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A new monotonicity formula for the spatially homogeneous Landau equation with Coulomb potential and its applications

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Abstract. We describe a time-dependent functional involving the relative entropy and the $\dot{H}^1$ seminorm, which decreases along solutions to the spatially homogeneous Landau equation with Coulomb potential. The study of this monotone functional sheds light on the competition between dissipation and nonlinearity for this equation. It enables us to obtain new results concerning regularity/blowup issues for the Landau equation with Coulomb potential.

Keywords. Landau equation, Landau operator, degenerate diffusion, Coulomb interaction

1. Introduction

We consider the spatially homogeneous Landau equation with Coulomb potential

$$\partial_t f = Q(f, f)(v), \tag{1.1}$$

complemented with initial data $f_0 = f_0(v) \geq 0$. Here $f := f(t, v) \geq 0$ stands for the distribution of particles that at time $t \in \mathbb{R}_+$ have velocity $v \in \mathbb{R}^3$. The *Landau operator* (with *Coulomb potential*) Q is a bilinear operator acting only on the velocity variable v. It reads

$$Q(g, h) = \nabla \cdot ([a * g]\nabla h - [a * \nabla g]h) \tag{1.2}$$

with

$$a(z) = |z|^{-1}\left(\mathrm{Id} - \frac{z \otimes z}{|z|^2}\right). \tag{1.3}$$

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This equation, first obtained by Landau in 1936, is used to describe the evolution in time of a (spatially homogeneous) plasma due to collisions between charged particles under the Coulomb potential.

Introducing the quantity

$$b_i(z) := \sum_{j=1}^{3} \partial_j a_{ij}(z) = -2z_i|z|^{-3}, \tag{1.4}$$

the Landau operator with Coulomb potential can also be written as

$$\begin{aligned} Q(f,f) &= \sum_{i=1}^{3} \partial_i \Big(\sum_{j=1}^{3} (a_{ij} * f)\partial_j f - (b_i * f) f \Big) \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3} (a_{ij} * f)\partial_{ij} f + 8\pi f^2, \end{aligned} \tag{1.5}$$

where we have used the identity $\sum_{i=1}^{3} \partial_i b_i(z) = -8\pi\delta_0(z)$.

1.1. Basic properties of the equation and notations

The weak counterpart of the Landau operator Q, for a suitable test function φ, is the following:

$$\begin{aligned} \int_{\mathbb{R}^3} Q(f,f)(v)\varphi(v)\,dv &= -\frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} \iint_{\mathbb{R}^3\times\mathbb{R}^3} a_{ij}(v-v_*) \\ &\times \left\{ \frac{\partial_j f}{f}(v) - \frac{\partial_j f}{f}(v_*) \right\} \{\partial_i \varphi(v) - \partial_i \varphi(v_*)\} f(v) f(v_*)\,dv\,dv_*. \end{aligned} \tag{1.6}$$

From this formula, we can obtain the fundamental properties of the Landau operator Q. The operator indeed conserves (at the formal level) mass, momentum and energy, more precisely

$$\int_{\mathbb{R}^3} Q(f,f)(v)\varphi(v)\,dv = 0 \quad \text{for } \varphi(v) = 1, v_i, |v|^2/2,\ i = 1,2,3. \tag{1.7}$$

We also deduce from (1.6) the entropy structure of the operator (still at the formal level) by taking the test function $\varphi(v) = \log f(v)$, that is, we define

$$\begin{aligned} D(f) &:= -\int_{\mathbb{R}^3} Q(f,f)(v)\log f(v)\,dv \\ &= \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} \iint_{\mathbb{R}^3\times\mathbb{R}^3} a_{ij}(v-v_*) \\ &\times \left\{ \frac{\partial_i f}{f}(v) - \frac{\partial_i f}{f}(v_*) \right\} \left\{ \frac{\partial_j f}{f}(v) - \frac{\partial_j f}{f}(v_*) \right\} f(v) f(v_*)\,dv\,dv_*. \end{aligned} \tag{1.8}$$

Note that $D(f) \geq 0$ since the matrix a is (semidefinite) positive. Note also that for any f such that $D(f) = 0$, it can be shown (see [9] and [13] for a rigorous statement and proof) that f is a *Maxwellian distribution*, that is, $f = \mu_{\rho,u,T}$ with

$$\mu_{\rho,u,T}(v) = \frac{\rho}{(2\pi T)^{3/2}} e^{-\frac{|v-u|^2}{2T}}, \tag{1.9}$$

where $\rho \geq 0$ is the density, $u \in \mathbb{R}^3$ is the mean velocity and $T > 0$ is the temperature of the plasma. They are defined by

$$\begin{aligned} \rho &= \int_{\mathbb{R}^3} f(v)\, dv, \quad u = \frac{1}{\rho} \int_{\mathbb{R}^3} v f(v)\, dv, \\ T &= \frac{1}{3\rho} \int_{\mathbb{R}^3} |v-u|^2 f(v)\, dv. \end{aligned} \tag{1.10}$$

Thanks to the conservation of mass, momentum and energy, we have (when $f := f(t,v)$ is a solution of (1.1)–(1.3) and ρ, u, T are as defined above, at the formal level),

$$\forall t \geq 0, \quad \rho(t) = \rho(0), \quad u(t) = u(0), \quad T(t) = T(0), \tag{1.11}$$

which implies that the parameters ρ, u, T are constant (along solutions of (1.1)–(1.3)).

Denoting (when $f := f(t,v)$ is a solution of (1.1)–(1.3)) by

$$\begin{aligned} H(t) &:= H(f|\mu_{\rho,u,T})(t) \\ &:= \int_{\mathbb{R}^3} \left(f(t,v) \log\left(\frac{f(t,v)}{\mu_{\rho,u,T}} \right) - f(t,v) + \mu_{\rho,u,T} \right) dv, \end{aligned} \tag{1.12}$$

the relative entropy with respect to $\mu_{\rho,u,T}$ (defined by (1.9), (1.10)), we see that (still at the formal level)

$$\frac{d}{dt} H(t) = -D(f(t,\cdot)) \leq 0. \tag{1.13}$$

Note that in the above definition, $H(t)$ differs from the usual (nonrelative) entropy $\int f(t,v) \log f(t,v)\, dv$ only by a constant, thanks to the identities (1.11).

Throughout this paper, we shall assume that $f_0 \geq 0$ and $f_0 \in L^1_2 \cap L \log L(\mathbb{R}^3)$. Furthermore, without loss of generality, we shall also assume that f_0 satisfies the normalization identities

$$\int_{\mathbb{R}^3} f_0(v)\, dv = 1, \quad \int_{\mathbb{R}^3} f_0(v) v\, dv = 0, \quad \int_{\mathbb{R}^3} f_0(v)|v|^2\, dv = 3, \tag{1.14}$$

which can be rewritten as $\rho(0) = 1$, $u(0) = 0$, $T(0) = 1$. Finally, we denote by

$$\mu(v) := (2\pi)^{-3/2} e^{-|v|^2/2} \tag{1.15}$$

the Maxwellian distribution (centred reduced Gaussian) with the same mass, momentum and energy as f_0 satisfying (1.14).

Next we introduce some function spaces which will be used throughout the paper:

• Let $\langle v\rangle := (1+|v|^2)^{1/2}$ denote the Japanese bracket. For any $p \in [1,+\infty[$, $l \in \mathbb{R}$, the L^p_l norm is defined by

$$\|f\|^p_{L^p_l} := \int_{\mathbb{R}^3} |f(v)|^p \langle v\rangle^{pl}\, dv.$$

• The following quantity, for functions in $L\log L$, is written as if it were a norm:

$$\|f\|_{L\log L} := \int_{\mathbb{R}^3} |f|\log(1+|f|)\, dv.$$

• For any $p \in (1,\infty)$, $q \in [1,+\infty]$, the standard *Lorentz space* $L^{p,q}$ is defined by the norm

$$\|f\|_{L^{p,q}} := \begin{cases} \left(\int_0^\infty (t^{1/p} f^{**}(t))^q \frac{dt}{t}\right)^{1/q}, & 1 \le q < \infty, \\ \sup_{t>0} t^{1/p} f^{**}(t), & q = \infty, \end{cases} \tag{1.16}$$

where $f^{**}(t) := t^{-1}\int_0^t f^*(s)\,ds$, and f^* is the decreasing rearrangement of f. When $l \in \mathbb{R}$ we also denote the weighted Lorentz norm by

$$\|f\|_{L^{p,q}_l} := \|f(\cdot)\langle\cdot\rangle^l\|_{L^{p,q}}.$$

More details on Lorentz spaces including the case when $p=1$, $p=\infty$ can be found in the Appendix.

• The homogeneous Sobolev $\dot{H}^m$ norm with $m \in \mathbb{R}$ is defined by

$$\|f\|^2_{\dot{H}^m} := \int_{\mathbb{R}^3} |\xi|^{2m} |\hat{f}(\xi)|^2\, d\xi,$$

while the weighted inhomogeneous Sobolev H^m_l norm with $m \in \mathbb{N}$, $l \in \mathbb{R}$ is defined by

$$\|f\|^2_{H^m_l} := \sum_{|\alpha|\le m} \int_{\mathbb{R}^3} |\partial^\alpha (f\langle v\rangle^l)|^2\, dv.$$

1.2. Short review of the Landau equation with Coulomb potential

We briefly review the works on the Landau equation with Coulomb potential (1.1)–(1.3).

• **Existence and uniqueness of solutions:** In [39], Villani proved the global existence of so-called H-solutions for (1.1)–(1.3) when the initial data have finite mass, energy and entropy. The key part of the proof lies in the use of the entropy dissipation $D(f)$, rewritten as

$$D(f(t)) = 2\iint \frac{1}{|v-v_*|} |\Pi(v-v_*)\nabla_{v-v_*}\sqrt{f(t,v)f(t,v_*)}|^2\, dv\, dv_*. \tag{1.17}$$

Here $\Pi(z)\nabla := (\mathrm{Id} - \frac{z}{|z|}\otimes\frac{z}{|z|})\nabla$ is called the *weak projection gradient* (see [23,39]). In all generality (when an estimate for $\nabla_v f$ is not available), $\Pi(v-v_*)\nabla_{v-v_*}$ is not equal to $\Pi(v-v_*)\nabla_v - \Pi(v-v_*)\nabla_{v_*}$. This means that the construction of approximations to an H-solution plays a significant role. When the solutions are *well-constructed* (that is, using a suitable approximation process), we have

$$\Pi(v-v_*)\nabla_{v-v_*} = \Pi(v-v_*)\nabla_v - \Pi(v-v_*)\nabla_{v_*}. \tag{1.18}$$

We refer the readers to [23] for more details. When (1.18) holds, we can use the estimate for the entropy dissipation $D(f)$ in [9] to show that an H-solution is a weak solution of the equation. More precisely, there is an explicitly computable constant $C_0 = C_0(\bar{H}) > 0$ such that, for all (normalized) $f \geq 0$ satisfying $H(f) \leq \bar{H}$, the following inequality holds:

$$\|f\|_{L^3_{-3}} \leq C_0(1 + D(f)). \tag{1.19}$$

Therefore, we know that such an H-solution of (1.1)–(1.3) lies in $L^1_{\mathrm{loc}}([0,\infty); L^3_{-3}(\mathbb{R}^3))$, and this estimate is sufficient to show that it is indeed a weak solution in the usual sense.

Fournier [14] showed that uniqueness holds for the solutions of (1.1)–(1.3) lying in $L^\infty_{\mathrm{loc}}([0,\infty); L^1_2(\mathbb{R}^3)) \cap L^1_{\mathrm{loc}}([0,\infty); L^\infty(\mathbb{R}^3))$, and this result implies a local well-posedness result assuming further that the initial data lie in $L^\infty(\mathbb{R}^3)$, thanks to the local existence result of Arsen'ev–Peskov [3] for such initial data. We also refer to [7] for uniqueness of higher integrable solutions, and to [26] for the study of an equation sharing significant features with (1.1)–(1.3).

In the spatially inhomogeneous context, we quote [38] for the existence of renormalized solutions and [18, 21] for the global well-posedness near Maxwellian and the local well-posedness in weighted Sobolev spaces. We finally refer to [5] for a general perturbation result, and to [24, 25] for conditional regularity results.

• **Long time behavior:** In a perturbative and spatially inhomogeneous framework, Guo and Strain [35] (see also [5]) proved for solutions of (1.1)–(1.3) the stretched exponential decay to equilibrium in a high-order Sobolev space with fast decay in the velocity variable. For (uniformly in time) *a priori* smooth solutions with large initial data, L.D. and Villani [12] proved the algebraic convergence to equilibrium.

In the homogeneous setting, Carrapatoso, L.D. and L.H. proved the following result which plays an essential role in the present paper:

Theorem 1.1 (Cf. [4, Theorem 2 and Lemma 8]). *Let $f_0 \in L\log L(\mathbb{R}^3) \cap L^1_\ell(\mathbb{R}^3)$ with $\ell > 19/2$ satisfy the normalization (1.14), and consider a (well-constructed) weak (or H-) solution f to (1.1)–(1.3) with initial datum f_0. Then for any strictly positive $\beta < \frac{2\ell^2-25\ell+57}{9(\ell-2)}$, there exists some computable constant $C_\beta > 0$ (depending only on β and $K > 0$ such that $\|f_0\|_{L^1_\ell(\mathbb{R}^3)} + \|f_0\|_{L\log L(\mathbb{R}^3)} \leq K$) such that the relative entropy satisfies*

$$\forall t \geq 0, \quad H(t) \leq C_\beta(1+t)^{-\beta}. \tag{1.20}$$

Moreover, for all $\ell > 2$, there exists $C_\ell > 0$ (which only depends on ℓ and K) such that

$$\forall t > 0, \quad \|f(t,\cdot)\|_{L^1_\ell(\mathbb{R}^3)} \leq C_\ell(1+t). \tag{1.21}$$

• **Functional estimates:** In [9], it is shown that (for normalized $f \geq 0$) the following estimate holds:

$$D(f) + 1 \geq C_{D,1} \| \sqrt{f} \|^2_{H^1_{-3/2}}, \tag{1.22}$$

where $C_{D,1} > 0$ depends only on an upper bound of $H(f|\mu)$.

Using the precise Sobolev embedding inequality

$$\| f \|_{L^{6,2}} \leq C \| \nabla f \|_{L^2}$$

(see [1]) and the O'Neil inequality in Lorentz spaces (see Proposition 6.2 in the Appendix), we end up with the following inequality (holding for normalized $f \geq 0$):

$$D(f) + 1 \geq C_{D,2} \| f \|_{L^{3,1}_{-3}}, \tag{1.23}$$

where $C_{D,2} > 0$ depends only on an upper bound of $H(f|\mu)$.

We refer to [4, 9, 13] for variants of (1.23).

• **Partial regularity issue:** Very recently Golse, Gualdani, Imbert and Vasseur [15] proved that the set of singular times for (suitable) weak solutions of the spatially homogeneous Landau equation with Coulomb potential has Hausdorff dimension at most $1/2$ if the initial data has all polynomial moments. The key ingredient of the proof is the application of De Giorgi's method to a suitable scaled solution. They also observed that the solution to the Landau equation with Coulomb interaction enjoys a scaling property which is similar to that of the 3D incompressible Navier–Stokes equation. This explains the link between the bound on the Hausdorff dimension of the set of singular times in both equations. We also mention the papers [16, 17] where Gualdani and Guillen provide estimates which are useful to understand the issues of regularity blowup and the role played by the various terms in the Landau equation with Coulomb potential.

1.3. Main result

A very challenging problem for the (spatially homogeneous) Landau equation with Coulomb potential (1.1)–(1.3) is to answer whether smoothness is propagated for all positive times, or if some blowup may occur in finite time. If a blowup appears, a further challenging issue is to understand what really happens at the blowup time (cf. Villani's monograph [40, Chapter 5, §1.3 (2)]). The main result of this paper provides new partial answers to the first question, while another result deals with the second question.

Our main result is concerned with the new monotonicity formula for equation (1.1)–(1.3) announced in the title, and its byproducts:

Theorem 1.2. *Let $f_0 \in L\log L(\mathbb{R}^3) \cap L^1_{55}(\mathbb{R}^3) \cap \dot{H}^1(\mathbb{R}^3)$ be a nonnegative initial datum satisfying the normalization* (1.14)*. Then there exist* (*explicitly computable*) *constants $B^*, C_6 > 0$, $k_2 > 7/2$, $k > 0$* (*depending only on K satisfying $\|f_0\|_{L^1_{55}(\mathbb{R}^3)} + \|f_0\|_{L\log L(\mathbb{R}^3)} \leq K$*) *such that the following three statements hold:*

(i) (Monotonicity of a functional) *Let* $T > 0$ *and denote by* $f := f(t, v)$ *a smooth and quickly decaying* ($C_t^2(\mathcal{S})$) *nonnegative solution on* $[0, T]$ *to* (1.1)–(1.3) *with initial datum* f_0. *Define* $h := f - \mu$, *where* μ *is given by* (1.15) (*recall also that* $H(t)$ *is the relative entropy given by* (1.12)). *Then the following a priori estimate* (*which we call the* monotonicity property) *holds for* $t \in [0, T]$*:*

$$\frac{d}{dt}\Big[H(t) - \tfrac{5}{2}\big(\|h(t)\|_{\dot{H}^1}^2 + B^*(1+t)^{-k_2+1}\big)^{-2/5}\Big] + C_6(1+t)^k \le 0. \tag{1.24}$$

(ii) (Global regularity for initial data below threshold) *If in addition we have* $H(0)(\|h(0)\|_{\dot{H}^1}^2 + B^*)^{2/5} \le 5/2$, *then* (1.1)–(1.3) *admits a* (*unique*) *global and strong* (*that is, lying in* $L^\infty(\mathbb{R}_+; H^1(\mathbb{R}^3))$) *nonnegative solution satisfying*

$$\forall t > 0, \quad \|h(t)\|_{\dot{H}^1}\left(H(t) + \frac{C_6}{k+1}[(1+t)^{1+k} - 1]\right)^{5/4} \le (2/5)^{-5/4}, \tag{1.25}$$

where h, μ *and* H *are as in* (i).

(iii) (No blowup after a finite time) *If finally* $H(0)(\|h(0)\|_{\dot{H}^1}^2 + B^*)^{2/5} > 5/2$, *set*

$$T^* := \left(\frac{1+k}{C_6}\Big[H(0) - \tfrac{5}{2}[\|h(0)\|_{\dot{H}^1}^2 + B^*]^{-2/5}\Big] + 1\right)^{\frac{1}{k+1}} - 1. \tag{1.26}$$

Then one can construct a global weak (or H*-) nonnegative solution of* (1.1)–(1.3) *such that for* $t > T^*$, *it becomes global and strong* (*that is, it lies in* $L^\infty((T^*, \infty); H^1(\mathbb{R}^3))$), *and satisfies the estimates*

$$H(t)[\|h(t)\|_{\dot{H}^1}^2 + B^*(1+t)^{1-k_2}]^{-2/5} \le 5/2, \tag{1.27}$$

$$\|h(t)\|_{\dot{H}^1}\left(\frac{C_6}{k+1}[(1+t)^{1+k} - (1+T^*)^{1+k}]\right)^{5/4} \le (2/5)^{-5/4}, \tag{1.28}$$

where h, μ *and* H *are as in* (i).

1.3.1. Comment on the monotonicity formula (1.24). To the best of our knowledge, inequality (1.24) of Theorem 1.2 (i) is a new monotonicity formula for the (smooth solutions of the) Landau equation with Coulomb potential. The explicit increasing rate $C_6(1+t)^k$ comes from the dissipation effect of the equation. We denote the monotone functional by

$$\mathcal{M}(t) := H(t) - \tfrac{5}{2}\big(\|h(t)\|_{\dot{H}^1}^2 + B^*(1+t)^{-k_2+1}\big)^{-2/5}, \tag{1.29}$$

and notice that the differential inequality (1.24) formally allows $\|h(t)\|_{\dot{H}^1}$ to blow up. The global dynamics of $\mathcal{M}(t)$ described by (1.24) gives clues about the global dynamics for the original solution:

- When $\mathcal{M}(t_0)$ is below its critical value, (that is, $\mathcal{M}(t_0) \le 0$, cf. comment below), the solution to equation (1.1)–(1.3) after time t_0 will remain bounded in $\dot{H}^1$ and converge to the equilibrium; this is indicated in Theorem 1.2 (ii). We call this the *stable regime*.

- When $\mathcal{M}(t)$ is above its critical value (that is, $\mathcal{M}(t_0) \geq 0$), some blowup may occur, but there exists a computable time T^* (strictly greater than the blowup time if it occurs) such that $\mathcal{M}(t)$ gets inside the stable regime for any $t > T^*$; this is indicated in Theorem 1.2 (iii).

Note that in Theorem 1.2 (i), the differential inequality (1.24) will be shown to rigorously hold for (smooth and quickly decaying when $|v| \to \infty$) solutions to an approximate problem (problem (2.42) described in §2.6) for (1.1)–(1.3). Finally, when it is integrated with respect to time (see (5.8)), it is shown in the proof of Proposition 1.3 (in §1.4) that it also rigorously holds for strong (that is, lying in $L^\infty_t(H^1_m)$ for m large enough) solutions to the original equation (1.1)–(1.3), such as those appearing (on suitable time intervals) in Propositions 1.1 and 1.2 (in §1.4).

