{z} = \mathbf{a} + t\mathbf{b}$. That is, $\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \cdots \leq \lambda_{|\mathbf{d}|} \leq \mu_{|\mathbf{d}|}$, where $\{\lambda_j\}_j$ and $\{\mu_j\}_j$ are the roots of the restrictions of $q + \sum_{j=1}^n \partial_{z_j} q$ and q, respectively.*

Proof. We use the theory of stability preservers by Borcea and Brändén, see Theorem 1.3 in [9]. The *symbol* of the operator $\mathcal{D}(q) = q + z_{n+1} \sum_{j=1}^n \partial_{z_j} q$ is

$$\mathcal{D}((\mathbf{z}+\mathbf{w})^{\mathbf{d}}) = (\mathbf{z}+\mathbf{w})^{\mathbf{d}} \Big(1 + z_{n+1} \sum_{j=1}^n d_j (z_j + w_j)^{-1}\Big).$$

We can see by inspection that this polynomial is stable. If $\operatorname{Im}(z_j) > 0$ and $\operatorname{Im}(w_j) > 0$ for all j, then we have that $\operatorname{Im}(-(z_{n+1})^{-1}) > 0$ and $\operatorname{Im}((z_j + w_j)^{-1}) < 0$ for all j, so $\sum_{j=1}^n d_j (z_j + w_j)^{-1} \neq -(z_{n+1})^{-1}$, and therefore $\mathcal{D}((\mathbf{z}+\mathbf{w})^{\mathbf{d}}) \neq 0$. This shows that the symbol $\mathcal{D}((\mathbf{z}+\mathbf{w})^{\mathbf{d}}) \in \mathbb{R}[z_1, \dots, z_n, z_{n+1}, w_1, \dots, w_n]$ is stable with respect to $\mathcal{H}_+$, and by the same argument, it is also stable with respect to $\mathcal{H}_-$. Then, by Theorem 1.3 in [9], the linear operation $\mathcal{D}$ preserves stability.

The statement of interlacing then follows from Lemma 1.8 in [9]. ∎

Lemma 3.9. *Let $q \in \mathfrak{S}_{\mathbf{d}}$, $I \subseteq [n]$, and let q_I denote restriction of q to $y_j = 0$ and $z_j = 1$ for all $j \notin I$. Then $q_I \in \mathfrak{S}_{\mathbf{d}_I}$, i.e., q_I is nonzero and $q_I(\mathbf{1}_I, \mathbf{z}_I)$ is stable with respect to $\mathcal{H}_+$ and $\mathcal{H}_-$. If additionally $q \in \mathfrak{S}^\circ_{\mathbf{d}}$, then $q_I \in \mathfrak{S}^\circ_{\mathbf{d}_I}$, i.e., q_I and ∇q_I have no common zeros in $(\mathbb{P}^1(\mathbb{R}))^I$.*

Proof. Note that $1 = \operatorname{coeff}(q, \mathbf{z}^{\mathbf{d}}) = q(\mathbf{0}, \mathbf{1}) = q_I(\mathbf{0}_I, \mathbf{1}_I)$, showing that q_I is nonzero and has $1 = \operatorname{coeff}(q_I, \prod_{i \in I} z_i^{d_i})$. Note that $\prod_{j \notin I} z_j^{d_j} \cdot q_I(\mathbf{1}_I, \mathbf{z}_I)$ is the initial form of $q(1, \mathbf{z})$ with respect to the vector $(\mathbf{0}_I, \mathbf{1}_{[n]\setminus I})$, and so $q_I(\mathbf{1}_I, \mathbf{z}_I)$ is stable by Proposition 3.3 (a).

Suppose that q_I is zero at a point $[\mathbf{a}_I : \mathbf{b}_I] \in (\mathbb{P}^1(\mathbb{R}))^I$. We will show that for some $i \in I$, $\partial_{y_i} q$ or $\partial_{z_i} q_I$ is nonzero at this point. Note that if $a_k = 0$ for some $k \in I$, then we can replace I with $I' = \{i \in I : a_i \neq 0\}$, which is non-empty by assumption. If there is some $i \in I'$ for which $\partial_{y_i} q_{I'}$ or $\partial_{z_i} q_{I'}$ is non-zero at $[\mathbf{a}_{I'} : \mathbf{b}_{I'}]$, this proves the claim. Therefore we may assume that for all $i \in I$, $a_i \neq 0$ and take $a_i = 1$. Moreover, by replacing $q(\mathbf{y}, \mathbf{z})$ with its substitution of $z_i \mapsto z_i + b_i y_i$ for all $i \in I$, we can assume that $b_i = 0$ for all $i \in I$.

Since $q \in \mathfrak{S}^\circ_{\mathbf{d}}$, there is some derivative $\partial_{y_j} q$ or $\partial_{z_j} q$ that is nonzero at $[\mathbf{a} : \mathbf{b}]$, where $\mathbf{a} = (\mathbf{1}_I, \mathbf{0}_{[n]\setminus I})$ and $\mathbf{b} = (\mathbf{0}_I, \mathbf{1}_{[n]\setminus I})$. If $j \in I$, we are done, so take $j \notin I$ and assume by contradiction that all derivatives with respect to variables labeled by I are zero. Since q is homogeneous of degree d_j in (y_j, z_j), then $y_j \partial_{y_j} q + z_j \partial_{z_j} q = d_j q$. Since q and y_j both vanish at this point and z_j does not, we see that it must be $\partial_{y_j} q$ that is nonzero at $[\mathbf{a} : \mathbf{b}]$.

Consider the polynomial

$$\tilde{q}(s, t) = q(\mathbf{a} - t e_j, \mathbf{b} + s \mathbf{1}_I) \in \mathbb{R}[s, t].$$

We claim that this polynomial is stable. To see this, note the upper halfplane is invariant under $\varphi(z) = -1/z$. Let $\varphi_I(\mathbf{z})$ be the vector with i-th entry $\varphi(z_i)$ if $i \notin I$ or z_i otherwise, so that $\prod_{j\notin I} z_j^{d_j} \cdot p(\varphi_I(\mathbf{z}))$ is stable with respect to upper and lower halfplanes if and only if $p(\mathbf{z})$ is. The polynomial

$$\prod_{j\notin I}(-z_j)^{d_j} \cdot q(\mathbf{1}, \varphi_I(\mathbf{z})) = q((\mathbf{1}_I, -\mathbf{z}_{[n]\setminus I}), (\mathbf{z}_I, \mathbf{1}_{[n]\setminus I})) \in \mathbb{R}[\mathbf{z}]$$

is stable. We then obtain the polynomial $\tilde{q}$ as a further restriction of $z_j = t$, $z_k = 0$ for $k \in [n]\setminus(I \cup \{j\})$ and $z_i = s$ for all $i \in I$. It follows that $\tilde{q}$ is stable by Proposition 3.3 (b).

Note that $\tilde{q}(0,0) = q(\mathbf{a},\mathbf{b}) = 0$. Moreover, we have that

$$\partial_s\tilde{q}|_{(s,t)=(0,0)} = \Big(\sum_{i\in I}\partial_{z_i}q\Big)\Big|_{(\mathbf{y},\mathbf{z})=(\mathbf{a},\mathbf{b})} = 0, \quad \text{and}$$
$$\partial_t\tilde{q}|_{(s,t)=(0,0)} = (-\partial_{y_j}q)|_{(\mathbf{y},\mathbf{z})=(\mathbf{a},\mathbf{b})} \neq 0.$$

The polynomial $\tilde{q}(s,0) = q(\mathbf{a}, \mathbf{b} + s\mathbf{1}_I)$ has leading term $s^{\sum_{i\in I} d_i}$, since $q(\mathbf{0},\mathbf{1}) = 1$, and so is nonzero. Let k be the smallest integer for which $\partial_s^k\tilde{q}|_{(s,t)=(0,0)}$ is nonzero. By the arguments above, k exists and $k \geq 2$. This means that all monomials $s^\alpha t^\beta$ appearing in $\tilde{q}$ with non-zero coefficients either have $\beta \geq 1$ or $\alpha \geq k \geq 2$. In particular, we have that $\langle(-1,-k),(\alpha,\beta)\rangle \leq -k$, with equality if and only if $(\alpha,\beta) = (k,0)$ or $(\alpha,\beta) = (0,1)$. We conclude that the initial form $\mathrm{in}_{(-1,-k)}\tilde{q} = as^k + bt$ for some non-zero coefficients $a, b \in \mathbb{R}^*$. By Proposition 3.3 (a), it is stable with respect to both upper and lower half-planes. However, since $k \geq 2$, there is some $c \in \mathcal{H}_+$ such that $c^k = -\frac{b}{a}i$, and so $(s,t) = (c,i) \in \mathcal{H}_+^2$ is a root, contradicting stability. ∎

Proposition 3.10. *For any $q \in \mathfrak{S}_{\mathbf{d}}$ and $\lambda > 0$, $\mathcal{D}_\lambda(q) \in \mathfrak{S}_{\mathbf{d}}$ and $\mathcal{D}_\lambda^{|\mathbf{d}|}(q) \in \mathfrak{S}_{\mathbf{d}}^\circ$.*

Proof. By Proposition 3.8, the operation $q \mapsto q + \lambda\sum_{i=1}^n \partial_{z_i}q$ preserves stability of polynomials in $\mathbb{R}[\mathbf{z}]$. We need to show that $\mathcal{D}_\lambda^{|\mathbf{d}|}q$ has no common zeros with its gradient on $(\mathbb{P}^1(\mathbb{R}))^n$. By the univariate case discussed above, $\mathcal{D}_\lambda^{|\mathbf{d}|}q(\mathbf{1}, \mathbf{b} + t\mathbf{1}) \in \mathbb{R}[t]$ has simple roots for all $\mathbf{b} \in \mathbb{R}^n$. It follows that if $\mathcal{D}_\lambda^{|\mathbf{d}|}q(\mathbf{1},\mathbf{z})$ vanishes at $\mathbf{z} \in \mathbb{R}^n$, then its gradient does not. This shows that $\mathcal{D}_\lambda^{|\mathbf{d}|}q(\mathbf{1},\mathbf{z})$ and its gradient have no common zeros of the form $[\mathbf{a}:\mathbf{b}]$ where $a_j \neq 0$ for all j. Assume by contradiction that $\mathcal{D}_\lambda^{|\mathbf{d}|}q(\mathbf{1},\mathbf{z})$ and its gradient have some common zero $[\mathbf{a}:\mathbf{b}] \in (\mathbb{P}^1(\mathbb{R}))^n$, and let $I = \{i : a_i \neq 0\}$, so $I \neq [n]$. Note that we can assume $b_j = 1$ for all $j \notin I$. If $I = \emptyset$, then $[\mathbf{a}:\mathbf{b}] = [\mathbf{0}:\mathbf{1}]$, at which $\mathcal{D}_\lambda^{|\mathbf{d}|}q(\mathbf{0},\mathbf{1}) = \mathrm{coeff}(\mathcal{D}_\lambda^{|\mathbf{d}|}q, \mathbf{z}^{\mathbf{d}}) \neq 0$. Therefore, $\emptyset \subsetneq I \subsetneq [n]$.

Let $q_I \in \mathbb{R}[y_i, z_i : i \in I]$ denote the restriction of q to $y_j = 0$ and $z_j = 1$ for $j \notin I$. Note that the operator $\mathcal{D}_\lambda$ commutes with the restriction to $y_j = 0$ and $z_j = 1$. That is,

$$(\mathcal{D}_\lambda q)|_{\{y_j=0,z_j=1:j\notin I\}} = \Big(q + \lambda\sum_{i=1}^n y_i\partial_{z_i}q\Big)\Big|_{\{y_j=0,z_j=1:j\notin I\}}$$
$$= q_I + \lambda\sum_{i\in I} y_i\partial_{z_i}q_I = \mathcal{D}_\lambda q_I.$$

In particular, $\mathcal{D}_\lambda q_I(\mathbf{a}_I, \mathbf{1}_I) = \mathcal{D}_\lambda q_I(\mathbf{1}_I, \mathbf{a}_I^{-1}) = 0$. Since $q_I \in \mathfrak{S}_{\mathbf{d}_I}$, by Lemma 3.9, and it has total degree $|\mathbf{d}_I| \leq |\mathbf{d}|$, then the argument above shows that $\mathcal{D}_\lambda q_I \in \mathfrak{S}_{\mathbf{d}_I}$ and that the gradient of $\mathcal{D}_\lambda^{|\mathbf{d}|}(q_I)(1, \mathbf{z}_I)$ cannot vanish at the zero $(\mathbf{1}_I, \mathbf{a}_I^{-1})$. Hence, there must be some nonzero derivative of $\mathcal{D}_\lambda^{|\mathbf{d}|}(q_I)$ at $[\mathbf{1}_I : \mathbf{a}_I^{-1}] = [\mathbf{a}_I : \mathbf{b}_I]$, which gives a nonzero derivative of $\mathcal{D}_\lambda^{|\mathbf{d}|}(q)$ at $[\mathbf{a} : \mathbf{b}]$. ∎

Proposition 3.11. *Both $\mathfrak{S}_{\mathbf{d}}$ and $\mathfrak{S}_{\mathbf{d}}^\circ$ are contractible, and $\mathfrak{S}_{\mathbf{d}}$ equals the closure (in the Euclidean topology on $\mathbb{R}[\mathbf{y}, \mathbf{z}]_{\mathbf{d}}$) of $\mathfrak{S}_{\mathbf{d}}^\circ(\mathbb{R})$.*

Proof. The proof follows the proof of the main theorem in [18]. For $\mu \in \mathbb{R}$, consider the linear operator G_μ on $\mathbb{R}[\mathbf{y}, \mathbf{z}]_{\mathbf{d}}$ defined by $G_\mu(q) = q(\mu\mathbf{y}, \mathbf{z})$. This operator preserves both stability and the coefficient of $\mathbf{z}^{\mathbf{d}}$.

For $\lambda \in [0, 1]$, consider the map $\mathcal{D}_{1-\lambda}^{|\mathbf{d}|} G_\lambda$. This map preserves stability and, for $\lambda \neq 1$, the image of $\mathfrak{S}_{\mathbf{d}}$ under this map belongs to $\mathfrak{S}_{\mathbf{d}}^\circ$. For $\lambda = 1$, we get the identity map $\mathcal{D}_0^{|\mathbf{d}|} G_1(q) = q$, and for $\lambda = 0$ we get $\mathcal{D}_1^{|\mathbf{d}|}(\mathbf{z}^{\mathbf{d}}) \in \mathfrak{S}_{\mathbf{d}}^\circ(\mathbb{R})$. Therefore, this gives a deformation retraction of both $\mathfrak{S}_{\mathbf{d}}$ and $\mathfrak{S}_{\mathbf{d}}^\circ$ onto the point $\mathcal{D}_1^{|\mathbf{d}|}(\mathbf{z}^{\mathbf{d}})$. ∎

Proposition 3.12. *The interior of $\mathfrak{S}_{\mathbf{d}}$ in $\{q \in \mathbb{R}[\mathbf{y}, \mathbf{z}]_{\mathbf{d}} : \operatorname{coeff}(q, \mathbf{z}^{\mathbf{d}}) = 1\}$ is not empty, and in particular, it contains $\mathfrak{S}_{\mathbf{d}}^\circ$.*

Proof. Suppose that $q \in \mathfrak{S}_{\mathbf{d}}^\circ$, so that q and its gradient have no common zeros in $(\mathbb{P}^1(\mathbb{R}))^n$. Let $(S^1)^n = \{(\mathbf{y}, \mathbf{z}) \in \mathbb{R}^{2n} : y_j^2 + z_j^2 = 1 \forall j\}$ and let $V = \{(\mathbf{y}, \mathbf{z}) \in (S^1)^n : \mathcal{D}_1 q(\mathbf{y}, \mathbf{z}) = 0\}$, where $\mathcal{D}_1 q$ is $\mathcal{D}_\lambda q$ at $\lambda = 1$. Consider the set of polynomials

$$U = \{g \in \mathbb{R}[\mathbf{y}, \mathbf{z}]_{\mathbf{d}} : \operatorname{coeff}(g, \mathbf{z}^{\mathbf{d}}) = 1, g(\mathbf{y}, \mathbf{z}) q(\mathbf{y}, \mathbf{z}) > 0 \text{ for all } (\mathbf{y}, \mathbf{z}) \in V\}.$$

The set V is compact, since $(S^1)^n$ is compact, and so $\min_{(\mathbf{y},\mathbf{z}) \in V} g(\mathbf{y}, \mathbf{z}) q(\mathbf{y}, \mathbf{z})$ is continuous in the coefficients of g, which means that U is open in $\{g \in \mathbb{R}[\mathbf{y}, \mathbf{z}]_{\mathbf{d}} : \operatorname{coeff}(g, \mathbf{z}^{\mathbf{d}}) = 1\}$. We claim that $q \in U$ and $U \subseteq \mathfrak{S}_{\mathbf{d}}^\circ$.

To see that $q \in U$, it suffices to show that q and $\mathcal{D}_1 q$ have no common zeros in $(\mathbb{P}^1(\mathbb{R}))^n$. We first check this for the points in the affine chart $\mathbf{y} = \mathbf{1}$. Suppose that $q(\mathbf{1}, \mathbf{b}) = 0$ for $[\mathbf{1} : \mathbf{b}] \in (\mathbb{P}^1(\mathbb{R}))^n$, so by assumption, there is some j for which $\partial_{z_j} q(\mathbf{1}, \mathbf{b})$ is nonzero. By Proposition 3.3, all of the the nonzero entries of $\{\partial_{z_i} q(\mathbf{1}, \mathbf{b}) : i = 1, \ldots, n\}$ have the same phase, which implies that $\sum_{i=1}^n \partial_{z_i} q(\mathbf{1}, \mathbf{b})$ is nonzero. Since $q(\mathbf{1}, \mathbf{b}) = 0$, it follows that $\mathcal{D}_1 q = q + \sum_{i=1}^n \partial_{z_i} q$ is nonzero at $[\mathbf{1} : \mathbf{b}]$.

For any arbitrary point $[\mathbf{a} : \mathbf{b}] \in (\mathbb{P}^1(\mathbb{R}))^n$, let $I = \{i \in [n] : a_i \neq 0\}$, which by assumption is non-empty. By Lemma 3.9, q_I is stable and has no common zeros with its gradient on $(\mathbb{P}^1(\mathbb{R}))^I$. The argument above shows that q_I and $\mathcal{D}_1 q_I$ cannot both be zero at $[\mathbf{a}_I : \mathbf{b}_I]$, and so q and $\mathcal{D}_1 q$ cannot both be zero at $[\mathbf{a} : \mathbf{b}]$.

To see that $U \subseteq \mathfrak{S}_{\mathbf{d}}^\circ$, consider $\mathbf{a} \in \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}_+^n$, let $\{\lambda_j\}_j$ denote the roots of $\mathcal{D}_1 q(\mathbf{1}, \mathbf{a} + \mathbf{b}t)$, and let $\{s_j\}_j$ denote the roots of $q(\mathbf{1}, \mathbf{a} + \mathbf{b}t)$. These roots are distinct by the argument above.

By Proposition 3.8, $\lambda_1 < \mu_1 < \lambda_2 < \cdots < \lambda_{|\mathbf{d}|} < \mu_{|\mathbf{d}|}$. In particular, q must alternate signs on the roots of $\mathcal{D}_1 q(\mathbf{1}, \mathbf{a} + \mathbf{b}t)$. If $g \in U$, then $g(\mathbf{1}, \mathbf{a} + \mathbf{b}t) \in \mathbb{R}[t]$ has degree $|\mathbf{d}|$ with positive leading coefficient $\mathbf{b}^{\mathbf{d}}$, and it alternates signs on the roots of $\mathcal{D}_1 q(\mathbf{1}, \mathbf{a} + \mathbf{b}t)$. Hence it has $|\mathbf{d}|$ distinct real roots. As this holds for any $\mathbf{a} \in \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}_+^n$, then $g \in \mathfrak{S}_{\mathbf{d}}^\circ$. See, for example, Section 2.3 and 2.4 in [22]. ∎

We can modify this using the Möbius transformations ϕ from (3.1) to translate these results to $\mathrm{LY}_{\mathbf{d}}$. For any $\mathbf{x} \in [0, 2\pi)^n$, define

$$\mathrm{LY}_{\mathbf{d}}(\mathbf{x}) = \{p \in \mathrm{LY}_{\mathbf{d}} : p(\exp(i\mathbf{x})) \neq 0\}, \quad \text{and}$$
$$\mathrm{LY}^{\circ}_{\mathbf{d}}(\mathbf{x}) = \{p \in \mathrm{LY}_{\mathbf{d}}(\mathbf{x}) : p \text{ and } \nabla p \text{ have no common zeros in } \mathbb{T}^n\}.$$

One can check that $q \in \mathbb{C}[\mathbf{y}, \mathbf{z}]_{\mathbf{d}}$ belongs to $\mathfrak{S}_{\mathbf{d}}$ (respectively, $\mathfrak{S}^{\circ}_{\mathbf{d}}$) if and only if $\phi^{-1} \cdot q(\mathbf{1}, \mathbf{z})$ belongs to $\mathrm{LY}_{\mathbf{d}}(\mathbf{x})$ (respectively, $\mathrm{LY}^{\circ}_{\mathbf{d}}(\mathbf{x})$) for ϕ defined using the angles $e^{ix_1}, \ldots, e^{ix_n}$.

