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The condenser quasicentral modulus

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Abstract. We introduced the quasicentral modulus to study normed ideal perturbations of operators. It is a limit of condenser quasicentral moduli in view of a recently noticed analogy with capacity in nonlinear potential theory. We prove here some basic properties of the condenser quasicentral modulus and compute a simple example. We also discuss some associated noncommutative variational problems. Part of the results are in the more general setting of a semifinite von Neumann algebra.

1. Introduction

The quasicentral modulus [8–10] plays a key role in the study of Hilbert space operators modulo normed ideals (see our surveys [11, 12]). This paper is a sequel to [13]. In [13] we made the case that the quasicentral modulus is a noncommutative analog of capacity in nonlinear potential theory, where the first-order Sobolev spaces use general rearrangement invariant norms of the gradients. One consequence is that the quasicentral modulus becomes a limit quantity of condenser quasicentral moduli. Note that the condenser quasicentral moduli are usually finite and non-zero also in situations where the quasicentral modulus can take only the values 0 and ∞, like in the case of the p-classes when $p > 1$. Another new feature is that we will often deal with the more general case of separable semifinite factors or von Neumann algebras, that is, not only with the type I case of the algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on the Hilbert space $\mathcal{H}$ and note also that even the case when $\mathcal{H}$ is finite dimensional is no longer a trivial case.

Concerning the nonlinear potential theory capacity with which we observed an analogy, see [2]; for more references, see [13]. Prior to this, we had noticed connections with Yamasaki hyperbolicity [14] and with the noncommutative potential theory based on Dirichlet forms [1].

Can the analogy with nonlinear potential theory be further extended? A way to achieve this may be via noncommutative variational problems. We take a small first

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step in this direction introducing variational problems related to the condenser. We observe that computations in the case of the p-classes naturally lead to noncommutative analogs of the p-Laplace equation.

Besides the introduction and references, there are eight more sections. Section 2 is about preliminaries and basic definitions. Section 3 contains some general properties of the condenser quasicentral modulus in the semifinite setting. In particular, we prove that under certain conditions the condenser quasicentral modulus is symmetric with respect to switching the projections which define the condenser. We also give a result about the behavior with respect to certain conditional expectations. Section 4 gives a lower bound for the condenser quasicentral modulus in the case of the algebra of bounded operators on a Hilbert space. This is analogous to the lower bound in [10] for the quasicentral modulus. Section 5 is the computation of an example arising from the bilateral shift operator. In Section 6, we adapt and generalize to our semifinite setting the result in [9] about the largest reducing projection on which the quasicentral modulus vanishes. In the analogy with nonlinear potential theory, this is a special noncommutative polar set. Section 7 deals with variants of the quasicentral condenser modulus. Section 8 is about noncommutative variational problems. We make some general remarks about minimizers for the condenser problem.

2. Preliminaries and definitions

We introduce here the framework in which we will work, especially related to normed ideal/symmetric operator space norms [3–6], and we recall the definition of the quantities we introduced in [13].

By $(\mathcal{M}, \rho)$ we will denote a von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a separable complex Hilbert space and ρ is a faithful normal semifinite trace on $\mathcal{M}$. We will assume that $\mathcal{M}$ is either atomic, that is, it is generated by its minimal projections, or that it is diffuse and $\rho(I) = \infty$, in particular, there are no minimal projections in this case. Thus, $\mathcal{M}$ could be, for instance, a type I or a type II_∞ factor with its trace, but it could also be for instance $L^\infty(S, d\sigma)$, where the measure sigma has no atoms and is not finite or it could be $\ell^\infty(X)$ with the measure giving mass 1 to each singleton subset (the traces are those corresponding to the measures).

We will denote by $\mathrm{Proj}(\mathcal{M})$ the selfadjoint projections in $\mathcal{M}$ and by $\mathcal{P}(\mathcal{M})$ or simply $\mathcal{P}$ the set of $P \in \mathrm{Proj}(\mathcal{M})$ so that $\rho(P) < \infty$. By $\mathcal{R}$ we will denote the set of $x \in \mathcal{M}$ for which there is $P \in \mathcal{P}$ so that $xP = x$, that is the ideal of operators of finite ρ-rank in $\mathcal{M}$. Further, $\mathcal{R}_1^+$ will stand for the positive contractions in $\mathcal{R}$, that is, $\{a \in \mathcal{R} \mid 0 \le a \le I\}$. It will also be convenient to introduce the set

$$\Lambda = \{L \in L^1(\mathcal{M}, \rho) \mid 0 \le L \le I\}.$$

Thus, we have $\mathcal{P} \subset \mathcal{R}_1^+ \subset \Lambda \subset \mathcal{M}$.

If $x \in \mathcal{M}$, the generalized singular values (see [5]) are

$$\mu(t, x) = \inf\{\|A(I - P)\| \mid P \in \mathcal{P},\ \rho(P) \leq t\},$$

where $t > 0$ and $\mu(x)$ will denote the function

$$(0, \infty) \ni x \to \mu(t, x) \in [0, \infty).$$

On $L^1(\mathcal{M}, \rho) \cap \mathcal{M}$ we will consider a norm $|\cdot|_{\mathcal{J}}$ to which we will refer as the symmetric operator norm. We will assume that $|\cdot|_{\mathcal{J}}$ satisfies the following conditions.

(1) $\mu(x) \leq \mu(y) \Rightarrow |x|_{\mathcal{J}} \leq |y|_{\mathcal{J}}$ which has among its consequences, that if $a, b \in \mathcal{M}$, $x \in \mathcal{M} \cap L^1(\mathcal{M}, \rho)$, then $|axb|_{\mathcal{J}} \leq \|a\||x|_{\mathcal{J}}\|b\|$.

(2) $C_1 \min(|x|_1, \|x\|) \leq \|x\|_{\mathcal{J}} \leq C_2(|x|_1 + \|x\|)$ for some constants $C_1, C_2 \in (0, \infty)$.

(3) If $P_n \in \mathcal{P}$ are so that $\rho(P_n) \to 0$ as $n \to \infty$, then $|P_n|_{\mathcal{J}} \to 0$. This condition is trivially satisfied if $\mathcal{M}$ is a type I factor because it is meaningless, there being no non-zero P_n as above. The condition is equivalent to $x_n \in \mathcal{M} \cap L^1(\mathcal{M}, \rho)$, $\|x_n\| \leq 1$, $|x_n|_1 \to 0 \Rightarrow |x_n|_{\mathcal{J}} \to 0$. The condition can also be put in the form: there is an increasing function $\phi : (0, \infty) \to (0, \infty)$ so that $\lim_{t \to 0} \phi(t) = 0$ for which we have $\|x\| \leq 1$, $x \in L^1(\mathcal{M}, \rho) \Rightarrow |x|_{\mathcal{J}} \leq \phi(|x|_1)$. Note also that if $(\mathcal{M}, \rho)$ is diffuse and the condition is not satisfied, then there must be a constant $C \in (0, \infty)$ so that $|x|_{\mathcal{J}} \geq C\|x\|$.

(4) Considering the von Neumann algebra $\mathcal{M} \otimes \mathfrak{M}_n$ endowed with the trace $\rho \otimes \mathrm{Tr}_n$, the norm $|\cdot|_{\mathcal{J}}$ on $\mathcal{M} \cap L^1(\mathcal{M}, \rho)$, identified with a subspace of $\mathcal{M} \otimes e_{11}$ has an extension to a norm, that we will still denote by $|\cdot|_{\mathcal{J}}$ on $(L^1(\mathcal{M}, \rho) \cap \mathcal{M}) \otimes \mathfrak{M}_n = L^1(\mathcal{M} \otimes \mathfrak{M}_n, \rho \otimes \mathrm{Tr}_n) \cap (\mathcal{M} \otimes \mathfrak{M}_n)$ satisfying the analogues of (1)–(3). (Here, $\mathfrak{M}_n$ denotes the $n \times n$ matrices, $e_{i,j}$ are the matrix units and Tr_n is the trace.)

We opted for this ad hoc way of introducing $|\cdot|_{\mathcal{J}}$ instead of a discussion starting with operator spaces [5] which would have taken us farther than the more modest aim of this paper. We will also denote by $\mathcal{J}$ the completion of $L^1(\mathcal{M}, \rho) \cap \mathcal{M}$ with respect to the norm $|\cdot|_{\mathcal{J}}$, which is an $\mathcal{M}$ bimodule. In the case of $\mathcal{B}(\mathcal{H})$, $\mathcal{J}$ identifies with a normed ideal in $\mathcal{B}(\mathcal{H})$ consistent with the notation we used in previous papers (see [12]). Note also that property 4. in the symmetric operator spaces setting, actually follows from the general relation of symmetric function spaces, symmetric sequence spaces and symmetric operator spaces (see [5, Section 2.5], [5, Theorem 2.5.3 and the discussion of Questions 2.5.4 and 2.5.5]).

Definition 2.1. Let $P, Q \in \mathcal{P}$, $PQ = 0$ and let $\tau = (T_j)_{1 \leq j \leq n}$, $T_j \in \mathcal{M}$, $1 \leq j \leq n$. The condenser quasicentral modulus with respect to the symmetric operator norm

$|\cdot|_{\mathcal{J}}$ is the number

$$k_{\mathcal{J}}(\tau; P, Q) = \inf\{\max_{1\le j\le n} |[T_j, A]|_{\mathcal{J}} \mid A \in \mathcal{R}_1^+, AP = P, AQ = 0\}.$$

Similarly, if $\alpha = (\alpha_j)_{1\le j\le n}$ is an n-tuple of automorphisms of $\mathcal{M}$ which preserve ρ we define

$$k_{\mathcal{J}}(\alpha; P, Q) = \inf\{\max_{1\le j\le n} |\alpha_j(A) - A|_{\mathcal{J}} \mid A \in \mathcal{R}_1^+, AP = P, AQ = 0\}.$$

We remark that if $P_1, Q_1 \in \mathcal{P}$, $P_1 Q_1 = 0$ and $P \le P_1$, $Q \le Q_1$ then

$$\{A \in \mathcal{R}_1^+ \mid AP = P, AQ = 0\} \supset \{A \in \mathcal{R}_1^+ \mid AP_1 = P_1, AQ_1 = 0\}$$

so that

$$k_{\mathcal{J}}(\tau; P, Q) \le k_{\mathcal{J}}(\tau; P_1, Q_1),$$
$$k_{\mathcal{J}}(\alpha; P, Q) \le k_{\mathcal{J}}(\alpha; P_1, Q_1).$$

Definition 2.2. Let $P, Q \in \mathrm{Proj}(\mathcal{M})$, $PQ = 0$ and let τ, α, $|\cdot|_{\mathcal{J}}$ be like in Definition 2.1. We define

$$k_{\mathcal{J}}(\tau; P, Q) = \sup\{k_{\mathcal{J}}(\tau; P', Q') \mid P', Q' \in \mathcal{P}, P' \le P, Q' \le Q\},$$
$$k_{\mathcal{J}}(\alpha; P, Q) = \sup\{k_{\mathcal{J}}(\alpha; P', Q') \mid P', Q' \in \mathcal{P}, P' \le P, Q' \le Q\}.$$

Further, we define

$$k_{\mathcal{J}}(\tau; P) = k_{\mathcal{J}}(\tau; P, 0),$$
$$k_{\mathcal{J}}(\alpha; P) = k_{\mathcal{J}}(\alpha; P, 0)$$

and the quasicentral moduli of τ and of α

$$k_{\mathcal{J}}(\tau) = k_{\mathcal{J}}(\tau; I),$$
$$k_{\mathcal{J}}(\alpha) = k_{\mathcal{J}}(\alpha : I).$$

If $\mathcal{M}$ needs to be specified, we will write $k_{\mathcal{J},\mathcal{M}}(\tau; P, Q)$, and so on.

