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Improved estimates for the sharp interface limit of the stochastic Cahn–Hilliard equation with space-time white noise

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Abstract. We study the sharp interface limit of the stochastic Cahn–Hilliard equation with cubic double-well potential and additive space-time white noise $\varepsilon^\sigma \dot{W}$, where $\varepsilon > 0$ is an interfacial width parameter. We prove that, for a sufficiently large scaling constant $\sigma > 0$, the stochastic Cahn–Hilliard equation converges to the deterministic Mullins–Sekerka/Hele-Shaw problem for $\varepsilon \to 0$. The convergence is shown in suitable fractional Sobolev norms as well as in the L^p-norm for $p \in (2, 4]$ in spatial dimension $d = 2, 3$. This generalizes the existing result for the space-time white noise to dimension $d = 3$ and improves the existing results for smooth noise, which were so far limited to $p \in (2, \frac{2d+8}{d+2}]$ in spatial dimension $d = 2, 3$. As a byproduct of the analysis of the stochastic problem with space-time white noise, we identify minimal regularity requirements on the noise which allow convergence to the sharp interface limit in the $\mathbb{H}^1$-norm and also provide improved convergence estimates for the sharp interface limit of the deterministic problem.

1. Introduction

We consider the stochastic Cahn–Hilliard equation with additive space-time white noise

$$\begin{aligned} du^\varepsilon &= \Delta\Big(-\varepsilon\Delta u^\varepsilon + \frac{1}{\varepsilon} f(u^\varepsilon)\Big)dt + \varepsilon^\sigma dW(t) && \text{in } \mathcal{D}_T := (0,T)\times\mathcal{D}, \\ \partial_{\vec{n}} u^\varepsilon &= \partial_{\vec{n}}\Delta u^\varepsilon = 0 && \text{on } (0,T)\times\partial\mathcal{D}, \\ u^\varepsilon(0) &= u_0^\varepsilon && \text{in } \mathcal{D}, \end{aligned} \tag{1}$$

where $\sigma > 0$, $T > 0$ are fixed constants and $\varepsilon > 0$ is a (small) interfacial width parameter. For simplicity we take $\mathcal{D} = (0,1)^d$ to be the unit cube in $\mathbb{R}^d$, $d = 2, 3$, with $\vec{n}$ the outer unit normal to $\partial\mathcal{D}$ and W the space-time white noise. The nonlinearity f in (1) ensures that asymptotically the solutions $\{u^\varepsilon\}_{\varepsilon>0}$ of (1) remain within the physically meaningful range $-1 \le u^\varepsilon \le 1$. One of the most widely used choices is $f(u) = F'(u) = u^3 - u$, where $F(u) = \frac{1}{4}(u^2-1)^2$ is a double-well potential with minima at ± 1.

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Formally, equation (1) can be equivalently written in the mixed form

$$\begin{aligned} du^{\varepsilon} &= \Delta w^{\varepsilon} dt + \varepsilon^{\sigma} dW(t) \quad \text{in } \mathcal{D}_T, \\ w^{\varepsilon} &= -\varepsilon \Delta u^{\varepsilon} dt + \frac{1}{\varepsilon} f(u^{\varepsilon}) \quad \text{in } \mathcal{D}_T, \end{aligned} \tag{2}$$

where w^{ε} is the so-called chemical potential. For sufficiently smooth noise and data the chemical potential w^{ε} has sufficient regularity so that the formulation in (2) can be made rigorous (cf. [2]).

The Cahn–Hilliard equation is a prototype model for mass-conservative phase separation and coarsening phenomena in binary alloys [7, 8]. The solution u^{ε} of (1) is an order parameter which approaches ± 1 in the regions occupied by the pure phases. The pure phases are separated by a thin layer where $|u^{\varepsilon}| < 1$, the so-called diffuse interface, with thickness proportional to the (small) interfacial width parameter ε. It has been observed in [12] that the simulation results obtained with the stochastic Cahn–Hilliard equation with the space-time white noise are in better agreement with physical experiments than those obtained by deterministic simulations.

In the deterministic setting (i.e., when $W \equiv 0$ in (1)) the Cahn–Hilliard equation reads as

$$\begin{aligned} \partial_t u_D^{\varepsilon} &= \Delta w_D^{\varepsilon} \quad \text{in } \mathcal{D}_T, \\ w_D^{\varepsilon} &= -\varepsilon \Delta u_D^{\varepsilon} + \frac{1}{\varepsilon} f(u_D^{\varepsilon}) \quad \text{in } \mathcal{D}_T. \end{aligned} \tag{3}$$

The sharp interface limit of the deterministic Cahn–Hilliard equation has been analyzed in [1], where it was shown that, for $\varepsilon \to 0$, the function w_D^{ε} tends to a function v, which together with the free boundary $\Gamma_t := \lim_{\varepsilon \to 0} \Gamma_t^{\varepsilon}$ (where $\Gamma_t^{\varepsilon} := \{x \in \mathcal{D} : u_D^{\varepsilon}(t,x) = 0\}$, $t \in (0,T)$) satisfies the Mullins–Sekerka/Hele-Shaw problem

$$\begin{aligned} \Delta v &= 0 \quad \text{in } \mathcal{D} \setminus \Gamma_t, \\ \partial_{\vec{n}} v &= 0 \quad \text{on } \partial\mathcal{D}, \\ v &= \lambda H \quad \text{on } \Gamma_t, \\ \mathcal{V} &= \frac{1}{2}(\partial_{\vec{n}_\Gamma} v^+ - \partial_{\vec{n}_\Gamma} v^-) \quad \text{on } \Gamma_t, \\ \Gamma_0 &= \Gamma_{00}, \end{aligned} \tag{4}$$

where H is the mean curvature of Γ_t, $\mathcal{V}$ is the normal velocity of the interface, $\vec{n}_\Gamma$ is the unit normal vector to Γ_t, and v^+, v^- are respectively the restriction of v on $\mathcal{D}_t^+$, $\mathcal{D}_t^-$ (the exterior and interior of Γ_t in $\mathcal{D}$).

In the stochastic setting with trace class noise, it is shown in [4] that, for suitable scaling of the noise, the sharp interface limit of the Cahn–Hilliard equation is the deterministic Hele-Shaw model given by (4) (cf. [3] for the case of multiplicative noise). The work [2] studies convergence of the numerical approximation of the stochastic Cahn–Hilliard equation with smooth noise to the sharp interface limit and also obtains the first result on uniform pointwise convergence to the deterministic Hele-Shaw problem in (4). The sharp interface limit in (4) of the Cahn–Hilliard equation with singular space-time white noise has been studied in [6]. We note that all aforementioned results (also including the present work) rely on the spectral estimate for the deterministic problem (cf. [1]), the use of which requires appropriate scaling of the noise with respect to the interfacial width parameter ε. We also mention the recent work [5], which employs a (discrete) stochastic counterpart of the principal eigenvalue problem to derive a posteriori error estimates for the numerical approximation of the stochastic Cahn–Hilliard equation and [13], where a stochastic Hele-Shaw problem is obtained as the sharp interface limit of the stochastic Cahn–Hilliard equation with (unscaled) smooth-in-time noise.

Throughout this paper we assume that for a given smooth closed hypersurface $\Gamma_{00} \subset \mathcal{D}$, the Hele-Shaw problem in (4) admits a smooth solution $(v, \{\Gamma_t\}_{t\in[0,T]})$. Under this assumption, it is possible to construct an approximation $(u_{\mathsf{A}}^{\varepsilon}, w_{\mathsf{A}}^{\varepsilon})$ of (3) that satisfies

$$
\begin{aligned}
\partial_t u_{\mathsf{A}}^{\varepsilon} &= \Delta w_{\mathsf{A}}^{\varepsilon} \quad \text{in } \mathcal{D}_T,\\
w_{\mathsf{A}}^{\varepsilon} &= -\varepsilon \Delta u_{\mathsf{A}}^{\varepsilon} + \frac{1}{\varepsilon} f(u_{\mathsf{A}}^{\varepsilon}) + r_{\mathsf{A}}^{\varepsilon} \quad \text{in } \mathcal{D}_T,
\end{aligned}
\tag{5}
$$

with the same boundary conditions as in (1) (cf. [1, (4.30)]). Furthermore, for any $K > 0$ and

$$
k > (d+2)\frac{d^2+6d+10}{4d+16},
$$

the following estimates hold (cf. [1, Theorems 2.1 and 4.12] and [1, (4.30)]):

$$
\|r_{\mathsf{A}}^{\varepsilon}\|_{C(\mathcal{D}_T)} \le C\varepsilon^{K-2}, \quad \|w_{\mathsf{A}}^{\varepsilon} - v\|_{C(\mathcal{D}_T)} \le C\varepsilon, \tag{6a}
$$

$$
\|u_D^{\varepsilon} - u_{\mathsf{A}}^{\varepsilon}\|_{L^p(0,T;\mathbb{L}^p)} \le C\varepsilon^k \quad \text{for } p \in \left(2, \frac{2d+8}{d+2}\right], \tag{6b}
$$

where the constant $C \ge 0$ is independent of ε.

For $d = 2$, the best possible space in (6b) is $L^3(0,T;\mathbb{L}^3)$ and for $d = 3$, the best space in (6b) is $L^{\frac{14}{5}}(0,T;\mathbb{L}^{\frac{14}{5}})$. This convergence result is suboptimal in the case of the double-well potential where $u^{\varepsilon}, u_{\mathsf{A}}^{\varepsilon} \in L^4(0,T;\mathbb{L}^4)$.

In the stochastic setting with trace-class noise one has the following error estimates (cf. [4, Theorem 3.10]):

$$
\mathbb{P}\left(\left\{\|u^{\varepsilon} - u_{\mathsf{A}}^{\varepsilon}\|_{L^p(0,T;\mathbb{L}^p)} \le C\varepsilon^{\gamma}\right\}\right) \ge 1 - C_l\varepsilon^l, \tag{7}
$$

$$
\mathbb{P}\left(\left\{\|u^{\varepsilon} - u_{\mathsf{A}}^{\varepsilon}\|^2_{L^\infty(0,T;\mathbb{H}^{-1})} \le C\varepsilon^{g(\sigma,\gamma)}\right\}\right) \ge 1 - C_l\varepsilon^l, \tag{8}
$$

$$
\mathbb{P}\left(\left\{\|u^{\varepsilon} - u_{\mathsf{A}}^{\varepsilon}\|^2_{L^2(0,T;\mathbb{H}^{1})} \le C\varepsilon^{h(\sigma,\gamma)}\right\}\right) \ge 1 - C_l\varepsilon^l, \tag{9}
$$

for suitable $\gamma > 0$, $l > 0$, where $u_{\mathsf{A}}^{\varepsilon}$ is the (deterministic) solution of (5) and p is as in (6b). Hence, as in the deterministic setting, the best spaces in which the convergence for the stochastic Cahn–Hilliard equation takes place in $d = 2$ and $d = 3$ are $L^3(0,T;\mathbb{L}^3)$ and $L^{\frac{14}{5}}(0,T;\mathbb{L}^{\frac{14}{5}})$, respectively.

The sharp interface limit of the two-dimensional stochastic Cahn–Hilliard equation driven by singular space-time white noise was recently analyzed in [6] where error estimates (7) (with $p = 3$) and (8) were obtained. Due to regularity restrictions, an analogue of error estimate (9) is not available in the case of space-time white noise.

In the previous works on the sharp interface limits of the deterministic and the stochastic Cahn–Hilliard equation [1, 4], the following inequality (see [1, Lemma 2.2]) was employed to estimate the nonlinearity:

$$-\int_{\mathcal{D}} \varepsilon^{-1}\mathcal{N}(u_{\mathrm{A}}^{\varepsilon}, v)v \le C\varepsilon^{-1}\|v\|_{\mathbb{L}^p}^p, \quad p \in (2,3], \tag{10}$$

where $\mathcal{N}(u, v) := f(u+v) - f(u) - f'(u)v$. The above estimate is combined with dimension-dependent interpolation inequalities which yield suboptimal dimension-dependent estimates for the double-well nonlinearity. In the stochastic case with the space-time white noise, an analogous approach restricts the analysis to spatial dimension $d = 2$ (cf. [6]).

The (probabilistically) strong variational solution of the stochastic Cahn–Hilliard equation enjoys the following regularity:

$$\text{(i)} \quad u \in L^4(\Omega; C([0,T], \mathbb{L}^4)) \cap L^2(\Omega; L^2(0,T; \mathbb{H}^1))$$

for sufficiently regular trace class noise (see [9, Proposition 2.2]) and

$$\text{(ii)} \quad u \in L^4(\Omega; C([0,T], \mathbb{L}^4)) \cap L^2(\Omega; L^2(0,T; \mathbb{H}^{2-\frac{d}{2}-\vartheta}))$$

for arbitrary $\vartheta > 0$ for space-time white noise (see Theorem 3.1 below). In the present work, instead of employing general formula (10), we estimate the double-well nonlinearity by an explicit calculation and use a new interpolation inequality (Lemma 4.5 below). This approach yields error estimates which are optimal with respect to the aforementioned regularity of the solution of the (stochastic) Cahn–Hilliard equation and also allows us to generalize the analysis to the case of the space-time white noise in dimension $d = 3$.

The main contributions of the present paper are the following:

(i) We prove (6) for $p \in (2,4]$ for any $d = 2, 3$; see Theorem 6.1. This improves [1, Theorem 2.1] for the double-well potential.

(ii) We prove (7) and (8) for stochastic Cahn–Hilliard equation driven by space-time white noise in dimension $d = 2, 3$ with $p \in (2,4]$; see Theorem 4.1.

(iii) We derive an analogue of error estimate (9) (see Theorem 4.1) in fractional Sobolev spaces

$$\mathbb{P}\big(\{\|u^{\varepsilon} - u_{\mathrm{A}}^{\varepsilon}\|^2_{L^2(0,T;\mathbb{H}^{2-\frac{d}{2}-\vartheta})} \le C\varepsilon^{\frac{\gamma}{3}-4}\}\big) \ge 1 - C_{\delta,\eta}\varepsilon^{\delta+\eta} - C_{\vartheta,\kappa}\varepsilon^{\vartheta+\kappa},$$

for any $\vartheta, \delta > 0$ and $\kappa, \eta \ge 0$. We observe that for $d = 2$, this leads to an error estimate almost in $\mathbb{H}^1$ and for $d = 3$, this leads to an error almost in $\mathbb{H}^{\frac{1}{2}}$. Note that it is not clear whether an error estimate in $L^2(0,T;\mathbb{H}^1)$ is achievable in the low regularity setting of space-time white noise.