Definition 3.13. Define an involution of polynomials in $\mathbb{C}[\mathbf{z}]_{\leq \mathbf{d}}$ by

$$p(z_1, \ldots, z_n) \mapsto p^{\dagger}(z_1, \ldots, z_n) = \mathbf{z}^{\mathbf{d}}\, \overline{p(\overline{z_1}^{-1}, \ldots, \overline{z_n}^{-1})}, \quad \text{namely}$$
$$\sum_{\boldsymbol{\alpha}} a_{\boldsymbol{\alpha}}\, \mathbf{z}^{\boldsymbol{\alpha}} \mapsto \sum_{\boldsymbol{\alpha}} \overline{a_{\boldsymbol{\alpha}}}\, \mathbf{z}^{\mathbf{d}-\boldsymbol{\alpha}} = \sum_{\boldsymbol{\alpha}} \overline{a_{\mathbf{d}-\boldsymbol{\alpha}}}\, \mathbf{z}^{\boldsymbol{\alpha}},$$

and define the set of polynomials in $\mathbb{C}[\mathbf{z}]_{\leq \mathbf{d}}$ that are invariant under the involution

$$\mathbb{C}[\mathbf{z}]^{\mathrm{in}}_{\leq \mathbf{d}} := \{p \in \mathbb{C}[\mathbf{z}]_{\leq \mathbf{d}} : p = p^{\dagger}\} = \Big\{ \sum_{\boldsymbol{\alpha} \leq \mathbf{d}} a_{\boldsymbol{\alpha}}\, \mathbf{z}^{\boldsymbol{\alpha}} : a_{\boldsymbol{\alpha}} = \overline{a_{\mathbf{d}-\boldsymbol{\alpha}}} \text{ for all } \boldsymbol{\alpha} \Big\},$$

and the set of polynomials for which $p^{\dagger}$ is a scalar multiple of p by

$$\begin{aligned}\mathbb{C} \cdot \mathbb{C}[\mathbf{z}]^{\mathrm{in}}_{\leq \mathbf{d}} &:= \big\{cp : c \in \mathbb{C}, p \in \mathbb{C}[\mathbf{z}]^{\mathrm{in}}_{\leq \mathbf{d}}\big\} \\ &= \big\{p \in \mathbb{C}[\mathbf{z}]_{\leq \mathbf{d}} : p^{\dagger} = cp \text{ for some } c \text{ with } |c| = 1\big\}.\end{aligned}$$

The next lemma is straightforward.

Lemma 3.14. *The set $\mathbb{C}[\mathbf{z}]^{\mathrm{in}}_{\leq \mathbf{d}}$ is a real vector space of dimension $\prod_{j=1}^{n}(d_j + 1)$, spanned by $(\mathbf{z}^{\boldsymbol{\alpha}} + \mathbf{z}^{\mathbf{d}-\boldsymbol{\alpha}})$ and $i(\mathbf{z}^{\boldsymbol{\alpha}} - \mathbf{z}^{\mathbf{d}-\boldsymbol{\alpha}})$ for $\boldsymbol{\alpha} \leq \mathbf{d}$. The set $\mathbb{C} \cdot \mathbb{C}[\mathbf{z}]^{\mathrm{in}}_{\leq \mathbf{d}}$ is a semialgebraic set of dimension $1 + \dim(\mathbb{C}[\mathbf{z}]^{\mathrm{in}}_{\leq \mathbf{d}})$ in the $(2\prod_{j=1}^{n}(d_j + 1))$-dimensional real vector space $\mathbb{C}[\mathbf{z}]_{\leq \mathbf{d}}$, from which it inherits the Euclidean topology.*

Remark 3.15. Note that from the polynomial $q = e^{ix} p$ with $x \in [0, \pi)$ and $p \in \mathbb{C}[\mathbf{z}]^{\mathrm{in}}_{\leq \mathbf{d}}$, we can uniquely determine x and p. Namely, $e^{ix} = (e^{ix}/e^{-ix})^{1/2} = (q(\mathbf{z})/q^{\dagger}(\mathbf{z}))^{1/2}$ and $p = e^{-ix} q$.

The image of $\mathbb{C} \cdot \mathbb{R}[\mathbf{y}, \mathbf{z}]_{\mathbf{d}}$ under the map $q \mapsto \phi^{-1} \cdot q(\mathbf{1}, \mathbf{z})$ coincides with $\mathbb{C} \cdot \mathbb{C}[\mathbf{z}]^{\mathrm{in}}_{\leq \mathbf{d}}$:

$$\mathbb{C} \cdot \mathbb{C}[\mathbf{z}]^{\mathrm{in}}_{\leq \mathbf{d}} = \Big\{ c \sum_{\boldsymbol{\alpha} \leq \mathbf{d}} a_{\boldsymbol{\alpha}} (-i)^{|\boldsymbol{\alpha}|}\, (\mathbf{z} + \exp(i\mathbf{x}))^{\boldsymbol{\alpha}} (\mathbf{z} - \exp(i\mathbf{x}))^{\mathbf{d}-\boldsymbol{\alpha}} : a_{\boldsymbol{\alpha}} \in \mathbb{R}, c \in \mathbb{C} \Big\}.$$

Note that for $p(\mathbf{z}) = \sum_{\boldsymbol{\alpha} \leq \mathbf{d}} a_{\boldsymbol{\alpha}} (-i)^{|\boldsymbol{\alpha}|} (\mathbf{z} + \exp(i\mathbf{x}))^{\boldsymbol{\alpha}} (\mathbf{z} - \exp(i\mathbf{x}))^{\mathbf{d}-\boldsymbol{\alpha}}$ with $a_{\boldsymbol{\alpha}} \in \mathbb{R}$, using the notation $\mathbf{z}^{-1} = (1/z_1, \ldots, 1/z_n)$, we have

$$\begin{aligned} p^{\dagger}(\mathbf{z}) &= \mathbf{z}^{\mathbf{d}} \sum_{\boldsymbol{\alpha} \leq \mathbf{d}} a_{\boldsymbol{\alpha}} (i)^{|\boldsymbol{\alpha}|}\, (\mathbf{z}^{-1} + \exp(-i\mathbf{x}))^{\boldsymbol{\alpha}}\, (\mathbf{z}^{-1} - \exp(-i\mathbf{x}))^{\mathbf{d}-\boldsymbol{\alpha}} \\ &= (\exp(-i\mathbf{x}))^{\mathbf{d}} \sum_{\boldsymbol{\alpha} \leq \mathbf{d}} a_{\boldsymbol{\alpha}} (i)^{|\boldsymbol{\alpha}|} (\exp(i\mathbf{x}) + \mathbf{z})^{\boldsymbol{\alpha}} (\exp(i\mathbf{x}) - \mathbf{z})^{\mathbf{d}-\boldsymbol{\alpha}} = (-\exp(-i\mathbf{x}))^{\mathbf{d}}\, p(\mathbf{z}), \end{aligned}$$

and so $cp \in \mathbb{C}[\mathbf{z}]^{\mathrm{in}}_{\leq \mathbf{d}}$ for $c = (i \exp(-i\mathbf{x}/2))^{\mathbf{d}}$.

Definition 3.16. Let $\mathcal{D}_{\lambda,\mathbf{x}}: \mathbb{C}[\mathbf{z}]_{\mathbf{d}} \to \mathbb{C}[\mathbf{z}]_{\mathbf{d}}$ denote the linear operator corresponding to $\mathcal{D}_\lambda$, and fix a tuple of Möbius transformations $\phi = (\phi_1, \dots, \phi_n)$ where ϕ_j is defined as in (3.1), with $\theta = x_j$ for $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$. Namely, $p \mapsto (\phi^{-1} \circ \mathcal{D}_\lambda \circ \phi \cdot p^{\mathrm{hom}})|_{\mathbf{y}=\mathbf{1}}$, where $p^{\mathrm{hom}} = \mathbf{y}^{\mathbf{d}} p(z_1/y_1, \dots, z_n/y_n)$. Explicitly, for

$$p(\mathbf{z}) = \sum_{\boldsymbol{\alpha} \le \mathbf{d}} a_{\boldsymbol{\alpha}} (-i)^{|\boldsymbol{\alpha}|} (\mathbf{z} + \exp(i\mathbf{x}))^{\boldsymbol{\alpha}} (\mathbf{z} - \exp(i\mathbf{x}))^{\mathbf{d}-\boldsymbol{\alpha}},$$

we have

$$\mathcal{D}_{\lambda,\mathbf{x}} p(\mathbf{z}) = p(\mathbf{z}) + \lambda \sum_{j=1}^{n} \sum_{\boldsymbol{\alpha} \le \mathbf{d}} \alpha_j \, a_{\boldsymbol{\alpha}} (-i)^{|\boldsymbol{\alpha}|} \, (\mathbf{z} + \exp(i\mathbf{x}))^{\boldsymbol{\alpha} - e_j} \, (\mathbf{z} - \exp(i\mathbf{x}))^{\mathbf{d}-\boldsymbol{\alpha}}.$$

Corollary 3.17. *For any $p \in \mathrm{LY}_{\mathbf{d}}(\mathbf{x})$ and $\lambda > 0$ and ϕ defined as above, $\mathcal{D}_{\lambda,\mathbf{x}}(p) \in \mathrm{LY}_{\mathbf{d}}(\mathbf{x})$ and $(\mathcal{D}_{\lambda,\mathbf{x}})^{|\mathbf{d}|}(p) \in \mathrm{LY}^{\circ}_{\mathbf{d}}(\mathbf{x})$. The interior of $\mathrm{LY}_{\mathbf{d}}(\mathbf{x})$ in the Euclidean topology on $\mathbb{C} \cdot \mathbb{C}[\mathbf{z}]^{\mathrm{in}}_{\le \mathbf{d}}$ is nonempty and contains $\mathrm{LY}^{\circ}_{\mathbf{d}}(\mathbf{x})$. Moreover, $\mathrm{LY}_{\mathbf{d}}(\mathbf{x})$ is contained in the closure of $\mathrm{LY}^{\circ}_{\mathbf{d}}(\mathbf{x})$.*

Proof. Note that $p \in \mathrm{LY}_{\mathbf{d}}(\mathbf{x})$ (respectively, $\mathrm{LY}^{\circ}_{\mathbf{d}}(\mathbf{x})$) if and only if the homogenization of $\phi \cdot p$ belongs to $\mathbb{C}^* \mathfrak{S}_{\mathbf{d}}$ (respectively, $\mathbb{C}^* \mathfrak{S}^{\circ}_{\mathbf{d}}(\mathbf{x})$). The result then follows from Propositions 3.10, 3.11, and 3.12. ■

Remark 3.18. The set of Lee–Yang polynomials is connected but not contractible, even in $\mathbb{P}(\mathbb{C}[\mathbf{z}]^{\mathrm{in}}_{\le \mathbf{d}})$. For example, the set of univariate Lee–Yang polynomials p of degree-one, modulo global scaling, is parametrized by $z - e^{i\theta}$, for $\theta \in [0, 2\pi]$, showing this set to be a circle.

Remark 3.19. The proof of Proposition 3.11 gives an explicit contraction of $\mathrm{LY}_{\mathbf{d}}(\mathbf{x})$ (modulo scaling) to a polynomial $p^* \in \mathrm{LY}^{\circ}_{\mathbf{d}}(\mathbf{x})$, namely $\phi^{-1} \circ \mathcal{D}_1^{|\mathbf{d}|} \mathbf{z}^{\mathbf{d}}$, which we can explicitly compute. The space of real stable polynomials is contracted to

$$\mathcal{D}_1^{|\mathbf{d}|} \mathbf{z}^{\mathbf{d}} = \Big(1 + \sum_{j=1}^{n} y_j \partial_{z_j}\Big)^{|\mathbf{d}|} \cdot \mathbf{z}^{\mathbf{d}} = \sum_{\boldsymbol{\alpha}} \binom{|\mathbf{d}|}{\boldsymbol{\alpha}} \mathbf{y}^{\boldsymbol{\alpha}} \, \partial^{\boldsymbol{\alpha}}_{\mathbf{z}} \mathbf{z}^{\mathbf{d}} = \sum_{\boldsymbol{\alpha} \le \mathbf{d}} \binom{|\mathbf{d}|}{\boldsymbol{\alpha}} \frac{\mathbf{d}!}{\boldsymbol{\alpha}!} \mathbf{y}^{\boldsymbol{\alpha}} \, \mathbf{z}^{\mathbf{d}-\boldsymbol{\alpha}},$$

where the sum in the third term is taken over all $\boldsymbol{\alpha} \in \mathbb{Z}^n_{\ge 0}$ with $|\boldsymbol{\alpha}| \le |\mathbf{d}|$, and where

$$\binom{|\mathbf{d}|}{\boldsymbol{\alpha}} = \frac{|\mathbf{d}|!}{(|\mathbf{d}| - |\boldsymbol{\alpha}|)! \, \alpha_1! \cdots \alpha_n!} \quad \text{and} \quad \frac{\mathbf{d}!}{\boldsymbol{\alpha}!} = \prod_{j=1}^{n} \Big(\frac{d_j!}{\alpha_j!}\Big).$$

Taking ϕ as in (3.1), we find that $\mathbb{P}(\mathrm{LY}_{\mathbf{d}}(\mathbf{x}))$ is contracted to

$$p^*(\mathbf{z}) = \phi^{-1} \cdot \mathcal{D}_1^{|\mathbf{d}|} \mathbf{z}^{\mathbf{d}} = \sum_{\boldsymbol{\alpha} \le \mathbf{d}} \binom{|\mathbf{d}|}{\boldsymbol{\alpha}} \frac{\mathbf{d}!}{\boldsymbol{\alpha}!} \, (\mathbf{z} - \exp(i\mathbf{x}))^{\boldsymbol{\alpha}} \, (-i(\mathbf{z} + \exp(i\mathbf{x})))^{\mathbf{d}-\boldsymbol{\alpha}}.$$

As above, let $\mathbb{C}[\mathbf{z}]^{\mathrm{in}}_{\le \mathbf{d}}$ denote the real vector space of polynomials in $\mathbb{C}[\mathbf{z}]_{\le \mathbf{d}}$ that are invariant under the involution $\sum_{\boldsymbol{\alpha}} a_{\boldsymbol{\alpha}} \mathbf{z}^{\boldsymbol{\alpha}} \mapsto \sum_{\boldsymbol{\alpha}} \overline{a_{\mathbf{d}-\boldsymbol{\alpha}}} \, \mathbf{z}^{\boldsymbol{\alpha}}$.

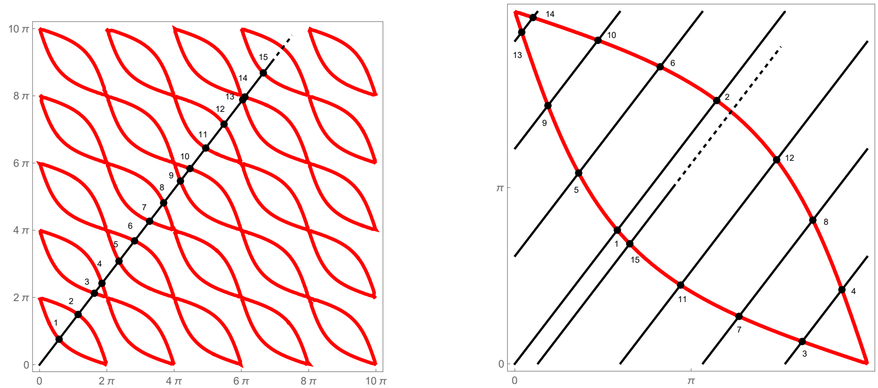


Figure 5. (Left) The zero set of $p(e^{ix}, e^{iy})$ (in red) and the line $(x, y) = t\ell$ (in black) as in Figure 1. (Right) Σ_p, the zero set modulo 2π, and the line $(x, y) = t\ell \bmod 2\pi$.

Theorem 3.20. *For any $\mathbf{d} \in \mathbb{Z}^n_{\geq 0}$, the set of Lee–Yang polynomials $\mathrm{LY}_{\mathbf{d}}$ is a full-dimensional semialgebraic subset of $\mathbb{C} \cdot \mathbb{C}[\mathbf{z}]^{\mathrm{in}}_{\leq \mathbf{d}}$. That is, $\dim(\mathrm{LY}_{\mathbf{d}}) = \prod_{j=1}^n (d_j + 1) + 1$. Its interior in $\mathbb{C} \cdot \mathbb{C}[\mathbf{z}]^{\mathrm{in}}_{\leq \mathbf{d}}$ is nonempty, and contains*

$$\mathrm{LY}^\circ_{\mathbf{d}} = \{p \in \mathrm{LY}_{\mathbf{d}} : p \text{ and } \nabla p \text{ have no common zeros in } \mathbb{T}^n\},$$

and $\mathrm{LY}_{\mathbf{d}}$ is contained in the closure of $\mathrm{LY}^\circ_{\mathbf{d}}$.

Proof. Note that the set

$$\{(p, \mathbf{a}, \mathbf{b}) \in \mathbb{C} \cdot \mathbb{C}[\mathbf{z}]^{\mathrm{in}}_{\leq \mathbf{d}} \times \mathbb{R}^n \times \mathbb{R}^n : p(\mathbf{a} + i\mathbf{b}) = 0$$
$$\text{and } ((a_j^2 + b_j^2 < 1 \forall j) \text{ or } (a_j^2 + b_j^2 > 1 \forall j))\}$$

is semialgebraic. By the Tarski–Seidenberg theorem, its projection on to $\mathbb{C} \cdot \mathbb{C}[\mathbf{z}]^{\mathrm{in}}_{\leq \mathbf{d}}$ is also semialgebraic, as is the complement of the image of this projection, $\mathrm{LY}_{\mathbf{d}}$.

Suppose that $p \in \mathrm{LY}_{\mathbf{d}}$ and fix $\mathbf{x} \in [0, 2\pi)^n$ with $p(\exp(i\mathbf{x})) \neq 0$. Then $p \in \mathrm{LY}_{\mathbf{d}}(\mathbf{x})$ and we invoke Corollary 3.17. If p and ∇p have no common zeros in $\mathbb{T}^n$, then p belongs to $\mathrm{LY}^\circ_{\mathbf{d}}(\mathbf{x})$, which is contained in the interior of $\mathrm{LY}_{\mathbf{d}}(\mathbf{x}) \subseteq \mathrm{LY}_{\mathbf{d}}$ in $\mathbb{C} \cdot \mathbb{C}[\mathbf{z}]^{\mathrm{in}}_{\leq \mathbf{d}}$. Otherwise, p is contained in the closure of $\mathrm{LY}^\circ_{\mathbf{d}}(\mathbf{x}) \subseteq \mathrm{LY}^\circ_{\mathbf{d}}$. ∎

4. The torus zero set Σ_p

It is a simple, yet fruitful, observation that the zeros of $x \mapsto p(\exp(ix\ell))$ correspond to intersection points of the line $\{x\ell \bmod 2\pi : x \in \mathbb{R}\} \subset \mathbb{R}^n/2\pi\mathbb{Z}^n$ with the zero set

$$\Sigma_p := \{\mathbf{x} \in \mathbb{R}^n/2\pi\mathbb{Z}^n : p(\exp(i\mathbf{x})) = 0\}.$$

See Figure 5. In particular, certain properties of $\mu_{p,\ell}$ are determined by the structure of Σ_p, regardless of the choice of $\ell \in \mathbb{R}^n_+$ with $\mathbb{Q}$-independent entries.

Lemma 4.1 (Dimension and singularity). *Given $p \in \mathrm{LY}_{\mathbf{d}}(n)$, its torus zero set $\Sigma_p \subset \mathbb{R}^n/2\pi\mathbb{Z}^n$ is a real analytic variety of dimension $n-1$, and the following hold.*

(1) *The set of singular points, $\mathrm{sing}(\Sigma_p)$, is a subvariety of dimension at most $n-2$. If p has no square factors (i.e., if it is square free), then $\mathbf{x} \in \Sigma_p$ is singular if and only if $\nabla p|_{\mathbf{z}=\exp(i\mathbf{x})} = 0$, or equivalently, its multiplicity is $\mathrm{m}(\mathbf{x}) > 1$.*

(2) *Every irreducible factor of p is a Lee–Yang polynomial. If p is irreducible, then Σ_p is irreducible in the following sense: the zero set of $q(\exp(i\mathbf{x}))$ in Σ_p, for any polynomial $q \in \mathbb{C}[z_1, \dots, z_n]$, is a subvariety of smaller dimension (at most $n-2$ dimensional), unless p is a factor of q, in which case $q(\exp(i\mathbf{x}))$ vanishes on Σ_p.*

Proof. Since the real and imaginary parts of $F(\mathbf{x}) = p(\exp(i\mathbf{x}))$ are real analytic, then Σ_p is a real analytic variety. As such, its singular set $\mathrm{sing}(\Sigma_p)$ is subvariety of lower dimension. To see why Σ_p is $n-1$ dimensional, let $p \in \mathrm{LY}_{\mathbf{d}}(n)$ and let Z_p denote its zero set in $\mathbb{C}^n$. As seen in [1], if p is Lee–Yang, then $Z_p \cap \mathbb{T}^n$ has real dimension $n-1$ and therefore $\dim(\Sigma_p) = n-1$ by the homeomorphism $\mathbf{x} \mapsto \exp(i\mathbf{x})$ between them. Moreover, $Z_p \cap \mathbb{T}^n$ is Zariski dense in Z_p, according to [1], which proves part (2).