In case $(\mathcal{M}, \rho) = (\mathcal{B}(\mathcal{H}), \mathrm{Tr})$, this definition of $k_{\mathcal{J}}(\tau)$ is equivalent to the definition we used in our earlier work, as we already pointed out in [13].

It will also be useful to have a technical result about replacing $\mathcal{R}_1^+$ by the larger set Λ in the above definitions.

Lemma 2.1. *Let $P, Q \in \mathcal{P}$, τ, α be as in Definition 2.1. We have*

$$k_{\mathcal{J}}(\tau; P, Q) = \inf\{\max_{1\le j\le n} |[T_j; A]|_{\mathcal{J}} \mid A \in \Lambda, AP = P, AQ = 0\},$$

$$k_{\mathcal{J}}(\alpha; P, Q) = \inf\{\max_{1\le j\le n} |\alpha_j(A) - A|_{\mathcal{J}} \mid A \in \Lambda, AP = P, AQ = 0\}.$$

Proof. Lemma is an immediate consequence of the fact that

$$\Xi = \{A \in \mathcal{R}_1^+ \mid AP = P, AQ = 0\}$$

is a dense subset with respect to the topology defined by the norm $|\cdot|_{\mathcal{J}}$ of the set

$$\Theta = \{A \in \Lambda \mid AP = P, AQ = 0\}.$$

This in turn is seen as follows:

$$\Xi = P + (I - P - Q)\mathcal{R}_1^+(I - P - Q),$$
$$\Theta = P + (I - P - Q)\Lambda(I - P - Q)$$

so that the proof reduces to the proof of the density of $\mathcal{R}_1^+$ in Λ. If $X \in \Lambda$ then let P_j be the spectral projection of X for $[1/j, 1]$. We then have $P_j \in \mathcal{P}$ and $\|XP_j - X\| \to 0$, $|XP_j - X|_1 \to 0$ so that $XP_j \in \mathcal{R}_1^+$ and by condition (2), we have $|X - XP_j|_{\mathcal{J}} \to 0$. ■

3. Some general properties

In this section we derive some basic properties of the quasicentral modulus.

Proposition 3.1. *Assume $k_{\mathcal{J}}(\tau) = 0$ and $P, Q \in \mathrm{Proj}(\mathcal{M})$ are so that $PQ = 0$. Then,*

$$k_{\mathcal{J}}(\tau; P, Q) = k_{\mathcal{J}}(\tau; Q, P).$$

Similarly, if instead of τ we have α with $k_{\mathcal{J}}(\alpha) = 0$, then

$$k_{\mathcal{J}}(\alpha; P, Q) = k_{\mathcal{J}}(\alpha; Q, P).$$

Proof. Since $k_{\mathcal{J}}(\cdot\,; P, Q)$ for general P, Q is the sup of such quantities with $P, Q \in \mathcal{P}$, it suffices to prove the proposition under the additional assumption that $P, Q \in \mathcal{P}$. Moreover, by symmetry, it clearly suffices to prove that

$$k_{\mathcal{J}}(\cdot\,; P, Q) \geq k_{\mathcal{J}}(\cdot\,; Q, P).$$

If $\varepsilon > 0$, there is

$$A \in P + (I - P - Q)\mathcal{R}_1^+(I - P - Q)$$

so that

$$\max_{1 \leq j \leq n} |[A, T_j]|_{\mathcal{J}} \leq k_{\mathcal{J}}(\tau; P, Q) + \varepsilon.$$

Then, we have

$$\max_{1 \leq j \leq n} |[(I - A), T_j]|_{\mathcal{J}} = \max_{1 \leq j \leq n} |[A, T_j]|_{\mathcal{J}} \leq k_{\mathcal{J}}(\tau; P, Q) + \varepsilon.$$

Clearly, $0 \leq I - A \leq I$, $(I - A)P = 0$, $(I - A)Q = Q$ and we would be done if there were not the problem that $I - A$ is not in $\mathcal{R}$ in general. Since $k_{\mathcal{J}}(\tau) = 0$ we also have $k_{\mathcal{J}}(\tau; P + Q) = 0$ so that there is $B \in \mathcal{R}_1^+$ such that $B(P + Q) = P + Q$ and $\max_{1 \leq j \leq n} |[B, T_j]|_{\mathcal{J}} < \varepsilon$. Let $F = B(I - A)B$. We have

$$F \in \mathcal{R}_1^+, \quad FP = 0, \quad FQ = Q$$

and

$$|[F, T_j]|_{\mathcal{J}} \leq 2|[B, T_j]|_{\mathcal{J}} + |[I - A, T_j]|_{\mathcal{J}} < 3\varepsilon.$$

Thus,

$$k_{\mathcal{J}}(\tau; P, Q) + 3\varepsilon \geq k_{\mathcal{J}}(\tau; Q, P)$$

and $\varepsilon > 0$ being arbitrary, 3ε is as good as ε here.

The case of the n-tuple of automorphisms α is dealt with along the same lines. If $A \in P + (I - P - Q)\mathcal{R}_1^+(I - P - Q)$ and

$$\max_{1 \leq j \leq n} |\alpha_j(A) - A|_{\mathcal{J}} \leq k_{\mathcal{J}}(\alpha; P, Q) + \varepsilon,$$

then we have

$$|\alpha_j(I - A) - (I - A)|_{\mathcal{J}} = |\alpha_j(A) - A|_{\mathcal{J}}$$

and $(I - A)P = 0$, $(I - A)Q = Q$. Choosing B as in the previous case, we consider $F = B(I - A)B$, and we have

$$|\alpha_j(F) - F|_{\mathcal{J}} \leq 2|\alpha_j(B) - B|_{\mathcal{J}} + |\alpha_j(A) - A|_{\mathcal{J}}.$$

This leads then to

$$k_{\mathcal{J}}(\alpha; P, Q) + 3\varepsilon \geq k_{\mathcal{J}}(\alpha; Q, P),$$

and so on. ■

The framework for the next result involves a von Neumann subalgebra $\mathcal{N} \subset \mathcal{M}$ so that $\rho|\mathcal{N}$ is semifinite, in which case $\mathcal{R}(\mathcal{N})$ is weakly dense in $\mathcal{N}$. Let E be the conditional expectation of $\mathcal{M}$ onto $\mathcal{N}$ with respect to ρ (see [7, Proposition 2.36]). If the n-tuple of automorphisms α is so that $\alpha_j(\mathcal{N}) = \mathcal{N}$, $1 \leq j \leq n$, then in view of the assumption $\rho \circ \alpha_j = \rho$, $1 \leq j \leq n$, we will have that

$$\alpha_j|\mathcal{N} \circ E = E \circ \alpha_j, \quad 1 \leq j \leq n.$$

We will also assume that

$$x \in L^1(\mathcal{M}) \cap \mathcal{M} \Rightarrow |E(x)|_{\mathcal{J}} \leq |x|_{\mathcal{J}}.$$

Since the symmetric operator norm $|\cdot|_{\mathcal{J}}$ was introduced in an ad hoc way, without going into the theory of spaces of operators, we prefer to treat this as an assumption.

Proposition 3.2. *Let $\mathcal{N}$ be a von Neumann subalgebra of $\mathcal{M}$ so that $\rho|\mathcal{N}$ is semifinite and let E be the conditional expectation of $\mathcal{M}$ onto $\mathcal{N}$ with respect to ρ. Assume also that the n-tuple of automorphisms α, which preserve ρ is so that $\alpha_j(\mathcal{N}) = \mathcal{N}$, $1 \le j \le n$. Let further $P, Q \in \mathrm{Proj}(\mathcal{N})$ be so that $PQ = 0$. Then, we have*

$$k_{\mathcal{J},\mathcal{M}}(\alpha; P, Q) = k_{\mathcal{J},\mathcal{N}}(\alpha|\mathcal{N}; P, Q).$$

Proof. We first prove the statement in case $P, Q \in \mathcal{P}(\mathcal{N})$. The inequality

$$k_{\mathcal{J},\mathcal{N}}(\alpha|\mathcal{N}; P, Q) \ge k_{\mathcal{J},\mathcal{M}}(\alpha; P, Q)$$

is obvious because the LHS is an inf over a subset of the set the inf of which is the RHS.

We have $E(\Lambda(\mathcal{M})) = \Lambda(\mathcal{N})$, so that

$$E(P + (I - P - Q)\Lambda(\mathcal{M})(I - P - Q)) = P + (I - P - Q)\Lambda(\mathcal{N})(I - P - Q).$$

Thus, if

$$A \in P + (I - P - Q)\Lambda(\mathcal{M})(I - P - Q),$$

then we have

$$|\alpha_j(E(A)) - E(A)|_{\mathcal{J}} = |E(\alpha_j(A)) - A|_{\mathcal{J}} \le |\alpha_j(A) - A|_{\mathcal{J}},$$

and hence,

$$k_{\mathcal{J},\mathcal{N}}(\alpha|\mathcal{N}; P, Q) \le k_{\mathcal{J},\mathcal{M}}(\alpha; P, Q).$$

This concludes the proof in the case of $P, Q \in \mathcal{P}(\mathcal{N})$.

Assume now that we only have $P, Q \in \mathrm{Proj}(\mathcal{N})$. Then, we get that

$$\begin{aligned} &k_{\mathcal{J},\mathcal{M}}(\alpha; P, Q) \\ &\quad = \sup\{k_{\mathcal{J},\mathcal{M}}(\alpha; P_1; Q_1) \mid P_1 \le P, Q_1 \le Q, P_1, Q_1 \in \mathcal{P}(\mathcal{M})\} \\ &\quad \ge \sup\{k_{\mathcal{J},\mathcal{M}}(\alpha; P_1, Q_1) \mid P_1 \le P, Q_1 \le Q, P_1, Q_1 \in \mathcal{P}(\mathcal{N})\} \\ &\quad = \sup\{k_{\mathcal{J},\mathcal{N}}(\alpha \mid \mathcal{N}; P_1, Q_1) \mid P_1 \le P, Q_1 \le Q, P_1, Q_1 \in \mathcal{P}(\mathcal{N})\} \\ &\quad = k_{\mathcal{J},\mathcal{N}}(\alpha \mid \mathcal{N}; P, Q). \end{aligned}$$

We still must prove that if $P_1, Q_1 \in \mathcal{P}(\mathcal{M})$, $P_1 \le P$, $Q_1 \le Q$, then there are $P_2, Q_2 \in \mathcal{P}(\mathcal{N})$, $P_2 \le P$, $Q_2 \le Q$ so that

$$k_{\mathcal{J},\mathcal{M}}(\alpha; P_1, Q_1) \le k_{\mathcal{J},\mathcal{M}}(\alpha; P_2, Q_2) + \eta$$

for a given $\eta > 0$.

