J. Comb. Algebra 8 (2024), 111–119
DOI 10.4171/JCA/79

Dimension expanders via quiver representations

Markus Reineke

Abstract. We relate the notion of dimension expanders to quiver representations and their general subrepresentations and use this relation to establish sharp existence results.

1. Introduction

Let F be a field, and let $\varepsilon > 0$ be a real number. An ε-expander is a tuple $(V, T_1, \dots, T_k)$, consisting of a finite-dimensional F-vector space V, together with linear operators $T_1, \dots, T_k$ on V such that, for all subspaces $U \subset V$ of dimension $\dim U \le \frac{1}{2} \dim V$, we have

$$\dim\left(U + \sum_{i=1}^{k} T_i(U)\right) \ge (1 + \varepsilon) \dim U.$$

This is a linear algebra analogue of the notion of expander graph [7]. It is proven in [8] for fields of characteristic zero, and in [1, 2] for finite fields, that there exist k and a *fixed* $\varepsilon > 0$ such that ε-expanders $(V, T_1, \dots, T_k)$ exist for *all* dimensions of the F-vector space V.

In the present article, we sharpen this existence result and determine the optimal expansion coefficient ε for F an algebraically closed field (of arbitrary characteristic).

Theorem 1.1. *Let F be an algebraically closed field, let $k \ge 2$, and define*

$$\varepsilon_k = \left(k + 1 - \sqrt{k^2 - 2k + 5}\right) \Big/ 2.$$

Then, there exist ε-expanders $(V, T_1, \dots, T_k)$ in all dimensions of V if and only if $\varepsilon \le \varepsilon_k$.

We will translate this result into a property of dimension vectors of subrepresentations of general representations of generalized Kronecker quivers, which we will derive from results of Schofield [9]. Since these results address Zariski-open properties of quiver representations of infinitely many dimension types, our proof will be non-constructive and will require the base field to be algebraically closed.

2020 Mathematics Subject Classification. Primary 16G20; Secondary 05C48.
Keywords. Dimension expander, quiver representation, general subrepresentation.

We will review the necessary quiver techniques in Section 2. In Section 3, we describe the dimension vectors of general representations of generalized Kronecker quivers. The applications to dimension expanders are derived in Section 4; namely, Theorem 1.1 is derived from the more general Theorem 4.3. Finally, in Section 5, we discuss potential generalizations to arbitrary quivers.

2. Recollections on quiver representations

From now on, let F be an algebraically closed field. Let Q be a finite quiver with set of vertices Q_0 and arrows written as $\alpha : i \to j$ for $i, j \in Q_0$, which we assume to be acyclic. We define the Euler form of Q on $\mathbb{Z}Q_0$ by

$$\langle \mathbf{d}, \mathbf{e} \rangle = \sum_{i \in Q_0} d_i e_i - \sum_{\alpha : i \to j} d_i e_j$$

for $\mathbf{d} = (d_i)_i$ and $\mathbf{e} = (e_i)_i$ in $\mathbb{Z}Q_0$. We consider the category $\mathrm{rep}_F\, Q$ of finite-dimensional F-representations of Q, which is an abelian F-linear hereditary finite length category. Its Grothendieck group identifies with $\mathbb{Z}Q_0$ by associating to a representation V its dimension vector $\mathbf{dim}V$, and its homological Euler form is given by the Euler form, that is,

$$\dim \mathrm{Hom}(V, W) - \dim \mathrm{Ext}^1(V, W) = \langle \mathbf{dim}V, \mathbf{dim}W \rangle$$

for all representations V and W.