For part (1), suppose that p is square free, so that the singular points of Z_p are exactly the points in Z_p where $\nabla p = 0$ (if p has square factors, this criterion fails at zeros of any multiple factor), or equivalently with multiplicity > 1. Due to Proposition 3.4 with $\varphi(x) = e^{ix}$, $\mathbf{x} \in \mathrm{sing}(\Sigma_p)$ if and only if $\mathbf{z} = \exp(i\mathbf{x}) \in \mathrm{sing}(Z_p)$. ■

4.1. The layers structure of Σ_p

It was shown in Lemma 4.14 of [2] that, for Lee–Yang polynomials arising from quantum graphs, Σ_p is the union of $2n$ layers, each homeomorphic to $(0, 2\pi]^{n-1}$. These special polynomials are square free and have $\mathbf{d} = (2, 2, \dots, 2)$, so $2n = |\mathbf{d}|$. In this section, it is shown that for any $p \in \mathrm{LY}_{\mathbf{d}}$, Σ_p is the union of $|\mathbf{d}|$ such layers, and in the case of polynomials with square factors, multiplicities should be taken into account.

Proposition 4.2 (Layers structure). *Given $p \in \mathrm{LY}_{\mathbf{d}}(n)$, Σ_p is the union of $|\mathbf{d}|$ layers,*

$$\Sigma_p = \bigcup_{j=1}^{|\mathbf{d}|} \Sigma_{p,j}.$$

Each layer is homeomorphic to $(0, 2\pi]^{n-1}$ through the parameterization $\varphi_j : (0, 2\pi]^{n-1} \to \Sigma_{p,j}$ given by

$$\varphi_j(\mathbf{y}) := (\mathbf{y}, 0) + \theta_j(\mathbf{y}, 0)\mathbf{1} \mod 2\pi,$$

where $\mathbf{1} = (1, 1, \dots, 1)$ and $\theta_j : \mathbb{R}^n \to \mathbb{R}$ is a continuous function. Each φ_j is real analytic on the open set $\varphi_j^{-1}(\mathrm{reg}(\Sigma_p)) \subset (0, 2\pi]^{n-1}$, which has full Lebesgue measure. The multiplicity of $\mathbf{x}$ as a zero of $p(\exp(i\mathbf{x}))$ is equal to the number of layers $\Sigma_{p,j}$ containing $\mathbf{x}$. In particular, if p is square free, then

$$\mathrm{sing}(\Sigma_p) = \bigcup_{1 \le i < j \le |\mathbf{d}|} \Sigma_{p,i} \cap \Sigma_{p,j}.$$

See Figure 6 for example of the layers structure of Σ_p for p from Example 1.2.

Remark 4.3 (Square factors, overlaps, and multiplicities). Suppose that $p = \prod_{j=1}^N q_j^{c_j}$, where $(q_j)_{j=1}^N$ are the distinct irreducible factors, each raised to the power $c_j \in \mathbb{N}$. Define the reduced polynomial $p^{\mathrm{red}} := \prod_{j=1}^N q_j$, so that it is square free and has the same zero set as p, so $\Sigma_{p^{\mathrm{red}}} = \Sigma_p$, but the total degree of p^{red} may be smaller, in which case $\Sigma_{p^{\mathrm{red}}}$ would have fewer layers than Σ_p. This means that the layers coming from p must overlap, resulting in multiplicity. Note that a given layer $\Sigma_{p,j}$ might comprise of pieces of the varieties of several different irreducible factors of p, each coming with their own multiplicities, which can differ.

To prove Proposition 4.2, we introduce the continuous *phase functions* $\theta_j : \mathbb{R}^n \to \mathbb{R}$, for $j = 1, \dots, |\mathbf{d}|$, in Proposition 4.5 below.

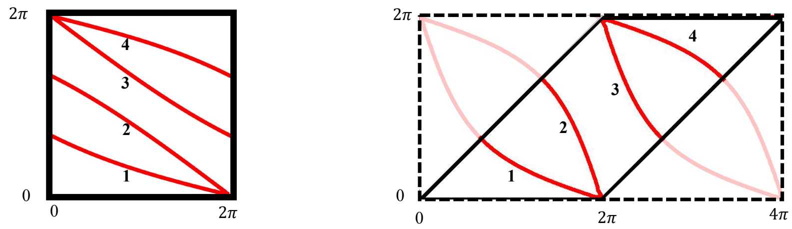


Figure 6. (Right) The four layers of Σ_p, presented in the tilted fundamental domain, for $p \in \mathrm{LY}_{(2,2)}$ given in Example 1.2. (Left) The graphs of $\theta_j(y,0)$ for $y \in (0, 2\pi]$.

Definition 4.4. Given $p = \sum_{\boldsymbol{\alpha}} a_{\boldsymbol{\alpha}} \mathbf{z}^{\boldsymbol{\alpha}} \in \mathrm{LY}_{\mathbf{d}}(n)$ and $\mathbf{x} \in \mathbb{R}^n$, define the univariate polynomial $p_{\mathbf{x}}(s) \in \mathbb{C}[s]$ by

$$p_{\mathbf{x}}(s) = p(se^{ix_1}, \dots, se^{ix_n}) = \sum_{j=0}^{|\mathbf{d}|} \Big(\sum_{|\boldsymbol{\alpha}|=j} a_{\boldsymbol{\alpha}}\, e^{i\langle \mathbf{x}, \boldsymbol{\alpha}\rangle} \Big) s^j. \tag{4.1}$$

The polynomial $p_{\mathbf{x}}$ has degree $|\mathbf{d}|$, with leading term $a_{\mathbf{d}}\, e^{i\langle \mathbf{x}, \mathbf{d}\rangle} s^{|\mathbf{d}|}$, and all roots on the unit circle, say $(e^{i\theta_j(\mathbf{x})})_{j=1}^{|\mathbf{d}|}$. Let $\mathrm{m}(\theta_j, \mathbf{x})$ denote the multiplicity of $e^{i\theta_j(\mathbf{x})}$ as a root of $p_{\mathbf{x}}$, which agrees with the multiplicity of $\mathbf{x}$ as a zero of $p(\exp(i\mathbf{x}))$ when $e^{i\theta_j(\mathbf{x})} = 1$, by Lemma 3.6.

Proposition 4.5 (Phase functions). *Given $p \in \mathrm{LY}_{\mathbf{d}}(n)$, its* phase functions *are $|\mathbf{d}|$ continuous functions $\theta_j : \mathbb{R}^n \to \mathbb{R}$, such that $(e^{i\theta_1(\mathbf{x})}, \dots, e^{i\theta_{|\mathbf{d}|}(\mathbf{x})})$ are the roots of $p_{\mathbf{x}}$, ordered as follows: $\theta_1(\mathbf{x}) \le \cdots \le \theta_{|\mathbf{d}|}(\mathbf{x}) \le \theta_1(\mathbf{x}) + 2\pi$, for all $\mathbf{x} \in \mathbb{R}^n$. Let $\widehat{\Sigma}_p$ denote the lift of Σ_p to $\mathbb{R}^n$, so that*

$$p(\exp(i\mathbf{x})) = a_{\mathbf{d}}\, e^{i\langle \mathbf{d}, \mathbf{x}\rangle} \prod_{j=1}^{|\mathbf{d}|} (1 - e^{i\theta_j(\mathbf{x})}) \quad \textit{and} \quad \widehat{\Sigma}_p = \bigcup_{j=1}^{|\mathbf{d}|} \theta_j^{-1}(2\pi\mathbb{Z}). \tag{4.2}$$

The phase functions enjoy the following properties:

(1) *Each θ_j satisfies $\theta_j(\mathbf{x} + t\mathbf{1}) = \theta_j(\mathbf{x}) - t$, for all $\mathbf{x} \in \mathbb{R}^n$ and $t \in \mathbb{R}$. More generally, θ_j is monotonically decreasing when restricted to lines in any non-negative direction $\ell \in \mathbb{R}^n_{\ge 0}$, with upper and lower bounds on the slope $-t\ell_{\max} \le \theta_j(\mathbf{x} + t\ell) - \theta_j(\mathbf{x}) \le -t\ell_{\min}$, where $\ell_{\min}$ and $\ell_{\max}$ are the minimal and maximal entries of ℓ.*

(2) *Each* θ_j *is real analytic on* $\operatorname{reg}(\widehat{\Sigma}_p)$. *It is also real analytic around any* $\mathbf{x} \in \mathbb{R}^n$ *which is not a discontinuity point of* $\operatorname{m}(\theta_j, \mathbf{x})$*, the multiplicity of* $e^{i\theta(\mathbf{x})}$ *as a root of* $p_{\mathbf{x}}$. *The discontinuity set of* $\mathbf{x} \mapsto \operatorname{m}(\theta_j, \mathbf{x})$*, denoted by* $M_j \subset \mathbb{R}^n$*, is a closed set of dimension* $\dim(M_j) \leq n-1$*, and* $\operatorname{sing}(\widehat{\Sigma}_p) = \bigcup_{j=1}^{|\mathbf{d}|} (\theta_j^{-1}(2\pi\mathbb{Z}) \cap M_j)$.

(3) *The sum of the phase functions is linear in* $\mathbf{x} \in \mathbb{R}^n$:

$$\sum_{j=1}^{|\mathbf{d}|} \theta_j(\mathbf{x}) = \langle \mathbf{d}, \mathbf{x} \rangle + \sum_{j=1}^{|\mathbf{d}|} \theta_j(0).$$

(4) *Translations by the lattice* $2\pi\mathbb{Z}^n$ *act on the ordered tuple* $(\theta_1, \ldots, \theta_{|\mathbf{d}|})$ *by*

$$\theta(\mathbf{x} + 2\pi \boldsymbol{n}) \equiv \sigma^{\langle \mathbf{d}, \boldsymbol{n} \rangle} \theta(\mathbf{x}) \mod 2\pi,$$

for all $\boldsymbol{n} \in \mathbb{Z}^n$*, where* σ *is the permutation* $(1, 2, \ldots, |\mathbf{d}|) \mapsto (|\mathbf{d}|, 1, 2, \ldots, |\mathbf{d}| - 1)$.

Remark 4.6. The choice of such phase functions is not unique. However, given any $\mathbf{x}_0 \in \mathbb{R}^n$ which is a zero of $p(\exp(i\mathbf{x}))$ of multiplicity $m < |\mathbf{d}|$, there is a unique choice of phase functions as in Proposition 4.5, such that

$$0 = \theta_1(\mathbf{x}_0) = \cdots = \theta_m(\mathbf{x}_0) < \theta_{m+1}(\mathbf{x}_0) \leq \cdots \leq \theta_{|\mathbf{d}|}(\mathbf{x}_0) < 2\pi.$$

The proof of Proposition 4.5 includes a proof of Remark 4.6.

Proof of Proposition 4.5. Fix an arbitrary $\mathbf{x}_0 \in \mathbb{R}^n$ such that $p(\exp(i\mathbf{x}_0)) = 0$ with multiplicity $m < |\mathbf{d}|$, so that $s = 1$ is a root of $p_{\mathbf{x}_0}(s)$ of multiplicity m, by Lemma 3.6. Let $(s_j(\mathbf{x}_0))_{j=1}^{|\mathbf{d}|}$ denote the roots of $p_{\mathbf{x}_0}$, so we can write $s_j(\mathbf{x}_0) = e^{i\theta_j(\mathbf{x}_0)}$ such that

$$0 = \theta_1(\mathbf{x}_0) = \cdots = \theta_m(\mathbf{x}_0) < \theta_{m+1}(\mathbf{x}_0) \leq \cdots \leq \theta_{|\mathbf{d}|}(\mathbf{x}_0) < 2\pi.$$

The roots of a univariate polynomial changes continuously with its coefficients, as a result of Rouché's theorem. The coefficients of $p_{\mathbf{x}}$ are analytic in $\mathbf{x} \in \mathbb{R}^n$, so the roots of $p_{\mathbf{x}_0}$ can extend continuously to the roots $(s_j(\mathbf{x}))_{j=1}^{|\mathbf{d}|}$ of $p_{\mathbf{x}}$ for any $\mathbf{x} \in \mathbb{R}^n$, since $\mathbb{R}^n$ is simply connected, and we may do it while maintaining their counter-clockwise ordering. Each $s_j\colon \mathbb{R}^n \to S^1$ can be lifted to a (unique) continuous function $\theta_j\colon \mathbb{R}^n \to \mathbb{R}$ with $\theta_j(\mathbf{x}_0)$ as prescribed above. Since the roots were kept in a counterclockwise order throughout $\mathbb{R}^n$, the relation $\theta_1 \leq \cdots \leq \theta_{|\mathbf{d}|} \leq \theta_1 + 2\pi$ holds everywhere. Since the leading coefficient of $p_{\mathbf{x}}$ is $a_{\mathbf{d}} e^{i\langle \mathbf{d}, \mathbf{x} \rangle}$, as stated in Definition 4.4, we may write $p_{\mathbf{x}}(s) = a_{\mathbf{d}} e^{i\langle \mathbf{d}, \mathbf{x} \rangle} \prod_{j=1}^{|\mathbf{d}|} (s - e^{i\theta_j(\mathbf{x})})$. In particular,

$$p(\exp(i\mathbf{x})) = p_{\mathbf{x}}(1) = a_{\mathbf{d}} e^{i\langle \mathbf{d}, \mathbf{x} \rangle} \prod_{j=1}^{|\mathbf{d}|} (1 - e^{i\theta_j(\mathbf{x})}).$$

Since $\widehat{\Sigma}_p$ is the zero set of $p(\exp(i\mathbf{x}))$ in $\mathbb{R}^n$, then it is the union of $\theta_j^{-1}(2\pi\mathbb{Z})$.

The univariate polynomial changes along the line $\{\mathbf{x} + t\mathbf{1} : t \in \mathbb{R}\}$, for $\mathbf{x} \in \mathbb{R}^n$, by

$$p_{\mathbf{x}+t\mathbf{1}}(s) := p(se^{i(t+x_1)}, \ldots, se^{i(t+x_n)}) = p_{\mathbf{x}}(se^{it}).$$

Together with the continuity and ordering of the phase functions, this gives

$$\theta_j(\mathbf{x}+t\mathbf{1}) = \theta_j(\mathbf{x}) - t. \tag{4.3}$$

Proof of (2). The function $\mathbf{x} \mapsto \mathrm{m}(\theta_j, \mathbf{x})$ is integer valued, so it is continuous at a point if it is constant in a neighborhood of that point. Therefore M_j, its set of discontinuity points, is closed. Let $p_{\mathbf{x}}^{(k)}(s)$ denote the k-th derivative (in s) of $p_{\mathbf{x}}(s)$. Given a point $\mathbf{x} \in \mathbb{R}^n \setminus M_j$ with $m = \mathrm{m}(\theta_j, \mathbf{x})$, every $\mathbf{x}'$ in some small neighborhood of $\mathbf{x}$ satisfies $p_{\mathbf{x}'}^{(k)}(s_j(\mathbf{x}')) = 0$ for all $k < m$ and $p_{\mathbf{x}'}^{(m)}(s_j(\mathbf{x}')) \neq 0$. Then, s_j is analytic around $\mathbf{x}$, by the implicit function theorem for analytic functions, as the $s(\mathbf{x}')$ solution of $p_{\mathbf{x}'}^{(m-1)}(s) = 0$ around the point $(s, \mathbf{x}') = (s_j(\mathbf{x}), \mathbf{x})$. We conclude that s_j is analytic on $\mathbb{R}^n \setminus M_j$, and therefore θ_j is real analytic on the same domain. Since $\theta_j(\mathbf{x}+t\mathbf{1}) = \theta_j(\mathbf{x}) - t$ holds for all j simultaneously, then $\mathrm{m}(\theta_j, \mathbf{x}+t\mathbf{1}) = \mathrm{m}(\theta_j, \mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$ and $t \in \mathbb{R}$. In particular, $\mathrm{m}(\theta_j, \mathbf{x}')$ is locally constant around a point $\mathbf{x} \in \theta_j^{-1}(2\pi k) \subset \widehat{\Sigma}_p$, namely $\mathbf{x} \notin M_j$, if and only if it is constant in some neighborhood of $\mathbf{x}$ in the level set $\theta_j^{-1}(2\pi k)$. Since the multiplicity $\mathrm{m}(\mathbf{x})$ of $\mathbf{x}$ as a zero of $p(\exp(i\mathbf{x}))$ agrees with $\mathrm{m}(\theta_j, \mathbf{x})$ for $\mathbf{x} \in \theta_j^{-1}(2\pi\mathbb{Z}) \subset \Sigma_p$, by Lemma 3.6, and the discontinuity set of $\mathrm{m}(\mathbf{x})$ over $\widehat{\Sigma}_p$ is exactly $\mathrm{sing}(\widehat{\Sigma}_p)$, we conclude that $\mathrm{sing}(\widehat{\Sigma}_p) = \bigcup_{j=1}^{|\mathbf{d}|}(\theta_j^{-1}(2\pi\mathbb{Z}) \cap M_j)$. Next we show that $\bigcup_{j=1}^{|\mathbf{d}|} M_j$ is the projection of an analytic variety of dimension $n-1$, from which it follows that $\dim(M_j) \leq n-1$ for each M_j. By (4.3), as discussed above, each M_j is invariant under translations in direction $\mathbf{1}$. In particular, using (4.3) again, $\mathbf{x} \in \bigcup_{j=1}^{|\mathbf{d}|} M_j$ if and only if $\mathbf{x}+t\mathbf{1} \in \mathrm{sing}(\widehat{\Sigma}_p)$ for some $t \in \mathbb{R}$. According to Lemma 4.1, $\mathrm{sing}(\widehat{\Sigma}_p)$ is an analytic variety of dimension at most $n-2$, so $\{(\mathbf{x}, t) \in \mathbb{R}^n \times \mathbb{R} : \mathbf{x}+t\mathbf{1} \in \mathrm{sing}(\widehat{\Sigma}_p)\}$ is an analytic variety of dimension at most $n-1$, and $\bigcup_{j=1}^{|\mathbf{d}|} M_j$ is the projection of this variety to $\mathbb{R}^n$ and it is closed since each M_j is. We conclude that $\bigcup_{j=1}^{|\mathbf{d}|} M_j$ is a closed subanalytic set with dimension at most $n-1$ (locally around any point), see [8] for the definitions.

Proof of (1). We claim that $\nabla\theta_j(\mathbf{x}) \in \mathbb{R}^n_{\leq 0}$ for all j and all $\mathbf{x} \in \mathbb{R}^n \setminus \bigcup_{j=1}^{|\mathbf{d}|} M_j$. To see this, let $\mathbf{x} \in \mathbb{R}^n \setminus \bigcup_{j=1}^{|\mathbf{d}|} M_j$, and since $\nabla\theta_j|_{\mathbf{x}} = \nabla\theta_j|_{\mathbf{x}+t\mathbf{1}}$ by (4.3), we may assume that $\theta_j(\mathbf{x}) = 0$. In particular, $\mathbf{x} \in \mathrm{reg}(\widehat{\Sigma}_p)$. Note that $\widehat{\Sigma}_p$ can also be written as the zero set of $F^{\mathrm{red}}(\mathbf{x}) = p^{\mathrm{red}}(\exp(i\mathbf{x}))$ for the reduced polynomial p^{red} of p. Since $\mathbf{x} \in \mathrm{reg}(\widehat{\Sigma}_p)$, then it has multiplicity one as a zero of $F^{\mathrm{red}}(\mathbf{x})$, and there is a well defined normal vector to $\widehat{\Sigma}_p$ at $\mathbf{x}$, which is proportional to both $\nabla\theta_j(\mathbf{x})$ and $\nabla F^{\mathrm{red}}(\mathbf{x})$. According to Proposition 3.5, the nonzero coordinates of $\nabla F^{\mathrm{red}}(\mathbf{x})$ have the same phase, and therefore the nonzero coordinates of $\nabla\theta_j(\mathbf{x}) \in \mathbb{R}^n$ all have the same sign. Since (4.3) gives $\nabla\theta_j(\mathbf{x}) \cdot \mathbf{1} = -1$, we find that $\nabla\theta_j(\mathbf{x}) \in \mathbb{R}^n_{\leq 0}$.

It follows that $\theta_j(\mathbf{x}_1) \geq \theta_j(\mathbf{x}_2)$ whenever $\mathbf{x}_2 - \mathbf{x}_1 \in \mathbb{R}^n_{\geq 0}$. To see why, we may use continuity to assume that both $\mathbf{x}_1$ and $\mathbf{x}_2$ lie in the open dense set $\mathbb{R}^n \setminus X$ for $X = \bigcup_{j=1}^{|\mathbf{d}|} M_j$. Consider all possible smooth curves $\varphi\colon [1,2] \to \mathbb{R}^n$ with $\varphi(1) = \mathbf{x}_1, \varphi(2) = \mathbf{x}_2$ and $\varphi'(t) \in \mathbb{R}^n_{\geq 0}$ for all t. For such φ, the composition $\theta_j \circ \varphi$ is continuous for all t, and smooth with non-positive derivative as long as $\varphi(t) \notin X$. Since X is a closed subanalytic set of dimension at most $n-1$, there exists such φ that either intersects X transversely in a discrete set of points, or does not intersect X at all, by Theorem 1.2 in [8] and dimension count. For such φ, $\theta(\varphi(2)) \geq \theta(\varphi(1))$.