If $A \in P_2 + (I - P_2 - Q_2)\Lambda(\mathcal{M})(I - P_2 - Q_2)$, where $P_2 \leq P$, $Q_2 \leq Q$, $P_2, Q_2 \in \mathcal{P}(\mathcal{N})$, we consider

$$B = P_1 + (I - P_1 - Q_1)A(I - P_1 - Q_1)$$

so that

$$B \in P_1 + (I - P_1 - Q_1)\Lambda(\mathcal{M})(I - P_1 - Q_1).$$

Since $\rho|\mathcal{N}$ is semifinite, there are $P_2, Q_2 \in \mathcal{P}(\mathcal{N})$, $P_2 \leq P$, $Q_2 \leq Q$ so that

$$|(I - P_2)P_1|_1 < \varepsilon, \quad |(I - Q_2)Q_1|_1 < \varepsilon,$$

which also implies

$$|P_1(I - P_2)|_1 < \varepsilon, \quad |Q_1(I - Q_2)|_1 < \varepsilon.$$

Then, we have

$$\begin{aligned} &|(I - P_1 - Q_1)(I - P_2 - Q_2) - (I - P_2 - Q_2)|_1 \\ &\quad = |(P_1 + Q_1)(I - P_2 - Q_2)|_1 = |P_1(I - P_2) + Q_1(I - Q_2)|_1 < 2\varepsilon. \end{aligned}$$

If $X \in \mathcal{M}$, $\|X\| \leq 1$, this gives

$$\begin{aligned} &|(I - P_1 - Q_1)(I - P_2 - Q_2)X(I - P_2 - Q_2)(I - P_1 - Q_1) \\ &\quad - (I - P_2 - Q_2)X(I - P_2 - Q_2)|_1 < 4\varepsilon \end{aligned}$$

so that in particular

$$\begin{aligned} |B - A|_1 &\leq 4\varepsilon + |P_1 - P_2 + (I - P_1 - Q_1)P_2(I - P_1 - Q_1)|_1 \\ &= 4\varepsilon + |P_1 - P_2 + (I - P_1)P_2(I - P_1)|_1 \\ &= 4\varepsilon + |P_1 - P_1P_2 - P_2P_1 + P_1P_2P_1|_1 \\ &\leq 5\varepsilon + |P_1P_2P_1 - P_2P_1|_1 \\ &= 5\varepsilon + |-P_1(I - P_2)P_1 + (I - P_2P_1)P_1|_1 \leq 7\varepsilon. \end{aligned}$$

We will use the fact that $|\alpha_j(x)|_{\mathcal{J}} = |x|_{\mathcal{J}}$, which is a consequence of $\mu(\alpha_j(x)) = \mu(x)$, which in turn follows from $\rho \circ \alpha_j = \rho$.

On the other hand,

$$|\alpha_j(B) - B|_{\mathcal{J}} - |\alpha_j(A) - A|_{\mathcal{J}} \leq |\alpha_j(A - B) - (A - B)|_{\mathcal{J}} \leq 2|A - B|_{\mathcal{J}}.$$

In view of condition (3) satisfied by $|\cdot|_{\mathcal{J}}$, we have

$$|A - B|_{\mathcal{J}} \leq 2\phi(2^{-1}|A - B|_1) \leq 2\phi(4\varepsilon).$$

This in turn gives

$$k_{\mathcal{J},\mathcal{M}}(\alpha; P_1, Q_1) \le k_{\mathcal{J},\mathcal{M}}(\alpha; P_2, Q_2) + 4\phi(4\varepsilon),$$

which concludes the proof. ■

There is also a similar result for n-tuples τ of selfadjoint operators instead of the n-tuple α of automorphisms. Since the proof is along the same lines, we will leave out many details.

Proposition 3.3. *Let $\mathcal{N}$ be a von Neumann subalgebra of $\mathcal{M}$ so that $\rho|\mathcal{N}$ is semifinite and let E be the conditional expectation of $\mathcal{M}$ onto $\mathcal{N}$ with respect to ρ. Let further τ be a n-tuple of selfadjoint elements in $\mathcal{N}$ and let $P, Q \in \mathrm{Proj}(\mathcal{N})$ be so that $PQ = 0$. Then, we have*

$$k_{\mathcal{J},\mathcal{M}}(\tau; P, Q) = k_{\mathcal{J},\mathcal{N}}(\tau; P, Q).$$

Sketch of Proof. We first deal with $P, Q \in \mathcal{P}(\mathcal{N})$. Obviously, we have

$$k_{\mathcal{J},\mathcal{N}}(\tau; P, Q) \ge k_{\mathcal{J},\mathcal{M}}(\tau; P, Q).$$

On the other hand,

$$E(\Lambda(\mathcal{M})) = \Lambda(\mathcal{N}),$$

and if

$$A \in P + (I - P - Q)\Lambda(\mathcal{M})(I - P - Q),$$

then

$$|[T_j, E(A)]|_{\mathcal{J}} = |E([T_j, A])|_{\mathcal{J}} \le |[T_j, A]|_{\mathcal{J}},$$

which gives

$$k_{\mathcal{J},\mathcal{N}}(\tau; P, Q) \le k_{\mathcal{J},\mathcal{M}}(\tau; P, Q).$$

This concludes the proof in case $P, Q \in \mathcal{P}(\mathcal{N})$.

If we only have $P, Q \in \mathrm{Proj}(\mathcal{N})$, then the preceding immediately gives

$$k_{\mathcal{J},\mathcal{M}}(\tau; P, Q) \ge k_{\mathcal{J},\mathcal{N}}(\tau; P, Q).$$

We still must show that if $P_1, Q_1 \in \mathcal{P}(\mathcal{M})$, $P_1 \le P$, $Q_1 \le Q$, then there are $P_2, Q_2 \in \mathcal{P}(\mathcal{N})$, $P_2 \le P$, $Q_2 \le Q$ so that

$$k_{\mathcal{J},\mathcal{M}}(\tau; P_1, Q_1) \le k_{\mathcal{J},\mathcal{N}}(\tau; P_2, Q_2) + \eta$$

for a given $\eta > 0$.

Since $\rho|\mathcal{N}$ is semifinite, there are $P_2, Q_2 \in \mathcal{P}(\mathcal{N})$ so that $P_2 \le P$, $Q_2 \le Q$ and

$$|(I - P_2)P_1|_1 < \varepsilon, \quad |(I - Q_2)Q_1|_1 < \varepsilon.$$

If

$$A \in P_2 + (I - P_2 - Q_2)\Lambda(\mathcal{M})(I - P_2 - Q_2),$$

we consider

$$B = P_1 + (I - P_1 - Q_1)A(I - P_1 - Q_1) \in P_1 + (I - P_1 - Q_1)\Lambda(\mathcal{M})(I - P_1 - Q_1).$$

Then, we have

$$|B - A|_1 \leq 7\varepsilon$$

so that

$$|B - A|_{\mathcal{J}} \leq 2\phi(2^{-1}|B - A|_1) \leq 2\phi(4\varepsilon).$$

It follows that

$$|[\tau, B]|_{\mathcal{J}} - |[\tau, A]|_{\mathcal{J}} \leq 2\|\tau\|\phi(4\varepsilon).$$

This in turn shows that

$$k_{\mathcal{J},\mathcal{M}}(\tau; P_1, Q_1) \leq k_{\mathcal{J},\mathcal{N}}(\tau; P_2, Q_2) + 2\|\tau\|\phi(4\varepsilon).$$

An appropriate choice of $\varepsilon > 0$ concludes the proof. ■

Taking $P = I$ and $Q = 0$ in Propositions 3.2 and 3.3, we have the following.

Corollary 3.1. *Let $\mathcal{N}$ be a von Neumann subalgebra of $\mathcal{M}$ so that $\rho|\mathcal{N}$ is semifinite. If $\alpha = (\alpha_j)_{1\leq j\leq n}$ is an n-tuple of automorphisms which preserve ρ, $\alpha_j(\mathcal{N}) = \mathcal{N}$, $1 \leq j \leq n$ and $\tau = (T_j)_{1\leq j\leq n}$ is an n-tuple of selfadjoint elements of $\mathcal{N}$, then we have*

$$k_{\mathcal{J},\mathcal{M}}(\alpha) = k_{\mathcal{J},\mathcal{N}}(\alpha|\mathcal{N}), \quad k_{\mathcal{J},\mathcal{M}}(\tau) = k_{\mathcal{J},\mathcal{N}}(\tau).$$

Remark 3. If $\mathcal{M} = \mathcal{M}_1 \otimes \mathcal{M}_2$, where $(\mathcal{M}_1, \sigma)$ is a type II_1-factor and $\mathcal{M}_2$ is a factor of type I_∞ so that $\rho = \sigma \otimes \mathrm{Tr}$, and if $\tau = I \otimes \tau_2$, where τ_2 is a n-tuple of selfadjoint operators in $\mathcal{M}_2 = \mathcal{B}(\mathcal{H}_2)$ for some Hilbert space $\mathcal{H}_2$, the preceding corollary gives

$$k_{\mathcal{J}}(\tau) = k_{\mathcal{J},\mathcal{M}_2}(\tau_2).$$

This shows in particular that the examples of quasicentral modulus of n-tuples in $\mathcal{B}(\mathcal{H}_2)$ give automatically examples of quasicentral modulus in type II_∞ factors by taking $I \otimes \tau_2$ in $\mathcal{M}_1 \otimes \mathcal{B}(\mathcal{H}_2)$. One can proceed in a similar way for n-tuples of automorphisms (for the type I_∞ case, the next proposition shows that this reduces to the quasicentral modulus for n-tuples of unitary operators).

Let us also make a very simple observation about the case of n-tuples of unitary operators. If $u = (U_j)_{1\leq j\leq n}$ is an n-tuple of unitary elements of $\mathcal{M}$ we denote by $\mathrm{Ad}\, u = (\mathrm{Ad}\, U_j)_{1\leq j\leq n}$ the n-tuple of inner automorphisms, where $(\mathrm{Ad}\, U_j)(x) = U_j x U_j^*$. Consider also a map $\varepsilon : \{1, \ldots, n\} \to \{1, *\}$ and let then $u^\varepsilon = (U_j^{\varepsilon(j)})_{1\leq j\leq n}$.

We have

$$\begin{aligned}|[U_j, A]|_{\mathcal{J}} &= |[U_j, A]U_j^*|_{\mathcal{J}} = |(\operatorname{Ad} U_j)(A) - A|_{\mathcal{J}} \\ &= |U_j^*[U_j, A]|_{\mathcal{J}} = |(\operatorname{Ad} U_j^*)(A) - A|_{\mathcal{J}} = |[U_j^*, A]|_{\mathcal{J}}.\end{aligned}$$

This immediately implies the following proposition.