(iv) We identify minimal regularity properties of the noise required for the $\mathbb{H}^1$ convergence given in (9) to hold; see Section 5. The condition is weaker than the one required in [4, Assumption 3.1].

We adopt the approach of [6, 9], which is based on introducing the stochastic convolution in (14) and studying the translated solution $Y^\varepsilon := u^\varepsilon - u^\varepsilon_{\mathsf{A}} - Z^\varepsilon$. We derive an a priori estimate for the translated solution Y^ε in Lemma 4.1. In order to obtain estimates that are robust with respect to the interfacial width parameter ε (i.e., to avoid the use of Gronwall's lemma), we employ the lower bound of the principal eigenvalue of the linearized (deterministic) Cahn–Hilliard equation (cf. [1, Theorem 3.1]). In order to overcome the barrier $p \leq \frac{2d+8}{d+2}$ we make use of a new interpolation inequality (see Lemma 4.5). To deal with the low regularity of the space-time white noise, we benefit from the smoothing properties of the semigroup generated by the bi-Laplacian Δ^2. Consequently, we obtain an estimate for convolution (14) in fractional Sobolev spaces in Lemma 3.2, which allows us to derive error estimate (40) below.

The paper is organized as follows: we introduce the notation and preliminary results in Section 2. Some useful regularity properties of the variational solution of (1) are presented in Section 3. The sharp interface limit of the stochastic Cahn–Hilliard equation is analyzed in Section 4. The corresponding results for more regular noise are summarized in Section 5 and the deterministic problem is analyzed in Section 6.

2. Notations and preliminaries

By $\mathbb{L}^p := L^p(\mathcal{D})$ we denote the standard Lebesgue space of p-th order integrable functions on $\mathcal{D}$. The $\mathbb{L}^2$ inner product is denoted by $(\cdot,\cdot)$ and the associated norm by $\|\cdot\|$. For $g \in \mathbb{L}^2$, we denote by $m(g)$ the average of g, given by $m(g) := \frac{1}{|\mathcal{D}|}\int_{\mathcal{D}} g(x)dx$. We also write $\mathbb{L}^2_0 := \{g \in \mathbb{L}^2 : m(g) = 0\}$. For $s \in \mathbb{R}$, we denote the standard Sobolev space on $\mathcal{D}$ by $\mathbb{H}^s := H^s(\mathcal{D})$.

The Neumann Laplace operator $-\Delta$ with domain $D(-\Delta) = \{u \in \mathbb{H}^2 : \frac{\partial u}{\partial \vec{n}} = 0 \text{ on } \partial\mathcal{D}\}$ is self-adjoint and positive and has compact resolvent. We consider an orthonormal basis of $\mathbb{L}^2$ consisting of eigenvectors $\{e_j\}_{j\in\mathbb{N}^d}$ of the Neumann Laplacian, with corresponding eigenvalues (λ_j) such that $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots \to +\infty$. Note that for $k = (k_1, \cdots, k_d) \in \mathbb{N}^d$, λ_k satisfies $\lambda_k \simeq |k|^2$, where $|k|^2 = \lambda_1^2 + \cdots + \lambda_d^2$ and it holds that

$$\sum_{j\in\mathbb{N}^d} \lambda_j^\alpha < +\infty \quad \text{iff} \quad \alpha < -\frac{d}{2}. \tag{11}$$

For $s \in \mathbb{R}$ and $u \in \mathbb{L}^2$, we define the fractional Laplacian $(-\Delta)^s$ as

$$(-\Delta)^s u = \sum_{j\in\mathbb{N}^d} \lambda_j^s u_j e_j \quad \text{for } u = \sum_{j\in\mathbb{N}^d} u_j e_j, \tag{12}$$

where the domain of $(-\Delta)^{\frac{s}{2}}$ is given by

$$D((-\Delta)^{\frac{s}{2}}) := \Big\{u = \sum_{j\in\mathbb{N}^d} u_j e_j \ : \ \sum_{j\in\mathbb{N}^d} \lambda_j^s u_j^2 < \infty\Big\}.$$

We introduce the seminorm and semiscalar product

$$|v|_s = \|(-\Delta)^{\frac{s}{2}} v\| \quad \text{and} \quad (u,v)_s = ((-\Delta)^{\frac{s}{2}} u, (-\Delta)^{\frac{s}{2}} v), \quad u, v \in D((-\Delta)^{\frac{s}{2}})$$

as well as the norm

$$\|v\|_s = (|v|_s^2 + m^2(v))^{\frac{1}{2}}, \quad v \in D((-\Delta)^{\frac{s}{2}}).$$

For $s \in [0,2]$, the norm $\|\cdot\|_s$ is equivalent to the usual norm on $\mathbb{H}^s$ and $D((-\Delta)^{\frac{s}{2}})$ is a closed subspace of $\mathbb{H}^s$ (see, e.g., [9, Section 2.1]).

The term W in (1) is the space-time white noise, which is formally represented as

$$W(x,t) = \sum_{j\in\mathbb{N}^d} \beta_j(t) e_j(x), \tag{13}$$

where the β_j are independent and identically distributed standard Brownian motions on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_t, \mathbb{P})$ (cf. [9]). Note that the space-time white noise enjoys the zero-mean property, that is, it holds that $m(W) = 0$.

We make use of the following spectral estimate of the deterministic problem (cf. [1, Proposition 3.1]):

Proposition 2.1. *Let $u_{\mathrm{A}}^{\varepsilon}$ be the approximation in (5). Then, for all $w \in \mathbb{H}^1$ with $\int_{\mathcal{D}} w dx = 0$, the following holds:*

$$\int_{\mathcal{D}} \Big(\varepsilon|\nabla w|^2 + \frac{1}{\varepsilon} f'(u_{\mathrm{A}}^{\varepsilon}) w^2\Big) \geq -C_0 \|w\|_{\mathbb{H}^{-1}}^2,$$

where $C_0 \geq 0$ is a constant independent of w and ε.

3. Existence and regularity of the solution

In this section we summarize existence and regularity properties of the solution of the stochastic Cahn–Hilliard equation in (1) with space-time white noise.

We introduce the stochastic convolution

$$Z^{\varepsilon}(t) := \varepsilon^{\sigma} \int_0^t e^{-(t-s)\varepsilon\Delta^2} dW(s) = \varepsilon^{\sigma} \sum_{i\in\mathbb{N}^d} \int_0^t e^{-\lambda_i^2(t-s)\varepsilon} e_i d\beta_i(s), \quad t \in [0,T]. \tag{14}$$

The following two lemmas will be useful in the rest of this paper:

Lemma 3.1. *For any $p \in [1, \infty)$, there exists a constant $C = C(p) \geq 0$ such that*

$$\mathbb{E}\big[\sup_{t\in[0,T]} \|Z^\varepsilon(t)\|^p_{\mathbb{L}^p}\big] \leq C(p)\varepsilon^{(\sigma-\frac{1}{2})p}.$$

Proof. Taking the expectation of both sides of (14) yields

$$\mathbb{E}\big[\sup_{t\in[0,T]} \|Z^\varepsilon(t)\|^p_{\mathbb{L}^p}\big] = \varepsilon^{\sigma p}\int_{\mathcal{D}} \mathbb{E}\Big[\sup_{t\in[0,T]}\Big|\sum_{i\in\mathbb{N}^d}\int_0^t e^{-\lambda_i^2(t-s)\varepsilon}d\beta_i(s)e_i(x)\Big|^p\Big]dx.$$

Using the Burkholder-Davis—Gundy (BDG) inequality [10, Theorem 4.36] and the uniform boundedness of $(e_i)_{i\in\mathbb{N}}$, we obtain

$$\begin{aligned}\mathbb{E}\big[\sup_{t\in[0,T]} \|Z^\varepsilon(t)\|^p_{\mathbb{L}^p}\big] &\leq C(p)\varepsilon^{\sigma p}\int_{\mathcal{D}}\Big(\sum_{i\in\mathbb{N}^d}\int_0^T \big(e^{-\lambda_i^2(T-s)\varepsilon}\big)^2 ds\Big)^{\frac{p}{2}}dx\\ &\leq C(p)\varepsilon^{\sigma p}\int_{\mathcal{D}}\Big(\sum_{i\in\mathbb{N}^d}\int_0^T e^{-2\lambda_i^2(T-s)\varepsilon}ds\Big)^{\frac{p}{2}}\\ &\leq C(p)\varepsilon^{(\sigma-\frac{1}{2})p}\int_{\mathcal{D}}\Big(\sum_{i\in\mathbb{N}^d}\frac{1}{\lambda_i^2}\Big)^{\frac{p}{2}}dx \leq C(p)\varepsilon^{(\sigma-\frac{1}{2})p},\end{aligned}$$

where in the last step we used the fact that $\sum_{i\in\mathbb{N}^d}\lambda_i^{-2} < \infty$, since $d = 2, 3$; see (11). ∎

Lemma 3.2. *For any $\vartheta > 0$, $p \geq 2$, there is a constant $C \geq 0$ such that*

$$\mathbb{E}\big[\sup_{t\in[0,T]} \|(-\Delta)^{1-\frac{d}{4}-\frac{\vartheta}{2}}Z^\varepsilon(t)\|^p\big] \leq C\varepsilon^{(\sigma-\frac{1}{2})p}.$$

Proof. From (14) and (12), it follows that

$$(-\Delta)^{1-\frac{d}{4}-\frac{\vartheta}{2}}Z^\varepsilon(t) = \varepsilon^\sigma\sum_{i\in\mathbb{N}^d}\int_0^t \lambda_i^{1-\frac{d}{4}-\frac{\vartheta}{2}}e^{-\lambda_i^2(t-s)\varepsilon}e_i\,d\beta_i(s), \quad t \in [0, T]. \tag{15}$$

Taking the $\mathbb{L}^2$-norm in (15), raising to power p, using the embedding $\mathbb{L}^p \hookrightarrow \mathbb{L}^2$ ($p \geq 2$), taking the supremum and the expectation, and using the BDG inequality [10, Proposition 4.36], it follows that

$$\begin{aligned}&\mathbb{E}\big[\sup_{t\in[0,T]} \|(-\Delta)^{1-\frac{d}{4}-\frac{\vartheta}{2}}Z^\varepsilon(t)\|^p\big] \leq C\,\mathbb{E}\big[\sup_{t\in[0,T]} \|(-\Delta)^{1-\frac{d}{4}-\frac{\vartheta}{2}}Z^\varepsilon(t)\|^p_{\mathbb{L}^p}\big]\\ &\quad\leq C\varepsilon^{\sigma p}\int_{\mathcal{D}}\mathbb{E}\Big[\sup_{t\in[0,T]}\Big|\sum_{i\in\mathbb{N}^d}\int_0^t \lambda_i^{1-\frac{d}{4}-\frac{\vartheta}{2}}e^{-\lambda_i^2(t-s)\varepsilon}d\beta_i(s)e_i(x)\Big|^p\Big]dx\\ &\quad\leq C\varepsilon^{\sigma p}\int_{\mathcal{D}}\Big(\sum_{i\in\mathbb{N}^d}\int_0^T \lambda_i^{2-\frac{d}{2}-\vartheta}e^{-2\lambda_i^2(T-s)\varepsilon}ds\Big)^{\frac{p}{2}}dx\\ &\quad\leq C\varepsilon^{(\sigma-\frac{1}{2})p}\int_{\mathcal{D}}\Big(\sum_{i\in\mathbb{N}^d}\lambda_i^{-\frac{d}{2}-\vartheta}\Big)^{\frac{p}{2}}dx \leq C\varepsilon^{(\sigma-\frac{1}{2})p},\end{aligned}$$

where in the last step we used the fact that $\sum_{i\in\mathbb{N}^d}\lambda_i^{-\frac{d}{2}-\vartheta} < \infty$; see (11). ∎

Theorem 3.1. *Let $u_0^\varepsilon \in \mathbb{H}^{-1}$. Then, there exists a unique strong variational solution u^ε of* (1) *such that*

$$u^\varepsilon \in L^2(\Omega; C([0,T];\mathbb{H}^{-1})) \cap L^2(\Omega; L^2(0,T;\mathbb{H}^{2-\frac{d}{2}-\vartheta})) \cap L^4(\Omega; L^4(0,T;\mathbb{L}^4)),$$

Furthermore, for any $p \geq 2$, it holds that

$$\begin{aligned}\mathcal{E}_p(u^\varepsilon) &:= \mathbb{E}\Big[\|u^\varepsilon\|^p_{L^\infty(0,T;\mathbb{H}^{-1})} + \varepsilon^{\frac{p}{2}}\|(-\Delta)^{1-\frac{d}{4}-\frac{\vartheta}{2}}u^\varepsilon\|^p_{L^2(0,T;\mathbb{L}^2)} + \frac{1}{\varepsilon^{\frac{p}{2}}}\|u^\varepsilon\|^{2p}_{L^4(0,T;\mathbb{L}^4)}\Big]\\ &\leq C(\varepsilon^{-\frac{p}{2}} + \varepsilon^{(\sigma-\frac{1}{2})p} + \varepsilon^{(2\sigma-\frac{3}{2})p}),\end{aligned} \tag{16}$$

where $\vartheta > 0$ is any arbitrary small number.

