Rev. Mat. Iberoam. 40 (2024), no. 3, 987–1021
DOI 10.4171/RMI/1456

The dynamics of conformal Hamiltonian flows: dissipativity and conservativity

Simon Allais and Marie-Claude Arnaud

Abstract. We study in detail the dynamics of conformal Hamiltonian flows that are defined on a conformal symplectic manifold (this notion was popularized by Vaisman in 1976). We show that they exhibit some conservative and dissipative behaviours. We also build many examples of various dynamics that show simultaneously their difference and resemblance with the contact and symplectic case.

1. Introduction

Symplectic dynamics models many conservative movements. Yet, other phenomena are dissipative and require another setting. This is the case of the damped mechanical systems: they are modelled by conformal Hamiltonian dynamics, which alter the symplectic form up to a scaling factor.

This notion of conformal symplectic dynamics can be placed in a broader context. To define such a dynamics, we only need to know in charts an equivalence class of 2-forms for the relation $\omega_1 \sim \omega_2$, where $\omega_1 \sim \omega_2$ if $\omega_1 = f\omega_2$ for some non-vanishing function f. A manifold endowed with such an equivalence class of local 2-forms, one of them being closed, is called a conformal symplectic manifold, a notion popularized by Vaisman in [12]. An equivalent notion is the notion of conformal structure (M, η, ω), a manifold M endowed with a 1-form called the Lee form and a 2-form called the conformal form, such that $\mathrm{d}\omega - \eta \wedge \omega = 0$. A proof of the equivalence of the two notions is given in [2].

We will study autonomous conformal Hamiltonian flows (CHF in short) $(\varphi_s)_{s\in\mathbb{R}}$ of compact manifolds. If $H\colon M \to \mathbb{R}$ is a C^2 function, the associated conformal Hamiltonian vector field X is defined by $\iota_X\omega = \mathrm{d}H - H\eta$. The CHF alter the conformal form up to a non-constant scaling factor. As the volume ω^n can increase or decrease at different points of the manifold under the action of the dynamics, we can expect different behaviours, some of them being *conservative*, e.g., completely elliptic periodic orbits, invariant foliations with compact leaves, and some other being *dissipative*, e.g., attractors or repulsors.

Mathematics Subject Classification 2020: 37C05 (primary); 34D45, 37C70, 53E50, 34C45 (secondary).
Keywords: smooth mappings and diffeomorphisms, attractors of solutions to ordinary differential equations, attractors and repellers of smooth dynamical systems and their topological structure, flows related to symplectic and contact structures, invariant manifolds for ordinary differential equations.

A precise definition of what we call conservative or dissipative requires the introduction of a notion related to the shape of the orbits. The *winding* of a point $x \in M$ through time is defined as the map $t \mapsto r_t(x)$ (r stands for “rotation”),

$$r_t(x) := \int_0^t \eta(\partial_s \varphi_s(x))\, \mathrm{d}s, \quad \forall t \in \mathbb{R}.$$

Then $\varphi_t^* \omega = e^{r_t} \omega$, see Lemma 2.1, and a point $x \in M$ is

- either (positively) dissipative when $\lim_{t \to +\infty} |r_t(x)| = +\infty$;
- or (positively) conservative.

Our main result, Theorem 3.1, asserts that for every CHF (φ_t^H), if $\mathcal{D}_+$ is the set of positively dissipative points and if $\mathcal{C}_+$ the set of positively conservative points, then up to a set of zero volume, $\mathcal{C}_+$ coincides with the set of positively recurrent points, and then $\mathcal{D}_+$ with the set of positively non-recurrent points. Also, the ω-limit set $\omega(x)$ of every $x \in \mathcal{D}_+$ is contained in $\{H = 0\}$.

Some examples of conservative and dissipative points are:

- every attractor intersects $\{H = 0\}$, has non-trivial homology and almost every point in its basin of attraction that does not belong to the attractor is in $\mathcal{D}_+$, Corollary 3.3.
- If x is a periodic point that is not a critical point of H, then
  - when $x \in \mathcal{C}_+$, the first return map to a Poincaré section preserves a closed 2-form and a foliation into (local) hypersurfaces;
  - when $x \in \mathcal{D}_+$, then $H(x) = 0$ and the first return map to a Poincaré section alters a certain closed 2-form up to a constant factor that is different from 1.
- Every fixed point of the flow is conservative. Observe that this implies that on a compact conformal symplectic manifold, a gradient flow of a Morse function cannot be a conformal Hamiltonian flow. Indeed, for such a gradient flow, one fixed point x is a hyperbolic attractor, and this implies for every $t > 0$ that $0 > r_t(x) = 0$, a contradiction.

We will provide also an example of wild conservative points: points that are recurrent, in $\{H \neq 0\}$, but whose ω-limit set intersects $\{H = 0\}$, see Section 3.5. Hence these points satisfy $\liminf_{t \to +\infty} |r_t(x)| = +\infty$. The origin of most of our examples is contact geometry. In particular, in Section 2.4, we introduce a notion of twisted conformal symplectization that is crucial to the elaboration of examples and counterexamples.

In a similar way, switching H to $-H$, the set $\mathcal{C}_-$ of negatively conservative points is the set of $x \in M$ such that $\lim_{t \to -\infty} |r_t(x)| = +\infty$ and $\mathcal{D}_- = M \setminus \mathcal{C}_-$ is the set of negatively dissipative points. We prove in Proposition 3.4 that $\mathcal{C}_-$ and $\mathcal{C}_+$ are always equal up to a set of zero volume. It is a priori not true that for a general flow, the set of positively recurrent points is equal to the set of negatively recurrent points up to a set of volume zero. Of course, there exist flows (as gradient flows are) for which the set of positively recurrent points is equal to the set of negatively recurrent points. But this is not true for every flow. For example, consider an irrational number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, a function $\eta\colon \mathbb{T}^2 \to [0, 1]$ that vanishes only at $(0, 0)$, and the vector field $X\colon \mathbb{T}^2 \to \mathbb{R}^2$ that is defined by $X(\theta) = \eta(\theta)(1, \alpha)$. The associated flow has a fixed point, the positive orbit of $(1, \alpha)$ is dense and so $(1, \alpha)$ is positively recurrent, but not negatively recurrent because its α-limit set is the fixed point. Observe that the points that do not belong to this orbit are negatively recurrent, hence the set of negatively recurrent points has full Lebesgue measure.

A CHF (φ_t) is (positively) conservative when $\mathcal{C}_+ = M$, and dissipative when $\mathcal{C}_+$ has zero volume; it is negatively conservative when $\mathcal{C}_- = M$, and negatively dissipative when $\mathcal{C}_-$ has zero volume. We highlight a strong relation between the topology of $\{H = 0\}$ and the property of being conservative: when $\{H = 0\}$ has a neighbourhood V such that $\int_\gamma \eta = 0$ for every loop $\gamma\colon \mathbb{T} \to V$, then (φ_t^H) is conservative, Section 4.2. This contains the case where H does not vanish, Section 4.1. But there exist some examples of conservative CHF that are not in this case, Section 4.5. As the non-vanishing property is open in C^0-topology, we obtain C^0-open sets Hamiltonians H such that the associated CHF flows are conservative.

Among the conservative CHF, the Lee flows are those that correspond to the Hamiltonian $H = 1$ for some choice of representative (η, ω) of the conformally symplectic structure. They are an extension of the Reeb flows from the contact setting, see Subsection 1.1 for reminders in contact and symplectic dynamics. We will provide in every dimension examples of Lee flows

- that are transitive, Section 4.3; this is different from the Hamiltonian symplectic case, where the level sets of H are preserved;
- that have no periodic orbits, Section 4.4; Weinstein conjecture in the contact setting and Arnol'd conjecture in the symplectic setting assert the existence of periodic orbits. This example emphasizes one difference between the CHF and the Reeb flows as well as the symplectic Hamiltonian flows.

We will give a 2-dimensional example of Lee flow that is minimal (Section 1.2.2), but we do not know if there is such an example in higher dimension.

We will give in Part 5.2.1 of Section 5.2 an example of dissipative CHF, with one normally hyperbolic attractor that is a Lagrangian submanifold, one normally hyperbolic repulsor that is also a Lagrangian submanifold, and the remaining part of the manifold that is filled with heteroclinic connections. This gives a C^1-open set of CHF that are dissipative. See also Section 1.2.1.

There also exist C^1-open sets of CHF such that both $\mathcal{C}_+$ and $\mathcal{D}_+$ have positive volume. This happens when there is a normally hyperbolic periodic attractor and one non-degenerate local minimum of $e^\theta H$, where θ is a local primitive of η.

Another feature of the conformally symplectic dynamics is that they preserve isotropy (this is even a characterization of these dynamics). This is a common point with symplectic dynamics and contact dynamics. Therefore, we extend or amend some classical results for the invariant submanifolds of Hamiltonian flows. In Section 2.5, we prove that the CHF have a codimension 1 invariant foliation, and explain in Section 6.2 the relation for a submanifold between being tangent to this foliation, being invariant and being isotropic (or coisotropic). This is reminiscent of Hamilton–Jacobi equation in the usual Hamiltonian setting.

We deduce that on a conformal cotangent bundle (see Section 2.3), a Lagrangian invariant graph is necessarily contained in the zero level set, which is a major difference with the usual Hamiltonian setting.

Motivated by the result of Herman [6] in the exact symplectic setting, which asserts that every invariant torus on which the dynamics is C^1-conjugate to a minimal rotation is isotropic, we consider tori $\mathcal{T}$ that are invariant by a CHF and such that the restricted dynamics is topologically conjugate to a rotation. In Section 6.2, we recall the definition of the asymptotic cycle of an invariant measure, and introduce in a similar way the asymptotic cycle for flows on tori that are C^0-conjugate to a not necessarily minimal rotation. We prove that if the product of the cohomology class of the Lee form by the asymptotic cycle of $\mathcal{T}$ is nonzero, then $\mathcal{T}$ is isotropic. In particular, when the cohomology class of the Lee form is rational and when the rotation is minimal, the invariant torus is isotropic.

1.1. Reminders of Hamiltonian dynamics in the contact and symplectic cases and comparison with CH flows

We recall that a symplectic manifold $(M^{(2n)}, \omega)$ is a manifold endowed with a non-degenerate closed 2-form. A (cooriented) contact manifold $(N^{(2n+1)}, \ker\alpha)$ is a manifold N endowed with a hyperplane distribution (called the contact distribution) described as the kernel of a contact 1-form α, i.e., such that $\alpha \wedge (\mathrm{d}\alpha^n)$ is a volume form. If α is a contact form on N, there exists a unique vector field R_α (the Reeb vector field) on N defined by the equations

$$\begin{cases} \iota_{R_\alpha}\mathrm{d}\alpha = 0, \\ \alpha(R_\alpha) = 1. \end{cases}$$

A C^2 function $H\colon N, M \to \mathbb{R}$ is called a Hamiltonian. The Hamiltonian vector field X associated to H is defined by

- $\iota_X\omega = \mathrm{d}H$ in the symplectic case;
- $\iota_X(\mathrm{d}\alpha) = \mathrm{d}H(R_\alpha)\alpha - \mathrm{d}H$ and $\alpha(X) = H$ in the contact case.

In the symplectic case, the Hamiltonian flow preserves the symplectic form ω and the Hamiltonian H. It also preserves the volume ω^n. Hence, when M is compact, by the Poincaré recurrence theorem, the set of positively and negatively recurrent points has full volume.

In the contact case, we have

$$(\varphi_t^*\alpha)_x = \exp\Big(\int_0^t \mathrm{d}H(R_\alpha(\varphi_s(x)))\,\mathrm{d}s\Big)\alpha_x, \quad \forall x \in N.$$

When $H \equiv 1$, $X = R_\alpha$ is the Reeb vector field and α and $\mathrm{d}\alpha$ are preserved by the flow.

The conformal Hamiltonian may preserve a non-degenerate 2-form and a volume, and thus look very similar to the symplectic dynamics for this reason, but this can happen

- without preserving the Hamiltonian function (e.g., with a dense orbit, see Subsection 1.2.2);
- on manifolds that have no symplectic structure.

They can be dissipative, and this is reminiscent of some contact dynamics, even if a conformally symplectic manifold is even dimensional and a contact manifold is odd dimensional. This is the reason for which the twisted conformal symplectizations are elaborated on contact manifolds.

1.2. 2-dimensional examples

1.2.1. A dissipative example. Let us discuss a simple two-dimensional dissipative example that illustrates some of our results. Let $(M,\eta,\omega) = (\mathbb{T}^2, \mathrm{d}x, \mathrm{d}x \wedge \mathrm{d}y)$, where $\mathbb{T}^2$ denotes the 2-torus $\mathbb{R}^2/\mathbb{Z}^2$, and let $H\colon \mathbb{T}^2 \to \mathbb{R}$ be the Hamiltonian function $H(x,y) = \sin(2\pi y)$. The associated Hamiltonian vector field is $2\pi\cos(2\pi y)\partial_x + \sin(2\pi y)\partial_y$. We have pictured integral curves of the associated dynamics on Figure 1.

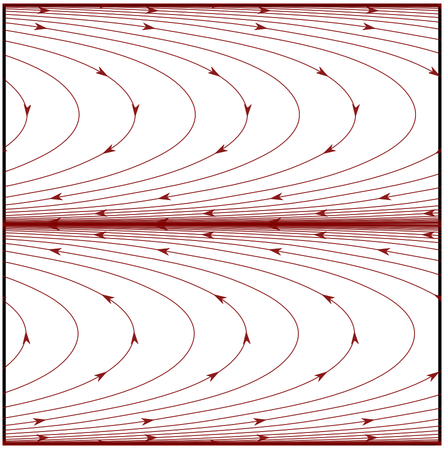

Figure 1. Dynamics of $H(x,y) = \sin(2\pi y)$ in the fundamental domain $[0,1]^2$.

In this figure, we see that the only level set of H that is preserved is $\{H=0\}$ and that it has two connected components: an attractive circle and a repelling one. Such a picture can be drawn in any dimension: *if the Lee form is not exact, there exist Hamiltonian flows with attractive or repelling hyperbolic orbits* (cf. Proposition 5.7). Attractors (or repellers) are not necessarily contained in $\{H=0\}$: we consider the covering map $\pi\colon \mathbb{R}/\mathbb{Z}\times\mathbb{R}/2\mathbb{Z} \to \mathbb{T}^2$. Then $(\mathbb{R}/\mathbb{Z}\times\mathbb{R}/2\mathbb{Z}, \pi^*\mathrm{d}x, \pi^*(\mathrm{d}x\wedge\mathrm{d}y))$ is a conformal symplectic structure. The Hamiltonian flow of $H\circ\pi$ is the lift of the flow of H, with two attracting periodic orbits and two repulsive orbits. The domain $\mathbb{T}\times[1/2,3/2]$, whose boundary is the union of the two attracting cycles, is an attractor that is not contained in $\{H\circ\pi=0\}$. However, as we will see, attractors always intersect $\{H=0\}$. On Figure 1, we see that the attractor is winding in the x's direction. In general, *the Lee form is not exact in any neighbourhood of the intersection of an attractor with* $\{H=0\}$ (cf. Corollary 3.3). In particular, an attractor cannot be finite and must intersect $\{H=0\}$.

1.2.2. A conservative example. In the opposite direction, let us point out the existence of conformal Hamiltonian dynamics that preserve the symplectic form ω but the behaviour of which nonetheless differs from the symplectic Hamiltonian case. As a simple 2-dimensional example, let us consider the 2-torus $\mathbb{T}^2$ endowed with its canonical area form $\omega = \mathrm{d}x\wedge\mathrm{d}y$ once again. Let us fix $a,b\in\mathbb{R}$ and choose the Lee form $\eta := a\,\mathrm{d}x + b\,\mathrm{d}y$. The Hamiltonian flow of $H\equiv 1$, which is called the Lee flow associated to the representative of the conformally symplectic structure (the *gauge*) (η,ω), is $\varphi_t(x,y) = (x+bt, y-at)$. If a and b are rationally independent, this flow is minimal. This is a striking difference with autonomous Hamiltonian flows of symplectic manifolds, where trajectories are never dense and there usually are plenty of periodic orbits. In general, we prove that *there exist topologically transitive Lee flows in any dimension* and that *there exist Lee flows without periodic orbit in any dimension* (cf. Propositions 4.2 and 4.3). In both cases, the Lee form η is not completely resonant (i.e., the set of its integrals along the loops is a dense subgroup of $\mathbb{R}$), which is necessary in order to have dense trajectories. One could ask whether there always is a periodic orbit when η is completely resonant, but this is a hard question: answering it would give a proof to the Weinstein conjecture (i.e., the existence of a periodic orbit for any Reeb flow of a closed contact manifold).