Proposition 3.4. *Let $u = (U_j)_{1\le j\le n}$ be an n-tuple of unitary elements of $\mathcal{M}$, let $\varepsilon : \{1,\ldots,n\} \to \{1, *\}$, and let $P, Q \in \operatorname{Proj}(\mathcal{M})$, $PQ = 0$. Then, we have*

$$k_{\mathcal{J}}(u; P, Q) = k_{\mathcal{J}}(u^{\varepsilon}; P, Q) = k_{\mathcal{J}}(\operatorname{Ad} u^{\varepsilon}; P, Q).$$

4. The lower bound

In this section, we assume $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and that $|\cdot|_{\mathcal{J}}$ is the symmetric norm arising from a norming function Φ, so that we will write $|\cdot|_{\Phi}, k_{\Phi}(\cdots)$ instead of $|\cdot|_{\mathcal{J}}$, $k_{\mathcal{J}}(\cdots)$. By Φ^* we will denote the dual norming function so that if $\mathfrak{S}_{\Phi}^{(0)}$ is the closure of $\mathcal{R}$ in the norm $|\cdot|_{\Phi}$, then $\mathfrak{S}_{\Phi^*}$, the set of compact operators K so that $|K|_{\Phi^*} = \sup\{|KP|_{\Phi^*} | P \in \mathcal{P}\} < \infty$, is its dual with respect to the trace pairing [3]. By $\mathcal{C}_1$ we will denote the trace class.

Proposition 4.1. *Assume $\tau = \tau^*$, that is $T_j = T_j^*$, $1 \le j \le n$ and let $P, Q \in \mathcal{P}$, $PQ = 0$. Let further*

$$\Omega = \Big\{X_j = X_j^*, 1 \le j \le n \mid i\sum_j [T_j, X_j] \in (\mathcal{B}(\mathcal{H}))_+ + \mathcal{C}_1, \sum_j |X_j|_{\Phi^*} = 1\Big\}.$$

Then, we have

$$\begin{aligned}k_{\Phi}(\tau; P, Q) \ge \sup\Big\{&\operatorname{Tr} \mathrm{PYP} - \operatorname{Tr}((I-P-Q)Y(I-P-Q))_- | Y \\ &= i\sum_j [T_j, X_j],\ (X_j)_{1\le j\le n} \in \Omega\Big\}\end{aligned}$$

and equality holds if $k_{\Phi}(\tau; P, Q) > 0$.

Proof. We will first prove $\ge$, and then assuming $k_{\Phi}(\tau; P, Q) > 0$, we will prove $\le$, which will yield the equality stated above. We start with $0 \le B \le I - P - Q$, $B \in \mathcal{R}_1^+$ and $(X_j)_{1\le j\le n} \in \Omega$, and we will show that

$$\max_{1\le j\le n} |[P + B, T_j]|_{\Phi} \ge \operatorname{Tr} \mathrm{PYP} - \operatorname{Tr}((I-P-Q)Y(I-P-Q))_-,$$

where

$$Y = i\sum_j [T_j, X_j].$$

We have

$$\begin{aligned}\operatorname{Tr}((P+B)Y) &= \operatorname{Tr} \mathrm{PYP} + \operatorname{Tr}(B(I-P-Q)Y(I-P-Q)) \\ &= \operatorname{Tr} \mathrm{PYP} + \operatorname{Tr}(B((I-P-Q)Y(I-P-Q))_+) \\ &\quad - \operatorname{Tr}(B((I-P-Q)Y(I-P-Q))_-) \\ &\geq \operatorname{Tr} \mathrm{PYP} - \operatorname{Tr}(B((I-P-Q)Y(I-P-Q))_-) \\ &= \operatorname{Tr} \mathrm{PYP} - \operatorname{Tr}(Z^{1/2} B Z^{1/2}),\end{aligned}$$

where

$$Z = ((I-P-Q)Y(I-P-Q))_-.$$

Since $0 \leq B \leq I$ this gives

$$\operatorname{Tr}((P+B)Y) \geq \operatorname{Tr} \mathrm{PYP} - \operatorname{Tr} Z.$$

On the other hand,

$$\begin{aligned}|\operatorname{Tr}(P+B)Y| &= \Big|\operatorname{Tr}\Big(i(P+B)\sum_j [T_j, X_j]\Big)\Big| \\ &= \Big|\operatorname{Tr}\Big(\sum_j [P+B, T_j] X_j\Big)\Big| \\ &\leq \Big(\max_{1\leq j\leq n} |[P+B, T_j]|_\Phi\Big) \sum_j |X_j|_{\Phi^*} \\ &= \max_{1\leq j\leq n} |[P+B, T_j]|_\Phi.\end{aligned}$$

Hence,

$$\operatorname{Tr} \mathrm{PYP} - \operatorname{Tr}((I-P-Q)Y(I-P-Q))_- \leq \max_{1\leq j\leq n} |[P+B, T_j]|_\Phi$$

for all $B \in \mathcal{R}_1^+, 0 \leq B \leq I - P - Q$.

This gives

$$\operatorname{Tr} \mathrm{PYP} - \operatorname{Tr}((I-P-Q)Y(I-P-Q))_- \leq k_\Phi(\tau; P, Q)$$

since $k_\Phi(\tau; P, Q)$ is the inf of the $\max_{1\leq j\leq n} |[T_j, P+B]|_\Phi$ when $B \in \mathcal{R}_1^+$ satisfies $0 \leq B \leq I - P - Q$. This concludes the proof of $\geq$.

Assume now that $k_\Phi(\tau; P, Q) > 0$. To prove $\leq$, we will consider the real Banach space $(\mathfrak{S}_{\Phi,h}^{(0)})^n$ of n-tuples $(X_j)_{1\leq j\leq n}$, $X_j = X_j^* \in \mathfrak{S}_\Phi^{(0)}$ with the norm $\max_{1\leq j\leq n} |X_j|_\Phi$ and two disjoint convex subsets of this Banach space. The first is the open ball centered at 0 of radius $k_\Phi(\tau; P, Q)$. The second is

$$\{(i[T_j, A])_{1\leq j\leq n} | A \in P + (I-P-Q)\mathcal{R}_1^+(I-P-Q)\}.$$

The two convex sets are disjoint and the first is open so that there is

$$(X_j)_{1\le j\le n} \in ((\mathfrak{S}^{(0)}_{\Phi,h})^n)^* = (\mathfrak{S}_{\Phi^*,h})^n$$

separating the two and having norm 1. Thus, we have

$$\sum_j |X_j|_{\Phi^*} = 1$$

and

$$\sum_j \operatorname{Tr}(i[T_j, A]X_j) \ge k_\Phi(\tau; P, Q)$$

for all $A \in P + (I - P - Q)\mathcal{R}_1^+(I - P - Q)$. This gives

$$\sum_j \operatorname{Tr}(i[T_j, P + B]X_j) \ge k_\Phi(\tau; P, Q)$$

for all $B \in \mathcal{R}_1^+$, $0 \le B \le I - P - Q$. The LHS equals

$$\operatorname{Tr}\Big((P + B)i \sum_j [X_j, T_j]\Big) = \operatorname{Tr} \mathrm{PYP} + \operatorname{Tr}(B(I - P - Q)Y(I - P - Q)),$$

where

$$Y = i \sum_j [X_j, T_j].$$

Remark that $(I-P-Q)Y(I-P-Q)-Y$ is a finite-rank operator of rank $\le 2\operatorname{Tr}(P+Q)$, and its norm is $\le 4\sum_j \|T_j\|\|X_j\| \le 4\max_{1\le j\le n}\|T_j\|$. This implies that

$$\inf\{\operatorname{Tr} CY | C \in \mathcal{R}_1^+\} \ge \text{const} + \inf\{\operatorname{Tr} BY | B \in (I - P - Q)\mathcal{R}_1^+(I - P - Q)\} > -\infty$$

so that

$$Y \in (\mathcal{B}(\mathcal{H}))_+ + \mathcal{C}_1.$$

Thus, we have also proved that

$$\inf\{\operatorname{Tr} \mathrm{PYP} + \operatorname{Tr}(B(I - P - Q)Y(I - P - Q)) | B \in \mathcal{R}_1^+\} \ge k_\Phi(\tau; P, Q).$$

The above inf is precisely

$$\operatorname{Tr} \mathrm{PYP} - \operatorname{Tr}(((I - P - Q)Y(I - P - Q))_-),$$

which is the result we wanted to prove. ■

There is an analogue of Proposition 4.1 for unitary operators or equivalently for the corresponding inner automorphisms. The proof being along the same lines is left as an exercise for the reader.

Proposition 4.2. *Let u be an n-tuple of unitary operators and let $P, Q \in \mathcal{P}$, $PQ = 0$. Let further*

$$\Omega = \Big\{X_j = X_j^* \in \mathfrak{S}_{\Phi^*}, 1 \le j \le n \mid \sum_j ((\operatorname{Ad} U_j)(X_j) - X_j) \in \mathcal{B}(\mathcal{H})_+ + \mathcal{C}_1, \sum_j |X_j|_{\Phi^*} = 1\Big\}.$$

Then, we have

$$\begin{aligned} &k_\Phi(u : P, Q) \\ &= k_\Phi(\operatorname{Ad} u; P, Q) \\ &\ge \sup\Big\{\operatorname{Tr} \mathrm{PYP} - \operatorname{Tr}((I - P - Q)Y(I - P - Q))_- \mid Y = \sum_j ((\operatorname{Ad} U_j)(X_j) - X_j), \\ &\qquad (X_j)_{1 \le j \le n} \in \Omega\Big\}, \end{aligned}$$

and if $k_\Phi(u; P, Q) > 0$, equality holds.

5. An example

Let U be the bilateral shift operator on $\mathcal{H} = \ell^2(\mathbb{Z})$, $Ue_j = e_{j+1}$, where $\{e_j\}_{j\in\mathbb{Z}}$ is the canonical orthonormal basis. If $f : \mathbb{Z} \to \mathbb{C}$ is a bounded function, we will denote by $D(f)$ the diagonal operator in $\mathcal{H}$ with respect to the canonical basis. In case $f : \mathbb{R} \to \mathbb{C}$ we will write $D(f)$ for $D(f|\mathbb{Z})$. Moreover, if $\omega \subset \mathbb{Z}$, P_ω will denote the projection $D(\chi_\omega)$. In case of a singleton $\{j\}$, we will write P_j instead of $P_{\{j\}}$.

Here, we will compute

$$k_p(U; P_M, P_N),$$

where $M, N \subset \mathbb{Z}$ are two disjoint finite nonempty subsets, By Propositions 3.2 and 3.4 this would be equivalent to a problem in $\ell^\infty(\mathbb{Z})$, that is a problem on the Cayley graph of $\mathbb{Z}$. We will not use this explicitly, though all our computations will be around two operators $B = D(f)$ and $X = D(g)$.