Proof. The proof of the existence and the uniqueness, as well as the proof of the fact that u^ε belongs to $L^2(\Omega; C([0,T];\mathbb{H}^{-1}))$, can be found in [9, Theorem 2.1]. To prove that u^ε belongs to $L^4(\Omega; L^4(0,T;\mathbb{L}^4))$ and to $L^2(\Omega; L^2(0,T;\mathbb{H}^{2-\frac{d}{2}-\vartheta}))$, we set $\widehat{u}^\varepsilon(t) := u^\varepsilon(t) - Z^\varepsilon(t)$. Then, $\widehat{u}^\varepsilon(t)$ satisfies the following random PDE:

$$\begin{aligned}\frac{d}{dt}\widehat{u}^\varepsilon(t) &= -\varepsilon\Delta^2\widehat{u}^\varepsilon(t) + \frac{1}{\varepsilon}\Delta f(\widehat{u}^\varepsilon(t) + Z^\varepsilon(t)), \quad t \in (0,T],\\ \widehat{u}^\varepsilon(0) &= u_0^\varepsilon.\end{aligned}$$

Testing the above equation with $(-\Delta)^{-1}\widehat{u}^\varepsilon(t)$ yields

$$\frac{1}{2}\frac{d}{dt}\|\widehat{u}^\varepsilon(t)\|^2_{\mathbb{H}^{-1}} + \varepsilon\|\nabla\widehat{u}^\varepsilon(t)\|^2 + \frac{1}{\varepsilon}(f(\widehat{u}^\varepsilon(t) + Z^\varepsilon(t)), \widehat{u}^\varepsilon(t)) = 0.$$

Using the fact that $(f(v), v) \geq \frac{1}{2}\|v\|^4_{\mathbb{L}^4} - C$, $v \in \mathbb{L}^4$, it follows that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\widehat{u}^\varepsilon(t)\|^2_{\mathbb{H}^{-1}} + \varepsilon\|\nabla\widehat{u}^\varepsilon(t)\|^2 + \frac{1}{2\varepsilon}\|\widehat{u}^\varepsilon(t) + Z^\varepsilon(t)\|^4_{\mathbb{L}^4}&\\ \leq \frac{C}{\varepsilon} + \frac{1}{\varepsilon}|(f(\widehat{u}^\varepsilon(t) + Z^\varepsilon(t)), Z^\varepsilon(t))|.&\end{aligned} \tag{17}$$

Noting that $|f(x)| \leq 2|x|^3 + C_1$, using Hölder's and Young's inequalities and the embedding $\mathbb{L}^4 \hookrightarrow \mathbb{L}^1$, we deduce that

$$\begin{aligned}&|(f(\widehat{u}^\varepsilon(t) + Z^\varepsilon(t)), Z^\varepsilon(t))|\\ &\leq 2\int_{\mathcal{D}}|\widehat{u}^\varepsilon(t) + Z^\varepsilon(t)|^3|Z^\varepsilon(t)|dx + C_1\int_{\mathcal{D}}|Z^\varepsilon(t)|dx\\ &\leq 2\Big(\int_{\mathcal{D}}|\widehat{u}^\varepsilon(t) + Z^\varepsilon(t)|^4dx\Big)^{\frac{3}{4}}\Big(\int_{\mathcal{D}}|Z^\varepsilon(t)|^4dx\Big)^{\frac{1}{4}} + C_1\int_{\mathcal{D}}|Z^\varepsilon(t)|dx\\ &\leq \frac{1}{4}\int_{\mathcal{D}}|\widehat{u}^\varepsilon(t) + Z^\varepsilon(t)|^4dx + C\int_{\mathcal{D}}|Z^\varepsilon(t)|^4dx + C_1\int_{\mathcal{D}}|Z^\varepsilon(t)|dx\\ &\leq \frac{1}{4}\|\widehat{u}^\varepsilon(t) + Z^\varepsilon(t)\|^4_{\mathbb{L}^4} + C\|Z^\varepsilon(t)\|^4_{\mathbb{L}^4} + C.\end{aligned} \tag{18}$$

Substituting (18) into (17) and absorbing $\frac{1}{4\varepsilon}\|\widehat{u}^\varepsilon(t)+Z^\varepsilon(t)\|_{\mathbb{L}^4}^4$ into the left-hand side yields

$$\frac{1}{2}\frac{d}{dt}\|\widehat{u}^\varepsilon(t)\|_{\mathbb{H}^{-1}}^2+\varepsilon\|\nabla\widehat{u}^\varepsilon(t)\|^2+\frac{1}{4\varepsilon}\|\widehat{u}^\varepsilon(t)+Z^\varepsilon(t)\|_{\mathbb{L}^4}^4\le\frac{C}{\varepsilon}+\frac{C}{\varepsilon}\|Z^\varepsilon(t)\|_{\mathbb{L}^4}^4. \quad (19)$$

Integrating (19) over $[0,t]$ and taking the supremum over $[0,T]$ yields

$$\begin{aligned}\sup_{t\in[0,T]}\|\widehat{u}^\varepsilon(t)\|_{\mathbb{H}^{-1}}^2&+\varepsilon\int_0^T\|\nabla\widehat{u}^\varepsilon(s)\|^2ds+\frac{1}{4\varepsilon}\int_0^T\|\widehat{u}^\varepsilon(s)+Z^\varepsilon(s)\|_{\mathbb{L}^4}^4ds\\&\le\|\widehat{u}^\varepsilon(0)\|_{\mathbb{H}^{-1}}^2+\frac{C}{\varepsilon}+\frac{C}{\varepsilon}\int_0^T\|Z^\varepsilon(s)\|_{\mathbb{L}^4}^4ds. \quad (20)\end{aligned}$$

Taking the expectation on both sides of (20) and using Lemma 3.1 yields

$$\begin{aligned}\mathbb{E}\big[\sup_{t\in[0,T]}\|\widehat{u}^\varepsilon(t)\|_{\mathbb{H}^{-1}}^2\big]&+\varepsilon\int_0^T\mathbb{E}[\|\nabla\widehat{u}^\varepsilon(s)\|^2]ds+\frac{1}{4\varepsilon}\int_0^T\mathbb{E}[\|\widehat{u}^\varepsilon(s)+Z^\varepsilon(s)\|_{\mathbb{L}^4}^4]ds\\&\le\|u_0^\varepsilon\|_{\mathbb{H}^{-1}}^2+\frac{CT}{\varepsilon}+\frac{C}{\varepsilon}\int_0^T\mathbb{E}[\|Z^\varepsilon(s)\|_{\mathbb{L}^4}^4]ds\le C(1+\varepsilon^{-1}+\varepsilon^{4\sigma-3}). \quad (21)\end{aligned}$$

The proof of the fact that $u^\varepsilon\in L^4(\Omega;L^4(0,T;\mathbb{L}^4))$ then follows from (21) by using the triangle inequality and Lemma 3.1.

Using the fact that $(-\Delta)^{-\alpha}$ is bounded in $\mathbb{L}^2$ for $\alpha>0$, the equivalence of norms $\|\cdot\|_s$ and the usual norm of $\mathbb{H}^s$ for $s>0$ (see Section 2), and Poincaré's inequality, it follows that

$$\begin{aligned}\|(-\Delta)^{1-\frac{d}{4}-\frac{\vartheta}{2}}\widehat{u}(s)\|&\le\|(-\Delta)^{\frac{1}{2}-\frac{d}{4}-\frac{\vartheta}{2}}\|_{\mathcal{L}(\mathbb{L}^2)}\|(-\Delta)^{\frac{1}{2}}\widehat{u}(s)\|\\&\le C\|(-\Delta)^{\frac{1}{2}}\widehat{u}(s)\|\le C\|\nabla\widehat{u}(s)\|, \quad (22)\end{aligned}$$

where we used that $\frac{1}{2}-\frac{d}{4}-\frac{\vartheta}{2}<0$ (since $d=2,3$ and $\vartheta>0$).

The proof of the fact that $u^\varepsilon\in L^2(\Omega;L^2(0,T;\mathbb{H}^{2-\frac{d}{2}-\vartheta}))$ follows from (21) and (22) by using the triangle inequality and Lemma 3.2.

To prove (16), we start from (20). Omitting the terms involving the norms $\|\nabla\cdot\|$ and $\|\cdot\|_{\mathbb{L}^4}$ on the left-hand side, raising the resulting inequality to power $\frac{p}{2}$, taking the expectation on both sides, and using Hölder's inequality, the embedding $\mathbb{L}^r\hookrightarrow\mathbb{L}^s$, $s\le r$, and Lemma 3.1 yields

$$\begin{aligned}\mathbb{E}\big[\|\widehat{u}^\varepsilon\|_{\mathbb{L}^\infty(0,T;\mathbb{H}^{-1})}^p\big]&\le\|u_0^\varepsilon\|_{\mathbb{H}^{-1}}^2+C\varepsilon^{-\frac{p}{2}}+\frac{C}{\varepsilon^{\frac{p}{2}}}\mathbb{E}\Big(\int_0^T\|Z^\varepsilon(s)\|_{\mathbb{L}^4}^4ds\Big)^{\frac{p}{2}}\\&\le\|u_0^\varepsilon\|_{\mathbb{H}^{-1}}^2+C\varepsilon^{-\frac{p}{2}}+\frac{C}{\varepsilon^{\frac{p}{2}}}\int_0^T\mathbb{E}\|Z^\varepsilon(s)\|_{\mathbb{L}^4}^{2p}ds\\&\le\|u_0^\varepsilon\|_{\mathbb{H}^{-1}}^2+C\varepsilon^{-\frac{p}{2}}+\frac{C}{\varepsilon^{\frac{p}{2}}}\int_0^T\mathbb{E}\|Z^\varepsilon(s)\|_{\mathbb{L}^{2p}}^{2p}ds\\&\le\|u_0^\varepsilon\|_{\mathbb{H}^{-1}}^2+C\varepsilon^{-\frac{p}{2}}+C\varepsilon^{(2\sigma-\frac{3}{2})p}.\end{aligned}$$

Repeating the argument above (i.e., dropping appropriate terms on the left-hand side in (20), raising the resulting inequality to power $\frac{p}{2}$, and using Hölder's inequality, (22), and Lemma 3.1), we arrive at

$$\begin{aligned}&\frac{1}{\varepsilon^{\frac{p}{2}}}\mathbb{E}\big[\|\widehat{u}^{\varepsilon}+Z^{\varepsilon}\|^{2p}_{L^4(0,T;\mathbb{L}^4)}+\varepsilon^{\frac{p}{2}}\|(-\Delta)^{1-\frac{d}{4}-\frac{\vartheta}{2}}\widehat{u}^{\varepsilon}\|^{p}_{L^2(0,T,\mathbb{L}^2)}\big]\\&\qquad\le\|u_0^{\varepsilon}\|^2_{\mathbb{H}^{-1}}+C\varepsilon^{-\frac{p}{2}}+C\varepsilon^{(2\sigma-\frac{3}{2})p}.\end{aligned}$$

Summing the two preceding estimates, using Lemmas 3.1 and 3.2 completes the proof of (16). ∎

4. Sharp interface limit of the stochastic problem

Recall that the solution $u_{\mathrm{A}}^{\varepsilon}$ of (5) is constructed in [1]. We set $R^{\varepsilon}:=u^{\varepsilon}-u_{\mathrm{A}}^{\varepsilon}$. From (1) and (5), it follows that R^{ε} satisfies the stochastic PDE (SPDE)

$$\begin{aligned}dR^{\varepsilon}=-\varepsilon\Delta^2R^{\varepsilon}dt+\frac{1}{\varepsilon}\Delta(f(u_{\mathrm{A}}^{\varepsilon}+R^{\varepsilon})-f(u_{\mathrm{A}}^{\varepsilon}))dt-\Delta r_{\mathrm{A}}^{\varepsilon}dt+\varepsilon^{\sigma}dW\quad&\text{in }\mathcal{D}_T,\\ \partial_{\vec{n}}R^{\varepsilon}=\partial_{\vec{n}}\Delta R^{\varepsilon}=0\quad&\text{on }\partial\mathcal{D},\\ R^{\varepsilon}(0)=0\quad&\text{in }\mathcal{D}.\end{aligned}$$

We set $Y^{\varepsilon}:=R^{\varepsilon}-Z^{\varepsilon}$. Note that (14) implies that $dZ^{\varepsilon}=-\varepsilon\Delta^2Z^{\varepsilon}+\varepsilon^{\sigma}dW$. Hence, we deduce that Y^{ε} satisfies $\mathbb{P}$-almost surely the following random PDE:

$$\begin{aligned}\frac{d}{dt}Y^{\varepsilon}=-\varepsilon\Delta^2Y^{\varepsilon}+\frac{1}{\varepsilon}\Delta(f(u_{\mathrm{A}}^{\varepsilon}+Y^{\varepsilon}+Z^{\varepsilon})-f(u_{\mathrm{A}}^{\varepsilon}))-\Delta r_{\mathrm{A}}^{\varepsilon}\quad&\text{in }\mathcal{D}_T,\\ \partial_{\vec{n}}Y^{\varepsilon}=\partial_{\vec{n}}\Delta Y^{\varepsilon}=0\quad&\text{on }\partial\mathcal{D},\\ Y^{\varepsilon}(0)=0\quad&\text{in }\mathcal{D}.\end{aligned}\tag{23}$$

In the next lemma we derive an estimate for the solution of RPDE (23).