1.3. Structure of the article

In Section 2, we introduce the notions of conformal symplectic manifold and conformal Hamiltonian dynamics and prove some of their properties. Then we provide some examples: the conformal cotangent bundle, the twisted conformal symplectization, and describe the invariant foliation.

In Section 3, we characterize the global conservative-dissipative decomposition of the dynamics in term of recurrence. We prove the almost everywhere coincidence of the behaviours in the past and in the future. We also prove that the boundedness of the winding number implies the existence of invariant measures. We also provide an example of orbits that are conservative and have a strange oscillating behaviour.

In Section 4, we prove that some topological conditions on $\{H = 0\}$ imply that the dynamics is conservative. We give some examples of such dynamics that are transitive and some others that have no periodic orbit, and also an example of a conservative dynamics for which the topological condition for $\{H = 0\}$ is not satisfied.

In Section 5, we begin by studying some ergodic measures whose support is dissipative. Then we give examples of dissipative dynamics with Lagrangian attractors and repulsors, and also examples with periodic attractors and repulsors. We also prove sufficient conditions implying that some connected component of $\{H = 0\}$ cannot be an attractor.

In Section 6, we give some condition that implies that a component of $\{H = 0\}$ is in the closure of a non-compact leaf of the invariant distribution. Then we study invariant submanifolds from different points of view: their position relatively to the invariant foliation, and when they are rotational tori, the relations between their asymptotic cycle and their isotropy.

Finally, there is an appendix dealing with isotropic submanifolds.

2. Preliminaries

2.1. Conformal symplectic manifolds

Given a closed 1-form η, the associated Lichnerowicz–De Rham differential d_η is defined on the differential forms α by $\mathrm{d}_\eta\alpha := \mathrm{d}\alpha - \eta \wedge \alpha$. It satisfies $\mathrm{d}_\eta^2 = 0$ and $\mathrm{d}_{\eta+\mathrm{d}f}\alpha = e^f\,\mathrm{d}_\eta(e^{-f}\alpha)$. If $\mathrm{d}_\eta\alpha = 0$, one says that α is η-closed.

Given an even dimensional manifold M, a conformal symplectic structure is an equivalence class of couples (η, ω), where η is a closed 1-form of M and ω is a non-degenerate

2-form that is η-closed, two such couples (η_i, ω_i), $i \in \{1,2\}$, being equivalent if there exists a map $f: M \to \mathbb{R}$ such that $\eta_2 = \eta_1 + \mathrm{d}f$ and $\omega_2 = e^f \omega_1$. A conformal symplectic manifold is an even dimensional manifold M endowed with a conformal symplectic structure; we will often work with a specific representative (η, ω) and write (M, η, ω) the conformal symplectic manifold. A notion that does not depend on the specific choice of representative (η, ω) is called gauge invariant or well defined up to gauge equivalence. The closed 1-form η is called the Lee form of (M, η, ω), its cohomology class $[\eta] \in H^1(M; \mathbb{R})$ is gauge invariant. A conformal symplectomorphism $\varphi: (M_1, \eta_1, \omega_1) \to (M_2, \eta_2, \omega_2)$ is a diffeomorphism $\varphi: M_1 \to M_2$ such that $\varphi^* \eta_2 = \eta_1 + \mathrm{d}f$ and $\varphi^* \omega_2 = e^f \omega_1$ for some $f: M_1 \to \mathbb{R}$ (this notion is gauge invariant). When $\dim M \geq 4$, the second equality implies the first one.

Similarly to the symplectic case, a submanifold N of a conformal symplectic manifold (M, η, ω) is called isotropic if $TN \subset TN^\omega$, coisotropic if $TN^\omega \subset TN$, and Lagrangian if $TN = TN^\omega$ (where E^ω denotes the ω-orthogonal bundle of the bundle E); these notions are gauge invariant.

A symplectic manifold (M, ω) has a natural conformal symplectic structure $(0, \omega)$ (which is the same as $(0, \lambda\omega)$ for $\lambda \in \mathbb{R}^*$); conversely, a conformal symplectic structure (η, ω) comes from a symplectic structure if and only if η is exact.

2.2. Hamiltonian dynamics

Given a map $H: M \to \mathbb{R}$ defined on a conformal symplectic manifold (M, η, ω), we define its associated Hamiltonian vector field X by $\iota_X \omega = \mathrm{d}_\eta H$; conversely, H is the Hamiltonian of X. When the cohomology class of η is not 0, then H is unique. This matching Hamiltonian-vector field does depend on the choice of representative (η, ω), but not the algebra of Hamiltonian vector fields: the previous vector field X is the same as the one induced by $e^f H$ for the Lee form $\eta + \mathrm{d}f$. One can extend this definition to time-dependent Hamiltonian maps, but we will focus on autonomous Hamiltonians in this paper. When $H \equiv 1$, the associated vector field L^η is called the Lee vector field of η and its flow is called the Lee flow.

Let us assume that the vector field X associated with H is complete. Let us denote by (φ_t) its flow, and let

$$r_t^H(x) := \int_0^t \eta(X \circ \varphi_s(x))\, \mathrm{d}s, \quad \forall x \in M, \forall t \in \mathbb{R}.$$

When the choice of H is clear, we set $r_t := r_t^H$.

Lemma 2.1. *Given a complete Hamiltonian flow (φ_t) on (M, η, ω) associated with a Hamiltonian H, for all $t \in \mathbb{R}$,*

$$\varphi_t^* \omega = e^{r_t} \omega, \quad \varphi_t^* \mathrm{d}_\eta H = e^{r_t} \mathrm{d}_\eta H, \quad H \circ \varphi_t = e^{r_t} H \quad \text{and} \quad \varphi_t^* \eta = \eta + \mathrm{d}r_t.$$

In particular, the level set $\{H = 0\}$ is invariant under the flow, and $\frac{1}{H}\omega$ is an invariant 2-form on $\{H \neq 0\}$.

Proof. By taking the Lie derivative of ω,

$$\mathcal{L}_X \omega = \mathrm{d}(\mathrm{d}_\eta H) + \iota_X(\eta \wedge \omega) = \eta \wedge \mathrm{d}H + \eta(X)\omega - \eta \wedge (\iota_X \omega)$$

Thus,

$$\mathcal{L}_X\omega = \eta \wedge \mathrm{d}H + \eta(X)\omega - \eta \wedge \mathrm{d}H + \eta \wedge \eta H = \eta(X)\omega,$$

which implies the first equality of the statement.

We deduce that

$$\begin{aligned}\varphi_t^*(\mathrm{d}_\eta H) = \varphi_t^*(\iota_X\omega) = \omega(X \circ \varphi_t, \mathrm{d}\varphi_t\cdot) = \omega(\mathrm{d}\varphi_t X, \mathrm{d}\varphi_t\cdot)\\ = \iota_X(\varphi_t^*\omega) = \iota_X(e^{r_t}\omega) = e^{r_t}\,\mathrm{d}_\eta H.\end{aligned}$$

Injecting X in $\iota_X\omega = \mathrm{d}_\eta H$, one gets that $\mathrm{d}H \cdot X = \eta(X)H$, which implies that $h(t) := H \circ \varphi_t(x)$, for a fixed $x \in M$, satisfies $h'(t) = \eta(X \circ \varphi_t(x))h(t)$, and the third statement follows. Finally, the last statement is due to $\mathcal{L}_X\eta = \mathrm{d}(\eta(X))$. ■

We remark that the relations $\varphi_t^*\omega = e^{r_t}\omega$ and $\varphi_t^*\eta = \eta + \mathrm{d}r_t$ are also satisfied in the time-dependent setting. This indeed implies that conformal Hamiltonian diffeomorphisms are conformal symplectomorphisms.

2.3. Conformal cotangent bundles

Given a manifold L endowed with a closed 1-form β, one can define a conformal symplectic structure on T^*L denoted $T^*_\beta L$ in the following way. Let $\pi\colon T^*L \to L$ be the cotangent bundle map and let λ be the associated Liouville form: $\lambda_{(q,p)} \cdot \xi := p(\mathrm{d}\pi \cdot \xi)$. The conformal structure defining $T^*_\beta L$ is $(\eta, \omega) := (\pi^*\beta, -\mathrm{d}_\eta\lambda)$. The neighbourhood of the 0-section of $T^*_\beta L$ is a model of a neighbourhood of a Lagrangian embedding of L pulling back the Lee form to β (see Section A.2).

Let us recall how one can canonically extend diffeomorphisms and flows of M to conformal symplectomorphisms and Hamiltonian flows of $T^*_\beta M$. Let $f\colon M \to N$ be a diffeomorphism; one can symplectically extend it to $\hat{f}\colon T^*M \to T^*N$ by the well-known formula

$$\hat{f}(q,p) = (f(q), p \circ \mathrm{d}f_q^{-1}), \quad \forall (q,p) \in T^*M.$$

Now, if the diffeomorphism $f\colon M \to N$ satisfies $f^*\beta = \alpha + \mathrm{d}r$, for closed 1-forms α, β and some map $r\colon M \to \mathbb{R}$, the extension $\hat{f}\colon T^*_\alpha M \to T^*_\beta N$ defined by

$$\hat{f}(q,p) = (f(q), e^{r(q)} p \circ \mathrm{d}f_q^{-1}), \quad \forall (q,p) \in T^*M,$$

is conformally symplectic. Indeed, let us denote by π_M, π_N the associated cotangent bundle maps, and by λ_M, λ_N, the associated Liouville forms. Then $\hat{f}^*\lambda_N = e^{r\circ\pi_M}\lambda_M$:

$$(\hat{f}^*\lambda_N)_{(q,p)} \cdot \xi = e^{r(q)} p \circ \mathrm{d}f^{-1} \circ \mathrm{d}\pi_N \circ \mathrm{d}\hat{f} \cdot \xi = e^{r(q)} p \circ \mathrm{d}\pi_M \cdot \xi,$$

as $\pi_N \circ \hat{f} = f \circ \pi_M$. We deduce $\hat{f}^*(\mathrm{d}_{\pi_N^*\beta}\lambda_N) = e^{r\circ\pi_M}\mathrm{d}_{\pi_M^*\alpha}\lambda_M$:

$$\begin{aligned}\hat{f}^*(\mathrm{d}\lambda_N - \pi_N^*\beta \wedge \lambda_N) &= \mathrm{d}(e^{r\circ\pi_M}\lambda_M) - \pi_M^*(\alpha + \mathrm{d}r) \wedge (e^{r\circ\pi_M}\lambda_M)\\ &= e^{r\circ\pi_M}(\mathrm{d}\lambda_M - \pi_M^*\alpha \wedge \lambda_M).\end{aligned}$$

Now given a flow $f_t : M \to M$, with $f_0 = \mathrm{id}$, of associated vector field X_t, one has $f_t^*\beta = \beta + \mathrm{d}r_t$, with $r_t(q) := \int_0^t \beta(X_s \circ f_s(q))\,\mathrm{d}s$, so the associated conformal symplectic flow $(\hat{f}_t)$ is well defined, and one checks that it corresponds to the Hamiltonian flow of $H_t(q,p) = p(X_t(q))$.

2.4. Twisted conformal symplectizations

A large class of conformal symplectic manifold that are non-symplectic is given by the conformal symplectizations of contact manifolds. Let (Y^{2n+1}, α) be a manifold endowed with a contact form α (i.e., a 1-form satisfying $\alpha \wedge (\mathrm{d}\alpha)^n \neq 0$), its conformal symplectization $\mathcal{S}^{\mathrm{conf}}(Y, \alpha)$ is the manifold $Y \times S^1$ endowed with the structure $(\eta = -\mathrm{d}\theta, \omega = -\mathrm{d}_\eta(\pi^*\alpha))$ where $S^1 = \mathbb{R}/\mathbb{Z}$, whereas $\theta\colon Y \times S^1 \to S^1$ and $\pi\colon Y \times S^1 \to Y$ are the canonical projections. The conformal symplectization only depends on the oriented contact distribution $\ker\alpha$. Indeed, when $(\eta', \omega') = (\eta - \mathrm{d}f, -\mathrm{d}_{\eta'}(e^{-f}\alpha))$, the map $(x, \theta) \mapsto (x, \theta - f(x))$ is a conformally symplectic diffeomorphism between (ω, η) and (ω', η').

Given a closed 1-form β of Y, we also define the β-twisted conformal symplectization of (V, α) by replacing η in the previous definition with $\eta = \pi^*\beta - \mathrm{d}\theta$; we denote it $\mathcal{S}^{\mathrm{conf}}_\beta(Y, \alpha)$. We check that ω is non-degenerate by showing that ω^{n+1} does not vanish:

$$
\begin{aligned}
(-1)^{n+1}\omega^{n+1} &= (n+1)(\mathrm{d}\theta - \pi^*\beta) \wedge \pi^*\alpha \wedge (\mathrm{d}(\pi^*\alpha))^n \\
&= (n+1)\,\mathrm{d}\theta \wedge \pi^*(\alpha \wedge (\mathrm{d}\alpha)^n) \neq 0;
\end{aligned}
$$

the second equality comes from the fact that $\beta \wedge \alpha \wedge (\mathrm{d}\alpha)^n = 0$ for a degree reason, and the contact hypotheses implies the non-vanishing of the last expression. When the choice of the contact form α is clear, the couple $(\eta, \omega) = (\pi^*\beta - \mathrm{d}\theta, -\mathrm{d}_\eta(\pi^*\alpha))$ as well as the associated Lee vector field and Hamilton equations will be implicitly chosen or referred to as standard.

Let us show how the study of conformal Hamiltonian dynamics also informs us about contact Hamiltonian dynamics, see also Proposition 5.6. We recall that the contact Hamiltonian vector field X associated with the contact Hamiltonian map $H\colon Y \to \mathbb{R}$ is defined by

$$
\begin{cases}
\alpha(X) = H, \\
\iota_X \mathrm{d}\alpha = (\mathrm{d}H \cdot R)\alpha - \mathrm{d}H,
\end{cases}
\tag{2.1}
$$

where R is the Reeb vector field defined by $\alpha(R) = 1$ and $\iota_R\,\mathrm{d}\alpha = 0$ (the Hamiltonian vector field associated with $H \equiv 1$).

Lemma 2.2. *Let $H\colon Y \to \mathbb{R}$ be a contact Hamiltonian map of the contact manifold (Y, α) with fixed contact form α associated with the Reeb vector field R, and let X be the associated Hamiltonian vector field. The conformal Hamiltonian vector field on $\mathcal{S}^{\mathrm{conf}}_\beta(Y, \alpha)$ associated with $\widetilde{H}\colon (x, \theta) \mapsto H(x)$ is*

$$
\widetilde{X} = X \oplus (\beta(X) - \mathrm{d}H \cdot R)\partial_\theta \in TY \oplus TS^1.
$$

In particular, the standard Lee vector field is $L := R \oplus \beta(R)\partial_\theta$.

Proof. Let us first derive the expression of the Lee vector field L. Let $\eta := \pi^*\beta - \mathrm{d}\theta$ be the Lee form and $\omega := -\mathrm{d}_\eta(\pi^*\alpha)$ be associated symplectic form. Let us write $L = V \oplus f\partial_\theta$, V being a vector field of Y and $f\colon Y \to \mathbb{R}$. Since $\iota_L\omega = -\eta$, one has $\eta(L) = 0$, that is $f = \beta(V)$. Developing the Lee equation, one then gets

$$
\iota_L \mathrm{d}(\pi^*\alpha) + \alpha(V)(\pi^*\beta - \mathrm{d}\theta) = \pi^*\beta - \mathrm{d}\theta.
$$

By identification, $\alpha(V) = 1$ and $\iota_V \mathrm{d}\alpha = \beta - \alpha(V)\beta = 0$, therefore $V = R$. The general case is also deduced by identification, once we have remarked that

$$\eta(\widetilde{X}) = \omega(\widetilde{X}, L) = \mathrm{d}\widetilde{H} \cdot L - \widetilde{H}\eta(L) = \mathrm{d}H \cdot R. \qquad \blacksquare$$

Therefore, the conformal Hamiltonian flow (Φ_t) of $\mathcal{S}_\beta^{\mathrm{conf}}(Y, \alpha)$ lifting the contact Hamiltonian flow (φ_t) is

$$\Phi_t(x, \theta) = (\varphi_t(x), \theta + \rho_t(x) - r_t(x)),$$

where

$$r_t(x) = r_t^{\widetilde{H}}(x, \theta) = \int_0^t (\mathrm{d}H(\varphi_s(x)) \cdot R)\,\mathrm{d}s \quad \text{and} \quad \rho_t(x) = \int_0^t \beta(\partial_s \varphi_s(x))\,\mathrm{d}s.$$

The expression of r_t is consistent with the following general fact for conformal Hamiltonian vector fields X of (M, η, ω):

$$\eta(X) = \omega(X, L) = \mathrm{d}H \cdot L - \eta(L)H = \mathrm{d}H \cdot L,$$

where L is the Lee vector field. An isotropic embedding $i\colon L \hookrightarrow (Y, \alpha)$ is by definition an embedding such that $i^*\alpha = 0$, and it is Legendrian when the dimension of L is maximal: $2\dim L + 1 = \dim Y$. One can associate to every isotropic submanifold $L \subset Y$ the isotropic lift $L \times S^1 \subset \mathcal{S}_\beta^{\mathrm{conf}}(Y, \alpha)$. Therefore, dynamical properties of contact Hamiltonians can be deduced from properties of conformal Hamiltonians "by projection $\mathcal{S}_\beta^{\mathrm{conf}}(Y, \alpha) \to Y$". See Part 5.2.1 of Section 5.2.