Now let $\mathbf{x} \in \mathbb{R}^n$, $t \in \mathbb{R}$, and $\ell \in \mathbb{R}^n_{\geq 0}$. Consider the three points $\mathbf{x}_1 = \mathbf{x} + t\ell_{\min}\mathbf{1}$, $\mathbf{x}_2 = \mathbf{x} + t\ell$, and $\mathbf{x}_3 = \mathbf{x} + t\ell_{\max}\mathbf{1}$, so $\mathbf{x}_3 - \mathbf{x}_2 \in \mathbb{R}^n_{\geq 0}$ and $\mathbf{x}_2 - \mathbf{x}_1 \in \mathbb{R}^n_{\geq 0}$, which gives

$$\theta_j(\mathbf{x} + t\ell_{\max}\mathbf{1}) \leq \theta_j(\mathbf{x} + t\ell) \leq \theta_j(\mathbf{x} + t\ell_{\min}\mathbf{1}),$$

and therefore, using (4.3),

$$\theta_j(\mathbf{x}) - t\ell_{\max} \leq \theta_j(\mathbf{x} + t\ell) \leq \theta_j(\mathbf{x}) - t\ell_{\min}.$$

Proof of (3). Recall that $p_{\mathbf{x}}(s) = a_{\mathbf{d}}\, e^{i\langle \mathbf{d},\mathbf{x}\rangle} \prod_{j=1}^{|\mathbf{d}|}(s - e^{i\theta_j(\mathbf{x})})$, and by substituting $s = 0$ we get

$$p(\mathbf{0}) = p_{\mathbf{x}}(0) = a_{\mathbf{d}}(-1)^{|\mathbf{d}|}\, e^{i(\langle \mathbf{d},\mathbf{x}\rangle + \sum_{j=1}^{|\mathbf{d}|}\theta_j(\mathbf{x}))} \neq 0, \quad \text{for all } \mathbf{x} \in \mathbb{R}^n.$$

Since $\langle \mathbf{d}, \mathbf{x}\rangle + \sum_{j=1}^{|\mathbf{d}|}\theta_j(\mathbf{x})$ is continuous and $e^{i(\langle \mathbf{d},\mathbf{x}\rangle + \sum_{j=1}^{|\mathbf{d}|}\theta_j(\mathbf{x}))} = (-1)^{|\mathbf{d}|}\frac{p(\mathbf{0})}{a_{\mathbf{d}}}$ is constant, then

$$\langle \mathbf{d}, \mathbf{x}\rangle + \sum_{j=1}^{|\mathbf{d}|}\theta_j(\mathbf{x}) = \sum_{j=1}^{|\mathbf{d}|}\theta_j(\mathbf{0}), \quad \text{for all } \mathbf{x} \in \mathbb{R}^n.$$

Proof of (4). To prove that $\theta(\mathbf{x} + 2\pi\mathbf{n}) \equiv \sigma^{\langle \mathbf{d},\mathbf{n}\rangle}\theta(\mathbf{x}) \mod 2\pi$ holds for all $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{n} \in \mathbb{Z}^n$, where σ is the permutation $(1, 2, \ldots, |\mathbf{d}|) \mapsto (|\mathbf{d}|, 1, 2, \ldots, |\mathbf{d}| - 1)$ and $\theta(\mathbf{x}) = (\theta_1(\mathbf{x}), \ldots, \theta_{|\mathbf{d}|}(\mathbf{x}))$, it is enough to consider standard basis vectors, namely $\mathbf{n} = \mathbf{e}_i$. We only consider $\mathbf{n} = \mathbf{e}_1$, but the proof holds for every $\mathbf{e}_i$. For every $\mathbf{x} \in \mathbb{R}^n$, the polynomials $p_{\mathbf{x}}$ and $p_{\mathbf{x}+2\pi\mathbf{e}_1}$ are equal by Definition 4.4, so their roots are equal as a set but may have different counterclockwise numbering, which means that $\theta(\mathbf{x} + 2\pi\mathbf{e}_1) = \sigma^r\theta(\mathbf{x}) + 2\pi\mathbf{k}$ for some integer $0 \leq r \leq |\mathbf{d}|$ and $\mathbf{k} \in \mathbb{Z}^n$, that may a-priori depend on $\mathbf{x}$. Notice that if the roots of $p_{\mathbf{x}}$ are all simple, then r and $\mathbf{k}$ are uniquely determined; however, if all the roots have multiplicity two, for example, then there can be two choice r and $r + 1$. Nevertheless, as of $p_{\mathbf{x}}$ and $p_{\mathbf{x}+2\pi\mathbf{e}_1}$ change continuously in $\mathbf{x}$ in the same manner, then there is a continuous (hence constant) choice of r and $\mathbf{k}$. It is therefore enough to show that $r = d_1$ for some point $\mathbf{x}_0$ that minimise $\min_{j\leq|\mathbf{d}|}\mathrm{m}(\theta_j, \mathbf{x})$, and as this quantity is invariant to translations in direction $\mathbf{1}$, then we may take $\mathbf{x}_0 \in \widehat{\Sigma}_p$. Let $m = \mathrm{m}(\mathbf{x}_0) < |\mathbf{d}|$, and by Remark 4.6, we may assume that

$$0 = \theta_1(\mathbf{x}_0) = \cdots = \theta_m(\mathbf{x}_0) < \theta_{j+1}(\mathbf{x}_0) \leq \cdots \leq \theta_{|\mathbf{d}|}(\mathbf{x}_0) < 2\pi.$$

Let r and $\mathbf{k} = (k_1, \ldots, k_n)$ be such that $\theta(\mathbf{x}_0 + \mathbf{e}_1) = \sigma^r\theta(\mathbf{x}_0) + 2\pi\mathbf{k}$. Using part (3) and the fact that the sum of $\sigma^r\theta(\mathbf{x}_0)$ and $\theta(\mathbf{x}_0)$ is the same, we get

$$\sum_{j=1}^{\mathbf{d}} k_j = \sum_{j=1}^{\mathbf{d}}\theta(\mathbf{x}_0 + \mathbf{e}_1) - \sum_{j=1}^{\mathbf{d}}\theta(\mathbf{x}_0) = -2\pi d_1.$$

By part (1), $\theta_j(\mathbf{x}_0 + \mathbf{e}_1) - \theta_j(\mathbf{x}_0) \in [0, 2\pi]$. Since $2\pi k_j = \theta_j(\mathbf{x}_0 + \mathbf{e}_1) - \theta_{j'}(\mathbf{x}_0)$ for some j', and $|\theta_{j'}(\mathbf{x}_0) - \theta_j(\mathbf{x}_0)| < 2\pi$, then $k_j \in \{0, -1\}$ for all j. The equation for the sum above implies that $\mathbf{k}$ has exactly d_1 entries equal to -1, and the rest are zero.

Denote $v := \sigma^r\theta(\mathbf{x}_0)$ so that

$$v = (\theta_{|\mathbf{d}|-r+1}(\mathbf{x}_0), \theta_{|\mathbf{d}|-r+2}(\mathbf{x}_0), \ldots, \theta_{|\mathbf{d}|}(\mathbf{x}_0), \theta_1(\mathbf{x}_0), \ldots, \theta_{|\mathbf{d}|-r}(\mathbf{x}_0)).$$

Then $v_1 \le \cdots \le v_r$ and $v_{r+1} \le \cdots \le v_{|\mathbf{d}|}$, with $v_{r+1} = 0 < v_r < 2\pi$, while $v + 2\pi\mathbf{k}$ is ordered increasingly. We conclude that $k_j = -1$ for $j \le r$ and $k_j = 0$ for $j > r$, which means that $r = d_1$. This proves part (4). ■

We are now in position to prove Proposition 4.2 using Proposition 4.5.

Proof of Proposition 4.2. Consider the linear transformation $\mathcal{T}(\mathbf{y}, t) = (\mathbf{y}, 0) + t\mathbf{1}$, and the quotient map $\pi\colon \mathbb{R}^n \to \mathbb{R}^n/2\pi\mathbb{Z}^n$. Consider $\Omega := \mathcal{T}((0, 2\pi]^n)$, which is a fundamental domain of $2\pi\mathbb{Z}^n$, so $\pi\colon \Omega \to \mathbb{R}^n/2\pi\mathbb{Z}^n$ is bijective. The map $\mathbf{y} \mapsto \mathcal{T}(\mathbf{y}, \theta_j(\mathbf{y}, 0))$ is continuous with a continuous inverse $\mathbf{x} \mapsto (x_1 - x_n, \ldots, x_{n-1} - x_n)$, since θ_j is continuous by Proposition 4.5, and therefore $\varphi_j(\mathbf{y}) = \pi(\mathcal{T}(\mathbf{y}, \theta_j(\mathbf{y}, 0)))$ is a homeomorphism between $(0, 2\pi]^{n-1}$ and its image, which we denote by $\Sigma_{p,j}$.

Notice that $\theta_j(\mathcal{T}(\mathbf{y}, t)) = \theta_j(\mathbf{y}, 0) - t$ by part (1) of Proposition 4.5, so

$$\theta_j(\mathcal{T}(\mathbf{y}, t)) \in 2\pi\mathbb{Z} \iff \varphi_j(\mathbf{y}) = \pi(\mathcal{T}(\mathbf{y}, t)). \tag{4.4}$$

($\Sigma_{p,j} \subset \Sigma_p$) Given $\mathbf{y} \in (0, 2\pi]^{n-1}$, let $t = \theta_j(\mathbf{y}, 0)$, so that $\theta_j(\mathcal{T}(\mathbf{y}, t)) = \theta_j(\mathbf{y}, 0) - t = 0$. Therefore, $\mathcal{T}(\mathbf{y}, t) \in \widehat{\Sigma}_p$ which means that $\pi(\mathcal{T}(\mathbf{y}, t)) = \varphi(\mathbf{y}) \in \Sigma_p$ by (4.4).

($\Sigma_p \subset \bigcup_{j=1}^{|\mathbf{d}|} \Sigma_{p,j}$) Consider $\Omega := \mathcal{T}((0, 2\pi]^n)$, which is a fundamental domain of $2\pi\mathbb{Z}^n$, so $\pi\colon \Omega \to \mathbb{R}^n/2\pi\mathbb{Z}^n$ is bijective, and therefore any $\mathbf{x} \in \Sigma_p \subset \mathbb{R}^n/2\pi\mathbb{Z}^n$ has a unique point $(\mathbf{y}, t) \in (0, 2\pi]^{n-1} \times (0, 2\pi]$ for which $\pi(\mathcal{T}(\mathbf{y}, t)) = \mathbf{x}$. In such case, $\mathcal{T}(\mathbf{y}, t) \in \widehat{\Sigma}_p$ so $\theta_j(\mathcal{T}(\mathbf{y}, t)) \in 2\pi\mathbb{Z}$ for some j, and $\varphi_j(\mathbf{y}) = \pi(\mathcal{T}(\mathbf{y}, t)) = \mathbf{x}$ by (4.4).

(Multiplicity and singularity) Let $\mathbf{x} = \pi(\mathcal{T}(\mathbf{y}, t)) \in \Sigma_p$ as above. The number of layers containing $\mathbf{x}$ is the number of j's for which $\varphi_j(\mathbf{y}) = \mathbf{x}$, which are those j's for which $e^{i\theta_j(\mathcal{T}(\mathbf{y},t))} = 1$. This is exactly $\mathrm{m}(\mathbf{x})$, the multiplicity of $\mathbf{x}$ as a zero of $p(\exp(i\mathbf{x}))$, by Lemma 3.6. If p is square free, then $\mathbf{x} \in \operatorname{sing}(\Sigma_p) \iff \mathrm{m}(\mathbf{x}) > 1 \iff \mathbf{x} \in \Sigma_{p,i} \cap \Sigma_{p,j}$ for some $i \ne j$, by Lemma 4.1.

(Real analyticity) Suppose that $\varphi_j(\mathbf{y}_0) = \pi(\mathcal{T}(\mathbf{y}_0, t)) \in \operatorname{reg}(\Sigma_p)$. Then $\mathcal{T}(\mathbf{y}_0, t) \in \operatorname{reg}(\widehat{\Sigma}_p)$, which means that θ_j is real analytic around $\mathcal{T}(\mathbf{y}_0, t) = (\mathbf{y}_0, 0) + t\mathbf{1}$, according to Proposition 4.5. Therefore, θ_j is real analytic around $(\mathbf{y}_0, 0)$, due to the shift $\theta_j((\mathbf{y}, 0) + v) = \theta_j(\mathcal{T}(\mathbf{y}, t) + v) + t$ for all $v \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^n$. It follows that $\mathbf{y} \mapsto \mathcal{T}(\mathbf{y}, \theta_j(\mathbf{y}, 0))$ is real analytic around $\mathbf{y}_0$, and therefore so is φ_j. ■

We may now relate the $\operatorname{mingap}(p)$ given in Definition 1.11 to the phase functions $(\theta_j)_{j=1}^{|\mathbf{d}|}$ defined in Proposition 4.5.

Lemma 4.7. *Let $p \in \mathrm{LY}_{\mathbf{d}}(n)$ and let $(\theta_j)_{j=1}^{|\mathbf{d}|}$ be its phase functions. Then*

$$\operatorname{mingap}(p) = \min[\theta_{j+1}(\mathbf{x}) - \theta_j(\mathbf{x})] \quad \textit{over } 1 \le j \le |\mathbf{d}| \textit{ and } \mathbf{x} \in [0, 2\pi]^n,$$

where we set $\theta_{|\mathbf{d}|+1} = \theta_1 + 2\pi$ by convention. In particular, $\operatorname{mingap}(p) > 0$ *if and only if p has no square factors and* $\operatorname{sing}(\Sigma_p) = \emptyset$ *(equivalently, $\nabla p(\mathbf{z}) \ne 0$ for any $\mathbf{z} \in \mathbb{T}^n$ with $p(\mathbf{z}) = 0$).*

Proof. By definition, if $\mathbf{z} = \exp(i\mathbf{x}) \in \mathbb{T}^n$, then $(e^{i\theta_j(\mathbf{x})})_{j=1}^{|\mathbf{d}|}$ are the roots of $p_{\mathbf{z}}$, ordered cyclically with multiplicity, and so $\operatorname{mingap}(p_{\mathbf{z}})$ is the minimum of $\theta_{j+1}(\mathbf{x}) - \theta_j(\mathbf{x})$ over $1 \le j \le |\mathbf{d}|$. Since this difference is invariant to $\mathbf{x} \mapsto \mathbf{x} + t\mathbf{1}$, we see that $\operatorname{mingap}(p_{\mathbf{z}}) = 0$ if and only if $\theta_{j+1}(\mathbf{x} + t\mathbf{1}) = \theta_j(\mathbf{x} + t\mathbf{1}) = 0$ for some t, which means that $e^{it}\mathbf{z} = \exp(\mathbf{x} + t\mathbf{1})$ is a multiple zero of p (multiplicity two or higher) by Lemma 3.6. Since $\operatorname{mingap}(p)$ is the minimum of $\operatorname{mingap}(p_{\mathbf{z}})$ over $\mathbf{x} \in [0, 2\pi]^n$, then $\operatorname{mingap}(p) > 0$ if and only if $\operatorname{mingap}(p_{\mathbf{z}}) > 0$ for all $\mathbf{z} \in \mathbb{T}^n$ which happens if and only if p has no multiple zeros in $\mathbb{T}^n$, which happens if and only if p has no square factors and $\operatorname{sing}(\Sigma_p) = \emptyset$. ■

5. Zeros density. Proof of Theorem 1.9

When the polynomial p arises from a quantum graph, then Theorem 1.9 holds by the proof of Weyl's law for quantum graphs in Lemma 3.7.4 of [6]. In such case, $p(\mathbf{z}) = \det(1 - D(z)S)$, where S is some orthogonal matrix and $D(z) = \operatorname{diag}(z_1, \ldots, z_n, z_1, \ldots, z_n)$. The proof for a general Lee–Yang polynomial p is similar. The roots $(e^{i\theta_j(\mathbf{x})})_{j=1}^{|\mathbf{d}|}$ of the univariate polynomial $p_{\mathbf{x}}(s)$ replace the eigenvalues of $D(\exp(i\mathbf{x}))S$.

Proof of Theorem 1.9. Let $p \in \mathrm{LY}_{\mathbf{d}}$, and consider the phase functions $(\theta_j)_{j=1}^{|\mathbf{d}|}$ described in Proposition 4.5. Given $\ell \in \mathbb{R}_+^n$, a point $x \in \mathbb{R}$ is a zero of $f(x) = p(\exp(ix\ell))$ of multiplicity m if and only if exactly m of the phase functions satisfy $\theta_j(x\ell) \in 2\pi\mathbb{Z}$, by Lemma 3.6 and Proposition 4.5. The number of zeros of $p(\exp(ix\ell))$, counted with multiplicities, in an interval $[a, b] \subset \mathbb{R}$ is therefore

$$\mu_{p,\ell}([a,b]) = \sum_{j=1}^{|\mathbf{d}|} |\{x \in [a,b] : \theta_j(x\ell) \in 2\pi\mathbb{Z}\}|.$$

According to part (1) of Proposition 4.5, $\theta_j(a\ell) > \theta_j(b\ell)$ for each j, and the map $x \mapsto \theta_j(x\ell)$ is a bijection between $[a, b]$ and the interval $[\theta_j(b\ell), \theta_j(a\ell)] \subset \mathbb{R}$, which has length $\theta_j(a\ell) - \theta_j(b\ell)$. Therefore,

$$|\{x \in [a,b] : \theta_j(x\ell) \in 2\pi\mathbb{Z}\}| = \left|[\theta_j(b\ell), \theta_j(a\ell)] \cap 2\pi\mathbb{Z}\right| = \frac{\theta_j(a\ell) - \theta_j(b\ell)}{2\pi} + \mathrm{err}_j,$$

with $|\mathrm{err}_j| \le 1$. Let $\mathrm{err} := \sum_{j=1}^{|\mathbf{d}|} \mathrm{err}_j$. Then $|\mathrm{err}| \le |\mathbf{d}|$ and

$$\mu_{p,\ell}([a,b]) = \sum_{j=1}^{|\mathbf{d}|} \frac{\theta_j(a\ell) - \theta_j(b\ell)}{2\pi} + \mathrm{err} = \frac{\langle \mathbf{d}, \ell\rangle}{2\pi} |b - a| + \mathrm{err}.$$

In the last equality, we used part (3) of Proposition 4.5. This proves part (1) of the theorem, by substituting $[a, b] = [x, x + T]$ and $\mathrm{err}(x, T) = \mathrm{err}$.

For part (2) of the theorem, let $x_{j+1} > x_j$ be consecutive zeros of $f(x)$, and consider an arbitrary interval $I \subset (x_j, x_{j+1})$, so

$$0 = \mu_{p,\ell}(I) \le \frac{\langle \mathbf{d}, \ell\rangle}{2\pi} |I| + \mathrm{err} \quad \Rightarrow \quad |I| \le 2\pi \frac{|\mathrm{err}|}{\langle \mathbf{d}, \ell\rangle} \le 2\pi \frac{|\mathbf{d}|}{\langle \mathbf{d}, \ell\rangle},$$

and $|I|$ can get arbitrarily close to $x_{j+1} - x_j$. ■

6. Ergodic dynamics on Σ_p

To prove the existence of a gap distribution for the eigenvalues of a quantum graph, Barra and Gaspard introduced an ℓ-dependent "first return" dynamical system on Σ_p, for the associated Lee–Yang polynomial p, which is uniquely ergodic when ℓ is $\mathbb{Q}$-linearly independent [4]. The same holds for any Lee–Yang p, as shown in this section.

Given $\ell \in \mathbb{R}^n_+$, consider the linear flow on $\mathbb{R}^n/2\pi\mathbb{Z}^n$ induced by the constant vector field ℓ. That is, the flow at time t is a map $\mathbf{x} \mapsto \mathbf{x} + t\ell \bmod 2\pi$ from $\mathbb{R}^n/2\pi\mathbb{Z}^n$ to itself. The minimal $t > 0$ for which a point $\mathbf{x} \in \Sigma_p$ gets back to Σ_p is called the *first return time* $\tau_\ell(\mathbf{x})$, and $\mathbf{x} \mapsto \mathbf{x} + \tau_\ell(\mathbf{x})\,\ell \bmod 2\pi$ is a map from Σ_p to itself that defines a dynamical system.

Remark 6.1. Throughout this subsection, we omit the "mod 2π" when it is clear from the context.

