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Orbital stability of the black soliton for the quintic Gross–Pitaevskii equation

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Abstract. In this work, a proof of the orbital stability of the black soliton solution of the quintic Gross–Pitaevskii equation in one spatial dimension is obtained. We first build and show explicitly black and dark soliton solutions and we prove that the corresponding Ginzburg–Landau energy is coercive around them by using some orthogonality conditions related to perturbations of the black and dark solitons. The existence of suitable perturbations around black and dark solitons satisfying the required orthogonality conditions is deduced from an implicit function theorem. In fact, these perturbations involve dark solitons with sufficiently small speeds and some proportionality factors arising from the explicit expression of their spatial derivative.

1. Introduction

In this work, we consider the one-dimensional quintic Gross–Pitaevskii equation (quintic GP, for short)

$$
\begin{cases} iu_t + u_{xx} = (|u|^4 - 1)u, \quad (t,x) \in \mathbb{R}^2, \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}
$$

where u is a complex-valued function and the initial data u_0 satisfies the boundary condition

$$
\lim_{|x|\to+\infty} |u_0(x)|^2 = 1. \tag{1.2}
$$

From the physical point of view, it is interesting to look for solutions $u(t,x)$ of (1.1) satisfying the boundary condition (1.2) for all $t \geq 0$.

This is a *defocusing* nonlinear Schrödinger equation modeling, for example, ultra-cold dilute Bose gases in highly elongated traps. More specifically, it describes dynamics of weak density modulations of one-dimensional bosonic clouds (Tonks–Girardeau gases) when the tight transverse confinement potential is turned off. In fact, (1.1) in the case

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of one-dimensional atomic strings allows to explain many fermionic properties arising in one-dimensional chains of bosons, phenomena usually named as *bosonic fermionization*. See [7, 14, 17, 19] and references therein for a complete background on the physical phenomena accounted for by this quintic defocusing model.

The quintic GP equation is phase (also called $U(1)$ invariance) and translation invariant, meaning that if u is a solution of (1.1), then

$$e^{i\theta}u(t,x+a), \quad a\in\mathbb{R},\ \theta\in\mathbb{R},$$

is also a solution of (1.1). The quintic GP (1.1) also bears Galilean invariance, namely,

$$e^{i(cx/2-c^2t/4)}\,u(t,x-ct), \quad c\in\mathbb{R},$$

but this will not be used in our approach. Note moreover that in (1.2) the asymptotic value 1 can be changed to any number $\zeta>0$ without loss of generality by rescaling the values of u through $v=\zeta u(\zeta^4 t,\zeta^2 x)$. Under this change, (1.1) recasts as

$$iv_t+v_{xx}=(|v|^4-\zeta^4)\,v, \quad (t,x)\in\mathbb{R}^2.$$

Furthermore, and as far as we know, the quintic GP (1.1)–(1.2) is a non-integrable hamiltonian model (see [6, 23]), with well-known low order conservation laws for *regular solutions*, such as the mass

$$M[u](t):=\int_{\mathbb{R}}(1-|u|^2)\,dx=M[u](0)$$

and the *classical* energy

$$E_1[u](t):=\int_{\mathbb{R}}\Big(|u_x|^2-\frac{1}{3}(1-|u|^6)\Big)\,dx=E_1[u](0).$$

In this work, a crucial role will be played by the so-called quintic *Ginzburg–Landau energy*, which is given by

$$E_2[u]=E_1[u]+M[u],$$

or more explicitly, by

$$E_2[u](t):=\int_{\mathbb{R}}\Big(|u_x|^2+\frac{1}{3}(1-|u|^2)^2(2+|u|^2)\Big)\,dx, \tag{1.3}$$

which is also preserved along the flow. Another conserved quantity of (1.1) is the *momentum*, which in the context of solutions satisfying (1.2) can be suited in different forms. For example, for nonvanishing solutions (see [16]), it is written as[1]

$$P_1[u](t):=\int_{\mathbb{R}}\langle iu,u_x\rangle_{\mathbb{C}}\Big(1-\frac{1}{|u|^2}\Big)\,dx. \tag{1.4}$$

[1] Here $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ means complex product, as introduced in (2.8).

Moreover, considering vanishing solutions, in [3] it was introduced a renormalized version of the momentum (1.4), namely (here $u(t,x) = A(t,x)e^{i\varphi(t,x)}$),

$$P_2[u](t) := \lim_{R_1,R_2\to+\infty}\Big(\frac{1}{2}\int_{-R_1}^{R_2}\langle iu,u_x\rangle_{\mathbb{C}}\,dx - \frac{1}{2}(\varphi(R_2)-\varphi(R_1))\Big) \mod \pi.$$

Here, by *regular solutions* we will understand those solutions that belong to the energy space associated to (1.1):

$$\Sigma = \big\{u \in H^1_{\mathrm{loc}}(\mathbb{R}) : u_x \in L^2(\mathbb{R}) \text{ and } 1-|u|^4 \in L^2(\mathbb{R})\big\}. \tag{1.5}$$

Notice that if $u \in \Sigma$, then $1-|u|^2 \in L^2(\mathbb{R})$. Hence,

$$(1-|u|^2)^2(2+|u|^2) = (1-|u|^2)^2 + (1-|u|^2)(1-|u|^4) \in L^1(\mathbb{R}), \tag{1.6}$$

and $E_2[u]$ is well defined.

Some previous results on the Cauchy problem of (1.1) are well known in the literature. For example, local well-posedness in the context of a Zhidkov space, that is, $\{u \in L^\infty(\mathbb{R}) : \partial_x u \in L^2(\mathbb{R})\}$ was shown in [25] and global well-posedness of (1.1)–(1.2) was established in [9], where the following general model was considered:

$$iu_t + u_{xx} + f(|u|^2)u = 0, \tag{1.7}$$

with regular nonlinearity $f:\mathbb{R}^+ \to \mathbb{R}$ satisfying $f(1) = 0$ and $f'(1) < 0$. The model (1.7) includes, as particular cases, other important equations such as the following:

- Pure powers: $f(r) = 1 - r^p$, $p \in \mathbb{Z}^+$.
- Cubic case ($p = 1$): $f(r) = 1 - r$, the cubic Gross–Pitaevskii (cubic GP) equation.
- Cubic-quintic case: $f(r) = (r-1)(2a+1-3r)$ with $0 < a < 1$.
- Quintic case ($p = 2$): $f(r) = 1 - r^2$, the quintic GP equation (1.1).

More precisely, it was proved in Theorem 1.1 of [9] that the Cauchy problem for the quintic GP equation (1.1)–(1.2) is globally well-posed in the space

$$\phi + H^1(\mathbb{R}),$$

for any ϕ satisfying

$$\phi \in C^2_b(\mathbb{R}), \quad \phi' \in H^2(\mathbb{R}), \quad |\phi|^2 - 1 \in L^2(\mathbb{R}).$$

See [2, 11] and [10] for further reading on these generalized Schrödinger models.

Concerning solutions, complex constants with modulus one are the simplest solutions contained in (1.1). Moreover, with respect to particular soliton solutions, and specifically to the stability of solitonic waves for (1.7), the situation is well understood in the case of the cubic GP equation, profiting its integrable character (see [24]). Indeed, it is well known that the black soliton of the cubic GP is

$$v_0(x) = \tanh\Big(\frac{x}{\sqrt{2}}\Big),$$

which is a stationary, i.e., time independent wave solution. Furthermore, the study of orbital and asymptotic stability for $\nu_0(x)$ was considered in several works [3, 8, 12, 13]. Besides that, for some cases of the cubic-quintic model ($f(r) = (r-1)(2a+1-3r)$), the stability of traveling solitonic bubbles was shown in [18]. See [1,3,20] for more details on these models. Finally, [4] dealt with stability (and instability) problems for stationary and subsonic traveling waves giving an explicit condition on a general C^2 nonlinearity f in the NLS model. Once in this work we have obtained exact traveling wave solitons (also named as dark solitons), Theorem 24 in [4] can be applied to study the orbital stability of stationary solutions (i.e., black solutions) but in another metric, well adapted to the Σ space (1.5), different from the metric used in the current work.

In comparison with the cubic GP, the non-integrability of the quintic GP equation makes the search of solutions even harder as well as the rigorous study of the analytical properties related to them. Actually, and as far as we know, the *black soliton* solution for the quintic GP was discovered in [14, equation (12)]. Besides that, we present in this work the explicit expression of this solution as well as its formal derivation (see Section 2). Namely, the black soliton of (1.1) is given by

$$\phi_0(x) = \sqrt{2}\,\frac{\tanh(x)}{\sqrt{3-\tanh^2(x)}}, \tag{1.8}$$

which is a solution of

$$\phi'' + (1-\phi^4)\phi = 0, \tag{1.9}$$

the corresponding differential equation describing stationary real waves of (1.1) with $u(0,x) = \phi(x)$ (see Section 2 for further details). Therefore, it is natural to question whether, in the case of the quintic GP, the stability of ϕ_0 is preserved in some sense. In fact, the main result of this work is the following (see Section 5 for a more detailed version and proof of this result):

Theorem 1.1. *The black soliton solution ϕ_0 given in* (1.8) *of the quintic GP equation* (1.1) *is orbitally stable in a subspace of the energy space Σ, see* (1.5)*.*

The black soliton (1.8), stationary by nature, belongs to a greater family of traveling waves. As far as we know, an explicit and correct expression of a traveling wave family of solutions for the quintic GP (1.1) is missing in the literature. In fact, we show in this work that the quintic GP (1.1)–(1.2) also bears explicit traveling-wave solutions. These waves, with the form

$$u(t,x) = \Phi_c(x-ct),$$

are currently known as *dark solitons*, a reminiscent terminology coming from nonlinear optics (see [15]). The function Φ_c satisfies the complex nonlinear ordinary differential equation

$$\Phi_c'' - ic\,\Phi_c' + (1-|\Phi_c|^4)\,\Phi_c = 0. \tag{1.10}$$

Indeed, for $|c| < 2$, we are able to obtain the following explicit family of dark solitons:

$$\Phi_c(\xi) = \frac{i\mu_1(c) + \mu_2(c)\tanh(\kappa(c)\xi)}{\sqrt{2}\sqrt{1+\mu(c)\tanh^2(\kappa(c)\xi)}}, \tag{1.11}$$

with $\xi = x - ct$, where

$$\kappa \equiv \kappa(c) = \frac{\sqrt{4-c^2}}{2}, \tag{1.12}$$

$$\mu_1 \equiv \mu_1(c) = \frac{3c^2 - 4 + 2\sqrt{3c^2+4}}{\sqrt{18c^2 - 8 + (3c^2+4)^{3/2}}},$$

$$\mu_2 \equiv \mu_2(c) = \frac{3c\sqrt{4-c^2}}{\sqrt{18c^2 - 8 + (3c^2+4)^{3/2}}},$$

and where $\mu \equiv \mu(c)$ satisfies the constraint relation

$$\frac{\mu_1^2 + \mu_2^2}{2 + 2\mu} = 1 \tag{1.13}$$

for all $|c| < 2$, which comes from (1.2). Therefore, μ is explicitly

$$\mu \equiv \mu(c) = \frac{3c^2 + 20 - 8\sqrt{4+3c^2}}{3(-4+c^2)}. \tag{1.14}$$

Note that

$$\lim_{c\to 0} \mu_1 = 0, \qquad \lim_{c\to 0^{\pm}} \mu_2 = \pm\frac{2}{\sqrt{3}}$$

and, from (1.13),

$$\lim_{c\to 0} \mu = -\frac{1}{3}, \quad \text{with } -\frac{1}{3} \le \mu \le 0.$$

Also notice that, as a consequence of the above limits, we get

$$\lim_{c\to 0^{\pm}} \Phi_c(x) = \pm\Phi_0(x) = \pm\sqrt{2}\,\frac{\tanh(x)}{\sqrt{3 - \tanh^2(x)}}. \tag{1.15}$$

Finally, hereafter, since $\pm\Phi_c$ are both solutions of (1.10), it is better to consider a c-smooth continuation of (1.8) in the following way:

$$\phi_c = \begin{cases} \Phi_c, & c \ge 0, \\ -\Phi_c, & c < 0. \end{cases} \tag{1.16}$$

Hence,

$$\lim_{c\to 0^{\pm}} \phi_c = \phi_0.$$

Remark 1.2. The main ingredient in the orbital stability proof is the use of the associated family of complex dark profiles ϕ_c whose real parts are odd functions with respect to the speed c and which are laterally approximated to the stationary black solution when $c \to 0$, as shown in (1.15).

Our motivation to deal with the orbital stability of black solitons with this particular quintic nonlinearity (note that the cubic GP case was approached in [13]) comes firstly from the physical relevance that this model has in quantum gases as we said above. We were also motivated to prove this orbital stability result getting rid of a hydrodynamical formulation because this approach only describes non-vanishing solutions, and therefore excluding the black soliton solution. From a specific mathematical point of view, we avoid technical issues coming from the non-integrable character of the model and therefore not being allowed to use classical integrability methods.

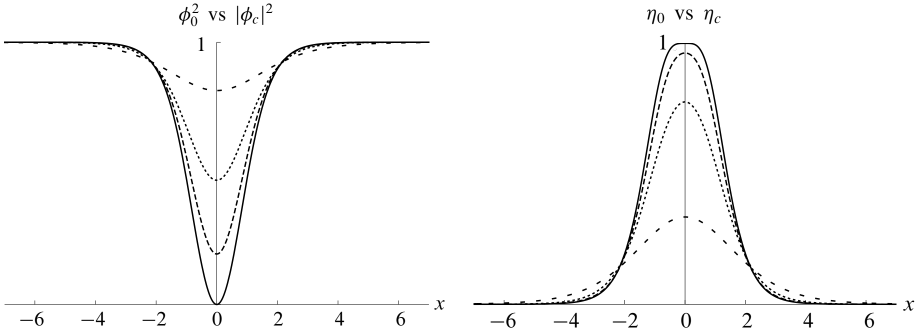

Figure 1. Left: graphics for the black soliton ϕ_0^2 (full line), see (1.8), against several profiles of dark solitons $|\phi_c|^2$, see (1.16), corresponding to dashed ($c = 0.75$), dotted ($c = 1.25$), and dashed but less segmented ($c = 1.75$). Right: the nonlinear weight η_0 (full line) against several weights η_c, see (2.5), varying c similarly.

More specifically, our proof establishes the coercivity of the functional $\mathcal{Q}_c[\mathfrak{z}]$, defined in (3.37), when the function $\mathfrak{z}$ satisfies suitable orthogonality conditions well adapted to this specific quintic nonlinearity. These orthogonality conditions are guaranteed by the introduction of modulation parameters (see Proposition 4.2) and are needed to perturb a stationary object as the black soliton of the quintic GP. This approach has the advantage to show a better control on the perturbation with respect to the black soliton.

Besides that, we are able to explicitly obtain (despite the nonintegrable nature of the model) dark solitons (traveling wave kinks) of the quintic GP, being one of the special cases where it is still possible to get these solutions (the other one is the cubic GP). The knowledge about dark solitons of quintic GP allowed us to perform precise perturbations on the black soliton.

Summarizing, we highlight here the main results involved in the proof of the orbital stability of the black soliton of the quintic GP equation (1.1). Our proof introduces new theoretical and technical tools, with respect to the integrable cubic GP equation [13] or more recently with respect to systems of cubic GP equations [5]. These tools are specially suited to deal with the associated nonlinear solutions of (1.1), namely, the black soliton ϕ_0, see (1.8), and the dark soliton ϕ_c profile, see (2.4). Specifically, we introduce the following:

- A new family of traveling wave solutions ϕ_c, see (1.16), close to the stationary black soliton ϕ_0, for the quintic GP equation (1.1). Obtaining non-constant solutions of this non-integrable equation is not a simple task, and even more in the case when one has to solve a coupled nonlinear ODE system, see (1.10). Only by proposing a suitable ansatz and a careful tuning of the free parameters allowed us to obtain them. Just, compare these solutions of the quintic GP equation with the corresponding ones of the cubic GP equation, where a simple complex constant translation gives the traveling family. See Section 2.2 for further reading.

- A modified metric d_c. This is a weighted metric with nonlinear weight ϕ_c^3 as it is dictated from the coercivity estimates that we need to prove on the quintic Ginzburg–Landau energy on black and dark solitons. See (2.12) for a precise definition of d_c, and also Propositions 3.2–3.4.
- New functional spaces, in order to correctly measure the distance between black and dark solitons and their perturbations $\mathfrak{z}$. See Section 2.3 for details.
- New orthogonality conditions which are associated to perturbations of the black and dark solitons, see (3.15) and (3.35), and which are specially adapted to the spectral properties of the quintic GP equation.

In this work, we were able to overcome several technical issues coming from the nonlinear functional structure of the quintic GP and its black and dark solitons, by working in a small speed region $|c| < \mathfrak{c}$.

Moreover, the apparent structural difference between black solitons in the cubic GP and the quintic GP is reflected in many identities and related functions around these black (and dark) solitons, e.g., the quintic Ginzburg–Landau energy E_2, see (1.3), or the spatial derivative ϕ_0'.

The strategy we used for the proof of the orbital stability result for the black soliton ϕ_0 of (1.1) was focused to first show that the quintic Ginzburg–Landau energy E_2 is coercive around the black and dark solitons. This was done by using some orthogonality relations based on perturbations $\mathfrak{z}$ of the black and dark solitons and arising from the particular spectral problem related to (1.1), suitable nonlinear identities and some proper Gagliardo–Nirenberg estimates on functions of the black and dark solitons of (1.1).

We notice that the orthogonality conditions arising from the coercivity result (Proposition 3.2) on the black soliton ϕ_0 do not include a linear term appearing after the expansion of E_2, see (1.3), around the black and dark solitons, and therefore we must deal with this remaining linear term along the proof, estimating it in a suitable way to obtain the expected bounds, in contrast with previous approaches (see [13]) where their natural orthogonality conditions imposed its cancellation.

After that main step, we continued with Proposition 4.2, proving, through a modulation of parameters, the existence of suitable perturbations $\mathfrak{z}$ of the dark soliton which satisfy the orthogonality conditions defined in (3.35).

Finally, note that related with the orbital stability is the concept of asymptotic stability, which essentially states the convergence of perturbations of the black soliton to a special element in the tubular neighborhood generated by its symmetries, e.g., phase and translation invariances. A detailed study on the asymptotic stability of the black soliton (1.8) of the quintic GP equation (1.1) is currently being made and it will appear elsewhere.

1.1. Final remarks

Our work does not get an orbital stability result for dark solitons (1.16) in d_c metric (2.12) for speeds close to 0. However, this kind of stability for dark profiles can be obtained in an alternative metric as the one used in Theorem 1.1 of [18]. In fact, computing directly

$$P_1[\phi_c] = \frac{\mu_1\mu_2}{\sqrt{\mu}}\arctan(\sqrt{\mu}) - 2\arctan\Big(\frac{\mu_2}{\mu_1}\Big),$$

we get

$$\frac{dP_1}{dc}[\phi_c] < 0,$$

as it can also be seen in Figure 2.

Note that (1.1) is phase invariant, and therefore, since

$$e^{i\tau_0}\phi_c \xrightarrow{c\to 0} e^{i\tau_0}\phi_0, \qquad \tau_0 \in (0, 2\pi),$$

we also have orbital stability for this phase transformed family of black solitons.

The quintic NLS is given as

$$iv_t + v_{xx} - |v|^4 v = 0.$$

The application to this model is rather direct, because it only involves the introduction of a rotation in time transformation $u = e^{it}v$ to connect (1.1) with the quintic NLS.

Note that some recent works (see [21, 22]) have approached another NLS model with modified dispersion terms, and dealing with orbital stability of black solitons using dark solitons with small speed, close to 0, but without an explicit expression of them and resorting to symmetries to simplify the coercivity analysis.

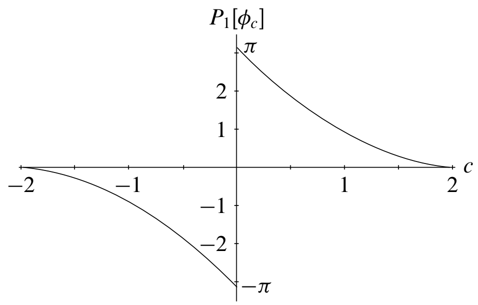

Figure 2. Momentum P_1, see (1.4), at ϕ_c, see (1.16).

1.2. Structure of the paper

In Section 1, we introduce the problem and the main result. In Section 2, we obtain the black and dark solitons of (1.1) and describe some properties and nonlinear identities and norms based on them. In Section 3, we present the coercivity properties of the quintic Ginzburg–Landau energy E_2 around black and dark solitons. In Section 4, we study the existence and time growth of some modulation parameters associated to black and dark solitons. Finally, in Section 5, we prove the main theorem on the orbital stability of the black soliton of (1.1), gathering the results obtained in the previous sections.

2. Derivation of black and dark solitons for the quintic GP

In this section, we explain the derivation of the black and dark solutions given in (1.8) and (1.16). The following basic result will be useful for obtaining the black family (1.8).

Lemma 2.1. *Let $b > 0$. Then*

$$\int_0^y \frac{ds}{(b-s^2)\sqrt{s^2+2b}} = \frac{1}{2b\sqrt{3}} \ln\left(\frac{\sqrt{2b+y^2}+\sqrt{3}y}{\sqrt{2b+y^2}-\sqrt{3}y}\right),$$

for all $|y| < \sqrt{b}$.

See Appendix B for a proof of this identity.

2.1. Derivation of black solitons

Using (1.9), we get, after multiplication by ϕ',

$$\phi(x)\phi'(x) + \phi''(x)\phi'(x) = \phi^5(x)\phi'(x),$$

and then

$$\frac{d}{dx}\left[\phi(x)^2 + (\phi'(x))^2 - \frac{\phi^6(x)}{3}\right] = 0,$$

which yields

$$\phi(x)^2 + (\phi'(x))^2 - \frac{\phi^6(x)}{3} = K_0.$$

From the boundary conditions at infinity in (1.2), we conclude that $K_0 = 2/3$ and we obtain the following first order ODE:

$$(\phi')^2 = \frac{1}{3}\phi^6 - \phi^2 + \frac{2}{3} = \frac{1}{3}(1-\phi^2)^2(2+\phi^2). \tag{2.1}$$

Assuming that $\phi' > 0$ and integrating, we get

$$\int_{x_0}^x \frac{\phi'(\tilde{x})\,d\tilde{x}}{\sqrt{(1-\phi(\tilde{x})^2)^2\,(\phi(\tilde{x})^2+2)}} = \frac{x-x_0}{\sqrt{3}},$$

where we consider $x_0 = \phi^{-1}(0)$. Then, making the change $s = \phi(\tilde{x})$, we have

$$\int_0^{\phi(x)} \frac{ds}{\sqrt{(1-s^2)^2\,(s^2+2)}} = \frac{x-x_0}{\sqrt{3}}.$$

Without loss of generality we can assume $x_0 = 0$. Since $|\phi(x)| < 1$, using Lemma 2.1, it follows that

$$\frac{1}{2\sqrt{3}} \ln\left(\frac{\sqrt{2+\phi^2(x)}+\sqrt{3}\,\phi(x)}{\sqrt{2+\phi^2(x)}-\sqrt{3}\,\phi(x)}\right) = \frac{x}{\sqrt{3}},$$

which yields

$$\frac{\phi(x)}{\sqrt{2+\phi^2(x)}} = \frac{1}{\sqrt{3}}\frac{e^{2x}-1}{e^{2x}+1} = \frac{1}{\sqrt{3}}\tanh(x),$$

and consequently

$$\phi(x) = \sqrt{2}\,\frac{\tanh(x)}{\sqrt{3-\tanh^2(x)}},$$

which in fact is the unique (up to symmetries of the equation) non-trivial stationary solution of the quintic GP (1.1)–(1.2) and named as *black* soliton.