2.5. The invariant distribution $\mathcal{F}$

In the conformal setting, the Hamiltonian map H is not an integral of motion. But the (singular) distribution $\mathcal{F}(H) := \ker \mathrm{d}_\eta H$ ($\mathcal{F}$ in short) is still invariant since $\varphi_t^* \mathrm{d}_\eta H = e^{r_t} \mathrm{d}_\eta H$ (Lemma 2.1). Moreover, we have

$$\mathrm{d}\big(\mathrm{d}_\eta H\big) = \eta \wedge \mathrm{d}H = \eta \wedge \mathrm{d}_\eta H,$$

hence by Frobenius theorem, at every regular point, the Pfaffian distribution $\ker \mathrm{d}_\eta H$ is integrable.

However, in dynamical systems with dissipative behaviours, their regular leaves are often non-compact (the important exception being $\{H = 0\}$). Let us describe the major properties of $\mathcal{F}$.

Lemma 2.3. *Let $H\colon M \to \mathbb{R}$ be a Hamiltonian map on a conformally symplectic manifold. If $\gamma\colon [0,1] \to M$ is a path tangent to $\mathcal{F}(H)$, then $H(\gamma(1)) = e^{\int_\gamma \eta} H(\gamma(0))$.*

Proof. Similarly to the proof of Lemma 2.1, $h := H \circ \gamma$ satisfies $h' = \eta(\dot{\gamma})h$. $\blacksquare$

Corollary 2.4. *Let $H\colon M \to \mathbb{R}$ be a Hamiltonian map on a conformally symplectic manifold. Every connected submanifold $L \subset M$ tangent to $\mathcal{F}(H)$ is either included in $\{H = 0\}$ or in $\{H \neq 0\}$. In the case when the pull-back of the Lee form to L is not exact, L is included in $\{H = 0\}$.*

In the symplectic case, regular levels of H admit invariant volume forms (see e.g. Section I.8 of [3]), the following proposition generalizes this phenomenon.

Proposition 2.5. *Let (M, η, ω) be a $2n$-dimensional closed conformal symplectic manifold, and let $H\colon M \to \mathbb{R}$ be a Hamiltonian, the flow of which is (φ_t). Let $i\colon \Sigma \hookrightarrow M$ be an embedded leaf tangent to $\mathcal{F}$. Then there exists a volume form μ of Σ such that $\varphi_t^*\mu = e^{(n-1)r_t}\mu$. Moreover, there exists a $(2n-1)$-form μ_0 on M such that $\mu = i^*\mu_0$ and $\mu_0 \wedge \mathrm{d}_\eta H = \omega^n$ in some neighbourhood of Σ.*

As an example, let us consider the foliation $\mathcal{F} = \ker \eta$ induced by the Lee flow corresponding to $H \equiv 1$. In the case where the Lee form η is integral (that is $\int_\gamma \eta$ only takes integral values on loops γ), a leaf F of $\mathcal{F}$ is embedded provided that $\eta_x \neq 0$ for all $x \in F$. Indeed, in this case, the leaves of $\mathcal{F}$ are the level sets of a map $M \to \mathbb{R}/\mathbb{Z}$, $x \mapsto \int_{x_0}^x \eta$ mod $\mathbb{Z}$, so they are automatically topologically closed subsets.

Proof. Let μ_1 be a $(2n-1)$-form on M such that $i^*\mu_1$ is a volume form of Σ (which is oriented by $\mathrm{d}_\eta H$). By assumption, $(\mathrm{d}_\eta H)_x \neq 0$ for $x \in \Sigma$ whereas $i^*\mathrm{d}_\eta H = 0$ so $(\mu_1 \wedge \mathrm{d}_\eta H)_x \neq 0$ for every $x \in \Sigma$. Since Σ is an embedded leaf, there exists an open neighbourhood U of Σ on which $\mu_1 \wedge \mathrm{d}_\eta H$ does not vanish. There exists $f\colon M \to \mathbb{R}$ that does not vanish on U such that $f\mu_1 \wedge \mathrm{d}_\eta H = \omega^n$ restricted to U. Let us show that $\mu_0 := f\mu_1$ and the volume form $\mu := i^*\mu_0$ are the desired forms.

Let us recall that

$$\mathcal{L}_X\omega = \eta(X)\omega \quad \text{and} \quad \mathcal{L}_X \mathrm{d}_\eta H = \eta(X)\mathrm{d}_\eta H$$

(see Lemma 2.1 or its proof). Let us apply $\mathcal{L}_X$ to the equation $\mu_0 \wedge \mathrm{d}_\eta H = \omega^n$:

$$(\mathcal{L}_X\mu_0) \wedge \mathrm{d}_\eta H + \mu_0 \wedge \eta(X)\mathrm{d}_\eta H = n\eta(X)\omega^n.$$

Therefore,

$$(\mathcal{L}_X\mu_0) \wedge \mathrm{d}_\eta H = (n-1)\eta(X)\mu_0 \wedge \mathrm{d}_\eta H$$

so that

$$i^*(\mathcal{L}_X\mu_0) = (n-1)\eta(X)i^*\mu_0.$$

Since the flow (φ_t) preserves Σ,

$$i^*(\mathcal{L}_X\mu_0) = \mathcal{L}_X(i^*\mu_0),$$

and the conclusion follows. ∎

Corollary 2.6. *Let $H\colon M \to \mathbb{R}$ be a Hamiltonian map on a closed conformally symplectic manifold. Every embedded leaf of $\mathcal{F}(H)$ outside $\{H = 0\}$ admits an invariant volume form*

Proof. Let μ be the volume form associated with Σ by Proposition 2.5. Since $\varphi_t^*\mu = e^{(n-1)r_t}\mu$ and $H \circ \varphi_t = e^{r_t}H$ (by Lemma 2.1), the volume form μ/H^{n-1} is invariant. ∎

When the embedded leaf of $\mathcal{F}$ is not compact, this invariant volume can be unbounded.

3. A global decomposition of the phase space: conservative versus dissipative

3.1. Multiple notions of attractors

Let us introduce three notions of attractors that will be used in different parts of this article.

- An invariant compact subset $A \subset M$ is a *weak attractor* if there exists an open subset $U \supset A$, called a basin of attraction of A, such that $\bigcup_{x\in U} \omega(x) \subset A$, where $\omega(x)$ is the omega-limit set of x. The basin of attraction is not necessarily unique.
- A subset $A \subset M$ is a *strong attractor* if there exists an open subset $U \supset A$, such that $\forall t > 0, \varphi_t(\overline{U}) \subset U$ and $A = \bigcap_{t>0} \varphi_t(U)$ (which implies that A is compact and invariant).
- An invariant closed submanifold $N \subset M$ is *normally hyperbolically attractive* if there exist a tubular neighbourhood $V = j(N \times [-\varepsilon_0, \varepsilon_0])$ of N, where $j: N \times [-\varepsilon_0, \varepsilon_0] \hookrightarrow M$ is an embedding, $\tau > 0$ and $a \in (0,1)$ such that $\varphi_\tau^H(V) \subset \mathrm{Int}(V)$, and if we denote $V_\varepsilon = j(N \times [-\varepsilon, \varepsilon])$, then
$$\forall \varepsilon \in (0, \varepsilon_0], \quad \varphi_\tau^H(V_\varepsilon) \subset V_{a\varepsilon}.$$

Observe that a strong attractor is always a weak attractor.

3.2. The conservative-dissipative decomposition

Let $H: M \to \mathbb{R}$ be a Hamiltonian map on a conformally symplectic manifold. We have defined in the introduction the partition in invariant sets $M = \mathcal{C}_+(H) \sqcup \mathcal{D}_+(H)$ (in short, $\mathcal{C}_+ \sqcup \mathcal{D}_+$), with

$$\mathcal{C}_+ := \left\{ x \in M \mid \liminf_{\substack{t\to+\infty \\ t\in\mathbb{R}}} |r_t(x)| < +\infty \right\} = \left\{ x \in M \mid \liminf_{\substack{p\to+\infty \\ p\in\mathbb{N}}} |r_p(x)| < +\infty \right\} \tag{3.1}$$

and

$$\mathcal{D}_+ := \left\{ x \in M \mid \lim_{\substack{t\to+\infty \\ t\in\mathbb{R}}} |r_t(x)| = +\infty \right\}. \tag{3.2}$$

The equality between both definitions of $\mathcal{C}_+$ is due to $|\partial_t r_t(x)| \le \|\eta(X)\|_\infty < +\infty$ (see Lemma 2.1). In a similar way, one defines the partition $M = \mathcal{C}_- \sqcup \mathcal{D}_-$ by replacing "$t \to +\infty$" by "$t \to -\infty$" in the definitions.

Theorem 3.1. *Let $H: M \to \mathbb{R}$ be a Hamiltonian map on a closed conformally symplectic manifold. On the one hand, up to a set with zero Lebesgue measure, the set of positively recurrent points of the Hamiltonian dynamics coincides with $\mathcal{C}_+$. Moreover, for every embedded leaf Σ included in $\{H \neq 0\}$ with a proper inclusion map, up to a set with zero $(2n-1)$-dimensional volume, $\mathcal{C}_+ \cap \Sigma$ coincides with the set of positively recurrent points in Σ.*

On the other hand, the ω-limit set of every point in $\mathcal{D}_+$ is in $\{H = 0\}$. Almost every point in $\mathcal{D}_+$ is in $\{H \neq 0\}$. If $x \in \mathcal{D}_+ \cap \{H \neq 0\}$, then $r_t(x) \to -\infty$ as $t \to +\infty$ and every neighbourhood of $\omega(x) = A$ contains a closed curve γ such that $\int_\gamma \eta \neq 0$. In particular, A is infinite.

Proof. First, let us show that $\mathcal{C}_+$ coincides with the set of positively recurrent points up to a negligible set. Let us remark that ω^n-almost every point of $\{H = 0\}$ is trivially recurrent and in $\mathcal{C}_+$: every point of the subset $\{H = 0\} \cap \{\mathrm{d}H = 0\}$ is fixed by the dynamics whereas $\{H = 0\} \cap \{\mathrm{d}H \neq 0\}$ is negligible.

Let us now show that almost every point of $\mathcal{C}'_+ := \mathcal{C}_+ \cap \{H \neq 0\}$ is recurrent. According to Lemma 2.1, $H \circ \varphi_t = e^{r_t} H$, so

$$\mathcal{C}'_+ = \Big\{x \in M \mid \limsup_{\substack{p \to +\infty \\ p \in \mathbb{N}}} |H(\varphi_p(x))| \neq 0\Big\}.$$

For $k \in \mathbb{N}^*$, let us define the following compact sets:

$$\mathcal{H}_k := \Big\{x \in M \mid |H(x)| \geq \frac{1}{k}\Big\}.$$

Then $\mathcal{C}'_+$ is the increasing union of the $\mathcal{C}'_k$'s defined by

$$\mathcal{C}'_k := \mathcal{H}_k \cap \bigcap_{N \in \mathbb{N}} \bigcup_{\substack{p \geq N \\ p \in \mathbb{N}}} \varphi_p^{-1}(\mathcal{H}_k), \quad \forall k \in \mathbb{N}^*.$$

For each $k \in \mathbb{N}^*$, there is a well-defined first-return measurable map $f_k \colon \mathcal{C}'_k \to \mathcal{C}'_k$ given by $f_k(x) := \varphi_{n(x)}(x)$, where $n(x) := \min\{p \in \mathbb{N}^* \mid \varphi_p(x) \in \mathcal{H}_k\}$. Since the 2-form ω/H of $\{H \neq 0\}$ is preserved by φ_p for all $p \in \mathbb{N}$ (Lemma 2.1), the measurable maps f_k's are preserving the measure $\nu \colon A \mapsto \int_A \omega^n / H^n$. Since, for $k \in \mathbb{N}^*$, $\mathcal{C}'_k$ has a countable basis of open sets and a measure $\nu(\mathcal{C}'_k) \leq \nu(\mathcal{H}_k) \leq k^n \omega^n(M)$ which is finite by compactness of M, the Poincaré recurrence theorem implies that almost every point of $\mathcal{C}'_k$ is recurrent for f_k. Therefore, almost every point of $\mathcal{C}'_+$ is recurrent.

It remains to prove that almost every point of $\mathcal{D}_+$ is not positively recurrent. Since $H \circ \varphi_t = e^{r_t} H$ by Lemma 2.1, a point $x \in M$ satisfying $r_t(x) \to +\infty$ must be in $\{H = 0\}$ $\cap \{\mathrm{d}H \neq 0\}$, which is a negligible set. Hence if $x \in \mathcal{D}_+ \cap \{H \neq 0\}$, $r_t(x) \to -\infty$ as $t \to +\infty$ and then $\lim_{t \to +\infty} H(\varphi_t x) = 0$ and x is not positively recurrent. Therefore, $\mathcal{C}_+$ coincides with the set of positively recurrent points of M up to a negligible set.

Now, let $\Sigma \subset \{H \neq 0\}$ be an embedded leaf of $\mathcal{F}$ with a proper inclusion map, and let us prove that $\mathcal{C}_+ \cap \Sigma$ coincides with the set of positively recurrent points of Σ, up to a negligible set of Σ. We have seen that no point in $\mathcal{D}_+ \cap \Sigma$ is positively recurrent. Since $\mathcal{C}_+ \cap \Sigma = \mathcal{C}'_+ \cap \Sigma$, it is enough to prove that almost every point of $\mathcal{C}'_k \cap \Sigma$ is recurrent, for all $k \in \mathbb{N}^*$. Let μ be the volume form associated with Σ by Proposition 2.5. Then the measure $\nu_\Sigma \colon A \mapsto \int_A \mu / H^{n-1}$ defined on Σ is preserved by the first return maps $f_k|_{\Sigma \cap \mathcal{C}'_k}$. Since $\Sigma \hookrightarrow \{H \neq 0\}$ is proper, the $\Sigma \cap \mathcal{H}_k$ are compact, so the $\nu_\Sigma(\Sigma \cap \mathcal{C}'_k)$'s are finite.

Finally, let us prove that if $x \in \mathcal{D}_+ \cap \{H \neq 0\}$, every neighbourhood V of $\omega(x)$ contains a closed curve γ such that $\int_\gamma \eta \neq 0$. By compactness of $\omega(x)$, one can assume that V is a finite union of path-connected contractible open sets V_j. Let $K > 0$ be such that $|\int_\gamma \eta| < K$ for every $\gamma \colon [0,1] \to V_j$ and every j (where η denotes the Lee form). Let $T > 0$ be such that for all $t \geq T$, $\varphi_t(x) \in V$, and let j_0 be such that there exist arbitrarily large t's satisfying $\varphi_t(x) \in V_{j_0}$. Let $t_1 > t_0 > T$ be such that $r_{t_0}(x) - r_{t_1}(x) > K$ and $\varphi_{t_i}(x) \in V_{j_0}$ for $i \in \{0, 1\}$. Then, concatenating $t \mapsto \varphi_t(x)$, $t \in [t_0, t_1]$, with a path of V_{j_0} connecting $\varphi_{t_1}(x)$ to $\varphi_{t_0}(x)$, one gets a loop $\gamma \colon I \to V$ satisfying $\int_\gamma \eta \neq 0$. The conclusion follows. ∎

We recall that $U \subset M$ is a wandering set if $\exists T > 0, \forall t \geq T, \varphi_t(U) \cap U = \emptyset$.

Corollary 3.2. *Let $H\colon M \to \mathbb{R}$ be a Hamiltonian map on a closed conformally symplectic manifold. Let U be a wandering set of the Hamiltonian dynamics. Then almost every point of U belongs to $\mathcal{D}_+ \cap \mathcal{D}_-$ and satisfies* $\lim_{t\to+\infty} H(\varphi_t^H(x)) = \lim_{t\to-\infty} H(\varphi_t^H(x)) = 0$.