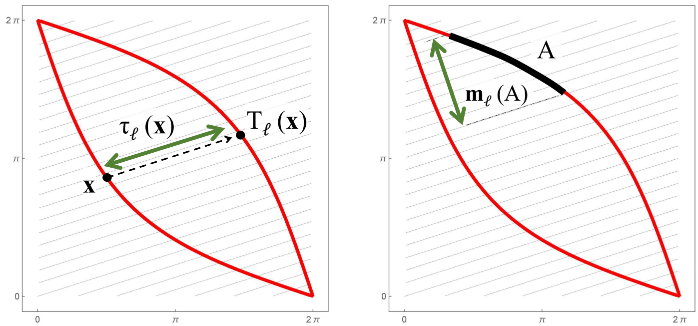


Figure 7. Illustration of T_ℓ, τ_ℓ, and the measure $\mathbf{m}_\ell$, as in Definition 6.2, for the Lee–Yang polynomial p from Example 1.2 and $\ell = (\pi, 1)$. In the background, the line $(x, y) = t\ell \bmod 2\pi$ for $t \in [0, 44]$.

Definition 6.2 (Dynamical system on Σ_p). Let $p \in \mathrm{LY}_{\mathbf{d}}(n)$ and $\ell \in \mathbb{R}^n_+$. The *first-return time* $\tau_\ell\colon \Sigma_p \to \mathbb{R}_+$ and the *first-return map* $T_\ell\colon \Sigma_p \to \Sigma_p$ are defined by

$$\tau_\ell(\mathbf{x}) := \min\{t > 0 : \mathbf{x} + t\ell \in \Sigma_p\} \quad \text{and} \quad T_\ell(\mathbf{x}) := \mathbf{x} + \tau_\ell(\mathbf{x})\,\ell.$$

The measure $\mathbf{m}_\ell$ is a Borel measure on Σ_p defined for any Borel subset $A \subset \Sigma_p$ by

$$\mathbf{m}_\ell(A) := \lim_{\varepsilon \to 0} \frac{\mathrm{vol}_n(A_{\varepsilon\ell})}{2\varepsilon}, \quad \text{with} \quad A_{\varepsilon\ell} := \{\mathbf{x} + t\ell : \mathbf{x} \in A,\ |t| < \varepsilon\},$$

where vol_n is n-dimensional volume (Lebesgue measure) in $\mathbb{R}^n/2\pi\mathbb{Z}^n$.

Definition 6.3. A bounded function $h\colon \Sigma_p \to \mathbb{C}$ is called *Riemann integrable* if its discontinuity set has zero volume in Σ_p, with respect to the $n-1$ dimensional volume form induced by the n-dimensional volume form on $\mathbb{R}^n/2\pi\mathbb{Z}^n$.

Recall that if p has a decomposition into distinct irreducible factors $p = \prod_{j=1}^M q_j^{c_j}$, then the reduced polynomial is $p^{\mathrm{red}} := \prod_{j=1}^M q_j$ and its multi-degree is denoted by $\mathbf{d}^{\mathrm{red}}$. Let $\mathrm{m}(\mathbf{x})$ denote the multiplicity of $\exp(i\mathbf{x})$ as a zero of p.

Theorem 6.4 (Unique ergodicity). *Let $p \in \mathrm{LY}_{\mathbf{d}}(n)$, let $\ell \in \mathbb{R}^n_+$ with $\mathbb{Q}$-linearly independent entries, and fix an arbitrary point $\mathbf{x}_0 \in \Sigma_p$. Let $(x_j)_{j\in\mathbb{Z}}$ denote the zeros of $f(x) = p(\exp(i(\mathbf{x}_0 + x\ell)))$, ordered increasingly with multiplicities, and consider $(T^j_\ell(\mathbf{x}_0))_{j\in\mathbb{Z}}$, the T_ℓ orbit of $\mathbf{x}_0$. Then the averages of any bounded Riemann integrable $h\colon \Sigma_p \to \mathbb{C}$ over the orbit $(T^j_\ell(\mathbf{x}_0))_{j\in\mathbb{N}}$, and over the sequence $(\mathbf{x}_0 + x_j\ell)_{j\in\mathbb{N}}$, are independent of $\mathbf{x}_0$ and are given by*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} h(T^j_\ell(\mathbf{x}_0)) = \frac{1}{(2\pi)^{n-1}\langle \mathbf{d}^{\mathrm{red}}, \ell\rangle}\int_{\Sigma_p} h(\mathbf{x})\, d\mathbf{m}_\ell(\mathbf{x}) \tag{6.1}$$

$$\lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} h(\mathbf{x}_0 + x_j\ell) = \frac{1}{(2\pi)^{n-1}\langle \mathbf{d}, \ell\rangle}\int_{\Sigma_p} \mathrm{m}(\mathbf{x})h(\mathbf{x})\, d\mathbf{m}_\ell(\mathbf{x}), \tag{6.2}$$

where $\mathbf{m}_\ell(\Sigma_p) = (2\pi)^{n-1}\langle \mathbf{d}^{\mathrm{red}}, \ell\rangle$ and $\int_{\Sigma_p} \mathrm{m}(\mathbf{x})\, d\mathbf{m}_\ell(\mathbf{x}) = (2\pi)^{n-1}\langle \mathbf{d}, \ell\rangle$.

For Lee–Yang polynomials associated to quantum graphs, this is shown in [4, 7, 12]. A proof for any Lee–Yang polynomial is provided for completeness.

Proof. Let $\{x_i\}_{i\in\mathbb{N}}$ denote the positive zeros of $f(x) = p(\exp(i(\mathbf{x}_0 + x\ell)))$ ordered *with multiplicity*, and let $x_0 = 0$, since $\mathbf{x}_0 \in \Sigma_p$. Let $\{k_i\}_{i\in\mathbb{N}}$ denote the *distinct* zeros of f, ordered *without multiplicity*, with $k_0 = 0$, so that $T^i_\ell(\mathbf{x}_0) = \mathbf{x}_0 + k_i\ell$ and $\mathrm{m}(T^i_\ell(\mathbf{x}_0))$ is the multiplicity of k_i as a zero of f for all $i \in \mathbb{N}$. The first step of the proof is showing that for any bounded Riemann integrable $h\colon \Sigma_p \to \mathbb{C}$,

$$\lim_{R\to\infty} \frac{1}{R}\sum_{k_i\le R} h(T^i_\ell(\mathbf{x}_0)) = \frac{1}{(2\pi)^n}\int_{\Sigma_p} h\, d\mathbf{m}_\ell. \tag{6.3}$$

Consider a layer $\Sigma_{p,j}$ as in Proposition 4.2, and let $h = \chi_A$ be the indicator function of a Borel set $A \subset \Sigma_{p,j}$ with boundary of zero volume in Σ_p. The set $A_{\varepsilon\ell} = \{\mathbf{x} + t\ell : (\mathbf{x}, t) \in A \times [-\varepsilon, \varepsilon]\}$ is then a Borel set with boundary of zero volume in $\mathbb{R}^n/2\pi\mathbb{Z}^n$. Since ℓ has $\mathbb{Q}$-linearly independent entries, the Kronecker–Weyl theorem gives

$$\frac{\mathrm{vol}_n(A_{\varepsilon\ell})}{(2\pi)^n} = \lim_{R\to\infty} \frac{\mathrm{length}(\{t \in [0, R] : \mathbf{x}_0 + t\ell \in A_{\varepsilon\ell}\})}{R}. \tag{6.4}$$

Let $\mathcal{A} = \{k_i : T^i_\ell(\mathbf{x}_0) \in A\} \subset \mathbb{R}$, so that $\theta_j(\mathbf{x}_0 + k_i\ell) \in 2\pi\mathbb{Z}$ for all $k_i \in \mathcal{A}$ since $A \subset \Sigma_{p,j}$. The function $t \mapsto \theta_j(\mathbf{x}_0 + t\ell)$ is strictly monotone with uniform upper and lower bounds on its slope, by Proposition 4.5 part (1), so $\mathcal{A}$ is uniformly discrete, and therefore, for small enough $\varepsilon > 0$, the 2ε-intervals $[k_i - \varepsilon, k_i + \varepsilon]$ for $k_i \in \mathcal{A}$ are mutually disjoint. The set $\{t \in [0, R] : \mathbf{x}_0 + t\ell \in A_{\varepsilon\ell}\}$ is the intersection of these disjoint 2ε-intervals with $[0, R]$, so up to an error of 2ε, its lengths is $2\varepsilon\,|\mathcal{A} \cap [0, R]| = 2\varepsilon \sum_{k_i\le R} h(T^i_\ell(\mathbf{x}_0))$. Substituting this estimate into (6.4) gives

$$\frac{\mathrm{vol}_n(A_{\varepsilon\ell})}{(2\pi)^n} = \lim_{R\to\infty}\Big(2\varepsilon\frac{1}{R}\sum_{k_i\le R} h(T^i_\ell(\mathbf{x}_0)) + \frac{1}{R}O(\varepsilon)\Big) = 2\varepsilon \lim_{R\to\infty}\frac{1}{R}\sum_{k_i\le R} h(T^i_\ell(\mathbf{x}_0)). \tag{6.5}$$

Dividing both sides by 2ε and taking $\varepsilon \to 0$ proves (6.3) for the indicator function $h = \chi_A$. Both sides of (6.3) are linear in h, so it holds for any step function $\sum_{j=1}^N c_j \chi_{A_j}$ such that the sets $A_j \subset \Sigma_p$ are Borel with boundary of zero volume in Σ_p. Such functions can approximate (in the sup-norm) any non-negative bounded Riemann integrable function from below and above to any given precision, by taking the upper and lower Darboux sums, as they converge to the Riemann integral of h. We conclude that (6.3) holds for any bounded Riemann integrable function $h\colon \Sigma_p \to \mathbb{C}$, as it can be written as $h = h_1 - h_2 + i(h_3 - h_4)$ so that each h_j is real non-negative, bounded, and Riemann integrable, and hence can be approximated by step functions, for which (6.3) holds.

The second step is calculating $\mathbf{m}_\ell(\Sigma_p)$ and $\int_{\Sigma_p} \mathrm{m}(\mathbf{x})\, d\mathbf{m}_\ell(\mathbf{x})$. The sum of multiplicities of distinct zeros up to T is the number of repeated zeros up to R, $\sum_{k_i \le R} \mathrm{m}(T_\ell^i(\mathbf{x}_0)) = |\{x_i < R\}|$, which is equal to $\frac{\langle \mathbf{d}, \ell\rangle}{2\pi} R + O(1)$, by Theorem 1.9, and applying (6.3) to $h = m$ gives

$$\frac{1}{(2\pi)^n} \int_{\Sigma_p} \mathrm{m}(\mathbf{x})\, d\mathbf{m}_\ell(\mathbf{x}) = \lim_{R\to\infty} \frac{1}{R} \sum_{k_i \le T} \mathrm{m}(T_\ell^i(\mathbf{x}_0)) = \lim_{R\to\infty} \frac{|\{x_i < R\}|}{R} = \frac{\langle \mathbf{d}, \ell\rangle}{2\pi}.$$

It follows that $\int_{\Sigma_p} \mathrm{m}(\mathbf{x})\, d\mathbf{m}_\ell(\mathbf{x}) = (2\pi)^{n-1}\langle \mathbf{d}, \ell\rangle$, and by replacing p with p^{red} we get that $\mathbf{m}_\ell(\Sigma_p) = \int_{\Sigma_p} d\mathbf{m}_\ell = (2\pi)^{n-1}\langle \mathbf{d}^{\mathrm{red}}, \ell\rangle$. To see why, notice that the torus zero set of p^{red} is equal to Σ_p, with the same measure $\mathbf{m}_\ell$, but with multiplicity function which is one for every $\mathbf{x} \in \mathrm{reg}(\Sigma_p)$. The complement has $\mathbf{m}_\ell(\mathrm{sing}(\Sigma_p)) = 0$, since $\dim(A) \le n-2$ for $A = \mathrm{sing}(\Sigma_p)$, which means that $\dim(A_{\varepsilon\ell}) \le n-1$ and so $\mathrm{vol}_n(A_{\varepsilon\ell}) = 0$.

To prove (6.1), apply (6.3) twice and divide the two limits:

$$\frac{\int_{\Sigma_p} h\, d\mathbf{m}_\ell}{\mathbf{m}_\ell(\Sigma_p)} = \lim_{R\to\infty} \frac{\sum_{k_i \le R} h(T_\ell^i(\mathbf{x}_0))}{|\{k_i \le R\}|} = \lim_{N\to\infty} \frac{\sum_{i=1}^N h(T_\ell^i(\mathbf{x}_0))}{N}.$$

Since

$$\sum_{x_i \le R} h(\mathbf{x}_0 + x_i \ell) = \sum_{k_i \le R} \mathrm{m}(T_\ell^i(\mathbf{x}_0))\, h(T_\ell^i(\mathbf{x}_0)),$$

the same argument gives

$$\frac{\int_{\Sigma_p} \mathrm{m}(\mathbf{x})\, h(\mathbf{x})\, d\mathbf{m}_\ell(\mathbf{x})}{\int_{\Sigma_p} \mathrm{m}(\mathbf{x})\, d\mathbf{m}_\ell(\mathbf{x})} = \lim_{R\to\infty} \frac{\sum_{x_i \le R} h(\mathbf{x}_0 + x_i \ell)}{|\{x_i \le R\}|} = \lim_{N\to\infty} \frac{\sum_{i=1}^N h(\mathbf{x}_0 + x_i \ell)}{N}. \quad \blacksquare$$

6.1. Properties of τ_ℓ and $\mathbf{m}_\ell$

The gap distributions in Section 9 are defined in terms of τ_ℓ and $\mathbf{m}_\ell$. The needed properties of τ_ℓ and $\mathbf{m}_\ell$ are stated in the next two lemmas.

In what follows, consider $\mathrm{reg}(\Sigma_p)$ as a smooth Riemannian manifold with volume form $d\sigma$, induced by $d\mathrm{vol}_n$ in $\mathbb{R}^n/2\pi\mathbb{Z}^n$, and the normal vector field $\hat{\mathbf{n}}$ with $\hat{\mathbf{n}}(\mathbf{x}) \in \mathbb{R}^n_{\ge 0}$ for all $\mathbf{x} \in \mathrm{reg}(\Sigma_p)$, as guaranteed by Proposition 3.5. The $n-1$ form with dx_j missing is denoted by $dx_1 \wedge dx_2 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n$.

Lemma 6.5. *The measure $\mathbf{m}_\ell$ is absolutely continuous with respect to $d\sigma$, the volume form on* $\operatorname{reg}(\Sigma_p)$*, with a strictly positive distribution*

$$d\mathbf{m}_\ell = \langle \hat{\mathbf{n}}, \ell \rangle \, d\sigma = \sum_{j=1}^{n} \ell_j (-1)^{j+1} dx_1 \wedge dx_2 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n. \tag{6.6}$$

For each layer $\Sigma_{p,j}$, with parameterization $\varphi_j \colon (0, 2\pi]^{n-1} \to \Sigma_{p,j}$ as in Proposition 4.2, and for every measurable $h \colon \Sigma_{p,j} \to \mathbb{C}$,

$$\int_{\Sigma_{p,j}} h(\mathbf{x}) \, d\mathbf{m}_\ell(\mathbf{x}) = -\int_{(0,2\pi]^{n-1}} h(\varphi_j(\mathbf{y})) \langle \nabla\theta_j(\mathbf{y}, 0), \ell \rangle \, d\mathbf{y}, \tag{6.7}$$

and in particular, for $\ell = \mathbf{1}$,

$$\int_{\Sigma_{p,j}} h(\mathbf{x}) \, d\mathbf{m}_{\mathbf{1}}(\mathbf{x}) = \int_{(0,2\pi]^{n-1}} h(\varphi_j(\mathbf{y})) \, d\mathbf{y}. \tag{6.8}$$

Proof of Lemma 6.5. It was shown in the proof of Theorem 6.4 that $\mathbf{m}_\ell(\operatorname{sing}(\Sigma_p)) = 0$, so $\mathbf{m}_\ell$ is supported on $\operatorname{reg}(\Sigma_p)$. To show (6.6), it is enough to consider a small open set $A \subset \operatorname{reg}(\Sigma_p)$. If A is sufficiently small, for $\varepsilon > 0$ sufficiently small, we can choose local coordinates $\kappa = (\kappa_1, \ldots, \kappa_{n-1})$ such that $d\kappa = d\sigma$, which extend to local coordinates in a neighborhood of $A_{\varepsilon\ell}$ by adding a coordinate t in the normal direction $\hat{\mathbf{n}}$. The fact that $d\sigma$ is induced from $d\mathrm{vol}_n$ means that $d\mathrm{vol}_n = d\kappa \, dt$. Therefore, $d\kappa = 2\varepsilon \int_A \langle \hat{\mathbf{n}}(\kappa), \ell \rangle \, d\kappa > 0$, using that $\ell \in \mathbb{R}^n_+$ and $\hat{\mathbf{n}}(\kappa) \in \mathbb{R}^n_{\geq 0}$ for all $\kappa \in A$. We conclude that $\hat{\mathbf{n}}(\kappa) \in \mathbb{R}^n_{\geq 0}$ for all $\kappa \in A$. We conclude that

$$\mathbf{m}_\ell(A) := \lim_{\varepsilon \to 0} \frac{\mathrm{vol}_n(A_{\varepsilon\ell})}{2\varepsilon} = \int_A \langle \hat{\mathbf{n}}, \ell \rangle \, d\sigma.$$

By definition, the form $\langle \hat{\mathbf{n}}, \ell \rangle \, d\sigma$ agrees with the $n-1$ form

$$\omega = \sum_{j=1}^{n} \ell_j (-1)^{j+1} \, dx_1 \wedge dx_2 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n$$

when restricted to $\operatorname{reg}(\Sigma_p)$.

We are left with deducing (6.7) from (6.6) by simple change of variables. Let $\mathbf{y} \in (0, 2\pi]^{n-1}$ be such that $\varphi_j(\mathbf{y}) \in \operatorname{reg}(\Sigma_p)$, and let $D = D\varphi_j|_{\mathbf{y}}$ be the $n \times (n-1)$ matrix of derivatives whose (s, i)th entry is $\partial(\varphi_j)_s / \partial y_i|_{\mathbf{y}}$. Then the change of variables formula for $\mathbf{x} = \varphi_j(\mathbf{y})$ is

$$\sum_{k=1}^{n} \ell_k (-1)^{k+1} \, dx_1 \wedge dx_2 \wedge \cdots \wedge \widehat{dx_k} \wedge \cdots \wedge dx_n = \sum_{j=k}^{n} \ell_k (-1)^{k+1} D_k \, d\mathbf{y},$$

where D_j denotes the $(n-1) \times (n-1)$ minor of D obtained by removing the j-th row. Adding ℓ as a column vector gives an $n \times n$ matrix $M = (D \ \ell)$, whose determinant is exactly $\det(M) = \sum_{k=1}^{n} \ell_k (-1)^{k+1} D_k$, by expanding according to the column ℓ. We need to show that $\det(M) = -\langle \nabla\theta_j(\mathbf{y}, 0), \ell \rangle$. Let $v = (\partial\theta_j(\mathbf{y}, 0)/\partial y_1, \ldots, \partial\theta_j(\mathbf{y}, 0)/\partial y_{n-1})$

$\in \mathbb{R}^{n-1}$, so that the entries of D are $D_{s,i} = v_i$ if $i \neq s$, and $D_{i,i} = v_i + 1$, since $\varphi_j(\mathbf{y}) = (\mathbf{y}, 0) + \theta_j(\mathbf{y}, 0)\mathbf{1}$. Subtracting the last row of M from all other gives the matrix

$$\tilde{M} = \begin{pmatrix} \mathrm{id}_{n-1} & \tilde{\ell} \\ v & \ell_n \end{pmatrix}, \quad \text{with } \tilde{\ell} = (\ell_1 - \ell_n, \ell_2 - \ell_n, \ldots, \ell_{n-1} - \ell_n),$$

so that $\det(M) = \det(\tilde{M}) = \ell_n - \langle v, \tilde{\ell}\rangle$, using the Schur complement in the last equality. Notice that $\langle v, \tilde{\ell}\rangle = \langle \nabla\theta_j(\mathbf{y}, 0), \ell - \ell_n\mathbf{1}\rangle = \langle \nabla\theta_j(\mathbf{y}, 0), \ell\rangle - \ell_n$ since $\langle \nabla\theta_j(\mathbf{y}, 0), \mathbf{1}\rangle = -1$ by part (1) of Proposition 4.5 . We conclude that $\det(M) = -\langle \nabla\theta_j(\mathbf{y}, 0), \ell\rangle$, which proves (6.7), and (6.8) follows from $\langle \nabla\theta_j(\mathbf{y}, 0), \mathbf{1}\rangle = -1$ again. ■

For the next lemma, let $p \in \mathrm{LY}_{\mathbf{d}}(n)$ and, for any $\ell \in \mathbb{R}^n_+$, consider the first-return-time $\tau_\ell \colon \Sigma_p \to \mathbb{R}_+$ introduced in Definition 6.2.