For $\mathbf{d} \in \mathbb{N}Q_0$, we fix F-vector spaces V_i of dimension d_i for all $i \in Q_0$. We define the representation space

$$R_{\mathbf{d}}(Q) = \bigoplus_{\alpha : i \to j} \mathrm{Hom}_F(V_i, V_j),$$

whose points $(f_\alpha)_\alpha$ we identify with the corresponding representation of Q on the vector spaces V_i. On the F-vector space $R_{\mathbf{d}}(Q)$, the reductive linear algebraic group

$$G_{\mathbf{d}} = \prod_{i \in Q_0} \mathrm{GL}(V_i)$$

acts linearly via

$$(g_i)_i (f_\alpha)_\alpha = (g_j f_\alpha g_i^{-1})_{\alpha : i \to j}$$

such that the $G_{\mathbf{d}}$-orbits $\mathcal{O}_V$ in $R_{\mathbf{d}}(Q)$ naturally correspond to the isomorphism classes $[V]$ of F-representations V of Q of dimension vector $\mathbf{d}$.

For $\mathbf{e} \leq \mathbf{d}$ componentwise, the subset of $R_{\mathbf{d}}(Q)$ of all representations V admitting a subrepresentation of dimension vector $\mathbf{e}$ is Zariski-closed. Therefore, almost all representations in $R_{\mathbf{d}}(Q)$ (that is, those in a Zariski-dense subset) admit a subrepresentation of dimension vector $\mathbf{e}$ if and only if all representations in $R_{\mathbf{d}}(Q)$ do so. In this case, we write $\mathbf{e} \hookrightarrow \mathbf{d}$. There is a recursive numerical criterion for this notion due to Schofield in characteristic zero, generalized to positive characteristic by Crawley–Boevey.

Theorem 2.1 ([3,9]). *We have* $\mathbf{e} \hookrightarrow \mathbf{d}$ *if and only if* $\langle \mathbf{e}', \mathbf{d} - \mathbf{e} \rangle \geq 0$ *for all* $\mathbf{e}' \hookrightarrow \mathbf{e}$.

3. General subrepresentations of representation of generalized Kronecker quivers

Our main result in this section, which will directly apply to dimension expanders, is a non-recursive description of the relation $\mathbf{e} \hookrightarrow \mathbf{d}$ for generalized Kronecker quivers $K(m) = 1 \overset{(m)}{\rightrightarrows} 2$, given by two vertices 1, 2, and $m \geq 2$ arrows from 1 to 2. We prepare this description by some preliminary results.

Lemma 3.1. *For dimension vectors of* $K(m)$, *we have* $(e_1, e_2) \hookrightarrow (d_1, d_2)$ *if and only if* $(d_2 - e_2, d_1 - e_1) \hookrightarrow (d_2, d_1)$.

Proof. For a representation V of $K(m)$ given by an m-tuple $f_1, \dots, f_m : V_1 \to V_2$ of linear maps, we denote by V^* the representation $f_1^*, \dots, f_m^* : V_2^* \to V_1^*$. This obviously defines a duality on $\mathrm{rep}_F\, K(m)$. Assume that a general representation V of dimension vector $\mathbf{d}$ admits a subrepresentation U of dimension vector $\mathbf{e}$. Then, dually, a general representation V^* of dimension vector (d_2, d_1) admits a factor representation of dimension vector (e_2, e_1), whose kernel is a subrepresentation of dimension vector $(d_2 - e_2, d_1 - e_1)$. This finishes the proof. ■

We now extend the Euler form $\langle \_\, , \_ \rangle$ of $Q = K(m)$ to $\mathbb{R}Q_0$. We fix $0 \neq \mathbf{d} = (d_1, d_2) \in \mathbb{N}Q_0$ such that $0 \geq \langle \mathbf{d}, \mathbf{d} \rangle = d_1^2 + d_2^2 - m d_1 d_2$. In particular, $d_1, d_2 \geq 1$, and

$$\left(m - \sqrt{m^2 - 4}\right)\Big/2 \leq d_2/d_1 \leq \left(m + \sqrt{m^2 - 4}\right)\Big/2 =: \beta.$$