Let $a = \inf(M \cup N)$ and $b = \sup(M \cup N)$ and let $a \le a_1 < b_1 \le a_2 < b_2 \le \cdots \le a_m < b_m \le b$ be so that the (a_j, b_j) are the maximal open intervals so that $(a_j, b_j) \cap (M \cup N) = \varnothing$ and the endpoints a_j and b_j are in different sets of the partition of $M \cup N$ into M and N. Let also $h \in \mathbb{N}$. We define a continuous function $f : \mathbb{R} \to \mathbb{R}$ as follows. First, we require that

$$f|M \equiv 1, \quad f|N \equiv 0, \quad f|(-\infty, a - h] \equiv 0, \quad f|[b + h, \infty) \equiv 0.$$

Then, on each open interval at the endpoints of which f has been defined, we extend the definition by linearity on the interval. Thus, f will be piecewise linear with respect to the partition $-\infty < a - h < a \le a_1 < b_1 \le a_2 < b_2 \le \cdots \le a_m < b_m \le b < b + h < \infty$. The intervals on which this function is not constant are the $[a_j, b_j]$ and possibly also $[a - h, a]$ and $[b, b + h]$ depending on whether $a, b \in M$ or not. Thus, if $B = D(f)$ the list of non-zero singular values of $B - \mathrm{Ad}(U)(B)$ consists of $b_j - a_j$ times the number $(b_j - a_j)^{-1}$ for each interval $[a_j, b_j]$ and each of the intervals $[a - h, a]$, $[b, b + h]$ may contribute h times the number h^{-1} depending on whether $a, b \in M$. This gives

$$|B - \mathrm{Ad}(U)(B)|_p^p = \sum_j (b_j - a_j)^{1-p} + h^{1-p} \cdot \sharp(\{a, b\} \cap M).$$

Since

$$|B - \mathrm{Ad}(U)(B)|_p \ge k_p(\mathrm{Ad}(U); P_M, P_N),$$

we get the following upper bound. If $p = 1$, we have

$$m + \sharp(\{a, b\} \cap M) \ge k_1(\mathrm{Ad}(U); P_M, P_N),$$

while if $1 < p < \infty$, we have

$$\Big(\sum_j (b_j - a_j)^{1-p}\Big)^{1/p} \ge k_p(\mathrm{Ad}(U); P_M, P_N)$$

because h being arbitrary we can take the limit as $h \to \infty$.

To get the lower bound using Proposition 4.2, we construct an operator $X = D(g)$. Let $\varepsilon(j) = -1$ if $a_j \in M$ and $b_j \in N$ and let $\varepsilon(j) = +1$ if $a_j \in N$ and $b_j \in M$ and observe that $\varepsilon(j) = -\varepsilon(j + 1)$. If $p > 1$, we define

$$g = c \cdot \sum_j \varepsilon(j)(b_j - a_j)^{-p/q} \cdot \chi_{[a_j, b_j)},$$

where

$$c = \Big(\sum_j (b_j - a_j)^{1-p}\Big)^{-1/q}.$$

We have

$$|X|_q^q = c^q \cdot \sum_j (b_j - a_j) \cdot (b_j - a_j)^{-p} = 1.$$

If $Y = \mathrm{Ad}(U)(X) - X$, we have

$$Y = c \cdot \sum_j \varepsilon(j)(b_j - a_j)^{-p/q}(P_{b_j} - P_{a_j}).$$

From Proposition 4.2, we get the lower bound

$$\operatorname{Tr} P_M Y P_M - \operatorname{Tr}((I - P_M - P_N)Y(I - P_M - P_N))_-.$$

Since $b_j, a_j \in (M \cup N)$, the second term in the lower bound is zero, so we need only compute $\operatorname{Tr} P_M Y P_M$. If $\varepsilon(j) = -1$ we have $a_j \in M$ and $b_j \in N$, while if $\varepsilon(j) = +1$, we have $a_j \in N$ and $b_j \in M$. This gives that $\operatorname{Tr} P_M(\varepsilon(j)(P_{b_j} - P_{a_j})) = 1$ for all indices j. It follows that

$$\operatorname{Tr} P_M Y P_M = c \cdot \sum_j (b_j - a_j)^{-p/q}$$

and since $p/q = 1/p$, we have

$$\operatorname{Tr} P_M Y P_M = \Big(\sum_j (b_j - a_j)^{1-p}\Big)^{1/p},$$

where we used $1 - 1/q = 1/p$. Thus, the lower and the upper bound are equal if $p > 1$.

To obtain the lower bound when $p = 1$, we will consider

$$g = -\varepsilon(1)\chi_{(-\infty,a)} - \varepsilon(m)\chi_{[b,\infty)} + \sum_j \varepsilon(j)\chi_{[a_j,b_j)}$$

and $X = D(g)$. Then,

$$Y = \operatorname{Ad}(U)(X) - X = -\varepsilon(1)P_a + \varepsilon(m)P_b + \sum_j \varepsilon(j)(P_{b_j} - P_{a_j}).$$

Again, the lower bound reduces to computing

$$\operatorname{Tr} P_M Y P_M = \operatorname{Tr} P_M(-\varepsilon(1)P_a + \varepsilon(m)P_b)P_M + m.$$

It is also easy to see that

$$\operatorname{Tr} P_M(-\varepsilon(1)P_a + \varepsilon(m)P_b)P_M = \sharp(\{a,b\} \cap M)$$

so that also in this case the lower and upper bounds we found for $k_\Phi(\operatorname{Ad}(U); P_M, P_N)$ are equal.

Summing up and using Proposition 3.4, we have proved the following result.

Proposition 5.1. *With the notation introduced above, we have*

$$k_1(U; P_M, P_N) = m + \sharp(\{a,b\} \cap M)$$

and if $1 < p < \infty$, *we have*

$$k_p(U; P_M, P_N) = \Big(\sum_j (b_j - a_j)^{1-p}\Big)^{1/p}.$$

6. The singular projection and the regular projection

We adapt and generalize to our semifinite setting the facts in [9] about the largest projection on which the quasicentral modulus vanishes.

The following lemma is based on an argument we used in the proof of Proposition 3.2.

Lemma 6.1. *Let ϕ be the function in property* 3 *of the $\mathcal{J}$-norm. If $P_1, P_2, Q_1, Q_2 \in \mathcal{P}$, $P_1 Q_1 = P_2 Q_2 = 0$, $\|T_j\| \le C$, $1 \le j \le n$, and $\varepsilon > 0$ is so that*

$$|P_1 - P_2|_1 < \varepsilon, \quad |Q_1 - Q_2|_1 < \varepsilon,$$

then we have

$$|k_{\mathcal{J}}(\tau; P_1, Q_1) - k_{\mathcal{J}}(\tau; P_2, Q_2)| \le 4C\phi(6\varepsilon),$$
$$|k_{\mathcal{J}}(\alpha; P_1, Q_1) - k_{\mathcal{J}}(\alpha; P_2, Q_2)| \le 4\phi(6\varepsilon).$$

Proof. Let $X \in \mathcal{R}_1^+$, and let

$$A = (4C)^{-1}[T_j, P_1 + (I - P_1 - Q_1)X(I - P_1 - Q_1)],$$
$$B = (4C)^{-1}[T_j; P_2 + (I - P_2 - Q_2)X(I - P_2 - Q_2)].$$

Then, we have $\|A - B\| \le 1$ and $|A - B|_1 \le 6\varepsilon$ so that

$$|A - B|_{\mathcal{J}} \le \phi(6\varepsilon).$$

Taking into account the way we defined $k_{\mathcal{J}}(\tau; P, Q)$ using a max over j and then an inf over X, this gives

$$|k_{\mathcal{J}}(\tau; P_1, Q_1) - k_{\mathcal{J}}(\tau; P_2, Q_2)| \le 4C\phi(6\varepsilon).$$

For automorphisms, we use the same argument with A, B defined now to be

$$A = 4^{-1}(\alpha_j(P_1 + (I - P_1 - Q_1)X(I - P_1 - Q_1)) - (P_1 + (I - P_1 - Q_1)X(I - P_1 - Q_1)))$$

and

$$B = 4^{-1}(\alpha_j(P_2 + (I - P_2 - Q_2)X(I - P_2 - Q_2)) - (P_2 + (I - P_2 - Q_2)X(I - P_2 - Q_2))).$$

■

Proposition 6.1. *Assume that*

$$B_m \in \mathcal{R}, \quad B_m \ge 0, \quad w - \lim_{m \to \infty} B_m = B$$

and that

$$\lim_{m\to\infty} \max_{1\le j\le n} |[T_j, B_m]|_{\mathcal{J}} = 0.$$

Then, if F is the support projection of B (i.e., $E(B; (0,\infty))$), we have $[T_j, F] = 0$, $1 \le j \le n$, and

$$k_{\mathcal{J}}(\tau; F) = 0.$$

Proof. Replacing B_m by FB_mF, $\mathcal{M}$ by $F\mathcal{M}F|F\mathcal{H}$, τ by $\tau|F\mathcal{H}$, etc., it is easily seen that the proof reduces to the case when $F = I$, that is, Ker $B = 0$. Recall also that $k_{\mathcal{J}}(\tau; I) = k_{\mathcal{J}}(\tau)$. So, we need to prove that $k_{\mathcal{J}}(\tau; P) = 0$ if $P \in \mathcal{P}$.

Remark also that we may assume that

$$s - \lim_{m\to\infty} B_m = B.$$

Indeed, we may pass from the initial B_m's to a subsequence so that

$$s - \lim_{m\to\infty} m^{-1}(B_{i_1} + \cdots + B_{i_m}) = B$$

and replace B_m by $m^{-1}(B_{i_1} + \cdots + B_{i_m})$.

Now, we show how to complete the proof when B is invertible, that is $E(B; [0,\varepsilon)) = 0$ for some $\varepsilon > 0$ and then go back to the general situation Ker $B = 0$. Let $h : \mathbb{R} \to [0,1]$ be a C^∞-function which is 0 on $(-\infty, 0]$ and 1 on $[\varepsilon, \infty)$. Then, we have

$$s - \lim_{m\to\infty} h(B_m) = h(B) = 1$$

and

$$\lim_{m\to\infty} \max_{1\le j\le n} |[h(B_m), T_j]|_{\mathcal{J}} = 0.$$

Hence, replacing B_m by $h(B_m)$, we may assume that $B_m \in \mathcal{R}_1^+$ and B_m converges strongly to I. Let $P \in \mathcal{P}$. Remark then that

$$\lim_{m\to\infty} |B_m - (P + (I-P)B_m(I-P))|_1 = 0.$$

This follows from $|P - PB_mP|_1 \to 0$ and $|(I-P)B_mP|_1 \to 0$ which in turn follow from the strong convergences $P - PB_mP \to 0$ and $(PB_m(I-P)B_mP)^{1/2} \to 0$ in the finite von Neumann algebra $P\mathcal{M}P|P|\mathcal{H}$ endowed with the finite trace which is the restriction of ρ.