Lemma 4.1. *The following estimate holds $\mathbb{P}$-almost surely for the solution of (23):*

$$\begin{aligned}&\|Y^{\varepsilon}(t)\|^2_{\mathbb{H}^{-1}}+\varepsilon^4\int_0^t\|\nabla Y^{\varepsilon}(s)\|^2ds+\frac{13}{8\varepsilon}\int_0^t\|Y^{\varepsilon}(s)\|^4_{\mathbb{L}^4}ds\\&\quad\le\frac{C}{\varepsilon}\int_0^t\|Y^{\varepsilon}(s)\|^3_{\mathbb{L}^3}ds+\frac{C}{\varepsilon}\int_0^t\|Z^{\varepsilon}(s)\|^{\frac{4}{3}}_{\mathbb{L}^{\frac{4}{3}}}ds+\frac{C}{\varepsilon}\int_0^t\|Z^{\varepsilon}(s)\|^2ds\\&\qquad+\frac{C}{\varepsilon}\int_0^t\|Z^{\varepsilon}(s)\|^{\frac{8}{3}}_{\mathbb{L}^{\frac{8}{3}}}ds+\frac{C}{\varepsilon}\int_0^t\|Z^{\varepsilon}(s)\|^4_{\mathbb{L}^4}ds\\&\qquad+C\varepsilon^{\frac{1}{2}}\int_0^t\|r_{\mathrm{A}}^{\varepsilon}(s)\|^{\frac{3}{2}}_{C(\mathcal{D})}ds\qquad\textit{for } t\in[0,T].\end{aligned}$$

Proof. We fix $\omega\in\Omega$ and consider $Y^{\varepsilon}\equiv Y^{\varepsilon}(\omega)$. Testing (23) with $(-\Delta)^{-1}Y^{\varepsilon}$ yields

$$\frac{1}{2}\frac{d}{dt}\|Y^{\varepsilon}\|^2_{\mathbb{H}^{-1}}+\varepsilon\|\nabla Y^{\varepsilon}(t)\|^2+\frac{1}{\varepsilon}(f(u_{\mathrm{A}}^{\varepsilon}+Y^{\varepsilon}+Z^{\varepsilon})-f(u_{\mathrm{A}}^{\varepsilon}),Y^{\varepsilon})-(r_{\mathrm{A}}^{\varepsilon},Y^{\varepsilon})=0.\tag{24}$$

Recall that $f(s) = s^3 - s$. A straightforward computation yields

$$f(a) - f(b) = (a-b)f'(a) + (a-b)^3 - 3(a-b)^2 a, \quad a, b \in \mathbb{R}. \tag{25}$$

Using (25), we obtain

$$\begin{aligned}
&(f(u_{\mathsf{A}}^\varepsilon + Y^\varepsilon + Z^\varepsilon) - f(u_{\mathsf{A}}^\varepsilon + Z^\varepsilon), Y^\varepsilon) \\
&\quad = -(f(u_{\mathsf{A}}^\varepsilon + Z^\varepsilon) - f(u_{\mathsf{A}}^\varepsilon + Y^\varepsilon + Z^\varepsilon), Y^\varepsilon) \\
&\quad = (f'(u_{\mathsf{A}}^\varepsilon + Z^\varepsilon)Y^\varepsilon, Y^\varepsilon) + \|Y^\varepsilon\|_{\mathbb{L}^4}^4 + 3((Y^\varepsilon)^3, u_{\mathsf{A}}^\varepsilon + Z^\varepsilon) \\
&\quad = (f'(u_{\mathsf{A}}^\varepsilon)Y^\varepsilon, Y^\varepsilon) + ((f'(u_{\mathsf{A}}^\varepsilon + Z^\varepsilon) - f'(u_{\mathsf{A}}^\varepsilon))Y^\varepsilon, Y^\varepsilon) \\
&\qquad + \|Y^\varepsilon\|_{\mathbb{L}^4}^4 + 3((Y^\varepsilon)^3, u_{\mathsf{A}}^\varepsilon + Z^\varepsilon).
\end{aligned}$$

Using the preceding identity, we rewrite

$$\begin{aligned}
&(f(u_{\mathsf{A}}^\varepsilon + Y^\varepsilon + Z^\varepsilon) - f(u_{\mathsf{A}}^\varepsilon), Y^\varepsilon) \\
&\quad = (f(u_{\mathsf{A}}^\varepsilon + Y^\varepsilon + Z^\varepsilon) - f(u_{\mathsf{A}}^\varepsilon + Z^\varepsilon), Y^\varepsilon) + (f(u_{\mathsf{A}}^\varepsilon + Z^\varepsilon) - f(u_{\mathsf{A}}^\varepsilon), Y^\varepsilon) \\
&\quad = (f'(u_{\mathsf{A}}^\varepsilon)Y^\varepsilon, Y^\varepsilon) + ((f'(u_{\mathsf{A}}^\varepsilon + Z^\varepsilon) - f'(u_{\mathsf{A}}^\varepsilon))Y^\varepsilon, Y^\varepsilon) + \|Y^\varepsilon\|_{\mathbb{L}^4}^4 \\
&\qquad + 3((Y^\varepsilon)^3, u_{\mathsf{A}}^\varepsilon + Z^\varepsilon) + (f(u_{\mathsf{A}}^\varepsilon + Z^\varepsilon) - f(u_{\mathsf{A}}^\varepsilon), Y^\varepsilon).
\end{aligned}$$

Substituting the identity above into (24) leads to

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|Y^\varepsilon(t)\|_{\mathbb{H}^{-1}}^2 + \varepsilon\|\nabla Y^\varepsilon(t)\|^2 + \frac{1}{\varepsilon}(f'(u_{\mathsf{A}}^\varepsilon)Y^\varepsilon, Y^\varepsilon) + \frac{1}{\varepsilon}\|Y^\varepsilon\|_{\mathbb{L}^4}^4 \\
&\quad \le \frac{1}{\varepsilon}|((f'(u_{\mathsf{A}}^\varepsilon + Z^\varepsilon) - f'(u_{\mathsf{A}}^\varepsilon))Y^\varepsilon, Y^\varepsilon)| + \frac{3}{\varepsilon}|((Y^\varepsilon)^3, u_{\mathsf{A}}^\varepsilon + Z^\varepsilon)| \\
&\qquad + \frac{1}{\varepsilon}|(f(u_{\mathsf{A}}^\varepsilon + Z^\varepsilon) - f(u_{\mathsf{A}}^\varepsilon), Y^\varepsilon)| + |(r_{\mathsf{A}}^\varepsilon, Y^\varepsilon)| \\
&\quad =: I + II + III + IV.
\end{aligned} \tag{26}$$

Noting $f'(a) - f'(b) = 3(a-b)(a+b)$, using the uniform boundedness of $u_{\mathsf{A}}^\varepsilon$ (cf. (6)) and Hölder's and Young's inequalities yields

$$\begin{aligned}
I &= \frac{1}{\varepsilon}|((f'(u_{\mathsf{A}}^\varepsilon + Z^\varepsilon) - f'(u_{\mathsf{A}}^\varepsilon))Y^\varepsilon, Y^\varepsilon)| \le \frac{C}{\varepsilon}\int_{\mathcal{D}} |Z^\varepsilon|(Y^\varepsilon)^2 dx \\
&\quad + \frac{C}{\varepsilon}\int_{\mathcal{D}} (Z^\varepsilon)^2 (Y^\varepsilon)^2 dx \\
&\le \frac{C}{\varepsilon}\|Y^\varepsilon\|_{\mathbb{L}^4}^2\|Z^\varepsilon\| + \frac{C}{\varepsilon}\|Y^\varepsilon\|_{\mathbb{L}^4}^2\|Z^\varepsilon\|_{\mathbb{L}^4}^2 \le \frac{1}{16\varepsilon}\|Y^\varepsilon\|_{\mathbb{L}^4}^4 + \frac{C}{\varepsilon}\|Z^\varepsilon\|^2 + \frac{C}{\varepsilon}\|Z^\varepsilon\|_{\mathbb{L}^4}^4.
\end{aligned}$$

Using the uniform boundedness of $u_{\mathsf{A}}^\varepsilon$, inequalities (6), and Hölder's and Young's inequalities leads to

$$\begin{aligned}
II &= \frac{3}{\varepsilon}|((Y^\varepsilon)^3, u_{\mathsf{A}}^\varepsilon + Z^\varepsilon)| \le \frac{C}{\varepsilon}\|Y^\varepsilon\|_{\mathbb{L}^3}^3 + \frac{3}{\varepsilon}|((Y^\varepsilon)^3, Z^\varepsilon)| \\
&\le \frac{C}{\varepsilon}\|Y^\varepsilon\|_{\mathbb{L}^3}^3 + \frac{3}{\varepsilon}\|Y^\varepsilon\|_{\mathbb{L}^4}^3\|Z^\varepsilon\|_{\mathbb{L}^4} \le \frac{C}{\varepsilon}\|Y^\varepsilon\|_{\mathbb{L}^3}^3 + \frac{1}{16\varepsilon}\|Y^\varepsilon\|_{\mathbb{L}^4}^4 + \frac{C}{\varepsilon}\|Z^\varepsilon\|_{\mathbb{L}^4}^4.
\end{aligned}$$

Using (25), (6), and Hölder's and Young's inequalities yields

$$
\begin{aligned}
III &= \frac{1}{\varepsilon}|(f(u_{\mathsf{A}}^{\varepsilon}+Z^{\varepsilon})-f(u_{\mathsf{A}}^{\varepsilon}),Y^{\varepsilon})| \\
&\leq \frac{1}{\varepsilon}\int_{\mathcal{D}}|Z^{\varepsilon}||f'(u_{\mathsf{A}}^{\varepsilon})||Y^{\varepsilon}| + \frac{1}{\varepsilon}\int_{\mathcal{D}}|Z^{\varepsilon}|^3|Y^{\varepsilon}| + \frac{3}{\varepsilon}\int_{\mathcal{D}}|Z^{\varepsilon}|^2|u_{\mathsf{A}}^{\varepsilon}||Y^{\varepsilon}| \\
&\leq \frac{C}{\varepsilon}\|Y^{\varepsilon}\|_{\mathbb{L}^4}\|Z^{\varepsilon}\|_{\mathbb{L}^{\frac{4}{3}}} + \frac{1}{\varepsilon}\|Y^{\varepsilon}\|_{\mathbb{L}^4}\|Z^{\varepsilon}\|_{\mathbb{L}^4}^3 + \frac{C}{\varepsilon}\|Y^{\varepsilon}\|_{\mathbb{L}^4}\|Z^{\varepsilon}\|_{\mathbb{L}^{\frac{8}{3}}}^2 \\
&\leq \frac{1}{16\varepsilon}\|Y^{\varepsilon}\|_{\mathbb{L}^4}^4 + \frac{C}{\varepsilon}\|Z^{\varepsilon}\|_{\mathbb{L}^{\frac{4}{3}}}^{\frac{4}{3}} + \frac{C}{\varepsilon}\|Z^{\varepsilon}\|_{\mathbb{L}^{\frac{8}{3}}}^{\frac{8}{3}} + \frac{C}{\varepsilon}\|Z^{\varepsilon}\|_{\mathbb{L}^4}^4.
\end{aligned}
$$

Using the embedding $\mathbb{L}^3 \hookrightarrow \mathbb{L}^1$ and Young's inequality, it follows that

$$
\begin{aligned}
IV = |(r_{\mathsf{A}}^{\varepsilon}, Y^{\varepsilon}(t))| &\leq \|r_{\mathsf{A}}^{\varepsilon}\|_{C(\mathcal{D})}\|Y^{\varepsilon}(t)\|_{\mathbb{L}^1} \leq C\|r_{\mathsf{A}}^{\varepsilon}\|_{C(\mathcal{D})}\|Y^{\varepsilon}(t)\|_{\mathbb{L}^3} \\
&\leq C\varepsilon^{\frac{1}{2}}\|r_{\mathsf{A}}^{\varepsilon}\|_{C(\mathcal{D})}^{\frac{3}{2}} + \frac{C}{\varepsilon}\|Y^{\varepsilon}(t)\|_{\mathbb{L}^3}^3.
\end{aligned}
$$

Substituting the above estimates of I, II, III, and IV into (26) leads to

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|Y^{\varepsilon}\|_{\mathbb{H}^{-1}}^2 + \varepsilon\|\nabla Y^{\varepsilon}(t)\|^2 + \frac{1}{\varepsilon}(f'(u_{\mathsf{A}}^{\varepsilon})Y^{\varepsilon},Y^{\varepsilon}) + \frac{13}{16\varepsilon}\|Y^{\varepsilon}\|_{\mathbb{L}^4}^4 \\
&\quad\leq \frac{C}{\varepsilon}\|Y^{\varepsilon}\|_{\mathbb{L}^3}^3 + \frac{C}{\varepsilon}\|Z^{\varepsilon}\|_{\mathbb{L}^{\frac{4}{3}}}^{\frac{4}{3}} + \frac{C}{\varepsilon}\|Z^{\varepsilon}\|^2 + \frac{C}{\varepsilon}\|Z^{\varepsilon}\|_{\mathbb{L}^{\frac{8}{3}}}^{\frac{8}{3}} + \frac{C}{\varepsilon}\|Z^{\varepsilon}\|_{\mathbb{L}^4}^4 \\
&\qquad + C\varepsilon^{\frac{1}{2}}\|r_{\mathsf{A}}^{\varepsilon}\|_{C(\mathcal{D})}^{\frac{3}{2}}.
\end{aligned} \tag{27}
$$

Using Proposition 2.1, we deduce from (27) that

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|Y^{\varepsilon}\|_{\mathbb{H}^{-1}}^2 + \frac{13}{16\varepsilon}\|Y^{\varepsilon}\|_{\mathbb{L}^4}^4 &\leq \frac{C}{\varepsilon}(\|Y^{\varepsilon}\|_{\mathbb{L}^3}^3 + \|Z^{\varepsilon}\|_{\mathbb{L}^{\frac{4}{3}}}^{\frac{4}{3}} + \|Z^{\varepsilon}\|^2 + \|Z^{\varepsilon}\|_{\mathbb{L}^{\frac{8}{3}}}^{\frac{8}{3}} + \|Z^{\varepsilon}\|_{\mathbb{L}^4}^4) \\
&\quad + C\varepsilon^{\frac{1}{2}}\|r_{\mathsf{A}}^{\varepsilon}\|_{C(\mathcal{D})}^{\frac{3}{2}} + C_0\|Y^{\varepsilon}\|_{\mathbb{H}^{-1}}^2.
\end{aligned} \tag{28}
$$

By noting that $f'(x) = 3x^2 - 1$, it follows from (27) that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|Y^{\varepsilon}\|_{\mathbb{H}^{-1}}^2 + \varepsilon\|\nabla Y^{\varepsilon}(t)\|^2 + \frac{13}{16\varepsilon}\|Y^{\varepsilon}\|_{\mathbb{L}^4}^4 \\
&\quad\leq \frac{1}{\varepsilon}\|Y^{\varepsilon}\|^2 + \frac{C}{\varepsilon}\|Y^{\varepsilon}\|_{\mathbb{L}^3}^3 + \frac{C}{\varepsilon}\|Z^{\varepsilon}\|_{\mathbb{L}^{\frac{4}{3}}}^{\frac{4}{3}} + \frac{C}{\varepsilon}\|Z^{\varepsilon}\|^2 + \frac{C}{\varepsilon}\|Z^{\varepsilon}\|_{\mathbb{L}^{\frac{8}{3}}}^{\frac{8}{3}} \\
&\qquad + \frac{C}{\varepsilon}\|Z^{\varepsilon}\|_{\mathbb{L}^4}^4 + C\varepsilon^{\frac{1}{2}}\|r_{\mathsf{A}}^{\varepsilon}\|_{C(\mathcal{D})}^{\frac{3}{2}}.
\end{aligned} \tag{29}
$$

Multiplying (29) by ε^3 and (28) by $1-\varepsilon^3$, summing up the resulting inequalities yields