For fixed $x \in [0, d_1]$, we consider the function

$$q_x(y) = \langle (x, y), (d_1 - x, d_2 - y) \rangle$$

on $[0, d_2]$ and denote by $c_{\mathbf{d}}$ the smaller of its two zeroes. The explicit form

$$c_{\mathbf{d}}(x) = \left(mx + d_2 - \sqrt{(mx - d_2)^2 + 4x(d_1 - x)}\right)\Big/2$$

shows existence. In particular, $c_{\mathbf{d}}(x) \leq (mx + d_2)/2$, and $mx + d_2 - c_{\mathbf{d}}(x)$ is the larger zero of q_x. We have $q_x(y) \geq 0$ for $c_{\mathbf{d}}(x) \leq y \leq mx + d_2 - c_{\mathbf{d}}(x)$, and $q_x(y) \leq 0$ otherwise. We have the following estimate.

Lemma 3.2. *If* $\langle \mathbf{d}, \mathbf{d} \rangle \leq 0$*, we have* $d_2/d_1 \cdot x \leq c_{\mathbf{d}}(x) \leq \min(mx, d_2)$.

Proof. We have

$$q_x(d_2/d_1 \cdot x) = \langle x/d_1 \cdot \mathbf{d}, \mathbf{d} - x/d_1 \cdot \mathbf{d} \rangle = x(d_1 - x)/d_1^2 \cdot \langle \mathbf{d}, \mathbf{d} \rangle \leq 0$$

by assumption. Thus, the first inequality follows, since $c_{\mathbf{d}}(x) \leq (mx + d_2)/2$, once we know that $d_2/d_1 \cdot x \leq (mx + d_2)/2$. If $d_2/d_1 \leq m/2$, this holds trivially. Otherwise, we use $d_2 \leq md_1$ to estimate

$$(d_2/d_1 - m/2)x \leq (d_2/d_1 - m/2)d_1 = d_2/2 + (d_2 - md_1)/2 \leq d_2/2,$$

and again, the desired estimate follows.

For the second inequality, we calculate

$$q_x(mx) = \langle x \cdot (1, m), \mathbf{d} - x \cdot (1, m) \rangle = x(d_1 - x) \geq 0;$$

thus,

$$c_{\mathbf{d}}(x) \leq mx \leq mx + d_2 - c_{\mathbf{d}}(x),$$

which finishes the proof. ∎

Lemma 3.3. *If* $d_2 > \beta d_1$ *and* $\mathbf{e} \hookrightarrow \mathbf{d}$*, then* $e_2 > \beta e_1$.

Proof. $\mathbf{e} \hookrightarrow \mathbf{d}$ implies $\langle \mathbf{e}, \mathbf{d} - \mathbf{e} \rangle \geq 0$ by Schofield's criterion; thus, $e_2 \geq c_{\mathbf{d}}(e_1)$ by definition of $c_{\mathbf{d}}$. It thus suffices to prove that $c_{\mathbf{d}}(x) > \beta x$ provided that $d_2 > \beta d_1$. Since

$$\beta^2 - m\beta + 1 = 0,$$

we have

$$q_x(\beta x) = x(d_1 - (m - \beta)d_2) < xd_1(1 - m\beta + \beta^2) = 0,$$

from which we can conclude that $c_{\mathbf{d}}(x) > \beta x$ provided that $\beta x \leq mx + d_2 - c_{\mathbf{d}}(x)$. But

$$\beta x < mx \leq mx + d_2 - c_{\mathbf{d}}(x),$$

since $c_{\mathbf{d}}(x) \leq d_2$. ∎

We can now derive the main result of this section.

Proposition 3.4. *If* $Q = K(m)$ *is the* m*-arrow Kronecker quiver and* $\langle \mathbf{d}, \mathbf{d} \rangle \leq 0$*, then for* $\mathbf{e} \leq \mathbf{d}$ *the following are equivalent:*

(1) $\mathbf{e} \hookrightarrow \mathbf{d}$*;*

(2) $\langle \mathbf{e}, \mathbf{d} - \mathbf{e} \rangle \geq 0$*;*

(3) $e_2 \geq c_{\mathbf{d}}(e_1)$.