Since also clearly $\|B_m - (P + (I-P)B_m(I-P))\| \le 2$, we infer that also

$$|B_m - (P + (I-P)B_m(I-P))|_{\mathcal{J}} \to 0$$

as $m \to \infty$. This then gives

$$|[T_j, P + (I-P)B_m(I-P)]|_{\mathcal{J}} \to 0,$$

which then finally implies

$$k_{\mathcal{J}}(\tau; P) = 0.$$

Returning to the general case, where only Ker $B = 0$ is assumed, the result we have obtained thus far is easily seen to give that

$$k_{\mathcal{J}}(\tau; E(B, (\varepsilon, \|B\|))) = 0$$

if $\varepsilon > 0$. The proof is then completed by observing that given $P \in \mathcal{P}$ we can find $P_k \in \mathcal{P}$, $P_k \leq E(B; (1/k, \|B\|))$, so that

$$\lim_{k \to \infty} |P_k - P|_1 = 0$$

and use Lemma 6.1.

We may take P_k to be the left support projection of $E(B; (1/k, \|B\|))P$, that is the projection onto the closure of the range of this operator. ■

There is an entirely analogous result for automorphisms which we record as the next proposition, the proof of which is omitted, being only a slight variation on the preceding proof.

Proposition 6.2. *Assume that*

$$B_m \in \mathcal{R}, \quad B_m \geq 0, \quad w - \lim_{m \to \infty} B_m = B$$

and that

$$\lim_{m \to \infty} \max_{1 \leq j \leq n} |[\alpha_j(B_m) - B_m]|_{\mathcal{J}} = 0.$$

Then, if F is the support projection of B (that is $E(B; (0, \infty))$) *we have $\alpha_j(F) = F$, $1 \leq j \leq n$ and*

$$k_{\mathcal{J}}(\alpha; F) = 0.$$

Corollary 6.2. *Let $P_1, P_2 \in \mathcal{P}$. Then, we have*

$$k_{\mathcal{J}}(\tau; P_1) = k_{\mathcal{J}}(\tau; P_2) = 0 \Rightarrow k_{\mathcal{J}}(\tau; P_1 \vee P_2) = 0$$

and

$$k_{\mathcal{J}}(\alpha; P_1) = k_{\mathcal{J}}(\alpha : P_2) = 0 \Rightarrow k_{\mathcal{J}}(\alpha; P_1 \vee P_2) = 0.$$

Proof. We will prove only the first assertion, the proof of the second being completely analogous. If $k_{\mathcal{J}}(\tau; P_1) = k_{\mathcal{J}}(\tau; P_2) = 0$, there exist $A_m, C_m \in \mathcal{R}_1^+$ and $A, C \in \mathcal{M}$ so that

$$\begin{aligned} A_m P_1 &= P_1, & C_m P_2 &= P_2 \\ w - \lim_{m \to \infty} A_m &= A, & w - \lim_{m \to \infty} C_m &= C, \\ \lim_{m \to \infty} |[A_m, T_j]|_{\mathcal{J}} &= 0, & \lim_{m \to \infty} |[C_m, T_j]|_{\mathcal{J}} &= 0, \quad 1 \leq j \leq n. \end{aligned}$$

Since $AP_1 = P_1$, we have that $P_1\mathcal{H}$ and $\operatorname{Ker} A$ are orthogonal. Similarly, $P_2\mathcal{H}$ and $\operatorname{Ker} C$ are orthogonal. On the other hand, $\operatorname{Ker}(A + C) = \operatorname{Ker} A \cap \operatorname{Ker} C$ because A and C are ≥ 0. Thus, $\operatorname{Ker}(A + C)$ is orthogonal to $(P_1 \vee P_2)\mathcal{H}$. Applying Proposition 6.1 to the sequence $B_m = A_m + C_m$, we get the desired result. ■

Proposition 6.3. *Given τ, there exists a projection $E^0_{\mathcal{J}}(\tau) \in \operatorname{Proj}(\mathcal{M})$ so that*

$$P \in \operatorname{Proj}(\mathcal{M}), \quad k_{\mathcal{J}}(\tau; P) = 0 \Leftrightarrow P \leq E^0_{\mathcal{J}}(\tau).$$

The projection $E^0_{\mathcal{J}}(\tau)$ is unique, in particular, if β is an automorphism of $\mathcal{M}$ which preserves ρ and $\beta(\tau) = (\tau)$, then we have $\beta(E^0_{\mathcal{J}}(\tau)) = E^0_{\mathcal{J}}(\tau)$. Similarly, given α, there exists a projection $E^0_{\mathcal{J}}(\alpha) \in \operatorname{Proj}(\mathcal{M})$ so that

$$P \in \operatorname{Proj}(\mathcal{M}), \quad k_{\mathcal{J}}(\alpha; P) = 0 \Leftrightarrow P \leq E^0_{\mathcal{J}}(\alpha).$$

Moreover, the projection $E^0_{\mathcal{J}}(\alpha)$ is unique, in particular, if β is an automorphism of $\mathcal{M}$ which preserves ρ and $\beta \circ \alpha_j = \alpha_j \circ \beta$, $1 \leq j \leq n$, then we have $\beta(E^0_{\mathcal{J}}(\alpha)) = E^0_{\mathcal{J}}(\alpha)$.

Proof. We will only prove the first half of the statement, the arguments being very similar for the two cases. Moreover, in view of the definition of $k_{\mathcal{J}}(\tau; P)$, when $P \in \operatorname{Proj}(\mathcal{M})$, it is easily seen that what we must prove is that

$$E = \bigvee\{P \in \mathcal{P} \mid k_{\mathcal{J}}(\tau; P) = 0\} \Rightarrow k_{\mathcal{J}}(\tau; E) = 0.$$

The Hilbert space $\mathcal{H}$ being separable, there is a sequence P_i, $i \in \mathbb{N}$ so that $k_{\mathcal{J}}(\tau; P_i) = 0$, $i \in \mathbb{N}$ and $E = \bigvee\{P_i \mid i \in \mathbb{N}\}$. Let $E_i = P_1 \vee \cdots \vee P_i$. By Corollary 6.2, we have $k_{\mathcal{J}}(\tau; E_i) = 0$. Since $E_i \in \mathcal{P}$, there is $B_i \in \mathcal{R}^+_1$ so that $B_i E_i = E_i$ and $|[T_j, B_i]|_{\mathcal{J}} < 1/i$, $1 \leq j \leq n$. We can replace the increasing sequence E_i by a subsequence and assume that the B_i's are weakly convergent to some B. Then, $BE = E$, and we can apply Proposition 6.1 to infer that $k_{\mathcal{J}}(\tau; E(B; (0, \infty))) = 0$ which implies $k_{\mathcal{J}}(\tau; E) = 0$ since $E \leq E(B; (0, \infty))$. ■

We will call $E^0_{\mathcal{J}}(\tau)$, $E^0_{\mathcal{J}}(\alpha)$ the $\mathcal{J}$-singular projection of τ and, respectively, α. We will also use the notation $E_{\mathcal{J}}(\tau) = I - E^0_{\mathcal{J}}(\tau)$, $E_{\mathcal{J}}(\alpha) = I - E^0_{\mathcal{J}}(\alpha)$ and call $E_{\mathcal{J}}(\tau)$, $E_{\mathcal{J}}(\alpha)$ the $\mathcal{J}$-regular projection of τ and α, respectively.

In [9], in the case of $\mathcal{B}(\mathcal{H})$ and of a normed ideal given by a norming function Φ we had called a projection P which is τ-invariant Φ well behaved if $k_\Phi(\tau|P\mathcal{H}) = 0$. This is equivalent to $k_\Phi(\tau; P) = 0$, and we think that Φ-singular, the terminology we introduce here, is perhaps a better term for this.

Corollary 6.3. *We have*

$$\left[T_j, E^0_{\mathcal{J}}(\tau)\right] = 0, \quad 1 \leq j \leq n.$$

Similarly, we have

$$\alpha_j(E^0_{\mathcal{J}}(\alpha)) = E^0_{\mathcal{J}}(\alpha), \quad 1 \le j \le n.$$

Proof. Also, here, we will give only the proof of the first assertion, the proof of the second being along the same lines.

It is clear that it suffices to prove that if $P \in \mathcal{P}$ is so that $k_{\mathcal{J}}(\tau; P) = 0$, then there is $P' \in \mathrm{Proj}(\mathcal{M})$ so that $P' \ge P$, $k_{\mathcal{J}}(\tau; P') = 0$, $[P', T_j] = 0$, $1 \le j \le n$. Indeed, since $k_{\mathcal{J}}(\tau; P) = 0$, there are $B_m \in \mathcal{R}^+_1$ so that $B_m P = P$ and

$$\lim_{m\to\infty} \max_{1\le j\le n} |[T_j, B_m]|_{\mathcal{J}} = 0.$$

Passing to a subsequence, we can assume that

$$w - \lim_{m\to\infty} B_m = B,$$

and we will then have $BP = P$ and $[B, T_j] = 0$, $1 \le j \le n$. It follows from Proposition 6.1 that $P' = E(B; (0, \infty))$ has the desired properties. ■

Proposition 6.4. *If $A_m = A_m^* \in \mathcal{R}$ are so that $\|A_m\| \le C$ for all $m \in \mathbb{N}$ and*

$$\lim_{m\to\infty} \max_{1\le j\le n} |[A_m, T_j]|_{\mathcal{J}} = 0,$$

then we have

$$s - \lim_{m\to\infty} A_m E_{\mathcal{J}}(\tau) = 0.$$

Similarly, if $A_m = A_m^* \in \mathcal{R}$ are so that $\|A_m\| \le C$ for all $m \in \mathbb{N}$ and

$$\lim_{m\to\infty} \max_{1\le j\le n} |\alpha_j(A_m) - A_m|_{\mathcal{J}} = 0,$$

then we have

$$s - \lim_{m\to\infty} A_m E_{\mathcal{J}}(\alpha) = 0.$$

Proof. We will prove only the first assertion, the proof of the second being along the same lines. Let $B_m = E_{\mathcal{J}}(\tau) A_m^2 E_{\mathcal{J}}(\tau)$ so that we will have to prove that

$$w - \lim_{m\to\infty} B_m = 0.$$

Assuming the contrary and passing to a subsequence of this bounded sequence, we will have

$$w - \lim_{m\to\infty} B_m = B \ne 0.$$

Then, B_m and B satisfy the assumptions of Proposition 6.1. It follows that the projection $E(B; (0, \infty)) \ne 0$ is so that $k_{\mathcal{J}}(\tau; E(B; (0, \infty))) = 0$, and hence, by Proposition 6.3, $E(B; (0, \infty)) \le E^0_{\mathcal{J}}(\tau)$, while obviously $E(B; (0, \infty)) \le E_{\mathcal{J}}(\tau)$. This contradiction concludes the proof. ■

7. Variants

We briefly discuss here modifications of the definition of the condenser quasicentral modulus quantities along lines, which for the quasicentral modulus we already pointed out in [8]. We use instead of the max comparable devices in the definitions. This is in preparation for the next section, where the variants may have some advantages.