Corollary 3.3. *Let $H\colon M \to \mathbb{R}$ be a Hamiltonian map on a closed conformally symplectic manifold. Let A be a weak attractor of the Hamiltonian dynamics with basin U (see Section 3.1). Then for almost every point x of $U \setminus A$, $r_t(x) \to -\infty$ as $t \to +\infty$. In particular, the Lee form is not exact in any neighbourhood of $A \cap \{H = 0\}$.*

As a consequence, an attractor (or repeller) of (φ_t) is never a finite set.

Proof of Corollary 3.3. By definition, the points of $U \setminus A$ are not recurrent, so almost every point of $U \setminus A$ is in $\mathcal{D}_+$ by Theorem 3.1. The same proposition implies the other results. ∎

3.3. Almost everywhere coincidence of past and future

We have of course that $\mathcal{C}_-$ coincides with the set of negatively recurrent points, up to a set with zero volume. What is surprising is that the set of negatively recurrent points coincide with the set of positively recurrent points up to a set with zero volume.

Proposition 3.4. *Let $H\colon M \to \mathbb{R}$ be a Hamiltonian map on a closed conformally symplectic manifold. The set $\mathcal{C}_+$ coincides with $\mathcal{C}_-$ up to a set with zero volume.*

Moreover, for every embedded leaf Σ included in $\{H \neq 0\}$ with a proper inclusion map, up to a set with zero $(n-1)$-dimensional volume, $\mathcal{C}_+ \cap \Sigma$ and $\mathcal{C}_- \cap \Sigma$ coincide.

However, it is possible to construct Hamiltonian dynamics for which $\mathcal{C}_+ \neq \mathcal{C}_-$; see Remark 3.8 below.

Proof. We will prove that up to a set with zero volume, $\mathcal{C}_+ \subset \mathcal{C}_-$, and we will deduce the first part of Proposition 3.4. We keep the notation of the proof of Theorem 3.1. The first return map $f_k\colon \mathcal{C}'_k \to \mathcal{C}'_k$ preserves the finite volume $\frac{1}{H^n}\omega^n$, hence almost every point of $\mathcal{C}'_k$ is negatively recurrent for f_k. This implies that up to a set with zero volume, $\mathcal{C}'_+$ and hence $\mathcal{C}_+$ is a subset of $\mathcal{C}_-$.

The proof of the second part is similar. ∎

3.4. Boundedness of r_t and invariant measures

Let us assume that $L \subset M$ is an invariant measurable set of the dynamics on which $(t,x) \mapsto r_t(x)$ is a bounded map $\mathbb{R} \times L \to \mathbb{R}$ (in particular, $L \subset \mathcal{C}_+ \cap \mathcal{C}_-$). Inspired by the proof of Theorem 5.1.13 in [7], let us define the bounded measurable map $h\colon L \to \mathbb{R}$ as

$$h(x) := \sup_{t\in\mathbb{R}} r_t(x). \tag{3.3}$$

As the map $(t,x) \mapsto r_t(x)$ is continuous, we have also $h(x) := \sup_{t\in\mathbb{Q}} r_t(x)$, hence h is measurable.

Then $h \circ \varphi_t = h - r_t$, so that for instance Lemma 2.1 implies that, restricted to L,

$$\varphi_t^*(e^h\omega) = e^h\omega, \quad \varphi_t^*(e^h \mathrm{d}_\eta H) = e^h \mathrm{d}_\eta H \quad \text{and} \quad (e^h H)\circ\varphi_t = e^h H, \quad \forall t \in \mathbb{R}.$$

Corollary 3.5. *Let $H\colon M \to \mathbb{R}$ be a Hamiltonian map on a conformally symplectic manifold. An invariant measurable set $L \subset M$ of positive measure on which $(t,x) \mapsto r_t(x)$ is bounded admits an invariant Borel measure of positive density.*

Proof. The measure $A \mapsto \int_A e^{nh}\omega^n$ is positive and invariant. ■

Corollary 3.6. *Let $H\colon M \to \mathbb{R}$ be a Hamiltonian map on a closed conformally symplectic manifold. If L is an embedded leaf of $\mathcal{F}$ on which $(t,x) \mapsto r_t(x)$ is bounded, then it admits an invariant Borel measure of positive density.*

Proof. The desired measure is $A \mapsto \int_A e^{(n-1)h}\mu$, where μ is given by Proposition 2.5. ■

We will see in Section 4.2 that when η is exact in the neighbourhood of $\{H = 0\}$, the flow is conservative and Corollaries 3.5 and 3.6 apply.

One of the dynamical importance of these corollaries is signified by Poincaré's recurrence theorem: if the invariant measures in question are also finite, almost every point of the invariant set is recurrent. However, with the exception of regular leaves of $\mathcal{F}$ included in $\{H = 0\}$, the recurrence can also be deduced from Theorem 3.1.

3.5. An oscillating behaviour

In this subsection, we give an example of a flow possessing orbits included in $\mathcal{C}_+$ that are in $\{H \neq 0\}$, positively recurrent and whose ω-limit set intersects $\{H = 0\}$. Hence they have an unbounded associated winding $t \in [0, +\infty) \mapsto r_t(x)$.

Proposition 3.7. *There exists a Hamiltonian map $H\colon M \to \mathbb{R}$ on some closed conformal symplectic manifold, the flow of which satisfies*

$$\limsup_{t\to+\infty} r_t(x) > \liminf_{t\to+\infty} r_t(x) = -\infty,$$

for some point $x \in \{H \neq 0\}$.

In order to prove this proposition, let us briefly recall the statement of the shadowing lemma for flows (see e.g. Theorem 18.1.6 in [7]). Let (φ_t) be a smooth flow on a Riemannian manifold M, the infinitesimal generator of which is X_t. A differentiable curve $c\colon I \to \mathbb{R}$, $I \subset \mathbb{R}$ interval, is called an ε-orbit if $\|\dot c(t) - X_t(c(t))\| \leq \varepsilon$ for all $t \in I$. A differentiable curve $c\colon I \to \mathbb{R}$ is said to be δ-shadowed by the orbit $(\varphi_t(x))_{t\in J}$ if there exists $s\colon J \to I$ with $|s' - 1| < \delta$ such that $d(c(s(t)), \varphi_t(x)) < \delta$ for all $t \in J$ (d denoting the Riemannian distance). The shadowing lemma states that, given a hyperbolic set Λ of (φ_t), there is a neighbourhood $U \supset \Lambda$ so that, for every $\delta > 0$, there is an $\varepsilon > 0$ such that every ε-orbit included in U is δ-shadowed by an orbit of (φ_t).

Proof. Let Σ be a closed hyperbolic surface, let us denote $\pi\colon T^1\Sigma \to \Sigma$ the associated unit tangent bundle, and let β be a non-exact closed 1-form of Σ. Let us denote (G_t) the geodesic flow on $T^1\Sigma$ and X the associated vector field. Let (M, η, ω) be a conformal symplectic closed manifold associated with $(T^1\Sigma, \pi^*\beta)$ by Lemma A.2 in Appendix A: that is, one may assume that $L := T^1\Sigma \times S^1$ is a Lagrangian submanifold of M and that the restriction of η to this submanifold is $\alpha := \pi^*\beta - \mathrm{d}\theta$ (we identify $\pi^*\beta$ with its pull-back by the projection by a slight abuse of notation). Let W be a Weinstein neighbourhood of L: identifying the 0-section of $T^*_\alpha L$ with L, one can see W as a neighbourhood of the 0-section of $T^*_\alpha L$ (see Theorem 2.11 in [2] or Section A.2). Let us identify the vector field X of $T^1\Sigma$ with the vector field $X \oplus 0$ of L, and let $H: M \to \mathbb{R}$ be a Hamiltonian function satisfying $H(q,p) = p(X(q))$ on $W \subset T^*_\alpha L$ (shrinking W if necessary). Let us prove that H satisfies the statement of the proposition.

Let us first find an orbit $(\gamma, \dot\gamma): \mathbb{R}_+ \to T^1\Sigma$ of the geodesic flow (G_t) such that

$$\begin{cases} \exists K > 0, \forall t > 0, \quad \int_{\gamma|[0,t]} \beta \le K, \\ \limsup_{t\to+\infty} \int_{\gamma|[0,t]} \beta > \liminf_{t\to+\infty} \int_{\gamma|[0,t]} \beta = -\infty. \end{cases} \tag{3.4}$$

Such an orbit can be found applying the shadowing lemma to (G_t). Indeed, let us fix $\delta \in (0,1)$ and take an $\varepsilon > 0$ associated by the shadowing lemma. Let $a: \mathbb{R}/T\mathbb{Z} \to \Sigma$ be a closed geodesic of unit speed such that $\int_a \beta > 0$ (up to reparametrization, such an a can be obtained as a minimizer of the energy functional among loops homotopic to a loop b satisfying $\int_b \beta > 0$). By topological transitivity of (G_t), there exists an $\varepsilon/2$-orbit $(c, \dot c): [0, T'] \to T^1\Sigma$ such that $\dot c(0) = \dot a(0)$ and $\dot c(T') = -\dot a(0)$. By successively concatenating c or c^{-1} with higher and higher iterations of a and a^{-1}, one can thus build an ε-orbit $(\tilde\gamma, \dot{\tilde\gamma}): \mathbb{R}_+ \to T^1\Sigma$ satisfying the conditions in (3.4), where γ is replaced with $\tilde\gamma$. The shadowing lemma applied to this ε-orbit gives us the desired γ.

According to Section 2.3, on $W \subset T^*_\alpha L$, the Hamiltonian flow (φ_t) of H takes the form

$$\varphi_t(q,p;z) = (G_t(q), e^{r_t(q)} p \circ (\mathrm{d}G_t(q))^{-1}; z),$$

where $(q,p) \in T^*(T^1\Sigma)$, $z \in T^*S^1$ and

$$r_t(q) := \int_0^t \pi^*\beta(\partial_s G_s(q))\,\mathrm{d}s,$$

as long as $\varphi_s(q,p;z)$ stays inside W for $s \in [0,t]$. As (G_t) is Anosov, one has the bundle decomposition $T(T^1\Sigma) = E^s \oplus \mathbb{R}X \oplus E^u$, which is preserved by (G_t) with $\mathrm{d}G_t \cdot X = X \circ G_t$. Let $q \mapsto P_q$ be the section of $T^*(T^1\Sigma)$ vanishing on $E^s \oplus E^u$ and such that $P(X) \equiv 1$; it satisfies $P_q \circ (\mathrm{d}G_t(q))^{-1} = P_{G_t(q)}$ for all q. For fixed $z \in T^*S^1$ and $\lambda > 0$, let us consider the $\mathbb{R}_+$-orbit generated by $(\dot\gamma(0), \lambda P_{\dot\gamma(0)}; z)$ (where γ satisfies (3.4)). By the first condition of (3.4), $r_t(\dot\gamma(0))$ is bounded from above so this orbit keeps inside W for a sufficiently small λ. The second condition of (3.4) implies the statement for $x = (\dot\gamma(0), \lambda P_{\dot\gamma(0)}; z)$ (the orbit is in $\{H \ne 0\}$ since $P(X) \equiv 1$). ■

Remark 3.8 (An example where $\mathcal{C}_+ \ne \mathcal{C}_-$). Adapting the construction made in the above proof, it is not hard to obtain an orbit that is negatively dissipative and positively conservative, i.e., such that $\mathcal{C}_- \ne \mathcal{C}_+$.

4. Global conservative behaviours

As we have seen in Corollary 3.3, a necessary condition for attractors to appear is the non-exactness of the Lee form in the neighbourhood of $\{H = 0\}$. Here, we study the opposite case: a Hamiltonian flow (φ_t) of H on a closed conformal symplectic manifold (M^{2n}, η, ω) in the case where η is exact in the neighbourhood of the invariant set $\{H = 0\}$. That is, we assume that there exists an open set U containing $\{H = 0\}$ such that $[\eta|_U] = 0$ in $H^1(U; R)$. This hypothesis is thus gauge invariant.

4.1. When H does not vanish

Let us first assume that H does not vanish, and denote X_H^η its associated vector field for the Lee form η. Possibly reversing time, we will assume that H is positive. Since $X_H^\eta = X_{e^f H}^{\eta + \mathrm{d}f}$, by setting $f = -\log \circ H$, we see that X_H^η is the Lee vector field of $\eta' = \eta - \mathrm{d}(\ln \circ H)$. Therefore, Hamiltonian flows of non-vanishing H are Lee flows.

We now assume that $H \equiv 1$ for the choice of gauge (η, ω), so that the vector field is L^η. Since $\eta(L^\eta) = 0$, $r_t \equiv 0$ and $\mathcal{C}_+ = \mathcal{C}_- = M$, i.e., the flow is positively and negatively conservative with the terminology given in the introduction. Thus almost every point is positively and negatively recurrent.

Lemma 2.1 implies that ω is preserved by the flow. Let us point out that this flow is not conjugated to a symplectic flow in general, since one can have $H^2(M; \mathbb{R}) = 0$ (e.g., the conformal symplectization of the contact sphere $(\mathbb{S}^{2n-1}, \frac{1}{2}(x\mathrm{d}y - y\mathrm{d}x))$). The volume form ω^n is preserved, so almost every point is recurrent according to Poincaré's recurrence theorem. More precisely, almost every point of a proper embedded leaf of $\mathcal{F}$ is recurrent according to Theorem 3.1 and Corollary 3.6. Let us remark that in the case where η is completely resonant, (i.e., the subgroup $\{\int_\gamma \eta; \gamma : S^1 \to M\}$ is discrete), there exist $k \in \mathbb{R}^*$ and a map $\theta\colon M \to \mathbb{R}/k\mathbb{Z}$ such that $\eta = \mathrm{d}\theta$ and the invariant foliation $(\theta^{-1}(s))_{s \in \mathbb{R}/k\mathbb{Z}}$ has compact leaves.

4.2. When η is exact in the neighbourhood of $\{H = 0\}$

Let us move on to the general case: there exists an open neighbourhood U of $\{H = 0\}$ on which $\eta|_U = \mathrm{d}\theta$ for some $\theta\colon U \to \mathbb{R}$.

Proposition 4.1. *Let $H\colon M \to \mathbb{R}$ be a Hamiltonian map on a closed conformally symplectic manifold, the Lee form of which is exact in some neighbourhood of $\{H = 0\}$. Then the map $(t, x) \mapsto r_t(x)$ is bounded on $\mathbb{R} \times M$.*

Proof. Let $\varepsilon_0 > 0$ be such that the neighbourhood $V_0 := H^{-1}([-\varepsilon_0, \varepsilon_0])$ is included in U, and let $V := H^{-1}([-\varepsilon, \varepsilon])$ for some $\varepsilon \in (0, \varepsilon_0)$. Let

$$A := \max_{V_0} \theta - \min_{V_0} \theta, \quad b := \inf_{M\setminus V} |H| \quad \text{and} \quad B := \sup_{M\setminus V} |H|.$$

We will show that

$$|r_t(x)| \leq 2A + \log(B/b), \quad \forall (t, x) \in \mathbb{R} \times M.$$

Let $(t,x) \in \mathbb{R} \times M$. If $\varphi_s(x) \in V_0$ for all $s \in [0,t]$ ($[t,0]$ if $t<0$), then $|r_t(x)| \leq A$. If $x \in M \setminus V$ and $\varphi_t(x) \in M \setminus V$, then $b/B \leq |H(\varphi_t(x))/H(x)| \leq B/b$, so $|r_t(x)| \leq \log(B/b)$ according to Lemma 2.1.

If $x \in V$ and $\varphi_t(x) \notin V$, we assume $t > 0$, and let $t_0 \in [0,t]$ be such that $\varphi_s(x) \in V_0$ for all $s \in [0,t_0]$ and $\varphi_{t_0}(x) \in M \setminus V$. Then $|r_{t_0}(x)| \leq A$, whereas $|r_{t-t_0}(\varphi_{t_0}(x))| \leq \log(B/b)$ by the above case, implying $|r_t(x)| \leq A + \log(B/b)$. The same is symmetrically true for $t < 0$ and with x and $\varphi_t(x)$ intertwined.