Proof. Without loss of generality, we assume $d_1 \le d_2$ using the duality Lemma 3.1. Obviously, (1) implies (2) implies (3): if $\mathbf{e} \hookrightarrow \mathbf{d}$ then, by Schofield's criterion applied to $\mathbf{e}' = \mathbf{e}$, we have $\langle \mathbf{e}, \mathbf{d} - \mathbf{e}\rangle \ge 0$, and by definition of the function $c_{\mathbf{d}}$, this implies that $e_2 \ge c_{\mathbf{d}}(e_1)$. Conversely, assume that this inequality holds, and let $\mathbf{e}' \hookrightarrow \mathbf{e}$. We want to prove that $\langle \mathbf{e}', \mathbf{d} - \mathbf{e}\rangle \ge 0$; then $\mathbf{e} \hookrightarrow \mathbf{d}$ follows from Schofield's criterion. We first assume that $\langle \mathbf{e}, \mathbf{e}\rangle \ge 1$; thus, $e_2 > \beta e_1$ or $e_2 < (m-\beta)e_1$. Since $e_2 \ge c_{\mathbf{d}}(e_1) \ge d_2/d_1 \cdot e_1 \ge e_1$, by assumption and Lemma 3.2, we have $e_2 > \beta e_1$ since $\beta \ge 1$. By Lemma 3.3, we find $e_2' > \beta e_1'$, and thus

$$\langle \mathbf{e}', \mathbf{d}-\mathbf{e}\rangle = e_1'(d_1-e_1-m(d_2-e_2)) + e_2'(d_2-e_2) > e_1'(d_1-e_1-(m-\beta)(d_2-e_2)).$$

Since $d_2 \le \beta d_1$, we have $d_2 - e_2 < \beta(d_1 - e_1)$; thus, $d_1 - e_1 > (m-\beta)(d_2 - e_2)$, proving the claim. Now, we assume that $\langle \mathbf{e}, \mathbf{e}\rangle \le 0$. By Lemma 3.2, we have

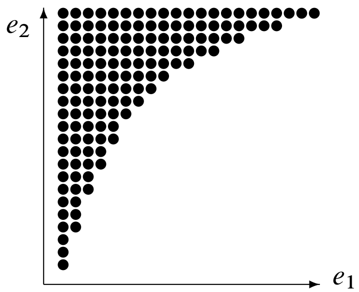
$$e_2' \ge c_{\mathbf{e}}(e_1') \ge e_2/e_1 \cdot e_1',$$

and thus

$$\begin{aligned}
\langle \mathbf{e}', \mathbf{d} - \mathbf{e}\rangle &= e_1'(d_1 - e_1 - m(d_2 - e_2)) + e_2'(d_2 - e_2) \\
&\ge e_1'/e_1 \cdot (e_1(d_1 - e_1) - me_1(d_2 - e_2) + e_2(d_2 - e_2)) \\
&= e_1'/e_1 \cdot \langle \mathbf{e}, \mathbf{d} - \mathbf{e}\rangle \ge 0,
\end{aligned}$$

again proving the claim. ∎

The following graph shows the set of all $\mathbf{e} \hookrightarrow (20, 20)$ for $m = 4$:



4. Application to dimension expanders

We generalize the definition of dimension expanders of Section 1, following [5], to a notion of expander representation. Proposition 3.4 then almost immediately yields sharp existence results.

Definition 4.1. Let $0 < \delta < 1$ and $\varepsilon > 0$, and let V and W be non-zero finite-dimensional F-vector spaces. We call a representation $f_1, \ldots, f_m : V \to W$ of $K(m)$ a (δ, ε)-expander representation if, for all subspaces $0 \neq U \subset V$ such that $\frac{\dim U}{\dim V} \leq \delta$, we have

$$\dim \sum_{k=1}^{m} f_k(U) \geq (1+\varepsilon) \cdot \frac{\dim W}{\dim V} \cdot \dim U.$$

The following lemma translates the existence of expander representations to properties of dimension vectors of subrepresentations of general representations.