An important observation is that the black soliton ϕ_0, see (1.8), has a definite variational structure. More precisely, considering the quintic Ginzburg–Landau energy $E_2[u]$ defined in (1.3) as the corresponding Lyapunov functional, and considering a small perturbation $\mathfrak{z}$ of the black soliton ϕ_0, namely, a $\mathfrak{z} \in \mathcal{H}_0(\mathbb{R})$ with $\mathcal{H}_0(\mathbb{R}) \subset H^1_{\text{loc}}(\mathbb{R})$ to be defined in (2.9), we get, after a power expansion in $\mathfrak{z}$ of E_2 (see (1.3))

$$E_2[\phi_0+\mathfrak{z}] = E_2[\phi_0] - 2\mathrm{Re}\Big[\int_{\mathbb{R}} \bar{\mathfrak{z}}(\phi_0'' + (1-|\phi_0|^4)\phi_0)\Big] + \mathcal{O}(\mathfrak{z}^2).$$

Because the first variation of E_2 vanishes for (1.9), the black soliton is characterized as critical point of the functional E_2 associated to the quintic GP (1.1). In fact, it is easy to see that

$$E_2[\phi_0] := 2\sqrt{3}\,\mathrm{arctanh}\Big(\frac{1}{\sqrt{3}}\Big). \tag{2.2}$$

Moreover, it is possible to state the following minimality's characterization on the black soliton solution.

Proposition 2.2 (Lemma 2.6 in [3]). *Let E_2 be as in (1.3) and let ϕ_0 be the black soliton solution (1.8). Then we have*

$$E_2[\phi_0] = \inf\Big\{E_2[\phi] : \phi \in H^1_{\text{loc}}(\mathbb{R}),\ \inf_{x\in\mathbb{R}} |\phi(x)| = 0\Big\}.$$

Moreover, if $E_2[\phi] < E_2[\phi_0]$, then $\inf_{x\in\mathbb{R}}|\phi(x)| > 0$.

Proof. This result is essentially contained in Lemma 2.6 of [3], where the black soliton case for the cubic GP was considered. The extension to the quintic GP case, once we work with the energy E_2, is direct and does not require additional steps. We therefore skip the details. ∎

2.2. Derivation of dark solitons

Once obtained the black soliton (1.8), the detailed construction of the dark soliton solution (1.16) to (1.10) is presented in Appendix A. A sketch of the derivation is the following: bearing in mind that (1.10) reduces to (1.9) at $c = 0$, we propose a suitable *ansatz* like

$$\Phi_c(x) = \frac{ia_1 + a_2\tanh(kx)}{\sqrt{1+a_3\tanh^2(kx)}}, \tag{2.3}$$

with a_1, a_2, a_3 and k as free parameters to be determined in order that (2.3) is actually a solution of (1.10) which satisfies the asymptotic behavior

$$\lim_{x\to\pm\infty} |\Phi_c(x)|^2 = 1.$$

Hence, substituting (2.3) into (1.10) and after lengthy manipulations, we get (1.16), with $\mu_1 = \sqrt{2}a_1$, $\mu_2 = \sqrt{2}a_2$ and $a_3 = \mu$, satisfying the relation (1.13) and $k = \kappa$ as in (1.12).

Finally, we introduce the notion of *dark profile*.

Definition 2.3 (Dark profile). Let $c \in (-2,2)$ and $x_0 \in \mathbb{R}$ be fixed parameters. We define the complex-valued dark profile ϕ_c with speed $c \neq 0$ as follows:

$$\phi_c(x) := \phi_c(x; c, x_0) = \operatorname{sgn}(c)\, \frac{i\mu_1(c) + \mu_2(c)\tanh(\kappa(c)(x+x_0))}{\sqrt{2}\sqrt{1+\mu(c)\tanh^2(\kappa(c)(x+x_0))}}. \tag{2.4}$$

Remark 2.4. Note that the profile ϕ_c is the standard profile associated to the dark soliton solution (1.16). Note, moreover, that although ϕ_c is not an exact solution of (1.1), it can be interpreted as follows: for each $(t,x) \in \mathbb{R}^2$,

$$(t,x) \mapsto \phi_c(x; c, x_0 - ct)$$

is an exact dark soliton solution of (1.1) moving with speed c.

2.3. Preliminaries

First of all, we introduce the following notation for the nonlinear weights:

$$\eta_0(x) = 1 - \phi_0^4(x) \quad \text{and} \quad \eta_c(x) = 1 - |\phi_c(x)|^4, \tag{2.5}$$

and for the real and imaginary parts of the dark soliton:

$$R_c(x) = \operatorname{Re}\phi_c(x) = \frac{\mu_2 \tanh(\kappa x)}{\sqrt{2}\sqrt{1+\mu\tanh^2(\kappa x)}}, \tag{2.6}$$

$$I_c(x) = \operatorname{Im}\phi_c(x) = \frac{\mu_1}{\sqrt{2}\sqrt{1+\mu\tanh^2(\kappa x)}}. \tag{2.7}$$

To simplify the notation, we shall also denote

$$\langle f, g\rangle_{\mathbb{C}} = \operatorname{Re}(f\bar{g}). \tag{2.8}$$

Moreover, we define the following functional spaces: given $c \in (-2,2)$, we consider the weighted Sobolev space

$$\mathcal{H}_c(\mathbb{R}) := \big\{ f \in C^0(\mathbb{R},\mathbb{C}) : f' \in L^2(\mathbb{R}) \text{ and } \eta_c^{1/2} f \in L^2(\mathbb{R}) \big\}, \tag{2.9}$$

with the norm

$$\|f\|_{\mathcal{H}_c} := \Big(\int_{\mathbb{R}} |f'|^2 + \eta_c |f|^2 \Big)^{1/2}.$$

We will also use $\mathcal{H}_c^{\mathrm{real}}(\mathbb{R})$ to denote the set of real-valued functions in $\mathcal{H}_c(\mathbb{R})$, that is,

$$\mathcal{H}_c^{\mathrm{real}}(\mathbb{R}) = \big\{ f \in C^0(\mathbb{R},\mathbb{R}) : f' \in L^2(\mathbb{R}) \text{ and } \eta_c^{1/2} f \in L^2(\mathbb{R}) \big\}.$$

Using the exponential decay of η_c, we can check that the space $\mathcal{H}_c$ does not depend on the velocity c when $|c| \leq \mathfrak{c}$, for some $\mathfrak{c}$ small enough. Even more, the norms $\|\cdot\|_{\mathcal{H}_c}$

are equivalent with $\|\cdot\|_{\mathcal{H}_0}$. For further details, see Lemma 2.6. Therefore, hereafter we simplify the notation using the identification

$$\mathcal{H} := \mathcal{H}_c \quad \text{and} \quad \mathcal{H}^{\text{real}} := \mathcal{H}_c^{\text{real}}, \tag{2.10}$$

for all $|c| < \mathfrak{c}$. Besides that, we define a proper subset of $\mathcal{Z}(\mathbb{R}) \subsetneq \mathcal{H}(\mathbb{R})$, namely,

$$\mathcal{Z}(\mathbb{R}) := \{u \in \mathcal{H}(\mathbb{R}) : 1 - |u|^4 \in L^2(\mathbb{R})\}, \tag{2.11}$$

which has metric structure with the distance

$$d_c(u_1, u_2) := \left(\|u_1 - u_2\|^2_{\mathcal{H}_c} + \|\phi_c^3(|u_1|^2 - |u_2|^2)\|^2_{L^2}\right)^{1/2}, \tag{2.12}$$

for all $|c| < \mathfrak{c}$. Also notice that if $u \in \mathcal{Z}(\mathbb{R})$, from the computations in (1.6), we see that the energy E_2 defined in (1.3) is well defined for elements in $\mathcal{Z}(\mathbb{R})$.

Remark 2.5. As in the theory developed in [13] in the context of the cubic GP model, here we also have that the unique global solution u of (1.1) with initial data $u_0 \in \mathcal{Z}(\mathbb{R})$ remains continuous from $\mathbb{R}$ to $\mathcal{Z}(\mathbb{R})$ endowed with the metric structure induced by d_c defined in (2.12).

2.4. Nonlinear identities and estimates for black and dark solitons

Now we present some nonlinear identities related to the black soliton and dark soliton profile (1.8) and (2.4), respectively, which shall be useful along the work. Firstly, we note that from (2.1), we get

$$\phi_0'(x) = \frac{1}{\sqrt{3}}(1 - \phi_0^2(x))\sqrt{2 + \phi_0^2(x)}. \tag{2.13}$$

In comparison, the dark soliton satisfies the following identity:

$$\phi_c'(x) = \frac{1}{\sqrt{2}} \frac{\kappa \operatorname{sech}^2(\kappa x)}{(1 + \mu \tanh^2(\kappa x))^{3/2}} (\mu_2 - i\mu\mu_1 \tanh(\kappa x)),$$

and which shows the localized character of ϕ_c'.

Notice that from (1.9), (2.1) and (2.2), the black soliton solution (1.8) satisfies the identity

$$\begin{aligned}\|\phi_0\|^2_{\mathcal{H}_0} = \int_{\mathbb{R}} (\eta_0 \phi_0^2 + (\phi_0')^2)\, dx &= \int_{\mathbb{R}} \left(-\phi_0 \phi_0'' + \frac{1}{3}(1 - |\phi_0|^2)^2(2 + |\phi_0|^2)\right) dx \\ &= E_2[\phi_0] = 2\sqrt{3}\operatorname{arctanh}\left(\frac{1}{\sqrt{3}}\right),\end{aligned}$$

and by direct calculation, we have also $\|\eta_0\|^2_{L^2(\mathbb{R})} = 2\sqrt{3}\operatorname{arctanh}(1/\sqrt{3})$. Hence,

$$\|\phi_0\|^2_{\mathcal{H}_0(\mathbb{R})} = \|\eta_0\|^2_{L^2(\mathbb{R})} = E_2[\phi_0] = 2\sqrt{3}\operatorname{arctanh}\left(\frac{1}{\sqrt{3}}\right). \tag{2.14}$$

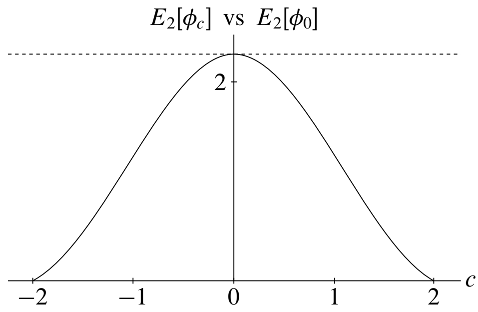

Figure 3. Comparison between the quintic Ginzburg–Landau energy (1.3) of the dark (full line), see (2.15), and black (dotted), see (2.2), soliton solutions of (1.1).

Coming back to (1.16) and using (1.14) and (1.13), we get that the explicit quintic Ginzburg–Landau energy (1.3) of the *dark* soliton (1.16) is

$$E_2[\phi_c] := \frac{s_1 + s_2 \operatorname{arctanh}(\sqrt{|\mu|})}{32\kappa\mu^2}, \tag{2.15}$$

with

$$\begin{aligned} s_1 &:= \big(2\mu(\mu+3) - \mu_2^2(\mu-1)\big)\big(\mu_2^4 - 4\mu_2^2\mu + 4\mu(\mu+\kappa^2)\big), \\ s_2 &:= \frac{1}{3\sqrt{|\mu|}}\big(12\mu\kappa^2\big(\mu_1^2(\mu-3)\mu + \mu_2^2(3\mu-1)\big) \\ &\qquad - (\mu_1^2-2)^2\big(\mu_2^2(3\mu^2-2\mu+3) - 2\mu(3\mu^2+10\mu-9)\big)\big). \end{aligned}$$

On the other hand, we get the right convergence of (2.15) to (2.14) when the speed c goes to 0, namely (see Figure 3),

$$\lim_{c\to 0} E_2[\phi_c] = E_2[\phi_0] = 2\sqrt{3}\operatorname{arctanh}\Big(\frac{1}{\sqrt{3}}\Big). \tag{2.16}$$

In fact, the expansion of (2.15) around ϕ_0 up to c^2 order is

$$E_2[\phi_c] = E_2[\phi_0] - \frac{1}{4}(3 + E_2[\phi_0])\,c^2 + \mathcal{O}(c^3), \tag{2.17}$$

and therefore, for c small enough one gets

$$E_2[\phi_c] - E_2[\phi_0] \geq -2\sqrt{3}\,c^2. \tag{2.18}$$

For the sake of completeness, we also show here the following related amounts:

$$d_0^2(\phi_0,\phi_c) := \|\phi_0 - \phi_c\|_{\mathcal{H}_0}^2 + \|\phi_0^3(|\phi_c|^2 - \phi_0^2)\|_{L^2}^2. \tag{2.19}$$

We now have that if $|c| < \mathfrak{c}$, with some $\mathfrak{c} \ll 1$,

$$\|\phi_0 - \phi_c\|_{\mathcal{H}_0}^2 = \mathcal{O}(c^2) \tag{2.20}$$

and

$$\|\phi_0^3(|\phi_c|^2 - \phi_0^2)\|_{L^2}^2 = \mathcal{O}(c^4).$$

Therefore, we get that

$$d_0^2(\phi_0, \phi_c) = \mathcal{O}(c^2). \tag{2.21}$$

In the following lines, we show some interesting computations, which are justified in Appendix D. Firstly, we have that

$$\left\| \frac{\phi_c'}{\sqrt{\eta_c}} \right\|_{L^2}^2 \leq \frac{\pi}{3\sqrt{3}} + \frac{2}{\sqrt{3}} \operatorname{arccotanh}(\sqrt{3}), \tag{2.22}$$

for all $|c| < 2$. On the other hand, because $-1/3 \leq \mu < 0$ for all $|c| < 2$, we have

$$\| I_c \|_{L^\infty} = \frac{\mu_1}{\sqrt{2 + 2\mu}} = \mathcal{O}(c), \tag{2.23}$$

with I_c defined in (2.7). We also have the following useful estimates:

$$\left\| \frac{|\phi_c|^2 - \phi_0^2}{\sqrt{\eta_0}} \right\|_{L^2} = \mathcal{O}(c^2), \tag{2.24}$$

$$\left\| \frac{\phi_0 \eta_0 - R_c \eta_c}{\sqrt{\eta_0}} \right\|_{L^2} = \mathcal{O}(c^2), \tag{2.25}$$

$$\left\| \frac{\eta_c |\phi_c|^2 - \eta_0 \phi_0^2}{\sqrt{\eta_0}} \right\|_{L^2} = \mathcal{O}(c^2), \tag{2.26}$$

$$\left\| \frac{\eta_c |\phi_c|^2 R_c^2 - \eta_0 \phi_0^4}{\sqrt{\eta_0}} \right\|_{L^2} = \mathcal{O}(c^2), \tag{2.27}$$

$$\left\| \frac{|\phi_c|^2 - \phi_0^2}{(1 + x^2)\eta_c} \right\|_{L^\infty}^2 = \mathcal{O}(c^2), \tag{2.28}$$

for all $|c| \leq \mathrm{c}$, with some $\mathrm{c} \ll 1$. Also we have the uniform pointwise estimate

$$|\phi_0(x)| \lesssim |\phi_c(x)| \quad \text{for all } x \in \mathbb{R},\ |c| \leq \mathrm{c},\ \mathrm{c} \ll 1. \tag{2.29}$$

For more details on these L^2 and L^∞-norms, see Appendices D.1 and D.2, respectively.

The next estimate will be useful in subsequent technical results on perturbations of the black soliton ϕ_0, and therefore we present a brief proof of it.

Lemma 2.6 (Equivalent norms). *Let ϕ_0 and ϕ_c be the black soliton and dark soliton profile* (1.8) *and* (2.4)*, respectively. Then there exists $\mathrm{c} \in (0, 2)$ such that*

$$\int_{\mathbb{R}} \left| |\phi_c|^4 - \phi_0^4 \right| |\mathfrak{z}|^2 \, dx \lesssim c^2 \|\mathfrak{z}\|_{\mathcal{H}_{c^*}}^2, \tag{2.30}$$

for all $|c|$, $|c^*| < \mathrm{c}$ and any $\mathfrak{z} \in \mathcal{H}_{c^*}$. Furthermore, we conclude that $\mathcal{H}_c \equiv \mathcal{H}_0$ for all $|c| < \mathrm{c}$, and there exist positive constants σ_1 and σ_2 such that

$$\sigma_1 \|\mathfrak{z}\|_{\mathcal{H}_0}^2 \leq \|\mathfrak{z}\|_{\mathcal{H}_c}^2 \leq \sigma_2 \|\mathfrak{z}\|_{\mathcal{H}_0}^2. \tag{2.31}$$

Proof. From (2.28) and using the identity $\mathfrak{z}(x) = \mathfrak{z}(0) + \int_0^x \mathfrak{z}'(\tilde{x})\,d\tilde{x}$, which implies

$$|\mathfrak{z}(x)| \le |\mathfrak{z}(0)| + |x|^{1/2}\|\mathfrak{z}'\|_{L^2},$$

we have that

$$\begin{aligned}
(2.32)\qquad &\int_{\mathbb{R}} ||\phi_c|^4 - \phi_0^4|\,|\mathfrak{z}|^2\,dx \\
&\le 2\int_{\mathbb{R}} ||\phi_c|^2 - \phi_0^2|\,|\mathfrak{z}|^2\,dx \lesssim c^2 \int_{\mathbb{R}} (1+x^2)\eta_c(x)|\mathfrak{z}(x)|^2\,dx \\
&\lesssim c^2\Big(|\mathfrak{z}(0)|^2 \underbrace{\int_{\mathbb{R}} (1+x^2)\eta_c(x)\,dx}_{\text{(I)}} + \|\mathfrak{z}'\|_{L^2}^2 \underbrace{\int_{\mathbb{R}} (1+x^2)\eta_c(x)|x|\,dx}_{\text{(II)}}\Big).
\end{aligned}$$

Now we show that the last two integrals are uniformly bounded in c, with $|c| \le 1$. Firstly, to do this we observe that

$$\max\{1+x^2,\ (1+x^2)|x|\} \lesssim \cosh(\kappa(c)x),$$

for all $x \in \mathbb{R}$ and $|c| \le 1$. Furthermore, due to the exponential decay of $\eta_c(x)$, one gets

$$\begin{aligned}
\text{(I)} + \text{(II)} &\lesssim \int_{\mathbb{R}} \cosh(\kappa(c)x)\,\eta_c(x)\,dx \\
&= \int_{\mathbb{R}} \cosh(\kappa(c)x)\left(1 - \frac{(\mu_1^2 + \mu_2^2\tanh^2(\kappa(c)x))^2}{4(1+\mu\tanh^2(\kappa(c)x))^2}\right)dx \\
&= \frac{\pi}{4\sqrt{2}}\,\frac{12 - 4\mu_1^2 - \mu_1^4}{\kappa(c)\sqrt{\mu_1^2+\mu_2^2}}.
\end{aligned}$$

So, using this control, from (2.32), we conclude that

$$(2.33)\qquad \int_{\mathbb{R}} ||\phi_c|^4 - \phi_0^4|\,|\mathfrak{z}|^2\,dx \lesssim c^2(|\mathfrak{z}(0)|^2 + \|\mathfrak{z}\|_{\mathcal{H}_{c^*}}^2).$$

To estimate $|\mathfrak{z}(0)|^2$, we consider a cut-off function $\chi \in C^\infty(\mathbb{R}, [0,1])$ such that

$$\chi = 1 \quad \text{on } [-1,1] \qquad \text{and} \qquad \chi = 0 \quad \text{on } \mathbb{R}\setminus[-2,2].$$

Then, using that the functions

$$\chi(x)/\sqrt{\eta_{c^*}(x)} \quad \text{and} \quad \chi'(x)/\sqrt{\eta_{c^*}(x)}$$

are bounded on $\mathbb{R}$ (uniformly for $|c^*| < \mathrm{c}$), combined with the Sobolev embedding, we have

$$(2.34)\qquad |\mathfrak{z}(0)|^2 \le \|\chi\mathfrak{z}\|_{L^\infty}^2 \lesssim \|\chi\mathfrak{z}\|_{L^2}(\|\chi'\mathfrak{z}\|_{L^2} + \|\chi\mathfrak{z}'\|_{L^2}) \lesssim \|\mathfrak{z}\|_{\mathcal{H}_{c^*}}^2.$$

Thus, (2.30) follows by substituting (2.34) into (2.33). Finally, in view of (2.30), and using the relation

$$\|\mathfrak{z}\|_{\mathcal{H}_c}^2 - \|\mathfrak{z}\|_{\mathcal{H}_0}^2 = \int_{\mathbb{R}} (\phi_0^4 - |\phi_c|^4)|\mathfrak{z}|^2\,dx,$$

we check that $\mathcal{H}_c \equiv \mathcal{H}_0$ for all $|c| < \mathrm{c}$, and further we have (2.31). ∎

3. Coercivity of the quintic Ginzburg–Landau energy

In this section we establish that the quintic Ginzburg–Landau energy E_2 given in (1.3) is coercive around ϕ_0 and ϕ_c solitons respectively. First of all, we establish some preliminary notation and results.