The last case is when $x \in V$ and $\varphi_t(x) \in V$; we assume $t > 0$ (the other case is symmetrical), and $\varphi_{s_0}(x) \notin V_0$ for some $s_0 \in [0,t]$. One can find $t_1 < s_0 < t_2$ such that $H(\varphi_{t_1}(x)) = H(\varphi_{t_2}(x)) = \varepsilon$ and $\varphi_s(x) \in V$ for $s \in [0,t_1] \cup [t_2,t]$. By Lemma 2.1, $r_{t_2}(x) = r_{t_1}(x)$, and the conclusion follows from the first case treated. ∎

Therefore, according to the conservative-dissipative decomposition of Section 3.2, $M = \mathcal{C}_- = \mathcal{C}_+$ and almost every point of M or an embedded leaf of $\mathcal{F}$ in $\{H \neq 0\}$ is positively and negatively recurrent. Moreover, according to Corollaries 3.5 and 3.6, M and every embedded leaf of $\mathcal{F}$ in $\{H \neq 0\}$ admit an invariant Borel measure of positive density. In particular, almost every point of a closed regular leaf of $\mathcal{F}$ in $\{H = 0\}$ is also positively and negatively recurrent.

4.3. A topologically transitive Lee flow

Our goal is to provide examples of topologically transitive Lee flow in every dimension.

We have seen in Section 1.2.2 that in dimension 2, there are very simple examples of minimal Lee flows. We recall this. Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ denote the 2-torus with canonical coordinates $x, y \in \mathbb{R}/\mathbb{Z}$. Let us fix $a, b \in \mathbb{R}$ and let us endow $\mathbb{T}^2$ with the conformal symplectic structure $(\eta, \omega) = (a\mathrm{d}x + b\mathrm{d}y, \mathrm{d}x \wedge \mathrm{d}y)$, the Lee flow of which is $\varphi_t(x,y) = (x + bt, y - at)$. This flow is minimal if and only if a and b are rationally independent.

One way to extend this example is to remark that in the case $a = -1$, it corresponds to the β-twisted conformal symplectization of the contact manifold $(S^1, \mathrm{d}y)$ with $\beta = b\mathrm{d}y$.

Proposition 4.2. *Let (Y,α) be a closed connected contact manifold with a fixed contact form, the Reeb flow of which is Anosov and possesses a periodic orbit γ such that $\int_\gamma \beta$ is irrational for some closed 1-form β. Then, the standard Lee flow of $S_\beta^{\mathrm{conf}}(Y,\alpha)$ is topologically transitive.*

Such a (Y,α) can be found in every dimension. Indeed, let us first recall that according to the Anosov theorem, the geodesic flow of any Riemannian manifold with negative sectional curvature is Anosov. Then, let N be a closed Riemannian manifold with negative sectional curvature and a non-trivial real homology group of degree 1: $H^1(N;\mathbb{R}) \neq 0$. Let thus β' be a non-exact closed 1-form such that $\int_c \beta'$ is irrational for some loop c. Let $\pi: T^1N \to N$ be the unit sphere bundle of N endowed with its usual contact structure (the Reeb flow of which is the geodesic flow); then the β-twisted conformal symplectization of $Y := T^1N$ with $\beta := \pi^*\beta'$ satisfies the hypothesis of the statement. Indeed, by taking a minimum of the energy functional among loops homotopic to c, one gets a closed geodesic homotopic to c, so a periodic orbit γ of the geodesic flow such that $\int_\gamma \beta = \int_c \beta'$.

However, Lee flows induced by these choices of (Y,α) have a lot of periodic orbits in dimension $2n \geq 4$, so they are not minimal.

Proof of Proposition 4.2. Let (φ_t) be the Reeb flow of Y. According to Lemma 2.2, the Lee flow (Φ_t) of $\mathcal{S}^{\mathrm{conf}}_\beta(Y,\alpha) = Y \times S^1$ takes the following form: for all $(x,\theta) \in Y \times S^1$ and $t \in \mathbb{R}$,

$$\Phi_t(x,\theta) = (\varphi_t(x), \theta + \rho_t(x)), \quad \text{where } \rho_t(x) := \int_0^t \beta(\partial_s \varphi_s(x))\,\mathrm{d}s.$$

In order to show topological transitivity, it is enough to prove that for every pair of product non-empty open sets $U_i \times V_i \subset Y \times S^1$, $i \in \{1,2\}$, there is some $(x,\theta) \in U_1 \times V_1$ and some $t \in \mathbb{R}$ such that $\Phi_t(x,\theta) \in U_2 \times V_2$ (see e.g. Lemma 1.4.2 in [7]). We can assume that the V_i are arcs of length $\ell > 0$.

By hypothesis, there exists a point $y \in Y$ such that $\varphi_{t_2}(y) = y$ for some $t_2 > 0$ and $\rho_{t_2}(y)$ is irrational.

We choose $x_i \in U_i$ for $i = 1,2$. Let $\delta > 0$ be so small that if a curve $c\colon [a,b] \to Y$ with $c(a) = x_1$ and $c(b) = x_2$ is δ-shadowed by an orbit $\nu\colon [a',b'] \to Y$, then $\nu(a') \in U_1$, $\nu(b') \in U_2$ and $|\int_c \beta - \int_\nu \beta| < \ell/3$.

By assumption, (φ_t) is a topologically transitive Anosov flow (contact Anosov flows on connected manifolds are topologically mixing, see Theorem 18.3.6 in [7]). Let $\varepsilon > 0$ be associated with δ by the shadowing lemma, so that every ε-orbit of (φ_t) is δ-shadowed by an actual orbit (see the paragraph below Proposition 3.7). By transitivity, there exist $\varepsilon/2$-orbits $c_1\colon [0,t_1] \to Y$ and $c_3\colon [0,t_3] \to Y$, with $c_1(0) = x_1$, $c_1(t_1) = y$ and $c_3(0) = y$, $c_3(t_3) = x_2$. Then (up to a small deformation at the connecting points), the concatenated paths $\nu_k := c_1 \cdot c_2^k \cdot c_3$, $k \in \mathbb{N}$, are ε-orbits for (φ_t). Let $R_k := \int_{\nu_k} \beta \bmod 1 \in S^1$. Since $\int_{c_2} \beta$ is irrational, (R_k) is dense in S^1. Let us fix $k \in \mathbb{N}$ such that $\theta + [R_k - \ell/3, R_k + \ell/3] \in V_2$ for some $\theta \in V_1$ (the length of the arcs V_i's being $> \ell$). Applying the shadowing lemma to ν_k, we find an orbit $\nu\colon [0,T] \to Y$ such that $\nu(0) \in U_1$, $\nu(T) \in U_2$ and $\int_\nu \beta \bmod 1$ is $\ell/3$-close to R_k, so that $\theta + \int_\nu \beta \in V_2$. ∎

4.4. A Lee flow with no periodic orbit

Relaxing the transitivity hypothesis, one can easily produce a Lee flow without periodic orbits.

Proposition 4.3. *Let $\mathbb{T}^n$ be the flat n-torus with canonical coordinates $x_i \in \mathbb{R}/\mathbb{Z}$, and let $\beta := a_1\mathrm{d}x_1 + \cdots + a_n\mathrm{d}x_n$ for some fixed $a_i \in \mathbb{R}$. The standard Lee flow of $\mathcal{S}^{\mathrm{conf}}_\beta(T^1\mathbb{T}^n)$ does not have any periodic orbit if and only if the family $(1, a_1, \ldots, a_n)$ is rationally independent.*

This flow is clearly not minimal and, in general, there is not much hope for the standard Lee flow of a closed twisted conformal symplectization to be minimal in dimension $2n \geq 4$. Indeed, the Weinstein conjecture states that every Reeb flow of a closed contact manifold (Y,α) should possess a closed orbit γ, so $\gamma \times S^1$ would be a closed invariant set of the standard Lee flow of the twisted conformal symplectizations of (Y,α).

Proof of Proposition 4.3. The Reeb flow of $T^1\mathbb{T}^n \simeq \mathbb{T}^n \times \mathbb{S}^{n-1}$ is $\varphi_t(x,v) := (x + tv, v)$. The associated $\rho_t(x,v) := \int_0^t \beta(\partial_s\varphi_s(x,v))\,\mathrm{d}s \bmod 1$ satisfies $\rho_t(x,v) = \sum_i a_i t v_i \bmod 1$. Therefore, a point $(x,v,\theta) \in T^1\mathbb{T}^n \times S^1$ is a τ-periodic point of the Lee flow if and only if $\tau v \in \mathbb{Z}^n$ and $\sum_i a_i \tau v_i \in \mathbb{Z}$. ∎

4.5. A conservative behaviour with $\eta_{|\{H=0\}}$ non-exact

Proposition 4.4. *Let $\mathbb{T}^n$ be the flat n-torus with canonical coordinates $q_i \in \mathbb{R}/\mathbb{Z}$, and let us consider on $\mathcal{S}^{\mathrm{conf}}(T^1\mathbb{T}^n)$ the Hamiltonian $H(q_1,\dots,q_n,p_1,\dots,p_n,\theta) = p_1$. The Hamiltonian flow is*

$$\varphi_t^H(q_1,\dots,q_n,p_1,\dots,p_n,\theta) = (q_1+t,q_2,\dots,q_n,p_1,\dots,p_n,\theta).$$

This flow preserves the conformal 2-form, and the zero level $\{H=0\}$ contains a loop γ such that

$$\int_\gamma \eta \neq 0.$$

Proof. The contact form is the restriction of the Liouville 1-form λ to $T^1\mathbb{T}^n$, and the Reeb vector field R at (q,p) is $(p,0)$. Hence $dH\cdot R = 0$ and the contact Hamiltonian flow X_H satisfies $\iota_{X_H} d\lambda = -dp_1$ and $X_H = (1,0,\dots,0)$. As $dH\cdot R = 0$, we deduce from Lemma 2.2 that the conformal Hamiltonian vector field is $(1,0,\dots,0)$. ∎

5. Dissipative behaviours

5.1. Dissipative ergodic measures

Let ν be an ergodic measure, and let us denote

$$\bar{r}(\nu) := \int_M \eta(X)\,\mathrm{d}\nu.$$

The following proposition is the ergodic counterpart of Corollary 2.4.

Proposition 5.1. *Let $H\colon M\to\mathbb{R}$ be a Hamiltonian map on a closed conformally symplectic manifold. Given an ergodic measure ν of the Hamiltonian dynamics, $\nu(\{H=0\})\in\{0,1\}$, and in the case where $\bar{r}(\nu)\neq 0$, the support of ν is included in $\{H=0\}$.*

Proof. The first statement is obvious since $\{H=0\}$ is an invariant set. If $\bar{r}(\nu)\neq 0$ and $\operatorname{supp}\nu \not\subset \{H=0\}$, there exists $x\in\{H\neq 0\}$ such that $r_t(x)\sim t\bar{r}(\nu)$ as $t\to\pm\infty$, according to the Birkhoff ergodic theorem. However, H is bounded and $H\circ\varphi_t = e^{r_t}H$ (Lemma 2.1), a contradiction. ∎

Let us recall the result of Liverani–Wojtkowski [14] about the Lyapunov spectrum of conformally symplectic cocycles. We state the results in the invertible case. Let (M,ν) be a probability space with an invertible ergodic map $T\colon M\to M$, and let $A\colon M\to \mathrm{GL}(\mathbb{R}^{2n})$ be a measurable map such that both measurable maps $\log_+\|A^{\pm1}\|$ are integrable (this is independent on the choice of norm on $\mathrm{GL}(\mathbb{R}^{2n})$). We define the so-called cocycle $(A^m)_{m\in\mathbb{Z}}$ as the family of measurable maps $M\to\mathrm{GL}(\mathbb{R}^{2n})$ given by $A^m(x) := A(T^{m-1}x)\cdots A(x)$ for $m\in\mathbb{Z}$. According to the Oseledets multiplicative ergodic theorem, there exists real numbers $\lambda_1<\cdots<\lambda_s$ called the Lyapunov exponents of A, and an associated decomposition of $\mathbb{R}^{2n}$ (that we will call the Lyapunov decomposition of $\mathbb{R}^{2n}$) in linear subspaces $F_1(x)\oplus\cdots\oplus F_s(x)$ defined for ν-almost every $x\in M$, such that

$$\lim_{m\to\pm\infty}\frac{1}{m}\log\|A^m(x)v\| = \lambda_k,\quad \forall v\in F_k(x), \forall k\in\{1,\dots,s\}.$$

The positive integer $d_k := \dim F_k(x)$ does not depend on x, and is called the multiplicity of λ_k. These multiplicities satisfy

$$\sum_{k=1}^{s} d_k \lambda_k = \int_M \log|\det A| \, \mathrm{d}\nu.$$

Liverani–Wojtkowski showed a symmetry of the Lyapunov spectrum in the case where A takes its values in the conformally symplectic linear group $\mathrm{CSp}(2n) := \mathrm{CSp}(\mathbb{R}^{2n}, \omega_0)$. A conformally symplectic linear map $S \in \mathrm{CSp}(E, \omega)$ is a linear map of a symplectic linear space (E, ω) satisfying $S^*\omega = \beta\omega$ for some $\beta \in \mathbb{R}^*$ called the conformal factor of S.

Theorem 5.2 (Theorem 1.4 in [14]). *Let (M, ν) be a probability space with an invertible ergodic map $T\colon M \to M$, and let $A\colon M \to \mathrm{CSp}(2n)$ be a measurable cocycle such that $\log_+ \|A^{\pm 1}\|$ are integrable. Let $\beta\colon M \to \mathbb{R}^*$ be such that $A(x)^*\omega_0 = \beta(x)\omega_0$ for all $x \in M$. Then we have the following symmetry of the Lyapunov spectrum $\lambda_1 < \cdots < \lambda_s$ of A:*

$$\lambda_k + \lambda_{s-k+1} = b, \quad \text{where } b := \int_M \log|\beta| \, \mathrm{d}\nu,$$

for every $k \in \{1, \ldots, s\}$. Moreover, the subspace $F_1 \oplus \cdots \oplus F_{s-k}$ is the ω_0-orthogonal subspace of $F_1 \oplus \cdots \oplus F_k$.

Let us come back to our setting and consider an ergodic measure ν of $M^{(2n)}$ for the Hamiltonian flow (φ_t). Let us fix a Riemannian metric g on M and let us consider a measurable symplectic trivialization of TM, that is a measurable (not necessarily continuous) map $TM \to M \times \mathbb{R}^{2n}$ such that, for all $x \in M$, its restriction to $T_x M$ maps its image in $\{x\} \times \mathbb{R}^{2n}$ and induces an isomorphism of symplectic vector spaces $(T_x M, \omega_x) \xrightarrow{\simeq} (\mathbb{R}^{2n}, \omega_0)$ (such measurable maps always exist by taking the union of suitable local regular symplectic trivializations of TM). By applying this measurable symplectic trivialization of TM, one naturally extends the Oseledets multiplicative ergodic theorem for measurable maps $A\colon M \to \mathrm{GL}(\mathbb{R}^{2n})$ such that $\log_+ \|A^{\pm 1}\|$ are integrable for ν to measurable sections $A\colon M \to \mathrm{GL}(TM)$ of the fiber bundle $\mathrm{GL}(TM)$ such that $\log_+ \|A^{\pm 1}\|$ are integrable, where $\|\cdot\|$ is the Riemannian operator norm associated with g. Now, the specific section $A\colon x \mapsto \mathrm{d}\varphi_1(x)$ satisfies the integrability condition and the cocycle A^m corresponds to $\mathrm{d}\varphi_m$ for $m \in \mathbb{Z}$. The associated Lyapunov exponents $\lambda_1 < \cdots < \lambda_s$ define the Lyapunov exponents of the flow (φ_t) for the ergodic measure ν. By compactness of M, $t \mapsto \partial_t(\log\|\mathrm{d}\varphi_t(x)v\|)$ is bounded, $x \in M$ and $v \in T_x M \setminus \{0\}$ being fixed, so

$$\lim_{\substack{m \to \pm\infty \\ m \in \mathbb{Z}^*}} \frac{1}{m} \log\|\mathrm{d}\varphi_m(x)v\| = \lim_{\substack{t \to \pm\infty \\ t \in \mathbb{R}^*}} \frac{1}{t} \log\|\mathrm{d}\varphi_t(x)v\|,$$

for every $(x, v) \in TM$ for which one of the limit is defined.

Corollary 5.3. *Let ν be an ergodic measure of some Hamiltonian flow (φ_t) of a closed conformal symplectic manifold M. Let $\lambda_1 < \cdots < \lambda_s$ be the associated Lyapunov spectrum, and $F_1, \ldots, F_s$ the associated Lyapunov decomposition of TM. For every $k \in \{1, \ldots, s\}$,*

$$\lambda_k + \lambda_{s-k+1} = \bar{r}(\nu)$$

and the subbundle $F_1 \oplus \cdots \oplus F_{s-k}$ is the ω-orthogonal subspace of $F_1 \oplus \cdots \oplus F_k$.