Lemma 4.2. *For fixed integers $m, d_1, d_2 \geq 1$ and real numbers $0 < \delta < 1$, $\varepsilon > 0$, there exists a (δ, ε)-expander representation of $K(m)$ of dimension vector (d_1, d_2) if and only if for all $(e_1, e_2) \hookrightarrow (d_1, d_2)$ such that $e_1 \leq \delta \cdot d_1$, we have*

$$e_2 \geq (1+\varepsilon) \cdot \frac{d_2}{d_1} \cdot e_1.$$

Proof. Assume that there exists such an expander representation M given by $f_1, \ldots, f_m : V \to W$, and assume that $(e_1, e_2) \hookrightarrow (d_1, d_2)$. Then, in particular, M admits a subrepresentation of dimension vector (e_1, e_2); that is, there exists a subspace $U \subset V$ of dimension e_1 such that $\sum_k f_k(U)$ is of dimension at most e_2. On the other hand, $\sum_k f_k(U)$ is at least of dimension $(1+\varepsilon) \cdot \frac{d_2}{d_1} \cdot \dim U$. The claimed inequality for e_2 follows. Conversely, assume that the numerical condition is satisfied. Then, the set $S_{(e_1,e_2)} \subset R_{(d_1,d_2)}(K(m))$ of representations admitting a subrepresentation of dimension vector (e_1, e_2) is a proper Zariski-closed subset whenever $e_2 < (1+\varepsilon) \cdot \frac{d_2}{d_1} \cdot e_1$. Thus, the union of all these finitely many proper closed subsets is again a proper subset, and any representation in its complement is a (δ, ε)-expander representation by definition. ∎

This allows us to establish the following sharp existence result.

Theorem 4.3. *Fix an integer $m \geq 1$, real numbers $0 < \delta < 1$ and $\varepsilon > 0$, and a rational α such that*

$$\alpha^2 - m\alpha + 1 < 0 \quad \textit{and} \quad m\delta + \alpha - 2\alpha\delta > 0.$$

Define

$$\varepsilon_m(\alpha, \delta) = \frac{m\delta + \alpha - 2\alpha\delta - \sqrt{(m\delta - \alpha)^2 + 4\delta(1-\delta)}}{2\alpha\delta} > 0.$$

Then, there exist (δ, ε)-expander representations of $K(m)$ for all dimension vectors (d_1, d_2) such that $d_2/d_1 = \alpha$ if and only if $\varepsilon \leq \varepsilon_m(\alpha, \delta)$.

Proof. The assumptions on α ensure that $\varepsilon_m(\alpha, \delta) > 0$ by a straightforward calculation. We consider dimension vectors $\mathbf{d}$ such that $d_2/d_1 = \alpha$; in particular, $\langle \mathbf{d}, \mathbf{d} \rangle < 0$. By the previous lemma and Proposition 3.4, we have the following.

There exists a (δ, ε)-expander representation of $K(m)$ of dimension vector $\mathbf{d}$ if and only if $e_2 \geq (1+\varepsilon)\alpha e_1$ for all $e_1 \leq \delta d_1$ and all $e_2 \geq c_{\mathbf{d}}(e_1)$, or, equivalently, if $\lceil c_{\mathbf{d}}(x) \rceil \geq (1+\varepsilon)\alpha x$ for all integral $x \leq \delta \cdot d_1$.

This implies the following.

There exist (δ, ε)-expander representations of $K(m)$ for all dimension vectors (d_1, d_2) such that $d_2/d_1 = \alpha$ if and only if $\lceil c_{\mathbf{d}}(x) \rceil \geq (1+\varepsilon)\alpha x$ for all dimension vectors $\mathbf{d} = (d_1, d_2)$ such that $d_2/d_1 = \alpha$ and all integral $x \leq \delta d_1$.