1.3.2. Dissipation and nonlinearity. Our results shed some new light on the competition between dissipation and nonlinearity for the Landau equation with Coulomb potential. We present here the main ideas which are used in the proofs of Theorem 1.2 and the related results. In particular, we point out that the mechanisms which enable us to build global strong solutions to (1.1)–(1.3) when $\mathcal{M}(0) \leq 0$ (in Theorem 1.2 (ii)) and when $\|h(0)\langle\cdot\rangle^2\|_{\dot{H}^1}$ is small (in Proposition 1.2) are quite different.

Let us first recall that by a previous study of the large time behavior of the Landau equation with Coulomb potential (cf. [4]), the L^1 moments of $h = f - \mu$ decrease with a power law (cf. Theorem 1.1).

As a consequence, by interpolation, we see that the L^2 norm of $D^2 f$ typically increases in time. Roughly speaking, for some $C, k_1 > 0$, this dissipation is lower bounded in the following way:

$$\|\nabla^2 h\|^2_{L^2_{-3/2}} \geq C\|h\|^{-4/5}_{L^1_{15/4}} \|\nabla h\|^{14/5}_{L^2} \geq C(1+t)^{k_1}\|\nabla h\|^{14/5}_{L^2}.$$

• **Initial data far from equilibrium (in terms of $\dot{H}^1$ norm):** In this situation, the main challenge is to show that the nonlinear terms can be controlled. Indeed, by interpolation, the behavior of the nonlinear term with respect to the $\dot{H}^1$ energy is of the same order as the dissipation term in the following sense:

$$\text{Nonlinearity} \lesssim D(f)\|\nabla h\|^{14/5}_{L^2}.$$

More precisely, a slightly simplified version of estimate (2.37) reads

$$\frac{d}{dt}\|\nabla h\|^2_{L^2} + C_1(1+t)^{k_1}\|\nabla h\|^{14/5}_{L^2} \leq C_3 D(f)\|\nabla h\|^{14/5}_{L^2} + C_2(1+t)^{-k_2}.$$

It is expected that dissipation will dominate the nonlinear term after some time since $(1+t)^{k_1} \to +\infty$ as $t \to \infty$ and $\int_0^\infty D(f)(s)\,ds < +\infty$. The detailed arguments are in Section 2.

• **Initial data close to equilibrium (in terms of $\dot{H}^1$ norm):** In this situation, we have

$$\text{Tail of linear term plus nonlinearity} \lesssim \|\nabla h\|^4_{L^2} + \|\nabla h\|^2_{L^2},$$

as we can observe from (4.17) by neglecting the weights. Since we have assumed that the $\dot{H}^1$ norm of the initial data is sufficiently small, we see that competition occurs between $(1+t)^{k_1}\|\nabla h\|_{L^2}^{14/5}$ and $\|\nabla h\|_{L^2}^2$.

Suppose now that $\|\nabla h_0\|_{L^2}^2 \sim \epsilon$. Then the smallness of $\|\nabla h\|_{L^2}^2$ can be kept at least for an interval of time of length $|\log \epsilon|$ (cf. the proof of Proposition 1.2). This implies that at some point, dissipation will be lower bounded in the following way:

$$(1+t)^{k_1}\|\nabla h\|_{L^2}^{14/5} \geq C(1+|\log \epsilon|)^{k_1}\|\nabla h\|_{L^2}^{4/5}\|\nabla h\|_{L^2}^2.$$

Hence, when $\|\nabla h\|_{L^2}^{4/5}$ is not small, dissipation still prevails and prevents a blowup of the $\dot{H}^1$ norm. We refer the readers to Section 4 for detailed arguments. Note however that in the description above, weights are not taken into account, whereas they play a significant role in the proof of Proposition 1.2 below. Finally, we refer to [16] for extra considerations on competition between dissipation and nonlinearities.

1.3.3. No blowup after a finite time. This is a direct consequence of inequality (1.24) since after time T^*, the monotone functional $\mathcal{M}(t)$ will enter the stable regime (defined in Comment 1.4.1).

- If the solution has not blown up in $\dot{H}^1$ before time T^*, then the solution will remain strong (that is, will lie in $\dot{H}^1$) for all time thanks to inequality (1.28). Then thanks to the uniqueness result established in [14] and the regularity obtained in Proposition 1.1, the constructed solution is the unique strong solution with initial data f_0 satisfying the conditions stated in Theorem 1.2.
- Looking at definition (1.29), we see that $\mathcal{M}(t)$ is still well-defined if $\|h(t)\|_{\dot{H}^1} = \infty$. When such a blowup (in $\dot{H}^1$ norm) happens, the constructed solution is the unique strong solution before the first blowup time, and becomes strong again after time T^*. Note that in order to give a rigorous proof of these facts, we apply the estimates obtained in this paper to solutions of an approximate problem and then pass to the limit.

Finally, combining our result with the previous result in [15], we see that the set of singular times for weak solutions is included in a subset of the interval $[\mathcal{T}, T^*]$ whose Hausdorff dimension is at most $1/2$.

1.3.4. Dependence of the coefficients in the main theorem on L^1 moments. We can provide estimates for the explicit dependence of all coefficients in Theorem 1.2. Moreover, we can extend the validity of this theorem somewhat, when the initial data has less than 55 moments. Indeed, when $f_0 \in L^1_\ell$,

$$q_{\ell,\theta} := -\frac{2\ell^2 - 25\ell + 57}{18(l-2)}\left(1 - \frac{\theta}{\ell}\right) + \frac{\theta}{\ell},$$

and choose $\ell > 31$ and $\tau \in [31, \ell)$ such that $q_{\ell,99/4} > 7/4, q_{\ell,\tau} > 0$. Then one can check that it is possible to take $k := \min\{k_1, \frac{2}{5}k_2 - \frac{7}{5}\}$ with $k_1 = \frac{4}{5}q_{\ell,14/5}, k_2 = q_{\ell,99/4}$ in such a way that estimate (1.24) holds. In our main theorem, we have selected $\ell = 55$ and $\tau = 45$ for the sake of readability.

1.4. Additional results

Using variants of the estimates in the main theorem, it is possible to get more standard results of local (in time) well-posedness for large initial data (in $\dot{H}^1$ norm), and global (in time) well-posedness for small initial data (in $\dot{H}^1$ norm). It is also possible to give estimates concerning a possible blowup (of the $\dot{H}^1$ norm). These results are stated in the following three propositions, where we recall that μ is the Maxwellian given by (1.15), and we denote $h := f - \mu$ and $h_0 := f_0 - \mu$.

We begin with local well-posedness:

Proposition 1.1. *Let $f_0 \in L\log L(\mathbb{R}^3) \cap L^1_{55}(\mathbb{R}^3) \cap \dot{H}^1(\mathbb{R}^3)$ be a nonnegative initial datum satisfying the normalization* (1.14). *Then there exists a time $\mathcal{T} := \frac{5}{4}(\|h_0\|^2_{\dot{H}^1} + C_7^{-1})^{-4/5}$ (where $C_7 > 0$ only depends on K such that $\|f_0\|_{L^1_{55}} + \|f_0\|_{L\log L} \le K$) such that the Landau equation* (1.1)–(1.3) *admits a unique strong solution on the interval $[0, \mathcal{T}]$. By strong solution, we mean here that $f \in C([0,\mathcal{T}]; \dot{H}^1) \cap L^2([0,\mathcal{T}]; H^2_{-3/2})$.*

We then turn to global well-posedness for small initial data:

Proposition 1.2. *Let $f_0 \in L\log L(\mathbb{R}^3) \cap L^1_{55}(\mathbb{R}^3) \cap H^1_2(\mathbb{R}^3)$ be a nonnegative initial datum satisfying the normalization* (1.14)*, and $h_0 := f_0 - \mu$. Then there exists a (small) constant $\epsilon_0 > 0$ (depending only on $K > 0$ such that $\|f_0\|_{L^1_{55}} + \|f_0\|_{L\log L} \le K$) such that if $\|h_0\langle\cdot\rangle^2\|_{\dot{H}^1} \le \epsilon_0$, the Landau equation with Coulomb potential* (1.1)–(1.3) *admits a (unique) global smooth (that is, lying in $L^\infty([0,+\infty); H^1_2(\mathbb{R}^3))$) and nonnegative solution, denoted by $f := f(t,v)$. Moreover, (under the same assumption on the initial datum) there exists a constant $C > 0$ only depending on K such that (with the notation $h := f - \mu$)*

$$\|h(t,\cdot)\|_{H^1_2} \le C(1+t)^{-15/4}.$$

Finally, we give some clues about the behavior of solutions close to a potential blowup:

Proposition 1.3. *Let $f := f(t,v)$ be a nonnegative solution of the Landau equation with Coulomb potential* (1.1)–(1.3)*, corresponding to initial data satisfying the assumptions of Theorem* 1.2*. Suppose that $f \in L^\infty([0,t]; H^1(\mathbb{R}^3))$ for all $t \in [0,\bar{T}[$, and that $\|\nabla f(t)\|_{L^2(\mathbb{R}^3)}$ blows up at time $\bar{T}$. Then for $\bar{T} - t \ll 1$ and some explicitly computable constants $c, C, C_1, C_2 > 0$ (depending only on K satisfying $\|f_0\|_{L\log L} + \|f_0\|_{L^1_{55}} \le K$),*

$$\|h(t)\|_{\dot{H}^1} \ge C(H(t) - \bar{H})^{-5/4} \quad \text{with} \quad H(t) - \bar{H} \ge C(\bar{T} - t)(1+\bar{T})^{k+1};$$

$$\inf_{s\in[t,\bar{T}]} \|h(s)\|_{\dot{H}^1} \le \left(\mathcal{B}(c(\bar{T}-t))\frac{2(\bar{T}-t)}{C_1}(1+\bar{T})^{-(k_1+k_2)}\right)^{5/14},$$

where $\bar{H} := \lim_{t\to\bar{T}_-} H(t)$ and $\mathcal{B}(x) := C_2 x^{-13}\exp\{7x^{-450/14}\}$.

1.4.1. Description of the potential blowup. Proposition 1.3 describes a potential blowup phenomenon for solutions to the Landau equation with Coulomb potential. We recall that restrictions are given in [15, 17] on the possible appearance of such a blowup. Our lower bound for the blowup rate is given in terms of relative entropy. Our upper bound enables excluding a double exponential (that is, exponential of an exponential) growth of the H^1 norm of the solution close to the first blowup time. This bound heavily depends on the number of initial moments which are assumed.

1.5. Additional comments

1.5.1. Nonoptimality of the results. We notice that the lifespan of local well-posedness is not optimal in Proposition 1.1. For example, it can be extended by the effect of the dissipation term $C_6(1+t)^k$, which is not used in the proof of this proposition.

It is also possible to use Proposition 1.1 in order to relax the condition $\mathcal{M}(0) \le 0$ in Theorem 1.2 (ii), recalling $\mathcal{M}(t)$ is defined in (1.29). Using the fact that the Landau equation with Coulomb interaction admits a local solution $f \in C([0,\mathcal{T}]; \dot{H}^1)$ where $\mathcal{T}$ depends only on the initial data f_0 (see Proposition 1.1 for more details), this condition can be transformed into

$$\mathcal{M}(0) \le \frac{C_6}{k+1}((1+\mathcal{T})^{k+1} - 1). \tag{1.30}$$

Indeed, thanks to estimates (1.24) and (1.29), one gets

$$\mathcal{M}(\mathcal{T}) + C_6 \int_0^{\mathcal{T}} (1+t)^k \, dt \le \mathcal{M}(0),$$

so that $\mathcal{M}(\mathcal{T}) \le 0$ and we can use Theorem 1.2 (ii) starting at time $\mathcal{T}$ (the equation being invariant by translation in time).

1.5.2. Landau equation with very soft potentials. We can generalize the result of Theorem 1.2 to the Landau equation with very soft potential in the range $\gamma \in]-3,-2]$, that is, when

$$a(z) := |z|^{\gamma+2}\left(\mathrm{Id} - \frac{z \otimes z}{|z|^2}\right). \tag{1.31}$$

Indeed, the main difference in the proof from the Coulomb case is the estimate of the term $\iint_{|v-v_*|\le 1} f(v_*)|v-v_*|^{\gamma+1}|\nabla h(v)|\,|\nabla^2 h(v)|\,dv_*\,dv$, which appears in Proposition 2.3. The Coulomb potential case $\gamma = -3$ is critical in the sense that it requires (in order to close the differential inequality (1.24)) the use of the Lorentz space $L^{3,1}$, since

$$\iint_{|v-v_*|\le 1} f(v_*)|v-v_*|^{\gamma+1}|\nabla h(v)|\,|\nabla^2 h(v)|\,dv_*\,dv \le C\|f\|_{L^3}\|\nabla h\|_{L^2}\|\nabla^2 h\|_{L^2}$$

holds when $\gamma > -3$ but not when $\gamma = -3$.

We think therefore that when $\gamma \in]-3,-2]$, it is possible to avoid the use of Lorentz spaces and still get a closed inequality in the same spirit as (1.24).

We also believe that if $\gamma = -2 - \eta$ with $\eta > 0$ sufficiently small, then the equation will generate a global smooth and bounded solution if initially $\|\nabla h_0\|_{L^2}^2 \le (C_1\eta)^{-C_2\eta^{-1}} - C_3$ for some $C_1, C_2, C_3 > 0$ (depending on $H(0)$). This is coherent with the existing theory of existence of global strong solutions when $\gamma \in [-2, 0[$; cf. [41] for example.

1.5.3. Comparison with Leray's work for 3D incompressible Navier–Stokes. We recall that the 3D incompressible Navier–Stokes equations read

$$\begin{cases} \partial_t u + u \cdot \nabla u - \triangle u + \nabla p = 0, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0. \end{cases} \tag{1.32}$$

In the classical work [29] (see also [32] and references therein), Leray proved the following results:

(i) If $\|u_0\|_{L^2}\|\nabla u_0\|_{L^2} \ll 1$, the 3D incompressible Navier–Stokes equations admit global smooth solutions, which are nowadays called Leray solutions.

(ii) He also considered the potential blowup phenomenon. Using the lower bound of the blowup rate for the potential singularity, one can show that the set of singular times for suitable weak solutions has Hausdorff dimension at most $1/2$.

For a result about long-time regularity, we refer to [28, 37].

We are in a position to compare our results with Leray's.

If we consider that the relative entropy H plays for the Landau equation with Coulomb interaction the same role as the energy $\|u\|_{L^2}$ for the Navier–Stokes equations, it is natural to compare the Leray condition $\|u_0\|_{L^2}\|\nabla u_0\|_{L^2} \ll 1$ to the condition $\mathcal{M}(0) \le 0$, written in the form $H(0)(\|h(0)\|_{\dot{H}^1}^2 + B^*)^{2/5} \le 5/2$. We then see that as in the Navier–Stokes equations, the L^2 norm of a gradient of a solution plays a decisive role. Note however that no equivalent of the term B^* exists in Leray's condition for the Navier–Stokes equations, which constitutes a significant difference.

The condition $\mathcal{M}(0) \le 0$ includes the case in which the initial relative entropy $H(0)$ is small, while $\|h(0)\|_{\dot{H}^1}$ may be large. Note that such (normalized) initial data exist. Indeed, one can take initial data $f(0)$ close (in weighted L^1) to the Maxwellian μ, but having quick oscillations, so that $\|h(0)\|_{\dot{H}^1}$ is large (see Proposition 6.6 for a concrete example).

Note that in Proposition 1.3, we get not only a lower bound, but also an upper bound for the rate at which a potential blowup occurs. However, the lower bound is given in terms of relative entropy and thus probably cannot be used to estimate the size of singular times. We recall that the size of that set for the Landau equation with Coulomb potential is anyway estimated in [15].

1.5.4. Blowup of ∇f in $L^2(\mathbb{R}^3)$. We remark that studying the blowup of ∇f in $L^2(\mathbb{R}^3)$ is directly related to studying the blowup of f in $L^\infty(\mathbb{R}^3)$. The results of [33] imply that if ∇f is uniformly in time bounded in $L^2(\mathbb{R}^3)$ (with suitable bounds of L^1 moments), then f is also bounded, and therefore (1.1) is globally well-posed. On the other hand, if ∇f becomes unbounded at a certain time, at that time f itself has to become unbounded.

2. $\dot{H}^1$ estimate and the proof of Theorem 1.2

This section is devoted to the $\dot{H}^1$ estimate for the Landau equation with Coulomb potential, which leads to the monotonicity formula (1.24). We first provide a set of *a priori* estimates for the terms appearing in the equation (this is done in §§2.1–2.5). Then we show that all estimates rigorously hold by passing to the limit in an approximate problem (in §2.6), which enables us to complete the proof of Theorem 1.2.

2.1. *Decomposition of the derivative in time of the $\dot{H}^1$ norm of solutions to the Landau equation*

To make full use of the results on the long-time behavior of the solution (cf. Theorem 1.1), we write the Landau equation with Coulomb potential as follows (at the formal level), setting $h := f - \mu$, with μ defined by (1.15):

$$\partial_t h = Q(f, h) + Q(h, \mu). \tag{2.1}$$

Then we focus (at the formal level) on the $\dot{H}^1$ norm of h. We write the equation (for $k = 1, 2, 3$) satisfied by $\partial_k h$:

$$\partial_t(\partial_k h) = Q(f, \partial_k h) + Q(\partial_k f, h) + Q(\partial_k h, \mu) + Q(h, \partial_k \mu). \tag{2.2}$$

Then we multiply it by $\partial_k h$, integrate with respect to v, and sum over all k. This gives

$$\frac{1}{2}\frac{d}{dt}\|\nabla h\|_{L^2}^2 = I_1 + I_2 + I_3 + I_4, \tag{2.3}$$

where I_1, I_2, I_3 and I_4 are defined (and subdivided) as follows:

(1) $I_1 := \sum_{k=1}^3 \int_{\mathbb{R}^3} Q(f, \partial_k h)\partial_k h \, dv$. We also write

$$I_1 := -I_{1,1} + I_{1,2}, \tag{2.4}$$

where

$$I_{1,1} := \sum_{k=1}^3 \int_{\mathbb{R}^3} (a * f) : \nabla\partial_k h \otimes \nabla\partial_k h \, dv,$$
$$I_{1,2} := \sum_{k=1}^3 \int_{\mathbb{R}^3} (b * f) \cdot \nabla\partial_k h \partial_k h \, dv.$$

(2) $I_2 = \sum_{k=1}^3 \int_{\mathbb{R}^3} Q(\partial_k f, h)\partial_k h \, dv$. We also write

$$I_2 := -I_{2,1} + I_{2,2}, \tag{2.5}$$

where

$$\begin{aligned}
I_{2,1} &:= \sum_{k=1}^{3}\int_{\mathbb{R}^3}(a*\partial_k f):\nabla\partial_k h\otimes\nabla h\,dv=\sum_{k=1}^{3}\int_{\mathbb{R}^3}(\partial_k a*f):\nabla\partial_k h\otimes\nabla h\,dv,\\
I_{2,2} &:= \sum_{k=1}^{3}\int_{\mathbb{R}^3}(b*\partial_k f)\cdot\nabla\partial_k h\,h\,dv=\sum_{k=1}^{3}\int_{\mathbb{R}^3}(\partial_k b*f)\cdot\nabla\partial_k h\,h\,dv\\
&=\sum_{k=1}^{3}\sum_{i=1}^{3}\Big(-\int_{\mathbb{R}^3}(\partial_i\partial_k b_i*f)\partial_k h\,h\,dv-\int_{\mathbb{R}^3}(\partial_k b_i*f)\partial_k h\partial_i h\,dv\Big)\\
&=\sum_{k=1}^{3}\Big(-8\pi\int_{\mathbb{R}^3}f(h\partial_k^2 h+(\partial_k h)^2)\,dv+\int_{\mathbb{R}^3}(b*f)\cdot(\nabla h\partial_k^2 h+\nabla\partial_k h\partial_k h)\,dv\Big)\\
&\le\sum_{k=1}^{3}\Big(-8\pi\int_{\mathbb{R}^3}fh\partial_k^2 h\,dv+\int_{\mathbb{R}^3}(b*f)\cdot(\nabla h\partial_k^2 h+\nabla\partial_k h\partial_k h)\,dv\Big).
\end{aligned}$$

(3) $I_3 := \sum_{k=1}^3\int_{\mathbb{R}^3} Q(\partial_k h,\mu)\partial_k h\,dv$. We also write

$$I_3 := -I_{3,1} + I_{3,2}, \tag{2.6}$$

where $I_{3,1} := \sum_{k=1}^3\int_{\mathbb{R}^3}(a*\partial_k h):\nabla\partial_k h\otimes\nabla\mu\,dv$ and

$$\begin{aligned}
I_{3,2} &:= \sum_{k=1}^{3}\int_{\mathbb{R}^3}(b*\partial_k h)\cdot\nabla\partial_k h\mu\,dv\\
&=\sum_{k=1}^{3}\sum_{i=1}^{3}\Big(-\int_{\mathbb{R}^3}(\partial_i b_i*\partial_k h)\partial_k h\mu\,dv-\int_{\mathbb{R}^3}(b_i*\partial_k h)\partial_k h\partial_i\mu\,dv\Big)\\
&=\sum_{k=1}^{3}\Big(8\pi\int_{\mathbb{R}^3}\mu|\partial_k h|^2\,dv+\int_{\mathbb{R}^3}(b*h)\cdot(\partial_k h\partial_k\nabla\mu+\partial_k^2 h\nabla\mu)\,dv\Big).
\end{aligned}$$

(4) $I_4 := \sum_{k=1}^3\int_{\mathbb{R}^3} Q(h,\partial_k\mu)\partial_k h\,dv$. We also write

$$I_4 := -I_{4,1} + I_{4,2}, \tag{2.7}$$

where

$$I_{4,1} := \sum_{k=1}^{3}\int_{\mathbb{R}^3}(a*h):\nabla\partial_k h\otimes\nabla\partial_k\mu\,dv,\quad I_{4,2} := \sum_{k=1}^{3}\int_{\mathbb{R}^3}(b*h)\cdot\nabla\partial_k h\partial_k\mu\,dv.$$

In §§2.1–2.5, the *a priori* estimates are proven as if the functions considered are smooth and quickly decaying (when $|v|\to\infty$). They are used later for solutions of an approximate problem which have those properties.