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|Y^{\varepsilon}\|_{\mathbb{H}^{-1}}^2 + \varepsilon^4\|\nabla Y^{\varepsilon}(t)\|^2 + \frac{13}{16\varepsilon}\|Y^{\varepsilon}\|_{\mathbb{L}^4}^4 \\
&\quad\leq \varepsilon^2\|Y^{\varepsilon}\|^2 + \frac{C}{\varepsilon}\|Y^{\varepsilon}\|_{\mathbb{L}^3}^3 + \frac{C}{\varepsilon}\|Z^{\varepsilon}\|_{\mathbb{L}^{\frac{4}{3}}}^{\frac{4}{3}} + \frac{C}{\varepsilon}\|Z^{\varepsilon}\|_{\mathbb{L}^2}^2 + \frac{C}{\varepsilon}\|Z^{\varepsilon}\|_{\mathbb{L}^{\frac{8}{3}}}^{\frac{8}{3}} \\
&\qquad + \frac{C}{\varepsilon}\|Z^{\varepsilon}\|_{\mathbb{L}^4}^4 + C\varepsilon^{\frac{1}{2}}\|r_{\mathsf{A}}^{\varepsilon}\|_{C(\mathcal{D})}^{\frac{3}{2}} + C_0\|Y^{\varepsilon}\|_{\mathbb{H}^{-1}}^2.
\end{aligned} \tag{30}
$$

Using the interpolation inequality $\|v\|^2 \le \|v\|_{\mathbb{H}^{-1}}\|\nabla v\|$, $v \in \mathbb{H}^1$ and Young's inequality, we estimate

$$\varepsilon^2\|Y^\varepsilon\|^2 \le \varepsilon^2\|Y^\varepsilon\|_{\mathbb{H}^{-1}}\|\nabla Y^\varepsilon\| \le \frac{1}{2}\|Y^\varepsilon\|^2_{\mathbb{H}^{-1}} + \frac{\varepsilon^4}{2}\|\nabla Y^\varepsilon\|^2. \tag{31}$$

Substituting (31) into (30) yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|Y^\varepsilon\|^2_{\mathbb{H}^{-1}} + \frac{\varepsilon^4}{2}\|\nabla Y^\varepsilon(t)\|^2 + \frac{13}{16\varepsilon}\|Y^\varepsilon\|^4_{\mathbb{L}^4} \\
&\quad\le \frac{C}{\varepsilon}\|Y^\varepsilon\|^3_{\mathbb{L}^3} + \frac{C}{\varepsilon}\|Z^\varepsilon\|^{\frac{4}{3}}_{\mathbb{L}^{\frac{4}{3}}} + \frac{C}{\varepsilon}\|Z^\varepsilon\|^2_{\mathbb{L}^2} + \frac{C}{\varepsilon}\|Z^\varepsilon\|^{\frac{8}{3}}_{\mathbb{L}^{\frac{8}{3}}} + \frac{C}{\varepsilon}\|Z^\varepsilon\|^4_{\mathbb{L}^4} \\
&\qquad + C\varepsilon^{\frac{1}{2}}\|r^\varepsilon_{\mathsf{A}}\|^{\frac{3}{2}}_{C(\mathcal{D})} + C\|Y^\varepsilon\|^2_{\mathbb{H}^{-1}}.
\end{aligned}$$

Integrating the preceding inequality on $[0,t]$ and noting that $Y^\varepsilon(0) = 0$ leads to

$$\begin{aligned}
&\|Y^\varepsilon(t)\|^2_{\mathbb{H}^{-1}} + \varepsilon^4\int_0^t \|\nabla Y^\varepsilon(s)\|^2 ds + \frac{13}{8\varepsilon}\int_0^t \|Y^\varepsilon(s)\|^4_{\mathbb{L}^4} ds \\
&\quad\le \frac{C}{\varepsilon}\Big(\int_0^t \|Y^\varepsilon(s)\|^3_{\mathbb{L}^3} ds + \int_0^t \|Z^\varepsilon(s)\|^{\frac{4}{3}}_{\mathbb{L}^{\frac{4}{3}}} ds + \int_0^t \|Z^\varepsilon(s)\|^2_{\mathbb{L}^2} ds + \int_0^t \|Z^\varepsilon(s)\|^{\frac{8}{3}}_{\mathbb{L}^{\frac{8}{3}}} ds\Big) \\
&\qquad + \frac{C}{\varepsilon}\int_0^t \|Z^\varepsilon(s)\|^4_{\mathbb{L}^4} ds + C_0\int_0^t \|Y^\varepsilon(s)\|^2_{\mathbb{H}^{-1}} ds + C\varepsilon^{\frac{1}{2}}\int_0^t \|r^\varepsilon_{\mathsf{A}}(s)\|^{\frac{3}{2}}_{C(\mathcal{D})} ds.
\end{aligned}$$

The result follows after an application of Gronwall's lemma. ■

We introduce the space $\Omega_{\delta,\eta,\varepsilon} \subset \Omega$ such that

$$\Omega_{\delta,\eta,\varepsilon} = \big\{\omega \in \Omega \,:\, \|Z^\varepsilon\|_{C(\mathcal{D}_T)} \le C_1\varepsilon^{\sigma^*-2\delta-2\eta}\big\}, \tag{32}$$

with $\sigma^* := \sigma - \frac{1}{4}$. We also introduce the space $\widetilde{\Omega}_{\vartheta,\kappa,\varepsilon} \subset \Omega$ such that

$$\widetilde{\Omega}_{\vartheta,\kappa,\varepsilon} = \big\{\omega \in \Omega \,:\, \sup_{t\in[0,T]} \|(-\Delta)^{1-\frac{d}{4}-\frac{\vartheta}{2}} Z^\varepsilon(t)\|^2 \le C_1\varepsilon^{2\gamma-\vartheta-\kappa-1}\big\}.$$

Lemma 4.2. *For any $C_1, \delta > 0$, $\eta \ge 0$, there exists a constant $C_{\delta,\eta} = C(\sigma,\eta,C_1) > 0$ such that $\mathbb{P}[\Omega_{\delta,\eta,\varepsilon}] > 1 - C_{\delta,\eta}\varepsilon^{\delta+\eta}$.*

Proof. The proof is given in [6, Lemma 4.6] for $d = 2$ but a close inspection reveals that it is also valid for $d = 3$. ■

Lemma 4.3. *For any $C_1, \vartheta > 0$, $\kappa \ge 0$, there exists a constant $C_{\vartheta,\kappa} = C(\vartheta,\kappa,C_1) > 0$ such that $\mathbb{P}[\widetilde{\Omega}_{\vartheta,\kappa,\varepsilon}] \ge 1 - C_{\vartheta,\kappa}\varepsilon^{\vartheta+\kappa}$.*

Proof. Using Markov's inequality and Lemma 3.2, it holds that

$$\mathbb{P}[\widetilde{\Omega}^c_{\vartheta,\kappa,\varepsilon}] \le \frac{\mathbb{E}\big[\sup_{t\in[0,T]}\|(-\Delta)^{1-\frac{d}{4}-\frac{\vartheta}{2}} Z^\varepsilon(t)\|^2\big]}{C_1\varepsilon^{2\sigma-\vartheta-\kappa-1}} \le C_{\vartheta,\kappa}\varepsilon^{\vartheta+\kappa}.$$

The statement follows after noting that $\mathbb{P}[\widetilde{\Omega}_{\vartheta,\kappa,\varepsilon}] = 1 - \mathbb{P}[\widetilde{\Omega}^c_{\vartheta,\kappa,\varepsilon}]$. ■

We introduce the following stopping time:

$$T_\varepsilon = T \wedge \inf\Big\{t > 0 : \int_0^t \|Y^\varepsilon(s)\|_{\mathbb{L}^3}^3 ds > \varepsilon^\gamma\Big\}, \tag{33}$$

for some $\gamma > 0$, which will be specified later.

In the next lemma we derive an estimate of Y^ε up to the stopping time T_ε on $\Omega_{\delta,\eta,\varepsilon}$.

Lemma 4.4. *The following estimate holds for the solution of (23) for $\omega \in \Omega_{\delta,\eta,\varepsilon}$ and $t \leq T_\varepsilon$:*

$$\begin{aligned}\sup_{s\in[0,t]} \|Y^\varepsilon(s)\|_{\mathbb{H}^{-1}}^2 + \varepsilon^4 \int_0^t \|\nabla Y^\varepsilon(s)\|^2 ds + \frac{13}{8\varepsilon}\int_0^t \|Y^\varepsilon(s)\|_{\mathbb{L}^4}^4 ds \\ \leq C(\varepsilon^{\gamma-1} + \varepsilon^{\frac{4}{3}(\sigma^*-2\delta-2\eta)-1} + \varepsilon^{2(\sigma^*-2\delta-2\eta)-1} + \varepsilon^{\frac{8}{3}(\sigma^*-2\delta-2\eta)-1}) \\ + C(\varepsilon^{4(\sigma^*-2\delta-2\eta)-1} + \varepsilon^{\frac{3K-5}{2}}),\end{aligned}$$

where C is a positive constant independent of ε and T_ε.

Proof. From Lemma 4.1, using (6) and the embedding $C(\mathcal{D}) \hookrightarrow \mathbb{L}^q$, $q \geq 1$, as well as recalling the definitions of T_ε (see (33)) and $\Omega_{\delta,\eta,\varepsilon}$ (see (32)), yields for any $t < T_\varepsilon$

$$\begin{aligned}\sup_{s\in[0,t]} \|Y^\varepsilon(s)\|_{\mathbb{H}^{-1}}^2 + \varepsilon^4 \int_0^t \|\nabla Y^\varepsilon(s)\|^2 ds + \frac{13}{8\varepsilon}\int_0^t \|Y^\varepsilon(s)\|_{\mathbb{L}^4}^4 ds \\ \leq \frac{C}{\varepsilon}\int_0^t \|Y^\varepsilon(s)\|_{\mathbb{L}^3}^3 ds + \frac{C}{\varepsilon}\int_0^t \|Z^\varepsilon(s)\|_{\mathbb{L}^{\frac{4}{3}}}^{\frac{4}{3}} ds + \frac{C}{\varepsilon}\int_0^t \|Z^\varepsilon(s)\|_{\mathbb{L}^2}^2 ds \\ + \frac{C}{\varepsilon}\int_0^t \|Z^\varepsilon(s)\|_{\mathbb{L}^{\frac{8}{3}}}^{\frac{8}{3}} ds + \frac{C}{\varepsilon}\int_0^t \|Z^\varepsilon(s)\|_{\mathbb{L}^4}^4 ds \\ + C\varepsilon^{\frac{1}{2}}\int_0^t \|r_{\mathsf{A}}^\varepsilon(s)\|_{C(\mathcal{D})}^{\frac{3}{2}} ds \\ \leq C(\varepsilon^{\gamma-1} + \varepsilon^{\frac{4}{3}(\sigma^*-2\delta-2\eta)-1} + \varepsilon^{2(\sigma^*-2\delta-2\eta)-1} + \varepsilon^{\frac{8}{3}(\sigma^*-2\delta-2\eta)-1}) \\ + C(\varepsilon^{4(\sigma^*-2\delta-2\eta)-1} + \varepsilon^{\frac{3K-5}{2}}).\end{aligned}$$

∎

In order to show that $T_\varepsilon \equiv T$ on $\Omega_{\delta,\eta,\varepsilon}$, we make use of the following interpolation inequality:

Lemma 4.5. *For all $2 < r < 3$ and $\widetilde{C} > 0$, there exists a positive constant $C_{\mathcal{D}}$, independent of ε, such that for every $v \in \mathbb{H}^1 \cap \mathbb{L}_0^2$ and $\alpha \in \mathbb{R}$, it holds that*

$$\widetilde{C}\|v\|_{\mathbb{L}^3}^3 \leq \varepsilon^\alpha \|v\|_{\mathbb{L}^4}^4 + C_{\mathcal{D}}\frac{\widetilde{C}^{4-r}}{4-r}\varepsilon^{-\alpha(3-r)}\|v\|_{\mathbb{H}^{-1}}^{\frac{4-r}{2}}\|v\|_{\mathbb{H}^1}^{\frac{3r-4}{2}}.$$

Proof. We recall Young's inequality to be

$$ab \leq \frac{q-1}{q}a^{\frac{q}{q-1}} + \frac{b^q}{q}, \quad a, b > 0,\ q > 1.$$

For $2 < r < 3$, applying the preceding estimate with $q = 4 - r$ leads to

$$\widetilde{C}|v|^3 = \widetilde{C}\varepsilon^{\alpha\frac{3-r}{4-r}}(|v|^4)^{\frac{3-r}{4-r}}\varepsilon^{-\alpha\frac{3-r}{4-r}}|v|^{\frac{r}{4-r}} \le \varepsilon^{\alpha}|v|^4 + \frac{\widetilde{C}^{4-r}}{4-r}\varepsilon^{-\alpha(3-r)}|v|^r.$$

Integrating the above estimate over $\mathcal{D}$ leads to

$$\widetilde{C}\|v\|_{\mathbb{L}^3}^3 \le \varepsilon^{\alpha}\|v\|_{\mathbb{L}^4}^4 + \frac{\widetilde{C}^{4-r}}{4-r}\varepsilon^{-\alpha(3-r)}\|v\|_{\mathbb{L}^r}^r. \tag{34}$$

Let us recall the following interpolation inequality (see [11, Proposition 6.10]):

$$\|u\|_{\mathbb{L}^{q'}} \le \|u\|_{\mathbb{L}^{p'}}^{\lambda}\|u\|_{\mathbb{L}^{r'}}^{1-\lambda}, \quad u \in \mathbb{L}^{r'}, \quad p' < q' < r', \quad \lambda = \frac{p'}{q'}\frac{r'-q'}{r'-p'}.$$

Using the preceding interpolation inequality with $p' = 2$, $q' = r$, and $r' = 4$ yields

$$\|v\|_{\mathbb{L}^r}^r \le \|v\|_{\mathbb{L}^2}^{4-r}\|v\|_{\mathbb{L}^4}^{2r-4} \le C_{\mathcal{D}}\|v\|_{\mathbb{L}^2}^{4-r}\|v\|_{\mathbb{H}^1}^{2r-4}, \tag{35}$$

where in the last step we used the embedding $\mathbb{H}^1 \hookrightarrow \mathbb{L}^4$. Using the interpolation inequality $\|v\| \le \|v\|_{\mathbb{H}^{-1}}^{\frac{1}{2}}\|\nabla v\|^{\frac{1}{2}}$, it follows from (35) that

$$\|v\|_{\mathbb{L}^r}^r \le C_{\mathcal{D}}\|v\|_{\mathbb{H}^{-1}}^{\frac{4-r}{2}}\|v\|_{\mathbb{H}^1}^{\frac{3r-4}{2}}.$$

Substituting the preceding inequality into (34) completes the proof of the lemma. ∎

Below, we let r in Lemma 4.5 be such that $2 < r \le \frac{8}{3}$. Then, it holds that $\frac{1}{2} < \frac{3r-4}{4} \le 1$.