We first expand the energy E_2 in (1.3) around ϕ_0 given in (1.8). Let $\mathfrak{z} := \mathfrak{z}_1 + i\mathfrak{z}_2$, with $\mathfrak{z}_1, \mathfrak{z}_2 \in \mathbb{R}$, and define

$$\rho_0(\mathfrak{z}) := |\phi_0 + \mathfrak{z}|^2 - |\phi_0|^2 = 2\,\mathrm{Re}(\phi_0\,\bar{\mathfrak{z}}) + |\mathfrak{z}|^2 = 2\phi_0\,\mathfrak{z}_1 + |\mathfrak{z}|^2. \tag{3.1}$$

Then

$$\begin{aligned}
E_2[\phi_0 + \mathfrak{z}] &= \int_{\mathbb{R}} \Big[|\phi_0' + \mathfrak{z}_x|^2 + \frac{1}{3}(1 - |\phi_0 + \mathfrak{z}|^2)^2(2 + |\phi_0 + \mathfrak{z}|^2)\Big]\,dx \\
&= \int_{\mathbb{R}} \Big[(\phi_0')^2 + 2\,\mathrm{Re}(\phi_0'\,\bar{\mathfrak{z}}_x) + |\mathfrak{z}_x|^2 + \frac{1}{3}(1 - \phi_0^2 - \rho_0)^2(2 + \phi_0^2 + \rho_0)\Big]\,dx \\
&= E_2[\phi_0] - 2\,\mathrm{Re}\int_{\mathbb{R}} \bar{\mathfrak{z}}(\phi_0'' + \eta_0\,\phi_0)\,dx \\
&\quad + \int_{\mathbb{R}} (|\mathfrak{z}_x|^2 - \eta_0|\mathfrak{z}|^2)\,dx + \int_{\mathbb{R}} \Big(\phi_0^2\,\rho_0^2 + \frac{1}{3}\rho_0^3\Big)\,dx,
\end{aligned}$$

thus, using (1.9), we have

$$E_2[\phi_0 + \mathfrak{z}] - E_2[\phi_0] = 2Q_0[\mathfrak{z}] + \mathcal{N}_0[\mathfrak{z}], \tag{3.2}$$

where $Q_0[\mathfrak{z}]$ is the quadratic form

$$Q_0[\mathfrak{z}] := \frac{1}{2}\int_{\mathbb{R}} (|\mathfrak{z}_x|^2 - \eta_0|\mathfrak{z}|^2)\,dx,$$

and $\mathcal{N}_0[\mathfrak{z}]$ is the nonlinear term

$$\mathcal{N}_0[\mathfrak{z}] := \int_{\mathbb{R}} \Big(|\phi_0|^2\rho_0^2 + \frac{1}{3}\rho_0^3\Big)\,dx.$$

In the case $\mathfrak{z} = f$ is a real-valued function belonging to the space $\mathcal{H}^{\mathrm{real}}(\mathbb{R})$ (see (2.10)),

$$Q_0[f] := \frac{1}{2}\int_{\mathbb{R}} [(f')^2 - \eta_0 f^2]\,dx.$$

Then, considering now that $\mathcal{H}^{\mathrm{real}}(\mathbb{R})$ is endowed with the inner product

$$\langle f, g\rangle_0 := \int_{\mathbb{R}} (f'g' + \eta_0 f\,g)\,dx,$$

we have that $(\mathcal{H}^{\mathrm{real}}(\mathbb{R}), \langle\cdot,\cdot\rangle_0)$ is a Hilbert space with the induced norm

$$\|f\|_{\mathcal{H}_0}^2 = \int_{\mathbb{R}} [(f')^2 + \eta_0 f^2]\,dx.$$

For a fixed $f \in \mathcal{H}^{\mathrm{real}}(\mathbb{R})$, we have $g \mapsto \int_{\mathbb{R}} \eta_0 f\, g \in [\mathcal{H}^{\mathrm{real}}(\mathbb{R})]'$. Indeed,

$$\Big| \int_{\mathbb{R}} \eta_0 f\, g\, dx \Big| \le \|\eta_0^{1/2} f\|_{L^2} \|\eta_0^{1/2} g\|_{L^2} \le \|\eta_0^{1/2} f\|_{L^2} \|g\|_{\mathcal{H}_0}.$$

Therefore, by the Riesz theorem, there exists a bounded and self-adjoint operator T_0 such that

$$\langle T_0 f, g\rangle_0 = \int_{\mathbb{R}} \eta_0 f g\, dx \quad \text{for all } g \in \mathcal{H}^{\mathrm{real}}(\mathbb{R}), \tag{3.3}$$

and also

$$\|T_0 f\|_{\mathcal{H}_0} \le \|\eta_0^{1/2} f\|_{L^2}.$$

Moreover, the quadratic form Q_0 satisfies

$$Q_0[f] = \frac{1}{2} \int_{\mathbb{R}} [(f')^2 + \eta_0 f^2]\, dx - \int_{\mathbb{R}} \eta_0 f^2\, dx = \Big\langle \Big(\frac{1}{2}\mathbb{1} - T_0\Big) f, f \Big\rangle_0, \tag{3.4}$$

for all $f \in \mathcal{H}^{\mathrm{real}}(\mathbb{R})$.

Lemma 3.1 (Compactness of T_0). *The operator $T_0 \colon \mathcal{H}^{\mathrm{real}}(\mathbb{R}) \to \mathcal{H}^{\mathrm{real}}(\mathbb{R})$ is compact.*

Proof. Throughout the proof we will use M_j, $j = 1, 2 \dots, 6$, to denote some universal constants. Consider now a sequence $f_n \in \mathcal{H}^{\mathrm{real}}(\mathbb{R})$ such that

$$\|f_n\|_{\mathcal{H}_0}^2 = \|f_n'\|_{L^2}^2 + \|\eta_0^{1/2} f_n\|_{L^2}^2 \le M_1 \quad \text{for all } n \in \mathbb{N}. \tag{3.5}$$

Then we can assume that

$$f_n \rightharpoonup f^* \in \mathcal{H}^{\mathrm{real}}(\mathbb{R}) \quad \text{when } n \to \infty.$$

Claim 1: It holds that

$$\|\eta_0^{1/2} f_n\|_{H^1}^2 \le M_2 \quad \text{for all } n \in \mathbb{N}. \tag{3.6}$$

To obtain this estimate, we note that

$$\|\eta_0^{1/2} f_n\|_{H^1}^2 \backsimeq \|\eta_0^{1/2} f_n\|_{L^2}^2 + \|\eta_0^{1/2} f_n'\|_{L^2}^2 + \Big\| \frac{-2\phi_0^3 \phi_0'}{\eta_0^{1/2}} f_n \Big\|_{L^2}^2. \tag{3.7}$$

Now, using (2.13) and that $|\phi_0| \le 1$, we get

$$\frac{2|\phi_0|^3 |\phi_0'|}{\eta_0^{1/2}} = \frac{2}{\sqrt{3}} \frac{|\phi_0|^3 (2+\phi_0^2)^{1/2}}{(1+\phi_0^2)^{1/2}} (1-\phi_0^2)^{1/2} \le 2\eta_0^{1/2},$$

so we have

$$\Big\| \frac{-2\phi_0^3 \phi_0'}{\eta_0^{1/2}} f_n \Big\|_{L^2} \le 2 \|\eta_0^{1/2} f_n\|_{L^2}. \tag{3.8}$$

Then, using that $\eta_0(x) \le 3\,\mathrm{sech}^2(x)$, combined with (3.5), (3.7) and (3.8), we obtain the statement in (3.6) and so Claim 1 is proved. In particular, from (3.6), we conclude that

$$|f_n(0)| \le \|\eta_0^{1/2} f_n\|_{L^\infty} \le M_3 \quad \text{for all } n \in \mathbb{N}, \tag{3.9}$$

and also we can assume that

$$\eta_0^{1/2} f_n \to \eta_0^{1/2} f^* \in C^0_{\mathrm{loc}}(\mathbb{R}), \tag{3.10}$$

i.e., we get uniform convergence on compact subsets of $\mathbb{R}$.

Claim 2: It holds that

$$\|\eta_0^{1/4} f_n\|_{L^2} \le M_4 \quad \text{for all } n \in \mathbb{N}. \tag{3.11}$$

To prove this estimate we first observe that

$$f_n(x) = f_n(0) + \int_0^x f_n'(s)\,ds \quad \text{for all } n \in \mathbb{N},$$

which implies that

$$\eta_0^{1/4}|f_n(x)| \le \eta_0^{1/4}|f_n(0)| + |x|^{1/2}\,\eta_0^{1/4}\,\|f_n'\|_{L^2}.$$

Then, from (3.5), (3.9) and using the exponential decay of η_0, we get the estimate (3.11) in Claim 2. Now given $\varepsilon > 0$, due to the exponential decay of η_0, we can take $a_\varepsilon > 0$ such that

$$\eta_0^{1/2} < \varepsilon \quad \text{for all } |x| > a_\varepsilon.$$

Then, using (3.11), we have

$$\begin{aligned}\int_{|x|>a_\varepsilon} \eta_0 (f_n - f^*)^2\,dx &\le \varepsilon \int_{|x|>a_\varepsilon} \eta_0^{1/2}(f_n - f^*)^2\,dx \\ &\le \varepsilon \|\eta_0^{1/4}(f_n - f^*)\|_{L^2}^2 \le M_5\,\varepsilon.\end{aligned} \tag{3.12}$$

Then, from (3.10), one gets

$$\|\eta_0^{1/2}(f_n - f^*)\|_{L^\infty(|x|\le a_\varepsilon)} < \varepsilon \quad \text{for all } n > n_\varepsilon, \text{ for some } n_\varepsilon \gg 1.$$

Therefore, by using (3.11), we get

$$\begin{aligned}\int_{|x|\le a_\varepsilon} \eta_0 (f_n - f^*)^2\,dx &\le \varepsilon \int_{|x|\le a_\varepsilon} \eta_0^{1/2}|f_n - f^*|\,dx \\ &\le \varepsilon \|\eta_0^{1/4}\|_{L^2} \|\eta_0^{1/4}(f_n - f^*)\|_{L^2} \le M_6\,\varepsilon.\end{aligned} \tag{3.13}$$

Now, from (3.12) and (3.13), notice that

$$\int_{-\infty}^{+\infty} \eta_0 (f_n - f^*)^2\,dx \lesssim \varepsilon,$$

for all $n \gg n_\varepsilon$, so $\lim_{n\to\infty} \|\eta_0^{1/2}(f_n - f^*)\|_{L^2} = 0$. Finally, from (3.3), we have that

$$\|T_0(f_n - f^*)\|_{\mathcal{H}_0} \le \|\eta_0^{1/2}(f_n - f^*)\|_{L^2},$$

which implies

$$T_0 f_n \xrightarrow{n\to\infty} T_0 f^* \quad \text{in } \mathcal{H}^{\mathrm{real}}(\mathbb{R}),$$

and the proof is completed. ∎

Proposition 3.2 (Coercivity of E_2 around the black soliton). *Let $\mathfrak{z} \in \mathcal{H}(\mathbb{R})$ be such that the perturbation $\phi_0 + \mathfrak{z} \in \mathcal{Z}(\mathbb{R})$ and set $\rho_0 = 2\,\mathrm{Re}(\phi_0 \bar{\mathfrak{z}}) + |\mathfrak{z}|^2$ as in (3.1). Then there exists a universal positive constant $\Lambda_0 > 0$ such that*

$$E_2[\phi_0 + \mathfrak{z}] - E_2[\phi_0] \ge \Lambda_0(\|\mathfrak{z}\|^2_{\mathcal{H}_0} + \|\phi_0^3 \rho_0\|^2_{L^2} + \|\mathfrak{z}_2 \rho_0\|^2_{L^2}) - \frac{1}{\Lambda_0}\|\mathfrak{z}\|^3_{\mathcal{H}_0} \tag{3.14}$$

as long as

$$\int_{\mathbb{R}} \langle \eta_0, \mathfrak{z}\rangle_{\mathbb{C}} = 0, \quad \int_{\mathbb{R}} \langle i\eta_0, \mathfrak{z}\rangle_{\mathbb{C}} = 0 \quad \text{and} \quad \int_{\mathbb{R}} \langle i\phi_0 \eta_0, \mathfrak{z}\rangle_{\mathbb{C}} = 0. \tag{3.15}$$

The proof will be divided into 3 steps.

Step 1. There exists a constant $\Lambda_1 > 0$ such that

$$Q_0[f] \ge \Lambda_1 \langle f, f\rangle_0 = \Lambda_1 \int_{\mathbb{R}} [(f')^2 + \eta_0 f^2]\, dx,$$

for any function $f \in \mathcal{H}^{\mathrm{real}}(\mathbb{R})$ satisfying

$$\int_{\mathbb{R}} f\eta_0\, dx = 0 \quad \text{and} \quad \int_{\mathbb{R}} f\phi_0 \eta_0\, dx = 0.$$

Furthermore, $Q_0[f] \ge 0$ if only the first orthogonality condition is satisfied.

Proof of Step 1. Recall that from (3.4), we have $Q_0[f] = \langle(\frac{1}{2}\mathbb{1} - T_0)f, f\rangle_0 = \langle \widetilde{Q}_0 f, f\rangle_0$, where

$$\widetilde{Q}_0 := \frac{1}{2}\mathbb{1} - T_0.$$

Then, using the spectral theorem, there exists a sequence $\{\lambda_n\}$ of eigenvalues for $\widetilde{Q}_0$ with $\lim_{n\to+\infty} \lambda_n = 1/2$ and a Hilbert basis $\{e_n\}$ of $\mathcal{H}^{\mathrm{real}}(\mathbb{R})$ such that

$$\widetilde{Q}_0 e_n = \lambda_n e_n, \quad n \in \mathbb{N}.$$

Notice that

$$Q_0[f] \le \frac{1}{2}\langle f, f\rangle_0 \quad \text{for all } f \in \mathcal{H}_0^{\mathrm{real}}(\mathbb{R}),$$

consequently,

$$\widetilde{Q}_0 \le \frac{1}{2}\mathbb{1}, \quad (\lambda_n)_{n\in\mathbb{N}} \subset (-\infty, 1/2] \quad \text{and} \quad \lambda_n \nearrow \frac{1}{2}.$$

Now, let $\lambda \in (-\infty, 1/2]$ be an eigenvalue with f as the corresponding eigenfunction. Then, for all $g \in \mathcal{H}^{\mathrm{real}}(\mathbb{R})$, we have

$$\langle \widetilde{Q}_0 f, g\rangle_0 = \lambda\langle f, g\rangle_0.$$

So, from (3.3), it holds that

$$\frac{1}{2}\int_{\mathbb{R}} f'g'\,dx - \frac{1}{2}\int_{\mathbb{R}} \eta_0 f\, g\,dx = \lambda\Big[\int_{\mathbb{R}} f'g'\,dx + \int_{\mathbb{R}} \eta_0 f\, g\,dx\Big],$$

which yields

$$\int_{\mathbb{R}} [(1-2\lambda) f'' + (2\lambda+1)\eta_0 f]\, g\,dx = 0,$$

for all $g \in \mathcal{H}^{\text{real}}(\mathbb{R})$. Thus,

$$(1-2\lambda) f'' + (2\lambda+1)\eta_0 f = 0,$$

which implies

$$-f'' - \eta_0 f = \frac{4\lambda}{1-2\lambda}\,\eta_0 f,$$

and therefore,

- $f = 1 =: e_0$ is a solution for $\lambda = -1/2$,
- $f = \phi_0 =: e_1$ is a solution for $\lambda = 0$.

Note that, since ϕ_0 has exactly one zero, the Sturm–Liouville theory guarantees that $\lambda = -1/2$ is the only negative eigenvalue of $\widetilde{Q}_0$ with kernel given by the span(ϕ_0). More precisely,

$$\lambda_0 = -\frac{1}{2} < 0 = \lambda_1 < \lambda_2 < \cdots < \frac{1}{2},$$

and

$$\operatorname{Ker}\Big(\widetilde{Q}_0 + \frac{1}{2}\mathbb{1}\Big) = \mathbb{R}, \quad \operatorname{Ker}(\widetilde{Q}_0) = \mathbb{R}\cdot\phi_0.$$

Thus, expanding $f \in \mathcal{H}^{\text{real}}$ on the normalized basis of eigenfunctions

$$f = \sum_{n=0}^{+\infty} \langle f, \tilde{e}_n\rangle_0\, \tilde{e}_n, \quad \tilde{e}_n = \frac{e_n}{\|e_n\|_{\mathcal{H}_0}},$$

if $\langle f, 1\rangle_0 = \langle f, \phi_0\rangle_0 = 0$, we get $f = \sum_{n=2}^{+\infty}\langle f, \tilde{e}_n\rangle_0\, \tilde{e}_n$, and then

$$Q_0[f] = \langle \widetilde{Q}_0[f], f\rangle_0 = \sum_{n=2}^{+\infty} \lambda_n \langle f, \tilde{e}_n\rangle_0^2 \geq \lambda_2 \sum_{n=2}^{+\infty}\langle f, \tilde{e}_n\rangle_0^2 = \lambda_2 \langle f, f\rangle_0. \tag{3.16}$$

Hence, under the hypothesis

$$\langle f, 1\rangle_0 = \int_{\mathbb{R}} \eta_0 f\,dx = 0 \quad \text{and} \quad \langle f, \phi_0\rangle_0 = 2\int_{\mathbb{R}} f\,\phi_0\,\eta_0 = 0,$$

and taking $\Lambda_1 := \lambda_2$, we complete the proof of Step 1. ∎

Step 2: Let $\mathfrak{z} \in \mathcal{H}(\mathbb{R})$ fulfilling the orthogonality conditions (3.15). Then it follows that

$$E_2[\phi_0 + \mathfrak{z}] - E_2[\phi_0] \geq 2Q_0[\mathfrak{z}_1] + \frac{2}{3}\,\|\phi_0\,\rho_0\|_{L^2}^2 + \frac{1}{3}\|\mathfrak{z}_2\,\rho_0\|_{L^2}^2 + 2\Lambda_1\|\mathfrak{z}_2\|_{\mathcal{H}_0}^2,$$

with Λ_1 as in Step 1.

Proof of Step 2. Recall that, from the expansion of E_2 in (3.2), we have

$$E_2[\phi_0 + \mathfrak{z}] - E_2[\phi_0] = 2Q_0[\mathfrak{z}] + \mathcal{N}_0[\mathfrak{z}],$$

where $\mathfrak{z} = \mathfrak{z}_1 + i\mathfrak{z}_2$, $\rho_0 = 2\phi_0\mathfrak{z}_1 + |\mathfrak{z}|^2$ and Q_0 satisfying $Q_0[\mathfrak{z}] = Q_0[\mathfrak{z}_1] + Q_0[\mathfrak{z}_2]$. Applying Young's inequality, we obtain the estimate

$$\begin{aligned}
\mathcal{N}_0[\mathfrak{z}] &= \int_{\mathbb{R}} \Big(\phi_0^2\rho_0^2 + \frac{1}{3}\rho_0^3\Big)\,dx \\
&= \int_{\mathbb{R}} \phi_0^2\rho_0^2\,dx + \frac{1}{3}\int_{\mathbb{R}} (2\phi_0\mathfrak{z}_1 + |\mathfrak{z}|^2)\rho_0^2\,dx \\
&\geq \int_{\mathbb{R}} \phi_0^2\rho_0^2\,dx + \frac{1}{3}\int_{\mathbb{R}} |\mathfrak{z}|^2\rho_0^2\,dx - \Big|\frac{2}{3}\int_{\mathbb{R}} \phi_0\mathfrak{z}_1\rho_0^2\,dx\Big| \\
&\geq \frac{2}{3}\int_{\mathbb{R}} \phi_0^2\rho_0^2\,dx + \frac{1}{3}\int_{\mathbb{R}} (|\mathfrak{z}|^2 - \mathfrak{z}_1^2)\rho_0^2\,dx \\
&= \frac{2}{3}\int_{\mathbb{R}} \phi_0^2\rho_0^2\,dx + \frac{1}{3}\int_{\mathbb{R}} \mathfrak{z}_2^2\rho_0^2\,dx.
\end{aligned} \tag{3.17}$$

On the other hand, from Step 1, the first two orthogonality conditions in (3.15) imply that

$$Q_0[\mathfrak{z}_1] \geq 0 \quad \text{and} \quad Q_0[\mathfrak{z}_2] \geq 0, \tag{3.18}$$

while, in addition, the last orthogonality condition in (3.15) ensures that

$$Q_0[\mathfrak{z}_2] \geq \Lambda_1 \|\mathfrak{z}_2\|_{\mathcal{H}_0}^2, \tag{3.19}$$

where $\Lambda_1 := \lambda_2$ is the first positive eigenvalue obtained in Step 1. Then, putting the bounds given in (3.17), (3.18) and (3.19) into the expansion of E_2, we obtain the claimed estimate in Step 2. ■

Since $Q_0[\mathfrak{z}_1] \geq 0$, in order to complete the proof of Proposition 3.2, we remark that, bearing in mind the estimate in Step 2, we only need to show the coercivity property for the operator Q_0 on the full variable $\mathfrak{z}$. We will explain this in the next step.

Step 3. Now we proceed with the proof of (3.14).