Thus, Definition 2.1 is modified as follows:

$$\tilde{k}_{\mathcal{J}}(\tau; P, Q) = \inf\left\{\left|\left(\sum_{j=1}^{n}[T_j, A]^*[T_j, A]\right)^{\frac{1}{2}}\right|_{\mathcal{J}} \,\middle|\, A \in \mathcal{R}_1^+, AP = P, AQ = 0\right\}$$

$$\tilde{k}_{\mathcal{J}}(\alpha; P, Q) = \inf\left\{\left|\left(\sum_{j=1}^{n}(\alpha_j(A) - A)^2\right)^{\frac{1}{2}}\right|_{\mathcal{J}} \,\middle|\, A \in \mathcal{R}_1^+, AP = P,\ AQ = 0\right\}.$$

This is then extended also to Definition 2.2, and the further $k_{\mathcal{J}}$ quantities are replaced by $\tilde{k}_{\mathcal{J}}$ quantities. Remark that in essence this amounts to replacing

$$\max_{1 \le j \le n} |X_j|_{\mathcal{J}}$$

by

$$\left|\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}\right|_{\mathcal{J}},$$

since

$$\left(\sum_{1 \le j \le n} X_j^* X_j\right)^{\frac{1}{2}}$$

is the positive operator in the polar decomposition of the column operator matrix. We view here the $1 \times n$ matrices with entries in $\mathcal{M}$ as a subspace in $\mathcal{M} \otimes \mathfrak{M}_n$ and use item (4) from the properties of $|\cdot|_{\mathcal{J}}$ in the preliminaries.

Though we will mostly use $\tilde{k}_{\mathcal{J}}$ in this paper, it is also quite natural to consider a Dirac operator construction. This produces Dirac versions $k_{\mathcal{J}}^D(\tau; P, Q)$, etc., where

$$k_{\mathcal{J}}^D(\tau; P, Q) = \inf\left\{\left|\left(\sum_{1 \le j \le n}[T_j, A] \otimes e_j\right)\right|_{\mathcal{J}} \,\middle|\, A \in \mathcal{R}_1^+, AP = P, AQ = 0\right\}$$

with $e_1, \dots, e_n$ denoting Clifford matrices.

We have

$$k_{\mathcal{J}}(\tau; P, Q) \le \tilde{k}_{\mathcal{J}}(\tau; P, Q) \le n k_{\mathcal{J}}(\tau; P, Q).$$

This implies that a $k_{\mathcal{J}}$-condenser quantity is zero or infinity iff the corresponding $\tilde{k}_{\mathcal{J}}$-condenser quantity is zero or, respectively, infinity.

It is also easy to see that Lemma 2.1, Propositions 3.1, 3.2, and 3.3 still hold if $k_{\mathcal{J}}$ is replaced by $\tilde{k}_{\mathcal{J}}$. Note however that it may not be the case that Proposition 3.4 remains valid when we pass to $\tilde{k}_{\mathcal{J}}$.

8. Noncommutative variational remarks

The quasicentral modulus, in its different versions, is based on quantities for which minimization problems can be formulated:

$$\begin{aligned}
I_\tau(X) &= \max_{1\le j\le n} |[T_j, X]|_{\mathcal{J}},\\
I_\alpha(X) &= \max_{1\le j\le n} |\alpha_j(X) - X|_{\mathcal{J}},\\
\widetilde{I}_\tau(X) &= \Big|\Big(\sum_{1\le j\le n} [T_j, X]^*[T_j, X]\Big)^{\frac{1}{2}}\Big|_{\mathcal{J}},\\
\widetilde{I}_\alpha(X) &= \Big|\Big(\sum_{1\le j\le n} (\alpha_j(X) - X)^*(\alpha_j(X) - X)\Big)^{\frac{1}{2}}\Big|_{\mathcal{J}},\\
I_\tau^D(X) &= \Big|\sum_{1\le j\le n} [T_j, X]\otimes e_j\Big|_{\mathcal{J}},\\
I_\alpha^D(X) &= \Big|\sum_{1\le j\le n} (\alpha_j(X) - X)\otimes e_j\Big|_{\mathcal{J}},
\end{aligned}$$

where $\tau = \tau^*$ throughout this section. If $I(X) \in [0, \infty]$ denotes any of the above, remark that it is a differential seminorm with additional properties:

$$\begin{aligned}
I(X+Y) &\le I(X) + I(Y),\\
I(\lambda X) &= |\lambda| I(X),\\
I(XY) &\le I(X)\|Y\| + \|X\| I(Y),\\
I(X) &\le C|X|_{\mathcal{J}},\\
w - \lim_{m\to\infty} X_m = X &\Rightarrow \liminf_{m\to\infty} I(X_m) \ge I(X),
\end{aligned}$$

and with the exception of $\widetilde{I}_\tau$ and $\widetilde{I}_\alpha$, we also have $I(X^*) = I(X)$.

If $X = X^*$, then $\widetilde{I}_\tau$ and $\widetilde{I}_\alpha$ can also be written as follows:

$$\begin{aligned}
\widetilde{I}_\tau(X) &= \Big|\Big(-\sum_{1\le j\le n} [T_j, X]^2\Big)^{\frac{1}{2}}\Big|_{\mathcal{J}},\\
\widetilde{I}_\alpha(X) &= \Big|\Big(\sum_{1\le j\le n} (\alpha_j(X) - X)^2\Big)^{\frac{1}{2}}\Big|_{\mathcal{J}}.
\end{aligned}$$

Actually, $X = X^*$ is a quite natural condition when we set up variational problems.

Euler equations in case $\mathcal{J}$ is the p-class, $2 \leq p < \infty$, can be found for the power-scaled I^p when I does not include a max. These equations can be viewed as analogues of the p-Laplace equation. More precisely, let $X = X^*$ be such that

$$\frac{d}{d\varepsilon} I^p(X + \varepsilon B)|_{\varepsilon=0} = 0.$$

for all $B = B^* \in \mathcal{R}$, where $I(X) < \infty$.

In case $I = \widetilde{I}_\tau$, we have

$$\begin{aligned}
&\frac{d}{d\varepsilon}\rho\Big(\Big(-\sum_{1\le j\le n}[X+\varepsilon B, T_j]^2\Big)^{\frac{p}{2}}\Big)\Big|_{\varepsilon=0}\\
&= \frac{p}{2}\sum_{1\le k\le n}\rho\Big(-[B,T_k][X,T_k]\Big(-\sum_{1\le j\le n}[X,T_j]^2\Big)\Big)^{\frac{p}{2}-1}\\
&\quad -\Big[[B,T_k]\Big(-\sum_{1\le j\le n}[X,T_j]^2\Big)^{\frac{p}{2}-1}[X,T_k]\Big]\\
&= \frac{p}{2}\sum_{1\le k\le n}\rho\Big(B\Big(-\Big[T_k,[X,T_k]\Big(-\sum_{1\le j\le n}[X,T_j]^2\Big)^{\frac{p}{2}-1}\Big]\Big)\\
&\quad + B\Big(-\Big[T_k,\Big(-\sum_{1\le j\le n}[X,T_j]^2\Big)^{\frac{p}{2}-1}[X,T_k]\Big]\Big)\Big),
\end{aligned}$$

which gives

$$\sum_{1\le k\le n}\Big[T_k,[X,T_k]\Big(-\sum_{1\le j\le n}[X,T_j]^2\Big)^{\frac{p}{2}-1} + \Big(-\sum_{1\le j\le n}[X,T_j]^2\Big)^{\frac{p}{2}-1}[X,T_k]\Big] = 0. \tag{8.1}$$

Similarly, in case $I = \widetilde{I}_\alpha$, we have

$$\begin{aligned}
&\frac{d}{d\varepsilon}\rho\Big(\Big(\sum_{1\le j\le n}(\alpha_j(X+\varepsilon B)-(X+\varepsilon B))^2\Big)^{\frac{p}{2}}\Big)\Big|_{\varepsilon=0}\\
&= \frac{p}{2}\sum_{1\le k\le n}\rho\Big((\alpha_k(B)-B)(\alpha_k(X)-X)\Big(\sum_{1\le j\le n}(\alpha_j(X)-X)^2\Big)^{\frac{p}{2}-1}\\
&\quad + (\alpha_k(B)-B)\Big(\sum_{1\le j\le n}(\alpha_j(X)-X)^2\Big)^{\frac{p}{2}-1}(\alpha_k(X)-(X))\Big)\\
&= \frac{p}{2}\sum_{1\le k\le n}\rho((\alpha_k(B)-B)D_k)\\
&= \frac{p}{2}\sum_{1\le k\le n}\rho(B(\alpha_k^{-1}(D_k)-D_k)),
\end{aligned}$$

where

$$D_k = (\alpha_k(X) - X)\Big(\sum_{1 \le j \le n} (\alpha_j(X) - X)^2\Big)^{\frac{p}{2}-1} + \Big(\sum_{1 \le j \le n} (\alpha_j(X) - X)^2\Big)^{\frac{p}{2}-1}(\alpha_k(X) - X).$$

With this notation, we have

$$\sum_{1 \le k \le n} (\alpha_k^{-1}(D_k) - D_k) = 0. \tag{8.2}$$

Similar computations can be carried out in the Dirac case.

Remark 8.1. It is natural to view solutions of (8.1) as $\tau - p$-harmonic elements and solutions of (8.2) as $\alpha - p$-harmonic elements. A possible technical problem which may appear is that in order not to limit considerations to "bounded p-harmonic" elements it may be necessary to be able to handle the situation when X is an unbounded operator affiliated with $\mathcal{M}$.

Remark 8.2. In the case of automorphisms, if $I \in \mathcal{N} \subset \mathcal{M}$ is a von Neumann subalgebra so that $\rho|\mathcal{N}$ is semifinite and $\alpha_j(\mathcal{N}) = \mathcal{N}$, $1 \le j \le n$, let E be the conditional expectation of $\mathcal{M}$ onto $\mathcal{N}$ so that $\rho \circ E = \rho$. If $I(X)$ stands for $I_\alpha(X)$, $\widetilde{I}_\alpha(X)$ or $I_\alpha^D(X)$, it is easily seen that

$$I(X) \ge I(EX).$$

The definitions of the condenser quasicentral moduli $k_{\mathcal{J}}(\tau; P, Q)$, $k_{\mathcal{J}}(\alpha; P, Q)$, $\tilde{k}_{\mathcal{J}}(\tau; P, Q)$, $\tilde{k}_{\mathcal{J}}(\alpha; P, Q)$, $k_{\mathcal{J}}^D(\tau; P, Q)$, $k_{\mathcal{J}}^D(\alpha; P, Q)$, where $P, Q \in \mathcal{P}$, $PQ = 0$ suggest corresponding variational problems for the $I(X)$ quantities involving the convex sets

$$\mathcal{C}_{PQ}^0 = \{B \in \mathcal{R}_1^+ \mid BP = P,\ BQ = 0\},$$
$$\mathcal{C}_{PQ} = \{X \in \mathcal{M} \mid X = X^*,\ 0 \le X \le I,\ XP = P,\ XQ = 0\}.$$

The inf of $I(B)$ when $B \in \mathcal{C}_{PQ}^0$ gives the condenser quasicentral moduli, while $\mathcal{C}_{PQ}^0$ is weakly dense in $\mathcal{C}_{PQ}$, which is a weakly compact convex set.