Theorem 4.1. *Let $u_{\mathsf{A}}^{\varepsilon}$ be the solution of (5) with large enough K and let u^{ε} be the solution of (1) with initial value $u_{\mathsf{A}}^{\varepsilon}(0) = u^{\varepsilon}(0) = u_0^{\varepsilon} \in \mathbb{H}^{-1}$. For sufficiently small $\varepsilon > 0$, $\delta > 0$, $\alpha > 0$, $\eta \ge 0$ and any $2 < r \le \frac{8}{3}$, $\sigma, \gamma > 0$ that satisfy*

$$\gamma > \frac{(7-2\alpha)r + 6\alpha - 8}{r-2}, \quad \sigma > \frac{3}{4}\gamma + \frac{1}{4} + 2\delta + 2\eta,$$

there exist positive constants C and $C_{\delta,\eta}$ independent of ε such that the following hold:

$$\mathbb{P}\left(\left\{\|u^{\varepsilon} - u_{\mathsf{A}}^{\varepsilon}\|_{L^\infty(0,T;\mathbb{H}^{-1})}^2 \le C\varepsilon^{\gamma-1}\right\}\right) \ge 1 - C_{\delta,\eta}\varepsilon^{\delta+\eta}, \tag{36}$$

$$\mathbb{P}\left(\left\{\|w^{\varepsilon} - w_{\mathsf{A}}^{\varepsilon}\|_{L^1(0,T;\mathbb{H}^{-2})}^2 \le C\varepsilon^{\frac{\gamma}{3}-1}\right\}\right) \ge 1 - C_{\delta,\eta}\varepsilon^{\delta+\eta}, \tag{37}$$

$$\mathbb{P}\left(\left\{\|u^{\varepsilon} - u_{\mathsf{A}}^{\varepsilon}\|_{L^3(\mathcal{D}_T)} \le C\varepsilon^{\frac{\gamma}{3}}\right\}\right) \ge 1 - C_{\delta,\eta}\varepsilon^{\delta+\eta}, \tag{38}$$

$$\mathbb{P}\left(\left\{\|u^{\varepsilon} - u_{\mathsf{A}}^{\varepsilon}\|_{L^4(\mathcal{D}_T)} \le C\varepsilon^{\frac{\gamma}{4}}\right\}\right) \ge 1 - C_{\delta,\eta}\varepsilon^{\delta+\eta}. \tag{39}$$

Moreover, for any $\vartheta \in (0, 2 - \frac{d}{2}]$ and $\kappa \ge 0$, there exists a constant $C_{\vartheta,\kappa} > 0$ independent of ε such that

$$\mathbb{P}\left(\left\{\|u^{\varepsilon} - u_{\mathsf{A}}^{\varepsilon}\|_{L^2(0,T;\mathbb{H}^{2-\frac{d}{2}-\vartheta})}^2 \le C\varepsilon^{\frac{\gamma}{3}-4}\right\}\right) \ge 1 - C_{\delta,\eta}\varepsilon^{\delta+\eta} - C_{\vartheta,\kappa}\varepsilon^{\vartheta+\kappa}. \tag{40}$$

Remark 4.1. Since α can be arbitrarily small, the smallest possible value for γ in space dimension $d = 2, 3$ is $\gamma > 16$. Since δ and η can be arbitrarily small, the smallest choice for the noise scaling parameter is $\sigma > \frac{49}{4}$.

Remark 4.2. Theorem 4.1 remains true in the case of trace class noise. It improves [4, Theorem 3.10] in the case $f(u) = u^3 - u$ and $d = 3$, since the obtained error estimates are in $L^p(\mathcal{D}_T)$, $p \in (2, 4]$, that is, with p exceeding the barrier $p \leq \frac{2d+8}{d+2}$ prescribed in [4]. Note that, for sufficiently regular trace class noise, it is possible to derive an analogue of (40) in the $\mathbb{H}^1$-norm (cf. [4]). So far an analogous estimate for the stochastic Cahn–Hilliard equation with the space-time white noise was missing (cf. [6] for the case $d = 2$). Estimate (40) depends on the spatial dimension: for $d = 2$, it holds up to $\mathbb{H}^1$-norm, while for $d = 3$, it holds up to $\mathbb{H}^{\frac{1}{2}}$ (excluding the respective borderline cases). This underlines the low regularity of the space-time white noise. Assuming slightly better regularity of the noise (which is still lower than in [4, Assumption 3.1]), we achieve an estimate in $\mathbb{H}^1$ even in dimension $d = 3$; see Section 5 below.

Proof. For $a, b \in \mathbb{R}$, let $a \wedge b := \min\{a, b\}$. We set

$$\begin{aligned}\gamma_1 :=\ & (\gamma - 1) \wedge \Big(\frac{4}{3}(\sigma^* - 2\delta - 2\eta) - 1\Big) \wedge (2(\sigma^* - 2\delta - 2\eta) - 1) \wedge \Big(\frac{8}{3}(\sigma^* - 2\delta) - 1\Big) \\ & \wedge (4(\sigma^* - 2\delta - 2\eta) - 1) \\ =\ & (\gamma - 1) \wedge \Big(\frac{4}{3}(\sigma^* - 2\delta - 2\eta) - 1\Big).\end{aligned}$$

Recall that $\sigma^* = \sigma - \frac{1}{4}$. Since δ and η can be arbitrarily small, $\sigma^* > 2\delta + 2\eta$.

We aim to show that $T_\varepsilon(\omega) = T$ for $\omega \in \Omega_{\delta,\eta,\varepsilon}$. We proceed by contradiction and assume that $T_\varepsilon < T$ on $\Omega_{\delta,\eta,\varepsilon}$. We consider K large enough so that $\gamma_1 \leq \frac{3K-5}{2}$. Using Lemma 4.5 with $\alpha > 0$, Hölder's inequality and Lemma 4.4 yields for all $t \leq T_\varepsilon$

$$\begin{aligned}\int_0^t \|Y^\varepsilon(s)\|_{\mathbb{L}^3}^3 ds &\leq \varepsilon^\alpha \int_0^t \|Y^\varepsilon(s)\|_{\mathbb{L}^4}^4 ds + C\varepsilon^{r\alpha - 3\alpha} \int_0^t \|Y^\varepsilon(s)\|_{\mathbb{H}^{-1}}^{\frac{4-r}{2}} \|Y^\varepsilon(s)\|_{\mathbb{H}^1}^{\frac{3r-4}{2}} ds \\ &\leq C\varepsilon^{\gamma_1+1+\alpha} + C\varepsilon^{r\alpha-3\alpha}\Big(\sup_{s\in[0,t]} \|Y^\varepsilon(s)\|_{\mathbb{H}^{-1}}^2\Big)^{\frac{4-r}{4}} \int_0^t \|Y^\varepsilon(s)\|_{\mathbb{H}^1}^{\frac{3r-4}{2}} ds \\ &\leq C\varepsilon^{\gamma_1+1+\alpha} + C\varepsilon^{r\alpha-3\alpha}\varepsilon^{(\frac{4-r}{4})\gamma_1}\Big(\int_0^t \|Y^\varepsilon(s)\|_{\mathbb{H}^1}^2 ds\Big)^{\frac{3r-4}{4}} \\ &\leq C\varepsilon^{\gamma_1+1+\alpha} + C\varepsilon^{r\alpha-3\alpha}\varepsilon^{(\frac{4-r}{4})\gamma_1}\varepsilon^{4-3r}\varepsilon^{(\frac{3r-4}{4})\gamma_1} \\ &= C\varepsilon^{\gamma_1+1+\alpha} + C\varepsilon^{\frac{r}{2}\gamma_1+(\alpha-3)r+4-3\alpha}. \end{aligned} \tag{41}$$

The right-hand side of the above inequality is bounded by ε^γ for sufficiently small ε if $\gamma_1 + 1 + \alpha > \gamma$ and $\frac{r}{2}\gamma_1 + (\alpha - 3)r + 4 - 3\alpha > \gamma$.

For $\frac{4}{3}(\sigma^* - 2\delta - 2\eta) > \gamma$ (i.e., for sufficiently large $\sigma > \frac{3}{4}\gamma + \frac{1}{4} + 2\delta + 2\eta$), we have $\gamma_1 = \gamma - 1$ and the requirement $\frac{r}{2}\gamma_1 + (\alpha - 3)r + 4 - 3\alpha > \gamma$ is equivalent to $\gamma > \frac{(7-2\alpha)r+6\alpha-8}{r-2}$. Consequently, $\int_0^t \|Y^\varepsilon(s)\|_{\mathbb{L}^3}^3 ds \le \varepsilon^\gamma$ for all $t \le T_\varepsilon$ and $\omega \in \Omega_{\delta,\eta,\varepsilon}$, which contradicts the definition of T_ε. Hence, it holds that $T_\varepsilon \equiv T$ on $\Omega_{\delta,\eta,\varepsilon}$.

Recalling $R^\varepsilon = Y^\varepsilon + Z^\varepsilon$ and noting that $T_\varepsilon = T$ on $\Omega_{\delta,\eta,\varepsilon}$, we deduce from Lemmas 4.2 and 4.4 by the embedding $C(\mathcal{D}) \hookrightarrow \mathbb{H}^{-1}$ that on $\Omega_{\delta,\eta,\varepsilon}$, it holds that

$$\begin{aligned}\sup_{t\in[0,T]} \|R^\varepsilon(t)\|_{\mathbb{H}^{-1}}^2 &\le 2 \sup_{t\in[0,T]} \|Y^\varepsilon(t)\|_{\mathbb{H}^{-1}}^2 + 2C \sup_{t\in[0,T]} \|Z^\varepsilon(t)\|_{C(\mathcal{D})}^2 \\ &\le C\varepsilon^{\gamma_1} + C\varepsilon^{2\sigma^*-4\delta-4\eta} \le C\varepsilon^{\gamma-1} + C\varepsilon^{2\sigma^*-4\delta-4\eta} \le C\varepsilon^{\gamma-1},\end{aligned}$$

for $\gamma > \frac{(7-2\alpha)r+6\alpha-8}{r-2}$ and $\sigma > \frac{3}{4}\gamma + \frac{1}{4} + 2\delta + 2\eta$ (recall $\sigma^* = \sigma - \frac{1}{4}$). Hence, it follows that $\Omega_{\delta,\eta,\varepsilon} \subseteq \{\omega \in \Omega : \|R^\varepsilon\|_{L^\infty(0,T;\mathbb{H}^{-1})}^2 \le C\varepsilon^{\gamma-1}\}$. Consequently, Lemma 4.2 yields that

$$\mathbb{P}\big(\{\|R^\varepsilon\|_{L^\infty(0,T;\mathbb{H}^{-1})}^2 \le C\varepsilon^{\gamma-1}\}\big) \ge \mathbb{P}(\Omega_{\delta,\eta,\varepsilon}) \ge 1 - C\varepsilon^{\delta+\eta}.$$

This proves (36), since $R^\varepsilon = u^\varepsilon - u_{\mathrm{A}}^\varepsilon$.

Recalling that $R^\varepsilon = Y^\varepsilon + Z^\varepsilon$, it follows from (41) and Lemma 4.2 that on $\Omega_{\delta,\eta,\varepsilon}$ we have

$$\|R^\varepsilon\|_{L^3(0,T;\mathbb{L}^3)} \le C\varepsilon^{\frac{\gamma}{3}} + C\varepsilon^{\sigma^*-2\delta-2\eta} \le C\varepsilon^{\frac{\gamma}{3}}, \tag{42}$$

for any $\gamma > \frac{5r-4}{2(r-2)}$ and $\sigma \ge \frac{\gamma}{3} + 2\delta + 2\eta + \frac{1}{4}$. This implies that for such σ and γ, we have $\Omega_{\delta,\eta,\varepsilon} \subseteq \{\omega \in \Omega : \|R^\varepsilon\|_{L^3(0,T;\mathbb{L}^3)} \le C\varepsilon^{\frac{\gamma}{3}}\}$. Consequently, from Lemma 4.2, we deduce

$$\mathbb{P}\big(\{\|R^\varepsilon\|_{L^3(0,T;\mathbb{L}^3)} \le C\varepsilon^{\frac{\gamma}{3}}\}\big) \ge \mathbb{P}(\Omega_{\delta,\eta,\varepsilon}) \ge 1 - C\varepsilon^{\delta+\eta},$$

which yields (38).

From Lemmas 4.4 and 4.2, we deduce for $\gamma > \frac{(7-2\alpha)r+6\alpha-8}{r-2}$ and $\sigma > \frac{3}{4}\gamma + \frac{1}{4} + 2\delta + 2\eta$ that on $\Omega_{\delta,\eta,\varepsilon}$ it holds that

$$\|R^\varepsilon\|_{L^4(0,T;\mathbb{L}^4)} \le C\varepsilon^{\frac{\gamma_1+1}{4}} + C\varepsilon^{\sigma^*-2\delta-2\eta} \le C\varepsilon^{\frac{\gamma}{4}} + C\varepsilon^{\sigma^*-2\delta-2\eta} \le C\varepsilon^{\frac{\gamma}{4}}.$$

This implies $\Omega_{\delta,\eta,\varepsilon} \subseteq \{\omega \in \Omega : \|R^\varepsilon\|_{L^4(0,T;\mathbb{L}^4)} \le C\varepsilon^{\frac{\gamma}{4}}\}$. Hence, using Lemma 4.2 yields

$$\mathbb{P}\big(\{\|R^\varepsilon\|_{L^4(0,T;\mathbb{L}^4)} \le C\varepsilon^{\frac{\gamma}{4}}\}\big) \ge \mathbb{P}(\Omega_{\delta,\eta,\varepsilon}) \ge 1 - C\varepsilon^{\delta+\eta}.$$

This completes the proof of (39).