Proof of Step 3. We begin by estimating the term $\frac{2}{3}\|\phi_0\rho_0\|_{L^2}^2$ which appears in the lower estimate of the Step 2:

$$\frac{2}{3}\|\phi_0\rho_0\|_{L^2}^2 = \frac{2}{3}\|\phi_0^3\rho_0\|_{L^2}^2 + I, \tag{3.20}$$

where

$$\begin{aligned}
I &:= \frac{2}{3}\int_{\mathbb{R}} \eta_0\phi_0^2\rho_0^2\,dx \\
&= \frac{2}{3}\int_{\mathbb{R}} \eta_0\phi_0^2|\mathfrak{z}|^4\,dx + \frac{8}{3}\int_{\mathbb{R}} \eta_0\phi_0^4\mathfrak{z}_1^2\,dx + \frac{8}{3}\int_{\mathbb{R}} \eta_0\phi_0^3\mathfrak{z}_1|\mathfrak{z}|^2\,dx \\
&=: I_1 + I_2 + I_3 \geq I_2 - |I_3|.
\end{aligned} \tag{3.21}$$

Now, bearing in mind (1.9) and integrating by parts, we simplify I_3 as follows:

$$\begin{aligned} I_3 &= -\frac{8}{3}\int_{\mathbb{R}} \phi_0''\phi_0^2 \mathfrak{z}_1|\mathfrak{z}|^2\,dx \\ &= \frac{8}{3}\int_{\mathbb{R}} 2(\phi_0')^2\phi_0 \mathfrak{z}_1|\mathfrak{z}|^2\,dx + \frac{8}{3}\int_{\mathbb{R}} \phi_0'\phi_0^2(\mathfrak{z}_1|\mathfrak{z}|^2)'\,dx =: I_{3,1} + I_{3,2}. \end{aligned} \tag{3.22}$$

Using (2.13), the inequality $0 < 1-\phi_0^2 \le 1-\phi_0^4$ and a Gagliardo–Nirenberg inequality, we obtain

$$\begin{aligned} |I_{3,1}| &\le \frac{16}{9}\int_{\mathbb{R}} (1-\phi_0^2)^2\,(2+\phi_0^2)\,|\phi_0|\,|\mathfrak{z}_1|\,|\mathfrak{z}|^2\,dx \\ &\le \frac{16}{3}\,\|(1-\phi_0^2)^{2/3}\mathfrak{z}\|_{L^3}^3 \le \frac{16}{3}\,\|(1-\phi_0^2)^{1/2}\mathfrak{z}\|_{L^3}^3 \\ &\lesssim \|(1-\phi_0^2)^{1/2}\mathfrak{z}\|_{L^2}^{5/2}\,\|((1-\phi_0^2)^{1/2}\mathfrak{z})'\|_{L^2}^{1/2} \\ &\lesssim \|\eta_0^{1/2}\mathfrak{z}\|_{L^2}^{5/2}\left(\|\phi_0\phi_0'(1-\phi_0^2)^{-1/2}\mathfrak{z}\|_{L^2} + \|(1-\phi_0^2)\mathfrak{z}'\|_{L^2}\right)^{1/2} \\ &\lesssim \|\mathfrak{z}\|_{\mathcal{H}_0}^{5/2}\left(\|\phi_0\phi_0'(1-\phi_0^2)^{-1/2}\mathfrak{z}\|_{L^2} + \|\mathfrak{z}'\|_{L^2}\right)^{1/2} \\ &\lesssim \|\mathfrak{z}\|_{\mathcal{H}_0}^{5/2}\left(\|(1-\phi_0^2)^{1/2}\mathfrak{z}\|_{L^2} + \|\mathfrak{z}'\|_{L^2}\right)^{1/2} \lesssim \|\mathfrak{z}\|_{\mathcal{H}_0}^3, \end{aligned} \tag{3.23}$$

and in a similar way, we deduce

$$\begin{aligned} |I_{3,2}| &\le \frac{8}{3\sqrt{3}}\int_{\mathbb{R}} \phi_0^2(1-\phi_0^2)(2+\phi_0^2)^{1/2}\,|\mathfrak{z}_1'(3\mathfrak{z}_1^2+\mathfrak{z}_2^2) + 2\mathfrak{z}_1\mathfrak{z}_2\mathfrak{z}_2'|\,dx \\ &\le \frac{8}{3}\int_{\mathbb{R}} (1-\phi_0^2)\,|\mathfrak{z}_1'(3\mathfrak{z}_1^2+\mathfrak{z}_2^2) + 2\mathfrak{z}_1\mathfrak{z}_2\mathfrak{z}_2'|\,dx \\ &\lesssim \int_{\mathbb{R}} (1-\phi_0^2)\,(|\mathfrak{z}_1'| + |\mathfrak{z}_2'|)|\mathfrak{z}|^2\,dx \\ &\lesssim \||\mathfrak{z}_1'| + |\mathfrak{z}_2'|\|_{L^2}\,\|(1-\phi_0^2)^{1/2}\mathfrak{z}\|_{L^4}^2 \\ &\lesssim \|\mathfrak{z}\|_{\mathcal{H}_0}\,\|(1-\phi_0^2)^{1/2}\mathfrak{z}\|_{L^2}^{3/2}\,\|((1-\phi_0^2)^{1/2}\mathfrak{z})'\|_{L^2}^{1/2} \lesssim \|\mathfrak{z}\|_{\mathcal{H}_0}^3. \end{aligned} \tag{3.24}$$

Therefore, combining (3.20), (3.21), (3.22), (3.23) and (3.24), we get, for some positive number γ,

$$\frac{2}{3}\,\|\phi_0\rho_0\|_{L^2}^2 \ge \frac{2}{3}\,\|\phi_0^3\rho_0\|_{L^2}^2 + \frac{8}{3}\int_{\mathbb{R}} \eta_0\phi_0^4\mathfrak{z}_1^2\,dx - \gamma\|\mathfrak{z}\|_{\mathcal{H}_0}^3. \tag{3.25}$$

Now, we consider the real function $\tilde{\mathfrak{z}}_1 := \mathfrak{z}_1 - \langle \mathfrak{z}_1, e_1\rangle_0 e_1$, where $e_1 = \phi_0/\|\phi_0\|_{\mathcal{H}_0}$. Then, using the first orthogonality condition in (3.15), we have $\langle \tilde{\mathfrak{z}}_1, e_0\rangle_0 = \langle \tilde{\mathfrak{z}}_1, e_1\rangle_0 = 0$. Thus, the expansion of $\tilde{\mathfrak{z}}_1$ is given by

$$\tilde{\mathfrak{z}}_1 = \sum_{n=2}^{+\infty}\langle \tilde{\mathfrak{z}}_1, e_n\rangle_0\, e_n \quad \text{and} \quad Q_0[\mathfrak{z}_1] = Q_0[\tilde{\mathfrak{z}}_1].$$

Hence, from Step 1, it follows that

$$Q_0[\mathfrak{z}_1] \ge \lambda_2\|\mathfrak{z}_1 - \langle \mathfrak{z}_1, e_1\rangle_0 e_1\|_{\mathcal{H}_0}^2 = \lambda_2\|\mathfrak{z}_1\|_{\mathcal{H}_0}^2 - \lambda_2\langle \mathfrak{z}_1, e_1\rangle_0^2.$$

Now, for any number $0 < \nu < 1$ which will be chosen later, using the identities (2.14) and

$$\|\phi_0\|_{\mathcal{H}_0}^2 = 2\|\phi_0'\|_{L^2}^2 \quad \text{and} \quad \int_{\mathbb{R}} \eta_0 \, dx = 3,$$

combined with the Cauchy–Schwarz inequality, we have

$$\begin{aligned} Q_0[\mathfrak{z}_1] &\geq \lambda_2 \|\mathfrak{z}_1\|_{\mathcal{H}_0}^2 - \lambda_2 \langle \mathfrak{z}_1, e_1\rangle_0^2 \\ &= \lambda_2 \|\mathfrak{z}_1'\|_{L^2}^2 + \lambda_2 \|\eta_0^{1/2}\mathfrak{z}_1\|_{L^2}^2 \\ &\quad - \frac{\lambda_2}{\|\phi_0\|_{\mathcal{H}_0}^2}\Big((1-\nu)\int_{\mathbb{R}} \mathfrak{z}_1' \phi_0' \, dx + (1+\nu)\int_{\mathbb{R}} \eta_0 \mathfrak{z}_1 \phi_0 \, dx\Big)^2 \\ &\geq \lambda_2 \|\mathfrak{z}_1'\|_{L^2}^2 + \lambda_2 \|\eta_0^{1/2}\mathfrak{z}_1\|_{L^2}^2 \\ &\quad - \frac{2\lambda_2}{\|\phi_0\|_{\mathcal{H}_0}^2}\Big((1-\nu)^2 \|\mathfrak{z}_1'\|_{L^2}^2 \|\phi_0'\|_{L^2}^2 + 3(1+\nu)^2 \int_{\mathbb{R}} \eta_0 \mathfrak{z}_1^2 \phi_0^2 \, dx\Big) \\ &\geq \lambda_2 (1-(1-\nu)^2)\|\mathfrak{z}_1'\|_{L^2}^2 + \lambda_2 \|\eta_0^{1/2}\mathfrak{z}_1\|_{L^2}^2 - 6\lambda_2 \frac{(1+\nu)^2}{\|\phi_0\|_{\mathcal{H}_0}^2}\int_{\mathbb{R}} \eta_0 \mathfrak{z}_1^2 \phi_0^2 \, dx. \end{aligned} \tag{3.26}$$

Now, using the estimates $\phi_0^2 \leq 1/4 + \phi_0^4$ and $\lambda_2 < 1/2$ in (3.26) allows us to select a positive constant Λ_ν such that

$$\begin{aligned} Q_0[\mathfrak{z}_1] &\geq \lambda_2\big(1-(1-\nu)^2\big)\|\mathfrak{z}_1'\|_{L^2}^2 + \lambda_2 \|\eta_0^{1/2}\mathfrak{z}_1\|_{L^2}^2 \\ &\quad - \frac{3\lambda_2(1+\nu)^2}{2\|\phi_0\|_{\mathcal{H}_0}^2}\int_{\mathbb{R}} \eta_0 \mathfrak{z}_1^2 \, dx - \frac{6\lambda_2(1+\nu)^2}{\|\phi_0\|_{\mathcal{H}_0}^2}\int_{\mathbb{R}} \eta_0 \mathfrak{z}_1^2 \phi_0^4 \, dx \\ &\geq \lambda_2(1-(1-\nu)^2)\|\mathfrak{z}_1'\|_{L^2}^2 + \lambda_2\Big(1 - \frac{3(1+\nu)^2}{2\|\phi_0\|_{\mathcal{H}_0}^2}\Big)\|\eta_0^{1/2}\mathfrak{z}_1\|_{L^2}^2 \\ &\quad - \frac{3(1+\nu)^2}{\|\phi_0\|_{\mathcal{H}_0}^2}\int_{\mathbb{R}} \eta_0 \mathfrak{z}_1^2 \phi_0^4 \, dx \\ &\geq \Lambda_\nu \|\mathfrak{z}_1\|_{\mathcal{H}_0}^2 - \frac{3(1+\nu)^2}{\|\phi_0\|_{\mathcal{H}_0}^2}\int_{\mathbb{R}} \eta_0 \mathfrak{z}_1^2 \phi_0^4 \, dx, \end{aligned} \tag{3.27}$$

which holds under the restriction

$$\frac{3(1+\nu)^2}{2\|\phi_0\|_{\mathcal{H}_0}^2} < 1, \tag{3.28}$$

valid for small enough ν because $\|\phi_0\|_{\mathcal{H}_0}^2 \approx 2.28 > 3/2$ (see (2.14)).

On the other hand, from Step 2 and (3.25), we have

$$\begin{aligned} E_2[\phi_0 + \mathfrak{z}] - E_2[\phi_0] &\geq 2\lambda_2 \|\mathfrak{z}_2\|_{\mathcal{H}_0}^2 + \frac{2}{3}\|\phi_0^3 \rho_0\|_{L^2}^2 + \frac{1}{3}\|\mathfrak{z}_2 \rho_0\|_{L^2}^2 \\ &\quad + 2Q_0[\mathfrak{z}_1] + \frac{8}{3}\int_{\mathbb{R}} \eta_0 \phi_0^4 \mathfrak{z}_1^2 - \gamma \|\mathfrak{z}\|_{\mathcal{H}_0}^3. \end{aligned} \tag{3.29}$$

Then, substituting (3.27) in (3.29), we obtain the estimate

$$E_2[\phi_0+\mathfrak{z}] - E_2[\phi_0] \geq 2\Lambda_\nu \|\mathfrak{z}_1\|^2_{\mathcal{H}_0} + 2\lambda_2\|\mathfrak{z}_2\|^2_{\mathcal{H}_0} + \frac{2}{3}\,\|\phi_0^3\,\rho_0\|^2_{L^2} + \frac{1}{3}\|\mathfrak{z}_2\,\rho_0\|^2_{L^2} - \gamma\|\mathfrak{z}\|^3_{\mathcal{H}_0},$$

which holds if

$$\frac{6(1+\nu)^2}{\|\phi_0\|^2_{\mathcal{H}_0}} < \frac{8}{3}. \tag{3.30}$$

One can find such a ν because $\|\phi_0\|^2_{\mathcal{H}_0} \approx 2.28 > 9/4$.

Finally, since conditions (3.28) and (3.30) are satisfied for a small enough positive number ν, there exists a universal constant Λ_0 satisfying inequality (3.14), and the proof is complete. ■

Remark 3.3. We recall that in the case of cubic GP treated in [13], the corresponding black soliton (denoted as U_0) satisfies the relation $U_0' = \frac{1}{\sqrt{2}}(1-U_0^2)$, and hence the orthogonality condition

$$\langle f, 1\rangle_0 = \int_{\mathbb{R}} (1-U_0^2) f\,dx = \sqrt{2}\int_{\mathbb{R}} U_0' f\,dx = 0,$$

but in the case of the quintic GP (1.1) this relation is not satisfied anymore.

Now we perturb the black soliton ϕ_0 of (1.1), given in (1.8), with a function $u \in \mathcal{H}(\mathbb{R})$ belonging to the orbit generated by the symmetries of (1.1), namely,

$$\mathcal{U}_0(\alpha) := \big\{w \in \mathcal{H}(\mathbb{R}) : \inf_{(b,\iota)\in\mathbb{R}^2} \|e^{-i\iota}w(\cdot+b) - \phi_0\|_{\mathcal{H}_0(\mathbb{R})} < \alpha\big\}, \tag{3.31}$$

for some $\alpha > 0$. Then, given a function $u \in \mathcal{U}_0(\alpha)$, we can choose $(c, b, \iota) \in (-2,2)\times\mathbb{R}^2$ in such a way that

$$e^{-i\iota}u(\cdot+b) = \phi_c + \mathfrak{z},$$

with $\mathfrak{z}$ satisfying the orthogonality conditions (3.35) around the dark soliton.

Finally, note that we can define the following tubular subset of $\mathcal{U}_0(\alpha)$:

$$\mathcal{V}_0(\alpha) := \big\{v \in \mathcal{Z}(\mathbb{R}) : \inf_{(b,\iota)\in\mathbb{R}^2} d_0(e^{-i\iota}v(\cdot+b), \phi_0) < \alpha\big\} \subset \mathcal{U}_0(\alpha). \tag{3.32}$$

Coming back to the main question on the orbital stability of the black soliton, we use the coercivity of E_2 around the black soliton ϕ_0 to small perturbations around the dark soliton ϕ_c (see Proposition 3.4). In fact, the idea to introduce a dark soliton family in this argument is to give an extra degree of freedom which allows us to satisfy the third constraint in (3.2) rewritten as (3.35).

In that case, the situation is different with respect to the cubic GP equation, because we cannot assume the cancellation of the linear term $\langle i\phi_c', \mathfrak{z}\rangle_{\mathbb{C}}$ in our approach, given the orthogonality conditions arising naturally, from the particular structure of the associated spectral problem as we already saw in Proposition 3.2, e.g., (3.16). In fact, this extra technical difficulty introduced by the linear term $\langle i\phi_c', \mathfrak{z}\rangle_{\mathbb{C}}$ is overcome in Proposition 3.4, by using a previously computed L^2 norm, see (2.22).

Before establishing the next result, with (2.8), we fix the following notation:

$$\rho_c(\mathfrak{z}) := |\phi_c+\mathfrak{z}|^2 - |\phi_c|^2 = 2\langle\phi_c, \mathfrak{z}\rangle_{\mathbb{C}} + |\mathfrak{z}|^2 = 2\,\mathrm{Re}(\phi_c\,\bar{\mathfrak{z}}) + |\mathfrak{z}|^2. \tag{3.33}$$

Proposition 3.4 (Coercivity of E_2 around the dark soliton). *There exists* $\mathfrak{c} \in (0,2)$ *small enough*[2] *such that the following holds. For all* $|c| \leq \mathfrak{c}$ *and for any* $\mathfrak{z} \in \mathcal{H}(\mathbb{R})$ *satisfying*

$$\phi_c + \mathfrak{z} \in \mathcal{Z}(\mathbb{R}), \quad \text{with } \|\rho_c(\mathfrak{z})\|_{L^2} < C, \tag{3.34}$$

for some constant C *and the generalized orthogonality conditions*

$$\int_{\mathbb{R}} \langle \eta_c, \mathfrak{z}\rangle_{\mathbb{C}} = 0, \quad \int_{\mathbb{R}} \langle i\eta_c, \mathfrak{z}\rangle_{\mathbb{C}} = 0 \quad \text{and} \quad \int_{\mathbb{R}} \langle iR_c\eta_c, \mathfrak{z}\rangle_{\mathbb{C}} = 0, \tag{3.35}$$

there exists $\tilde{\Gamma} > 0$, *not depending on* $\mathfrak{c}$, *such that*

$$E_2[\phi_c + \mathfrak{z}] - E_2[\phi_0] \geq \tilde{\Gamma}(\|\mathfrak{z}\|^2_{\mathcal{H}_0} + \|\phi_c^3\rho_c\|^2_{L^2}) - \frac{1}{\tilde{\Gamma}}(c^2 + \|\mathfrak{z}\|^3_{\mathcal{H}_0}). \tag{3.36}$$

Remark 3.5. Note that the quadratic term $\|\mathfrak{z}_2\rho_c\|^2_{L^2}$ is not appearing in (3.36) because the lower bound is already guaranteed only with the current terms. Hereafter, for the sake of simplicity, we will not include such a quadratic term but note that by keeping it, we would recover (3.14) in the limit $c \to 0$.

Proof. First of all, we recall that $\mathfrak{z} = \mathfrak{z}_1 + i\mathfrak{z}_2 \in \mathcal{H}(\mathbb{R})$ and that by defining the quadratic form in $\mathfrak{z}$,

$$Q_c[\mathfrak{z}] := \frac{1}{2}\int_{\mathbb{R}} (|\mathfrak{z}_x|^2 - \eta_c|\mathfrak{z}|^2)\,dx, \tag{3.37}$$

and the nonlinear term

$$\mathcal{N}_c[\mathfrak{z}] := \int_{\mathbb{R}} \left(|\phi_c|^2\rho_c^2 + \frac{1}{3}\rho_c^3\right)dx,$$

we have

$$E_2[\phi_c + \mathfrak{z}] - E_2[\phi_c] = -c\int_{\mathbb{R}} 2\,\mathrm{Re}(i\phi_c'\bar{\mathfrak{z}})\,dx + 2Q_c[\mathfrak{z}] + \mathcal{N}_c[\mathfrak{z}]. \tag{3.38}$$

Also (see (2.16), (2.17) and (2.18)) we already know that for small c,

$$E_2[\phi_c] - E_2[\phi_0] = -\frac{1}{4}(3 + E_2[\phi_0])c^2 + \mathcal{O}(c^3) \geq -2\sqrt{3}\,c^2. \tag{3.39}$$

We recall (see (3.2)) that

$$E_2[\phi_0 + \mathfrak{z}] - E_2[\phi_0] = 2Q_0[\mathfrak{z}_1] + 2Q_0[\mathfrak{z}_2] + \mathcal{N}_0[\mathfrak{z}],$$

which implies

$$\begin{aligned} E_2[\phi_c + \mathfrak{z}] - E_2[\phi_0] \\ &= (E_2[\phi_c + \mathfrak{z}] - E_2[\phi_0 + \mathfrak{z}]) + 2Q_0[\mathfrak{z}_1] + 2Q_0[\mathfrak{z}_2] + \mathcal{N}_0[\mathfrak{z}]. \end{aligned} \tag{3.40}$$

[2] Note that $\mathfrak{c}$ is chosen as the minimum of the values that guarantee that some precise estimates in the proof hold for, e.g., (2.30) and (3.39).

We begin by computing the first term of the right-hand side of (3.40). Note that subtracting (3.38) and (3.2), we have

(3.41)

$$\begin{aligned} &E_2[\phi_c + \mathfrak{z}] - E_2[\phi_0 + \mathfrak{z}] \\ &\quad = E_2[\phi_c] - E_2[\phi_0] - c\int_{\mathbb{R}} 2\,\mathrm{Re}(i\phi_c'\bar{\mathfrak{z}})\,dx + \int_{\mathbb{R}} (|\phi_c|^4 - \phi_0^4)|\mathfrak{z}|^2\,dx + \Delta\mathcal{N}[\mathfrak{z}], \end{aligned}$$

where

$$\Delta\mathcal{N}[\mathfrak{z}] := \mathcal{N}_c[\mathfrak{z}] - \mathcal{N}_0[\mathfrak{z}] = \int_{\mathbb{R}} (|\phi_c|^2\rho_c^2 - \phi_0^2\rho_0^2)\,dx + \int_{\mathbb{R}} \frac{1}{3}(\rho_c^3 - \rho_0^3)\,dx.$$

Substituting (3.41) in the right-hand side of (3.40), we get

$$\begin{aligned} (3.42)\quad E_2[\phi_c + \mathfrak{z}] - E_2[\phi_0] &= 2Q_0[\mathfrak{z}_1] + 2Q_0[\mathfrak{z}_2] + \big(E_2[\phi_c] - E_2[\phi_0]\big) \\ &\quad + \mathcal{N}_c[\mathfrak{z}] - c\int_{\mathbb{R}} 2\,\mathrm{Re}(i\phi_c'\bar{\mathfrak{z}})\,dx + \int_{\mathbb{R}} \big(|\phi_c|^4 - \phi_0^4\big)|\mathfrak{z}|^2 dx. \end{aligned}$$

Hereinafter, unless otherwise noted, we shall consider the constant $\mathfrak{c}$ as defined in (2.31). Now we proceed to estimate the last three terms in (3.42), and we begin by the last integral. Using (2.30), we have

$$(3.43)\qquad \Big|\int_{\mathbb{R}} (|\phi_c|^4 - \phi_0^4)|\mathfrak{z}|^2\,dx\Big| \leq \int_{\mathbb{R}} \big||\phi_c|^4 - \phi_0^4\big||\mathfrak{z}|^2\,dx \leq \beta c^2\|\mathfrak{z}\|_{\mathcal{H}_0}^2,$$

for some positive constant β and all $|c| \leq \mathfrak{c}$. Now we continue estimating the linear term $-c\int_{\mathbb{R}} 2\,\mathrm{Re}(i\phi_c'\bar{\mathfrak{z}})\,dx$. In fact, we use (2.31) and (2.22), to get

$$(3.44)\qquad \Big|-c\int_{\mathbb{R}} 2\,\mathrm{Re}(i\phi_c'\bar{\mathfrak{z}})\,dx\Big| \leq 2|c|\Big\|\frac{\phi_c'}{\sqrt{\eta_c}}\Big\|_{L^2}\|\sqrt{\eta_c}\,\mathfrak{z}\|_{L^2} \leq \beta|c|\,\|\mathfrak{z}\|_{\mathcal{H}_0},$$

with a larger constant β if necessary, and all $|c| \leq \mathfrak{c}$. Now we estimate the nonlinear term $\mathcal{N}_c[\mathfrak{z}]$. Using that $|\mathrm{Re}(\phi_c\bar{\mathfrak{z}})| \leq (|\phi_c|^2 + |\mathfrak{z}|^2)/2$, we get

$$\begin{aligned} (3.45)\qquad \mathcal{N}_c[\mathfrak{z}] &= \int_{\mathbb{R}} |\phi_c|^2\rho_c^2\,dx + \frac{1}{3}\int_{\mathbb{R}} (2\,\mathrm{Re}(\phi_c\bar{\mathfrak{z}}) + |\mathfrak{z}|^2)\rho_c^2\,dx \\ &\geq \int_{\mathbb{R}} |\phi_c|^2\rho_c^2\,dx + \frac{1}{3}\int_{\mathbb{R}} |\mathfrak{z}|^2\rho_c^2\,dx - \frac{2}{3}\Big|\int_{\mathbb{R}} \mathrm{Re}(\phi_c\bar{\mathfrak{z}})\rho_c^2\,dx\Big| \\ &\geq \frac{2}{3}\int_{\mathbb{R}} |\phi_c|^2\rho_c^2\,dx. \end{aligned}$$