2.2. *Coercivity estimate for* $I_{1,1}$

In order to treat the term $I_{1,1}$, we introduce the following classical coercivity estimates (named here Proposition 2.1 and Corollary 2.1). Their proofs can be found in [2, Prop. 2.1, p. 4], or [33, Lemmas 3.1 and 3.2], or [4, 10].

Proposition 2.1. *For all* $j \in \{1,2,3\}$, $m \in \mathbb{R}$, $f \geq 0$, $p \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^3)$, *we have*

$$\int_{\mathbb{R}^3} |\nabla p(v)|^2 \langle v\rangle^{m-3}\, dv \leq 4\|f\|_{L^1_5(\mathbb{R}^3)} A_j(f)^{-2}$$
$$\times \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |v-v_*|^{-3}\{|v-v_*|^2-(v-v_*)\otimes(v-v_*)\} : \nabla p(v)\otimes\nabla p(v) f(v_*)\langle v\rangle^m\, dv\, dv_* \tag{2.8}$$

where $A_j(f) := \int_{\mathbb{R}^3} f v_j^2\, dv$.

Corollary 2.1. *Let* $f \geq 0$ *be such that* $\int_{\mathbb{R}^3} f(v)\, dv = 1$, $\int_{\mathbb{R}^3} f(v)|v|^2\, dv = 3$, *and such that* $\|f\|_{L^1_5} + \|f\|_{L\log L} \leq K$ *for some* $K > 0$. *Set* $h = f - \mu$. *Then there exists a constant* $C(K) > 0$ *(depending only on* K*) such that for all* $h \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$ *and* $m \in \mathbb{R}$,

$$\sum_{k=1}^{3}\int_{\mathbb{R}^3} (a*f) : \nabla\partial_k h\otimes\nabla\partial_k h\langle v\rangle^m\, dv \geq C(K)\|\nabla^2 h\|^2_{L^2_{m/2-3/2}(\mathbb{R}^3)}. \tag{2.9}$$

We now state another corollary which is an easy consequence of the above two results.

Corollary 2.2. *Let* $f \geq 0$ *be such that* $\int_{\mathbb{R}^3} f(v)\, dv = 1$, $\int_{\mathbb{R}^3} f(v)|v|^2\, dv = 3$, *and such that* $\|f\|_{L^1_5} + \|f\|_{L\log L} \leq K$ *for some* $K > 0$. *Set* $h = f - \mu$. *Then there exist constants* $C(K), C^*(K) > 0$, *depending only on* K, *such that for all* $h \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$ *and* $m \in \mathbb{R}$,

$$I_{1,1} \geq C(K)\|\nabla^2 h\|^2_{L^2_{-3/2}} + C(K)\|h\|^{-4/5}_{L^1_{15/4}}\|\nabla h\|^{14/5}_{L^2} - C^*(K)\|h\|^2_{L^1}. \tag{2.10}$$

Proof. Note first that by taking $m = 0$ in Corollary 2.1, we get

$$I_{1,1} \geq C(K)\|\nabla^2 h\|^2_{L^2_{-3/2}}.$$

Then using Proposition 6.4, with $m = 0$, we see that for some constant $C > 0$,

$$\|\nabla h\|_{L^2} \leq C\|h\|^{2/7}_{L^1_{15/4}}(\|h\|_{L^1} + \|\nabla^2 h\|_{L^2_{-3/2}})^{5/7}.$$

This inequality implies that (for some constant $C^* > 0$)

$$\|\nabla^2 h\|^2_{L^2_{-3/2}} \geq C\|h\|^{-4/5}_{L^1_{15/4}}\|\nabla h\|^{14/5}_{L^2} - C^*\|h\|^2_{L^1},$$

which yields (2.10). ∎

2.3. *Estimates for the remainder terms*

2.3.1. *Estimates for I_3 and I_4.*

Proposition 2.2. *Let $f \ge 0$ and $h = f - \mu$. Then for all $\eta \in]0,1[$,*

$$I_3 + I_4 \le C\eta^{-1}\|h\|^2_{L^2_2} + C\|\nabla h\|^2_{L^2} + \frac{\eta}{4}\|\nabla^2 h\|^2_{L^2_{-3/2}} \tag{2.11}$$

for some absolute constant $C > 0$.

Proof. Indeed, for some constant $C > 0$,

$$\begin{aligned} I_3 + I_4 &\le C\int_{\mathbb{R}^3} |\nabla h|^2 \mu\, dv + C\iint_{\mathbb{R}^6} |v-v_*|^{-2}|h_*|(|\nabla^2 h| + |\nabla h|)\mu^{1/2}\, dv_*\, dv \\ &\le C\int_{\mathbb{R}^3} |\nabla h|^2\, dv + \frac{\eta}{4}\|\nabla^2 h\|^2_{L^2_{-3/2}} \\ &\quad + C\eta^{-1}\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3} |v-v_*|^{-2}|h_*|\, dv_*\right)^2 \langle v\rangle^3 \mu\, dv \\ &\quad + C\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3} |v-v_*|^{-2}|h_*|\, dv_*\right)^2 \mu\, dv. \end{aligned}$$

Then we see that

$$\begin{aligned} &\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3} |v-v_*|^{-2}|h_*|\, dv_*\right)^2 \langle v\rangle^3 \mu\, dv \\ &\le 2\int_{\mathbb{R}^3}\left(\int_{|v-v_*|\le 1} |v-v_*|^{-2}|h_*|\, dv_*\right)^2 \langle v\rangle^3 \mu\, dv \\ &\quad + 2\int_{\mathbb{R}^3}\left(\int_{|v-v_*|\ge 1} |v-v_*|^{-2}|h_*|\, dv_*\right)^2 \langle v\rangle^3 \mu\, dv \\ &\le C\int_{v_*\in\mathbb{R}^3} |h_*|^2 \int_{|v-v_*|\le 1} |v-v_*|^{-2}\langle v\rangle^3 \mu\, dv\, dv_* + 2\left(\int_{\mathbb{R}^3} |h_*|\, dv_*\right)^2 \int_{\mathbb{R}^3} \langle v\rangle^3 \mu\, dv. \end{aligned}$$

We conclude thanks to the Cauchy–Schwarz inequality. ∎

Then we observe that for some constant $C > 0$,

$$\begin{aligned} I_{1,2} + I_2 &\le C\iint_{\mathbb{R}^6} |v-v_*|^{-2} f_* |\nabla h|\,|\nabla^2 h|\, dv_*\, dv \\ &\quad + C\int f|\nabla^2 h|\,|h|\, dv, \end{aligned} \tag{2.12}$$

and we define (with C being the same as above)

$$\mathcal{I} := C\iint_{\mathbb{R}^6} |v-v_*|^{-2} f_* |\nabla h|\,|\nabla^2 h|\, dv_*\, dv, \tag{2.13}$$

$$\mathcal{I}\mathcal{I} := C\int_{\mathbb{R}^3} f|\nabla^2 h|\,|h|\, dv. \tag{2.14}$$

2.3.2. Estimate for $\mathcal{J}$. We state

Proposition 2.3. *Let $f \geq 0$ (and $h = f - \mu$) be such that $\|f\|_{L^1_2} = 4$ and $\|f\|_{L^1_\tau(\mathbb{R}^3)} + \|f\|_{L\log L} \leq K$ for some $K > 0$, $\tau > 31$. Then for all $\eta \in]0,1[$ and $A > 1$,*

$$\mathcal{J} \leq \frac{\eta}{2}\|\nabla^2 h\|^2_{L^2_{-3/2}} + \big(C\eta + C(K)\eta^{-1}(\log A)^{-\frac{\tau-31}{5\tau}}\big)(D(f)+1)\|\nabla h\|^{14/5}_{L^2} + C(\eta + \eta^{-1} + \eta^{-3}A^2)\|\nabla h\|^2_{L^2_6} + C(K)\eta^{-1}(\log A)^{-\frac{\tau-31}{5\tau}}\|\nabla h\|^2_{L^2},$$

where $C(K) > 0$ is a constant depending only on K, and $C > 0$ is an absolute constant.

Proof. We write $\mathcal{J} = \mathcal{J}_1 + \mathcal{J}_2$ with

$$\begin{aligned} \mathcal{J}_1 &:= \iint_{|v-v_*|\leq 1} |v-v_*|^{-2} f_* |\nabla h|\,|\nabla^2 h|\,dv_*\,dv, \\ \mathcal{J}_2 &:= \iint_{|v-v_*|\geq 1} |v-v_*|^{-2} f_* |\nabla h|\,|\nabla^2 h|\,dv_*\,dv. \end{aligned} \tag{2.15}$$

We see that

$$\begin{aligned} \mathcal{J}_2 &\leq C \iint \langle v-v_*\rangle^{-2} f_* |\nabla h|\,|\nabla^2 h|\,dv_*\,dv \\ &\leq C \iint \langle v\rangle^{-2}\langle v_*\rangle^2 f_* |\nabla h|\,|\nabla^2 h|\,dv_*\,dv \\ &\leq C\|f\|_{L^1_2}\|\nabla h\|_{L^2}\|\nabla^2 h\|_{L^2_{-2}} \\ &\leq \frac{\eta}{4}\|\nabla^2 h\|^2_{L^2_{-2}} + C\eta^{-1}\|\nabla h\|^2_{L^2}. \end{aligned} \tag{2.16}$$

We now turn to $\mathcal{J}_1$. We first note that in the region $\{|v-v_*|\leq 1\}$ we have the estimate $\frac{1}{\sqrt{3}}\langle v\rangle \leq \langle v_*\rangle \leq \sqrt{3}\,\langle v\rangle$. Thus by the Cauchy–Schwarz inequality, we have

$$\mathcal{J}_1 \leq C\Bigg(\underbrace{\iint_{|v-v_*|\leq 1} |v-v_*|^{-2}(f_*\langle v_*\rangle^{-3})|\nabla h|\,|\nabla^2 h\langle v\rangle^{-3/2}|\,dv_*\,dv}_{=:\mathcal{J}_{1a}}\Bigg)^{1/2} \times \Bigg(\underbrace{\iint_{|v-v_*|\leq 1} |v-v_*|^{-2}(f_*\langle v_*\rangle^{6})|\nabla h|\,|\nabla^2 h\langle v\rangle^{-3/2}|\,dv_*\,dv}_{=:\mathcal{J}_{1b}}\Bigg)^{1/2}.$$

We observe that for $k \in]0,3[$, $1_{\{|\cdot|\leq 1\}}|\cdot|^{-k} \in L^{3/k,\infty}$. Therefore, we get

$$\mathcal{J}_{1a} \leq C\|f\|_{L^{3,1}_{-3}}\|\nabla h\|_{L^2}\|\nabla^2 h\|_{L^2_{-3/2}},$$

thanks to the O'Neil inequality (Proposition 6.2 in the Appendix).

Concerning $\mathcal{J}_{1b}$ (and for any $A > 1$), we split f into two parts: $f_A = f\chi(f/A)$ and $f^A := f - f_A$, where χ is a nonnegative C^1 function with $\chi = 1$ in B_1 and $\chi = 0$ outside of B_2. Thanks to this decomposition, we get

$$\mathcal{J}_{1b} \leq C\|f^A\|_{L^{3,1}_6}\|\nabla h\|_{L^2}\|\nabla^2 h\|_{L^2_{-3/2}} + CA\|\nabla h\|_{L^2_6}\|\nabla^2 h\|_{L^2_{-3/2}}.$$

Here we again use the O'Neil inequality (Proposition 6.2) for f^A, and the bound $f_A \le CA$.

Putting together the estimates for $\mathcal{I}_{1a}$ and $\mathcal{I}_{1b}$ yields

$$\begin{aligned}\mathcal{I}_1 &\le C\big(\|f\|^{1/2}_{L^{3,1}_{-3}}\|f^A\|^{1/2}_{L^{3,1}_6}\|\nabla h\|_{L^2}\|\nabla^2 h\|_{L^2_{-3/2}} \\ &\quad + A^{1/2}\|f\|^{1/2}_{L^{3,1}_{-3}}\|\nabla h\|^{1/2}_{L^2}\|\nabla h\|^{1/2}_{L^2_6}\|\nabla^2 h\|_{L^2_{-3/2}}\big) \\ &\le \frac{\eta}{4}\|\nabla^2 h\|^2_{L^2_{-3/2}} + \eta\|f\|^2_{L^{3,1}_{-3}}\|\nabla h\|^2_{L^2} \\ &\quad + C\big(\eta^{-1}\|f\|_{L^{3,1}_{-3}}\|f^A\|_{L^{3,1}_6}\|\nabla h\|^2_{L^2} + \eta^{-3}A^2\|\nabla h\|^2_{L^2_6}\big). \end{aligned} \tag{2.17}$$

In what follows, we estimate $\|f\|^2_{L^{3,1}_{-3}}\|\nabla h\|^2_{L^2}$ and $\|f\|_{L^{3,1}_{-3}}\|f^A\|_{L^{3,1}_6}\|\nabla h\|^2_{L^2}$.

Estimate of $\|f\|^2_{L^{3,1}_{-3}}\|\nabla h\|^2_{L^2}$. Remembering that $\|f\|_{L^1_2} = 4$ and using Proposition 6.3, we get the estimate

$$\|f\|_{L^{3,1}_{-3}} \le C\|f\langle\cdot\rangle^{-3}\|^{4/5}_{H^1} \le C(\|\nabla h\|^{4/5}_{L^2} + 1), \tag{2.18}$$

where we have used the interpolation estimate

$$\|h\|_{L^2} \le C(\|h\|_{L^1} + \|\nabla h\|_{L^2}) \le C(1 + \|\nabla h\|_{L^2}). \tag{2.19}$$

This yields

$$\|f\|^2_{L^{3,1}_{-3}}\|\nabla h\|^2_{L^2} \le C(\|\nabla h\|^{4/5}_{L^2} + 1)\|f\|_{L^{3,1}_{-3}}\|\nabla h\|^2_{L^2}. \tag{2.20}$$

We end up with the bound

$$\begin{aligned}\|f\|^2_{L^{3,1}_{-3}}\|\nabla h\|^2_{L^2} &\le C(\|f\|_{L^{3,1}_{-3}}\|\nabla h\|^{14/5}_{L^2} + \|f\|_{L^{3,1}_{-3}}\|\nabla h\|^2_{L^2}) \\ &\le C(D(f)\|\nabla h\|^{14/5}_{L^2} + \|\nabla h\|^{14/5}_{L^2} + \|\nabla h\|^2_{L^2}),\end{aligned}$$

where we have used the estimates (1.23) for the first term, and (2.18) for the second.

Estimate of $\|f\|_{L^{3,1}_{-3}}\|f^A\|_{L^{3,1}_6}\|\nabla h\|^2_{L^2}$. Thanks to the definition of f^A and the interpolation estimate (2.19), we first see that

$$\|f^A\|_{H^1} \le C\sqrt{1 + A^{-2}}\,(\|\nabla h\|_{L^2} + 1). \tag{2.21}$$

For $R > 0$ and $A > 1$, we know that

$$\|f^A\|_{L^1_{31}} \le R^{31}(\log A)^{-1}\int_{f\ge 1} f\log f\,dv + R^{-(\tau-31)}\|f\|_{L^1_\tau},$$

so that

$$\|f^A\|_{L^1_{31}} \le C(K)(\log A)^{-(\tau-31)/\tau}. \tag{2.22}$$

Using Proposition 6.3, one gets

$$\|f\|_{L^{3,1}_{-3}}\|f^A\|_{L^{3,1}_6}\|\nabla h\|^2_{L^2} \le C\|f\|_{L^{3,1}_{-3}}\|f^A\|^{1/5}_{L^1_{31}}\|f^A\|^{4/5}_{H^1}\|\nabla h\|^2_{L^2}.$$

Then, by (2.21) and (2.22),

$$\|f\|_{L^{3,1}_{-3}}\|f^A\|_{L^{3,1}_6}\|\nabla h\|^2_{L^2} \le C(K)(\log A)^{-(\tau-31)/(5\tau)}\|f\|_{L^{3,1}_{-3}}(\|\nabla h\|^{14/5}_{L^2} + \|\nabla h\|^2_{L^2}). \tag{2.23}$$

Using both estimates (1.23) and (2.18), we end up with

$$\|f\|_{L^{3,1}_{-3}}\|f^A\|_{L^{3,1}_6}\|\nabla h\|^2_{L^2} \le C(K)(\log A)^{-(\tau-31)/(5\tau)}[(D(f)+1)\|\nabla h\|^{14/5}_{L^2}+\|\nabla h\|^2_{L^2}]. \tag{2.24}$$

Finally, we see that

$$\begin{aligned}
\mathcal{I}_1 &\le \frac{\eta}{4}\|\nabla^2 h\|^2_{L^2_{-3/2}} + C\eta^{-3}A^2\|\nabla h\|^2_{L^2_6} + C\eta\big(D(f)\|\nabla h\|^{14/5}_{L^2} + \|\nabla h\|^{14/5}_{L^2} + \|\nabla h\|^2_{L^2}\big)\\
&\quad + C(K)\eta^{-1}(\log A)^{-\tau-31/5\tau}[(D(f)+1)\|\nabla h\|^{14/5}_{L^2} + \|\nabla h\|^2_{L^2}]\\
&\le \frac{\eta}{4}\|\nabla^2 h\|^2_{L^2_{-3/2}} + \big(C\eta + C(K)\eta^{-1}(\log A)^{-(\tau-31)/(5\tau)}\big)(D(f)+1)\|\nabla h\|^{14/5}_{L^2}\\
&\quad + C(\eta+\eta^{-3}A^2)\|\nabla h\|^2_{L^2_6} + C(K)\eta^{-1}(\log A)^{-(\tau-31)/(5\tau)}\|\nabla h\|^2_{L^2}.
\end{aligned} \tag{2.25}$$

We deduce the desired result by combining this estimate with the estimate for $\mathcal{I}_2$. ■

2.3.3. Estimate for $\mathcal{I}\mathcal{I}$. We now prove the following bound:

Proposition 2.4. *Consider $f \ge 0$ (and $h = f - \mu$) such that $\|f\|_{L^1_2} = 4$ and $\|f\|_{L^1_\tau} + \|f\|_{L\log L} \le K$ for some $K > 0$, $\tau > 31$. Then for all $\eta \in]0,1[$ and $A > 1$,*

$$\begin{aligned}
\mathcal{I}\mathcal{I} &\le \frac{\eta}{4}\|\nabla^2 h\|^2_{L^2_{-3/2}} + C(K)\eta^{-1}(\log A)^{-(\tau-31)/(5\tau)}(D(f)+1)\|\nabla h\|^{14/5}_{L^2}\\
&\quad + C\eta^{-1}A^2\|h\|^2_{L^2_{3/2}} + C(K)\eta^{-1}(\log A)^{-(\tau-31)/(5\tau)}\|\nabla h\|^2_{L^2},
\end{aligned}$$

where $C(K) > 0$ only depends on K (and τ), and $C > 0$ only depends on τ.

Proof. We recall that

$$\mathcal{I}\mathcal{I} \le \int f\,\partial_i^2 h\,h\,dv = \int f_A\partial_i^2 h\,h\,dv + \int f^A\partial_i^2 h\,h\,dv.$$

Then

$$\begin{aligned}
\mathcal{I}\mathcal{I} &\le CA\|\nabla^2 h\|_{L^2_{-3/2}}\|h\|_{L^2_{3/2}} + C\|f^A\|_{L^3_{3/2}}\|\nabla^2 h\|_{L^2_{-3/2}}\|h\|_{L^6}\\
&\le \frac{\eta}{4}\|\nabla^2 h\|^2_{L^2_{-3/2}} + C\eta^{-1}(A^2\|h\|^2_{L^2_{3/2}} + \|f\|_{L^3_{-3}}\|f^A\|_{L^3_6}\|\nabla h\|^2_{L^2}).
\end{aligned} \tag{2.26}$$

The desired result is obtained by using an estimate almost identical to that of (2.24). ■

2.3.4. Summary of the estimates for the remainder terms. We combine the results of Propositions 2.2–2.4:

Proposition 2.5. *Let $f \geq 0$ (and $h = f - \mu$) be such that $\|f\|_{L^1_2} = 4$ and $\|f\|_{L^1_\tau} + \|f\|_{L\log L} \leq K$ with $K > 0$, $\tau > 31$. Then for all $\eta \in]0,1[$ and $A > e$,*

$$I_{1,2} + I_2 + I_3 + I_4 \leq \eta \|\nabla^2 h\|^2_{L^2_{-3/2}} + \big(C\eta + C(K)\eta^{-1}(\log A)^{-(\tau-31)/(5\tau)}\big)(D(f)+1)\|\nabla h\|^{14/5}_{L^2} + C\eta^{-1}A^2\|h\|^2_{L^2_2} + C(K)(\eta^{-1} + \eta^{-3}A^2)\|\nabla h\|^2_{L^2_6},$$

where $C(K) > 0$ only depends on K and τ, and $C > 0$ only depends on τ.