Using the embedding $\mathbb{L}^1 \hookrightarrow \mathbb{H}^{-2}$, (25), the uniform boundedness of $u_{\mathrm{A}}^\varepsilon$ (cf. (6)), and (42), we obtain on $\Omega_{\delta,\eta,\varepsilon}$ that

$$\begin{aligned}\|f(u_{\mathrm{A}}^\varepsilon) - f(u^\varepsilon)\|_{L^1(0,T;\mathbb{H}^{-2})} &\le C\|f(u_{\mathrm{A}}^\varepsilon) - f(u^\varepsilon)\|_{L^1(0,T;\mathbb{L}^1)} \\ &\le C\|R^\varepsilon\|_{L^3(0,T;\mathbb{L}^3)} + C\|R^\varepsilon\|_{L^3(0,T;\mathbb{L}^3)}^2 + \|R^\varepsilon\|_{L^3(0,T;\mathbb{L}^3)}^3 \\ &\le C\varepsilon^{\frac{\gamma}{3}}.\end{aligned} \tag{43}$$

Recalling that $Y^\varepsilon = R^\varepsilon - Z^\varepsilon$, and using the embeddings $C(\mathcal{D}) \hookrightarrow \mathbb{L}^2$ and $\mathbb{L}^3 \hookrightarrow \mathbb{L}^2$, estimate (42) and Lemma 4.2 yield on $\Omega_{\delta,\eta,\varepsilon}$

$$\begin{aligned}\|\Delta(Y^\varepsilon + Z^\varepsilon)\|_{L^1(0,T;\mathbb{H}^{-2})} &\le C\|Y^\varepsilon\|_{L^1(0,T;\mathbb{L}^2)} + \|Z^\varepsilon\|_{C(\mathcal{D}_T)}\\ &\le C\|R^\varepsilon\|_{L^1(0,T;\mathbb{L}^3)} + \|Z^\varepsilon\|_{C(\mathcal{D}_T)}\\ &\le C\|R^\varepsilon\|_{L^3(0,T;\mathbb{L}^3)} + \|Z^\varepsilon\|_{C(\mathcal{D}_T)}\\ &\le C\varepsilon^{\frac{\gamma}{3}} + C\varepsilon^{\sigma^*-2\delta-2\eta} \le C\varepsilon^{\frac{\gamma}{3}}. \end{aligned} \tag{44}$$

Recalling that $w^\varepsilon - w^\varepsilon_{\mathsf{A}} = -\varepsilon\Delta(u^\varepsilon - u^\varepsilon_{\mathsf{A}}) + \frac{1}{\varepsilon}(f(u^\varepsilon) - f(u^\varepsilon_{\mathsf{A}}))$ and $R^\varepsilon = u^\varepsilon - u_{\mathsf{A}} = Y^\varepsilon + Z^\varepsilon$ and using (44), (43), and the fact that $0 < \varepsilon \le 1$, it follows that on $\Omega_{\delta,\eta,\varepsilon}$ we have

$$\|w^\varepsilon_{\mathsf{A}} - w^\varepsilon\|_{L^1(0,T;\mathbb{H}^{-2})} \le C\varepsilon^{\frac{\gamma}{3}} + C\varepsilon^{\frac{\gamma}{3}-1} \le C\varepsilon^{\frac{\gamma}{3}-1}.$$

Therefore, $\Omega_{\delta,\eta,\varepsilon} \subseteq \{\omega \in \Omega : \|w^\varepsilon_{\mathsf{A}} - w^\varepsilon\|_{L^1(0,T;\mathbb{H}^{-2})} \le C\varepsilon^{\frac{\gamma}{3}-1}\}$. Using Lemma 4.2 then yields

$$\mathbb{P}\left(\{\|w^\varepsilon_{\mathsf{A}} - w^\varepsilon\|_{L^1(0,T;\mathbb{H}^{-2})} \le C\varepsilon^{\frac{\gamma}{3}-1}\}\right) \ge \mathbb{P}(\Omega_{\delta,\eta,\varepsilon}) \ge 1 - C_{\delta,\eta}\varepsilon^{\delta+\eta}.$$

This completes the proof of (37).

Since for any $\vartheta > 0$, $1 - \frac{d}{4} - \frac{\vartheta}{2} \le \frac{1}{2}$, it follows from Lemmas 4.3 and 4.4 that

$$\begin{aligned}\|R^\varepsilon\|^2_{L^2(0,T;\mathbb{H}^{2-\frac{d}{2}-\vartheta})} &\le \|Y^\varepsilon\|^2_{L^2(0,T;\mathbb{H}^{2-\frac{d}{2}-\vartheta})} + \|Z^\varepsilon\|^2_{L^2(0,T;\mathbb{H}^{2-\frac{d}{2}-\vartheta})}\\ &\le \|Y^\varepsilon\|^2_{L^2(0,T;\mathbb{H}^1)} + \|Z^\varepsilon\|^2_{L^2(0,T;\mathbb{H}^{2-\frac{d}{2}-\vartheta})}\\ &\le C\varepsilon^{\gamma_1-4} + C\varepsilon^{2\sigma+\frac{\vartheta}{4}-1} \le C\varepsilon^{\frac{\gamma}{3}-4} + C\varepsilon^{2\sigma+\frac{\vartheta}{4}-1} \le C\varepsilon^{\frac{\gamma}{3}-4}\end{aligned}$$

on $\Omega_{\delta,\eta,\varepsilon} \cap \widetilde{\Omega}_{\vartheta,\kappa,\varepsilon}$. This implies

$$\Omega_{\delta,\eta,\varepsilon} \cap \widetilde{\Omega}_{\vartheta,\kappa,\varepsilon} \subseteq \left\{\omega \in \Omega : \|R^\varepsilon\|^2_{L^2(0,T;\mathbb{H}^{2-\frac{d}{2}-\vartheta})} \le C\varepsilon^{\frac{\gamma}{3}-4}\right\}.$$

Using Lemmas 4.2 and 4.3 and the identity $\Omega_{\delta,\eta,\varepsilon} = (\Omega_{\delta,\eta,\varepsilon} \cap \widetilde{\Omega}_{\vartheta,\kappa,\varepsilon}) \cup (\Omega_{\delta,\eta,\varepsilon} \cap \widetilde{\Omega}^c_{\vartheta,\kappa,\varepsilon})$ implies

$$\begin{aligned}\mathbb{P}\left(\{\|R^\varepsilon\|^2_{L^2(0,T;\mathbb{H}^{2-\frac{d}{2}-\vartheta})} \le C\varepsilon^{\frac{\gamma}{3}-4}\}\right) &\ge \mathbb{P}(\Omega_{\delta,\eta,\varepsilon} \cap \widetilde{\Omega}_{\vartheta,\kappa,\varepsilon})\\ &= \mathbb{P}(\Omega_{\delta,\eta,\varepsilon}) - \mathbb{P}(\Omega_{\delta,\eta,\varepsilon} \cap \widetilde{\Omega}^c_{\vartheta,\kappa,\varepsilon})\\ &\ge \mathbb{P}(\Omega_{\delta,\eta,\varepsilon}) - \mathbb{P}(\widetilde{\Omega}^c_{\vartheta,\kappa,\varepsilon})\\ &\ge 1 - C_{\delta,\eta}\varepsilon^{\delta+\eta} - C_{\vartheta,\kappa}\varepsilon^{\vartheta+\kappa},\end{aligned}$$

which yields (40). The proof of Theorem 4.1 is therefore complete. ∎

The corollary below provides an estimate of the difference between the solutions of the stochastic and the deterministic Cahn–Hilliard equation and implies convergence to the solution of the deterministic problem for $\varepsilon \to 0$ if $\delta + \eta > 1$ and $\vartheta + \kappa > 1$.

Corollary 4.1. Let the assumptions of Theorem 4.1 be fulfilled. Then, it holds that

$$\mathbb{E}\Big[\|u^\varepsilon - u_D^\varepsilon\|^2_{L^\infty(0,T;\mathbb{H}^{-1})} + \frac{1}{\varepsilon}\|u^\varepsilon - u_D^\varepsilon\|^4_{L^4(0,T;\mathbb{L}^4)} + \varepsilon^4\|u^\varepsilon - u_D^\varepsilon\|^2_{L^2(0,T;\mathbb{H}^{2-\frac{d}{2}-\vartheta})}\Big]$$
$$\leq C\varepsilon^{\gamma-1} + C_{\delta,\eta}\varepsilon^{\frac{\delta+\eta-1}{2}} + C_{\vartheta,\kappa}\varepsilon^{\frac{\vartheta+\kappa-1}{2}}.$$

Proof. From [1, Theorem 2.1] or [1, Theorem 4.1 and Remark 4.6], $u_{\mathrm{A}}^\varepsilon \in C^2(\overline{\mathcal{D}}_T) \cap \mathbb{L}_0^2$ and

$$\|u_{\mathrm{A}}^\varepsilon - u_D^\varepsilon\|^2_{L^\infty(0,T;\mathbb{H}^{-1})} + \|\nabla(u_{\mathrm{A}}^\varepsilon - u_D^\varepsilon)\|^2_{L^2(0,T;\mathbb{L}^2)} \leq C\varepsilon^{2\sigma}. \tag{45}$$

Testing (3) with $(-\Delta)^{-1}u_D^\varepsilon$, along the same lines as in the proof of (16), yields

$$\|u_D^\varepsilon\|^2_{L^\infty(0,T;\mathbb{H}^{-1})} + \frac{1}{\varepsilon}\|u_D^\varepsilon\|^4_{L^4(0,T;\mathbb{L}^4)} + \varepsilon\|\nabla u_D^\varepsilon\|^2_{L^2(0,T;\mathbb{L}^2)} \leq C\varepsilon^{-1}. \tag{46}$$

From [1, Theorem 2.3], we have $\|u_{\mathrm{A}}^\varepsilon - u_D^\varepsilon\|_{C^1(\mathcal{D}_T)} \leq C\varepsilon$. Using the triangle inequality, (46), and Theorem 6.1 below, it follows that

$$\mathcal{E}(u_{\mathrm{A}}^\varepsilon) := \|u_{\mathrm{A}}^\varepsilon\|^2_{L^\infty(0,T;\mathbb{H}^{-1})} + \frac{1}{\varepsilon}\|u_{\mathrm{A}}^\varepsilon\|^4_{L^4(0,T;\mathbb{L}^4)} + \varepsilon^4\|\nabla u_{\mathrm{A}}^\varepsilon\|^2_{L^2(0,T;\mathbb{L}^2)} \leq C\varepsilon^{-1}. \tag{47}$$

Next, we consider the subspace $\widetilde{\Omega}_1 \subset \Omega$ given by

$$\widetilde{\Omega}_1 = \Big\{\omega \in \Omega \;:\; \|u^\varepsilon - u_{\mathrm{A}}^\varepsilon\|^2_{L^\infty(0,T;\mathbb{H}^{-1})} + \frac{1}{\varepsilon}\|u^\varepsilon - u_{\mathrm{A}}^\varepsilon\|^4_{L^4(0,T;\mathbb{L}^4)}$$
$$+ \varepsilon^4\|(-\Delta)^{1-\frac{d}{4}-\frac{\vartheta}{2}}(u^\varepsilon - u_{\mathrm{A}}^\varepsilon)\|^2_{L^2(0,T;\mathbb{L}^2)} \leq C\varepsilon^{\gamma-1}\Big\}.$$

By Theorem 4.1, it holds $\mathbb{P}[\widetilde{\Omega}_1^c] \leq C_{\delta,\eta}\varepsilon^{\delta+\eta} + C_{\vartheta,\kappa}\varepsilon^{\vartheta+\kappa}$. We set

$$\mathtt{Err}_{\mathrm{A}} := \|u^\varepsilon - u_{\mathrm{A}}^\varepsilon\|^2_{L^\infty(0,T;\mathbb{H}^{-1})} + \frac{1}{\varepsilon}\|u^\varepsilon - u_{\mathrm{A}}^\varepsilon\|^4_{L^4(0,T;\mathbb{L}^4)}$$
$$+ \varepsilon^4\|(-\Delta)^{1-\frac{d}{4}-\frac{\vartheta}{2}}(u^\varepsilon - u_{\mathrm{A}}^\varepsilon)\|^2_{L^2(0,T;\mathbb{L}^2)}.$$

Using the Cauchy–Schwarz inequality, the triangle inequality, $\mathcal{E}_4(u_{\mathrm{A}}^\varepsilon) \leq C\,\mathcal{E}(u_{\mathrm{A}}^\varepsilon)^2$ (by the embedding $\mathbb{H}^1 \hookrightarrow \mathbb{H}^{2-\frac{d}{2}-\vartheta}$), (16) (with $p = 4$), and (47), it holds that

$$\begin{aligned}\mathbb{E}[\mathtt{Err}_{\mathrm{A}}] &= \int_\Omega \mathbb{1}_{\widetilde{\Omega}_1}\mathtt{Err}_A\, d\mathbb{P}(\omega) + \int_\Omega \mathbb{1}_{\widetilde{\Omega}_1^c}\mathtt{Err}_{\mathrm{A}}\, d\mathbb{P}(\omega)\\ &\leq C\varepsilon^{\gamma-1} + C(\mathbb{P}[\widetilde{\Omega}_1^c])^{\frac{1}{2}}(\mathcal{E}_4(u^\varepsilon) + \mathcal{E}(u_{\mathrm{A}}^\varepsilon)^2)^{\frac{1}{2}}\\ &\leq C\varepsilon^{\gamma-1} + C_{\delta,\eta}\varepsilon^{\frac{\delta+\eta-1}{2}} + C_{\vartheta,\kappa}\varepsilon^{\frac{\vartheta+\kappa-1}{2}}.\end{aligned} \tag{48}$$

The statement of the theorem then follows from (48), (45), and Theorem 6.1 by the triangle inequality. ∎

As a consequence of Theorem 4.1, we obtain $\mathbb{P}$-almost surely convergence of the solution u^ε of the stochastic Cahn–Hilliard equation to the solution of the deterministic Hele-Shaw problem in (4) in the sense that $\{(t,x) \in \mathcal{D}_T : t \in (0,T),\ \lim_{\varepsilon\to 0} u^\varepsilon(t,x) \to \pm 1\}$ respectively converge to the exterior and interior of the interface $\{\Gamma_t\}_{t\in(0,T)}$. The proof of the result follows along the lines of [6, Corollary 4.5].