Proof. We apply Theorem 5.2 to the section $x \mapsto \mathrm{d}\varphi_1(x)$ of the subbundle of conformally symplectic linear maps of TM, with associated conformal factor $\beta\colon x \mapsto e^{r_1(x)}$. We only need to prove that $b := \int_M \log|\beta| \mathrm{d}\nu$ equals $\bar{r}(\nu)$. By Fubini's theorem and the invariance of ν,

$$b = \int_M r_1 \mathrm{d}\nu = \int_M \int_0^1 \eta(X \circ \varphi_t(x))\,\mathrm{d}t\,\mathrm{d}\nu(x) = \int_0^1 \int_M \eta(X)\,\mathrm{d}\nu\,\mathrm{d}t = \bar{r}(\nu). \qquad \blacksquare$$

Let us remark that the fact that $F_1 \oplus \cdots \oplus F_{s-k}$ is the ω-orthogonal subspace of $F_1 \oplus \cdots \oplus F_k$ for every k implies that

$$F_k^{\omega} \cap F_{s-k+1} = 0, \quad \forall k \in \{1, \ldots, s\}, \tag{5.1}$$

ν-almost everywhere.

Corollary 5.4. *Let ν be an ergodic measure of a Hamiltonian flow (φ_t) of a closed conformal symplectic manifold M. There is a measurable sub-bundle F of the Lyapunov decomposition of TM which is transverse to $\mathcal{F}$ on which, for ν-almost every $x \in M$,*

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \|\mathrm{d}\varphi_t(x)v\| = \bar{r}(\nu), \quad \forall v \in F(x) \setminus \{0\}.$$

Proof. Let $\lambda_1 < \cdots < \lambda_s$ be the associated Lyapunov spectrum and let $F_1, \ldots, F_s$ be the associated decomposition of TM. Let X be the vector field of (φ_t). Since $\mathbb{R}X$ is invariant with $\mathrm{d}\varphi \cdot X = X \circ \varphi$, there is $k \in \{1, \ldots, s\}$ such that $\lambda_k = 0$ and $\mathbb{R}X \subset F_k$ ν-almost everywhere. Let us show that $F := F_{s-k+1}$ is the desired sub-bundle. According to Corollary 5.3, $\lambda_{s-k+1} = \bar{r}(\nu)$. According to (5.1), $\omega(X, v) \neq 0$ for some $v \in F \setminus \{0\}$ when $X \neq 0$. Since $\mathrm{d}_\eta H = \iota_X \omega$, the conclusion follows. $\blacksquare$

Let $r \in \{1, \ldots, s\}$ be the maximal integer such that $\lambda_r < 0$. According to the nonlinear ergodic theorem of Ruelle, see Theorem 6.3 in [10], for every $k \in \{1, \ldots, r\}$ and for ν-almost every $x \in M$, the set

$$V_k(x) := \left\{ y \in M \;\middle|\; \limsup_{t\to+\infty} \frac{1}{t} \log d(\varphi_t(x), \varphi_t(y)) \leq \lambda_k \right\},$$

where d denotes the Riemannian distance, is the image of $F_1(x) \oplus \cdots \oplus F_k(x)$ by a smooth injective immersion tangent to identity at x. Therefore, the last corollary implies the following statement.

Corollary 5.5. *Let ν be an ergodic measure of a Hamiltonian flow (φ_t) of a closed conformal symplectic manifold M such that $\bar{r}(\nu) < 0$ and such that* $\operatorname{supp}\nu$ *is included in a connected component Σ of $\{H = 0\}$ without critical point of H. For ν-almost every point x of Σ, there exists an immersed submanifold $V \subset M$ transverse to Σ and containing x such that*

$$\limsup_{t\to+\infty} \frac{1}{t} \log d(\varphi_t(x), \varphi_t(y)) \leq \bar{r}(\nu), \quad \forall y \in V.$$

5.2. Examples of isotropic attractors

5.2.1. Legendrian attractors. Contact Hamiltonian dynamical systems can provide examples of conformal dynamical systems by taking their lift to the conformal symplectization (which is closed if the contact manifold is closed).

Proposition 5.6. *Given any contact manifold, every closed Legendrian submanifold is a hyperbolic attractor* (*see Section* 3.1) *for some autonomous contact Hamiltonian flow.*

Proof. Let L be a closed Legendrian submanifold of a contact manifold. According to the contact Weinstein neighbourhood theorem, one can assume that L is the 0-section of $(T^*L \times \mathbb{R}, \mathrm{d}z - y\mathrm{d}x)$, with local coordinates $(x, y) \in T^*L$ and $z \in \mathbb{R}$ (see e.g. Corollary 2.5.9 and Example 2.5.11 in [4]). Given $H\colon T^*L \times \mathbb{R} \to \mathbb{R}$, the contact Hamilton equations (2.1) take the form

$$\begin{cases} \dot{x} = -\partial_y H, \\ \dot{y} = \partial_x H + y\partial_z H, \\ \dot{z} - y\dot{x} = H. \end{cases}$$

Choosing $H(x, y, z) = -z$, the flow is $\varphi_t(x, y, z) = (x, e^{-t}y, e^{-t}z)$. ∎

Let us give some explicit global examples. Let us first consider the standard contact sphere $(\mathbb{S}^{2n-1}, \frac{1}{2}(x\mathrm{d}y - y\mathrm{d}x))$. Since $(\mathbb{C}^n \setminus 0, \mathrm{d}x \wedge \mathrm{d}y)$ is the symplectization of the standard sphere, every contact Hamiltonian flow can be obtained in the following way: let $H\colon \mathbb{C}^n \setminus \{0\} \to \mathbb{R}$ be a positively 2-homogeneous Hamiltonian, the flow of which is (Φ_t). Then

$$\varphi_t(z) := \frac{\Phi_t(z)}{\|\Phi_t(z)\|}, \quad \forall z \in \mathbb{S}^{2n-1}, \forall t \in \mathbb{R},$$

defines a contact Hamiltonian flow of $\mathbb{S}^{2n-1}$. Let $H(x, y) := \frac{1}{2}(\|x\|^2 - \|y\|^2)$, so that $\Phi_t(x, y) = (\cosh(t)x + \sinh(t)y, \sinh(t)x + \cosh(t)y)$. The associated contact flow has one Legendrian attractor $L_+ := \{x = y\}$ and one Legendrian repeller $L_- := \{x = -y\}$, every point outside of them having its α-limit set inside L_- and its ω-limit set inside L_+.

Let us now consider a vector field X on some closed manifold M generating a flow (f_t). According to Section 2.3, this flow extends to a Hamiltonian flow $(\hat{f}_t)$ (identifying the 0-section with M) on T^*M which is fiberwise homogeneous: $\hat{f}_t(q, ap) = a\hat{f}_t(q, p)$, $\forall (q, p) \in T^*M$, $\forall a \in \mathbb{R}$. Let us endow M with a Riemannian metric; the flow $(\hat{f}_t)$ induces a contact Hamiltonian flow (φ_t) on the unit cotangent bundle $(S^*M, i^*\lambda)$ (λ being the Liouville form and $i\colon S^*M \hookrightarrow T^*M$ the inclusion) by

$$\varphi_t(q, p) := \frac{\hat{f}_t(q, p)}{\|\hat{f}_t(q, p)\|}, \quad \forall (q, p) \in S^*M.$$

A hyperbolically attracting (respectively, repelling) fixed point $x \in M$ of (f_t) corresponds to a normally hyperbolically attracting (respectively, repelling) Legendrian fiber S_x^*M of $(\hat{f}_t)$.

Let us remark that in both examples, one can directly work in the conformal symplectization by taking the flow induced by (Φ_t) (respectively, $(\hat{f}_t)$) on the quotient space $(\mathbb{C}^n \setminus \{0\})/(z \sim ez)$ (respectively, $(T^*M \setminus \{0\})/((q, p) \sim (q, ep))$), where $e := \exp(1)$.

5.2.2. Hyperbolic attractive and repulsive closed orbit in every non-symplectic manifold. Here, by a non-symplectic manifold, we mean a conformally symplectic manifold, the conformal structure of which is not $\sim (0,\omega)$.

Proposition 5.7. *Let (M,η,ω) be a conformally symplectic manifold and let $\gamma\colon S^1 \hookrightarrow M$ be an embedded loop such that $\int_\gamma \eta < 0$. There exists a Hamiltonian $H\colon M \to \mathbb{R}$ admitting γ as a hyperbolic attracting periodic orbit.*

In particular, every non-symplectic manifold admits a conformal Hamiltonian flow that has a hyperbolic attractive periodic orbit and a hyperbolic repulsive periodic orbit Hamiltonian.

Proof. Let us first remark that γ is included in an open Lagrangian submanifold. According to Corollary A.4 in Appendix A, using a cut-off function to define H globally, one can assume that $M = T^*_\beta L$ with γ included in the 0-section identified with L (in fact, $L \simeq \gamma \times \mathbb{R}^{n-1}$). By assumption, $r := \int_\gamma \beta < 0$. Let X be a complete vector field of L inducing a flow (f_t) for which γ is a 1-periodic hyperbolic orbit such that the eigenvalues μ of $\mathrm{d}f_1(\gamma(0))$ satisfy $e^r < \mu < 1$. Let $(\hat{f}_t)$ be the lifted Hamiltonian flow of $T^*_\beta L$ properly cut-off outside a neighbourhood of γ (see the end of Section 2.3). The differential of $\hat{f}_1$ at $\gamma(0)$ is equivalent to $\mathrm{d}f_1(\gamma(0)) \oplus e^r(\mathrm{d}f_1(\gamma(0)))^{-1}$, so its eigenvalues are in $(0,1)$. ∎

5.3. Connected components of $\{H = 0\}$ and attraction

Here we wonder if a connected component of $\{H=0\}$ can be an attractor. In Section 1.2.1, we gave a 2-dimensional example where a connected component of $\{H = 0\}$ is attractive. This is the only example that we know, and here we give conditions that ensure that such a component cannot be attractive. We will say that a subset Σ of M separates locally M in two connected components if in every neighbourhood V of Σ, there exists an open neighbourhood $U \subset V$ of Σ such that $U \setminus \Sigma$ has exactly two connected components. The manifold M being connected, we say that Σ separates globally M if $M \setminus \Sigma$ is not connected.

Proposition 5.8. *Let $H\colon M \to \mathbb{R}$ be a Hamiltonian map on a closed conformally symplectic manifold. Assume that Σ is an isolated connected component of $\{H = 0\}$ that separates globally and locally M in two connected components. Then Σ cannot be a strong attractor (see Section 3.1).*

Proof. Let us assume by contradiction that Σ satisfies the hypothesis of the statement and that it is a strong attractor. By definition, there exists an open neighbourhood U_0 of Σ such that $U_0 \cap \{H = 0\} = \Sigma$. As Σ separates locally M in two connected components, there exists a neighbourhood $U \subset U_0$ of Σ such that $U \setminus \Sigma$ has two connected components, U_- and U_+. We denote by $\varepsilon_\pm \in \{-1,1\}$ the sign of $H|_{U_\pm}$.

We know that $M \setminus \Sigma$ is not connected, and the boundary of each of its connected components intersects Σ, and thus contains U_- or U_+. This implies that $M \setminus \Sigma$ has exactly two connected components, M_- that contains U_-, and M_+ that contains U_+. We denote by $\varepsilon\colon M \setminus \Sigma \to \{-1,1\}$ the function such that $\varepsilon|_{M_\pm} = \varepsilon_\pm$.

We choose a smooth bump function $\chi\colon M \to [0,1]$ such that χ is equal to 1 in a neighbourhood of Σ and the support of χ is contained in U. Then let us define the Hamiltonian $K\colon M \to \mathbb{R}$ by $K = \chi H + (1-\chi)\varepsilon$. We have $\Sigma = \{K = 0\}$. The Hamiltonian flow of K coincides with the flow of H in a neighbourhood of Σ, and then Σ is also a strong attractor for (φ_t^K). If x is a generic point in the basin of attraction V of Σ for K but not in Σ, x is wandering. Observe that a wandering point is wandering for (φ_t^K) and (φ_t^{-K}). We deduce from Corollary 3.2 that $\lim_{t\to-\infty} K(\varphi_t^K(x)) = 0$, since x was taken generically. But as $\Sigma = \{K = 0\}$ is a strong attractor, this is not possible. Indeed, $x \notin \varphi_{t_0}^K(V)$ for some $t_0 > 0$, while $\varphi_\tau^K(x) \in \varphi_{t_0}^K(V)$ for some $\tau < 0$, so $\varphi_{-\tau}^K(\varphi_{t_0}^K V) \not\subset \varphi_{t_0}^K(V)$, a contradiction with $\varphi_{-\tau}^K(V) \subset V$. ∎

We do not know whether a similar statement is true without the separation assumption. We obtain the following result when we assume normal hyperbolic attraction.

Theorem 5.9. *Let us assume (M,η,ω) is a closed conformally symplectic manifold of dimension $2n \geq 4$, and let $H\colon M \to \mathbb{R}$ be a Hamiltonian map. Let Σ be a closed connected component of $\{H = 0\}$ without critical point of H. Then Σ cannot be hyperbolically normally attracting (see Section 3.1).*

The assumption on the dimension of M is crucial: a simple counterexample is discussed in Section 1.2.1 (see also Proposition 5.7).

Proof. Let us assume the hypothesis of the statement, except that M can have dimension 2. Let us moreover assume that Σ is hyperbolically normally attracting and reach the conclusion that $\dim M = 2n$ must be 2. Let us restrict ourselves to a neighbourhood of Σ. One can assume that $\Sigma = \{H = 0\}$ and that $M = \Sigma \times (-\varepsilon_0, \varepsilon_0)$ for some $\varepsilon_0 > 0$, with $H(x,y) = y$ for all $(x,y) \in \Sigma \times (-\varepsilon_0,\varepsilon_0)$, by a change of variables in a tubular neighbourhood of Σ. Let $V_\varepsilon := \Sigma \times (-\varepsilon, \varepsilon)$. Then Σ being normally hyperbolic means that one can assume that there exist $a \in (0,1)$ and $\tau > 0$ such that

$$\varphi_\tau(V_\varepsilon) \subset V_{a\varepsilon}, \quad \forall \varepsilon \in (0,\varepsilon_0). \tag{5.2}$$

According to Proposition 2.5 applied to the leaf Σ, there exists a volume form μ of Σ such that $\varphi_t^*\mu = e^{(n-1)r_t}\mu$, and which is the pull-back of a form μ_0 of M such that $\mu_0 \wedge \mathrm{d}y = \omega^n$ in the neighbourhood of Σ. Let $\pi\colon M \to \Sigma$ be the projection on the first factor. By decreasing ε_0, one can assume that $\pi^*\mu \wedge \mathrm{d}y$ does not vanish so that there exists a non-vanishing map $f\colon M \to \mathbb{R}_+$ such that $f\pi^*\mu \wedge \mathrm{d}y = \omega^n$. Since $\mu_0 \wedge \mathrm{d}y = \omega^n$ and μ_0 coincides with $\pi^*\mu$ on $T\Sigma$, $f|_\Sigma \equiv 1$. By (5.2),

$$\omega^n(\varphi_\tau(V_\varepsilon)) \leq \omega^n(V_{a\varepsilon}), \quad \forall \varepsilon \in (0,\varepsilon_0). \tag{5.3}$$

On the one hand,

$$\omega^n(V_\varepsilon) = \int_{x\in\Sigma} \Big(\int_{-\varepsilon}^{\varepsilon} f(x,y)\,\mathrm{d}y\Big)\mu_x \overset{\varepsilon\to 0}{\sim} 2\varepsilon \cdot \mu(\Sigma). \tag{5.4}$$

On the other hand,

$$\omega^n(\varphi_\tau(V_\varepsilon)) = \int_{V_\varepsilon} e^{nr_\tau}\,\omega^n \overset{\varepsilon\to 0}{\sim} 2\varepsilon \int_\Sigma e^{nr_\tau}\,\mu. \tag{5.5}$$

Therefore, (5.3), (5.4) and (5.5) imply

$$\int_{\Sigma} e^{nr_\tau}\mu \leq a\mu(\Sigma). \tag{5.6}$$

By Hölder's inequality,

$$\int_{\Sigma} e^{(n-1)r_\tau}\mu \leq \Big(\int_{\Sigma} \mathbf{1}^n\mu\Big)^{1/n}\Big(\int_{\Sigma} e^{nr_\tau}\mu\Big)^{(n-1)/n},$$

since $\varphi_\tau^*\mu = e^{(n-1)r_\tau}\mu$. It follows from (5.6) that

$$\mu(\Sigma) = \mu(\varphi_\tau(\Sigma)) \leq \mu(\Sigma)^{1/n}\, a^{(n-1)/n}\, \mu(\Sigma)^{(n-1)/n},$$

so $a^{(n-1)/n} \geq 1$. Since $a \in (0,1)$, this implies that $n = 1$, that is, $\dim M = 2$. ∎

6. Invariant distribution and submanifolds

Let us fix a Hamiltonian vector field X on (M, η, ω) associated to a Hamiltonian map H. In this section, we study properties related to its invariant distribution $\mathcal{F}$ introduced in Section 2.5.