Lemma 6.6. *For any fixed $\ell \in \mathbb{R}^n_+$, the map $\tau_\ell \colon \Sigma_p \to \mathbb{R}_+$ is bounded by $2\pi|\mathbf{d}|/\langle \mathbf{d}, \ell\rangle$ and satisfies the following.*

(1) *Given any pair of distinct consecutive zeros of $f(x) = p(\exp(ix\ell))$, say $x_{j+1} > x_j$,*

$$\tau_\ell(\mathbf{x}) = x_{j+1} - x_j \quad \textit{for } \mathbf{x} = x_j\ell \mod 2\pi.$$

If $x_{j+1} = x_j$, then $\mathbf{x} \in \mathrm{sing}(\Sigma_p)$, and

(2) *for any $\mathbf{x} \in \mathrm{sing}(\Sigma_p)$ and any $U \subset \Sigma_p$ neighborhood of $\mathbf{x}$,*

$$\{0, \tau_\ell(\mathbf{x})\} \subset \overline{\{\tau_\ell(\mathbf{x}) : \mathbf{x} \in \mathrm{reg}(\Sigma_p) \cap U\}}.$$

In particular, the infimum of $\tau_\ell(\mathbf{x})$ over $\mathbf{x} \in \mathrm{reg}(\Sigma_p)$ is 0 if and only if $\mathrm{sing}(\Sigma_p) = \emptyset$.

(3) *Assume p is square free (otherwise, replace p with p^{red}). Then the infimum of τ_ℓ is bounded by*

$$\frac{\mathrm{mingap}(p)}{\ell_{\max}} \leq \inf \tau_\ell \leq \frac{\mathrm{mingap}(p)}{\ell_{\min}},$$

where $\ell_{\max}$ and $\ell_{\min}$ denote, respectively, the largest and smallest entry of ℓ, and mingap *was defined in Definition 1.11 (see also Lemma 4.7).*

Moreover, if we let ℓ vary in $\mathbb{R}^n_+$,

(4) *the map $\tau(\mathbf{x}, \ell) := \tau_\ell(\mathbf{x})$ is continuous on $\mathrm{reg}(\Sigma_p) \times \mathbb{R}^n_+$ and is real analytic on the open subset $\{(\mathbf{x}, \ell) \in \mathrm{reg}(\Sigma_p) \times \mathbb{R}^n_+ : T_\ell(\mathbf{x}) \in \mathrm{reg}(\Sigma_p)\}$.*

Proof. Since p and the reduced polynomial p^{red} share the same torus zeros set, then they share the same τ_ℓ, and so we may assume that p is square free. We work with the lift of τ_ℓ from Σ_p to $\widehat{\Sigma_p}$. Abusing notation, we write $\tau_\ell(\mathbf{x}) = \tau_\ell(\mathbf{x} \bmod 2\pi)$ when $\mathbf{x} \in \widehat{\Sigma_p} \subset \mathbb{R}^n$, and similarly τ lifts to $\widehat{\Sigma_p} \times \mathbb{R}^n_+$. This means that

$$\tau(\mathbf{x}, \ell) = \tau_\ell(\mathbf{x}) = \min\{t > 0 : p(\exp(i(\mathbf{x} + t\ell))) = 0\}.$$

In particular, $\tau_\ell(x_j\ell) = x_{j+1} - x_j$ when $x_{j+1} > x_j$ are consecutive zeros of $p(\exp(ix\ell))$, which proves (1).

For the bound $\tau_\ell(\mathbf{x}) \leq 2\pi|\mathbf{d}|/\langle \mathbf{d}, \ell\rangle$, by replacing $p(\mathbf{z})$ with $p(e^{-ix_1}z_1, \ldots, e^{-ix_n}z_n)$ if needed, it is enough to consider $\tau_\ell(\mathbf{0})$ when $p(\exp(\mathbf{0})) = 0$, and to number the zeros of $f(x) = p(\exp(ix\ell))$ such that $0 = x_0 < x_1$. Then $\tau_\ell(\mathbf{0}) = x_1 - x_0 \leq 2\pi|\mathbf{d}|/\langle \mathbf{d}, \ell\rangle$ by Theorem 1.9.

For (2) and (4), the same argument allows us to assume that $p(\exp(\mathbf{0})) = 0$, namely $\mathbf{0} \in \widehat{\Sigma_p}$, and focus on a small neighborhood of $\mathbf{0}$. The two cases of $\mathbf{0}$ being a regular or a singular point of $\widehat{\Sigma_p}$ are treated separately.

Case $\mathbf{0} \in \mathrm{reg}(\widehat{\Sigma_p})$. If $\mathbf{0}$ is a regular point of $\widehat{\Sigma}_p$, then it is a zero of $p(\exp(i\mathbf{x}))$ of multiplicity one, since p is square free. The phase functions (defined in Proposition 4.5) can be chosen, according to Remark 4.6, such that

$$0 = \theta_1(\mathbf{0}) < \theta_2(\mathbf{0}) \le \cdots \le \theta_{|\mathbf{d}|}(\mathbf{0}) < 2\pi,$$

Taking $U \subset \widehat{\Sigma}_p$ a small enough neighborhood of $\mathbf{0}$, we can ensure that $\theta_1(\mathbf{x}) = 0$ and $\theta_j(\mathbf{x}) \in (0, 2\pi)$ for all $\mathbf{x} \in U$ and $j \ge 2$. In particular, the minimal $t > 0$ for which $p(\exp(i(\mathbf{x}+t\ell)) = 0$ must satisfy $\theta_2(\mathbf{x}+t\ell) = 0$, for any $(\mathbf{x}, \ell) \in U \times \mathbb{R}^n_+$, by the ordering and strict monotonicity of the phase functions as shown in Proposition 4.5. In such case, $\tau = \tau(\mathbf{x}, \ell)$ is the unique solution to $\theta_2(\mathbf{x} + \tau\ell) = 0$ and is therefore continuous in $(\mathbf{x}, \ell)$, by the continuity of $(\mathbf{x}, \ell, t) \mapsto \theta_2(\mathbf{x} + t\ell)$ and the implicit function theorem for monotone continuous functions. As a result, $(\mathbf{x}, \ell) \mapsto T_\ell(\mathbf{x}) = \mathbf{x} + \tau(\mathbf{x}, \ell)\ell$ is also continuous in $U \times \mathbb{R}^n_+$, and therefore the set $\Omega := \{(\mathbf{x}, \ell) \in U \times \mathbb{R}^n_+ : T_\ell(\mathbf{x}) \in \mathrm{reg}(\widehat{\Sigma}_p)\}$ is an open subset of $U \times \mathbb{R}^n_+$. If $(\mathbf{x}', \ell') \in \Omega$, then $\theta_2(\mathbf{x} + t\ell)$ is real analytic in $(\mathbf{x}, \ell, t)$ around $(\mathbf{x}', \ell', \tau(\mathbf{x}', \ell'))$, by Proposition 4.5, and so $\tau(\mathbf{x}, \ell)$ is real analytic around $(\mathbf{x}', \ell')$ by the implicit function theorem for real analytic functions, which proves (4).

Case $\mathbf{0} \in \mathrm{sing}(\widehat{\Sigma_p})$. If $\mathbf{0}$ is a singular point of $\widehat{\Sigma}_p$, then it has multiplicity $m = \mathrm{m}(\mathbf{0})$ as a zero of $p(\exp(i\mathbf{x}))$. Choose the phase functions, according to Remark 4.6, such that

$$0 = \theta_1(\mathbf{0}) = \cdots = \theta_m(\mathbf{0}) < \theta_{m+1}(\mathbf{0}) \le \cdots \le \theta_{|\mathbf{d}|}(\mathbf{0}) < 2\pi.$$

If $U \subset \widehat{\Sigma_p}$ is a small enough neighborhood of $\mathbf{0}$, then it has the form

$$U = \bigcup_{j=1}^{m} U_j, \quad \text{with } U_j := \{\mathbf{x} \in U : \theta_j(\mathbf{x}) = 0\},$$

since $\widehat{\Sigma}_p = \bigcup_{j=1}^{|\mathbf{d}|} \theta_j^{-1}(2\pi\mathbb{Z})$ and the phase function are continuous. Define $t_j(\mathbf{x}, \ell)$ as the unique t-solution to $\theta_j(\mathbf{x} + t\ell) = 0$. As before, t_j is continuous on $U \times \mathbb{R}^n_+$, and $\tau(\mathbf{0}, \ell) = t_{m+1}(\mathbf{0}, \ell)$ (where $\theta_{|\mathbf{d}|+1} = \theta_1 + 2\pi$ if $m = |\mathbf{d}|$). Furthermore, for any $j \le m$ and $\mathbf{x} \in U_j \cap \mathrm{reg}(\widehat{\Sigma}_p)$, $\theta_{j+1}(\mathbf{x}) > 0$, and so $\tau(\mathbf{x}, \ell) = t_{j+1}(\mathbf{x}, \ell)$. Consider a converging sequence $\mathbf{x}_n \to \mathbf{0}$, with $\mathbf{x}_n \in \mathrm{reg}(\widehat{\Sigma}_p)$ for all n, and by taking a subsequence if needed, we may assume $\mathbf{x}_n \in \mathrm{reg}(\widehat{\Sigma}_p) \cap U_j$ for all n, for some specific j. So $\tau(\mathbf{x}_n, \ell) = t_{j+1}(\mathbf{x}_n, \ell)$ for all n, and

$$\lim_{n\to\infty} \tau(\mathbf{x}_n, \ell) = t_{j+1}(\mathbf{0}, \ell) = \begin{cases} \tau(\mathbf{0}, \ell) & \text{if } j = m, \\ 0 & \text{if } 1 \le j < m-1, \end{cases}$$

by continuity of t_{j+1}, using that $\theta_{j+1}(\mathbf{0}) = 0$ when $j + 1 \le m$. This proves the first part of (2), and the fact that if $\mathrm{sing}(\Sigma_p) \ne \emptyset$, then the infimum of τ_ℓ over $\mathrm{reg}(\Sigma_p)$ is zero. On the other hand, if $\mathrm{sing}(\Sigma_p) = \emptyset$, then τ_ℓ is continuous on $\Sigma_p = \mathrm{reg}(\Sigma_p)$ and positive, and by compactness it has a positive minimum. This proves (2).

For (3), define $\tau_{\ell,j}(\mathbf{x})$ as the t solution to $\theta_{j+1}(\mathbf{x}+t\ell)=\theta_j(\mathbf{x})$, which is well defined an positive since the phase functions are ordered, continuous and monotone in positive directions. Then $\theta_{j+1}(\mathbf{x})-\theta_{j+1}(\mathbf{x}+\tau_{\ell,j}(\mathbf{x})\ell)=\theta_{j+1}(\mathbf{x})-\theta_j(\mathbf{x})$ and part (1) of Proposition 4.5 gives

$$\frac{\theta_{j+1}(\mathbf{x})-\theta_j(\mathbf{x})}{\ell_{\max}}\leq\tau_{\ell,j}(\mathbf{x})\leq\frac{\theta_{j+1}(\mathbf{x})-\theta_j(\mathbf{x})}{\ell_{\min}},$$

and by taking the minimum over all $\mathbf{x}\in\mathbb{R}^n$ and $1\leq j\leq|\mathbf{d}|$, we get

$$\frac{\operatorname{mingap}(p)}{\ell_{\max}}\leq\min_{\mathbf{x},j}\tau_{\ell,j}(\mathbf{x})\leq\frac{\operatorname{mingap}(p)}{\ell_{\min}},$$

using Lemma 4.7. Now, on the one hand, for any $\mathbf{x}\in\widehat{\Sigma_p}$, there is some j such that $\theta_{j-1}(\mathbf{x})=0\mod 2\pi$ and $\theta_j(\mathbf{x})>\theta_{j-1}(\mathbf{x})$, and so $\tau_\ell(\mathbf{x})=\tau_{\ell,j}(\mathbf{x})$, which means that $\min_{\mathbf{x},j}\tau_{\ell,j}(\mathbf{x})\leq\inf\tau_\ell$. If $\inf\tau_\ell=0$, then we are done. If $\inf\tau_\ell>0$, then $\operatorname{sing}(\Sigma_p)=\emptyset$, and therefore $\tau_{\ell,j}(\mathbf{x})>0$ for all $\mathbf{x}$ and j. By Proposition 4.5 (1), $\tau_{\ell,j}(\mathbf{x}+t\mathbf{1})=\tau_{\ell,j}(\mathbf{x})$ and we can choose t so that $\theta_{j-1}(\mathbf{x}+t\mathbf{1})=0\mod 2\pi$, in which case $\tau_\ell(\mathbf{x}+t\mathbf{1})=\tau_{\ell,j}(\mathbf{x})$. Therefore, $\inf\tau_\ell=\min_{\mathbf{x},j}\tau_{\ell,j}(\mathbf{x})$, which finishes the proof. ■

7. Proof of Theorem 1.5

Let $p\in\mathrm{LY}_{\mathbf{d}}(n)$ with decomposition $p=\prod_{j=1}^N q_j^{c_j}$ into distinct irreducible polynomials, and let $\ell\in\mathbb{R}^n_+$ with $\mathbb{Q}$-linearly independent entries. Each factor q_j is a Lee–Yang polynomial by definition. Let $m_p(x)$ denote the multiplicity of x as a zero of $f_p(x)=p(\exp(ix\ell))$, with $m_p(x)=0$ if $f_p(x)\neq0$, and similarly, let $m_j(x)$ denote the multiplicity with respect to $f_j(x)=q_j(\exp(ix\ell))$. Since $f(x)=\prod_{j=1}^N(f_j(x))^{c_j}$ and multiplicity of zeros is additive under multiplication of functions, then $\mathrm{m}(x)=\sum_{j=1}^N c_jm_j(x)$. As a result,

$$\mu_{p,\ell}=\sum_{x\in\Lambda}m_p(x)\delta_x=\sum_{j=1}^N c_j\sum_{x\in\Lambda_j}m_j(x)\delta_x=\sum_{j=1}^N c_j\,\mu_{q_j,\ell},$$

where Λ denotes the zero set of f and Λ_j the zero set of f_j. Clearly, $\Lambda=\bigcup_{j=1}^N\Lambda_j$. The proof of Theorem 1.5 follows from the next lemma and proposition, considering the case of p being irreducible and either binomial or not.

Lemma 7.1 (Binomial). *If the polynomial $p\in\mathrm{LY}_{\mathbf{d}}(n)$ is binomial, normalized such that $p(\mathbf{0})=1$, then $p(\mathbf{z})=1-e^{-i\varphi}\mathbf{z}^{\mathbf{d}}$ for some $\varphi\in\mathbb{R}$. In such case, for any $\ell\in\mathbb{R}^n_+$, the zeros of $f(x)=p(\exp(ix\ell))$ are simple and form an infinite arithmetic progression $\{\varphi+2\pi k/\langle\mathbf{d},\ell\rangle:k\in\mathbb{Z}\}$.*

Proof. If $p\in\mathrm{LY}_{\mathbf{d}}(n)$, then $p(\mathbf{0})\neq0$ and the coefficient of $\mathbf{z}^{\mathbf{d}}$ is non-zero. If it has only two monomials and $p(\mathbf{0})=1$, then $p(\mathbf{z})=1+a\,\mathbf{z}^{\mathbf{d}}$. Assume by contradiction that $|a|\neq1$; then for any $|\mathbf{d}|$-th root $\omega\in\mathbb{C}$ of a, the point $\mathbf{z}=(\omega,\ldots,\omega)$ will be a root of p in $\mathbb{D}^n$ or in $(\mathbb{C}\setminus\overline{\mathbb{D}})^n$, in contradiction to $p\in\mathrm{LY}_{\mathbf{d}}(n)$. Therefore $p(\mathbf{z})=1-e^{-i\varphi}\mathbf{z}^{\mathbf{d}}$, and so $f(x)=1-e^{i(\langle\mathbf{d},\ell\rangle x-\varphi)}$, for some $\varphi\in\mathbb{R}$. Hence, $f(x)=0\iff x-\varphi\in\frac{2\pi}{\langle\mathbf{d},\ell\rangle}\mathbb{Z}$, in which case $f'(x)\neq0$. ■

Proposition 7.2 (Non-binomial). *Let $p \in \mathrm{LY}_{\mathbf{d}}(n)$ be irreducible and non-binomial, let $\ell \in \mathbb{R}^n_+$ be $\mathbb{Q}$-linearly independent, and let $f(x) = p(\exp(ix\ell))$ have zero set Λ and multiplicities $(\mathrm{m}(x))_{x\in\Lambda}$. Then,*

(1) $\mathrm{m}(x) \le |\mathbf{d}|$ *for all $x \in \Lambda$ and* $\lim_{R\to\infty} \frac{|\{|x|<R\,:\,x\in\Lambda,\ \mathrm{m}(x)=1\}|}{|\{|x|<R\,:\,x\in\Lambda\}|} = 1$.

(2) *For any $N \in \mathbb{N}$ and any set $\Gamma \in \mathbb{R}$ with $\dim_{\mathbb{Q}}(\Gamma) = N$, $|\Lambda \cap \Gamma| \le c$, with uniform bound $c = c(|\mathbf{d}|, N)$ that only depends on $|\mathbf{d}|$ and N. In particular, $\dim_{\mathbb{Q}}(\Lambda) = \infty$.*

Proof of Proposition 7.2, part (1). The bound $\mathrm{m}(x) \le |\mathbf{d}|$ follows from part (1) of Theorem 1.9. Number the distinct zeros of $f(x)$ by $(k_j)_{j\in\mathbb{Z}}$, with $k_j > 0$ for $j > 0$ and $k_j < 0$ for $j < 0$. We need to show that

$$\lim_{N\to\infty} \frac{|\{-N \le j \le N : \mathrm{m}(k_j) > 1\}|}{2N} = 0.$$

Let $p^\dagger \in \mathrm{LY}_{\mathbf{d}}$ as in Definition 3.13, so that $p^\dagger$ is also irreducible, non-binomial, and has $p^\dagger(\exp(ix\ell)) = 0 \iff p(\exp(-ix\ell)) = 0$ with the same multiplicities, so it is enough to prove the one sided limit

$$\lim_{N\to\infty} \frac{|\{1 \le j \le N : \mathrm{m}(k_j) > 1\}|}{N} = 0.$$

By Lemma 3.6 and since p is irreducible, $\mathrm{m}(k_j) > 1$ if and only if $k_j\ell \in \mathrm{sing}(\Sigma_p)$. Notice that $k_j\ell = T_\ell^j(k_0\ell)$, using the fact that the k_j's are the distinct zeros. Let h be the indicator function of $\mathrm{sing}(\Sigma_p)$, so that $|\{1 \le j \le N : \mathrm{m}(k_j) > 1\}| = \sum_{j=1}^N h(T_\ell^j(k_0\ell))$. Then, h is bounded Riemann integrable, and Theorem 6.4 gives

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} h(T_\ell^j(k_0\ell)) \propto \int h(\mathbf{x})\, d\mathbf{m}_\ell(\mathbf{x}) = \mathbf{m}_\ell(\mathrm{sing}(\Sigma_p)) = 0. \qquad \blacksquare$$

The proof of part (2) in of Proposition 7.2 is a consequence of Theorem 1.2 in [13], often known as Lang's GM theorem. To state it, we consider $(\mathbb{C}^*)^n$ as multiplicative group, and it will be convenient to define the notions of *rank, division group,* and *algebraic torus cosets* in terms of the exponent map $\exp\colon \mathbb{C}^n \to (\mathbb{C}^*)^n$.

Definition 7.3. A subgroup $G \subset (\mathbb{C}^*)^n$ has *rank N* if N is the minimal integer for which $G = \{\exp(A\mathbf{k}) : \mathbf{k} \in \mathbb{Z}^N\}$ for some matrix $A \in \mathbb{C}^{n\times N}$. Its *division group* is defined by $\overline{G} = \{\exp(A\mathbf{k}) : \mathbf{k} \in \mathbb{Q}^N\}$ for the same A. An *algebraic torus* of dimension d in $(\mathbb{C}^*)^n$ has the form $H = \{\exp(B\mathbf{y}) : \mathbf{y} \in \mathbb{C}^d\}$ for some integer matrix $B \in \mathbb{Z}^{n\times d}$ of rank d. The *algebraic torus coset* $\mathbf{z}H$ for $\mathbf{z} = \exp(\mathbf{x})$ is the set $\mathbf{z}H = \{\exp(\mathbf{x} + A\mathbf{y}) \mid \mathbf{y} \in \mathbb{C}^d\}$, for the same matrix B. It also has dimension d.

Theorem (Theorem 1.2 in [13]). *Let $V \subset (\mathbb{C}^*)^n$ be an algebraic variety of dimension N and degree D, and let G be a subgroup of $(\mathbb{C}^*)^n$, of rank N, with division group $\overline{G}$. Then $\overline{G} \cap V$ is contained in a union of at most r algebraic torus cosets $\mathbf{z}_j H_j \subset V$ for*

$$r \le e^{(N+1)\left(6D\binom{n+D}{D}\right)^{\left(5D\binom{n+D}{D}\right)}}.$$

Lemma 7.4. *Suppose that $\mathbf{z}H \subset (\mathbb{C}^*)^n$ is an algebraic torus coset of dimension $d \le n-2$, and that $\ell \in \mathbb{R}^n$ has $\mathbb{Q}$-linearly independent entries. Then there is at most one $k \in \mathbb{R}$ such that* $\exp(ik\ell) \in \mathbf{z}H$.