Proof. This estimate is directly obtained from Propositions 2.2–2.4, remembering that $\eta < 1$, $\log A > 1$ and $-(\tau - 31)/(5\tau) < 0$. ■

2.3.5. Summary of the estimates for all terms. We now combine the results of Proposition 2.5 and Corollary 2.2. From now on, we typically denote by C^* constants which can be replaced by a larger constant, and by C constants which can be replaced by a smaller (strictly positive) constant. We get

Proposition 2.6. *Let $f \geq 0$ (and $h = f - \mu$) with $\int_{\mathbb{R}^3} f(v)\,dv = 1$, $\int_{\mathbb{R}^3} f(v)|v|^2\,dv = 3$, and $\|f\|_{L^1_\tau} + \|f\|_{L\log L} \leq K$ with $\tau > 31$. Then for all $\eta \in]0,1[$,*

$$I_1 + I_2 + I_3 + I_4 \leq -\frac{C(K)}{2}\|\nabla^2 h\|^2_{L^2_{-3/2}} - \frac{C(K)}{2}\|h\|^{-4/5}_{L^1_{15/4}}\|\nabla h\|^{14/5}_{L^2} + C^*(K)\eta^{-13}\exp(7\eta^{-\frac{10\tau}{\tau-31}})\|h\|^2_{L^1_{99/4}} + C^*(K)\eta(1 + D(f))\|\nabla h\|^{14/5}_{L^2}, \quad (2.27)$$

where $C(K), C^*(K) > 0$ only depend on K and τ.

Proof. Using Proposition 2.5 and Corollary 2.2, we see that

$$I_1 + I_2 + I_3 + I_4 \leq -C(K)\|\nabla^2 h\|^2_{L^2_{-3/2}} - C(K)\|h\|^{-4/5}_{L^1_{15/4}}\|\nabla h\|^{14/5}_{L^2} + C^*(K)\|h\|^2_{L^1} + \eta\|\nabla^2 h\|^2_{L^2_{-3/2}} + (C^*\eta + C^*(K)\eta^{-1}(\log A)^{-(\tau-31)/(5\tau)})(D(f)+1)\|\nabla h\|^{14/5}_{L^2} + C^*\eta^{-1}A^2\|h\|^2_{L^2_2} + C^*(K)(\eta^{-1} + \eta^{-3}A^2)\|\nabla h\|^2_{L^2_6}. \quad (2.28)$$

Using Proposition 6.4 for $m = 6$, we see that

$$\|\nabla h\|_{L^2_6} + \|h\|_{L^2_6} \leq C^*\|h\|^{2/7}_{L^1_{99/4}}\|h\|^{5/7}_{L^1} + C^*\|h\|^{2/7}_{L^1_{99/4}}\|\nabla^2 h\|^{5/7}_{L^2_{-3/2}},$$

so that thanks to Young's inequality, for any $\zeta > 0$,

$$\|\nabla h\|^2_{L^2_6} + \|h\|^2_{L^2_2} \leq C^*(1 + \zeta^{-7/2})\|h\|^2_{L^1_{99/4}} + C^*\zeta^{7/5}\|\nabla^2 h\|^2_{L^2_{-3/2}}.$$

Taking $\zeta := C^*\eta^{10/7}A^{-10/7}$, we see that

$$C^*\eta^{-1}A^2\|h\|_{L^2_2}^2 \le C^*\eta^{-1}A^2(1+\eta^{-5}A^5))\|h\|_{L^1_{99/4}}^2 + \frac{\eta}{2}\|\nabla^2 h\|_{L^2_{-3/2}}^2, \tag{2.29}$$

while taking $\zeta := C^*(K)\eta^{20/7}A^{-10/7}$ (and observing that $\eta^{-1} \le \eta^{-3}A^2$), we see that

$$C^*(K)(\eta^{-1}+\eta^{-3}A^2)\|\nabla h\|_{L^2_6}^2 \le C^*(K)\eta^{-3}A^2(1+\eta^{-10}A^5)\|h\|_{L^1_{99/4}}^2 + \frac{\eta}{2}\|\nabla^2 h\|_{L^2_{-3/2}}^2. \tag{2.30}$$

Making use of this bound in estimate (2.28), we see that

$$\begin{aligned} I_1+I_2+I_3+I_4 &\le (2\eta - C(K))\|\nabla^2 h\|_{L^2_{-3/2}}^2 - C(K)\|h\|_{L^1_{15/4}}^{-4/5}\|\nabla h\|_{L^2}^{14/5} \\ &\quad + [C^*(K) + C^*\eta^{-1}A^2(1+\eta^{-5}A^5) + C^*(K)\eta^{-3}A^2(1+\eta^{-10}A^5)]\|h\|_{L^1_{99/4}}^2 \\ &\quad + (C^*\eta + C^*(K)\eta^{-1}(\log A)^{-(\tau-31)/(5\tau)})(D(f)+1)\|\nabla h\|_{L^2}^{14/5}, \end{aligned} \tag{2.31}$$

so that when $\eta < C(K)/4$,

$$\begin{aligned} I_1+I_2+I_3+I_4 &\le -\frac{C(K)}{2}\|\nabla^2 h\|_{L^2_{-3/2}}^2 - C(K)\|h\|_{L^1_{15/4}}^{-4/5}\|\nabla h\|_{L^2}^{14/5} \\ &\quad + C^*(K)\eta^{-13}A^7\|h\|_{L^1_{99/4}}^2 \\ &\quad + (C^*\eta + C^*(K)\eta^{-1}(\log A)^{-(\tau-31)/(5\tau)})(D(f)+1)\|\nabla h\|_{L^2}^{14/5}. \end{aligned} \tag{2.32}$$

We now select $A > e$ such that $(\log A)^{-(\tau-31)/(10\tau)} = \eta$, and get (changing the names of the constants) estimate (2.27). ∎

2.4. Application of the estimates to solutions of the Landau equation

Lemma 2.1. *Let $\ell > 19/2$, $f_0 \in L^1_\ell \cap L\log L$ be a nonnegative function such that $\int_{\mathbb{R}^3} f_0(v)\,dv = 1$, $\int_{\mathbb{R}^3} f_0(v)|v|^2\,dv = 3$. Let $f := f(t,v)$ a weak (well-constructed) nonnegative solution to the Landau equation with Coulomb potential* (1.1)–(1.3)*, and $h = f - \mu$. Then for all $\theta \in [0,\ell]$ and $q < q_{\ell,\theta}$ with*

$$q_{\ell,\theta} := -\frac{2\ell^2 - 25\ell + 57}{18(l-2)}\left(1 - \frac{\theta}{\ell}\right) + \frac{\theta}{\ell}, \tag{2.33}$$

there exists $C > 0$ (depending on θ, ℓ and K such that $\|f_0\|_{L^1_l} + \|f_0\|_{L\log L} \le K$) such that

$$\forall t \ge 0, \quad \|h(t,\cdot)\|_{L^1_\theta} \le C(1+t)^q. \tag{2.34}$$

More specifically, if $\ell \ge 55$, then for some $r_1 > 7/4$, $r_2 > 0$,

$$\forall t \ge 0, \quad \|h(t,\cdot)\|_{L^1_{99/4}} \le C(1+t)^{-r_1}, \quad \|h(t,\cdot)\|_{L^1_{45}} \le C(1+t)^{-r_2}. \tag{2.35}$$

Proof. We first recall that thanks to Theorem 1.1, for $\beta < \frac{2\ell^2-25\ell+57}{9(\ell-2)}$, the relative entropy decays according to the inequality

$$\forall t > 0, \quad H(t) \le C_\beta (1+t)^{-\beta},$$

where $C_\beta > 0$ only depends on ℓ and K such that $\|f_0\|_{L^1_\ell} + \|f_0\|_{L\log L} \le K$.

Using the Cziszár–Kullback–Pinsker inequality (cf. [8, 27]), we see that

$$\forall t > 0, \quad \|h(t,\cdot)\|_{L^1} \le C_\beta (1+t)^{-\beta/2}.$$

Then (using Theorem 1.1 again) for all $\ell > 2$, there exists $C_\ell > 0$ (which only depends on ℓ and K such that $\|f_0\|_{L^1_2} + \|f_0\|_{L\log L} \le K$) such that

$$\forall t > 0, \quad \|h(t,\cdot)\|_{L^1_\ell} \le C_\ell (1+t).$$

Finally, we interpolate between the previous two inequalities, for $\theta \in [0,\ell]$:

$$\|h\|_{L^1_\theta} \le \|h\|_{L^1}^{1-\theta/\ell} \|h\|_{L^1_\ell}^{\theta/\ell} \le C(1+t)^q \tag{2.36}$$

for $q < q_{\theta,\ell}$, and $C > 0$ as described in the lemma.

The special case (when $\theta = 99/4$, or $\theta = 45$) is directly obtained thanks to this estimate. ∎

We now write the H^1 estimate that will yield the differential inequality (1.24).

Proposition 2.7. *Let $f_0 \in L^1_{55}(\mathbb{R}^3) \cap L\log L(\mathbb{R}^3)$ be a nonnegative function such that $\int_{\mathbb{R}^3} f_0(v)\,dv = 1$ and $\int_{\mathbb{R}^3} f_0(v)|v|^2\,dv = 3$. Let $f := f(t,v)$ be a nonnegative smooth and quickly decaying (when $|v| \to \infty$) $C^2_t(\mathcal{S})$ solution (on a time interval $[0,T]$) to the Landau equation (1.1)–(1.3), and $h = f - \mu$. Then for some $k_1 > 0$, $k_2 > 7/2$, $C_1, C_2, C_3 > 0$ (depending only on K such that $\|f_0\|_{L^1_{55}} + \|f_0\|_{L\log L} \le K$), the following differential inequality holds (on $[0,T]$) for all $\eta \in]0,1[$ sufficiently small (depending on K):*

$$\frac{d}{dt}\|\nabla h\|_{L^2}^2 + C_1(1+t)^{k_1}\|\nabla h\|_{L^2}^{14/5} \le \eta C_3 D(f)\|\nabla h\|_{L^2}^{14/5} + C_2\eta^{-13}\exp\{7\eta^{-450/14}\}(1+t)^{-k_2}. \tag{2.37}$$

Proof. We consider a smooth and quickly decaying (when $|v| \to \infty$) solution $f := f(t,v) \ge 0$ to (1.1)–(1.3) (on a given time interval $[0,T]$). According to Lemma 2.1 (more precisely to the special case described in this lemma), this solution is bounded in $L^1_{45}(\mathbb{R}^3)$ (with a bound controlled by K such that $\|f_0\|_{L^1_{55}} + \|f_0\|_{L\log L} \le K$).

Recalling the computation (2.3), which rigorously holds, we can use Proposition 2.6 with $\tau = 45$ (for a smooth solution to (1.1)–(1.3)), and we see that (for some $C, C_4, C_5 > 0$ depending only on K such that $\|f_0\|_{L^1_{55}} + \|f_0\|_{L\log L} \le K$),

$$\frac{d}{dt}\|\nabla h\|_{L^2}^2 + C\|h\|_{L^1_{15/4}}^{-4/5}\|\nabla h\|_{L^2}^{14/5} \le C_4\eta^{-13}\exp(7\eta^{-450/14})\|h\|_{L^1_{99/4}}^2 + C_5\eta(1+D(f))\|\nabla h\|_{L^2}^{14/5}. \tag{2.38}$$

Using again the special case described at the end of Lemma 2.1 (and observing that $\|h\|_{L^1_{15/4}} \le \|h\|_{L^1_{99/4}}$), we complete the proof of the differential inequality (2.37). ■

Thanks to Proposition 2.7, we can now reduce the main results in Theorem 1.2 to the analysis of some ordinary differential inequality.

2.5. *Analysis of a differential inequality*

We start with the following lemma, which corresponds to the special case $\eta = C_3^{-1}$, $B^* := \mathcal{B}(\eta)$, $\mathcal{B}(x) := C_2 x^{-13} \exp\{7x^{-450/14}\}$ of Proposition 2.7.

Lemma 2.2. *Let X, H be C^1 functions from $[0, T]$ to $\mathbb{R}_+$ (for $T \in]0, +\infty]$), $C_1, B^*, k_1 > 0$, $k_2 > 7/2$, and $D := -H'$ such that*

$$\forall t \in [0, T], \quad \frac{d}{dt} X(t)^2 + C_1 (1+t)^{k_1} X(t)^{14/5} \le D(t) X(t)^{14/5} + B^* (1+t)^{-k_2}. \tag{2.39}$$

Then for $k := \min\{\frac{2k_2-7}{5}, k_1\}$ and some constant $C_6 > 0$ depending only on C_1, B^*, k_2,

$$\forall t \in [0, T], \quad \frac{d}{dt}\left(H(t) - \tfrac{5}{2}[X(t)^2 + B^*(1+t)^{1-k_2}]^{-2/5}\right) + C_6 (1+t)^k \le 0. \tag{2.40}$$

Proof. We first observe that denoting $Y(t) := B^*(1+t)^{-k_2+1}$ and $c_1 := (B^*)^{-2/5}(k_2-2)$, we have

$$\forall t \in [0, T], \quad \frac{d}{dt} Y(t) + c_1 (1+t)^{2k_2 - 7/5} Y(t)^{7/5} \le -B^*(1+t)^{-k_2}.$$

Therefore for some $C_6 > 0$ depending only on C_1, B^*, k_2,

$$\begin{aligned} \forall t \in [0, T], \quad \frac{d}{dt}[X(t)^2 + B^*(1+t)^{1-k_2}] + C_6 (1+t)^k [X(t)^2 + B^*(1+t)^{1-k_2}]^{7/5} \\ \le D(t)[X(t)^2 + B^*(1+t)^{1-k_2}]^{7/5}. \end{aligned}$$

The differential inequality stated in the lemma is then obtained by dividing this differential inequality by $[X(t)^2 + B^*(1+t)^{1-k_2}]^{7/5}$. ■

Next we turn to the following consequence of Lemma 2.2:

Lemma 2.3. *Let X, H be C^1 functions from $[0, T]$ to $\mathbb{R}_+$ (for $T \in]0, +\infty]$), $C_1, B^*, k_1 > 0$, $k_2 > 7/2$, and $D := -H'$, such that the differential inequality (2.39) holds.*

- *If $H(0)[X(0)^2 + B^*]^{2/5} \le 5/2$, then for some constant $C_6 > 0$ depending only on C_1, B^*, k_2,*

$$\forall t \in [0, T], \quad X(t) \le (2/5)^{-5/4} \left(H(t) + \frac{C_6}{k+1}[(1+t)^{1+k} - 1] \right)^{-5/4};$$

- *If* $H(0)[X(0)^2 + B^*]^{2/5} > 5/2$, *then for*

$$T^* := \left(\frac{1+k}{C_6}\left[H(0) - \frac{5}{2}[X(0)^2 + B^*]^{-2/5}\right] + 1\right)^{1/k+1} - 1,$$

one has (for $T > T^*$*)* $H(T^*) \le \frac{5}{2}[X(T^*)^2 + B^*(1+T^*)^{1-k_2}]^{-2/5}$ *and for* $t \in [0, T - T^*]$,

$$X(T^* + t) \le (2/5)^{-5/4}\left(H(T^*) + \frac{C_6}{k+1}[(1 + T^* + t)^{1+k} - (1+T^*)^{1+k}]\right)^{-5/4}.$$

Proof. By integrating both sides of (2.40) on $[t_1, t_2], T > t_2 > t_1 \ge 0$, we see that

$$H(t_2) - \tfrac{5}{2}[X(t_2)^2 + B^*(1+t_2)^{1-k_2}]^{-2/5} + \frac{C_6}{1+k}[(1+t_2)^{1+k} - (1+t_1)^{1+k}] \le H(t_1) - \tfrac{5}{2}[X(t_1)^2 + B^*(1+t_1)^{1-k_2}]^{-2/5}. \tag{2.41}$$

Taking $t_1 = 0, t_2 = t$, and using the condition $\frac{5}{2}[X(0)^2 + B^*]^{-2/5} \ge H(0)$, we rewrite the above inequality as

$$\tfrac{5}{2}[X(t)^2 + B^*(1+t)^{1-k_2}]^{-2/5} \ge H(t) + \frac{C_6}{1+k}[(1+t)^{1+k} - 1].$$

From this, we get

$$X(t) \le \left[(2/5)^{-5/2}\left(H(t) + \frac{C_6}{k+1}[(1+t)^{1+k} - 1]\right)^{-5/2} - B^*(1+t)^{1-k_2}\right]^{1/2},$$

which proves the first result.

The second result follows from estimate (2.41) by taking $t_1 = 0$ and $t_2 = T^*$, and solving (for T^*) the equation

$$\frac{C_6}{1+k}[(1+T^*)^{1+k} - 1] = H(0) - \tfrac{5}{2}[X(0)^2 + B^*]^{-2/5}.$$

Now let $t_1 = T^*$ and $t_2 = T^* + t, t > 0$. Then $H(T^*) - \frac{5}{2}[X(T^*)^2 + B^*(1+T^*)^{1-k_2}]^{-2/5} \le 0$ implies that

$$X(T^* + t) \le (2/5)^{-5/4}\left(H(T^*) + \frac{C_6}{k+1}[(1 + T^* + t)^{1+k} - (1+T^*)^{1+k}]\right)^{-5/4},$$

which gives the estimate for X after time T^* described in the lemma. ∎

2.6. *End of the proof of Theorem 1.2*

We are now in a position to prove Theorem 1.2. We show that the *a priori* estimates obtained in §§2.1–2.5 can be used to build a solution to (1.1)–(1.3), thanks to their application to smooth solutions of an approximate equation.

We introduce therefore the unique solution $f^\epsilon := f^\epsilon(t,v) \ge 0$ to the approximate equation

$$\partial_t f^\epsilon = Q^\epsilon(f^\epsilon, f^\epsilon), \tag{2.42}$$

where Q^ϵ is defined by

$$Q^\epsilon(g,h) = \nabla_v \cdot ([a^\epsilon * g]\nabla_v h - [a^\epsilon * \nabla g]h)$$

with

$$a^\epsilon(z) = (|z|^2 + \epsilon^2)^{-1/2}\left(\mathrm{Id} - \frac{z \otimes z}{|z|^2}\right). \tag{2.43}$$

Thus we are still considering a Landau equation, but with a regularized cross section. We also introduce smooth and quickly decaying (when $|v| \to \infty$) initial data, converging when $\epsilon \to 0$ towards the original initial data f_0. The problem (2.42)–(2.43) satisfies the same conservation properties (propagation of nonnegativity, conservation of mass, momentum and kinetic energy, decay of entropy) as the original equation (1.1)–(1.3).

Next we briefly explain how to prove

Proposition 2.8. *For $\epsilon > 0$, estimates (1.24), (1.25) and (1.27)–(1.28) hold for the unique smooth ($C^2_t(\mathcal{S})$) solution of (2.42)–(2.43) (with smoothed initial data), with constants which do not depend on ϵ.*

Proof. Step 1. Since (for $\epsilon > 0$) there is no singularity in a^ϵ, equation (2.42)–(2.43) behaves (from the point of view of regularity) like the Landau equation with Maxwell molecules (that is, when $\gamma = 0$ in (1.31)). Hence, smoothness and moments can be proved to be propagated globally for this equation. This is easily checked by following the strategy used in [19,20]. Thus equation (2.42)–(2.43) admits a unique (global) smooth solution (the initial data being themselves smooth).

Step 2. Using [9, Theorem 3], we see that estimates (1.22) and (1.23) hold when a is replaced by a^ϵ, with a constant that does not depend on ϵ. It is then possible to show, using the same method as in [4], that the long-time behavior estimates are the same for the solution to (2.42)–(2.43) as those for the solution to the Landau equation with Coulomb potential (1.1)–(1.3). In other words, Theorem 1.1 holds for the unique smooth solutions to (2.42)–(2.43), with constants which do not depend on ϵ.

Step 3. We show that Proposition 2.7 holds for the unique smooth solutions of (2.42)–(2.43), with constants in the estimate which do not depend on ϵ.

This amounts to showing that the estimates in the proof still hold when a is replaced by a^ϵ. Noticing that $b^\epsilon_i(z) := \sum_{j=1}^3 \partial_j a^\epsilon_{ij}(z) = -2z_i|z|^{-2}(|z|^2+\epsilon^2)^{-1/2}$, $\sum_{i=1}^3 \partial_i b^\epsilon_i(z) = -2\epsilon^{-3}|\epsilon^{-1}z|^{-2}(|\epsilon^{-1}z|^2+1)^{-3/2}$, we see that $|a^\epsilon| \le |a|$, $|b^\epsilon_i| \le |b_i|$, and those inequalities can be used to show that the estimates from above in §§2.3, 2.4 can be reproduced with the same constants for the approximate problem as for the original problem.