Combining (3.39), (3.42)–(3.45) and Young's inequality, we obtain

$$\begin{aligned} (3.46)\quad &E_2[\phi_c + \mathfrak{z}] - E_2[\phi_0] \\ &\quad \geq 2Q_0[\mathfrak{z}_1] + 2Q_0[\mathfrak{z}_2] - 2\sqrt{3}\,c^2 + \frac{2}{3}\|\phi_c\rho_c\|_{L^2}^2 - \beta|c|\,\|\mathfrak{z}\|_{\mathcal{H}_0} - \beta c^2\|\mathfrak{z}\|_{\mathcal{H}_0}^2 \\ &\quad \geq 2Q_0[\mathfrak{z}_1] + 2Q_0[\mathfrak{z}_2] - 2\sqrt{3}\,c^2 + \frac{2}{3}\|\phi_c\rho_c\|_{L^2}^2 - \beta_1 c^2 - \beta_2\|\mathfrak{z}\|_{\mathcal{H}_0}^2 - \beta c^2\|\mathfrak{z}\|_{\mathcal{H}_0}^2, \end{aligned}$$

for all $|c| \leq \mathfrak{c}$ and β_2 to be fixed later. Now we split the components of the perturbation $\mathfrak{z} = \mathfrak{z}_1 + i\mathfrak{z}_2$ in the following way:

$$\mathfrak{z}_1 = \mathfrak{z}_1^* + \omega_1(c)\eta_0,$$
$$\mathfrak{z}_2 = \mathfrak{z}_2^* + \omega_2(c)\eta_0 + \omega_3(c)\phi_0\,\eta_0,$$

with ω_1, ω_2 and ω_3 real-valued functions chosen so that $\mathfrak{z}^* := \mathfrak{z}_1^* + i\mathfrak{z}_2^*$ satisfies the orthogonality conditions in (3.15). Thus, using (3.35), the functions ω_i satisfy the relations:

$$(3.47)\qquad \begin{cases} \displaystyle\int_{\mathbb{R}} \langle \eta_0 - \eta_c, \mathfrak{z}\rangle_{\mathbb{C}}\,dx = \int_{\mathbb{R}} \langle \eta_0, \mathfrak{z}\rangle_{\mathbb{C}}\,dx = \omega_1(c)\int_{\mathbb{R}} \eta_0^2\,dx, \\ \displaystyle\int_{\mathbb{R}} \langle i\eta_0 - i\eta_c, \mathfrak{z}\rangle_{\mathbb{C}}\,dx = \int_{\mathbb{R}} \langle i\eta_0, \mathfrak{z}\rangle_{\mathbb{C}}\,dx \\ \displaystyle\qquad = \omega_2(c)\int_{\mathbb{R}} \eta_0^2\,dx + \omega_3(c)\int_{\mathbb{R}} \phi_0\,\eta_0^2\,dx, \\ \displaystyle\int_{\mathbb{R}} \langle i\phi_0\,\eta_0 - iR_c\eta_c, \mathfrak{z}\rangle_{\mathbb{C}}\,dx = \int_{\mathbb{R}} \langle i\phi_0\,\eta_0, \mathfrak{z}\rangle_{\mathbb{C}}\,dx \\ \displaystyle\qquad = \omega_2(c)\int_{\mathbb{R}} \phi_0\,\eta_0^2\,dx + \omega_3(c)\int_{\mathbb{R}} \phi_0^2\,\eta_0^2\,dx, \end{cases}$$

and the system has a solution, because it has a nonvanishing determinant.[3]

Now, using (2.24) and (2.25), the integrals in the left-hand side of (3.47) can be estimated as follows:

$$\left|\int_{\mathbb{R}} \langle \eta_0 - \eta_c, \mathfrak{z}\rangle_{\mathbb{C}}\,dx\right| + \left|\int_{\mathbb{R}} \langle i\eta_0 - i\eta_c, \mathfrak{z}\rangle_{\mathbb{C}}\,dx\right| + \left|\int_{\mathbb{R}} \langle i\phi_0\,\eta_0 - iR_c\eta_c, \mathfrak{z}\rangle_{\mathbb{C}}\,dx\right| \lesssim c^2\|\sqrt{\eta_0}\,\mathfrak{z}\|_{L^2} \lesssim c^2\|\mathfrak{z}\|_{\mathcal{H}_0},$$

and therefore there exists a positive constant β_3 such that

$$(3.48)\qquad |\omega_1(c)| + |\omega_2(c)| + |\omega_3(c)| \leq \beta_3 c^2 \|\mathfrak{z}\|_{\mathcal{H}_0}.$$

Notice that, by using Step 1 of Proposition 3.2 applied to $\mathfrak{z}_1^*$ and $\mathfrak{z}_2^*$, combined with (3.48), we can obtain the following estimates:

$$(3.49)\qquad \begin{aligned} Q_0[\mathfrak{z}_1] &= Q_0[\mathfrak{z}_1^*] - \omega_1^2 Q_0[\eta_0] + \omega_1\Big(\int_{\mathbb{R}} \mathfrak{z}_1'\eta_0'\,dx - \int_{\mathbb{R}} \mathfrak{z}_1\eta_0^2\,dx\Big) \\ &\geq Q_0[\mathfrak{z}_1^*] - \omega_1^2|Q_0[\eta_0]| - |\omega_1|\big(\|\mathfrak{z}_1'\|_{L^2}\|\eta_0'\|_{L^2} + \|\mathfrak{z}_1\sqrt{\eta_0}\|_{L^2}\|\eta_0^{3/2}\|_{L^2}\big) \\ &\geq Q_0[\mathfrak{z}_1^*] - \beta_4 c^2\|\mathfrak{z}\|_{\mathcal{H}_0}^2, \end{aligned}$$

where $Q_0[\mathfrak{z}_1^*] \geq 0$, β_4 is a positive constant and $|c| < \mathfrak{c} < 1$. Analogously, since $\mathfrak{z}_2^*$ verifies the inequality $Q_0[\mathfrak{z}_2^*] \geq \Lambda_1\|\mathfrak{z}_2^*\|_{\mathcal{H}_0}^2$, we deduce that

$$(3.50)\qquad Q_0[\mathfrak{z}_2] \geq \Lambda_1\|\mathfrak{z}_2\|_{\mathcal{H}_0}^2 - \beta_4 c^2\|\mathfrak{z}\|_{\mathcal{H}_0}^2.$$

for a larger β_4 if necessary and for all $|c| < \mathfrak{c} < 1$.

[3] Note that for parity reasons, $\int_{\mathbb{R}} \phi_0\,\eta_0^2\,dx = 0$.

On the other hand, since $\mathfrak{z}_1^*$ is orthogonal to η_0 in L^2, the same arguments used to obtain (3.27) in Step 3 of Proposition 3.2 allow us to conclude the existence of positive Λ_ν such that

$$
\begin{aligned}
(3.51)\qquad Q_0[\mathfrak{z}_1^*] &\geq \Lambda_\nu \|\mathfrak{z}_1^*\|^2_{\mathcal{H}_0} - \frac{3(1+\nu)^2}{\|\phi_0\|^2_{\mathcal{H}_0}} \int_{\mathbb{R}} \eta_0\, {\mathfrak{z}_1^*}^2 \phi_0^4\, dx,\\
&\geq \Lambda_\nu \|\mathfrak{z}_1\|^2_{\mathcal{H}_0} - \beta_5 c^2 \|\mathfrak{z}\|^2_{\mathcal{H}_0} - \underbrace{\frac{3(1+\nu)^2}{\|\phi_0\|^2_{\mathcal{H}_0}} \int_{\mathbb{R}} \eta_0\, \mathfrak{z}_1^2 \phi_0^4\, dx}_{J[\mathfrak{z}_1]}
\end{aligned}
$$

with $0<\nu<1$ such that $\frac{3(1+\nu)^2}{2\|\phi_0\|^2_{\mathcal{H}_0}} < 1$.

Now, combining (3.49), (3.50) and (3.51), we have

$$
(3.52)\qquad Q_0[\mathfrak{z}_1] + Q_0[\mathfrak{z}_2] \geq \Lambda_\nu \|\mathfrak{z}_1\|^2_{\mathcal{H}_0} + \Lambda_1 \|\mathfrak{z}_2\|^2_{\mathcal{H}_0} - (2\beta_4 + \beta_5)c^2 \|\mathfrak{z}\|^2_{\mathcal{H}_0} - J[\mathfrak{z}_1],
$$

and by substituting (3.52) in (3.46), it follows that

$$
\begin{aligned}
(3.53)\qquad E_2[\phi_c + \mathfrak{z}] - E_2[\phi_0] \geq{}& 2\Lambda_\nu \|\mathfrak{z}_1\|^2_{\mathcal{H}_0} + 2\Lambda_1 \|\mathfrak{z}_2\|^2_{\mathcal{H}_0} + \frac{2}{3} \|\phi_c \rho_c\|^2_{L^2}\\
&- 2J[\mathfrak{z}_1] - \beta_2 \|\mathfrak{z}\|^2_{\mathcal{H}_0} - (\beta_1 + 2\sqrt{3})c^2 - \beta_6 c^2 \|\mathfrak{z}\|^2_{\mathcal{H}_0},
\end{aligned}
$$

where $\beta_6 = \beta + 4\beta_4 + 2\beta_5$. At this point, to control the effect of $J[\mathfrak{z}_1]$ on the lower bound of (3.53), we shall proceed in a similar way as in (3.20)–(3.21), estimating the following L^2 norm:

$$
\begin{aligned}
\frac{2}{3} \|\phi_c \rho_c\|^2_{L^2} \geq{}& \frac{2}{3} \|\phi_c^3 \rho_c\|^2_{L^2} + \frac{8}{3} \int_{\mathbb{R}} \eta_0 \phi_0^4 \mathfrak{z}_1^2\, dx + \frac{8}{3} \underbrace{\int_{\mathbb{R}} (\eta_c |\phi_c|^2 R_c^2 - \eta_0 \phi_0^4) \mathfrak{z}_1^2\, dx}_{J_a[\mathfrak{z}]}\\
&- \frac{8}{3} \underbrace{\int_{\mathbb{R}} \eta_c |\phi_c|^2 |R_c \mathfrak{z}_1 + I_c \mathfrak{z}_2|\, |\mathfrak{z}|^2\, dx}_{J_b[\mathfrak{z}]} - \frac{16}{3} \underbrace{\int_{\mathbb{R}} \eta_c |\phi_c|^2 |R_c I_c \mathfrak{z}_1 \mathfrak{z}_2|\, dx}_{J_c[\mathfrak{z}]},
\end{aligned}
$$

where R_c, I_c are defined in (2.6), (2.7). Using that $|\phi_c| \leq 1$, (2.23) and (2.31), one gets

$$
|J_c[\mathfrak{z}]| \lesssim c \|\mathfrak{z}\|^2_{\mathcal{H}_0}.
$$

On the other hand, since $|\phi_c| \leq 1$, we have

$$
\begin{aligned}
J_b[\mathfrak{z}] \leq \int_{\mathbb{R}} \eta_c |\phi_c|^3 |\mathfrak{z}|^3\, dx &\leq \int_{\mathbb{R}} (\eta_c |\phi_c|^2 - \eta_0 \phi_0^2) |\mathfrak{z}|^3\, dx + \int_{\mathbb{R}} \eta_0 \phi_0^2 |\mathfrak{z}|^3\, dx\\
&=: J_{b_1}[\mathfrak{z}] + J_{b_2}[\mathfrak{z}].
\end{aligned}
$$

Note that

$$
J_{b_2}[\mathfrak{z}] = -\int_{\mathbb{R}} \phi_0'' \phi_0 |\mathfrak{z}|^3\, dx,
$$

so in a similar way as for the integral I_3 in (3.22), we have

$$|J_{b_2}[\mathfrak{z}]| \lesssim \|\mathfrak{z}\|^3_{\mathcal{H}_0}.$$

Recalling that (see Appendix C for details)

$$\|\mathfrak{z}\|_{L^\infty} \lesssim (1 + \|\rho_c\|_{L^2})(1 + \|\mathfrak{z}\|_{\mathcal{H}_0}), \tag{3.54}$$

then using (3.34), we obtain

$$\|\mathfrak{z}\|_{L^\infty} \lesssim 1 + \|\mathfrak{z}\|_{\mathcal{H}_0},$$

and finally, by (2.26), we obtain

$$|J_{b_1}[\mathfrak{z}]| \lesssim (1 + \|\mathfrak{z}\|_{\mathcal{H}_0})^2 \int_{\mathbb{R}} \frac{\eta_c|\phi_c|^2 - \eta_0\phi_0^2}{\sqrt{\eta_0}} \sqrt{\eta_0}\,|\mathfrak{z}|\,dx \lesssim c^2(\|\mathfrak{z}\|_{\mathcal{H}_0} + \|\mathfrak{z}\|^3_{\mathcal{H}_0}).$$

Following the same procedure as for estimating $J_{b_1}[\mathfrak{z}]$ and using (2.27), we get

$$|J_a[\mathfrak{z}]| \lesssim c^2(\|\mathfrak{z}\|_{\mathcal{H}_0} + \|\mathfrak{z}\|^3_{\mathcal{H}_0}).$$

Therefore, collecting estimates J_a, J_b, J_c, we conclude that

$$\begin{aligned}\frac{2}{3}\|\phi_c\rho_c\|^2_{L^2} \geq \frac{2}{3}\|\phi_c^3\rho_c\|^2_{L^2} + \frac{8}{3}\int_{\mathbb{R}} \eta_c|\phi_c|^2 R_c^2\mathfrak{z}_1^2\,dx - \gamma_1\|\mathfrak{z}\|^3_{\mathcal{H}_0}\\ - \gamma_2(c + c^2)\|\mathfrak{z}\|^2_{\mathcal{H}_0} - \beta_7 c^2,\end{aligned} \tag{3.55}$$

for some positive numbers $\gamma_1, \gamma_2, \beta_7$. Finally, we fix β_2 such that $\beta_2 < \min(\Lambda_1, \Lambda_\nu)$ and $0 < \nu < 1$, smaller if necessary, such that

$$\frac{6(1+\nu)^2}{\|\phi_0\|^2_{\mathcal{H}_0}} < \frac{8}{3}.$$

Then, substituting (3.55) into (3.53), the second term on the right-hand side of (3.55) allows to control the integral $J[\mathfrak{z}_1]$, and consequently we can take a positive constant Γ_{c} such that

$$E_2[\phi_c + \mathfrak{z}] - E_2[\phi_0] \geq \Gamma_{\mathrm{c}}(\|\mathfrak{z}\|^2_{\mathcal{H}_0} + \|\phi_c^3\rho_c\|^2_{L^2}) - \frac{1}{\Gamma_{\mathrm{c}}}(c^2 + \|\mathfrak{z}\|^3_{\mathcal{H}_0}) \quad \text{for all } |c| < \mathrm{c}.$$

Notice that in the process of obtaining the constant Γ_{c}, we see that this coercivity constant is lower bounded by a constant $\tilde{\Gamma}$ when $\mathrm{c} \to 0$. In other words,

$$\Gamma_{\mathrm{c}} \geq \tilde{\Gamma} \quad \text{and} \quad -1/\Gamma_{\mathrm{c}} \geq -1/\tilde{\Gamma}$$

for all c in a small interval $(0, \tilde{\gamma})$. Hence, we get

$$E_2[\phi_c + \mathfrak{z}] - E_2[\phi_0] \geq \tilde{\Gamma}(\|\mathfrak{z}\|^2_{\mathcal{H}_0} + \|\phi_c^3\rho_c\|^2_{L^2}) - \frac{1}{\tilde{\Gamma}}(c^2 + \|\mathfrak{z}\|^3_{\mathcal{H}_0}),$$

as claimed in (3.36). ∎

4. Modulation of parameters

In order to apply the coercivity property of the quintic Ginzburg–Landau energy E_2 shown in Section 3, we have to ensure that the orthogonality relations hold.

In this section, we prove that there exist small perturbations $\mathfrak{z} \in \mathcal{H}(\mathbb{R})$ such that the orthogonality conditions (3.15) for the black soliton are satisfied. In fact, we will prove a more general result, valid for $c \neq 0$, and dealing with the *generalized orthogonality conditions* (3.35) for perturbations $\mathfrak{z} \in \mathcal{H}(\mathbb{R})$ around the dark soliton, thus obtaining the desired orthogonality conditions related with the black soliton (3.15) in the limit $c = 0$.

Firstly, and for the sake of completeness, we will present a preliminary result on the continuous dependence for the shift and phase parameters b, θ on the corresponding dark soliton profile.

Lemma 4.1. *Let $(c, a, \theta) \in (-\mathfrak{c}, \mathfrak{c}) \times \mathbb{R}^2$ and set $\phi_{c,a,\theta} := e^{i\theta}\phi_c(\cdot - a)$. Given a positive number δ, there exists a positive number $\tilde{\delta}$ such that if*

$$\|\phi_{c,b_1,\theta_1} - \phi_{c,b_2,\theta_2}\|_{\mathcal{H}_c} < \tilde{\delta},$$

then we have $|b_2 - b_1| + |e^{i\theta_2} - e^{i\theta_1}| < \delta$.

Proof. The proof runs exactly as in Lemma 2.1 of [13]. ■

Proposition 4.2 (Modulation). *Let $(c, a, \theta) \in (-\mathfrak{c}, \mathfrak{c}) \times \mathbb{R}^2$. There exist two positive numbers $\tilde{r}_c$ and $\tilde{s}_c$, depending continuously on c, for which there exist a map $(\tilde{c}, \tilde{a}, \tilde{\theta}) : B_{\mathcal{H}_0}(\phi_{c,a,\theta}; \tilde{r}_c) \to (-\mathfrak{c}, \mathfrak{c}) \times \mathbb{R}^2$ with $(\tilde{c}, \tilde{a}, \tilde{\theta})(\phi_{c,a,\theta}) = (c, a, \theta)$ and such that for any $w \in B_{\mathcal{H}_0}(\phi_{c,a,\theta}; \tilde{r}_c)$, the perturbation of the dark soliton profile*

$$\mathfrak{z} := e^{-i\tilde{\theta}(w)} w(\cdot + \tilde{a}(w)) - \phi_{\tilde{c}(w)}$$

satisfies the generalized orthogonality conditions (3.35). Also, $\tilde{c}$, $\tilde{a}$, $\tilde{\theta}$ are C^1-functions in $B_{\mathcal{H}_0}(\phi_{c,a,\theta}, \tilde{r}_c)$, and given any $w \in B_{\mathcal{H}_0}(\phi_{c,a,\theta}; \tilde{r}_c)$, the vector $(\tilde{c}, \tilde{a}, \tilde{\theta})(w)$ is the unique element in the ball $B((c, a, \theta); \tilde{s}_c) \subset \mathbb{R}^3$ satisfying (3.35).

Proof of Proposition 4.2. The proof of this result is a classical application of the implicit function theorem. We begin by considering the functional $F \colon \mathcal{H} \times (-\mathfrak{c}, \mathfrak{c}) \times \mathbb{R}^2 \to \mathbb{R}^3$, given by

$$F(w, \sigma, b, \iota) := \Big(\int_{\mathbb{R}} \langle \eta_\sigma, \mathfrak{z} \rangle_{\mathbb{C}}, \int_{\mathbb{R}} \langle i\eta_\sigma, \mathfrak{z} \rangle_{\mathbb{C}}, \int_{\mathbb{R}} \langle iR_\sigma \eta_\sigma, \mathfrak{z} \rangle_{\mathbb{C}} \Big), \quad \mathfrak{z} := e^{-i\iota} w(\cdot + b) - \phi_\sigma.$$

Notice that, similarly to the context of the cubic GP, see [13], the functional F has C^1-regularity. Recall now the notation introduced in Lemma 4.1:

$$\phi_{c,a,\theta} := e^{i\theta}\phi_c(\cdot - a). \tag{4.1}$$

Then

$$F(\phi_{c,a,\theta}, c, a, \theta) = \mathbf{0} \quad \text{for all } (c, a, \theta) \in (-\mathfrak{c}, \mathfrak{c}) \times \mathbb{R}^2,$$

where $\mathbf{0} := (0,0,0)$. On the other hand, we have that[4]

$$(4.2)\qquad \begin{cases} \partial_\sigma F(\phi_{c,a,\theta}, c, a, \theta) = \Big(0, \int_{\mathbb{R}} \langle i\eta_c, -\partial_c \phi_c\rangle_{\mathbb{C}}, 0\Big), \\ \partial_b F(\phi_{c,a,\theta}, c, a, \theta) = \Big(\int_{\mathbb{R}} \langle \eta_c, \partial_x \phi_c\rangle_{\mathbb{C}}, 0, \int_{\mathbb{R}} \langle iR_c\eta_c, \partial_x \phi_c\rangle_{\mathbb{C}}\Big), \\ \partial_\iota F(\phi_{c,a,\theta}, c, a, \theta) = \Big(\int_{\mathbb{R}} \langle \eta_c, -i\phi_c\rangle_{\mathbb{C}}, 0, \int_{\mathbb{R}} \langle iR_c\eta_c, -i\phi_c\rangle_{\mathbb{C}}\Big). \end{cases}$$

Let $\mathcal{F}(c)$ be the 3×3 matrix $\mathcal{F}(c) := (\partial_\sigma F, \partial_b F, \partial_\iota F)(\phi_{c,a,\theta}, c, a, \theta)$, which is a continuously differentiable function on the interval $c \in (-\mathfrak{c}, \mathfrak{c})$.

From (4.2), we have that for all $c \in (-\mathfrak{c}, \mathfrak{c})$ (see Appendix E for a detailed computation of $\det \mathcal{F}(c)$),

$$(4.3)\qquad \det \mathcal{F}(c) = -\int_{\mathbb{R}} \langle i\eta_c, -\partial_c \phi_c\rangle_{\mathbb{C}} \times \mathcal{D}(c) \neq 0,$$

where

$$\mathcal{D}(c) = \Big(\int_{\mathbb{R}} \langle \eta_c, \partial_x \phi_c\rangle_{\mathbb{C}} \int_{\mathbb{R}} \langle iR_c\eta_c, -i\phi_c\rangle_{\mathbb{C}} - \int_{\mathbb{R}} \langle \eta_c, -i\phi_c\rangle_{\mathbb{C}} \int_{\mathbb{R}} \langle iR_c\eta_c, \partial_x \phi_c\rangle_{\mathbb{C}}\Big).$$

Therefore, by the implicit function theorem, there exists a neighborhood

$$B_{\mathcal{H}_0}(\phi_{c,a,\theta}; \tilde{r}_c) \times B((c,a,\theta); \tilde{s}_c) \subset \mathcal{H} \times ((-\mathfrak{c}, \mathfrak{c}) \times \mathbb{R}^2)$$

and a unique C^1 map $(\tilde{c}, \tilde{a}, \tilde{\theta})\colon B_{\mathcal{H}_0}(\phi_{c,a,\theta}; \tilde{r}_c) \to B((c,a,\theta); \tilde{s}_c)$ such that

$$F(w, \tilde{c}(w), \tilde{a}(w), \tilde{\theta}(w)) = \mathbf{0},$$

for any $w \in B_{\mathcal{H}_0}(\phi_{c,a,\theta}; \tilde{r}_c)$, and consequently, we get (3.35). ∎

Before establishing the next result, we recall the neighborhood (3.31) of the orbit of ϕ_0, that is,

$$\mathcal{U}_0(\alpha) := \Big\{ w \in \mathcal{H}(\mathbb{R}) : \inf_{(b,\iota)\in\mathbb{R}^2} \|e^{-i\iota} w(\cdot + b) - \phi_0\|_{\mathcal{H}_0(\mathbb{R})} < \alpha \Big\},$$

where we split $e^{-i\iota} w(\cdot + b) = \phi_c + \mathfrak{z}$. By taking α smaller, if necessary, we can apply the well-known standard theory of modulation for the solution $u(\cdot) \in \mathcal{U}_0(\alpha)$ of the Cauchy problem (1.1).