In view of the weak lower semicontinuity property of $I(X)$, we have that the inf of $I(X)$ over $\mathcal{C}_{PQ}$ is attained at some point of $\mathcal{C}_{PQ}$. Note, however, that we only know that

$$\inf\{I(X) | X \in \mathcal{C}_{PQ}\} \le \inf\{I(X) \mid X \in \mathcal{C}_{PQ}^0\}.$$

Let $X_m \in \mathcal{C}_{PQ}^0$, $m \in \mathbb{N}$ be a sequence so that

$$\lim_{m \to \infty} I(X_m) = \inf\{I(X) | X \in \mathcal{C}_{PQ}^0\}$$

and which is weakly convergent

$$w - \lim_{m \to \infty} X_m = X_\infty,$$

which can be arranged by passing to a subsequence. We have $X_\infty \in \mathcal{C}_{PQ}$.

More can be said when $(\mathcal{J}, |\cdot|_\mathcal{J})$ is the p-class, $1 < p < \infty$, because then $(\mathcal{J} \otimes \mathfrak{M}_k, |\cdot|_\mathcal{J})$ is a uniformly convex Banach space. Assume moreover, $I(X)$ is one of $\widetilde{I}_\tau(X)$, $\widetilde{I}_\alpha(X)$, $I_\tau^D(X)$, $I_\alpha^D(X)$, i.e., there is no max in the definition of $I(X)$. Then, $I(X) = |\partial(X)|_\mathcal{J}$, where in each of the four cases $\partial(X)$ is

$$\begin{pmatrix} [X, T_1] \\ \vdots \\ [X, T_n] \end{pmatrix},$$

$$\begin{pmatrix} \alpha_1(X) - X \\ \vdots \\ \alpha_n(X) - X \end{pmatrix},$$

$$\sum_{1 \le j \le n} [X, T_j] \otimes e_j,$$

$$\sum_{1 \le j \le n} (\alpha_j(X) - X) \otimes e_j.$$

We have

$$I\left(\frac{1}{2}(X_p + X_q)\right) \ge \lim_{m \to \infty} I(X_m),$$

that is

$$\left|\frac{1}{2}(\partial(X_p) + \partial(X_q))\right|_\mathcal{J} \ge \lim_{m \to \infty} |\partial(X_m)|_\mathcal{J}.$$

In view of the uniform convexity, we have that the sequence $\partial(X_m)$, $m \in \mathbb{N}$, is convergent in the norm $|\cdot|_\mathcal{J}$. Since $\partial(X_m)$ is weakly convergent to $\partial(X_\infty)$, we infer that the limit in the norm $|\cdot|_\mathcal{J}$ equals the weak limit $\partial(X_\infty)$. It follows that

$$I(X_\infty) = \inf\{I(B) \mid B \in \mathcal{C}_{PQ}^0\}.$$

Assume that $X'_m \in \mathcal{C}_{PQ}^0$, $m \in \mathbb{N}$, is another sequence which is weakly convergent to X'_∞, and so, that

$$\lim_{m \to \infty} I(X'_m) = \inf\{I(B) \mid B \in \mathcal{C}_{PQ}^0\}.$$

Then,

$$\left|\frac{1}{2}(\partial(X_m) + \partial(X'_m))\right|_\mathcal{J} \ge \inf\{|\partial(B)|_\mathcal{J} \mid B \in \mathcal{C}_{PQ}^0\},$$

which by uniform convexity implies that

$$\lim_{m\to\infty} |\partial(X_m) - \partial(X'_m)|_{\mathcal{J}} = 0$$

so that

$$\partial(X_\infty) = \partial(X'_\infty).$$

The set $\ker\partial$ is a von Neumann subalgebra of $\mathcal{M}$. In the case of $\widetilde{I}_\tau$, I^D_τ it is $(\tau)' \cap \mathcal{M}$ the relative commutant of τ in $\mathcal{M}$, while in the case of $\widetilde{I}_\alpha$, I^D_α it is the fixed point algebra of the n-tuple of automorphisms α.

Summarizing, we have shown the following remark.

Remark 8.3. Assume $\mathcal{J}$ is the p-class, $1 < p < \infty$, and assume that $I(X)$ is one of the four quantities which do not involve a max. The weak limits of $\mathcal{C}^0_{PQ}$-sequences which minimize I over $\mathcal{C}^0_{PQ}$ form a weakly compact convex subset of $\mathcal{C}_{PQ}$ on which the value of I is the infimum of I over $\mathcal{C}^0_{PQ}$. Moreover, modulo the von Neumann subalgebra $\ker\partial$, the elements of this convex set are equal.

The preceding remark leaves open the question about equality of the infimum of I over $\mathcal{C}^0_{PQ}$ and $\mathcal{C}_{PQ}$. We can answer this in case $k_{\mathcal{J}}(\tau) = 0$ or, respectively, $k_{\mathcal{J}}(\alpha) = 0$. There is not a "no max" restriction on $I(X)$ for this. In this situation, there is a sequence $B_k \in \mathcal{R}^+_1$, $k \in \mathbb{N}$ so that $B_k(P+Q) = P+Q$, $s-\lim_{k\to\infty} B_k = I$ and $|[\tau, B_k]|_{\mathcal{J}} \to 0$ or, respectively, $|\alpha_j(B_k) - B_k|_{\mathcal{J}} \to 0$, $1 \le j \le n$ as $k \to \infty$ that is $I(B_k) \to 0$ for our choice of I. Let $X \in \mathcal{C}_{PQ}$ be so that

$$I(X) = \inf\{I(Y) \mid Y \in \mathcal{C}_{PQ}\}.$$

Then, $B_k X B_k \in \mathcal{C}^0_{PQ}$ and

$$s - \lim_{k\to\infty} B_k X B_k = X.$$

We have

$$I(X) \le I(B_k X B_k) \le I(X) + 2I(B_k).$$

This gives

$$\lim_{k\to\infty} I(B_k X B_k) = I(X).$$

Summarizing, we have the following remark.

Remark 8.4. Assume that $k_{\mathcal{J}}(\tau) = 0$ or, respectively, that $k_{\mathcal{J}}(\alpha) = 0$. Then, we have

$$I(X) = \inf\bigl\{I(B) \mid B \in \mathcal{C}^0_{PQ}\bigr\} = \inf\bigl\{I(Y) \mid Y \in \mathcal{C}_{PQ}\bigr\}$$

for some $X \in \mathcal{C}_{PQ}$.

Assume that $\mathcal{J}$ is the p-class, $2 \leq p < \infty$, and that $I(\cdot)$ involves no max. Let $X \in \mathcal{C}_{PQ}$ be so that

$$I(X) = \inf\{I(Y) \mid Y \in \mathcal{C}_{PQ}\}.$$

Let $P_1 = E(X;\{1\})$ and $Q_1 = E(X;\{0\})$. If $B \in \mathcal{R}_1^+$ and $\varepsilon \in [-1, 0]$, we have

$$X_\varepsilon = X + \varepsilon\big((P_1 - P) + (X - X^2)^{\frac{1}{2}}\big)B\big((P_1 - P) + (X - X^2)^{\frac{1}{2}}\big) \in \mathcal{C}_{PQ},$$

and hence,

$$\frac{d}{d\varepsilon} I^p(X_\varepsilon)|_{\varepsilon=0} \leq 0.$$

The computations preceding Remark 8.1 with B replaced by

$$\big((P_1 - P) + (X - X^2)^{\frac{1}{2}}\big)B\big((P_1 - P) + (X - X^2)^{\frac{1}{2}}\big)$$

give

$$\rho\big(\Theta\big((P_1 - P) + (X - X^2)^{\frac{1}{2}}\big)\big)B\big((P_1 - P) + (X - X^2)^{\frac{1}{2}}\big) \leq 0$$

if I is $\widetilde{I}_\tau$ or $\widetilde{I}_\alpha$ and Θ denotes the quantity appearing in (8.1) and, respectively, (8.2). Since $B \in \mathcal{R}_1^+$ is arbitrary, this implies

$$\big((P_1 - P) + (X - X^2)^{\frac{1}{2}}\big)\Theta\big((P_1 - P) + (X - X^2)^{\frac{1}{2}}\big) \leq 0.$$

This in turn is equivalent to

$$(I - P - Q_1)\Theta(I - P - Q_1) \leq 0.$$

Similarly, if $\varepsilon \in [0, 1]$, we have

$$X_\varepsilon = X + \varepsilon\big((Q_1 - Q) + (X - X^2)^{\frac{1}{2}}\big)B\big((Q_1 - Q) + (X - X^2)^{\frac{1}{2}}\big) \in \mathcal{C}_{PQ}$$

and proceeding as above, we get that

$$(I - P_1 - Q)\Theta(I - P_1 - Q) \geq 0.$$

Thus, in case $I = \widetilde{I}_\tau$, we have

$$\begin{aligned} &P_1, Q_1 \in \mathrm{Proj}(\mathcal{M}), \quad P \leq P_1, \quad Q \leq Q_1, \quad P_1 Q_1 = 0, \quad XP_1 = P_1, \quad XQ_1 = 0, \\ &(I - P - Q_1)\Theta(I - P - Q_1) \leq 0 \end{aligned} \tag{8.3}$$

and

$$(I - P_1 - Q)\Theta(I - P_1 - Q) \geq 0,$$

where

$$\Theta = \sum_{1 \leq k \leq n} \Big[T_k, [X, T_k]\Big(-\sum_{1 \leq j \leq n} [X, T_j]^2\Big)^{\frac{p}{2}-1} + \Big(-\sum_{1 \leq j \leq n} [X, T_j]^2\Big)^{\frac{p}{2}-1}[X, T_k]\Big].$$

In case $I = \widetilde{I}_\alpha$, we have

$$
\begin{aligned}
&P_1, Q_1 \in \mathrm{Proj}(\mathcal{M}), \quad P \le P_1, \quad Q \le Q_1, \quad P_1 Q_1 = 0, \quad XP_1 = P_1, \quad XQ_1 = 0 \\
&(I - P - Q_1)\Big(\sum_{1\le k\le n} (\alpha_k^{-1}(D_k) - D_k)\Big)(I - P - Q_1) \le 0
\end{aligned}
\tag{8.4}
$$

and

$$
(I - P_1 - Q)\Big(\sum_{1\le k\le n} (\alpha_k^{-1}(D_k) - D_k)\Big)(I - P_1 - Q) \ge 0,
$$

where

$$
\begin{aligned}
D_k = {} & (\alpha_k(X) - X)\Big(\sum_{1\le j\le n} (\alpha_j(X) - X)^2\Big)^{\frac{p}{2}-1} \\
& + \Big(\sum_{1\le j\le n} (\alpha_j(X) - X)^2\Big)^{\frac{p}{2}-1} (\alpha_k(X) - X).
\end{aligned}
$$

Similar computations can be carried out in the Dirac case.

Summarizing, we have proved the following remark.

Remark 8.5. Assume that $\mathcal{J}$ is the p-class, $2 \le p < \infty$, and that X is a minimizer of $I(\cdot)$ in $\mathcal{C}_{PQ}$. If I is $\widetilde{I}_\tau$ then X satisfies (8.3), and if I is $\widetilde{I}_\alpha$, then X satisfies (8.4). These conditions are compressions of noncommutative $\tau - p$-Laplace and, respectively, $\alpha - p$-Laplace inequalities.

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