We can then directly check by inspecting the proofs that the coercivity estimate appearing in Proposition 2.1 and Corollary 2.1 can be reproduced when a is replaced by a^ϵ, with constants that do not depend on ϵ.

Since for $\epsilon > 0$, the solution f^ϵ is smooth and quickly decaying when $|v| \to \infty$, the assumptions of Proposition 2.7 are fulfilled, so that estimate (2.37) holds (for this solution), with constants in the estimate which do not depend on ϵ.

Step 4. We can now apply Lemmas 2.2 and 2.3 to $X = \|\nabla h^\epsilon\|_{L^2}$, and obtain the estimates of Theorem 1.2 for the unique smooth solution of (2.42)–(2.43), with constants in the estimate which do not depend on ϵ. ∎

End of the proof of Theorem 1.2. Note first that Theorem 1.2 (i) is immediately obtained (without using the approximate problem) by the use of Proposition 2.7 and Lemma 2.2.

We now turn to parts (ii) and (iii). As in [9], we let f^ϵ be the unique smooth solution of (2.42)–(2.43) with initial data strongly converging to f^0. It is then possible to pass to the limit (in a weighted weak L^1 space, and up to extracting a subsequence) when $\varepsilon \to 0$ in f^ϵ, and get in this way a (well-constructed) weak solution f to the original equation (1.1)–(1.3) with initial data f^0.

Due to the convexity of $x \mapsto x \log x$ and the lower semicontinuity of weak convergence in $\dot{H}^1$, we obtain

$$H(t) \le \liminf_{\epsilon \to 0} H^\epsilon(t), \quad \|\nabla h\|_{L^2} \le \liminf_{\epsilon \to 0} \|\nabla h^\epsilon\|_{L^2},$$

where $H^\epsilon(t)$ is the relative entropy of f^ϵ at time t.

Thanks to these properties, we can pass to the limit in the following estimates:

- for the initial data under the threshold,

$$\forall t \ge 0, \quad \|h^\epsilon(t)\|_{\dot{H}^1} \le (2/5)^{-5/4} \left(H^\epsilon(t) + \frac{C_6}{k+1}[(1+t)^{1+k} - 1] \right)^{-5/4};$$

- for general suitable initial data and $t > T^*$,

$$H^\epsilon(t) \le \tfrac{5}{2}[\|h^\epsilon(t)\|^2_{\dot{H}^1} + B^*(1+t)^{1-k_2}]^{-2/5},$$
$$\|h^\epsilon(t)\|_{\dot{H}^1} \le (2/5)^{-5/4} \left(\frac{C_6}{k+1}[(1+t)^{1+k} - (1+T^*)^{1+k}] \right)^{-5/4}.$$

We thus conclude the proof of Theorem 1.2 (ii, iii).

∎

3. Local solutions: proof of Proposition 1.1

We start the proof of Proposition 1.1 with the following proposition, which is a variant of Proposition 2.6:

Proposition 3.1. *Let $f \geq 0$ (and $h = f - \mu$) be such that $\|f\|_{L^1_2(\mathbb{R}^3)} = 4$ and $\|f\|_{L^1_\tau} + \|f\|_{L\log L} \leq K$ with $K > 0$ and $\tau > 31$. Then*

$$I_1 + I_2 + I_3 + I_4 \leq -\frac{C(K)}{4}\|\nabla^2 h\|^2_{L^2_{-3/2}} - \frac{C(K)}{2}\|h\|^{-4/5}_{L^1_{15/4}}\|\nabla h\|^{14/5}_{L^2} - \frac{1}{2}I_{1,1} + C^*(K)\|h\|^2_{L^1_{99/4}} + \|\nabla h\|^{18/5}_{L^2}, \tag{3.1}$$

where $C^*(K), C(K) > 0$ only depend on K and τ.

Proof. Using estimates (2.18) and (2.20), we see that

$$\|f\|^2_{L^{3,1}_{-3}}\|\nabla h\|^2_{L^2} \leq C(\|\nabla h\|^2_{L^2} + \|\nabla h\|^{18/5}_{L^2}).$$

Then, recalling estimates (2.18) and (2.23), we also see that (for $A > e$)

$$\|f\|_{L^{3,1}_{-3}}\|f^A\|_{L^{3,1}_6}\|\nabla h\|^2_{L^2} \leq C(K)(\log A)^{-(\tau-31)/(5\tau)}(\|\nabla h\|^2_{L^2} + \|\nabla h\|^{18/5}_{L^2}).$$

Using the notation (2.13) and bounds (2.16) and (2.17), this leads to the bound (for all $\eta \in]0,1[$ and $A > e$)

$$\mathcal{J} \leq \frac{\eta}{2}\|\nabla^2 h\|^2_{L^2_{-3/2}} + \big(C\eta + C(K)\eta^{-1}(\log A)^{-(\tau-31)/(5\tau)}\big)\|\nabla h\|^{18/5}_{L^2} + C(\eta + \eta^{-1} + \eta^{-3}A^2)\|\nabla h\|^2_{L^2_6} + C(K)\eta^{-1}(\log A)^{-(\tau-31)/(5\tau)}\|\nabla h\|^2_{L^2}.$$

Using the notation (2.14) and estimate (2.26), we also get the estimate (for all $\eta \in]0,1[$ and $A > e$)

$$\mathcal{J}\mathcal{J} \leq \frac{\eta}{4}\|\nabla^2 h\|^2_{L^2_{-3/2}} + C(K)\eta^{-1}(\log A)^{-(\tau-31)/(5\tau)}\|\nabla h\|^{18/5}_{L^2} + C\eta^{-1}A^2\|h\|^2_{L^2_{3/2}} + C(K)\eta^{-1}(\log A)^{-(\tau-31)/(5\tau)}\|\nabla h\|^2_{L^2},$$

where $C(K) > 0$ only depends on K and τ, and $C > 0$ only depends on τ.

Recalling now estimates (2.11) and inequality (2.12) (together with notations (2.13) and (2.14)), and remembering that $\eta < 1$, $\log A > 1$ and $-(\tau - 31)/(5\tau) < 0$, we end up with the estimate

$$I_{1,2} + I_2 + I_3 + I_4 \leq \eta\|\nabla^2 h\|^2_{L^2_{-3/2}} + (C\eta + C(K)\eta^{-1}(\log A)^{-(\tau-31)/(5\tau)})\|\nabla h\|^{18/5}_{L^2} + C\eta^{-1}A^2\|h\|^2_{L^2_2} + C(K)(\eta^{-1} + \eta^{-3}A^2)\|\nabla h\|^2_{L^2_6}, \tag{3.2}$$

where $C(K) > 0$ only depend on K (and τ) and $C > 0$ only depends on τ.

Using estimates (2.10), (2.28) and (2.38), we see that (using C^* for constants which can be replaced by larger constants, and C for constants which can be replaced by smaller constants)

$$I_1 + I_2 + I_3 + I_4 \leq \big(2\eta - \tfrac{1}{2}C(K)\big)\|\nabla^2 h\|^2_{L^2_{-3/2}} - \tfrac{1}{2}C(K)\|h\|^{-4/5}_{L^1_{15/4}}\|\nabla h\|^{14/5}_{L^2} - \tfrac{1}{2}I_{1,1} + \big[\tfrac{1}{2}C^*(K) + C^*\eta^{-1}A^2(1 + \eta^{-5}A^5)) + C^*(K)\eta^{-3}A^2(1 + \eta^{-10}A^5)\big]\|h\|^2_{L^1_{99/4}} + (C^*\eta + C^*(K)\eta^{-1}(\log A)^{-(\tau-31)/(5\tau)})\|\nabla h\|^{18/5}_{L^2}, \tag{3.3}$$

so that when $\eta < C(K)/8$,

$$I_1 + I_2 + I_3 + I_4 \le -\frac{C(K)}{4}\|\nabla^2 h\|^2_{L^2_{-3/2}} - \frac{C(K)}{2}\|h\|^{-4/5}_{L^1_{15/4}}\|\nabla h\|^{14/5}_{L^2} - \tfrac{1}{2}I_{1,1}$$
$$+ C^*(K)\eta^{-13}A^7\|h\|^2_{L^1_{99/4}} + \big(C^*\eta + C^*(K)\eta^{-1}(\log A)^{-(\tau-31)/(5\tau)}\big)\|\nabla h\|^{18/5}_{L^2}. \tag{3.4}$$

Selecting $\eta > 0$ sufficiently small, and $A > e$ such that $(\log A)^{-(\tau-31)/(10\tau)} = \eta$, we see that estimate (3.1) holds. ∎

End of the proof of Proposition 1.1. We observe that inequality (3.1) still holds when the kernel of the Landau equation is replaced by the kernel of the approximate equation (2.42)–(2.43), with all constants independent of ϵ. Then, when f^ϵ (and $h^\epsilon = f^\epsilon - \mu$) is the unique smooth solution of (2.42)–(2.43) (with regularized initial data), and proceeding as in the proof of Proposition 2.7, we get the estimate

$$\frac{d}{dt}\|\nabla h^\epsilon\|^2_{L^2} + C_{12}\|\nabla^2 h^\epsilon\|^2_{L^2_{-3/2}} + \tfrac14 I^\epsilon_{1,1} + \tfrac14 C_{10}(1+t)^{k_1}\|\nabla h^\epsilon\|^{14/5}_{L^2}$$
$$\le \|\nabla h^\epsilon\|^{18/5}_{L^2} + C_{11}(1+t)^{-k_2}, \tag{3.5}$$

where $k_1 > 0$ and $k_2 > 7/2$ are defined as in Proposition 2.7, and $C_{10}, C_{11}, C_{12} > 0$ only depend on K such that $\|f_0\|_{L^1_{55}} + \|f_0\|_{L\log L} \le K$.

This differential inequality implies that

$$\frac{d}{dt}\|\nabla h^\epsilon\|^2_{L^2} \le \|\nabla h^\epsilon\|^{18/5}_{L^2} + C_{11},$$

so that

$$\frac{d}{dt}(\|\nabla h^\epsilon\|^2_{L^2} + C_{11}^{5/9}) \le (\|\nabla h^\epsilon\|^2_{L^2} + C_{11}^{5/9})^{9/5}.$$

Therefore, for $t \le \mathcal{T} := \frac54(\|\nabla h^\epsilon(0)\|^2_{L^2} + 2C_{11}^{5/9})^{-4/5}$,

$$\|\nabla h^\epsilon(t)\|^2_{L^2} \le \big[(\|\nabla h^\epsilon(0)\|^2_{L^2} + C_{11}^{5/9})^{-4/5} - \tfrac45 t\big]^{-5/4} - C_{11}^{5/9}. \tag{3.6}$$

Passing to the limit when $\epsilon \to 0$ as at the end of the proof of Theorem 1.2, we get the existence of a weak solution of the Landau equation (1.1)–(1.3) on $[0, \mathcal{T}]$ which is in fact strong in the sense that it lies in $L^\infty([0,\mathcal{T}]; H^1(\mathbb{R}^3))$. Note indeed that the first time of blowup (in H^1 norm) is *strictly* greater than $\mathcal{T}$ since part of the dissipative terms were not used in the differental inequality in order to get the bound (3.6).

We now focus on the regularity of the solution obtained, and the consequences concerning the issue of uniqueness. Using Theorem 1.1, we see that on the time interval $[0,\mathcal{T}]$, one has $h \in L^\infty_t(L^1_{55})$. Then the estimates (3.6) and (3.5) imply that $\nabla h \in L^\infty_t(L^2)$ and $\nabla^2 h \in L^2_t(L^2_{-3/2})$. Thanks to a Sobolev embedding, we see that $h \in L^\infty_t(L^6)$. Interpolating with the estimate stating that $h \in L^\infty_t(L^1_{55})$, we see that $h \in L^\infty_t(L^2_{22})$. Interpolating again this estimate with the statement $\nabla^2 h \in L^2_t(L^2_{-3/2})$, we see that $h \in L^{16/7}_t(H^{7/4}_{23/16})$.

Thanks to yet another Sobolev embedding, we find that $h \in L^2_t(L^\infty)$, which is sufficient to apply the stability result in [14], and get the uniqueness of the strong solution built above on the relevant time interval.

We finally prove that $f \in C([0,\mathcal{T}];\dot{H}^1)$. Using estimate (3.5), we see that $\nabla^2 h \in L^2([0,\mathcal{T}];L^2_{-3/2})$ and $I_{1,1} \in L^1([0,\mathcal{T}])$. Recalling identities (2.3), (2.4) and estimate (3.2), we observe that $\frac{d}{dt}\|\nabla h\|^2_{L^2} \in L^1([0,\mathcal{T}])$, so that $t \mapsto \|\nabla h(t)\|^2_{L^2}$ is continuous on $[0,\mathcal{T}]$.

Remembering the weak formulation (1.6) and the fact $\nabla h \in L^\infty_t(L^2)$, it is not difficult to check that $t \mapsto \int_{\mathbb{R}^3} \partial_i h(t,v)\phi(v)\,dv$ is continuous on $[0,\mathcal{T}]$ for any smooth and compactly supported function ϕ. We conclude that $h \in C([0,\mathcal{T}];\dot{H}^1)$ by combining the above facts. Indeed, thanks to the continuity of $t \mapsto \|\nabla h(t)\|^2_{L^2}$, we know that

$$\lim_{s\to t} \|\nabla(h(t)-h(s))\|^2_{L^2} = 2\|\nabla h(t)\|^2_{L^2} - 2\lim_{s\to t}(\nabla h(s),\nabla h(t)).$$

We conclude by approximating $\nabla h(t)$ in L^2 by a sequence $\phi_n \in C_c^\infty$. Note finally that the formula appearing in the definition of $\mathcal{T}$ in Proposition 1.1 is obtained by defining $C_7 := \frac{1}{2}C_{11}^{-5/9}$. ∎

4. Weighted H^1 estimates and proof of Proposition 1.2

The main goal of this section is to get estimates for weighted H^1 norms of solutions to (1.1)–(1.3), and then to use them in order to prove Proposition 1.2.

4.1. *Weighted $\dot{H}^1$ estimate*

Multiplying the equation for the derivatives of the Landau equation (1.1)–(1.3), that is (remembering that $h = f - \mu$ and that μ is the normalized Maxwellian given by (1.15)),

$$\partial_t(\partial_k h) = Q(f,\partial_k h) + Q(\partial_k f, h) + Q(\partial_k h,\mu) + Q(h,\partial_k \mu), \tag{4.1}$$

by $\langle v\rangle^m \partial_k h$, integrating with respect to v and summing for $k = 1,2,3$, we obtain (at the formal level)

$$\frac{1}{2}\frac{d}{dt}\|\nabla h\|^2_{L^2_{m/2}} = W_1 + W_2 + W_3 + W_4, \tag{4.2}$$

where W_1, W_2, W_3 and W_4 correspond to the terms of the right-hand side of (4.1).

We start our study by estimating the most significant terms, that is, W_1 and W_2.

4.1.1. Estimate for W_1 and W_2. The following proposition enables us to treat a large part of terms coming from W_1 and W_2:

Proposition 4.1. *Let f be a nonnegative function with normalization (1.14), and $h = f - \mu$. Then, for all $m \geq 0$ and some (absolute) constant $C > 0$,*

$$\iint [|v-v_*|^{-1} + |v-v_*|^{-2}] f(v_*)|\nabla h(v)|^2\langle v\rangle^m\,dv_*\,dv \leq C(1+\|\nabla h\|_{L^2})\|\nabla h\|^2_{L^2_{m/2}} \tag{4.3}$$

and

$$\iint |v-v_*|^{-2}|\nabla f(v_*)|\,|h(v)|\,|\nabla h(v)|\langle v\rangle^m\,dv_*\,dv \le C(1+\|\nabla h\|_{L^2}^2)\|h\|_{H^1_{m/2}}\|\nabla h\|_{L^2_{m/2}}. \quad (4.4)$$

Proof. For (4.3), we bound the integral over $|v-v_*|\le 1$ in the following way:

$$\begin{aligned}\iint_{|v-v_*|\le 1} &[|v-v_*|^{-1}+|v-v_*|^{-2}]f(v_*)|\nabla h(v)|^2\langle v\rangle^m\,dv_*\,dv\\ &\le \left\|(|\cdot|^{-1}_{|\cdot|\le 1}+|\cdot|^{-2}_{|\cdot|\le 1})*f\right\|_{L^\infty}\|\nabla h\|^2_{L^2_{m/2}}\\ &\le \left\||\cdot|^{-1}_{|\cdot|\le 1}+|\cdot|^{-2}_{|\cdot|\le 1}\right\|_{L^{6/5}}\|f\|_{L^6}\|\nabla h\|^2_{L^2_{m/2}}\\ &\le C\|\nabla f\|_{L^2}\|\nabla h\|^2_{L^2_{m/2}}\le C(1+\|\nabla h\|_{L^2})\|\nabla h\|^2_{L^2_{m/2}}. \quad (4.5)\end{aligned}$$

The integral over $|v-v_*|\ge 1$ satisfies

$$\iint_{|v-v_*|\ge 1}[|v-v_*|^{-1}+|v-v_*|^{-2}]f(v_*)|\nabla h(v)|^2\langle v\rangle^m\,dv_*\,dv\le C\|f\|_{L^1}\|\nabla h\|^2_{L^2_{m/2}}. \quad (4.6)$$

Now, estimate (4.3) is a consequence of (4.5) and (4.6).

For (4.4), using $\frac{1}{\sqrt{3}}\langle v\rangle\le\langle v_*\rangle\le\sqrt{3}\,\langle v\rangle$ when $|v-v_*|\le 1$, we see that the integral over $|v-v_*|\le 1$ is bounded by

$$\begin{aligned}\iint_{|v-v_*|\le 1}&|v-v_*|^{-2}|\nabla f(v_*)|\,|h(v)|\,|\nabla h(v)|\langle v\rangle^m\,dv_*\,dv\\ &\le \left\||\cdot|^{-2}_{|\cdot|\le 1}*|\langle\cdot\rangle^{m/2}\nabla f|\right\|_{L^3}\left\||h|\,|\nabla h|\langle\cdot\rangle^{m/2}\right\|_{L^{3/2}}\\ &\le\|\nabla f\|_{L^2_{m/2}}\left\||\cdot|^{-2}_{|\cdot|<1}\right\|_{L^{6/5}}\|h\|_{L^6}\|\nabla h\|_{L^2_{m/2}}\\ &\le C(1+\|\nabla h\|_{L^2_{m/2}})\|\nabla h\|_{L^2}\|\nabla h\|_{L^2_{m/2}}. \quad (4.7)\end{aligned}$$

Since $|\cdot|^{-2}_{|\cdot|\ge 1}$ lies in L^2, the integral over $|v-v_*|\ge 1$ is bounded in the following way:

$$\begin{aligned}\iint_{|v-v_*|\ge 1}&|v-v_*|^{-2}|\nabla f(v_*)|\,|h(v)|\,|\nabla h(v)|\langle v\rangle^m\,dv_*\,dv\\ &\le C\|\nabla f\|_{L^2}\|h\|_{L^2_{m/2}}\|\nabla h\|_{L^2_{m/2}}\\ &\le C\|h\|_{L^2_{m/2}}\|\nabla h\|_{L^2_{m/2}}+C\|\nabla h\|_{L^2}\|h\|_{L^2_{m/2}}\|\nabla h\|_{L^2_{m/2}}. \quad (4.8)\end{aligned}$$

We get (4.4) by collecting the bounds (4.7) and (4.8). ∎

Next we estimate the terms W_1 and W_2. We start with

Proposition 4.2. *Let $f \ge 0$ be such that $\int_{\mathbb{R}^3} f(v)\,dv = 1$, $\int_{\mathbb{R}^3} f(v)|v|^2\,dv = 3$, and $\|f\|_{L^1_{15/2}} + \|f\|_{L\log L} \le K$ for some $K > 0$. Set $h = f - \mu$. Then for all $m \ge 0$ and some constants $C^*(K)$, $C(K) > 0$ depending only on K,*

$$\begin{aligned} W_1 &:= \langle Q(f, \partial_k h), \langle v\rangle^m \partial_k h\rangle \\ &\le -\tfrac{7}{8} C(K)\|\nabla^2 h\|^2_{L^2_{m/2-3/2}} + C^*(K)(1 + \|\nabla h\|^2_{L^2})\|\nabla h\|^2_{L^2_{m/2}}. \end{aligned} \tag{4.9}$$

Proof. Using an integration by parts, we see that

$$\begin{aligned} W_1 &= \langle Q(f, \partial_k h), \langle\cdot\rangle^m \partial_k h\rangle \\ &= -\Big(\sum_{k,i,j} \int (a_{ij} * f)(\partial_j \partial_k h)[\partial_i(\langle\cdot\rangle^m \partial_k h)]\,dv\Big) \\ &\quad + \Big(\sum_{k,i} \int (b_i * f)(\partial_k h)[\partial_i(\langle\cdot\rangle^m \partial_k h)]\,dv\Big) \\ &= -\Big(\sum_k \int (a * f) : (\nabla\partial_k h) \otimes (\nabla\partial_k h)\langle\cdot\rangle^m\,dv - \frac{1}{2}\sum_{k,i}\int (b_i * f)[\partial_k h]^2 \partial_i\langle\cdot\rangle^m\,dv \\ &\quad - \frac{1}{2}\int \sum_{i,j,k} (a_{ij} * f)[\partial_k h]^2 \partial_j\partial_i\langle\cdot\rangle^m\,dv\Big) + \Big(\frac{1}{2}\sum_{k,i}\int (b_i * f)[\partial_k h]^2 \partial_i\langle\cdot\rangle^m\,dv \\ &\quad + 4\pi \sum_k \int f[\partial_k h]^2 \langle\cdot\rangle^m\,dv\Big). \end{aligned}$$