Corollary 4.3. *Let $\tilde{r}_0$ and $\tilde{s}_0$ be the constants established in Proposition 4.2 for the case $c = 0$, chosen in such a way that $\tilde{r}_0 \tilde{s}_0 < 1$. There exists $\alpha > 0$ such that for a given $w \in \mathcal{U}_0(\alpha)$, there exist numbers a_w and θ_w such that $w \in B_{\mathcal{H}_0}(\phi_{0,a_w,\theta_w}; \tilde{r}_0/2)$, and the map $(\tilde{c}, \tilde{a}, \tilde{\theta})$ established in Proposition 4.2 in each ball $B_{\mathcal{H}_0}(\phi_{0,a_w,\theta_w}; \tilde{r}_0/2)$ is well defined from the neighborhood $\mathcal{U}_0(\alpha)$ with values in $\mathbb{R}^2 \times \mathbb{R}/2\pi$. More precisely, the functions $\tilde{c}(w), \tilde{a}(w)$ and $\tilde{\theta}(w)$ (modulo 2π) do not depend on which (a,θ) parameters are chosen.*

[4]By parity arguments, some of the terms vanish in (4.2).

Proof. Taking $\alpha \leq \alpha_0 := \min\{\tilde{r}_0/2,\ \tilde{\delta}/4\}$ (with $\tilde{\delta}$ provided in Lemma 4.1 in the case $c = 0$), the proof follows in a similar way as it was done in the first part of the Step 2 in the proof of Proposition 2 in [13]. ■

Corollary 4.4. *Consider α as in Corollary 4.3 and let $u(t,\cdot)$ be the solution of (1.1)–(1.2) with initial data u_0 satisfying $d_0(u_0,\phi_0) < \alpha$. Then there exist $T > 0$ and mappings*

$$[-T,T] \ni t \mapsto (c(t), a(t), \theta(t)),$$

such that $F(u(t,\cdot), c(t), a(t), \theta(t)) = \mathbf{0}$.

Proof. As a direct consequence of the continuity of the quintic GP flow in $\mathcal{Z}(\mathbb{R})$, we can find $T > 0$ such that

$$\|u(t,\cdot) - \phi_0\|_{\mathcal{H}_0} < d_0(u_0,\phi_0) < \alpha \quad \text{for all } t \in [-T,T],$$

and consequently $u(t,\cdot) \in B_{\mathcal{H}_0}(\phi_0;\alpha) \subset \mathcal{U}_0(\alpha)$ for all $t \in [-T,T]$. So, from Corollary 4.3, we can define the mappings

$$t \mapsto c(t), \quad t \mapsto a(t), \quad t \mapsto \theta(t)$$

on $[-T,T]$ by setting $c(t) := \tilde{c}(u(t,\cdot))$, $a(t) := \tilde{a}(u(t,\cdot))$, $\theta(t) := \tilde{\theta}(u(t,\cdot))$. Moreover, the perturbation $\mathfrak{z}(t) = e^{-i\theta(t)}u(\cdot + a(t)) - \phi_{c(t)}$ satisfies, for all $t \in [-T,T]$,

$$\begin{aligned} &F(u(t,\cdot), c(t), a(t), \theta(t)) \\ &\quad = \Big(\int_{\mathbb{R}} \langle \eta_{c(t)}, \mathfrak{z}(t)\rangle_{\mathbb{C}}, \int_{\mathbb{R}} \langle i\eta_{c(t)}, \mathfrak{z}(t)\rangle_{\mathbb{C}}, \int_{\mathbb{R}} \langle iR_{c(t)}\eta_{c(t)}, \mathfrak{z}(t)\rangle_{\mathbb{C}} \Big) = \mathbf{0}. \end{aligned}$$ ■

Furthermore, using the definition in (4.1), we also have an estimate on the size of the modulation parameters involved in the perturbation

$$\mathfrak{z}(t,\cdot) = e^{-i\theta(t)}u(t,\cdot) - \phi_{c(t)}(\cdot - a(t)),$$

namely, the following result.

Corollary 4.5. *Let α be as given in Corollary 4.3, and let $u(t,\cdot)$, $c(t)$, $\theta(t)$ and $a(t)$ be as in Corollary 4.4. There exist positive constants K_0 and A_0 such that if for some $(a,\theta) \in \mathbb{R}^2$ and $0 < \varepsilon \leq \min\{1,\alpha\}$,*

$$\|u(t,\cdot) - \phi_{0,a,\theta}\|_{\mathcal{H}_0} = \|u(t,\cdot) - e^{i\theta}\phi_0(\cdot - a)\|_{\mathcal{H}_0} \leq \varepsilon, \quad t \in [-T,T], \tag{4.4}$$

then it follows that

$$|c(t)| + |a(t) - a| + |e^{i\theta(t)} - e^{i\theta}| \leq K_0\varepsilon \quad \text{and} \quad \|\mathfrak{z}(t,\cdot)\|_{\mathcal{H}_0} \leq A_0\sqrt{\varepsilon}. \tag{4.5}$$

Proof. First of all, note that all components in the mapping

$$w \in B_{\mathcal{H}_0}(\phi_{0,a,\theta};\alpha) \mapsto (\tilde{c}(w), \tilde{a}(w), \tilde{\theta}(w)) \in B((0,a,\theta);\tilde{s}_0)$$

are C^1-functions, and therefore Lipschitz continuous with Lipschitz constant K_0. So, from (4.4), we have that

$$\begin{aligned} |c(t)| + |a(t) - a| + |\theta(t) - \theta| &= |\tilde{c}(u(t,\cdot))| + |\tilde{a}(u(t,\cdot)) - a| + |\tilde{\theta}(u(t,\cdot)) - \theta| \\ &\leq K_0\|u(t,\cdot) - \phi_{0,a,\theta}\|_{\mathcal{H}_0} \leq K_0\varepsilon \end{aligned} \tag{4.6}$$

for all $t \in [-T,T]$. This implies the first estimate in (4.5).

On the other hand, using (2.20), we have that

$$
\begin{aligned}
(4.7)\qquad &\|\phi_{c(t),a(t),\theta(t)}-\phi_{0,a,\theta}\|^2_{\mathcal{H}_0}\\
&= \|e^{i\theta(t)}\phi_{c(t)}(\cdot-a(t))-e^{i\theta}\phi_0(\cdot-a)\|^2_{\mathcal{H}_0}\\
&\lesssim \|e^{i\theta(t)}\phi_{c(t)}(\cdot-a(t))-e^{i\theta(t)}\phi_0(\cdot-a(t))\|^2_{\mathcal{H}_0}\\
&\quad+\|e^{i\theta(t)}\phi_0(\cdot-a(t))-e^{i\theta}\phi_0(\cdot-a)\|^2_{\mathcal{H}_0}\\
&\lesssim c^2(t)+\|\phi_0(\cdot-a(t))-\phi_0(\cdot-a)\|^2_{\mathcal{H}_0}+|e^{i\theta(t)}-e^{i\theta}|^2\|\phi_0(\cdot-a)\|^2_{\mathcal{H}_0}\\
&\lesssim c^2(t)+\|\phi_0(\cdot-a(t))-\phi_0(\cdot-a)\|^2_{\mathcal{H}_0}+|\theta(t)-\theta|^2\|\phi_0\|^2_{\mathcal{H}_0}.
\end{aligned}
$$

Now, using the mean value theorem, there exist $\nu_i=\nu_i(t,x)\in(0,1)$, $i=1,2$, such that

$$
\begin{aligned}
(4.8)\qquad &\|\phi_0(\cdot-a(t))-\phi_0(\cdot-a)\|^2_{\mathcal{H}_0}\\
&=\int_{\mathbb{R}}\eta_0|\phi_0(\cdot-a(t))-\phi_0(\cdot-a)|^2+\int_{\mathbb{R}}|\phi_0'(\cdot-a(t))-\phi_0'(\cdot-a)|^2\\
&=|a(t)-a|^2\int_{\mathbb{R}}\eta_0|\phi_0'(\cdot-\nu_1a+(1-\nu_1)a(t))|^2\\
&\quad+|a(t)-a|\int_{\mathbb{R}}|\phi_0'(\cdot-a(t))-\phi_0'(\cdot-a)|\,|\phi_0''(\cdot-\nu_2a+(1-\nu_2)a(t)|\\
&\lesssim |a(t)-a|(\|\eta_0\|_{L^1}|a(t)-a|+2\|\phi_0'\|_{L^1})\lesssim |a(t)-a|^2+|a(t)-a|.
\end{aligned}
$$

Combining (4.7) and (4.8), we have

$$
(4.9)\quad \|\phi_{c(t),a(t),\theta(t)}-\phi_{0,a,\theta}\|^2_{\mathcal{H}_0}\le K\big(c(t)^2+|a(t)-a|^2+|a(t)-a|+|\theta(t)-\theta|^2\big),
$$

for some universal constant K for all $|c|<\mathfrak{c}$.

Now, from (4.6) and (4.9), and using that $\varepsilon<1$, one gets

$$
\begin{aligned}
(4.10)\qquad \|\mathfrak{z}(t,\cdot)\|_{\mathcal{H}_0}&=\|u(t,\cdot)-\phi_{c(t),a(t),\theta(t)}\|_{\mathcal{H}_0}\\
&\le \|u(t,\cdot)-\phi_{0,a,\theta}\|_{\mathcal{H}_0}\,\|\phi_{c(t),a(t),\theta(t)}-\phi_{0,a,\theta}\|_{\mathcal{H}_0}\\
&\le (1+\sqrt{K}(K_0+\sqrt{K_0}))\sqrt{\varepsilon},
\end{aligned}
$$

which yields the second estimate in (4.5) with $A_0=1+\sqrt{K}(K_0+\sqrt{K_0})$. ■

Now, we will determine the growth in time of the modulation parameters $c(t), a(t)$ and $\theta(t)$ for any $t\in[-T,T]$. We will first show the evolution equation satisfied by the perturbation

$$
\mathfrak{z}(t)\equiv\mathfrak{z}(t,\cdot)=e^{-i\theta(t)}u(t,\cdot+a(t))-\phi_{c(t)}(\cdot).
$$

Lemma 4.6 (Evolution equation for $\mathfrak{z}$). *Let $\mathfrak{z}(t)=e^{-i\theta(t)}u(t,\cdot+a(t))-\phi_{c(t)}(\cdot)$ be the perturbation of the dark soliton profile $\phi_{c(t)}(\cdot)$, see (2.4). Then we have that*

$$
(4.11)\quad \partial_t\mathfrak{z}(t):=-c'(t)\partial_c\phi_{c(t)}-i\theta'(t)(\phi_{c(t)}+\mathfrak{z}(t))+a'(t)(\partial_x\phi_{c(t)}+\partial_x\mathfrak{z}(t))+iZ(t),
$$

with

$$
(4.12)\quad Z(t):=\partial_{xx}\mathfrak{z}(t)+ic(t)\partial_x\phi_{c(t)}+\eta_{c(t)}\mathfrak{z}(t)-(\rho^2_{c(t)}+2|\phi_{c(t)}|^2\rho_{c(t)})(\phi_{c(t)}+\mathfrak{z}(t)).
$$

and $\rho_{c(t)}=\rho_{c(t)}(\mathfrak{z}(t))$.

Proof. First, consider the explicit time derivative of $\mathfrak{z}(t,\cdot)$:

$$\partial_t\mathfrak{z}(t) = -c'(t)\partial_c\phi_{c(t)} - i\theta'(t)(\phi_{c(t)} + \mathfrak{z}(t)) \\ + a'(t)(\partial_x\phi_{c(t)} + \partial_x\mathfrak{z}(t)) + e^{-i\theta(t)}\partial_t u(t,\cdot + a(t)).$$

Now computing the last term $\partial_t u(t,\cdot + a(t))$, bearing in mind that u fulfills (1.1), and also (2.5), (3.33) and that

$$|u|^4 = |\phi_{c(t)}|^4 + \rho_{c(t)}^2 + 2\rho_{c(t)}|\phi_{c(t)}|^2,$$

a direct calculation gives us (4.11). ∎

We now look for an expression for the growth in time of the modulation parameters $c(t)$, $a(t)$ and $\theta(t)$. In order to do that, we resort to the continuity of the quintic Gross–Pitaevskii flow in $\mathcal{Z}(\mathbb{R}) \subset \mathcal{H}(\mathbb{R})$. Specifically, if the initial data u_0 is chosen such that $d_0(u_0,\phi_0) < \alpha$, then we get a time T such that the corresponding solution $u(t,\cdot)$ along the quintic Gross–Pitaevskii flow belongs to $\mathcal{V}_0(\alpha)$, for any $t \in [-T,T]$, see (3.32).

We will see, in Section 5, that we can fix the smallness parameter α in such a way that the solution $u(t,\cdot)$ of the Cauchy problem (1.1) still belongs to $\mathcal{V}_0(\alpha)$ for all $t \in \mathbb{R}$.

Proposition 4.7 (Estimates on the growth of the modulation parameters). *There exist numbers $\alpha_1 > 0$ and $A_1(\alpha_1) > 0$ such that if the solution $u(t,\cdot)$ lies in $\mathcal{V}_0(\alpha_1)$ for any $t \in [-T,T]$, then the functions c, a and θ are $C^1([-T,T];\mathbb{R})$ and satisfy*

$$|c'(t)| + |a'(t)| + |\theta'(t)| \le A_1^2(\alpha_1)\|\mathfrak{z}(t,\cdot)\|_{\mathcal{H}_0} \quad \text{for all } t \in [-T,T]. \tag{4.13}$$

Proof. We differentiate with respect to time the three generalized orthogonality conditions (3.35) for perturbations around the dark soliton profile $\phi_{c(t)}$.

Since we need to compute the derivatives in time of the orthogonality conditions, we initially consider regular enough initial data, for example, $\partial_x u_0 \in H^2(\mathbb{R})$. In fact, with this regularity we can justify (4.14), (4.15) and (4.16) below. We consider initially α and u_0 as in Corollary 4.4 so that the solution $u(t,\cdot)$ belongs to $\mathcal{U}_0(\alpha)$ for all $t \in [-T,T]$, and then we can set the modulation parameters $(c(t),a(t),\theta(t)) \in (-\mathrm{c},\mathrm{c}) \times \mathbb{R}^2$ for any $t \in [-T,T]$.

Note that c, a and θ belong to $C^1([-T,T],\mathbb{R})$ by the chain rule theorem and moreover note that $\mathfrak{z}(t) \in C^1([-T,T],\mathcal{H}(\mathbb{R}))$, and therefore we can get (4.11). Therefore, differentiating the first orthogonality condition in (3.35), and with the notation m_{ij}, $i,j = 1,2,3$, for integrals independent of $\mathfrak{z}(t)$ and n_k, $k = 1,\dots,9$, for integrals with terms depending on $\mathfrak{z}(t)$, we get

$$\begin{aligned} &\partial_t \int_{\mathbb{R}} \langle \eta_{c(t)}, \mathfrak{z}(t)\rangle_{\mathbb{C}} \\ &= \int_{\mathbb{R}} \big(\langle c'(t)\partial_c\eta_{c(t)}, \mathfrak{z}(t)\rangle_{\mathbb{C}} + \langle \eta_{c(t)}, c'(t)\partial_c\mathfrak{z}(t)\rangle_{\mathbb{C}} + \langle \eta_{c(t)}, \partial_t\mathfrak{z}(t)\rangle_{\mathbb{C}}\big) \\ &= \int_{\mathbb{R}} \big(\langle c'(t)\partial_c\eta_{c(t)}, \mathfrak{z}(t)\rangle_{\mathbb{C}} + \langle \eta_{c(t)}, c'(t)\partial_c\mathfrak{z}(t)\rangle_{\mathbb{C}} \\ &+ \big\langle \eta_{c(t)}, -c'(t)\partial_c\phi_{c(t)} - i\theta'(t)(\phi_{c(t)} + \mathfrak{z}(t)) + a'(t)(\partial_x\phi_{c(t)} + \partial_x\mathfrak{z}(t)) + iZ(t)\big\rangle_{\mathbb{C}}\big). \end{aligned} \tag{4.14}$$

Thus,

$$\begin{aligned}
&\partial_t \int_{\mathbb{R}} \langle \eta_{c(t)}, \mathfrak{z}(t)\rangle_{\mathbb{C}} \\
&= a'(t)\Big(\int_{\mathbb{R}} \langle \eta_{c(t)}, \partial_x \phi_{c(t)}\rangle_{\mathbb{C}} + \int_{\mathbb{R}} \langle \eta_{c(t)}, \partial_x \mathfrak{z}(t)\rangle_{\mathbb{C}}\Big) \\
&\quad + c'(t)\Big(-\int_{\mathbb{R}} \langle \eta_{c(t)}, \partial_c \phi_{c(t)}\rangle_{\mathbb{C}} + \int_{\mathbb{R}} \langle \partial_c \eta_{c(t)}, \mathfrak{z}(t)\rangle_{\mathbb{C}} + \int_{\mathbb{R}} \langle \eta_{c(t)}, \partial_c \mathfrak{z}(t)\rangle_{\mathbb{C}}\Big) \\
&\quad + \theta'(t)\Big(\int_{\mathbb{R}} \langle \eta_{c(t)}, -i\phi_{c(t)}\rangle_{\mathbb{C}} + \int_{\mathbb{R}} \langle \eta_{c(t)}, -i\mathfrak{z}(t)\rangle_{\mathbb{C}}\Big) + \int_{\mathbb{R}} \langle \eta_{c(t)}, iZ(t)\rangle_{\mathbb{C}} \\
&= a'(t)(m_{11} + n_1) + c'(t)(n_2 - m_{12}) + \theta'(t)(m_{13} + n_3) + \int_{\mathbb{R}} \langle \eta_{c(t)}, iZ(t)\rangle_{\mathbb{C}} = 0.
\end{aligned}$$

Now, we differentiate the second orthogonality condition in (3.35), and we obtain

$$\begin{aligned}
&\partial_t \int_{\mathbb{R}} \langle i\eta_{c(t)}, \mathfrak{z}(t)\rangle_{\mathbb{C}} \\
&= a'(t)\Big(\int_{\mathbb{R}} \langle i\eta_{c(t)}, \partial_x \phi_{c(t)}\rangle_{\mathbb{C}} + \int_{\mathbb{R}} \langle i\eta_{c(t)}, \partial_x \mathfrak{z}(t)\rangle_{\mathbb{C}}\Big) \\
&\quad + c'(t)\Big(-\int_{\mathbb{R}} \langle i\eta_{c(t)}, \partial_c \phi_{c(t)}\rangle_{\mathbb{C}} + \int_{\mathbb{R}} \langle i\partial_c \eta_{c(t)}, \mathfrak{z}(t)\rangle_{\mathbb{C}} + \int_{\mathbb{R}} \langle i\eta_{c(t)}, \partial_c \mathfrak{z}(t)\rangle_{\mathbb{C}}\Big) \\
&\quad + \theta'(t)\Big(\int_{\mathbb{R}} \langle i\eta_{c(t)}, -i\phi_{c(t)}\rangle_{\mathbb{C}} + \int_{\mathbb{R}} \langle i\eta_{c(t)}, -i\mathfrak{z}(t)\rangle_{\mathbb{C}}\Big) + \int_{\mathbb{R}} \langle i\eta_{c(t)}, iZ(t)\rangle_{\mathbb{C}} \\
&= a'(t)(m_{21} + n_4) + c'(t)(n_5 - m_{22}) + \theta'(t)(m_{23} + n_6) + \int_{\mathbb{R}} \langle i\eta_{c(t)}, iZ(t)\rangle_{\mathbb{C}} = 0.
\end{aligned} \tag{4.15}$$

Finally, we differentiate the third orthogonality condition in (3.35), and we get

$$\begin{aligned}
&\partial_t \int_{\mathbb{R}} \langle iR_{c(t)}\eta_{c(t)}, \mathfrak{z}(t)\rangle_{\mathbb{C}} \\
&= \int_{\mathbb{R}} \langle iR_{c(t)}\eta_{c(t)}, \partial_t \mathfrak{z}(t)\rangle_{\mathbb{C}} + \int_{\mathbb{R}} \langle iR_{c(t)}\eta_{c(t)}, c'(t)\partial_c \mathfrak{z}(t)\rangle_{\mathbb{C}} \\
&\quad + \int_{\mathbb{R}} \langle ic'(t)\partial_c R_{c(t)}\eta_{c(t)} + ic'(t)R_{c(t)}\partial_c \eta_{c(t)}, \mathfrak{z}(t)\rangle_{\mathbb{C}} \\
&= a'(t)\Big(\int_{\mathbb{R}} \langle iR_{c(t)}\eta_{c(t)}, \partial_x \phi_{c(t)}\rangle_{\mathbb{C}} + \int_{\mathbb{R}} \langle iR_{c(t)}\eta_{c(t)}, \partial_x \mathfrak{z}(t)\rangle_{\mathbb{C}}\Big) \\
&\quad + c'(t)\Big(\int_{\mathbb{R}} \langle i\partial_c (R_{c(t)}\eta_{c(t)}), \mathfrak{z}(t)\rangle_{\mathbb{C}} + \int_{\mathbb{R}} \langle iR_{c(t)}\eta_{c(t)}, \partial_c \mathfrak{z}(t)\rangle_{\mathbb{C}} \\
&\quad - \langle iR_{c(t)}\eta_{c(t)}, \partial_c \phi_{c(t)}\rangle_{\mathbb{C}}\Big) \\
&\quad + \theta'(t)\Big(\int_{\mathbb{R}} \langle iR_{c(t)}\eta_{c(t)}, -i\phi_{c(t)}\rangle_{\mathbb{C}} + \int_{\mathbb{R}} \langle iR_{c(t)}\eta_{c(t)}, -i\mathfrak{z}(t)\rangle_{\mathbb{C}}\Big) \\
&\quad + \int_{\mathbb{R}} \langle iR_{c(t)}\eta_{c(t)}, iZ(t)\rangle_{\mathbb{C}}
\end{aligned} \tag{4.16}$$

$$= a'(t)(m_{31}+n_7) + c'(t)(n_8 - m_{32}) + \theta'(t)(m_{33}+n_9)$$
$$+ \int_{\mathbb{R}} \langle iR_{c(t)}\eta_{c(t)}, iZ(t)\rangle_{\mathbb{C}} = 0.$$

Gathering all three previous derivatives, we obtain the following linear system:

$$\mathcal{M}(c,\mathfrak{z})\begin{pmatrix} a'(t) \\ c'(t) \\ \theta'(t)\end{pmatrix} = \mathcal{B}(c,\mathfrak{z}), \tag{4.17}$$

with the matrix $\mathcal{M}(c,\mathfrak{z})$ defined by

$$\mathcal{M}(c,\mathfrak{z}) := \begin{pmatrix} m_{11}+n_1 & n_2-m_{12} & m_{13}+n_3 \\ m_{21}+n_4 & n_5-m_{22} & m_{23}+n_6 \\ m_{31}+n_7 & n_8-m_{32} & m_{33}+n_9 \end{pmatrix}, \tag{4.18}$$

and the matrix $\mathcal{B}(c,\mathfrak{z})$ written as follows:

$$\mathcal{B}(c,\mathfrak{z}) := \begin{pmatrix} -\int_{\mathbb{R}}\langle \eta_{c(t)}, iZ(t)\rangle_{\mathbb{C}} \\ -\int_{\mathbb{R}}\langle i\eta_{c(t)}, iZ(t)\rangle_{\mathbb{C}} \\ -\int_{\mathbb{R}}\langle iR_{c(t)}\eta_{c(t)}, iZ(t)\rangle_{\mathbb{C}} \end{pmatrix},$$

with $Z(t)$ as in (4.12). See Appendix E for a full expression of the computed matrix elements $m_{i,j}$, $i,j = 1,2,3$.