Proof. Let $B \in \mathbb{Z}^{n\times d}$ of rank d such that $H = \{\exp(B\mathbf{y}) : \mathbf{y} \in \mathbb{C}^d\}$, and suppose that both $\exp(ik\ell)$ and $\exp(ik'\ell)$ lie in $\mathbf{z}H$. Then, $\exp(i(k-k')\ell) \in H$ and therefore $(k-k')\ell = B\mathbf{y} + 2\pi\mathbf{k}$ for some $\mathbf{k} \in \mathbb{Z}^n$, $\mathbf{y} \in \mathbb{C}^d$. The left kernel of B in $\mathbb{C}^n$ contains an $(n-d)$-dimensional $\mathbb{Q}$-linear vector space of vectors orthogonal to $B\mathbf{y}$, so $\dim_{\mathbb{Q}}(B\mathbf{y}) \le d$, and therefore,

$$\dim_{\mathbb{Q}}((k-k')\ell) = \dim_{\mathbb{Q}}(B\mathbf{y} + 2\pi\mathbf{k}) \le d+1 < n.$$

However, if $k - k' \ne 0$, then $\dim_{\mathbb{Q}}((k-k')\ell) = \dim_{\mathbb{Q}}(\ell) = n$, a contradiction. ∎

Proposition 7.2, part (2). Let p, ℓ and Λ as in Proposition 7.2. Let $V \subset (\mathbb{C}^*)^n$ be the zero set of p in $(\mathbb{C}^*)^n$. The degree of V is finite and only depends on $|\mathbf{d}|$. Given $N \in \mathbb{N}$, let $\Gamma \subset \mathbb{R}$ of $\dim_{\mathbb{Q}}(\Gamma) = N$, so $\Gamma = \{\langle \mathbf{a}, \mathbf{k}\rangle : \mathbf{k} \in \mathbb{Q}^N\}$ for some $\mathbf{a} \in \mathbb{R}^N$. Define the matrix $A \in \mathbb{C}^{n\times N}$ whose j-th row is the vector $i\ell_j\mathbf{a} \in \mathbb{C}^N$, and let $G = \{\exp(A\mathbf{k}) : \mathbf{k} \in \mathbb{Z}^N\}$ so that its division group is the set $\overline{G} = \{\exp(A\mathbf{k}) : \mathbf{k} \in \mathbb{Q}^N\} = \{\exp(it\ell) : t \in \Gamma\}$. So

$$x \in \Lambda \cap \Gamma \iff \exp(ix\ell) \in \overline{G} \cap V.$$

Since G has rank at most N, Lang's GM theorem says that there are at most $r = r(|\mathbf{d}|, N)$ algebraic torus cosets $\mathbf{z}_i H_i \subset V$ such that $\overline{G} \cap V \subset \mathbf{z}_1 H_1 \cup \cdots \cup \mathbf{z}_r H_r$. In particular, any $x \in \Lambda \cap \Gamma$ satisfies $\exp(ix\ell) \in \mathbf{z}_i H_i$ for some i. An algebraic torus coset of dimension $n-1$ is the zero set of a binomial polynomial, and since p is irreducible and not binomial, then $\dim(\mathbf{z}_i H_i) \le \dim(V) - 1 = n-2$ for every $i = 1, \ldots, r$. By Lemma 7.4, each $\mathbf{z}_i H_i$ contains at most one point $\exp(ix\ell)$ for $x \in \mathbb{R}$. We conclude that $\Lambda \cap \Gamma$ contains at most r points. ∎

8. Proof of Theorem 1.12

Proof of Theorem 1.12. Suppose that $n \ge 2$. Say that $p \in \mathrm{LY}_{\mathbf{d}}(n)$ satisfies (i) if p and ∇p have no common zeros in $\mathbb{T}^n$, and satisfies (ii) if p has a non-binomial factor. Say that $\mu_{p,\ell}$ satisfies ($\star$) if it is non-periodic, with unit coefficients and has a uniformly discrete support. The proof of Theorem 1.12 consists of three parts.

Proof of the characterization.

((i)+(ii) $\Rightarrow$($\star$)) It follows from Theorem 1.5 that $\mu_{p,\ell}$ is non-periodic when ℓ is $\mathbb{Q}$-linearly independent and p satisfies (ii). It is left to show that if p satisfies (i), then $\mu_{p,\ell}$ has unit coefficients and uniformly discrete support for any $\ell \in \mathbb{R}^n_+$. Assume that p satisfies (i) and $\ell \in \mathbb{R}^n_+$. Property (i) is equivalent to $\mathrm{sing}(\Sigma_p) = \emptyset$ and $\mathrm{m}(\mathbf{x}) \equiv 1$ for all $\mathbf{x} \in \Sigma_p$. According to Lemma 3.6, this means that the multiplicities of the zeros of $f(x) = p(\exp(ix\ell))$, which are the coefficients in $\mu_{p,\ell}$, are all equal to one. According to Lemma 6.6, $\mathrm{sing}(\Sigma_p) = \emptyset$ implies that $r = \inf\{\tau_\ell(\mathbf{x}) : \mathbf{x} \in \Sigma_p\} > 0$. The zeros of f are distinct, so their gaps are given by τ_ℓ, as seen in Lemma 6.6, providing the uniform lower bound $x_{j+1} - x_j = \tau_\ell(x_j\ell) \ge r > 0$.

(($\star$) $\Rightarrow$ (i)+(ii)) Let $p \in \mathrm{LY}_{\mathbf{d}}(n)$ with $\mathbb{Q}$-linearly independent $\ell \in \mathbb{R}^n_+$, and assume that $\mu_{p,\ell}$ satisfies ($\star$). Let Λ be the support of $\mu_{p,\ell}$, so it is non-periodic and uniformly discrete. If p had only binomial factors, then Λ would be a union of infinite arithmetic progressions, by Theorem 1.5, and such a union is either periodic or it has gaps as small as we wish. We conclude that p satisfies (ii), and it is left to show (i), namely that $\mathrm{sing}(\Sigma_p) = \emptyset$ and $\mathrm{m}(\mathbf{x}) \equiv 1$. Let $(x_j)_{j\in\mathbb{Z}}$ be the zeros of $f(x) = p(\exp(ix\ell))$, ordered increasingly, so by ($\star$) they are all simple and $\tau_\ell(x_j\ell) = x_{j+1} - x_j \geq r > 0$ uniformly for some given $r > 0$. Note that $x_j\ell \in \mathrm{reg}(\Sigma_p)$ with $\mathrm{m}(x_j\ell) = 1$ for all $j \in \mathbb{Z}$, since every x_j has multiplicity one. The sequence $\{x_j\ell\}_{j\in\mathbb{Z}}$ is dense in $\mathrm{reg}(\Sigma_p)$ since ℓ is $\mathbb{Q}$-linearly independent, so $\mathrm{m}(\mathbf{x}) = 1$ for all $\mathbf{x} \in \mathrm{reg}(\Sigma_p)$ and $\inf\{\tau_\ell(\mathbf{x}) : \mathbf{x} \in \mathrm{reg}(\Sigma_p)\} = \inf\{\tau_\ell(x_j\ell) : j \in \mathbb{Z}\} \geq r > 0$, by continuity of τ_ℓ and m on $\mathrm{reg}(\Sigma_p)$. Then $\mathrm{sing}(\Sigma_p) = \emptyset$, by Lemma 6.6, which means that $\mathrm{m}(\mathbf{x}) \equiv 1$.

Proof of the explicit lower estimate.

Lemma 4.7 states that mingap > 0 if and only if p satisfies (i). If mingap $= 0$, the desired lower bound holds. If mingap > 0, then the desired inequality follows from parts (1) and (3) of Lemma 6.6.

Proof of the genericity.

By Theorem 3.20, For any $\mathbf{d} \in \mathbb{Z}^n_{>0}$, the subset $\mathrm{LY}^\circ_{\mathbf{d}} \subset \mathrm{LY}_{\mathbf{d}}$ of $p \in \mathrm{LY}_{\mathbf{d}}(n)$ that satisfy (i), is a semialgebraic open, dense subset of $\mathrm{LY}_{\mathbf{d}}(n)$. Furthermore, for any nonzero $p \in \mathrm{LY}_{\mathbf{d}}(n)$, we can chose $\mathbf{x} \in [0, 2\pi)^n$ for which $p(\exp(i\mathbf{x})) \neq 0$. By Corollary 3.17, for any $\lambda > 0$, the polynomial $(\mathcal{D}_{\lambda,\mathbf{x}})^{|\mathbf{d}|}p$ satisfies (i). As seen in Definition 3.16, every application of $\mathcal{D}_{\lambda,\mathbf{x}}$ contributes one to the degree of λ and so the result, $(\mathcal{D}_{\lambda,\mathbf{x}})^{|\mathbf{d}|}p$ can be expressed as a polynomial of degree $|\mathbf{d}|$ in λ.

For (ii), consider the set $B_{\boldsymbol{\alpha}}$ of polynomials $p \in \mathrm{LY}_{\mathbf{d}}(n)$ that has a binomial factor of multi-degree $\boldsymbol{\alpha} \leq \mathbf{d}, \boldsymbol{\alpha} \neq \mathbf{d}$. We will see that $B_{\boldsymbol{\alpha}}$ is a semialgebraic subset of $\mathrm{LY}_{\mathbf{d}}(n)$ of positive codimension. By Lemma 7.1, the binomial factor of p has the form $(1 + a\mathbf{z}^{\boldsymbol{\alpha}})$ for some $a \in \mathbb{C}^*$ with $|a| = 1$. Therefore $B_{\boldsymbol{\alpha}} = \{(1 + a\mathbf{z}^{\boldsymbol{\alpha}})q(\mathbf{z}) : |a| = 1,\ q \in \mathrm{LY}_{\mathbf{d}-\boldsymbol{\alpha}}\}$. From this and Theorem 3.20, we see that $B_{\boldsymbol{\alpha}}$ is semialgebraic of dimension

$$\dim(B_{\boldsymbol{\alpha}}) = 1 + \dim(\mathrm{LY}_{\mathbf{d}-\boldsymbol{\alpha}}) = 2 + \prod_{j=1}^{n}(d_j - \alpha_j + 1).$$

Since $\boldsymbol{\alpha} \neq \mathbf{0}$, there is some $\alpha_i \geq 1$. We then calculate that

$$\begin{aligned}\prod_{j=1}^{n}(d_j - \alpha_j + 1) &\leq (d_i - \alpha_i + 1)\prod_{j\neq i}(d_j + 1) \\ &= \prod_{j=1}^{n}(d_j + 1) - \alpha_i\prod_{j\neq i}(d_j + 1) < \prod_{j=1}^{n}(d_j + 1) - 1,\end{aligned}$$

using that $\alpha_i \prod_{j\neq i}(d_j + 1) \geq 2^{n-1} > 1$ since $n \geq 2$ and $d_j + 1 \geq 2$ for all j. This shows that $\dim(B_{\boldsymbol{\alpha}}) < \dim(\mathrm{LY}_{\mathbf{d}})$ for any $0 \lneq \boldsymbol{\alpha} \leq \mathbf{d}$.

Together, these show that the set of polynomials in $\mathrm{LY}_{\mathbf{d}}(n)$ satisfying (i) and (ii) is a semialgebraic, open dense subset of $\mathrm{LY}_{\mathbf{d}}(n)$. ∎

9. Gap distributions

The existence of a gap distribution $\rho_{p,\ell}$ was previously known for specific type of Lee–Yang polynomials, those for which the zeros of $p(\exp(ix\ell))$ are the square-root eigenvalues of a quantum graph that has n edges of lengths $\ell = (\ell_1, \dots, \ell_n)$, assuming these lengths are $\mathbb{Q}$-linearly independent [4, 7, 12]. The existence of a gap distribution of $\mu_{p,\ell}$ for any choice of Lee–Yang p and positive ℓ is proven in this section. In particular, this includes the case of quantum graphs with edge lengths not $\mathbb{Q}$-linearly independent.

Recall that if p has multi-degree $\mathbf{d}$ and it decomposes as $p = \prod_{j=1}^{N} q_j^{c_j}$ into distinct irreducible q_j's, then $p^{\text{red}} = \prod_{j=1}^{N} q_j$ is the reduced square-free polynomial; we denote its multi-degree by $\mathbf{d}^{\text{red}}$. In particular, $\mathbf{d}^{\text{red}} \leq \mathbf{d}$ element-wise, with equality if and only if p is square free. As seen in Lemma 6.6, if we number the zeros of $f(x) = p(\exp(ix\ell))$ increasingly with multiplicity, then the positive gaps are described by $\tau_\ell \colon \Sigma_p \to \mathbb{R}_+$,

$$x_{j+1} - x_j = \tau_\ell(x_j \ell) \quad \text{whenever } x_{j+1} \neq x_j, \tag{9.1}$$

as can be seen in Figure 5. To prove Theorem 1.14, let us define the measure $\nu_{p,\ell}$.

Definition 9.1. Let $p \in \mathrm{LY}_{\mathbf{d}}(n)$, $\ell \in \mathbb{R}_+^n$, and let $(\tau_\ell)_* \mathbf{m}_\ell$ denote the push-forward of $\mathbf{m}_\ell$ by τ_ℓ. Define the measure $\nu_{p,\ell}$ on $\mathbb{R}_{\geq 0}$ by

$$\nu_{p,\ell} := c_0 \delta_0 + c_\tau (\tau_\ell)_* \mathbf{m}_\ell, \quad \text{with } c_0 := \frac{\langle \mathbf{d} - \mathbf{d}^{\text{red}}, \ell \rangle}{\langle \mathbf{d}, \ell \rangle} \text{ and } c_\tau := \frac{1}{(2\pi)^{n-1} \langle \mathbf{d}, \ell \rangle}. \tag{9.2}$$

That is, for any continuous $f \colon \mathbb{R}_{\geq 0} \to \mathbb{C}$,

$$\int f \, d\nu_{p,\ell} := c_0 f(0) + c_\tau \int_{\Sigma_p} f(\tau_\ell(\mathbf{x})) \, d\mathbf{m}_\ell(\mathbf{x}). \tag{9.3}$$

Remark 9.2. The measure $\nu_{p,\ell}$ is normalized, $\int d\nu_{p,\ell} = 1$, since $\int d(\tau_\ell)_* \mathbf{m}_\ell = \mathbf{m}_\ell(\Sigma_p)$,

$$c_\tau = \frac{1}{\int_{\Sigma_p} \mathrm{m}(\mathbf{x}) \, d\mathbf{m}_\ell(\mathbf{x})} \quad \text{and} \quad c_0 = \frac{\int_{\Sigma_p} (\mathrm{m}(\mathbf{x}) - 1) \, d\mathbf{m}_\ell(\mathbf{x})}{\int_{\Sigma_p} \mathrm{m}(\mathbf{x}) \, d\mathbf{m}_\ell(\mathbf{x})} = 1 - c_\tau \mathbf{m}_\ell(\Sigma_p).$$

Proof of Theorem 1.14 and Theorem 1.15. Fix μ, an $\mathbb{N}$-FQ, and let $n \in \mathbb{N}$, $p \in \mathrm{LY}_{\mathbf{d}}(n)$, and $\ell \in \mathbb{R}_+^n$ with $\mathbb{Q}$-linearly independent entries, such that $\mu = \mu_{p,\ell}$, as guaranteed by [3]. Consider the decomposition $p = \prod_{j=1}^{N} q_j^{c_j}$ into distinct irreducible Lee–Yang polynomials. Let $(x_j)_{j \in \mathbb{Z}}$ be the zeros of $p(\exp(ix\ell))$, numbered increasingly with multiplicity.

The proofs of Theorem 1.14 and Theorem 1.15 interlace according to the following sequence of lemmas, which will be proven afterwards. For each, we take the assumptions listed above.

Lemma 9.3. *The gap distribution $\rho = \rho_{p,\ell}$ exists and is equal to $\nu_{p,\ell}$. That is, for any continuous function $f \colon \mathbb{R} \to \mathbb{C}$,*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(x_{j+1} - x_j) = \int f \nu_{p,\ell}.$$

Moreover, $\nu_{p,\ell} = \nu_{q,\ell}$ when $q(\mathbf{z}) := p(\exp(i\mathbf{x}_0)\mathbf{z})$ for any fixed $\mathbf{x}_0 \in \mathbb{R}^n$ (part (1) of Theorem 1.15).

For the average gap, Theorem 1.9 provides two estimates:

$$\mu_{p,\ell}([x_1, x_{N+1}]) = N + O(1), \quad \text{and}$$
$$\mu_{p,\ell}([x_1, x_{N+11}]) = \frac{\langle \mathbf{d}, \ell\rangle}{2\pi}(x_{N+1} - x_1) + O(1),$$

and their ratio as $N \to \infty$ gives the following.

Corollary 9.4 (Theorem 1.14, part (4)). *The average gap is the density inverse*

$$\mathbb{E}(\rho) := \lim_{N\to\infty} \frac{\sum_{n=1}^{N} x_{j+1} - x_j}{N} = \lim_{N\to\infty} \frac{x_{N+1} - x_j}{N} = \frac{2\pi}{\langle \mathbf{d}, \ell\rangle}.$$

Lemma 9.5 (Theorem 1.15, parts (2) and (3)). *The measure $\nu_{p,\ell}$ has an atom at $\Delta = 0$ if and only if p is not square free. It has an atom at $\Delta > 0$ if and only if some (not necessarily distinct) pair of factors, q_i and q_j, are related by $q_j(\mathbf{z}) = q_i(\exp(i\Delta\ell)\mathbf{z})$ for all $\mathbf{z}$. Moreover, if this holds and $q_i = q_j$, then q_i is binomial.*

Lemma 9.6. *The measure $\nu_{p,\ell}$ has no singular continuous part.*

Lemma 9.5 and Lemma 9.6 then give the following.

Corollary 9.7 (Theorem 1.14, part (1)). *The measure $\nu_{p,\ell}$ has finitely many atoms and no singular continuous part.*

Lemma 9.8 (Theorem 1.14, part (3)). *For any $\Delta = x_{j+1} - x_j$ and any open interval I that contains Δ, $\nu_{p,\ell}(I) > 0$.*

Together with Theorem 1.5, this gives the following.

Corollary 9.9 (Theorem 1.14, part (2)). *If $\mu_{p,\ell}$ is periodic, then $\nu_{p,\ell}$ is purely atomic. Conversely, if $\mu_{p,\ell}$ is not periodic, with support Λ, then at least one of the following holds:*

(1) *Λ contains two arithmetic progressions with periods Δ_1, Δ_2 such that $\Delta_1/\Delta_2 \notin \mathbb{Q}$.*

(2) $\dim_{\mathbb{Q}}(\Lambda) = \infty$.

Each one of these ensures that there are infinitely many gap values, hence the support of $\nu_{p,\ell}$ is not finite. In particular, $\nu_{p,\ell}$ must have an absolutely continuous part.

Once proven, these statements complete the proof of Theorems 1.14 and 1.15. ∎

Proof of Lemma 9.3. Let $f : \mathbb{R} \to \mathbb{C}$ be continuous, so the composition $f \circ \tau_\ell$ is bounded and Riemann integrable, since τ_ℓ is bounded and continuous on an open full measure set $\operatorname{reg}(\Sigma_p)$, see Lemma 6.6. Therefore, the function

$$h(\mathbf{x}) := \frac{\mathrm{m}(\mathbf{x}) - 1}{\mathrm{m}(\mathbf{x})} f(0) + \frac{1}{\mathrm{m}(\mathbf{x})} f(\tau_\ell(\mathbf{x}))$$

is bounded and Riemann integrable. By Theorem 6.4, we get

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} h(x_j \ell) = \frac{\langle \mathbf{d} - \mathbf{d}^{\mathrm{red}}, \ell\rangle}{\langle \mathbf{d}, \ell\rangle} f(0) + \frac{1}{(2\pi)^{n-1}\langle \mathbf{d}, \ell\rangle} \int_{\Sigma_p} f(\tau_\ell(\mathbf{x}))\, d\mathbf{m}_\ell(\mathbf{x})$$
$$= \int f\, d\nu_{p,\ell}.$$

Whenever $x_{j-1} < x_j = x_{j+1} = \cdots = x_{j+(m-1)} < x_{j+m}$, we have $\tau_\ell(x_i\ell) = x_{j+m} - x_{j+(m-1)}$ and $\mathrm{m}(x_i\ell) = m$ for all $i \in \{j, \ldots, j+m-1\}$, so

$$\sum_{i=j}^{j+m-1} h(x_i\ell) = (m-1)f(0) + f(x_{j+m} - x_{j+(m-1)}) = \sum_{i=j}^{j+m-1} f(x_{i+1} - x_i).$$

Therefore, given any $N \in \mathbb{N}$ such that $x_N < x_{N+1}$,

$$\frac{1}{N}\sum_{j=1}^{N} h(x_j\ell) = \frac{1}{N}\sum_{j=1}^{N} f(x_{j+1} - x_j). \tag{9.4}$$

The left-hand-side of (9.4) converges to $\int f\, d\nu_{p,\ell}$ as $N \to \infty$. The equality in (9.4) holds for infinitely many N values (those for which $x_N < x_{N+1}$) whose spacing is bounded by the maximum multiplicity $|\mathbf{d}|$, so according to Lemma A.1,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} f(x_{j+1} - x_j) = \int f\, d\nu_{p,\ell}.$$

Given any fixed $\mathbf{x}_0 \in \mathbb{R}^n/2\pi\mathbb{Z}^n$, let $q(\mathbf{z}) := p(\exp(i\mathbf{x}_0)\mathbf{z})$, and let $(t_j)_{j\in\mathbb{Z}}$ denote the repeated ordered zeros of $t \mapsto q(\exp(it\ell)) = p(\exp(i(\mathbf{x}_0 + t\ell)))$. Then, according to Theorem 6.4,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} f(x_{j+1} - x_j) = \lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} f(t_{j+1} - t_j),$$

namely $\nu_{p,\ell} = \nu_{q,\ell}$. ∎

Proof of Lemma 9.5. By Definition 9.1, $\nu_{p,\ell}$ has an atom at $\Delta = 0$ if and only if the multidegrees of p and p^{red} differ, which occurs if and only if p is not square free.