Thanks to Corollary 2.1, the first term of the expression above satisfies

$$\sum_k \int (a * f) : (\nabla\partial_k h) \otimes (\nabla\partial_k h)\langle\cdot\rangle^m\,dv \ge C(K)\|\nabla^2 h\|^2_{L^2_{m/2-3/2}}.$$

Then, thanks to Hölder's inequality and Sobolev embedding ($\dot{H}^1 \subset L^6$),

$$\begin{aligned} \Big|\sum_k \int f[\partial_k h]^2 \langle\cdot\rangle^m\,dv\Big| &\le C^*\|f\|_{L^3_{3/2}}\|\nabla h\|_{L^2_{m/2}}\|\nabla((\nabla h)\langle\cdot\rangle^{m/2-3/2})\|_{L^2} \\ &\le C^*\|f\|^{1/5}_{L^1_{15/2}}\|f\|^{4/5}_{L^6}\|\nabla h\|_{L^2_{m/2}}(\|\nabla^2 h\|_{L^2_{m/2-3/2}} + \|\nabla h\|_{L^2_{m/2-5/2}}) \\ &\le C^*(K)\|\nabla f\|^{4/5}_{L^2}\|\nabla h\|_{L^2_{m/2}}\|\nabla^2 h\|_{L^2_{m/2-3/2}} + C^*(K)\|\nabla f\|^{4/5}_{L^2}\|\nabla h\|^2_{L^2_{m/2}} \\ &\le \frac{C^*(K)}{8}\|\nabla^2 h\|^2_{L^2_{m/2-3/2}} + C^*(K)(1 + \|\nabla h\|_{L^2})^{8/5}\|\nabla h\|^2_{L^2_{m/2}} \\ &\quad + C^*(K)(1 + \|\nabla h\|_{L^2})^{4/5}\|\nabla h\|^2_{L^2_{m/2}}. \end{aligned}$$

Using the estimates above and (4.3), we end up with estimate (4.9). ∎

Proposition 4.3. *Let $f \geq 0$ be such that $\int_{\mathbb{R}^3} f(v)\,dv = 1$, $\int_{\mathbb{R}^3} f(v)|v|^2\,dv = 3$, and $\|f\|_{L^1_{45}} + \|f\|_{L\log L} \leq K$ for some $K > 0$. Set $h = f - \mu$. Let $C(K)$ be as in Proposition 4.2 and $0 \leq m \leq 76$. Then there exists some constant $C^*(K)$ depending only on K such that*

$$\begin{aligned} W_2 &:= \langle Q(\partial_k f, h), \langle v\rangle^m \partial_k h\rangle \\ &\leq \frac{3C(K)}{8}\|\nabla^2 h\|^2_{L^2_{m/2-3/2}} + C^*(K)(1+\|\nabla h\|^2_{L^2})\|h\|_{H^1_{m/2}}\|\nabla h\|_{L^2_{m/2}}. \end{aligned} \tag{4.10}$$

Proof. Using integration by parts, we see that

$$\begin{aligned} W_2 &= \langle Q(\partial_k f, h), \langle v\rangle^m \partial_k h\rangle \\ &= \sum_{k,i,j}\int -(\partial_k a_{ij} * f)(\partial_j h)[(\partial_i\partial_k h)\langle\cdot\rangle^m + (\partial_k h)\partial_i\langle\cdot\rangle^m]\,dv \\ &\quad + \sum_{k,i}\int (b_i * \partial_k f)(h)[(\partial_i\langle\cdot\rangle^m)\partial_k h + (\partial_i\partial_k h)\langle\cdot\rangle^m]\,dv. \end{aligned}$$

Using first (4.3) of Proposition 4.1, we obtain the estimate

$$\left|\sum_{k,i,j}\int(\partial_k a_{ij} * f)(\partial_j h)(\partial_k h)\langle\cdot\rangle^m\,dv\right| \leq C(1+\|\nabla h\|_{L^2})\|\nabla h\|^2_{L^2_{m/2}}.$$

Also, still treating separately $|v-v_*| \leq 1$ and $|v-v_*| > 1$, and observing that $|\cdot|^{-2}_{|\cdot|<1} \in L^{4/3}$, we compute

$$\begin{aligned} &\left|\sum_{k,i,j}\int(\partial_k a_{ij} * f)(\partial_j h)(\partial_i\partial_k h)\langle\cdot\rangle^m\,dv\right| \\ &\quad\leq C^*\big(\|f\|_{L^4_{3/2}}\|\nabla h\|_{L^2_{m/2}}\|\nabla^2 h\|_{L^2_{m/2-3/2}} + \|f\|_{L^1_2}\|\nabla h\|_{L^2_{m/2-1/2}}\|\nabla^2 h\|_{L^2_{m/2-3/2}}\big) \\ &\quad\leq C^*\big(\|f\|^{1/10}_{L^1_{15}}\|f\|^{9/10}_{L^6}\|\nabla h\|_{L^2_{m/2}} + \|\nabla h\|_{L^2_{m/2-1/2}}\big)\|\nabla^2 h\|_{L^2_{m/2-3/2}} \\ &\quad\leq \frac{C(K)}{8}\|\nabla^2 h\|^2_{L^2_{m/2-3/2}} + C^*(K)(1+\|\nabla h\|_{L^2})^{9/5}\|\nabla h\|^2_{L^2_{m/2}}. \end{aligned}$$

Then, thanks to Proposition 4.1 again,

$$\left|\sum_{k,i}\int(b_i * \partial_k f)(h)(\partial_i\langle\cdot\rangle^m)\partial_k h\,dv\right| \leq C^*(1+\|\nabla h\|^2_{L^2})\|h\|_{H^1_{m/2}}\|\nabla h\|_{L^2_{m/2}}.$$

Finally, we estimate $|\sum_{k,i}\int(b_i * \partial_k f)(h)(\partial_i\partial_k h)\langle\cdot\rangle^m\,dv|$. Recall that $f = h + \mu$. Thanks to integration by parts, we have

$$\begin{aligned} &\sum_{k,i}\int(b_i * \partial_k\mu)(h)(\partial_i\partial_k h)\langle\cdot\rangle^m\,dv \\ &\quad= -\sum_{k,i}\int(b_i * \partial_k\mu)(\partial_k h)\partial_i(h\langle\cdot\rangle^m)\,dv + 8\pi\sum_k\int(\partial_k\mu)(\partial_k h)(h\langle\cdot\rangle^m)\,dv. \end{aligned}$$

Therefore,

$$\left|\sum_{k,i}\int (b_i * \partial_k \mu)(h)(\partial_i\partial_k h)\langle\cdot\rangle^m \, dv\right| \le C^* \|h\|_{H^1_{m/2}} \|\nabla h\|_{L^2_{m/2}}.$$

We now turn to the term $|\sum_{k,i}\int (b_i * \partial_k h)(h)(\partial_i\partial_k h)\langle\cdot\rangle^m \, dv|$. The integral over $|v-v_*|\le 1$ is bounded by

$$\begin{aligned}
C^* &\iint_{|v-v_*|\le 1} |v-v_*|^{-2}|\nabla h(v_*)|\,|h(v)|\,|\nabla^2 h(v)|\langle v\rangle^m \, dv_*\, dv \\
&\le C^* \big\| |\cdot|^{-2}_{|\cdot|\le 1} * |\langle\cdot\rangle^{m/2}\nabla h|\big\|_{L^4} \big\| |h|\,|\nabla^2 h|\langle\cdot\rangle^{m/2}\big\|_{L^{4/3}} \\
&\le C^* \|\nabla h\|_{L^2_{m/2}} \big\| |\cdot|^{-2}_{|\cdot|<1}\big\|_{L^{4/3}} \|h\|_{L^4_{3/2}} \|\nabla^2 h\|_{L^2_{m/2-3/2}} \\
&\le C^* \|\nabla h\|_{L^2_{m/2}} \|h\|^{1/10}_{L^1_{15}} \|\nabla h\|^{9/10}_{L^2} \|\nabla^2 h\|_{L^2_{m/2-3/2}} \\
&\le \frac{C(K)}{8}\|\nabla^2 h\|^2_{L^2_{m/2-3/2}} + C^*(K)(1+\|\nabla h\|^2_{L^2})\|\nabla h\|^2_{L^2_{m/2}}.
\end{aligned}$$

Notice now that $|\cdot|^{-2}_{|\cdot|\ge 1} \in L^{14/9}$. Thus, the integral over $|v-v_*|\ge 1$ is bounded by

$$\begin{aligned}
C^* &\iint_{|v-v_*|\ge 1} |v-v_*|^{-2}|\nabla h(v_*)|\,|h(v)|\,|\nabla^2 h(v)|\langle v\rangle^m \, dv_*\, dv \\
&\le \big\| |\cdot|^{-2}_{|\cdot|\ge 1} * |\nabla h|\big\|_{L^7} \big\| |h|\,|\nabla^2 h|\langle\cdot\rangle^{m}\big\|_{L^{7/6}} \\
&\le \|\nabla h\|_{L^2} \big\| |\cdot|^{-2}_{|\cdot|\ge 1}\big\|_{L^{14/9}} \|h\|_{L^{14/5}_{m/2+3/2}} \|\nabla^2 h\|_{L^2_{m/2-3/2}} \\
&\le C^* \|\nabla h\|_{L^2} \|h\|^{8/35}_{L^1_{m/2+105/16}} \|\nabla h\|^{27/35}_{L^2_{m/2}} \|\nabla^2 h\|_{L^2_{m/2-3/2}} \\
&\le \frac{C(K)}{8}\|\nabla^2 h\|^2_{L^2_{m/2-3/2}} + C^*(K)(1+\|\nabla h\|^2_{L^2})\|\nabla h\|^2_{L^2_{m/2}},
\end{aligned}$$

since $m/2 + 105/16 < 45$ when $m \le 76$.

Finally, we get estimate (4.10) by regrouping all the estimates above. ■

4.1.2. Estimates for W_3 and W_4. We now estimate jointly the terms W_3 and W_4.

Proposition 4.4. *Let $f \ge 0$ be such that $\int_{\mathbb{R}^3} f(v)\,dv = 1$ and $\int_{\mathbb{R}^3} f(v)|v|^2\,dv = 3$. Then for all $m \ge 2$, and some (absolute) constant $C > 0$,*

$$\begin{aligned}
W_3 + W_4 &:= \langle Q(\partial_k h, \mu), \langle\cdot\rangle^m \partial_k h\rangle + \langle Q(h, \partial_k \mu)\langle\cdot\rangle^m \partial_k h\rangle \\
&\le C\|\nabla h\|_{L^2}\|h\|_{H^1_{m/2}}.
\end{aligned} \tag{4.11}$$

Proof. Using integrations by parts, we compute

$$\begin{aligned}
W_3 &= \langle Q(\partial_k h, \mu), \langle\cdot\rangle^m \partial_k h\rangle \\
&= \sum_{k,j}\int (b_j * \partial_k h)(\partial_j \mu)\langle\cdot\rangle^m \partial_k h\, dv + \sum_{k,i,j}\int (\partial_k a_{ij} * h)(\partial_i\partial_j\mu)\langle\cdot\rangle^m \partial_k h\, dv \\
&\quad + \sum_k 8\pi \int \mu|\partial_k h|^2\langle\cdot\rangle^m\, dv - \sum_{k,i}\int (b_i * \partial_k h)(\partial_i \mu)(\partial_k h)\langle\cdot\rangle^m\, dv,
\end{aligned}$$

and

$$\begin{aligned}W_4 &= \langle Q(h, \partial_k\mu), \langle\cdot\rangle^m \partial_k h\rangle\\ &= \sum_{k,j}\int (b_j * h)(\partial_j\partial_k\mu)(\partial_k h)\langle\cdot\rangle^m\, dv + \sum_{k,i,j}\int (a_{ij} * h)(\partial_i\partial_j\partial_k\mu)(\partial_k h)\langle\cdot\rangle^m\, dv\\ &\quad - \sum_{k,i}\Big(\int (b_i * \partial_i h)(\partial_k\mu)(\partial_k h)\langle\cdot\rangle^m\, dv + \int (b_i * h)(\partial_i\partial_k\mu)(\partial_k h)\langle\cdot\rangle^m\, dv\Big).\end{aligned}$$

Hence, using the elementary inequality $\langle v_*\rangle^2 1_{|v-v_*|\le 1} \le |v-v_*|^2 e^{v^2/4} 1_{|v-v_*|\le 1}$, we get

$$\begin{aligned}W_3 + W_4 &\le C\Big[\int |b| * |\nabla h|\,|\nabla h|\mu^{1/2} + \int |b| * |h|\,|\nabla h|\mu^{1/2} + \int |\nabla h|^2\mu^{1/2}\Big]\\ &\le C\|[|b|1_{|\cdot|\le 1}] * (|h| + |\nabla h|)\|_{L^2}\|\nabla h\|_{L^2}\\ &\quad + C\iint_{|v-v_*|\ge 1}\langle v_*\rangle^{-2}[|h(v_*)| + |\nabla h(v_*)|]|\nabla h(v)|\mu(v)^{1/2}\, dv\, dv_* + C\|\nabla h\|^2_{L^2}\\ &\le C\|\nabla h\|^2_{L^2} + C\|h\|_{L^2}\|\nabla h\|_{L^2} + C\|\nabla h\|_{L^2}(\|h\|_{L^2_1} + \|\nabla h\|_{L^2_1})\\ &\le C\|\nabla h\|_{L^2}(\|h\|_{L^2_{m/2}} + \|\nabla h\|_{L^2_{m/2}}) \le C\|\nabla h\|_{L^2}\|h\|_{H^1_{m/2}},\end{aligned}$$

remembering that $m \ge 2$. ∎

4.2. L^2 estimate

Proposition 4.5. *Let $f \ge 0$ be such that $\int_{\mathbb{R}^3} f(v)\, dv = 1$, $\int_{\mathbb{R}^3} f(v)|v|^2\, dv = 3$, and $\|f\|_{L^1_{45}} + \|f\|_{L\log L} \le K$ for some $K > 0$. Set $h = f - \mu$. Let $C(K)$ be as in Proposition 4.2. Then for all $m \ge 4$, and some constant $C^*(K)$ depending only on K*

$$\begin{aligned}&\langle Q(f,h), \langle\cdot\rangle^m h\rangle + \langle Q(h,\mu), \langle\cdot\rangle^m h\rangle\\ &\quad\le -C(K)\|\nabla h\|^2_{m/2-3/2} + C^*(K)(1 + \|\nabla h\|_{L^2})(\|\nabla h\|^2_{L^2_{m/2}} + \|h\|^2_{L^2_{m/2}}).\end{aligned} \tag{4.12}$$

Proof. Using integration by parts, we obtain the decomposition

$$\begin{aligned}\langle Q(f,h), \langle\cdot\rangle^m h\rangle &= -\int (a * f) : (\nabla h)\otimes(\nabla h)\langle\cdot\rangle^m\, dv\\ &\quad - \sum_{i,j}\int (a_{ij} * f)(\partial_j h)h(\partial_i\langle\cdot\rangle^m)\, dv + \sum_i \int (b_i * f)(h)[\partial_i(\langle\cdot\rangle^m h)]\, dv\\ &=: -\mathbb{E}_1 - \mathbb{E}_2 + \mathbb{E}_3.\end{aligned}$$

Using Proposition 2.1 (and keeping in mind the arguments used in the proof of Corollary 2.1), we see that

$$\mathbb{E}_1 \ge C(K)\|\nabla h\|^2_{m/2-3/2}. \tag{4.13}$$

Using then the same computations as in the proof of Proposition 4.1, we see that

$$\begin{aligned} |\mathbb{E}_2| &\le C\|\nabla f\|_{L^2}\|h\|_{L^2_{m/2}}\|\nabla h\|_{L^2_{m/2}} + C\|f\|_{L^1}\|h\|_{L^2_{m/2}}\|\nabla h\|_{L^2_{m/2}} \\ &\le C(1+\|\nabla h\|_{L^2})\|h\|_{L^2_{m/2}}\|\nabla h\|_{L^2_{m/2}}. \end{aligned} \tag{4.14}$$

Similarly,

$$|\mathbb{E}_3| \le C(1+\|\nabla h\|_{L^2})(\|h\|_{L^2_{m/2}}\|\nabla h\|_{L^2_{m/2}} + \|h\|^2_{L^2_{m/2}}). \tag{4.15}$$

Using again integration by parts, we also see that

$$\begin{aligned} &|\langle Q(h,\mu), \langle\cdot\rangle^m h\rangle| \\ &\quad = \Big|\sum_i \int \Big(\sum_j -(a_{ij}*h)(\partial_j\mu)[\partial_i(\langle\cdot\rangle^m h)] + (b_i*h)(\mu)[\partial_i(\langle\cdot\rangle^m h)]\Big)\,dv\Big| \\ &\quad \le C\int (|a|+|b|)*|h|(|h|+|\nabla h|)\mu^{1/2}\,dv \\ &\quad \le C\big\|[(|a|+|b|)1_{|\cdot|\le 1}]*|h|\big\|_{L^2}(\|h\|_{L^2}+\|\nabla h\|_{L^2}) \\ &\qquad + C\iint_{|v-v_*|\ge 1}\langle v_*\rangle^{-1}|h(v_*)|[|h(v)|+|\nabla h(v)|]\mu(v)^{1/4}\,dv\,dv_* \\ &\quad \le C(\|h\|_{L^2}+\|\nabla h\|_{L^2})\|h\|_{L^2_2}, \end{aligned} \tag{4.16}$$

where $|v-v_*|^{-1}1_{|v-v_*|\ge 1} \le \langle v_*\rangle^{-1}\langle v\rangle$ is used.

Collecting all terms and remembering that $m \ge 4$, we conclude the proof of Proposition 4.5. ∎

4.3. *End of the proof of Proposition 1.2*

To end the proof, we perform the computations for a smooth $C_t^2(\mathcal{S})$ solution $f \ge 0$ of the Landau equation with Coulomb potential (1.1)–(1.3). We should in fact repeat here the process of approximation presented in the proofs of Theorem 1.2 and Proposition 1.1. We do not write it for the sake of readability, since no new argument is used to deal with the approximate process.

We first observe that thanks to the assumptions of Theorem 1.2 and Lemma 2.1, there exists a constant $K > 0$ such that

$$\sup_{t>0}(\|f(t)\|_{L^1_{45}} + \|f(t)\|_{L\log L}) \le K.$$

Then we compute (for $4 \le m \le 76$) the quantity $\frac{1}{2}\frac{d}{dt}\|\nabla h\|^2_{L^2_{m/2}}$. By using the computations (4.1), (4.2) and Proposition 4.2–4.4, we end up with the estimate

$$\frac{1}{2}\frac{d}{dt}\|\nabla h\|^2_{L^2_{m/2}} + \frac{C(K)}{2}\|\nabla^2 h\|^2_{L^2_{m/2-3/2}} \le C^*(K)(1+\|\nabla h\|^2_{L^2})\|h\|_{H^1_{m/2}}\|\nabla h\|_{L^2_{m/2}}. \tag{4.17}$$

Then, multiplying (2.1) by $\langle\cdot\rangle^m$, and integrating with respect to v, we compute

$$\frac{1}{2}\frac{d}{dt}\|h\|^2_{L^2_{m/2}} = \langle Q(f,h), \langle\cdot\rangle^m h\rangle + \langle Q(h,\mu), \langle\cdot\rangle^m h\rangle. \tag{4.18}$$

Using Proposition 4.5 and (4.18), we get the differential inequality

$$\frac{1}{2}\frac{d}{dt}\|h\|^2_{L^2_{m/2}} + C(K)\|\nabla h\|^2_{m/2-3/2} \le C^*(K)(1+\|\nabla h\|_{L^2})(\|\nabla h\|^2_{L^2_{m/2}} + \|h\|^2_{L^2_{m/2}}). \tag{4.19}$$

Combining (4.19) and (4.17), we finally obtain the differential inequality

$$\frac{1}{2}\frac{d}{dt}\|h\|^2_{H^1_{m/2}} + \frac{C(K)}{2}\|\nabla h\|^2_{H^1_{m/2-3/2}} \le C^*(K)(1+\|\nabla h\|^2_{L^2})\|h\|^2_{H^1_{m/2}}. \tag{4.20}$$

We emphasize that from the proofs of Propositions 4.2–4.5, the constants $C^*(K)$, $C(K) > 0$ in the above inequality only depend on K such that $\|f_0\|_{L^1_{55}} + \|f_0\|_{L\log L} \le K$.