Note that, in the case of null perturbation in (4.18), and considering the limit case of $c = 0$, it turns out that $\mathcal{M}(0,0)$ has a nonvanishing determinant, namely,

$$\det \mathcal{M}(0,0) = \frac{8}{5}\, E_2[\phi_0],$$

and therefore $\mathcal{M}(0,0)$ is invertible. By using a continuity argument, we can select a small enough parameter $\alpha_1 < \alpha$ such that for small speeds and perturbations $(c,\mathfrak{z})$, the matrix $\mathcal{M}(c,\mathfrak{z})$ is still invertible. In fact, having in mind the Neumann series theorem, it is enough to consider $(c,\mathfrak{z})$ satisfying

$$\|\mathcal{M}(c,\mathfrak{z}) - \mathcal{M}(0,0)\|_{M_{2\times 2}(\mathbb{C})} \le \alpha_1 < \|\mathcal{M}^{-1}(0,0)\|^{-1}_{M_{2\times 2}(\mathbb{C})}.$$

Namely, choosing $\alpha_1 < \alpha$ small enough such that $u(t,\cdot) \in \mathcal{U}_0(\alpha_1)$, for all $t \in [-T,T]$, and therefore, from (4.5) in Proposition 4.2, it holds

$$\|\mathfrak{z}(t,\cdot)\|_{\mathcal{H}_0} + |c(t)| \le A_0\alpha_1,$$

with $\det \mathcal{M}(c,\mathfrak{z}) \ne 0$, and consequently the operator norm of its inverse is bounded by some positive number $A_1(\alpha_1)$. In the same way, the right-hand side of (4.17) is bounded as follows:

$$\|\mathcal{B}(c,\mathfrak{z})\|_{\mathbb{R}^3} \le A_1(\alpha_1)\|\mathfrak{z}(t,\cdot)\|_{\mathcal{H}_0},$$

for a suitable choice of the constant $A_1(\alpha_1)$. Therefore, from (4.17), we finally get that

$$|a'(t)| + |c'(t)| + |\theta'(t)| \le \left|\mathcal{M}(c,\mathfrak{z})^{-1}\cdot \mathcal{B}(c,\mathfrak{z})\right| \le A_1^2(\alpha_1)\|\mathfrak{z}(t,\cdot)\|_{\mathcal{H}_0}, \tag{4.19}$$

for all $t \in [-T,T]$.

Finally, we extend the above estimate (4.19) for general initial data $u_0 \in \mathcal{Z}(\mathbb{R})$. In fact, the flow of (1.1) is continuous with respect to initial data in $\mathcal{Z}(\mathbb{R})$ (see [9]). Moreover, from the continuity of the modulation parameters $c(t)$, $a(t)$ and $\theta(t)$, we have that the matrices $\mathcal{M}(c, \mathfrak{z})$ and $\mathcal{B}(c, \mathfrak{z})$ depend continuously on $u \in \mathcal{H}(\mathbb{R})$. Therefore, since the matrix $\mathcal{M}(c, \mathfrak{z})$ is invertible with an operator norm of its inverse depending on α_1, we can use a standard density argument to extend (4.17) to a general solution. Therefore, we get the continuous differentiability property of the modulation parameters $c(t)$, $a(t)$ and $\theta(t)$, and we obtain the corresponding estimates (4.13) from (4.17). ■

5. Proof of the main theorem

In this section we prove a detailed version of Theorem 1.1.

Theorem 5.1 (Orbital stability of the black soliton). *Let ϕ_0 be the black soliton (1.8) of the quintic GP equation (1.1). Given $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ and a positive constant A_* such that if the initial data u_0 verifies*

$$u_0 \in \mathcal{Z}(\mathbb{R}) \quad \text{and} \quad d_0(u_0, \phi_0) < \delta(\varepsilon),$$

then there exist functions $a, \theta \in C^1(\mathbb{R}, \mathbb{R})$ such that the solution u of the Cauchy problem for the quintic GP equation (1.1), with initial data u_0, satisfies

$$d_0(e^{-i\theta(t)} u(t, \cdot + a(t)), \phi_0) < \varepsilon \tag{5.1}$$

and

$$|a'(t)| + |\theta'(t)| < A_* \varepsilon$$

for any $t \in \mathbb{R}$.

Remark 5.2. With respect to the cubic case [13], a difference appears in the proof of the orbital stability theorem for black solitons of (1.1), that is, we could not achieve a Lipschitzian control of the metric, i.e.,

$$d_0(e^{-i\theta(t)} u(t, \cdot + a(t)), \phi_{c(t)}) \lesssim d_0(u_0, \phi_0) \quad \text{for all } c \in (0, \mathrm{c}). \tag{5.2}$$

The main reason is that if the momentum (see equation (1.27) on p. 313 of [13]) is used in our problem, a linear term

$$\int_{\mathbb{R}} \langle i\phi'_{c(t)}, \mathfrak{z}\rangle_{\mathbb{C}} \tag{5.3}$$

appears when expanding it around dark solitons $\phi_{c(t)}$. Unfortunately this term does not match with any orthogonality relation (3.34) and it cannot be bounded from above nor controlled in the right and proper way. This is a big difference with respect to the cubic GP case (see equation (1.7) on p. 314 of [13]) which allows one to get an upper bound on the speed $c(t)$ as shown in equation (1.3) on p. 314 of [13].

In our case, instead of (5.2), we get

$$d_0(e^{-i\theta(t)} u(t, \cdot + a(t)), \phi_{c(t)}) \lesssim d_0(u_0, \phi_0) + \mathrm{c}.$$

Moreover, and again in view of this technical issue with the linear term (5.3), we decided to change this uniform control on $c(t)$ on a fixed speed interval $(-\mathrm{c}, \mathrm{c})$ by using the following strategy: for each fixed $\varepsilon > 0$, we choose a suitable interval $(0, \mathrm{c}(\varepsilon))$ which allows us to select initial data in an appropriate ball with center ϕ_0 in the d_0 metric and such that the solution remains in the ε-neighborhood for all time by a bootstrap argument (note that (5.1) is not as strong as the corresponding statement in equation (1.9) on p. 308 of [13]). Obviously with this approach, once we reduce ε, the speed interval $(-\mathrm{c}, \mathrm{c})$ can also be reduced. As a consequence of our approach, we lose any possibility to say something about the orbital stability of the dark soliton solution.

Proof of theorem 5.1. In order to simplify the explanation, we show the proof for $t \geq 0$.

Let α, A_0 and α_1 be as in Corollary 4.3, Corollary 4.5 and Proposition 4.7, respectively. Consider now $\varepsilon > 0$ such that

$$0 < \varepsilon \leq \min\{1, \alpha\}, \quad 0 < \varepsilon < \alpha_1 \quad \text{and} \quad A_0 \varepsilon \ll 1.$$

Firstly, we take $u_0 \in \mathcal{Z}(\mathbb{R})$, see (2.11), such that $d_0(u_0, \phi_0) < \varepsilon/2$ and $u \in C(\mathbb{R}, \mathcal{Z}(\mathbb{R}))$ is the corresponding solution to (1.1).

Now we define

$$T^* := \sup\Big\{T > 0 : \text{for all } t \in [0, T],\ \inf_{(\iota,b)\in\mathbb{R}^2} d_0(e^{-i\iota}u(t, \cdot + b), \phi_0) < \varepsilon\Big\}, \tag{5.4}$$

and the idea is to use a contradiction argument under the assumption $T^* < \infty$ when $d_0(u_0, \phi_0)$ is small enough.

Note that since $d_0(u_0, \phi_0) < \varepsilon$, then, as a direct consequence of the continuity of the quintic GP flow in $\mathcal{Z}(\mathbb{R})$ with respect to the metric d_0, we can find $T_0 > 0$ such that

$$d_0(u(t, \cdot), \phi_0) < \varepsilon \quad \text{for all } t \in [0, T_0],$$

which, in particular, implies that T^* is well defined in (5.4). Furthermore,

$$u(t, \cdot) \in \mathcal{V}_0(\varepsilon) \subset \mathcal{U}_0(\varepsilon) \subset \mathcal{U}_0(\alpha) \quad \text{for all } t \in [0, T^*). \tag{5.5}$$

Consider now the functions $c(t), a(t), \theta(t)$ given in Corollary 4.4 and notice that, in view of (5.5), we can consider these functions defined on the whole interval $[0, T^*)$.

Now suppose that $T^* < +\infty$ and consider

$$\mathfrak{z}(t, \cdot) = e^{-i\theta(t)}u(t, \cdot + a(t)) - \phi_{c(t)}(\cdot), \quad t \in [0, T^*),$$

where $(c(t), a(t), \theta(t)) \in (-\mathrm{c}, \mathrm{c}) \times \mathbb{R}^2$.

Then, having in mind the global theory in [9], which guarantees that $\|\rho_c(\mathfrak{z})\|_{L^2}$ verifies (3.34), using the coercivity of E_2, see (1.3), around the dark soliton (3.36) in Proposition 3.4 and Corollary 4.5 with $(a, \theta) = (0, 0)$, we obtain

$$\begin{aligned}
&\|\mathfrak{z}(t, \cdot)\|_{\mathcal{H}_0}^2 + \|\phi_{c(t)}^3 \rho_{c(t)}(\mathfrak{z}(t, \cdot))\|_{L^2}^2 \\
&\leq \frac{1}{\tilde{\Gamma}^2}\big(\tilde{\Gamma}(E_2[\phi_{c(t)} + \mathfrak{z}] - E_2[\phi_0]) + c^2(t) + \|\mathfrak{z}(t, \cdot)\|_{\mathcal{H}_0}^3\big) \\
&= \frac{1}{\tilde{\Gamma}^2}\big(\tilde{\Gamma}(E_2[e^{-i\theta(t)}u(t, \cdot + a(t))] - E_2[\phi_0]) + c^2(t) + A_0\sqrt{\varepsilon}\|\mathfrak{z}(t, \cdot)\|_{\mathcal{H}_0}^2\big).
\end{aligned}$$

Selecting ε such that $A_0\sqrt{\varepsilon} < \frac{1}{2}\tilde{\Gamma}^2$ and using the conservation of the E_2 energy (1.3), one gets

$$\|\mathfrak{z}(t,\cdot)\|^2_{\mathcal{H}_0} + \|\phi^3_{c(t)}\rho_{c(t)}(\mathfrak{z}(t,\cdot))\|^2_{L^2} \le \frac{2}{\tilde{\Gamma}^2}(\tilde{\Gamma}(E_2[u_0] - E_2[\phi_0]) + c(t)^2) \tag{5.6}$$

for all $t \in [0, T^*)$. Now, from the expansion (3.2) with $\mathfrak{z} = u_0 - \phi_0$, there exists a positive constant $\tilde{k}_1$ such that

$$E_2[u_0] - E_2[\phi_0] \le \tilde{k}_1 d_0^2(u_0, \phi_0),$$

with $\tilde{k}_1$ independent of u_0. Then, putting this estimate into (5.6) and using (2.29), we have that there exists a universal positive constant $\tilde{K}$, not depending on $\mathfrak{c}$, such that

$$\begin{aligned}\|\mathfrak{z}(t,\cdot)\|^2_{\mathcal{H}_0} + \|\phi^3_0\rho_{c(t)}(\mathfrak{z}(t,\cdot))\|^2_{L^2} &\le \tilde{K}(\|\mathfrak{z}(t,\cdot)\|^2_{\mathcal{H}_0} + \|\phi^3_{c(t)}\rho_{c(t)}(\mathfrak{z}(t,\cdot))\|^2_{L^2})\\ &\le \frac{2\tilde{K}}{\tilde{\Gamma}^2}(\tilde{\Gamma}\tilde{k}_1 d_0^2(u_0,\phi_0) + c^2(t))\\ &\le \frac{2\tilde{K}\tilde{k}_1}{\tilde{\Gamma}}\, d_0^2(u_0,\phi_0) + \frac{2\tilde{K}}{\tilde{\Gamma}^2}\,\mathfrak{c}^2.\end{aligned}$$

So, we have

$$\begin{aligned}d_0(e^{-i\theta(t)}u(t,\cdot + a(t)), \phi_{c(t)}) &= (\|\mathfrak{z}(t,\cdot)\|^2_{\mathcal{H}_0} + \|\phi^3_0\rho_{c(t)}(\mathfrak{z}(t,\cdot))\|^2_{L^2})^{1/2}\\ &\le \Big(\frac{2\tilde{K}\tilde{k}_1}{\tilde{\Gamma}}\, d_0^2(u_0,\phi_0) + \frac{2\tilde{K}}{\tilde{\Gamma}^2}\,\mathfrak{c}^2\Big)^{1/2},\end{aligned}$$

and hence from (2.21) we have

$$\begin{aligned}d_0(e^{-i\theta(t)}u(t,\cdot + a(t)), \phi_0) &\le \Big(\frac{2\tilde{K}\tilde{k}_1}{\tilde{\Gamma}}\, d_0^2(u_0,\phi_0) + \frac{2\tilde{K}}{\tilde{\Gamma}^2}\,\mathfrak{c}^2\Big)^{1/2} + d_0(\phi_{c(t)}, \phi_0)\\ &\le \Big(\frac{2\tilde{K}\tilde{k}_1}{\tilde{\Gamma}}\Big)^{1/2} d_0(u_0,\phi_0) + \frac{\sqrt{2\tilde{K}}}{\tilde{\Gamma}}\,\mathfrak{c} + \tilde{k}_2\,\mathfrak{c},\end{aligned} \tag{5.7}$$

for some positive constant $\tilde{k}_2$. Now we reduce $\mathfrak{c}$, if necessary, so that

$$\Big(\frac{\sqrt{2\tilde{K}}}{\tilde{\Gamma}} + \tilde{k}_2\Big)\mathfrak{c} < \frac{\varepsilon}{4} \tag{5.8}$$

and we also consider u_0 satisfying

$$d(u_0,\phi_0) < \min\Big\{\frac{\varepsilon}{2}, \frac{\varepsilon}{4}\Big(\frac{\tilde{\Gamma}}{2\tilde{K}\tilde{k}_1}\Big)^{1/2}\Big\}. \tag{5.9}$$

Then, combining (5.7), (5.8) and (5.9), we get

$$d_0(e^{-i\theta(t)}u(t,\cdot + a(t)), \phi_0) < \frac{\varepsilon}{2} \quad \text{for all } t \in [0, T^*),$$

which contradicts the definition of $T^* < \infty$ in (5.4), due to the continuity of the flow of the solution $u(t,\cdot)$ with respect to the metric d_0. Then $T^* = \infty$ and the proof of (5.1) is completed.

Finally, from Proposition 4.7, one gets

$$\sup_{t\in\mathbb{R}} |a'(t)| + |\theta'(t)| < A_* \varepsilon,$$

for some positive constant A_*. This completes the proof of Theorem 5.1. ■

A. Proof of (1.11)

In order to prove that (1.11) is a solution of (1.10), we propose a suitable ansatz (see (2.3)):

$$\Phi_c(\xi) = \frac{ia_1 + a_2 \tanh(k\xi)}{\sqrt{1 + a_3 \tanh^2(k\xi)}}, \quad \xi = x - ct. \tag{A.1}$$

This ansatz must reduce to the black solution (1.8) when $c = 0$, therefore this implies that a_1 has to be dependent on c in some way. We make the following selection:

$$a_1 = c\tilde{a}_1 a_2,$$

with $\tilde{a}_1$ to be determined. Hence, we recast (A.1) as follows:

$$\Phi_c(\xi) = \frac{ic\tilde{a}_1 a_2 + a_2 \tanh(k\xi)}{\sqrt{1 + a_3 \tanh(k\xi)^2}}, \tag{A.2}$$

where $\tilde{a}_1, k, a_2, a_3$ are parameters to be determined imposing that (A.2) is a solution of (1.10). Therefore, substituting (A.2) into (1.10) and simplifying (here $X = \tanh(k\xi)$, $D = 1 + a_3 \tanh(k\xi)^2$), we get

$$\Phi_c'' - ic\Phi_c' + (1 - |\Phi_c|^4)\Phi_c = \frac{a_2}{D^{5/2}} \sum_{i=0}^{5} r_i X^i,$$

where r_i, $i = 0, \dots, 5$, are the following complex coefficients:

$$\begin{cases} r_0 = -ic(k + \tilde{a}_1 k^2 a_3 + \tilde{a}_1^5 c^4 a_2^4 - \tilde{a}_1), \\ r_1 = (-\tilde{a}_1 c^2 k a_3 + k^2(-3a_3 - 2) - \tilde{a}_1^4 c^4 a_2^4 + 1), \\ r_2 = -ic(2\tilde{a}_1 k^2(-a_3 - 2)a_3 - k(1 - a_3) + 2\tilde{a}_1^3 c^2 a_2^4 - 2\tilde{a}_1 a_3), \\ r_3 = -(-\tilde{a}_1 c^2 k a_3(1 - a_3) + k^2(-4a_3 - 2) + 2\tilde{a}_1^2 c^2 a_2^4 - 2a_3), \\ r_4 = -ic(\tilde{a}_1 a_2^4 - \tilde{a}_1 a_3^2 - a_3 k + \tilde{a}_1 a_3(3 + 2a_3)k^2), \\ r_5 = (-a_2^4 - a_3(k^2 - \tilde{a}_1 c^2 k a_3 - a_3)). \end{cases} \tag{A.3}$$

Now, we impose that

$$r_i = 0 \quad \text{for all } i = 0, \dots, 5, \tag{A.4}$$

and look for non-trivial solutions (i.e., $\phi \neq 0$). Starting with the last equation $r_5 = 0$, we get

$$a_2^4 = a_3(-k^2 + \tilde{a}_1 c^2 k a_3 + a_3). \tag{A.5}$$

Substituting the above value for a_2^4 into system (A.4), we get that the equation $r_4 = 0$ is solved for

$$a_3 = -\frac{-1 + 2k\tilde{a}_1}{\tilde{a}_1(2k + c^2\tilde{a}_1)}. \tag{A.6}$$

Therefore, with these values for a_2^4 and a_3, the group of (A.3) is recasted as follows:

$$\begin{aligned} r_0 &= -ickHM_0 = 0, \quad r_1 = -\frac{kH}{\tilde{a}_1} M_0 = 0, \\ r_2 &= -ickHM_1 = 0, \quad r_3 = -\frac{kH}{\tilde{a}_1} M_1 = 0, \quad r_4 = 0, \quad r_5 = 0, \end{aligned}$$

with

$$H = \frac{1 + c^2\tilde{a}_1^2}{(2k + c^2\tilde{a}_1)^2}$$

and

$$\begin{cases} M_0 = 6k^2 + 6\tilde{a}_1^4 c^4 k^2 + \tilde{a}_1 k(5c^2 - 4(k^2+1)) \\ \qquad\quad + \tilde{a}_1^3 c^2 k(-5c^2 + 4(k^2+1)) + \tilde{a}_1^2(c^4 - 4c^2(2k^2+1)), \\ M_1 = -8k^2 + 8\tilde{a}_1 k^3 + c^2(1 - 10\tilde{a}_1 k + 12\tilde{a}_1^2 k^2) - 4 + 8\tilde{a}_1 k. \end{cases} \tag{A.7}$$

Solving $M_1 = 0$, for $\tilde{a}_1$, we get (selecting, e.g., the + root)

$$\tilde{a}_1 = \frac{(5c^2 - 4) - 4k^2 + \sqrt{13c^4 + 8c^2(7k^2+1) + 16(k^2+1)^2}}{12c^2 k}. \tag{A.8}$$

Now, rewriting M_0 in (A.7) with $\tilde{a}_1$ as in (A.8), we get

$$M_0 = \frac{(4k^2 + c^2 - 4)}{144c^2k^2} m_0(c,k),$$

with

$$\begin{aligned} m_0(c,k) = \big[&16 - 16c^2 + 19c^4 + 32k^2 + 80c^2k^2 + 16k^4 \\ &+ (5c^2 - 4k^2 - 4)\sqrt{13c^4 + 8c^2(7k^2+1) + 16(k^2+1)^2}\,\big]. \end{aligned}$$

Finally, selecting $k = \frac{1}{2}\sqrt{4 - c^2}$, we get $M_0 = 0$, and we have solved system (A.4), and therefore (A.2) is a solution. Note that for these values of $\tilde{a}_1$ and k, the factor H is well defined; in fact,

$$H = \frac{6c^2 + (3c^2 - 4)\sqrt{3c^2 + 4} + 8}{c^2(\sqrt{3c^2+4} + 4)^2}.$$

In order to compare this solution with (1.16), we rewrite it as follows: firstly note that with this value of k, (A.8) and (A.6) reduce to

$$\tilde{a}_1 = \frac{3c^2 - 4 + 2\sqrt{4 + 3c^2}}{3c^2\sqrt{4 - c^2}} \tag{A.9}$$

and

$$a_3 = -\frac{3(4-c^2)(\sqrt{3c^2+4}-2)}{(4+\sqrt{3c^2+4})(3c^2-4+2\sqrt{3c^2+4})},$$

and hence, from (A.5), with the above values of $\tilde{a}_1$, k and a_3, and simplifying, we get (taking for instance a real + root, with the notation $\Delta = \sqrt{4-c^2}\sqrt{-3c^4+8c^2+16}$)

$$\text{(A.10)} \qquad \begin{aligned} a_2 &= \frac{3c(c^2-4)}{\sqrt{2}\sqrt{-18c^4+(3\Delta+80)c^2+4(\Delta-8)}} \\ &= \frac{3c\sqrt{4-c^2}}{\sqrt{2}\sqrt{3(\sqrt{3c^2+4}+6)c^2+4(\sqrt{3c^2+4}-2)}} \\ &= \frac{3c\sqrt{4-c^2}}{\sqrt{2}\sqrt{18c^2-8+(3c^2+4)^{3/2}}}. \end{aligned}$$

Therefore,

$$\sqrt{2}\,a_2 = \mu_2.$$

Now, from ansatz (A.2), and the values of (A.9) and (A.10), we get that

$$\text{(A.11)} \qquad \begin{aligned} c\tilde{a}_1 a_2 &= c \times \Big(\frac{3c^2+2\sqrt{3c^2+4}-4}{3c^2\sqrt{4-c^2}}\Big) \times \Big(\frac{3c\sqrt{4-c^2}}{\sqrt{2}\sqrt{18c^2-8+(3c^2+4)^{3/2}}}\Big) \\ &= \frac{3c^2-4+2\sqrt{3c^2+4}}{\sqrt{2}\sqrt{18c^2-8+(3c^2+4)^{3/2}}}, \end{aligned}$$

and hence

$$\sqrt{2}\,c\tilde{a}_1 a_2 = \mu_1.$$

Finally, note that $k = \kappa$, and with (A.11) and (A.10), we get

$$\frac{\mu_1^2+\mu_2^2}{2+2a_3} - 1 = 0,$$

and then $a_3 = \mu$.