Suppose that $\nu_{p,\ell}$ has an atom at $\Delta > 0$. Then, $(\tau_\ell)_*\mathbf{m}_\ell$ has an atom at Δ, which means that the level set $\tau_\ell^{-1}(\Delta)$ has positive measure $\mathbf{m}_\ell(\tau_\ell^{-1}(\Delta)) > 0$. The set $A := \mathrm{reg}(\Sigma_p) \cap T_\ell^{-1}(\mathrm{reg}(\Sigma_p))$ is an open subset of $\mathrm{reg}(\Sigma_p)$ of full $\mathbf{m}_\ell$ measure and τ_ℓ is real analytic on A by Lemma 6.6. Since $\mathbf{m}_\ell$ is absolutely continuous with respect to the volume measure on A, then $A \cap \tau_\ell^{-1}(\Delta)$ has positive volume, and therefore τ_ℓ is identically Δ on some open set $U \subset A$. By taking U sufficiently small, there are two (not necessarily distinct) irreducible factors of p, say q_1 and q_2, such that $q_1(\exp(i\mathbf{x})) = 0$ and $q_2(\exp(i(\mathbf{x} + \Delta\ell))) = 0$ for all $\mathbf{x} \in U$. It follows that $q_1(\mathbf{z}) = q_2(\exp(i\Delta\ell)\mathbf{z})$ for all $\mathbf{z} \in \mathbb{C}^n$ by part (3) of Lemma 4.1, since q_1 and q_2 are irreducible Lee–Yang polynomials.

Now, suppose that $q_1 = q_2$, so $q_1(\mathbf{z}) = q_1(\exp(i\Delta\ell)\mathbf{z})$ for all $\mathbf{z} \in \mathbb{C}^n$. In particular, if Λ is the zero set of $x \mapsto q_1(\exp(ix\ell))$, then for any $x \in \Lambda$ we have $x + \Delta \in \Lambda$, and as a result, $x + j\Delta \in \Lambda$ for any $j \in \mathbb{N}$. Since q_1 is irreducible Lee–Yang polynomial and $\ell \in \mathbb{R}_+^n$ has $\mathbb{Q}$-linearly independent entries, then q_1 must be binomial, by Theorem 1.5. ∎

Proof of Lemma 9.6. It follows from Lemma 9.5 that $\nu_{p,\ell}$ has finitely many atoms, say $(t_i)_{i=1}^N$, so that $\nu_{p,\ell} = \sum_{j=1}^n c_j\delta_{t_j} + \rho_{\mathrm{ac}}$, with ρ_{ac} being a continuous measure (no atoms). We now show that ρ_{ac} is absolutely continuous with respect to Lebesgue measure. Let

$$A := \mathrm{reg}(\Sigma_p) \cap T_\ell^{-1}(\mathrm{reg}(\Sigma_p)).$$

We use A_{nc} to denote the union of the connected components of A on which τ_ℓ is not constant. Then ρ_{ac} is (c_τ times) the push-forward of $\mathbf{m}_\ell$ by the restriction of τ_ℓ to A_{nc}, using that A has full measure. It is left to show that for any set $E \subset \mathbb{R}$ of Lebesgue measure zero, the set $A_{\mathrm{nc}} \cap \tau_\ell^{-1}(E)$ has zero $\mathbf{m}_\ell$ measure, or equivalently, due to Lemma 6.5, zero volume in $\mathrm{reg}(\Sigma_p)$.

Since τ_ℓ is real analytic on A and is not constant on any open set in A_{nc}, then the set $\Omega = \{\mathbf{x} \in A_{\mathrm{nc}} : \nabla\tau_\ell(\mathbf{x}) \neq 0\}$ is open in A_{nc} and its complement in A_{nc} has zero volume. By the definition of Ω, τ_ℓ has no critical points in Ω, which means that for any compact connected $K \subset \Omega$, the image of τ_ℓ over K is an interval $[a,b]$ and the level sets $K \cap \tau_\ell^{-1}(t)$ for $t \in [a,b]$ are homotopic to one another. In particular, if we let $\mathrm{area}(K,t) = \sigma_{n-2}(K \cap \tau_\ell^{-1}(t))$ denote the $(n-2)$-dimensional volume of the level set, induced by the volume form $d\sigma$ on $\mathrm{reg}(\Sigma_p)$, then $t \mapsto \mathrm{area}(K,t)$ is continuous in $t \in [a,b]$, and so it is bounded by some constant. Let C be the maximum of $\mathrm{area}(K,t)$ for $t \in [a,b]$, and $|\nabla\tau_\ell(\mathbf{x})|^{-1}$ for $\mathbf{x} \in K$. Then,

$$\int_{K\cap\tau_\ell^{-1}(E)} d\sigma \le C \int_{K\cap\tau_\ell^{-1}(E)} \|\nabla\tau_\ell(\mathbf{x})\|\, d\sigma(\mathbf{x}) = M \int_{t\in E} \mathrm{area}(K,t)\, dt \le C^2 \int_{t\in E} dt = 0,$$

using the co-area formula (or disintegration theorem) in the middle equality. It follows that $\mathbf{m}_\ell(K \cap \tau_\ell^{-1}(E)) = 0$ for any compact connected $K \subset \Omega$, hence

$$\rho_{\mathrm{ac}}(E) \propto \mathbf{m}_\ell(\Omega \cap \tau_\ell^{-1}(E)) = 0.$$

As this holds for any E of zero Lebesgue measure, ρ_{ac} is absolutely continuous. ■

Proof of Lemma 9.8. Let $\Delta = x_{j+1} - x_j$ for some arbitrary fixed choice of j, let $I \subset \mathbb{R}$ be any open interval with $\Delta \in I$, and consider the open set $U := \{\mathbf{x} \in \mathrm{reg}(\Sigma_p) : \tau_\ell(\mathbf{x}) \in I\}$. It is enough to show that $U \neq \emptyset$ to conclude that $\mathbf{m}_\ell(U) > 0$, by Lemma 6.5, and so

$$\nu_{p,\ell}(I) \ge c_\tau\, \mathbf{m}_\ell(U) > 0.$$

Consider two cases, according to whether $\Delta > 0$ or $\Delta = 0$.

Case $\Delta > 0$. Suppose $x_{j+1} > x_j$ and let $\mathbf{x} = x_j\ell \mod 2\pi$, so $\tau_\ell(\mathbf{x}) = \Delta$. If $\mathbf{x} \in \mathrm{reg}(\Sigma_p)$, then $\mathbf{x} \in U$. Otherwise, if $\mathbf{x} \in \mathrm{sing}(\Sigma_p)$, then $\Delta \in \overline{\{\tau_\ell(\mathbf{x}) : \mathbf{x} \in \mathrm{reg}(\Sigma_p)\}}$, by Lemma 6.6, which means that $U \neq \emptyset$.

Case $\Delta = 0$. Suppose $x_{j+1} = x_j$. If $\mathrm{sing}(\Sigma_p) \neq \emptyset$, then $\Delta = \inf_{\mathbf{x}\in\mathrm{reg}(\Sigma_p)} \tau_\ell(\mathbf{x})$ by Lemma 6.6, and so $U \neq \emptyset$. Otherwise, if $\mathrm{sing}(\Sigma_p) = \emptyset$, having $x_{j+1} = x_j$ means that p has a square factor, and so

$$\nu_{p,\ell}([\Delta - \varepsilon, \Delta + \varepsilon]) \ge \nu_{p,\ell}(\{0\}) = c_0 > 0.$$ ■

Proof of Corollary 1.16. Items (2) and (3) were already discussed in Theorem 1.5 and Lemma 9.5, respectively. For (1), if p is irreducible and not binomial, then its gap distribution cannot have any atoms by Theorem 1.15 part (3), and so it is absolutely continuous by Theorem 1.14 part (1).

Part (4) is a counting argument. Suppose that p has $N + M$ distinct irreducible factors, M of which are binomial. There can be three types of atoms according to Lemma 9.5:

(a) an atom at zero,

(b) an atom at positive $\Delta > 0$ coming from a pair of distinct non-binomial factors related by $q_i(\mathbf{z}) = q_j(\exp(i\Delta\ell)\mathbf{z})$,

(c) an atom at a positive $\Delta > 0$, coming from a pair of (not necessarily distinct) binomial factors related by $q_i(\mathbf{z}) = q_j(\exp(i\Delta\ell)\mathbf{z})$.

Notice that if q_i is binomial and q_j is non-binomial, then they cannot satisfy a relation of the form $q_i(\mathbf{z}) = q_j(\exp(i\Delta\ell)\mathbf{z})$, as such a relation means that the torus zero set Σ_{q_i}, which is a torus, is a translation of the torus zero set Σ_{q_j}, which is not a torus. It is left to bound the number of atoms of each type. There can be at most one (a) atom.

For atoms of type (b), notice that a pair of non-binomial factors cannot satisfy the relation $q_i(\mathbf{z}) = q_j(\exp(i\Delta\ell)\mathbf{z})$ for two different values of $\Delta > 0$, say $\Delta_1 \neq \Delta_2$. Otherwise, we get $q_j(\mathbf{z}) = q_j(\exp(i(\Delta_1 - \Delta_2)\ell)\mathbf{z})$ in contradiction to q_j being non-binomial. Therefore, there are at most $\binom{N}{2}$ atoms of type (b), one for each possible pair.

To bound the number of type (c) atoms, consider a pair of (not necessarily distinct) binomial factors related by $q_i(\mathbf{z}) = q_j(\exp(i\Delta\ell)\mathbf{z})$. In particular, q_i and q_j share the same multi-degree, say $\boldsymbol{\alpha}$. According to Lemma 7.1, the zero sets of $f_i(x) = q_i(\exp(ix\ell))$ and $f_j(x) = q_j(\exp(ix\ell))$ are arithmetic progressions of the same step size, say $\Lambda_i = \{a + 2\pi k/\langle\boldsymbol{\alpha}, \ell\rangle\}_{k\in\mathbb{Z}}$ and $\Lambda_j = \{a + \Delta + 2\pi k/\langle\boldsymbol{\alpha}, \ell\rangle\}_{k\in\mathbb{Z}}$ for some $a \in \mathbb{R}$. Suppose that p has exactly $M_{\boldsymbol{\alpha}}$ binomial factors with multi-degree $\boldsymbol{\alpha}$, and let $\Lambda_{\boldsymbol{\alpha}}$ denote the union of their arithmetic progressions defined above. Then $\Lambda_{\boldsymbol{\alpha}}$ is $2\pi/\langle\boldsymbol{\alpha}, \ell\rangle$ periodic with $M_{\boldsymbol{\alpha}}$ points in a period, and therefore at most $M_{\boldsymbol{\alpha}}$ gap values between consecutive points. By partitioning the M binomial factors according to their multi-degrees, we see that there are at most M atoms of type (c). We conclude that there are at most $\binom{N}{2} + M + 1$ atoms ■

Proof of Theorem 1.17. By Lemma 9.3, if $p \in \mathrm{LY}_{\mathbf{d}}(n)$ and $\ell \in \mathbb{R}^n_+$ has $\mathbb{Q}$-linearly independent entries, then $\rho_{p,\ell} = \nu_{p,\ell}$. It is left to show that $\nu_{p,\ell}$ is weakly continuous in ℓ, namely, that for any fixed continuous $f\colon \mathbb{R} \to \mathbb{C}$, the following integral is continuous in $\ell \in \mathbb{R}^n_+$:

$$\int f\, d\nu_{p,\ell} := c_0 f(0) + c_\tau \int_{\Sigma_p} f(\tau_\ell(\mathbf{x}))\, d\mathbf{m}_\ell(\mathbf{x}).$$

The weights c_0 and c_τ, given in Definition 9.1, are continuous in $\ell \in \mathbb{R}^n_+$, and the remaining integral can be written as

$$\int_{\Sigma_p} f(\tau_\ell(\mathbf{x}))\, d\mathbf{m}_\ell(\mathbf{x}) = \int_{\mathrm{reg}(\Sigma_p)} f(\tau_\ell(\mathbf{x}))\, d\mathbf{m}_\ell(\mathbf{x}) = \sum_{j=1}^{n} \ell_j \int_{\mathrm{reg}(\Sigma_p)} f(\tau_\ell(\mathbf{x}))\, d\mathbf{m}_{e_j}(\mathbf{x}),$$

using that $\mathbf{m}_\ell(\mathrm{sing}(\Sigma_p)) = 0$ in the first equality, and the linearity of $\mathbf{m}_\ell$ in ℓ (Lemma 6.5) in the second one. The integral $\int_{\mathrm{reg}(\Sigma_p)} f(\tau_\ell(\mathbf{x}))d\mathbf{m}_{e_j}(\mathbf{x})$ is continuous in ℓ because $(\mathbf{x}, \ell) \mapsto f(\tau_\ell(\mathbf{x}))$ is continuous over $\mathrm{reg}(\Sigma_p) \times \mathbb{R}^n_+$ by continuity of f and Lemma 6.6. ■

Let us now prove Theorem 1.19.

Proof of Theorem 1.19. Fix $p \in \mathrm{LY}_{\mathbf{d}}(n)$, and for any $\mathbf{x} \in \mathbb{R}^n$ let $p_{\mathbf{x}} \in \mathrm{LY}_{|\mathbf{d}|}(1)$ be the univariate polynomial $p_{\mathbf{x}}(s) = p(se^{ix_1}, se^{ix_2}, \ldots, se^{ix_n})$ whose degree is $|\mathbf{d}|$ and its roots lie on the unit circle. Let $\theta_j\colon \mathbb{R}^n \to \mathbb{R}$, for $j = 1, 2, \ldots, |\mathbf{d}|$, be the continuous phase functions given in Proposition 4.5, so that $(e^{i\theta_j(\mathbf{x})})_{j=1}^{|\mathbf{d}|}$ are the roots of $p_{\mathbf{x}}$ numbered (counter-clockwise) increasingly including multiplicity, and let $\theta_{|\mathbf{d}|+1} = \theta_1 + 2\pi$. We need to prove that for any continuous $f:\mathbb{R} \to \mathbb{C}$,

$$\int f\,d\nu_{p,\mathbf{1}} = \frac{1}{(2\pi)^n} \int_{\mathbf{x}\in[0,2\pi]^n} \Big[\frac{1}{|\mathbf{d}|} \sum_{j=1}^{|\mathbf{d}|} f(\theta_{j+1}(\mathbf{x}) - \theta_j(\mathbf{x}))\Big]\,d\mathbf{x},$$

where $\mathbf{1} = (1, 1, \dots, 1)$. Fix a continuous $f:\mathbb{R} \to \mathbb{C}$ and define

$$h(\mathbf{x}) := \frac{\mathrm{m}(\mathbf{x}) - 1}{\mathrm{m}(\mathbf{x})} f(0) + \frac{1}{\mathrm{m}(\mathbf{x})} f(\tau_\ell(\mathbf{x})).$$

Consider the layers $\Sigma_{p,j}$ and their parameterizations φ_j as defined in Proposition 4.2, so that the multiplicity $\mathrm{m}(\mathbf{x})$ counts the number of layers containing $\mathbf{x}$, so that

$$\int_{\Sigma_p} \mathrm{m}(\mathbf{x})\,h(\mathbf{x})\,d\mathbf{m_1}(\mathbf{x}) = \sum_{j=1}^{|\mathbf{d}|} \int_{\Sigma_{p,j}} \mathrm{m}(\mathbf{x})\,h(\mathbf{x})\,d\mathbf{m_1}(\mathbf{x}) = \sum_{j=1}^{|\mathbf{d}|} \int_{(0,2\pi]^{n-1}} h(\varphi_j(\mathbf{y}))\,d\mathbf{y},$$

using (6.8) from Lemma 6.5 in the last equality. As in the proof of Lemma 9.3, this gives

$$\int f\,d\nu_{p,\mathbf{1}} = \frac{1}{(2\pi)^{n-1}|\mathbf{d}|} \int_{\Sigma_p} \mathrm{m}(\mathbf{x})h(\mathbf{x})\,d\mathbf{m_1}(\mathbf{x}) = \frac{1}{(2\pi)^{n-1}|\mathbf{d}|} \sum_{j=1}^{|\mathbf{d}|} \int_{(0,2\pi]^{n-1}} h(\varphi_j(\mathbf{y}))\,d\mathbf{y}.$$

As seen in the proof of Lemma 6.6, if $\theta_{j+1}(\varphi_j(\mathbf{y})) > \theta_j(\varphi_j(\mathbf{y}))$, then $\tau_{\mathbf{1}}(\varphi_j(\mathbf{y}))$ is equal to the unique $t \in \mathbb{R}$ such that

$$\theta_{j+1}(\varphi_j(\mathbf{y}) + t\mathbf{1}) = \theta_j(\varphi_j(\mathbf{y})).$$

In such case, using part (1) of Proposition 4.5 and the definition of φ_j, we get

$$\tau_{\mathbf{1}}(\varphi_j(\mathbf{y})) = \theta_{j+1}(\mathbf{y}, 0) - \theta_j(\mathbf{y}, 0).$$

The number of j's for which $\theta_{j+1}(\mathbf{y}, 0) = \theta_j(\mathbf{y}, 0)$ is exactly $\sum_{j=1}^{|\mathbf{d}|}(\mathrm{m}(\varphi_j(\mathbf{y})) - 1)$, so

$$\sum_{j=1}^{|\mathbf{d}|} h(\tau_\ell(\varphi_j(\mathbf{y}))) = \sum_{j=1}^{|\mathbf{d}|} f(\theta_{j+1}(\mathbf{y}, 0) - \theta_j(\mathbf{y}, 0)),$$

for every $\mathbf{y}$, and integrating gives

$$\int f\,d\nu_{p,\mathbf{1}} = \frac{1}{(2\pi)^{n-1}|\mathbf{d}|} \int_{(0,2\pi]^{n-1}} \Big[\sum_{j=1}^{|\mathbf{d}|} f(\theta_{j+1}(\mathbf{y}, 0) - \theta_j(\mathbf{y}, 0))\,dy\Big]. \tag{9.5}$$

Let

$$g(\mathbf{x}) := \sum_{j=1}^{|\mathbf{d}|} f(\theta_{j+1}(\mathbf{x}) - \theta_j(\mathbf{x})),$$

and notice that g is continuous, satisfies $g((\mathbf{y},0)+t\mathbf{1}) = g(\mathbf{y},0)$ by part (1) of Proposition 4.5, and is 2π periodic by part (4) of Proposition 4.5, so

$$\int_{\mathbf{y}\in(0,2\pi]^{n-1}} g(\mathbf{y},0)\,d\mathbf{y} = \frac{1}{2\pi}\int_{t=0}^{2\pi}\int_{\mathbf{y}\in(0,2\pi]^{n-1}} g((\mathbf{y},0)+t\mathbf{1})\,d\mathbf{y}\,dt = \int_{\mathbf{x}\in(0,2\pi]^{n}} g(\mathbf{x})\,d\mathbf{x}.$$

The needed result follows:

$$\int f\,d\nu_{p,\mathbf{1}} = \frac{1}{(2\pi)^{n-1}|\mathbf{d}|}\int_{\mathbf{x}\in(0,2\pi]^{n}} \Big[\sum_{j=1}^{|\mathbf{d}|} f\,\big(\theta_{j+1}(\mathbf{x})-\theta_j(\mathbf{x})\big)\,d\mathbf{x}\Big] \qquad \blacksquare$$

A. Appendix

The next lemma is being used throughout the paper.

Lemma A.1. *Let $(a_n)_{n\in\mathbb{N}}$ be a bounded sequence $|a_n| < M$ and let $(s_n)_{n\in\mathbb{N}}$ be the sequence of partial averages,*

$$s_N := \frac{1}{N}\sum_{n=1}^{N} a_n.$$

Suppose that there exists a converging subsequence $\lim_{j\to\infty} s_{n_j} = L$, with a uniform spacing bound $n_{j+1} - n_j < M'$. Then, $\lim_{n\to\infty} s_n = L$.

Proof. Given any $n_j \le n' \le n_{j+1}$, the uniform spacing bound gives $n_j/n' \to 0$ as $j\to\infty$, and we have

$$|s_{n'} - \frac{n_j}{n'}s_{n_j}| = \frac{|a_{n_j+1} + a_{n_j+2} + \ldots + a_{n'}|}{n'} \le \frac{M'M}{n_j} \to 0 \ \text{ as } j\to\infty. \qquad \blacksquare$$

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