6.1. Holonomy of embedded leaves of $\mathcal{F}$

Let us study the holonomy of a regular leaf F of $\mathcal{F}$. By definition, one can find an open neighbourhood U of F on which $\mathcal{F}$ defines a non-singular foliation. The holonomy of F is well defined as the holonomy of F in U for this foliation.

Let us recall the definition of the holonomy $\pi_1(F) \to G$ of a leaf F of a foliation $\mathcal{G}$ of codimension p on a manifold N^n. We refer to [5]. A distinguished map $f : V \to \mathbb{R}^p$ of $\mathcal{G}$ is a map on a trivialization neighbourhood $V \simeq \mathbb{R}^p \times \mathbb{R}^{n-p}$ that factors by the projection $\mathbb{R}^p \times \mathbb{R}^{n-p} \to \mathbb{R}^p$. Given a point $z \in F$, let G be the group of germs of local homeomorphisms of $\mathbb{R}^p$ fixing 0 defined up to internal automorphisms (i.e., up to conjugacy by such germs). Given a loop $\gamma : S^1 \to F$ based at z and a germ of distinguished map f sending z to 0, there is a unique continuous lift (f_t) of γ in the space of germs of distinguished maps such that $f_0 = f$ and f_t sends $\gamma(t)$ to 0. There exists a unique germ $g : \mathbb{R}^p \to \mathbb{R}^p$ fixing the origin such that $f_1 = g \circ f_0$. This germ only depends on f and the homotopy class of γ. If one takes another germ f' of distinguished map at z, the same procedure will give a germ $g' : \mathbb{R}^p \to \mathbb{R}^p$ fixing the origin that is conjugated to g. Therefore, one defines the holonomy of F (based at z) as the morphism $\pi_1(F, z) \to G$ sending the class of the loop γ to the class of the germ g. The holonomy group of F is the image of the holonomy. Up to isomorphisms, these notions do not depend on the base point z (a leaf being path-connected).

Proposition 6.1. *Let $H : M \to \mathbb{R}$ be a Hamiltonian map on a conformally symplectic manifold. The holonomy of an embedded leaf outside $\{H = 0\}$ is trivial. Let $\Sigma \subset \{H = 0\}$ be a connected component of $\{H = 0\}$ without critical point of H. The holonomy of Σ is*

$$[\gamma] \mapsto \big[y \mapsto e^{\int_\gamma \eta} y\big].$$

In particular, the holonomy group of Σ is isomorphic to the subgroup $\langle [\eta], \pi_1(\Sigma)\rangle$ of $\mathbb{R}$.

Proof. One can prove this result by considering the global distinguished map $e^{-\theta} H \circ p$ defined on the universal cover $p\colon \widetilde{M} \to M$ with $\mathrm{d}\theta = p^*\eta$. Let us give a more intrinsic proof.

If F is an embedded leaf outside $\{H = 0\}$, the pull-back of the Lee form η to F is exact according to Corollary 2.4, so $\eta = \mathrm{d}\theta$ on a tubular neighbourhood U of F. Therefore, $\mathcal{F}$ is trivially fibered by $e^{-\theta} H$ in U and the holonomy is thus trivial.

Let Σ be a connected component of $\{H = 0\}$ without critical point. Let $i\colon \Sigma \hookrightarrow M$ be the inclusion map. If $i^*\eta$ is exact, the holonomy is trivial, as above. Otherwise, let us fix $z \in \Sigma$ such that $(i^*\eta)_z \neq 0$, which implies that $\ker \eta_z$ is transverse to $T_z\Sigma$. Let $T \subset M$ be an open connected 1-dimensional manifold containing z and tangent to $\ker \eta$. By shrinking T, one can assume that H induces an isomorphism $H|_T\colon T \to (-\varepsilon, \varepsilon)$ sending z to 0, for some $\varepsilon > 0$. Let us remark that there exists a distinguished map f in the neighbourhood of z such that $f|_T = H|_T$. Indeed, in the neighbourhood of z, let θ be such that $\theta(z) = 0$ and $\mathrm{d}\theta = \eta$; then $f := e^{-\theta} H$ is a distinguished map. Since T is tangent to $\ker \eta$, $\theta|_T \simeq 0$, so that $f|_T = H|_T$.

Let $\gamma\colon [0,1] \to \Sigma$ be a smooth loop based at z. According to [5], Section 2.5, for every $x \in W$ a connected neighbourhood of z in T, there are smooth paths $\gamma_x\colon [0,1] \to M$ tangent to $\mathcal{F}$ and C^0-close to γ such that $\gamma_x(0) = x$, $\gamma_x(1) \in T$, and the image of the holonomy $\pi_1(\Sigma, z) \to G$ at $[\gamma]$ is the class of the germ

$$y \mapsto H\big(\gamma_{H|_T^{-1}(y)}(1)\big),$$

(here we used that $H|_T = f|_T$, where f is a distinguished map). According to Lemma 2.3,

$$H\big(\gamma_{H|_T^{-1}(y)}(1)\big) = e^{\int_{\gamma_x} \eta} y, \quad \text{with} \quad x := H|_T^{-1}(y).$$

Since T is tangent to $\ker \eta$, by concatenating the path γ_x with the image of the segment $[H(\gamma_x(0)), H(\gamma_x(1))]$ under $H|_T^{-1}$, one gets a loop $\tilde{\gamma}_x$ such that $\int_{\tilde{\gamma}_x} \eta = \int_{\gamma_x} \eta$. Since γ_x is C^0-close to γ, one can reparametrize $\tilde{\gamma}_x$ such that this loop is C^0-close to γ, so $\tilde{\gamma}_x$ is homotopic to γ and the conclusion follows. ∎

A consequence is the following (see Section 2.5 in [5]).

Corollary 6.2. *Let $H\colon M \to \mathbb{R}$ be a Hamiltonian map on a closed conformally symplectic manifold. Let $\Sigma \subset \{H = 0\}$ be a connected component of $\{H = 0\}$ without critical point of H. If the pull-back of the Lee form to Σ is not trivial, there exist leaves of $\mathcal{F}$ different from Σ, the closure of which contains Σ.*

Examples 1.2.2 and 4.3 show that a non-compact leaf can go far away from $\{H = 0\}$.

6.2. Invariance and isotropy

Proposition 6.3. *Let $H\colon M \to \mathbb{R}$ be a Hamiltonian map on a conformally symplectic manifold. Let L be a submanifold of M.*

(1) *If L is isotropic and invariant, then it is tangent to $\mathcal{F}$.*
(2) *If L is coisotropic and tangent to $\mathcal{F}$, then it is invariant.*

(3) *If L is invariant and tangent to $\mathcal{F}$, then the pull-back of ω to L is degenerate. In particular, if L is of even dimension $2k$, the pull-back of ω^k to L is zero. An invariant surface tangent to $\mathcal{F}$ is thus isotropic.*

Proof. If L is an invariant isotropic submanifold, then the Hamiltonian vector field X is tangent to L, so $\forall v \in TL$, $\mathrm{d}_\eta H \cdot v = \omega(X, v) = 0$. Conversely, if L is coisotropic and tangent to $\mathcal{F}$, for $x \in L$, $T_xL \supset (T_xL)^\omega \supset (\mathcal{F}_x)^\omega = \mathbb{R}X(x)$ so L is invariant. If $i: L \hookrightarrow M$ is invariant and tangent to $\mathcal{F}$, then $X|_L$ is in the kernel of $i^*\omega$. ∎

Combining Proposition 6.3 and Corollary 2.4, one gets the following result.

Corollary 6.4. *Let $H: M \to \mathbb{R}$ be a Hamiltonian map on a closed conformally symplectic manifold. An isotropic invariant submanifold on which the pull-back of the Lee form is not exact is included in $\{H = 0\}$.*

Corollary 6.4 can be applied to Lagrangian graphs of $T^*_\beta Q$ for a non-exact closed 1-form β of Q. Indeed, for every β-closed 1-form α of Q, $q \mapsto \alpha_q$ defines a Lagrangian section $Q \hookrightarrow T^*_\beta Q$ pulling back the Lee form to the non-exact form β.

We are now interested in how the dynamics can force the isotropy.

Following [11], we recall that a point $x \in M$ is *quasi-regular* if for every continuous map $f: M \to \mathbb{R}$, the following limit exists:

$$\lim_{t\to+\infty} \frac{1}{t} \int_0^t f(\varphi_s x)\,\mathrm{d}s.$$

Then we can associate to every quasi-regular point its *asymptotic cycle* $A(x) \in H_1(M, \mathbb{R})$, which satisfies that, for every continuous closed 1-form ν on M,

$$\langle [\nu], A(x)\rangle = \lim_{t\to\infty} \frac{1}{t} \int_0^t \nu(X_H \circ \varphi_s(x))\,\mathrm{d}s.$$

Moreover, if μ is an invariant Borel probability by (φ_t), μ almost every point is quasi-regular and the asymptotic cycle $A(\mu) \in H_1(M, \mathbb{R})$ of μ is defined by

$$\langle [\nu], A(\mu)\rangle = \int \langle [\nu], A(x)\rangle\,\mathrm{d}\mu(x).$$

We have the following well-known fact.

Proposition 6.5. *Let $(R_{t\alpha})_{t\in\mathbb{R}}$ be the flow of rotations of $\mathbb{T}^n$ with vector $\alpha \in \mathbb{R}^n$ that is defined by*

$$R_{t\alpha}(\theta) = \theta + t\alpha.$$

We identify $H_1(\mathbb{T}^n)$ with $\mathbb{R}^n$ in the usual way. Then every point of $\mathbb{T}^n$ is quasi-regular, and the asymptotic cycle of every point of $\mathbb{T}^n$ and of every invariant probability measure is α.

Observe that when two flows $(f_t): M \to M$ and $(g_t): N \to N$ are conjugated via some homeomorphism $h: M \to N$, then the quasi-regular points of (g_t) are the h-images of the quasi-regular points of (f_t), and that when $x \in M$ is quasi-regular, we have

$$h_* A(x) = A(h(x)).$$

This allows us to introduce a notion of *rotational torus* $\mathcal{T}$ for a flow $(f_t): M \to M$. A rotational torus is a C^0-embedded torus $j: \mathbb{T}^m \hookrightarrow M$ such that $(j^{-1} \circ f_t \circ j)$ is a flow of rotation. When j is a C^1-embedding, $\mathcal{T}$ is a C^1-rotational torus. Thanks to Proposition 6.5, all the points of a rotational torus are quasi-regular with the same asymptotic cycle, that we denote by $A(\mathcal{T}) \in H_1(M)$, and every measure with support in $\mathcal{T}$ also has the same asymptotic cycle.

Let us prove a result that is reminiscent of a result of Herman in the symplectic setting [6].

Proposition 6.6. *Assume that $j(\mathbb{T}^m) = \mathcal{T}$ is a C^1-rotational torus for a conformal Hamiltonian flow (φ_t) of (M, η, ω). Then*

- *if the flow restricted to $\mathcal{T}$ is minimal, $\omega = \mathrm{d}_\eta \lambda$ is η-exact and $j^*\eta$ is exact, then $\mathcal{T}$ is isotropic*;
- *if $\mathcal{T}$ is not isotropic, then $\langle [\eta], A(\mathcal{T})\rangle = 0$.*

In particular, if the cohomological class of η is rational and nonzero and if the flow restricted to $\mathcal{T}$ is minimal, then $\mathcal{T}$ is isotropic.

Proof. We use the notation $R_{t\alpha} = j^{-1} \circ \varphi_t \circ j$.

Let us prove the first point. As $j^*\eta = \mathrm{d}f$ is exact, we have

$$\mathrm{d}(e^{-f} j^*\lambda) = e^{-f}(j^*\mathrm{d}\lambda - \mathrm{d}f \wedge j^*\lambda) = e^{-f} j^*(\mathrm{d}_\eta\lambda) = e^{-f} j^*\omega.$$

Hence $e^{-f} j^*\omega$ is exact. Observe that $j^*X_H = \alpha$. Hence $\forall x \in \mathbb{T}^m$, $\forall t \in \mathbb{R}$,

$$r_t(j(x)) = \int_0^t \eta(\varphi_s(j(x)))X_H(\varphi_s(j(x)))\,\mathrm{d}s = \int_0^t \mathrm{d}f(R_{s\alpha}(x))\alpha\,\mathrm{d}s = f(x+t\alpha) - f(x).$$

Because $(\varphi_t^H)^*\omega = e^{r_t}\omega$ and $\varphi_t^H \circ j = j \circ R_{t\alpha}$, we deduce

$$R_{t\alpha}^*(j^*\omega) = e^{r_t \circ j} j^*\omega,$$

and then

$$R_{t\alpha}^*(e^{-f} j^*\omega) = e^{-f} j^*\omega.$$

If we write

$$e^{-f} j^*\omega = \sum_{1 \le i < j \le m} a_{i,j}\,\mathrm{d}x_i \wedge \mathrm{d}x_j,$$

we deduce that every continuous function $a_{i,j}$ is invariant by $(R_{t\alpha})$, and then constant because the flow is minimal. The form $e^{-f} j^*\omega$ is constant and exact, it is then the zero form and $\mathcal{T}$ is isotropic.

Let us prove the second point. We assume that $\mathcal{T}$ is not isotropic. Hence $j^*\omega$ is not zero. There exists a sequence (t_n) of real numbers that tends to $+\infty$ and satisfies

$$\lim_{n\to\infty} t_n\alpha = 0 \quad \text{in } \mathbb{T}^m.$$

Then $(R_{t_n\alpha})$ tends to $\mathrm{id}_{\mathbb{T}^m}$ in C^1-topology. We deduce that $(R_{t_n\alpha}^*(j^*\omega))$ tends to $j^*\omega$. Moreover, we also have

$$R_{t_n\alpha}^*(j^*\omega) = e^{r_{t_n} \circ j} j^*\omega,$$

where, as $\varphi_s \circ j = j \circ R_{s\alpha}$ (so $X \circ j = \mathrm{d}j \cdot \alpha$),

$$r_{t_n} \circ j(x) = \int_0^{t_n} \eta(\mathrm{d}j(R_{s\alpha}(x))\alpha)\,\mathrm{d}s = t_n\langle [j^*\eta], \alpha\rangle + o(1).$$

As $j^*\omega$ is not the zero form, we have then $0 = \lim_{n\to\infty} r_{t_n}(x)$, which implies that $[j^*\eta]$ is orthogonal to α, i.e., $\langle [\eta], A(\mathcal{T})\rangle = 0$. ■

If we assume that the invariant torus is C^3, we can relax the hypothesis on the dynamics for the second point of Proposition 6.6, asking only a C^0-conjugacy. The main argument that we use is very similar to part 3 of [1].

Proposition 6.7. *Assume that $\mathcal{T}$ is a C^3-submanifold of a conformally symplectic manifold (M, η, ω) that is a C^0-rotational torus of a Hamiltonian flow (φ_t) of (M, η, ω) such that $\langle [\eta], A(\mathcal{T})\rangle \neq 0$. Then $\mathcal{T}$ is isotropic.*

In particular, if the cohomological class of η is rational and nonzero and if the flow restricted to $\mathcal{T}$ is minimal, then $\mathcal{T}$ is isotropic.

Proof of Proposition 6.7. We denote the canonical injection $\mathcal{T} \hookrightarrow M$ by j. We endow $\mathcal{T}$ with a Riemannian metric, and denote by $d_{\mathcal{G}}$ the distance along the leaves of the characteristic foliation $\mathcal{G}$ of $j^*\omega$. We assume that $\mathcal{T}$ is not isotropic. We denote the maximum rank of $j^*\omega$ by r and by U the open subset of $\mathcal{T}$

$$U = \{x \in \mathcal{T} \mid \operatorname{rank}(j^*\omega(x)) = r\}.$$

As $\varphi_t^*(j^*\omega) = e^{r_t} j^*\omega$, this set is invariant by the flow.