B. Proof of Lemma 2.1

The proof of this identity is made by quadratures. Making the change $s = \sqrt{2b}\tan\theta$, we get the following equalities for the indefinite integrals:

$$\text{(B.1)} \qquad \int \frac{ds}{(b-s^2)\sqrt{s^2+2b}} = \int \frac{\sec\theta\,d\theta}{b(1-2\tan^2\theta)} = \int \frac{\cos\theta\,d\theta}{b(1-3\sin^2\theta)}.$$

Now, by using the change $\rho = \sqrt{3}\sin\theta$, we obtain

$$\text{(B.2)} \qquad \int \frac{\cos\theta\,d\theta}{b(1-3\sin^2\theta)} = \int \frac{d\rho}{\sqrt{3}\,b(1-\rho^2)} = \frac{1}{2b\sqrt{3}}\ln\Big(\frac{1+\rho}{1-\rho}\Big).$$

Combining (B.1) and (B.2) the result follows from the fundamental theorem of calculus.

C. Proof of (3.54)

Having in mind that $\rho_c = |\phi_c + \mathfrak{z}|^2 - |\phi_c|^2 = 2\,\mathrm{Re}(\phi_c\,\bar{\mathfrak{z}}) + |\mathfrak{z}|^2$, it turns out that

$$|\phi_c + \mathfrak{z}|^2 = |\phi_c|^2 + \rho_c,$$

and therefore we have

$$\|\mathfrak{z}\|_{L^\infty} \lesssim (1 + \|\rho_c\|_{L^\infty}^{1/2}) \lesssim (1 + \|\rho_c\|_{L^\infty}). \tag{C.1}$$

On the other hand,

$$\|\rho_c\|_{L^\infty} \lesssim \|\rho_c\|_{L^2}^{1/2}\|\rho_c'\|_{L^2}^{1/2} \lesssim \|\rho_c\|_{L^2}^{1/2}(\|\mathfrak{z}\|_{L^\infty} + \|\mathfrak{z}'\|_{L^2} + \|\mathfrak{z}\|_{L^\infty}\|\mathfrak{z}'\|_{L^2})^{1/2}.$$

Hence, using Lemma 2.6, we get

$$\|\rho_c\|_{L^\infty} \lesssim \|\rho_c\|_{L^2}^{1/2}(\|\mathfrak{z}\|_{L^\infty}^{1/2} + \|\mathfrak{z}\|_{\mathcal{H}_0}^{1/2} + \|\mathfrak{z}\|_{L^\infty}^{1/2}\|\mathfrak{z}\|_{\mathcal{H}_0}^{1/2}). \tag{C.2}$$

Now, substituting (C.2) into (C.1) and using Young's inequality, we obtain

$$\|\mathfrak{z}\|_{L^\infty} \lesssim (1 + \|\rho_c\|_{L^2} + \|\rho_c\|_{L^2}^{1/2}\|\mathfrak{z}\|_{\mathcal{H}_0}^{1/2} + \|\rho_c\|_{L^2}\|\mathfrak{z}\|_{\mathcal{H}_0}) \lesssim (1 + \|\rho_c\|_{L^2})(1 + \|\mathfrak{z}\|_{\mathcal{H}_0}).$$

D. Computation of some L^2 and L^∞ norms

We collect some L^2 and L^∞ norms, needed along this work, in the following sections. Hereafter, we will consider by K the smallest of the constants that allow us to get the corresponding upper bound.

D.1. L^2 norms

We first compute the associated $\mathcal{H}_0$ norm in the distance d_0, see (2.19). By definition,

$$\|\phi_0 - \phi_c\|_{\mathcal{H}_0}^2 = \|\phi_0' - \phi_c'\|_{L^2}^2 + \|\sqrt{\eta_0}(\phi_0 - \phi_c)\|_{L^2}^2,$$

therefore we split the computation in two steps. First, we consider (with R_c in (2.6))

$$\|\phi_0' - \phi_c'\|_{L^2}^2 = \int_{\mathbb{R}} (\phi_0' - \phi_c')(\phi_0' - \bar{\phi}_c') = \int_{\mathbb{R}} ((\phi_0')^2 + |\phi_c'|^2 - 2R_c'\,\phi_0').$$

Then, expanding in c the last integrand, we note that this L^2 norm is bounded above, at small speeds $|c| \leq \mathfrak{c}$, with $\mathfrak{c} \ll 1$, by

$$\begin{aligned}\|\phi_0' - \phi_c'\|_{L^2}^2 &\leq K\int_{\mathbb{R}}\Big(-\frac{9(\tanh^2(x)\,\mathrm{sech}^4(x))}{8(\tanh^2(x) - 3)^3}\,c^2\Big)\,dx \\ &= \frac{K}{32}(12 - 5\sqrt{3}\log(2+\sqrt{3}))\,c^2.\end{aligned} \tag{D.1}$$

On the other hand, we consider

$$\|\sqrt{\eta_0}(\phi_0-\phi_c)\|_{L^2}^2=\int_{\mathbb{R}}\eta_0(\phi_0-\phi_c)(\phi_0-\bar{\phi}_c)=\int_{\mathbb{R}}\eta_0((\phi_0)^2+|\phi_c|^2-2R_c\phi_0),$$

which again behaves (proceeding as above), at small speeds $|c|\leq \mathfrak{c}$, with $\mathfrak{c}\ll 1$, as

$$\text{(D.2)}\qquad \|\sqrt{\eta_0}(\phi_0-\phi_c)\|_{L^2}^2\leq K\int_{\mathbb{R}}\Big(\frac{27(\tanh^4(x)+2\tanh^2(x)-3)}{8(\tanh^2(x)-3)^3}\,c^2\Big)\,dx$$
$$=K\,\frac{3}{32}(12-\sqrt{3}\log(2-\sqrt{3}))\,c^2.$$

Finally, summing (D.1) and (D.2) and simplifying, we get the $\mathcal{H}_0$ norm in (2.19):

$$\|\phi_0-\phi_c\|_{\mathcal{H}_0}^2\leq K\,\frac{c^2}{16}(24-\sqrt{3}\log(2+\sqrt{3})).$$

With respect to (2.22), we first compute $|\phi_c'/\sqrt{\eta_c}|^2$ as

$$\Big|\frac{\phi_c'}{\sqrt{\eta_c}}\Big|^2=\frac{\partial_x\phi_c\partial_x\bar{\phi}_c}{(\sqrt{\eta_c})^2}$$
$$=\frac{-2\kappa^2\operatorname{sech}^4(\kappa x)(\mu_1^2\mu^2\tanh^2(\kappa x)+\mu_2^2)}{(1+\mu\tanh^2(\kappa x))(\mu_1^4+2(\mu_1^2\mu_2^2-4\mu)\tanh^2(\kappa x)+(\mu_2^4-4\mu^2)\tanh^4(\kappa x)-4)}.$$

Therefore, integrating and having in mind the constraint relation (1.13), we have that (here $D_0=1+\mu\tanh^2(\kappa x)$)

$$\Big\|\frac{\phi_c'}{\sqrt{\eta_c}}\Big\|_{L^2}^2:=\int_{\mathbb{R}}\frac{-2\kappa^2\operatorname{sech}^4(\kappa x)(\mu_1^2\mu^2\tanh^2(\kappa x)+\mu_2^2)\,dx}{D_0(\mu_1^4+2(\mu_1^2\mu_2^2-4\mu)\tanh^2(\kappa x)+(\mu_2^4-4\mu^2)\tanh^4(\kappa x)-4)}$$
$$=\frac{-4\kappa}{(\mu_1^2-2)}\left(\sqrt{|\mu|}\operatorname{arctanh}(\sqrt{|\mu|})+\frac{(\mu_2^2+2\mu+\mu\mu_1^2)\arctan\Big(\sqrt{\frac{2\mu+\mu_2^2}{2+\mu_1^2}}\Big)}{\sqrt{2+\mu_1^2}\,\sqrt{2\mu+\mu_2^2}}\right)$$
$$\leq\frac{\pi}{3\sqrt{3}}+\frac{2}{\sqrt{3}}\operatorname{arccotanh}(\sqrt{3}).$$

With respect to the L^2-norms in (2.24) and (2.25), we get after an expansion in c, $|c|<\mathfrak{c}$, with $\mathfrak{c}\ll 1$ in the integrand of (2.24), that this L^2 norm is bounded above by

$$\Big\|\frac{|\phi_c|^2-\phi_0^2}{\sqrt{\eta_0}}\Big\|_{L^2}^2\leq K\int_{\mathbb{R}}\frac{3\operatorname{sech}^2(x)(\tanh^2(x)+9)^2}{64(\tanh^2(x)-3)^2(\tanh^2(x)+3)}\,c^4\,dx$$
$$\leq K\frac{c^4}{192}\,(36+\sqrt{3}\pi+12\sqrt{3}\log(2+\sqrt{3}))\leq\frac{K}{4}\,c^4,$$

and therefore we obtain (2.24). Now, in (2.25), expanding again in $|c|<\mathfrak{c}$, with $\mathfrak{c}\ll 1$, we get that

$$\Big\|\frac{\phi_0\eta_0-R_c\eta_c}{\sqrt{\eta_0}}\Big\|_{L^2}^2\leq K\int_{\mathbb{R}}\frac{3}{512}\,c^4\,\frac{(-1+\tanh^4(x))\tanh^2(x)}{(-3+\tanh^2(x))^5(3+\tanh^2(x))}\,dx$$
$$\leq K\,\frac{3}{512}\,c^4(630-8\sqrt{3}\pi-39\sqrt{3}\log(\sqrt{3}+2))\leq\frac{K}{4}\,c^4.$$

We now compute the L^2 norm in (2.26). Firstly, we write explicitly the integrand

$$\frac{(\eta_c|\phi_c|^2-\eta_0\phi_0^2)^2}{\eta_0}=\frac{\eta_c\frac{\mu_1^2+\mu_2^2\tanh^2(\kappa x)}{2(\mu\tanh^2(\kappa x)+1)}-\eta_0\frac{2\tanh^2(x)}{3-\tanh^2(x)}}{\left(1-\frac{4\tanh^4(x)}{(3-\tanh^2(x))^2}\right)}.$$

In fact, in the same small speed region $|c|<\mathrm{c}$, with $\mathrm{c}\ll 1$, we get, after an expansion of the above expression, that

$$\begin{aligned}\left\|\frac{\eta_c|\phi_c|^2-\eta_0\phi_0^2}{\sqrt{\eta_0}}\right\|_{L^2}^2&\leq K\int_{\mathbb{R}}\frac{81\,\mathrm{sech}^4(x)}{8(3-\tanh^2(x))^3}c^4\,dx\\&=K\frac{9c^4}{32}\sqrt{3}\log\Big(\frac{1}{2-\sqrt{3}}\Big)\leq\frac{K}{\sqrt{2}}c^4.\end{aligned}$$

We now compute the L^2 norm in (2.27) proceeding in the same way. In fact, after an expansion of the integrand in the small speed region $|c|<\mathrm{c}$, with $\mathrm{c}\ll 1$, we get

$$\left\|\frac{\eta_c|\phi_c|^2R_c^2-\eta_0\phi_0^4}{\sqrt{\eta_0}}\right\|_{L^2}^2\leq K\int_{\mathbb{R}}\frac{27\,\mathrm{sech}(x)\tanh^4(x)}{8(3-\tanh^2(x))^3}c^4\,dx\leq\frac{K}{2}c^4.$$

D.2. L^∞ norms

We now compute the L^∞ norm in (2.28). Expanding it in the small speed region $|c|<\mathrm{c}$, with $\mathrm{c}\ll 1$, we get

$$\frac{|\phi_c|^2-\phi_0^2}{(1+x^2)\eta_c}\leq\frac{(9-4x\tanh(x)+\tanh^2(x))}{8(1+x^2)(3+\tanh^2(x))^3}c^2,$$

uniformly in $x\in\mathbb{R}$, and whose maximum value $3/8$ is attained at $x=0$. Therefore, we get that

$$\left\|\frac{|\phi_c|^2-\phi_0^2}{(1+x^2)\eta_c}\right\|_{L^\infty}\leq\frac{3}{8}c^2.$$

Now we justify the uniform pointwise estimate in (2.29). First, we note that for any given $x\in\mathbb{R}$, we have

$$\frac{\sqrt{3}}{2}|x|=\kappa(1)|x|\leq\kappa(c)|x|\quad\text{for all }|c|\leq 1,\tag{D.3}$$

with κ defined in (1.12). Now observe that

$$\lim_{x\to 0^\pm}\frac{|\phi_0(x)|^2}{|\phi_0(\sqrt{3}x/2)|^2}=\frac{4}{3}\quad\text{and}\quad\lim_{x\to\pm\infty}\frac{|\phi_0(x)|^2}{|\phi_0(\sqrt{3}x/2)|^2}=1,$$

from which, we can conclude that

$$|\phi_0(x)|^2\lesssim\frac{\tanh^2(\sqrt{3}|x|/2)}{3-\tanh^2(\sqrt{3}|x|/2)}\quad\text{for all }x\in\mathbb{R}.\tag{D.4}$$

Now, selecting $s := \tanh(\sqrt{3}|x|/2)$ and using the fact that the function $s \mapsto \frac{s^2}{3-s^2}$ is increasing on the interval $[0,1]$, we have, by combining (D.3) and (D.4), that

$$|\phi_0(x)|^2 \lesssim \frac{\tanh^2(\kappa(c)|x|)}{3-\tanh^2(\kappa(c)|x|)} \quad \text{for all } (x,c) \in \mathbb{R} \times [-1,1]. \tag{D.5}$$

Finally, using that

$$\lim_{c\to 0^\pm} \mu_2(c) = \pm\frac{2}{\sqrt{3}} \quad \text{and} \quad -\frac{1}{3} \le \mu(c) \le 0,$$

we conclude, from (D.5), that there exists c $\ll 1$ such that

$$|\phi_0(x)|^2 \lesssim \frac{\mu_1^2(c) + \mu_2^2(c)\tanh^2(\kappa(c)|x|)}{2+2\mu(c)\tanh^2(\kappa(c)|x|)} \sim |\phi_c(x)|^2$$

for all $(x,c) \in \mathbb{R} \times (-\mathrm{c}, \mathrm{c})$.

E. Computation of det $\mathscr{F}(c)$ and matrix elements of $\mathscr{M}(c,z)$

First of all, we recall the expression of $\det \mathscr{F}(c)$ in (4.3):

$$\begin{aligned}\det \mathscr{F}(c) = &\int_{\mathbb{R}} \langle i\eta_c, -\partial_c \phi_c\rangle_{\mathbb{C}} \\ &\times \Big\{ \int_{\mathbb{R}} \langle \eta_c, -i\phi_c\rangle_{\mathbb{C}} \int_{\mathbb{R}} \langle iR_c\eta_c, \partial_x\phi_c\rangle_{\mathbb{C}} - \int_{\mathbb{R}} \langle \eta_c, \partial_x\phi_c\rangle_{\mathbb{C}} \int_{\mathbb{R}} \langle iR_c\eta_c, -i\phi_c\rangle_{\mathbb{C}} \Big\},\end{aligned} \tag{E.1}$$

for all $c \in (-2,2)$. Now, we compute the five different elements in (E.1) at $c = 0$. We start with the first factor in (E.1):

$$\int_{\mathbb{R}} \langle i\eta_c, -\partial_c\phi_c\rangle_{\mathbb{C}} = -\int_{\mathbb{R}} \mathrm{Re}(i\eta_c \partial_c \bar{\phi}_c), \tag{E.2}$$

and at $c = 0$, we get

$$-\mathrm{Re}(i\eta_c\partial_c\bar{\phi}_c)\big|_{c=0} = -\frac{9(\tanh^4(x) + 2\tanh^2(x) - 3)}{2\sqrt{2}\,\sqrt{3-\tanh^2(x)}\,(\tanh^2(x)-3)^2}.$$

Now, integrating the above expression, we obtain

$$\int_{\mathbb{R}} \langle i\eta_c, -\partial_c\phi_c\rangle_{\mathbb{C}}\big|_{c=0} = -2.$$

The second factor is

$$\int_{\mathbb{R}} \langle iR_c\eta_c, \partial_x\phi_c\rangle_{\mathbb{C}} = \int_{\mathbb{R}} \mathrm{Re}(iR_c\eta_c\partial_x\bar{\phi}_c), \tag{E.3}$$

and at $c = 0$, we have that

$$\mathrm{Re}(iR_c\eta_c\partial_x\bar{\phi}_c)\big|_{c=0} = 0,$$

and thus we get

$$\int_{\mathbb{R}} \langle iR_c\eta_c, \partial_x\phi_c\rangle_{\mathbb{C}}\big|_{c=0} = 0.$$

The corresponding third factor is

$$\int_{\mathbb{R}} \langle \eta_c, -i\phi_c\rangle_{\mathbb{C}} = \int_{\mathbb{R}} \mathrm{Re}(i\eta_c\bar{\phi}_c), \quad \text{with} \quad \mathrm{Re}(i\eta_c\bar{\phi}_c)\big|_{c=0} = 0. \tag{E.4}$$

Therefore, as above, we have

$$\int_{\mathbb{R}} \langle \eta_c, -i\phi_c\rangle_{\mathbb{C}}\big|_{c=0} = 0.$$

The fourth factor is

$$\int_{\mathbb{R}} \langle \eta_c, \partial_x\phi_c\rangle_{\mathbb{C}} = \int_{\mathbb{R}} \mathrm{Re}(\eta_c\partial_x\bar{\phi}_c), \tag{E.5}$$

with

$$\mathrm{Re}(\eta_c\partial_x\bar{\phi}_c)\big|_{c=0} = \frac{9\sqrt{2}\,(\tanh^4(x)+2\tanh^2(x)-3)\mathrm{sech}^2(x)}{\sqrt{3-\tanh^2(x)}\,(\tanh^2(x)-3)^3}.$$

Therefore, by integrating, we get

$$\int_{\mathbb{R}} \langle \eta_c, \partial_x\phi_c\rangle_{\mathbb{C}}\big|_{c=0} = \frac{8}{5}.$$

Finally, the last factor is

$$\int_{\mathbb{R}} \langle iR_c\eta_c, -i\phi_c\rangle_{\mathbb{C}} = \int_{\mathbb{R}} \mathrm{Re}(-R_c\eta_c\bar{\phi}_c), \tag{E.6}$$

with

$$\mathrm{Re}(-R_c\eta_c\bar{\phi}_c)\big|_{c=0} = -\frac{6(\tanh^2(x)(\tanh^4(x)+2\tanh^2(x)-3))}{(\tanh^2(x)-3)^3}.$$

By integrating, we get

$$\int_{\mathbb{R}} \langle iR_c\eta_c, -i\phi_c\rangle_{\mathbb{C}}\big|_{c=0} = -\frac{1}{2}\sqrt{3}\log(\sqrt{3}+2).$$

Finally, gathering the five terms above, we have that

$$\det\mathcal{F}(0) = -\frac{8\sqrt{3}}{5}\log(\sqrt{3}+2) = -\frac{8}{5}E_2[\phi_0].$$

Now, using a classical continuity argument, we get, for $c \in [0, \mathfrak{c})$, with smaller $\mathfrak{c} \lll 1$ if necessary, that

$$\det\mathcal{F}(c) \neq 0.$$

We now list here the computed matrix elements of $\mathcal{M}(c, z)$ in (4.18). By parity reasons, some terms vanish. Namely,

$$m_{11} = \int_{\mathbb{R}} \langle \eta_{c(t)}, \partial_x\phi_{c(t)}\rangle_{\mathbb{C}} = \text{(E.5)},$$

$$m_{12} = \int_{\mathbb{R}} \langle \eta_{c(t)}, \partial_c\phi_{c(t)}\rangle_{\mathbb{C}} = 0,$$

$$m_{13} = \int_{\mathbb{R}} \langle \eta_{c(t)}, -i\phi_{c(t)}\rangle_{\mathbb{C}} = \text{(E.4)}.$$

Now, we list the products coming from the second orthogonality condition in (3.35):

$$
\begin{aligned}
m_{21} &= \int_{\mathbb{R}} \langle i\eta_{c(t)}, \partial_x \phi_{c(t)}\rangle_{\mathbb{C}} = 0,\\
m_{22} &= \int_{\mathbb{R}} \langle i\eta_{c(t)}, \partial_c \phi_{c(t)}\rangle_{\mathbb{C}} = -(\text{E.2}),\\
m_{23} &= \int_{\mathbb{R}} \langle i\eta_{c(t)}, -i\phi_{c(t)}\rangle_{\mathbb{C}} = 0,
\end{aligned}
$$

and finally, the products coming from the third orthogonality relation of (3.35):

$$
\begin{aligned}
m_{31} &= \int_{\mathbb{R}} \langle iR_{c(t)}\eta_{c(t)}, \partial_x \phi_{c(t)}\rangle_{\mathbb{C}} = (\text{E.3}),\\
m_{32} &= \int_{\mathbb{R}} \langle iR_{c(t)}\eta_{c(t)}, \partial_c \phi_{c(t)}\rangle_{\mathbb{C}} = 0,\\
m_{33} &= \int_{\mathbb{R}} \langle iR_{c(t)}\eta_{c(t)}, -i\phi_{c(t)}\rangle_{\mathbb{C}} = (\text{E.6}).
\end{aligned}
$$

In the limit case when $(c = 0, z = 0)$, the matrix (4.18) has the simple expression:

$$
\mathcal{M}(0,0) := \begin{pmatrix} 8/5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -\frac{1}{2}E_2[\phi_0] \end{pmatrix}, \quad \text{with} \quad \det \mathcal{M}(0,0) = -\frac{8}{5}E_2[\phi_0].
$$

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