Thanks to Proposition 6.4, we know that, for some $C, C_3, C_4 > 0$, and some $k_3 > 2/5$ (we take $l = 55$, $\theta = 15/4 + 7$ and $q_{l,\theta} \sim -3.79$ with the notations of Lemma 2.1, then $k_3 > 3$),

$$\frac{C(K)}{2}\|\nabla h\|^2_{H^1_{1/2}} \ge C_3\|h\|^{14/5}_{H^1_2}\|h\|^{-4/5}_{L^1_{15/4+7}} - C\|h\|^2_{L^1_{-3/2}} \ge C_4(1+t)^{k_3}\|h\|^{14/5}_{H^1_2} - C\|h\|^2_{L^2_2}.$$

In the inequality above and in the rest of the proof, we do not make explicit the (existing) dependence of $C, C_3, C_4 > 0$, and $k_3 > 2/5$ on K.

Denoting $Y(t) := \|h(t)\|_{H^1_2}$, we therefore get (for some $C_5 > 0$ only depending on K)

$$\frac{d}{dt}Y(t)^2 + C_4(1+t)^{k_3}Y(t)^{14/5} \le C_5(Y(t)^4 + Y(t)^2). \tag{4.21}$$

Remembering that $\|h_0\|_{L^1_{45}}$ is bounded and that the initial condition is supposed to satisfy $\|h_0\langle\cdot\rangle^2\|_{\dot H^1} \le \epsilon_0 \ll 1$, we see that by interpolation, the differential inequality (4.21) is complemented with the initial datum $Y(0)^2 = \tilde\epsilon \ll 1$ (note that here and below, the way in which $\tilde\epsilon$ is small depends in fact (only) on K).

We now consider $T^* := \sup\{t > 0 \mid Y(t)^4 \le Y(t)^2\} = \sup\{t > 0 \mid Y(t) \le 1\}$. For $t \in [0, T^*]$, the differential inequality

$$\frac{d}{dt}Y(t)^2 \le 2C_5 Y(t)^2$$

holds. It implies that for all $t \in [0, T_1 := (2C_5)^{-1}|\log(\frac{1}{2}|\log\tilde\epsilon|^{-1}\tilde\epsilon)^{-1}|]$, the inequality $Y(t)^2 \le \frac{1}{2}|\log\tilde\epsilon|^{-1} \le 1$ also holds. Thus, $T^* \ge T_1$.

We now use a contradiction argument in order to show that solutions of (4.21) globally exist. If the set $\{t > 0 \mid Y(t)^2 = |\log\tilde\epsilon|^{-1}\}$ is empty, then this is automatically true. If not, we define $T^{**} := \inf\{t > 0 \mid Y(t)^2 = |\log\tilde\epsilon|^{-1}\}$. Then there exists a time T_2 defined by $T_2 := \sup\{t \le T^{**} \mid Y(t)^2 = \frac{1}{2}|\log\tilde\epsilon|^{-1}\}$. Because of the definition of T_1 and T_2, we see that $T^* > T^{**} > T_2 \geq T_1$, and $Y(t)^2|\log\tilde{\epsilon}| \in [1/2, 1]$ when $t \in [T_2, T^{**}]$. In particular, in the interval $[T_2, T^{**}]$, we have

$$\frac{d}{dt}Y(t)^2 + C_4(1+T_1)^{k_3}Y(t)^{4/5}Y(t)^2 \leq 2C_5Y(t)^2,$$

where

$$(1+|\log\tilde{\epsilon}|)^{k_3}|\log\tilde{\epsilon}|^{-2/5} = O_{\tilde{\epsilon}\to 0}[C_4(1+T_1)^{k_3}Y(t)^{4/5}].$$

This implies that if $k_3 > 2/5$ and $\tilde{\epsilon} > 0$ is sufficiently small (depending on K again), then $C_4(1+T_1)^{k_3}Y(t)^{4/5} \geq 2C_5$. Thus Y^2 is decreasing on the interval $[T_2, T^{**}]$, so that $Y(T^{**})^2 \leq Y(T_2)^2 = \frac{1}{2}|\log\tilde{\epsilon}|^{-1}$. This is not compatible with the definition of T^{**}, which entails that the set $\{t > 0 \mid Y(t)^2 = |\log\tilde{\epsilon}|^{-1}\}$ is empty. As a consequence, we get the global existence for solutions of (4.21), and those solutions moreover satisfy $\sup_{t\geq 0} Y(t)^2 \leq |\log\tilde{\epsilon}|^{-1}$.

The solutions therefore satisfy the following modified differential inequality:

$$\frac{d}{dt}Y(t)^2 + C_4(1+t)^{k_3}Y(t)^{14/5} \leq 2C_5Y(t)^2.$$

Splitting the interval $(0,\infty)$ into the two sets $\{t > 0 \mid C_4(1+t)^{k_3}Y(t)^{14/5} \leq 4C_5Y(t)^2\}$ and $\{t > 0 \mid C_4(1+t)^{k_3}Y(t)^{14/5} > 4C_5Y(t)^2\}$, we conclude that for some constant $C > 0$ (remembering that $k_3 > 3$)

$$Y(t) \leq C(1+t)^{-5k_3/4} \leq C(1+t)^{-15/4}.$$

We recall that the estimates obtained in this subsection hold for a smooth solution of the Landau equation (1.1)–(1.3), and that, as in Proposition 2.8, they also hold uniformly with respect to $\epsilon \in]0,1[$ for smooth solutions of the approximate equation (2.42)–(2.43), with suitably mollified initial datum (we recall that such solutions are known to exist and be unique). It is then possible to pass to the (weak weighted L^1) limit in the final estimate

$$\|h^\epsilon\|_{H^1_2} \leq C(1+t)^{-5k_3/4} \leq C(1+t)^{-15/4},$$

and get the existence of the unique strong global nonnegative solution to (1.1)–(1.3) announced in Proposition 1.2. The uniqueness is obtained thanks to a variant of the arguments used in the proofs of Theorem 1.2 and Proposition 1.1.

5. Investigation of a potential blowup

Here, we prove Proposition 1.3, which provides estimates describing the potential blowup (in $\dot{H}^1$) of solutions to (1.1)–(1.3).

We first present the following (abstract) lemma:

Lemma 5.1. *Let $\bar{T} > 0$, X, H be C^1 functions from $[0, \bar{T}[$ to $\mathbb{R}_+$, $C_1, C_3, k_1 > 0$, $k_2 > 7/2$, and $D := -H'$. Suppose that X is a solution to the following ordinary differential inequality for all $\eta > 0$ small enough:*

$$\frac{d}{dt}X(t)^2 + C_1(1+t)^{k_1}X(t)^{14/5} \le \eta C_3 D(t)X(t)^{14/5} + \mathcal{B}(\eta)(1+t)^{-k_2}, \tag{5.1}$$

and that $\lim_{t\to\bar{T}} X(t) = +\infty$. In the estimate above, $\mathcal{B}$ is a continuous decreasing nonnegative function.

Then the following quantitative estimates hold for some $c, C > 0$ (depending on C_1, C_3, k_1, k_2 and $\mathcal{B}$) and $k := \min\{\frac{2k_2-7}{5}, k_1\}$, when $\bar{T} - t > 0$ is small enough:

$$X(t) \ge C(H(t) - \bar{H})^{-5/4} \quad \text{while} \quad H(t) - \bar{H} \ge C(\bar{T} - t)(1+\bar{T})^k, \tag{5.2}$$

$$\inf_{s\in[t,\bar{T}]} X(s) \le \left(\mathcal{B}(c[\bar{T}-t])\frac{2}{C_1}(1+\bar{T})^{-(k_1+k_2)}\right)^{5/14}. \tag{5.3}$$

In the estimates above, we use the notation $\bar{H} := \lim_{t\to\bar{T}} H(t)$.

Proof. We can first use Lemma 2.2 with $\eta = C_3^{-1}$ (up to choosing $C_3 > 0$ large enough) and $B^* := \mathcal{B}(\eta)$. Estimate (2.40) implies that for $\delta > 0$ small enough, and some $C_6 > 0$ given by Lemma 2.2,

$$H(\bar{T}-\delta) + C_6\int_t^{\bar{T}-\delta}(1+s)^k ds \le H(t) - \tfrac{5}{2}[X(t)^2 + B^*(1+t)^{1-k_2}]^{-2/5}, \tag{5.4}$$

which is enough to get the first part of estimate (5.2), by letting $\delta \to 0$.

Using again estimate (5.4) and letting $\delta \to 0$, we see that

$$X(t)^2 + B^*(1+t)^{1-k_2} \ge \left[\tfrac{2}{5}(H(t) - \bar{H})\right]^{-5/2}.$$

By definition, $\lim_{t\to\bar{T}} H(t) = \bar{H}$ so that $(1+t)^{1-k_2} = o_{t\to\bar{T}}(H(t) - \bar{H})^{-5/2}$, and we get the second part of (5.2).

In order to prove estimate (5.3), we go back to assumption (5.1). Dividing it by $X^{-14/5}$, we get

$$-\frac{5}{2}\frac{d}{dt}X(t)^{-4/5} + C_1(1+t)^{k_1} \le \eta C_3 D(t) + \mathcal{B}(\eta)(1+t)^{-k_2}X(t)^{-14/5},$$

which gives

$$\tfrac{5}{2}X(t)^{-4/5} - \tfrac{5}{2}X(\bar{T}-\delta)^{-4/5} + C_1\int_t^{\bar{T}-\delta}(1+s)^{k_1}\,ds$$
$$\le \eta C_3 H(0) + \mathcal{B}(\eta)\int_t^{\bar{T}-\delta}(1+s)^{-k_2}X(s)^{-14/5}\,ds. \tag{5.5}$$

From this, we deduce (remember that $\lim_{t\to\bar{T}} X(t) = +\infty$)

$$\left(\sup_{s\in[t,\bar{T}]} X(s)^{-14/5}\right)(1+t)^{-k_2}(\bar{T}-t)\mathcal{B}(\eta) \ge C_1(1+t)^{k_1}(\bar{T}-t) - \eta C_3 H(0).$$

Let $\eta := c(\bar{T} - t)$, for $c > 0$ chosen small enough. Then for $\bar{T} - t$ small enough, we get

$$\sup_{s \in [t,\bar{T}]} X(s)^{-1} \geq \left((\mathcal{B}(c[\bar{T} - t]))^{-1} \frac{C_1}{2} (1 + \bar{T})^{k_1+k_2} \right)^{5/14},$$

which yields estimate (5.3). ∎

Remark 5.1. The entropy H plays an important role in the estimates, giving hints about the way that a possible blowup could occur for (1.1). We observe that this quantity is continuous (with respect to time, on $[0, \bar{T})$) under our assumptions (namely when $f := f(t, v)$ is a nonnegative solution to (1.1) lying in $C([0, \bar{T}); \dot{H}^1) \cap L^\infty_{loc}([0, \bar{T}); L^1_5)$). From the inequality

$$|a \log a - b \log b| \leq C_p |a - b|^{1/p} + |a - b| \log^+(|a - b|) + 2\sqrt{a \wedge b} \sqrt{|a - b|},$$

which is proved in Proposition 6.5 (for $p > 1$, $C_p = p/(e(p - 1))$ and $a \wedge b = \min\{a, b\}$), used when $p := 4/3$, we see that (for $0 \leq t_1, t_2 < \bar{T}$)

$$|H(t_1) - H(t_2)|$$
$$\leq C\left(\|f(t_1) - f(t_2)\|^{3/4}_{L^{3/2}_3} + \|f(t_1) - f(t_2)\|^2_{L^2} + \|f(t_1) - f(t_2)\|^{1/2}_{L^{3/2}} \|f(t_1) + f(t_2)\|^{1/2}_{L^{3/2}_3}\right).$$

Thanks to the interpolation inequalities (based on Hölder's inequality and Sobolev embeddings),

$$\|f\|_{L^{3/2}_3} \leq \|f\|^{3/5}_{L^1_5} \|f\|^{2/5}_{L^6} \leq C \|f\|^{3/5}_{L^1_5} \|\nabla f\|^{2/5}_{L^2}, \quad \|f\|_{L^2} \leq \|f\|^{2/5}_{L^1} \|\nabla f\|^{3/5}_{L^2},$$

we finally get the estimate (for some C depending on $\|f\|_{L^\infty_t(L^1_5)}$ and $\|f\|_{L^\infty_t(\dot{H}^1)}$, the norms being taken on $[0, \sup(t_1, t_2)]$)

$$|H(t_1) - H(t_2)|$$
$$\leq C(\|f(t_1) - f(t_2)\|^{3/10}_{\dot{H}^1} + \|f(t_1) - f(t_2)\|^{6/5}_{\dot{H}^1} + \|f(t_1) - f(t_2)\|^{1/5}_{\dot{H}^1}), \quad (5.6)$$

which is sufficient to conclude.

Proof of Proposition 1.3. We begin with the case when f is a smooth and quickly decaying (when $|v| \to \infty$) solution to (1.1)–(1.3) on a time interval $[0, \bar{T}[$. Thanks to estimate (2.37), we see that assumption (5.1) holds with $\mathcal{B}(x) := C_2 x^{-13} \exp\{7x^{-450/14}\}$. We can then apply Lemma 5.1 to $X(t) := \|\nabla f(t)\|_{L^2}$.

We now briefly explain how to prove Proposition 1.3 without assuming that f is smooth and quickly decaying (when $|v| \to \infty$). We consider a time interval on which $f \in L^\infty_t(H^1 \cap L^1_{55})$. We first observe that thanks to Proposition 1.1, we have $f \in L^2_t(H^2_{-3/2})$. Since $f \in L^\infty_t(L^1_{55})$, we see that thanks to Proposition 6.4, $f \in L^2_t(H^1_{12})$. Using now estimate (4.20) and the uniqueness result, we see that $f \in L^\infty_t(H^1_{19/2}) \cap L^2_t(H^2_8)$ on all compact subintervals of $[t_0, \bar{T}[$ where $t_0 > 0$.

Using the equation satisfied by second order derivatives of f and computing the time derivative of the square of the H^2 norm of f, we can use Corollary 2.1 and estimates like those in Propositions 4.1–4.4 to end up with the bound

$$\frac{1}{2}\frac{d}{dt}\|f\|^2_{\dot H^2} + C(K)\|\nabla f\|^2_{\dot H^2_{-3/2}} \le C(\|f\|^2_{H^2} + \|f\|^4_{L^2})\|f\|^2_{H^2_{9/2}}.$$

Since $f \in L^2_t(H^2_8)$, we see that $f \in L^\infty_t(H^2) \cap L^2_t(H^3_{-3/2})$ on all compact subintervals of $[t_0, \bar T[$ where $t_0 > 0$.

Using the estimates above for solutions f^ϵ of the approximate problem (2.42)–(2.43), we find that f^ϵ is bounded in $L^\infty(H^2)$ on any interval $[t_1, t_2] \subset]0, \bar T[$. Using also (5.6), this is sufficient to pass to the limit in the inequality

$$\mathcal{M}^\epsilon(t_2) + C_6\int_{t_1}^{t_2}(1+t)^k\,dt \le \mathcal{M}^\epsilon(t_1), \tag{5.7}$$

where $\mathcal{M}(t) = H^\epsilon(t) - \frac{5}{2}(\|h^\epsilon(t)\|^2_{\dot H^1} + B^*(1+t)^{-k_2+1})^{-2/5}$. We end up with the inequality (1.24) “integrated in time”:

$$\mathcal{M}(t_2) + C_6\int_{t_1}^{t_2}(1+t)^k\,dt \le \mathcal{M}(t_1). \tag{5.8}$$

The same construction can be used to obtain estimates (5.4) and (5.5) and conclude the proof of Proposition 1.3 when $f \in L^\infty_t(H^1 \cap L^1_{55})$ on all compact subintervals of $[0, \bar T[$. ■

6. Appendix

In this appendix, we present some results which are used in the paper. We start with interpolation results and properties of Lorentz spaces.

6.1. Dyadic decompositions

We start by recalling some aspects of the Littlewood–Paley decomposition. Let $B_{4/3} := \{x \in \mathbb{R}^3 \mid |x| < 4/3\}$ and $R_{3/4,8/3} := \{x \in \mathbb{R}^3 \mid 3/4 < |x| < 8/3\}$. Then one introduces two radially symmetric functions $\psi \in C_0^\infty(B_{4/3})$ and $\varphi \in C_0^\infty(R_{3/4,8/3})$ which satisfy

$$\psi, \varphi \ge 0, \quad \psi(x) + \sum_{j\ge 0}\varphi(2^{-j}x) = 1, \quad x \in \mathbb{R}^3. \tag{6.1}$$

The dyadic operator $\mathcal{P}_j$ is defined for $j \ge -1$ by

$$\mathcal{P}_{-1}f(x) := \psi(x)f(x), \quad \mathcal{P}_j f(x) := \varphi(2^{-j}x)f(x) \quad (j \ge 0).$$

We recall that $\mathcal{P}_j\mathcal{P}_k = 0$ if $|j-k| > N_0$ for some $N_0 \in \mathbb{N}$.

We present a norm based on the dyadic decomposition which is equivalent to the usual norm of the weighted Sobolev spaces $H^s_l(\mathbb{R}^3)$:

Proposition 6.1 ([22]). *Let $s, l \in \mathbb{R}$. Then for $f \in H^s_l$, $\sum_{k=-1}^\infty 2^{2kl}\|\mathcal{P}_k f\|^2_{H^s} \sim \|f\|^2_{H^s_l}$.*

6.2. *Definition, norms and quasi-norms of Lorentz spaces*

For the convenience of the readers and for self-containment, we collect some useful facts about Lorentz spaces from [1,34]. Consider $\mathbb{R}^n$ with Lebesgue measure $|\cdot|$. In Section 1, we have defined the norm in the Lorentz space $L^{p,q}$, $p \in [1,\infty[$, $q \in [1,\infty]$ (or $p = q = \infty$, using the convention $t^{1/\infty} = 1, t \geq 0$), by

$$\|f\|_{L^{p,q}} := \begin{cases} \left(\int_0^\infty (t^{1/p} f^{**}(t))^q \frac{dt}{t}\right)^{1/q}, & 1 \leq q < \infty, \\ \sup\limits_{t>0} t^{1/p} f^{**}(t), & q = \infty, \end{cases} \tag{1.16}$$

which is different from the commonly used definition

$$\|f\|^*_{L^{p,q}} := \begin{cases} \left(\int_0^\infty (t^{1/p} f^{*}(t))^q \frac{dt}{t}\right)^{1/q}, & 1 \leq q < \infty, \\ \sup\limits_{t>0} t^{1/p} f^{*}(t), & q = \infty. \end{cases} \tag{6.2}$$

Here

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds, \quad f^*(s) = \inf\{t \geq 0 : a_f(t) \leq s\},$$

where a_f is the distribution function of f given by

$$a_f(t) = |\{x \in \mathbb{R}^n : |f(x)| > t\}|.$$

For $p \in (1,\infty)$ and $q \in [1,\infty]$, we note that the functional $\|\cdot\|^*_{L^{p,q}}$ is a norm only when $q \leq p$ and a quasi-norm otherwise; on the other hand, $\|\cdot\|_{L^{p,q}}$ is always a norm. For $p \in (1,\infty)$ and $q \in [1,\infty]$, the following comparison inequality holds:

$$\|f\|^*_{L^{p,q}} \leq \|f\|_{L^{p,q}} \leq \frac{p}{p-1}\|f\|^*_{L^{p,q}}.$$

Clearly for $1 < p < \infty$, we have $\|f\|^*_{L^{p,p}} = \|f\|_{L^p}$ and thus $L^{p,p} = L^p$. For $p = 1$ the situation is different (see also [1, p. 224]); one can indeed check that

$$\|f\|_{L^{1,\infty}} = \sup_{t>0} t f^{**}(t) = \sup_{t>0}\int_0^t f^*(s)\,ds = \int_0^\infty f^*(s)\,ds = \|f\|_{L^1}.$$

Finally, for $p = \infty$ (see also [1, p. 224]), one can also check that

$$\|f\|_{L^{\infty,\infty}} = \sup_{t>0} f^{**}(t) = \sup_{t>0}\frac{1}{t}\int_0^t f^*(s)\,ds = f^*(0) = \|f\|_{L^\infty}.$$

6.3. *Inequalities and interpolation*

We begin with a Sobolev embedding theorem and the O'Neil inequality in Lorentz spaces.

Proposition 6.2 (see [1,31]). (i) *If $f \in H^1(\mathbb{R}^3)$, then $f \in L^{6,2}(\mathbb{R}^3)$ and*

$$\|f\|_{L^{6,2}(\mathbb{R}^3)} \leq C\|f\|_{H^1(\mathbb{R}^3)}.$$

(ii) *For* $p_1, p_2, q_1, q_2 \in [1,\infty]$ *with* $1/p = 1/p_1 + 1/p_2$ *and* $1/q \le 1/q_1 + 1/q_2$, *there exists a computable constant* C *depending only on* p_1, q_1, p_2, q_2 *such that*

$$\|fg\|_{L^{p,q}} \le C\|f\|_{L^{p_1,q_1}}\|g\|_{L^{p_2,q_2}}.$$

(iii) *If* $f \in L^{p,q_1}$, $g \in L^{p',q_2}$ *where* $p, p', q_1, q_2 \in [1,\infty]$ *are such that* $1/p + 1/p' = 1$ *and* $1/q_1 + 1/q_2 \ge 1$, *then* $f * g \in L^\infty$ *and*

$$\|f * g\|_{L^\infty} \le \|f\|_{L^{p,q_1}}\|g\|_{L^{p',q_2}}.$$

Next we will prove some useful interpolation inequalities which are widely used throughout the paper.