Corollary 4.2. There exists a subsequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} u^{\varepsilon_k} = 1 - 2\chi_{\mathcal{D}^-} \quad \text{in } L^p(0,T;\mathbb{L}^p) \text{ for } p \in (2,4],$$

$\mathbb{P}$-almost surely on Ω, where $\mathcal{D}^- := \{(t,x) \in \mathcal{D}_T;\ t \in (0,T),\ x \in \mathcal{D}_t^-\}$ and $\mathcal{D}_t^-$ is the interior of Γ_t in $\mathcal{D}$.

5. Limiting case with $\mathbb{H}^1$ spatial regularity

In this section we consider the stochastic Cahn–Hilliard equation with a slightly more regular noise

$$du^\varepsilon = \Delta\Big(-\varepsilon\Delta u^\varepsilon + \frac{1}{\varepsilon} f(u^\varepsilon)\Big)dt + \varepsilon^\sigma d\widetilde{W}(t) \quad \text{in } \mathcal{D}_T, \tag{49}$$

with the noise of the form

$$\widetilde{W}(t,x) = \sum_{i\in\mathbb{N}^d} q_i e_i(x)\beta_i(t), \qquad x \in \mathcal{D},\ t \in [0,T], \tag{50}$$

where $\{q_i\}_{i\in\mathbb{N}^d}$ are such that $q_i \approx \lambda_i^{\frac{1}{2}-\frac{d}{4}-\frac{\upsilon}{2}}$ and $\upsilon \in (0,1]$ can be arbitrarily small. The noise $\widetilde{W}$ is more regular than the space-time white noise W in (13); nevertheless, since $1 - \frac{d}{2} - \upsilon \geq -\frac{d}{2}$, the series given by (50) does not converge in $\mathbb{L}^2$.

We define the operator $Q : \mathbb{L}^2 \to \mathbb{L}^2$ as

$$Qu = \sum_{i\in\mathbb{N}^d} q_i (u, e_i) e_i, \quad \forall u \in \mathbb{L}^2.$$

Noting (12), we deduce that

$$\mathrm{Tr}((-\Delta)^{-1}Q) = \sum_{i\in\mathbb{N}^d} ((-\Delta)^{-1}Qe_i, e_i) = \sum_{i\in\mathbb{N}^d} q_i((-\Delta)^{-1}e_i, e_i) \approx \sum_{i\in\mathbb{N}^d} \lambda_i^{-\frac{1}{2}-\frac{d}{4}-\frac{\upsilon}{2}}.$$

For $d = 3$, the above identity implies that $\mathrm{Tr}((-\Delta)^{-1}Q) = \infty$; see (11). Hence, the condition given by [4, Assumption 3.1] is not satisfied for (50) in the case $d = 3$.

Similarly to Section 3, we introduce the stochastic convolution

$$\widetilde{Z}^\varepsilon(t) := \varepsilon^\sigma \int_0^t e^{-(t-s)\varepsilon\Delta^2} d\widetilde{W}(s) = \varepsilon^\sigma \sum_{i\in\mathbb{N}^d} q_i \int_0^t e^{-\lambda_i^2(t-s)\varepsilon} e_i \, d\beta_i(s), \quad t \in [0,T].$$

Analogously to Lemmas 3.1 and 3.2, for $1 \le p < \infty$, one can show the estimate

$$\mathbb{E}\big[\sup_{t\in[0,T]} \|\widetilde{Z}^{\varepsilon}(t)\|_{\mathbb{L}^p}^p\big] + \mathbb{E}\big[\sup_{t\in[0,T]} \|\nabla \widetilde{Z}^{\varepsilon}(t)\|^p\big] \le C(p)\varepsilon^{(\sigma-\frac{1}{2})}. \tag{51}$$

Owing to the better regularity properties of convolution (51), it is straightforward to modify the proof of Theorem 3.1 to show that the solution of (49) has the following regularity:

$$u^{\varepsilon} \in L^{\infty}(\Omega; C([0,T]; \mathbb{H}^{-1})) \cap L^2(\Omega; L^2(0,T; \mathbb{H}^1)) \cap L^4(\Omega; L^4(0,T; \mathbb{L}^4)).$$

Moreover, for any $p \ge 2$, the following estimate holds:

$$\begin{aligned}\widetilde{\mathcal{E}}_p(u^{\varepsilon}) &:= \mathbb{E}\Big[\|u^{\varepsilon}\|_{L^{\infty}(0,T;\mathbb{H}^{-1})}^p + \varepsilon^{\frac{p}{2}}\|\nabla u^{\varepsilon}\|_{L^2(0,T;\mathbb{L}^2)}^p + \frac{1}{\varepsilon^{\frac{p}{2}}}\|u^{\varepsilon}\|_{L^4(0,T;\mathbb{L}^4)}^{2p}\Big] \\ &\le C(\varepsilon^{-\frac{p}{2}} + \varepsilon^{(\sigma-\frac{1}{2})p} + \varepsilon^{(2\sigma-\frac{3}{2})p}).\end{aligned}$$

The remaining results of Section 4 hold true with the fractional Sobolev norm replaced by the $\mathbb{H}^1$ norm. In particular, (40) improves to

$$\mathbb{P}\Big(\big\{\|u^{\varepsilon} - u_{\mathsf{A}}^{\varepsilon}\|_{L^2(0,T;\mathbb{H}^1)}^2 \le C\varepsilon^{\frac{\gamma}{3}-4}\big\}\Big) \ge 1 - C_{\delta,\eta}\varepsilon^{\delta+\eta} - C_{\upsilon,\kappa}\varepsilon^{\upsilon+\kappa},$$

for any $\upsilon > 0$. The above estimate generalizes [4, Theorem 3.10] in the case $d = 3$ to the case of less regular noise.

Finally, we also obtain the following analogue of the estimate in Corollary 4.1:

$$\begin{aligned}\mathbb{E}\Big[\|u^{\varepsilon} - u_D^{\varepsilon}\|_{L^{\infty}(0,T;\mathbb{H}^{-1})}^2 + \frac{1}{\varepsilon}\|u^{\varepsilon} - u_D^{\varepsilon}\|_{L^4(0,T;\mathbb{L}^4)}^4 + \varepsilon^4\|u^{\varepsilon} - u_D^{\varepsilon}\|_{L^2(0,T;\mathbb{H}^1)}^2\Big] \\ \le C\varepsilon^{\gamma-1} + C_{\delta,\eta}\varepsilon^{\frac{\delta+\eta-1}{2}} + C_{\upsilon,\kappa}\varepsilon^{\frac{\upsilon+\kappa-1}{2}}.\end{aligned}$$

6. The deterministic problem

In this section we derive improved estimates for the sharp interface limit of the deterministic Cahn–Hilliard equation (see (3)).

Theorem 6.1. *Let $\varepsilon \in (0,1]$ be sufficiently small, $2 < r \le \frac{8}{3}$, and u_D^{ε} be the solution to the deterministic Cahn–Hilliard equation in (3). Then, for any $\alpha > 0$ (arbitrarily small), $\gamma > \frac{(7-2\alpha)r+6\alpha-8}{r-2}$, and K in (6) large enough so that $\frac{3}{2}(K-1) \ge \gamma$, there exists a constant $C = C(r) > 0$ independent of ε such that*

$$\begin{aligned}\sup_{t\in[0,T]} \|u_D^{\varepsilon}(t) - u_{\mathsf{A}}^{\varepsilon}(t)\|_{\mathbb{H}^{-1}}^2 + \varepsilon^4\|u_D^{\varepsilon} - u_{\mathsf{A}}^{\varepsilon}\|_{L^2(0,T;\mathbb{H}^1)}^2 + \frac{13}{8\varepsilon}\|u_D^{\varepsilon} - u_{\mathsf{A}}^{\varepsilon}\|_{L^4(0,T;\mathbb{L}^4)} \\ \le C\varepsilon^{\gamma-1}.\end{aligned}$$

Remark 6.1. Theorem 6.1 improves [1, Theorem 2.1] in the case $f(u) = u^3 - u$ and $d = 3$, in the sense that here we obtain error estimates in $L^p(0,T;\mathbb{L}^p)$, $p \in (2,4]$, that is, with p exceeding the barrier $p \leq \frac{2d+8}{d+2}$ prescribed in [1]. Note that even for $d = 2$, the error estimates in [1] are only in $L^3(0,T;\mathbb{L}^3)$, while we obtain an error estimates in $L^4(0,T;\mathbb{L}^4)$.

Proof. The proof goes along the same lines as the one of Theorem 4.1, so we only sketch some details. We set $R_D^\varepsilon := u_D^\varepsilon - u_{\mathsf{A}}^\varepsilon$, with u_D^ε being the solution to CHE (3). Then, R_D^ε satisfies the following PDE:

$$\frac{d}{dt}R_D^\varepsilon = -\varepsilon\Delta^2 R_D^\varepsilon + \frac{1}{\varepsilon}\Delta(f(R_D^\varepsilon + u_{\mathsf{A}}^\varepsilon) - f(u_{\mathsf{A}}^\varepsilon)) + \Delta r_{\mathsf{A}}^\varepsilon, \quad R_D^\varepsilon(0) = 0.$$

Testing the above equation with $(-\Delta)^{-1}R_D^\varepsilon$ leads to

$$\frac{1}{2}\frac{d}{dt}\|R_D^\varepsilon\|_{\mathbb{H}^{-1}}^2 + \varepsilon\|\nabla R_D^\varepsilon\|^2 + \frac{1}{\varepsilon}(f(R_D^\varepsilon + u_{\mathsf{A}}^\varepsilon) - f(u_{\mathsf{A}}^\varepsilon), R_D^\varepsilon) + (r_{\mathsf{A}}^\varepsilon, R_D^\varepsilon) = 0.$$

As in the proof of Lemma 4.1, we obtain

$$\begin{aligned}\|R_D^\varepsilon\|_{\mathbb{H}^{-1}}^2 &+ \varepsilon^4\int_0^t \|\nabla R_D^\varepsilon(s)\|^2 ds + \frac{13}{8\varepsilon}\int_0^t \|R_D^\varepsilon(s)\|_{\mathbb{L}^4}^4 ds \\ &\leq \frac{C}{\varepsilon}\int_0^t \|R_D^\varepsilon(s)\|_{\mathbb{L}^3}^3 ds + C\varepsilon^{\frac{1}{2}}\int_0^t \|r_{\mathsf{A}}^\varepsilon(s)\|_{C(\mathcal{D})}^{\frac{3}{2}} ds. \end{aligned} \tag{52}$$

We define

$$T_\varepsilon = T \wedge \inf\Big\{t \in [0,T] \, : \, \int_0^t \|R_D^\varepsilon(s)\|_{\mathbb{L}^3}^3 ds > \varepsilon^\gamma\Big\},$$

for some $\gamma > 0$, which will be specified later.

From (52), using (6a) and noting the definition of T_ε, it follows for all $t \leq T_\varepsilon$ that

$$\begin{aligned}\sup_{0\leq s\leq t}\|R_D^\varepsilon(s)\|_{\mathbb{H}^{-1}}^2 &+ \varepsilon^4\int_0^t \|\nabla R_D^\varepsilon(s)\|^2 ds + \frac{13}{8\varepsilon}\int_0^t \|R_D^\varepsilon(s)\|_{\mathbb{L}^4}^4 ds \\ &\leq C\varepsilon^{\gamma-1} + C\varepsilon^{\frac{3K-5}{2}} \leq C\varepsilon^{\gamma-1}, \end{aligned} \tag{53}$$

where we choose K large enough such that $\gamma - 1 \leq \frac{3K-5}{2}$.

Next, we show that $T_\varepsilon = T$ for suitable γ. We proceed by contradiction and assume that $T_\varepsilon < T$. Then, for all $t \leq T_\varepsilon$, using Lemma 4.5 with $\alpha > 0$ and (53) yields

$$\begin{aligned}\int_0^t \|R_D^\varepsilon(s)\|_{\mathbb{L}^3}^3 ds &\leq \varepsilon^\alpha\int_0^t \|R_D^\varepsilon(s)\|_{\mathbb{L}^4}^4 ds + C\varepsilon^{r\alpha-3\alpha}\int_0^t \|R_D^\varepsilon(s)\|_{\mathbb{H}^{-1}}^{\frac{4-r}{2}}\|R_D^\varepsilon(s)\|_{\mathbb{H}^1}^{\frac{3r-4}{2}} ds \\ &\leq C\varepsilon^{\gamma+\alpha} + C\varepsilon^{r\alpha-3\alpha}\sup_{s\in[0,t]}\|R_D^\varepsilon(s)\|_{\mathbb{H}^{-1}}^{\frac{4-r}{2}}\int_0^t \|R_D^\varepsilon(s)\|_{\mathbb{H}^1}^{\frac{3r-4}{2}} ds \\ &\leq C\varepsilon^{\gamma+\alpha} + C\varepsilon^{r\alpha-3\alpha}\varepsilon^{(\frac{4-r}{4})(\gamma-1)}\varepsilon^{4-3r}\varepsilon^{(\frac{3r-4}{4})(\gamma-1)} \\ &= C\varepsilon^{\gamma+\alpha} + C\varepsilon^{\frac{r}{2}\gamma+\frac{(2\alpha-7)r}{2}+4-3\alpha}. \end{aligned} \tag{54}$$

The right-hand side of (54) is bounded above by ε^γ for sufficiently small ε, if we have $\frac{r}{2}\gamma + \frac{(2\alpha-7)r}{2} + 4 - 3\alpha > \gamma$, that is, if $\gamma > \frac{(7-2\alpha)r+6\alpha-8}{r-2}$. Hence, for such values of K and γ, we have $\int_0^t \|R_D^\varepsilon(s)\|_{\mathbb{L}^3}^3 ds \le \varepsilon^\gamma$, which contradicts the definition of T_ε. Hence, $T_\varepsilon = T$. It then follows from (53) that

$$\sup_{0\le t\le T} \|R_D^\varepsilon(t)\|_{\mathbb{H}^{-1}}^2 + \varepsilon^4 \int_0^T \|\nabla R_D^\varepsilon(t)\|^2 dt + \frac{13}{18\varepsilon}\int_0^T \|R_D^\varepsilon(t)\|_{\mathbb{L}^4}^4 dt \le C\varepsilon^{\gamma-1},$$

which completes the proof. ∎

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