The function $c_{\mathbf{d}}(x)$ is concave on the interval $[0, d_2]$ since, by a straightforward calculation, its second derivative equals

$$c''_{\mathbf{d}}(x) = \frac{2\langle \mathbf{d}, \mathbf{d} \rangle}{((mx - d_2)^2 + 4x(d_1 - x))^{3/2}},$$

which is negative by assumption. Thus, in the interval $[0, \delta d_1]$, the fraction $c_{\mathbf{d}}(x)/x$ attains its minimum at δd_1. For $\rho \in [0, 1]$, we have

$$c_{\mathbf{d}}(\rho d_1)/(\alpha \rho d_1) = 1 + \varepsilon_m(\alpha, \rho);$$

thus, in the interval $[0, \delta]$, the function $\varepsilon_m(\alpha, \rho)$ of ρ attains its minimum at $\rho = \delta$. We thus find that the above existence condition is equivalent to

$$\lceil (1 + \varepsilon_m(\alpha, \rho))\alpha \rho d_1 \rceil \geq (1 + \varepsilon)\alpha \rho d_1$$

for all d_1 such that αd_1 is integral and all $\rho \in [0, \delta]$ such that ρd_1 is integral. This is clearly equivalent to $\varepsilon_m(\alpha, \rho) \geq \varepsilon$ for all $\rho \in [0, \delta]$, and this in turn to $\varepsilon_m(\alpha, \delta) \geq \varepsilon$. This finishes the proof. ∎

This result immediately implies Theorem 1.1 as the special case $m = k+1$, $\alpha = 1$, $\delta = 1/2$. Namely, in a general representation of $K(k+1)$ of dimension vector (d, d), the map representing the first arrow is invertible; thus, without loss of generality, the identity, and id, $T_1, \ldots, T_k : V \to V$ defines an expander representation if and only if $(V, T_1, \ldots, T_k)$ is a dimension expander; moreover, $\varepsilon_{k+1}(1, 1/2) = \varepsilon_k$.

5. Potential generalizations

We finish with a few remarks suggesting further directions.

The characterization of dimension vectors $\mathbf{e} \hookrightarrow \mathbf{d}$ of subrepresentations of general representation by the single quadratic equation $\langle \mathbf{e}, \mathbf{d} - \mathbf{e} \rangle \geq 0$ of Proposition 3.4 is special to the quivers $K(m)$. Namely, we have the following.

Example 5.1. For the complete bipartite three-vertex quiver $\bullet \Rightarrow \bullet \Leftarrow \bullet$, the dimension vector $\mathbf{d} = (3, 6, 5)$ is a Schur root (even belonging to the fundamental domain), and $\mathbf{e} = (3, 5, 1)$ fulfills $\langle \mathbf{e}, \mathbf{d} - \mathbf{e} \rangle \geq 0$ (even > 0), but $\mathbf{e} \not\hookrightarrow \mathbf{d}$.

It is natural to ask whether the explicit dimension expanders constructed in [8] using representations of $\mathrm{SL}_2(\mathbb{Z})$, and in [1] using monotone expanders, are already ε_k-expanders for the optimal expansion coefficients ε_k.

In another direction, dimension expanders were used in [4] to construct non-hyperfinite families of representations of generalized Kronecker quivers, and it would be interesting to know whether the present methods yield new insights about such families.

Representations $f_1, \ldots, f_m : V_1 \to V_2$ such that $\dim \sum_k f_k(U) > \frac{\dim V_2}{\dim V_1} \dim U$ for all proper non-zero subspaces $U \subset V_1$ are stable in the sense of geometric invariant theory [6]; thus, the (δ, ε)-expander property might be viewed as a quantitative form of stability. This point of view suggests a generalization to arbitrary quivers which we sketch in the following, leaving details to future work.