Proposition 6.3. *For* $m \in \mathbb{R}$, *and some constant* $C > 0$ *depending only on* m,

$$\|f\|_{L^{3,1}_m} \le C\|f\|^{1/5}_{L^1_{5m+1}}\|f\|^{4/5}_{H^1}.$$

Proof. We split the proof into two parts. The first step is devoted to showing that

$$\|f\|_{L^{3,1}} \le \|f\|^{1/5}_{L^1}\|f\|^{4/5}_{L^{6,2}} \le C\|f\|^{1/5}_{L^1}\|f\|^{4/5}_{H^1}.$$

By the definition of Lorentz spaces, one gets

$$\begin{aligned}\|f\|_{L^{3,1}} &= \int_0^\infty t^{1/3} f^{**}(t)\,\frac{dt}{t}\\ &\le \Big(\int_0^R (t^{1/6} f^{**}(t))^2\,\frac{dt}{t}\Big)^{1/2}\Big(\int_0^R t^{1/3}\,\frac{dt}{t}\Big)^{1/2} + \Big(\sup_{t>0} t f^{**}(t)\Big)\int_R^\infty t^{-2/3}\,\frac{dt}{t}\\ &\le \|f\|_{L^{6,2}} R^{1/6} + \|f\|_{L^{1,\infty}} R^{-2/3}.\end{aligned}$$

We conclude by optimizing R and by using the identity $\|f\|_{L^{1,\infty}} = \|f\|_{L^1}$(see §6.2 and [1, p. 224]).

In the next step, we extend the above result to the general case (the one with weights appearing in the norms) using a dyadic decomposition. We observe that

$$\begin{aligned}\|f\|_{L^{3,1}_m} &= \|f\langle\cdot\rangle^m\|_{L^{3,1}} \le \sum_{k=-1}^\infty \|\mathcal{P}_k f\langle\cdot\rangle^m\|_{L^{3,1}} \le C\sum_{k=-1}^\infty \|\mathcal{P}_k f\|_{L^{3,1}}\|\mathcal{P}_k\langle\cdot\rangle^m\|_{L^{\infty,\infty}}\\ &\le C\sum_{k=-1}^\infty \|\mathcal{P}_k f\|_{L^{3,1}} 2^{km} \le C\sum_{k=-1}^\infty (2^{5km}\|\mathcal{P}_k f\|_{L^1})^{1/5}\|\mathcal{P}_k f\|^{4/5}_{H^1}\\ &\le C\Big(\sum_{k=-1}^\infty 2^{5km/3}\|\mathcal{P}_k f\|^{1/3}_{L^1}\Big)^{3/5}\Big(\sum_{k=-1}^\infty \|\mathcal{P}_k f\|^2_{H^1}\Big)^{2/5},\end{aligned}$$

where we use the O'Neil inequality (Proposition 6.2) and the identity $\|f\|_{L^{\infty,\infty}} = \|f\|_{L^\infty}$ (see §6.2 and [1, p. 224]). From this together with the computation

$$\sum_{k=-1}^\infty 2^{\frac{5}{3}km}\|\mathcal{P}_k f\|^{1/3}_{L^1} \le C\sum_{k=-1}^\infty 2^{5km/3}2^{-(5m+1)k/3}\|f\|^{1/3}_{L^1_{5m+1}} \le C\|f\|^{1/3}_{L^1_{5m+1}},$$

we finally get the inequality

$$\|f\|_{L^{3,1}_m} \le C\|f\|^{1/5}_{L^1_{5m+1}} \|f\|^{4/5}_{H^1}. \qquad \blacksquare$$

Proposition 6.4. *For $m \in \mathbb{R}$,*

$$\|f\|_{H^1_m} \le C\|f\|^{2/7}_{L^1_{15/4+7m/2}} (\|f\|_{L^1_{-3/2}} + \|\nabla^2 f\|_{L^2_{-3/2}})^{5/7},$$

$$\|f\|_{H^1_m} \le C\|f\|^{2/7}_{L^1_{5/4+7m/2}} (\|f\|_{L^1_{-1/2}} + \|\nabla^2 f\|_{L^2_{-1/2}})^{5/7},$$

where $C > 0$ is a constant depending only on m.

Proof. We first claim that

$$\|f\|_{H^1} \le C\|f\|^{2/7}_{L^1}(\|f\|_{L^1} + \|\nabla^2 f\|_{L^2})^{5/7}.$$

Indeed, since $\|f\|^2_{H^1} \sim \int_{\mathbb{R}^3}(1+|\xi|)^2 \hat{f}(\xi)^2\, d\xi$, we have (for $R \ge 1$)

$$\|f\|^2_{H^1} \le C(R^5\|f\|^2_{L^1} + R^{-2}\|f\|^2_{H^2}).$$

We conclude by taking $R^7 = \|f\|^2_{H^2}/\|f\|^2_{L^1} + 1$, recalling that $\|f\|_{H^2} \sim \|f\|_{L^1} + \|\nabla^2 f\|_{L^2}$. Thanks to Proposition 6.1, we see that

$$\begin{aligned}
\|f\|^2_{H^1_m} &\sim \sum_{k=-1}^{\infty} 2^{2km}\|\mathcal{P}_k f\|^2_{H^1} \le C\sum_{k=-1}^{\infty} 2^{2km}\|\mathcal{P}_k f\|^{4/7}_{L^1}(\|\mathcal{P}_k f\|_{\dot{H}^2} + \|\mathcal{P}_k f\|_{L^1})^{10/7}\\
&\le C\Big(\sum_{k=-1}^{\infty} 2^{(7m+15/2)k}\|\mathcal{P}_k f\|^2_{L^1}\Big)^{2/7}\Big(\sum_{k=-1}^{\infty} 2^{-3k}(\|\mathcal{P}_k f\|^2_{\dot{H}^2} + \|\mathcal{P}_k f\|^2_{L^1})\Big)^{5/7}\\
&\le C\|f\|^{4/7}_{L^1_{15/4+7m/2}}(\|f\|_{L^1_{-3/2}} + \|\nabla^2 f\|_{L^2_{-3/2}})^{10/7}.
\end{aligned}$$

The proof of the second inequality is similar. $\blacksquare$

Proposition 6.5 ([23]). *For $a, b \ge 0$ and $1 < p < \infty$, the following inequality holds:*

$$|a\log a - b\log b| \le C_p|a-b|^{1/p} + |a-b|\log^+(|a-b|) + 2\sqrt{a\wedge b}\sqrt{|a-b|}, \qquad (6.3)$$

where $a \wedge b = \min\{a, b\}$, $C_p := \frac{p}{e(p-1)}$ and

$$\log^+|x| = \begin{cases} \log x & \text{if } x \ge 1,\\ 0 & \text{if } x < 1.\end{cases}$$

Proof. We first observe that

$$\log(1+x) \le \sqrt{x}, \quad x \ge 0; \quad |\log x| \le \frac{1}{e\alpha}x^{-\alpha}, \quad 0 < x \le 1, \alpha > 0. \qquad (6.4)$$

Then, let $q > 1$ satisfy $1/p + 1/q = 1$. In what follows, we assume that $a > b > 0$.

We first observe that

$$|a\log a - b\log b| \le (a-b)|\log a| + b\log\left(\frac{a}{b}\right).$$

Using estimate (6.4), we see that

$$b\log\left(\frac{a}{b}\right) = b\log\left(1+\frac{a-b}{b}\right) \le b\sqrt{\frac{a-b}{b}} = \sqrt{b}\sqrt{a-b}.$$

Next we compute

$$|\log a| = \left|\log\left((a-b)\left(1+\frac{b}{a-b}\right)\right)\right| \le \frac{q}{e}(a-b)^{-1/q} + \log^+(a-b) + \sqrt{\frac{b}{a-b}},$$

where in the case when $a-b\le 1$, we use estimate (6.4) with $\alpha = 1/q$. This gives

$$(a-b)|\log a| \le \frac{q}{e}(a-b)^{1/p} + (a-b)\log^+(a-b) + \sqrt{b}\sqrt{a-b},$$

which enables one to conclude. ∎

6.4. A remark on initial data

Finally we show that there exist initial data for Theorem 1.2 whose initial relative entropy $H(0)$ is not large, while their $\dot{H}^1$ norm is large. See also the last comment of Theorem 1.2 in the introduction.

Proposition 6.6. *Let $\epsilon, \eta \ll 1$ and $\eta := \epsilon^{11/9}$. Assume the Maxwellian μ and a smooth $\phi_0 \ge 0$ both satisfy the normalization (1.14). Then*

$$\begin{aligned} f_0(v) := {} & (1-\eta+\eta\epsilon^2)^{3/2} \\ & \times \left[(1-\eta)\mu((1-\eta+\eta\epsilon^2)^{1/2}v) + \eta\epsilon^{-3}\phi_0(\epsilon^{-1}(1-\eta+\eta\epsilon^2)^{1/2}v)\right] \end{aligned} \tag{6.5}$$

also satisfies the normalization (1.14), and

$$\mathcal{M}(0) := H(0) - \tfrac{5}{2}(\|h(0)\|^2_{\dot{H}^1} + B)^{-2/5} \le 0,$$

while $\|h_0\|_{\dot{H}^{1/2}} \sim \epsilon^{-7/9}$ (where $h_0 = f_0 - \mu$, and $H(0)$ is the relative entropy of f_0).

Proof. We check that f_0 satisfies the third condition of (1.14) since the other two are easier to check. Thanks to a change of variables,

$$\begin{aligned} &(1-\eta+\eta\epsilon^2)^{3/2}\Big[(1-\eta)\int \mu((1-\eta+\eta\epsilon^2)^{1/2}v)|v|^2\,dv \\ &\qquad\qquad + \eta\epsilon^{-3}\int \phi_0(\epsilon^{-1}(1-\eta+\eta\epsilon^2)^{1/2}v)|v|^2\,dv\Big] \\ &= \frac{1-\eta}{1-\eta+\eta\epsilon^2}\int \mu(v)|v|^2\,dv + \frac{\eta\epsilon^2}{1-\eta+\eta\epsilon^2}\int \phi_0(v)|v|^2\,dv = 3. \end{aligned}$$

Next let us estimate $\mathcal{M}(0)$. We first observe that $\eta\epsilon^{-3/2} \geq 1$. Then for any $s > 0$, $\|h_0\|_{\dot{H}^s} \sim \eta\epsilon^{-s-3/2}$. The relative entropy $H(0)$ is bounded from above by

$$
\begin{aligned}
H(0) &= \int_{\mathbb{R}^3} \left(\frac{f_0}{\mu} \log\left(\frac{f_0}{\mu}\right) - \frac{f_0}{\mu} + 1 \right) \mu \, dv \\
&\leq \int_{\mathbb{R}^3} \int_0^1 \left| \log\left(\frac{f_\theta}{\mu}\right) \right| \left| \frac{f_0}{\mu} - 1 \right| \mu \, d\theta \, dv
\end{aligned}
$$

for some $\theta \in [0, 1]$, with the notation $f_\theta = (1-\theta) f_0 + \theta\mu$. From now on we denote by C any strictly positive constant.

At points where $f_0 \geq \mu$, we see that

$$
|\log(f_\theta/\mu)| = \log(f_\theta/\mu) \leq C(|v|^2 + \log(\eta\epsilon^{-3})),
$$

while at points where $f_0 \leq \mu$,

$$
|\log(f_\theta/\mu)| = \log(\mu/f_\theta) \leq \log(\mu/f_0) \leq C(|v|^2 + 1).
$$

From these estimates, we deduce that

$$
H(0) \leq C(1 + \log(\eta\epsilon^{-3})) \| f_0 - \mu \|_{L^1_2} \leq C\eta(1 + \log(\eta\epsilon^{-3})).
$$

Thus,

$$
\mathcal{M}(0) \leq C\eta(1 + \log(\eta\epsilon^{-3})) - C\eta^{-4/5}\epsilon^2.
$$

Remembering that $\eta \sim \epsilon^{11/9}$, and $\epsilon \ll 1$, we see that

$$
\mathcal{M}(0) \leq C\epsilon^{11/9} \log(\epsilon^{-1}) - C\epsilon^{46/45} \leq 0.
$$

Finally, $\|h_0\|_{\dot{H}^{1/2}} \sim \epsilon^{-7/9}$, so that h_0 is a large initial datum for the Landau equation in $\dot{H}^{1/2}$ (the critical space for incompressible Navier–Stokes equations). ■

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References

[1] Adams, R. A., Fournier, J. J. F.: Sobolev Spaces. 2nd ed., Pure Appl. Math. (Amsterdam) 140, Elsevier (2003) Zbl 1098.46001 MR 2424078

[2] Alexandre, R., Liao, J., Lin, C.: Some a priori estimates for the homogeneous Landau equation with soft potentials. Kinet. Relat. Models **8**, 617–650 (2015) Zbl 1320.35104 MR 3375485

[3] Arsen'ev, A. A., Peskov, N. V.: The existence of a generalized solution of Landau's equation. Ž. Vyčisl. Mat. i Mat. Fiz. **17**, 1063–1068, 1096 (1977) (in Russian) Zbl 0355.35074 MR 470442

[4] Carrapatoso, K., Desvillettes, L., He, L.: Estimates for the large time behavior of the Landau equation in the Coulomb case. Arch. Ration. Mech. Anal. **224**, 381–420 (2017) Zbl 1390.35360 MR 3614751

[5] Carrapatoso, K., Mischler, S.: Landau equation for very soft and Coulomb potentials near Maxwellians. Ann. PDE **3**, art. 1, 65 pp. (2017) Zbl 1404.35043 MR 3625186

[6] Chapman, S., Cowling, T. G.: The Mathematical Theory of Non-Uniform Gases. Cambridge Univ. Press, Cambridge (1939) Zbl 0063.00782 MR 0001151

[7] Chern, J.-L., Gualdani, M.: Uniqueness of higher integrable solution to the Landau equation with Coulomb interactions. Math. Res. Lett., to appear

[8] Csiszár, I.: Information-type measures of difference of probability distributions and indirect observations. Studia Sci. Math. Hungar. **2**, 299–318 (1967) Zbl 0157.25802 MR 219345

[9] Desvillettes, L.: Entropy dissipation estimates for the Landau equation in the Coulomb case and applications. J. Funct. Anal. **269**, 1359–1403 (2015) Zbl 1325.35223 MR 3369941

[10] Desvillettes, L., Villani, C.: On the spatially homogeneous Landau equation for hard potentials. I. Existence, uniqueness and smoothness. Comm. Partial Differential Equations **25**, 179–259 (2000) Zbl 0946.35109 MR 1737547

[11] Desvillettes, L., Villani, C.: On the spatially homogeneous Landau equation for hard potentials. II. H-theorem and applications. Comm. Partial Differential Equations **25**, 261–298 (2000) Zbl 0951.35130 MR 1737548

[12] Desvillettes, L., Villani, C.: On the trend to global equilibrium for spatially inhomogeneous kinetic systems: the Boltzmann equation. Invent. Math. **159**, 245–316 (2005) Zbl 1162.82316 MR 2116276

[13] Desvillettes, L.: About Boltzmann's H Theorem for the Landau equation (Autour du Théorème H de Boltzmann). Séminaire Laurent Schwartz – EDP et Applications (2019–2020), exp. IX, 13 pp. Zbl 1497.35332

[14] Fournier, N.: Uniqueness of bounded solutions for the homogeneous Landau equation with a Coulomb potential. Comm. Math. Phys. **299**, 765–782 (2010) Zbl 1198.35274 MR 2718931

[15] Golse, F., Gualdani, M. P., Imbert, C., Vasseur, A.: Partial regularity in time for the space homogeneous Landau equation with Coulomb potential. Ann. Sci. École Norm. Sup., to appear; arXiv:1906.02841

[16] Gualdani, M., Guillen, N.: On A_p weights and the Landau equation. Calc. Var. Partial Differential Equations **58**, art. 17, 55 pp. (2019) Zbl 1404.35078 MR 3884792

[17] Gualdani, M. P., Guillen, N.: Estimates for radial solutions of the homogeneous Landau equation with Coulomb potential. Anal. PDE **9**, 1772–1809 (2016) Zbl 1378.35325 MR 3599518

[18] Guo, Y.: The Landau equation in a periodic box. Comm. Math. Phys. **231**, 391–434 (2002) Zbl 1042.76053 MR 1946444

[19] He, L.: Well-posedness of spatially homogeneous Boltzmann equation with full-range interaction. Comm. Math. Phys. **312**, 447–476 (2012) Zbl 1253.35093 MR 2917172

[20] He, L.: Asymptotic analysis of the spatially homogeneous Boltzmann equation: grazing collisions limit. J. Statist. Phys. **155**, 151–210 (2014) Zbl 1291.35162 MR 3180973

[21] He, L., Yang, X.: Well-posedness and asymptotics of grazing collisions limit of Boltzmann equation with Coulomb interaction. SIAM J. Math. Anal. **46**, 4104–4165 (2014) Zbl 1315.35140 MR 3505177

[22] He, L.-B.: Sharp bounds for Boltzmann and Landau collision operators. Ann. Sci. École Norm. Sup. (4) **51**, 1253–1341 (2018) Zbl 1428.35266 MR 3942041

[23] He, L.-B., Lu, X., Pulvirenti, M.: On semi-classical limit of spatially homogeneous quantum Boltzmann equation: weak convergence. Comm. Math. Phys. **386**, 143–223 (2021) Zbl 1469.82028 MR 4287184

[24] Henderson, C., Snelson, S.: C^∞ smoothing for weak solutions of the inhomogeneous Landau equation. Arch. Ration. Mech. Anal. **236**, 113–143 (2020) Zbl 1435.35101 MR 4072211

[25] Henderson, C., Snelson, S., Tarfulea, A.: Local existence, lower mass bounds, and a new continuation criterion for the Landau equation. J. Differential Equations **266**, 1536–1577 (2019) Zbl 1406.35401 MR 3906224

[26] Krieger, J., Strain, R. M.: Global solutions to a non-local diffusion equation with quadratic non-linearity. Comm. Partial Differential Equations **37**, 647–689 (2012) Zbl 1247.35087 MR 2901061

[27] Kullback, S.: A lower bound for discrimination information in terms of variation. IEEE Trans. Inform. Theory **4**, 126–127 (1967)

[28] Lemarié-Rieusset, P. G.: Recent Developments in the Navier–Stokes Problem. Chapman & Hall/CRC Res. Notes in Math. 431, Chapman & Hall/CRC, Boca Raton, FL (2002) Zbl 1034.35093 MR 1938147

[29] Leray, J.: Sur le mouvement d'un liquide visqueux emplissant l'espace. Acta Math. **63**, 193–248 (1934) Zbl 60.0726.05 MR 1555394

[30] Lifschitz, E. M., Pitaevskii, L. P.: Physical Kinetics. Pergamon Press., Oxford (1981)

[31] O'Neil, R.: Convolution operators and $L(p,q)$ spaces. Duke Math. J. **30**, 129–142 (1963) Zbl 0178.47701 MR 146673

[32] Ożański, W. S., Pooley, B. C.: Leray's fundamental work on the Navier-Stokes equations: a modern review of "Sur le mouvement d'un liquide visqueux emplissant l'espace". In: Partial Differential Equations in Fluid Mechanics, London Math. Soc. Lecture Note Ser. 452, Cambridge Univ. Press, Cambridge, 113–203 (2018) Zbl 1408.35126 MR 3838050

[33] Silvestre, L.: Upper bounds for parabolic equations and the Landau equation. J. Differential Equations **262**, 3034–3055 (2017) Zbl 1357.35066 MR 3582250

[34] Stein, E. M., Weiss, G.: Introduction to Fourier Analysis on Euclidean Spaces. Princeton Math. Ser. 32, Princeton Univ. Press, Princeton, NJ (1971) Zbl 0232.42007 MR 0304972

[35] Strain, R. M., Guo, Y.: Exponential decay for soft potentials near Maxwellian. Arch. Ration. Mech. Anal. **187**, 287–339 (2008) Zbl 1130.76069 MR 2366140

[36] Toscani, G., Villani, C.: Sharp entropy dissipation bounds and explicit rate of trend to equilibrium for the spatially homogeneous Boltzmann equation. Comm. Math. Phys. **203**, 667–706 (1999) Zbl 0944.35066 MR 1700142

[37] Tsai, T.-P.: Lectures on Navier–Stokes Equations. Grad. Stud. Math. 192, Amer. Math. Soc., Providence, RI (2018) Zbl 1414.35001 MR 3822765

[38] Villani, C.: On the Cauchy problem for Landau equation: sequential stability, global existence. Adv. Differential Equations **1**, 793–816 (1996) Zbl 0856.35020 MR 1392006

[39] Villani, C.: On a new class of weak solutions to the spatially homogeneous Boltzmann and Landau equations. Arch. Ration. Mech. Anal. **143**, 273–307 (1998) Zbl 0912.45011 MR 1650006

[40] Villani, C.: A review of mathematical topics in collisional kinetic theory. In: Handbook of Mathematical Fluid Dynamics, Vol. I, North-Holland, Amsterdam, 71–305 (2002) Zbl 1170.82369 MR 1942465

[41] Wu, K.-C.: Global in time estimates for the spatially homogeneous Landau equation with soft potentials. J. Funct. Anal. **266**, 3134–3155 (2014) Zbl 1296.35112 MR 3158719