So, let Q be a finite quiver, and let $\Theta, \kappa \in (\mathbb{R}Q_0)^*$ be linear functionals, with κ assuming positive values on positive vectors. We consider dimension vectors $\mathbf{d} \in \mathbb{N}Q_0$ for Q such that $\Theta(\mathbf{d}) = 0$ and real numbers $0 < \delta < 1$ and $\varepsilon > 0$.

Definition 5.2. A representation V of Q of dimension vector $\mathbf{dim} V = \mathbf{d}$ is called a (δ, ε)-expander relative to Θ, κ if, for all subrepresentations $U \subset V$ such that $\kappa(\mathbf{dim} U) \leq \delta \cdot \kappa(\mathbf{d})$, we have $\Theta(\mathbf{dim} U) \geq \varepsilon \cdot \kappa(\mathbf{dim} U)$. We say that (Q, Θ, κ) exhibits uniform expansion if, for all $0 < \delta < 1$, there exists $\varepsilon > 0$ such that there exist (δ, ε)-expander representations relative to Θ, κ for infinitely many dimension vectors $\mathbf{d}$ such that $\Theta(\mathbf{d}) = 0$.

The methods of Section 4, in particular Lemma 4.2 and Theorem 4.3, can be modified to work for general Θ, κ for the quivers $K(m)$, proving uniform expansion (in the sense of the definition) in this case. They can be further generalized to special classes of bipartite quivers, for which the problem can be effectively reduced to a convexity property as in the proof of Theorem 4.3. It is natural to conjecture that every wild quiver exhibits uniform expansion with respect to suitable Θ, κ.

Acknowledgements. The author is grateful to M. Bertozzi and K. Martinez, and to an anonymous referee, for many helpful comments on this manuscript.

Funding. This work is supported by the DFG SFB-TRR 191 “Symplectic structures in geometry, algebra and dynamics”.

References

[1] J. Bourgain, Expanders and dimensional expansion. *C. R. Math. Acad. Sci. Paris* **347** (2009), no. 7-8, 357–362 Zbl 1160.22006 MR 2537230

[2] J. Bourgain and A. Yehudayoff, Expansion in $\mathrm{SL}_2(\mathbb{R})$ and monotone expanders. *Geom. Funct. Anal.* **23** (2013), no. 1, 1–41 Zbl 1268.05103 MR 3037896

[3] W. Crawley-Boevey, Subrepresentations of general representations of quivers. *Bull. London Math. Soc.* **28** (1996), no. 4, 363–366 Zbl 0863.16008 MR 1384823

[4] S. Eckert, Wild kronecker quivers and amenability. 2020, arXiv:2011.02040

[5] V. Guruswami, N. Resch, and C. Xing, Lossless dimension expanders via linearized polynomials and subspace designs. *Combinatorica* **41** (2021), no. 4, 545–579 Zbl 1499.11360 MR 4328722

[6] A. D. King, Moduli of representations of finite-dimensional algebras. *Quart. J. Math. Oxford Ser. (2)* **45** (1994), no. 180, 515–530 Zbl 0837.16005 MR 1315461

[7] A. Lubotzky, Expander graphs in pure and applied mathematics. *Bull. Amer. Math. Soc. (N.S.)* **49** (2012), no. 1, 113–162 Zbl 1232.05194 MR 2869010

[8] A. Lubotzky and E. Zelmanov, Dimension expanders. *J. Algebra* **319** (2008), no. 2, 730–738 Zbl 1154.15002 MR 2381805

[9] A. Schofield, General representations of quivers. *Proc. London Math. Soc. (3)* **65** (1992), no. 1, 46–64 Zbl 0795.16008 MR 1162487

Received 24 February 2022; revised 26 April 2023.

Markus Reineke
Faculty of Mathematics, Ruhr University Bochum, Universitätsstr. 150, 44780 Bochum, Germany; markus.reineke@rub.de