A result of Proposition 6.5 is

$$r_n(x) = n\left(\langle [\eta], A(\mathcal{T})\rangle + o_{n\to\infty}(1)\right), \quad \forall x \in \mathcal{T}. \tag{6.1}$$

There are two cases:

- either $U = \mathcal{T}$ is compact; we choose $x \in U$ and $\mathcal{K} := U$;
- or $U \neq \mathcal{T}$. Then the closure of the orbit of a fixed point $x \in U$ is homeomorphic to a torus with dimension $k < m$. As $(\varphi_t|_{\mathcal{T}})$ is conjugate to a flow of rotation, there exists a compact invariant neighbourhood $\mathcal{K}$ of x in U that is homeomorphic to $\mathbb{T}^k \times [-1, 1]^{m-k}$.

In $\mathcal{K}$, we consider the characteristic foliation $\mathcal{G}$ of ω. We now follow the arguments and notation of [1] (except that $\mathcal{F}$ and the $\mathcal{F}_i$'s are here denoted $\mathcal{G}$ and $\mathcal{G}_i$). Our goal is to reach a contradiction by proving that the topological entropy of $(\varphi_t|_{\mathcal{T}})$ is positive. We use a finite covering of $\mathcal{K}$ by foliated charts $\mathcal{W}_1, \dots, \mathcal{W}_I$ in $\mathcal{U}$ and denote by $\mathcal{G}_i$ the foliation restricted to $\mathcal{W}_i$. Then there exists a constant $\mu > 0$ such that every $(m-r)$-submanifold $\mathcal{S}$ of $\mathcal{W}_i$ that intersects every leaf of $\mathcal{G}_i$ at most once satisfies $|\omega^{r/2}(\mathcal{S})| \leq \mu$.

Moreover, we may assume that there exists $\varepsilon > 0$ such that

(i) if x, y are in some $\mathcal{W}_i$, and such that $d_{\mathcal{G}}(x, y) < \varepsilon$, then x and y are in the same leaf of $\mathcal{W}_i$,

where $d_{\mathcal{G}}$ is the distance along the leaves.

We also have the existence of $\nu \in (0, \varepsilon)$ such that

$$d_{\mathcal{g}}(x,y) < \nu \;\Rightarrow\; d_{\mathcal{g}}(\varphi_{-1}(x), \varphi_{-1}(y)) < \varepsilon, \quad \forall x, y \in \mathcal{K}. \tag{6.2}$$

We then use a decomposition $(Q_j)_{1 \le j \le J}$ of $\mathcal{K}$ into submanifolds with corners that may intersect only along their boundary such that every Q_j is contained in at least one $\mathcal{W}_i$ that satisfies:

(ii) if $Q_j \subset \mathcal{W}_i$, then if $x, y \in Q_j$ are in the same leaf of $\mathcal{W}_i$, we have $d_{\mathcal{g}}(x,y) < \nu$.

If $\mathcal{S}$ is a piece of r-dimensional submanifold contained in some $Q_{j_0} \subset \mathcal{W}_{i_0}$ that is transverse to $\mathcal{G}$ and intersects every leaf of $\mathcal{G}_{i_0}$ at most once, let us consider $\mathcal{S}' = \varphi_1(\mathcal{S} \cap Q_{j_0}) \cap Q_{j_1}$ for some j_1. Then $\mathcal{S}'$ is also transverse to $\mathcal{G}$. Let $\mathcal{W}_{i_1}$ that contains Q_{j_1}, and let us assume that $x, y \in \mathcal{S}'$ are in a same leaf of $\mathcal{G}_{i_1}$. Because of (ii) and (6.2), $d_{\mathcal{g}}(\varphi_{-1}(x), \varphi_{-1}(y)) < \varepsilon$, and by (i), we have $\varphi_{-1}(x) = \varphi_{-1}(y)$ and $x = y$. Iterating this argument, we deduce that all the sets

$$\mathcal{S}' = \varphi_k(\mathcal{S} \cap Q_{j_0}) \cap \varphi_{k-1}(Q_{j_1}) \cap \cdots \cap Q_{j_k}$$

are such that if $Q_{j_k} \subset \mathcal{W}_{i_k}$, then $\mathcal{S}'$ intersects every leaf of $\mathcal{F}_{i_k}$ at most once, and thus $|\omega^{r/2}(\mathcal{S}')| \le \mu$.

If now N_k is the number of k-uples $(j_1, \dots, j_k)$ such that

$$\varphi_k(\mathcal{S} \cap Q_{j_0}) \cap \varphi_{k-1}(Q_{j_1}) \cap \cdots \cap Q_{j_k} \neq \emptyset,$$

then we have

$$|\omega^{r/2}(\varphi_k(\mathcal{S}))| \le N_k \mu. \tag{6.3}$$

By (6.1), we have

$$\omega^{r/2}(\varphi_k(\mathcal{S})) = \exp\left(k\,\frac{r}{2}\left(\langle [\eta], A(\mathcal{T})\rangle + o_{k\to\infty}(1)\right)\right)\omega^{r/2}(\mathcal{S}). \tag{6.4}$$

Combining (6.3) and (6.4), we deduce that

$$\limsup_{k\to\infty} \frac{1}{k} \log(N_k) \ge |\langle [\eta], A(\mathcal{T})\rangle| > 0$$

is a lower bound for the topological entropy of $(\varphi_t|_{\mathcal{T}})$. But this contradicts the fact that $(\varphi_t|_{\mathcal{T}})$ is C^0-conjugate to a flow of rotation and has zero entropy. ∎

A. Isotropic submanifolds

A.1. Isotropic embeddings

Lemma A.1. *Given a manifold N endowed with a closed 1-form β, there exists a Legendrian embedding of N in a contact manifold (V, ξ) endowed with a closed 1-form, the pull-back to N of which is β. This contact manifold may be chosen closed if N is closed.*

Proof. Given a submanifold M' of a Riemannian manifold (M, g), we denote $\nu^1 M' \subset T^1 M$ its unit normal bundle. Let us endow N and S^1 with Riemannian metrics and let us consider unit tangent bundle V of the product Riemannian manifold $(N \times S^1, g)$ endowed with its standard contact form. Let us recall that unit normal bundles of submanifolds of $N \times S^1$ are Legendrian submanifolds of V. Therefore, $\nu^1(N \times \{x\})$, $x \in S^1$ fixed, is a Legendrian submanifold, it is the disjoint union of two copies of N. We lift the closed 1-form β to V by pulling it back by the canonical projection $T^1(N \times S^1) \to N$. ∎

Lemma A.2. *Given a manifold N endowed with a closed 1-form β, there exists a Lagrangian embedding of $N \times S^1$ in a conformal exact symplectic manifold (M, η, ω) such that the pull-back of η to $N \times S^1$ is $\pi^*\beta - \mathrm{d}\theta$, where π is the projection on the first factor. This manifold M may be chosen closed if N is closed.*

Proof. Let (V, ξ) be a contact manifold as in Lemma A.1. We assume that $N \subset V$ and identify β with its extension to V provided by the same lemma. Then the β-twisted symplectization (M, η, ω) of V endowed with its standard Lee form satisfies the statement. ∎

A.2. Weinstein neighbourhood of isotropic submanifolds

We explicitly extend the usual Weinstein neighbourhood theorem for isotropic submanifolds ([13], Lecture 5) to the conformal setting following and adapting Section 2.5.2 of [4]. It can be seen as a specialization of the Darboux–Weinstein theorem proven by Otiman–Stanciu [9] in the conformally symplectic case (see Theorem A.5 below). The special case of Lagrangian submanifolds was already treated by Otiman–Stanciu, see Theorem 3.2 in [9] (see also Theorem 2.11 in [2]). The coisotropic analogue of this Weinstein neighbourhood theorem in the conformally symplectic setting has been studied by Lê–Oh in Section 4 of [8].

Let $Q^{(k)} \subset (M^{(2n)}, \eta, \omega)$ be an isotropic submanifold. Let us denote $T_Q M$ the restriction of the tangent bundle of M to Q and $TQ^\omega \subset T_Q M$ the ω-orthogonal bundle of TQ. Then the normal bundle $\pi \colon \nu Q \to Q$ can be non-canonically decomposed as

$$\nu Q = T_Q M / TQ \simeq TQ^\omega / TQ \oplus T_Q M / TQ^\omega. \tag{A.1}$$

In order to fix this decomposition, it can be useful to fix a complex structure J compatible with ω, i.e., such that $g := \omega(\cdot, J\cdot)$ defines a Riemannian metric. With respect to g, νQ is canonically isomorphic to the orthogonal vector bundle $TQ^\perp$, TQ^ω/TQ is isomorphic to $(TQ \oplus J(TQ))^\omega$, and $T_Q M/TQ^\omega$ to $J(TQ)$, so that the decomposition (A.1) takes the concrete form

$$TQ^\perp = (TQ \oplus J(TQ))^\omega \oplus J(TQ).$$

The fiber bundle $T_Q M/TQ^\omega$ is diffeomorphic to T^*Q under $(q, [v]) \mapsto \omega_q(v, \cdot)$. The fiber bundle

$$SN_{(M,\omega)}(Q) := TQ^\omega / TQ$$

is symplectic of rank $2(n-k)$ for the structure induced by ω, it is called the symplectic normal bundle of Q and depends on the isotropic embedding $Q \hookrightarrow (M, \omega)$. The Weinstein isotropic neighbourhood theorem asserts that the locally conformally symplectic structure of a small neighbourhood of the isotropic embedding of Q only depends on its symplectic normal bundle and the pull-back of the Lee form.

Theorem A.3 (Weinstein isotropic neighbourhood). *Let $Q_i \subset (M_i, \eta_i, \omega_i)$, $i \in \{1,2\}$, be closed isotropic submanifolds of conformally symplectic manifolds. Suppose there exists an isomorphism of symplectic bundles $\Phi\colon SN_{(M_1,\omega_1)}(Q_1) \xrightarrow{\simeq} SN_{(M_2,\omega_2)}(Q_2)$ covering a diffeomorphism $\phi\colon Q_1 \xrightarrow{\simeq} Q_2$ satisfying $\phi^*\eta_2 = \eta_1$ (by a slight abuse of notation, $\eta_i \in \Omega^1(Q_i)$ denotes the pull-back of $\eta_i \in \Omega^1(M_i)$). Then ϕ extends to a conformal symplectomorphism $\psi\colon U_1 \xrightarrow{\simeq} U_2$ defined on suitable neighbourhoods of Q_1 and Q_2 and such that the symplectic normal bundle isomorphism induced by* $\mathrm{d}\psi$ *along Q_1 is* (*symplectic*) *bundle homotopic to* Φ.

This implies the Weinstein Lagrangian neighbourhood theorem proven by Otiman–Stanciu (Theorem 3.2 in [9]): as the normal symplectic bundle of any Lagrangian submanifold L is of rank 0, the neighbourhood structure only depends on the pull-back β of the Lee form and one can take $T^*_\beta L$ as a local model. In this article, we are also interested in the following consequence.

Corollary A.4. *Let $Q \subset (M^{(2n)}, \eta, \omega)$ be a loop embedded in a conformally symplectic manifold, and denote by $\beta \in \Omega^1(Q)$ the pull-back of η. Then there exists a neighbourhood of Q that is conformally symplectomorphic to a neighbourhood of the subset $Q \times \{0\}$ of the zero-section $Q \times \mathbb{R}^{n-1}$ inside the cotangent bundle $T^*_{\beta\oplus 0}(Q \times \mathbb{R}^{n-1})$, through a symplectomorphism identifying Q with $Q \times \{0\}$.*

Proof of Corollary A.4. Since the group of symplectic matrices is connected in every dimension, any symplectic bundle over a loop is trivial. Therefore the requirements to apply Theorem A.3 to $Q \subset M$ and $Q \times \{0\} \subset T^*_{\beta\oplus 0}(Q \times \mathbb{R}^{n-1})$ are fulfilled. ∎

The proof is an adaptation of the symplectic case. We will use the following conformally symplectic version of the Darboux–Weinstein theorem proven by Otiman–Stanciu.

Theorem A.5 (Theorem 1.3 in [9]). *Let (M, η) be a manifold endowed with a closed 1-form and let $Q \subset M$ be a compact submanifold. Let us assume that there exist two η-conformal symplectic forms ω_0 and ω_1 agreeing on T_qM for all $q \in Q$. There exist two neighbourhoods U_0 and U_1 of Q, a diffeomorphism $\psi\colon U_0 \to U_1$ and a map $f\colon U_0 \to \mathbb{R}$ vanishing on Q such that $\psi|_Q = \mathrm{id}$, $\psi^*\omega_1 = e^f\omega_0$ and $\psi^*\eta = \eta + \mathrm{d}f$.*

Corollary A.6. *Let $Q_i \subset (M_i, \omega_i, \eta_i)$, $i \in \{1,2\}$, be closed submanifolds of locally conformally symplectic manifolds. Suppose there exists an isomorphism of symplectic bundles $\Phi\colon (T_{Q_1}M_1, \omega_1) \xrightarrow{\simeq} (T_{Q_2}M_2, \omega_2)$ covering a diffeomorphism $\phi\colon Q_1 \xrightarrow{\simeq} Q_2$ satisfying $\phi^*\eta_2 = \eta_1$ (and extending the bundle isomorphism $\mathrm{d}\phi\colon TQ_1 \xrightarrow{\simeq} TQ_2$). Then ϕ extends to a conformal symplectomorphism $\psi\colon U_1 \xrightarrow{\simeq} U_2$ defined on suitable neighbourhoods of Q_1 and Q_2 and such that the symplectic normal bundle isomorphism induced by* $\mathrm{d}\psi$ *along Q_1 is* (*symplectic*) *bundle homotopic to* Φ.

Proof of Corollary A.6. Let arbitrarily extend ϕ to a diffeomorphism $\tilde{\phi}\colon V_1 \xrightarrow{\simeq} V_2$ between tubular neighbourhoods of Q_1 and Q_2 in such a way that $\mathrm{d}\tilde{\phi}$ coincide with Φ on $T_{Q_1}M_1$ (such an extension can be found using auxiliary Riemannian metrics). Since V_1 retracts on Q_1, the closed 1-form $\eta' := \tilde{\phi}^*\eta_2$, that coincides with η_1 on Q_1, is cohomologous to η_1: $\eta' = \eta_1 + \mathrm{d}f$ for some $f\colon V_1 \to \mathbb{R}$. Treating each component separately, one can assume Q_1 to be connected, then $f|_{Q_1}$ is constant, so one can assume that f vanishes on Q_1. Then $\omega' := \tilde{\phi}^*\omega_2$ being η'-exact implies that $\omega := e^{-f}\omega'$ is η_1-exact. One can now conclude by applying Theorem A.5 on (M_1, η_1) with the two conformal forms ω_1 and ω which agree on T_qM_1 for $q \in Q_1$ by assumption (and since $\omega'_q = \omega_q$). The bundle homotopy asserted in the statement comes from the construction of ψ as the time 1 of an isotopy in the proof of Theorem A.5 (see Section 2 of [9]). ∎

Proof of Theorem A.3. Let us fix compatible almost complex structures on the M_i's and their induced compatible Riemannian metric. As explained in the beginning of the section, we then have a canonical bundle decomposition of $\nu Q_i \simeq TQ_i^{\perp}$ as $SN_{(M_i,\omega_i)}(Q_i) \oplus T^*Q_i$, so that the symplectic bundles $T_{Q_i}M_i$ split into two symplectic subbundles

$$T_{Q_i}M_i \simeq SN_{(M_i,\omega_i)}(Q_i) \oplus (TQ_i \oplus T^*Q_i).$$

Now the symplectic bundle isomorphism Φ covering ϕ extends to a symplectic isomorphism $\widetilde{\Phi}\colon T_{Q_1}M_1 \xrightarrow{\simeq} T_{Q_2}M_2$ covering ϕ by taking the direct sum of Φ with $(q; v, p) \mapsto (\phi(q); \mathrm{d}\phi \cdot v, p \circ (\mathrm{d}\phi)^{-1})$. One can then apply Corollary A.6 in order to conclude. ∎

Acknowledgements. The authors are grateful to Ana Rechtman for listening some preliminary versions of this work, discussing them and pointing out the link with classical results on foliations. Both authors also thank the anonymous referees for their numerous suggestions aiming at improving this article.

Funding. This work was partially supported by the project ANR-CE40-0014 CoSyDy: Conformally symplectic dynamics, beyond symplectic dynamics. The first author was supported by the postdoctoral fellowship of the Fondation Sciences Mathématiques de Paris.

References

[1] Arnaud, M.-C. and Fejoz, J.: Invariant submanifolds of conformal symplectic dynamics. *J. Éc. polytech. Math.* **11** (2024), 159–185.

[2] Chantraine, B. and Murphy, E.: Conformal symplectic geometry of cotangent bundles. *J. Symplectic Geom.* **17** (2019), no. 3, 639–661.

[3] Ekeland, I.: *Convexity methods in Hamiltonian mechanics*. Ergeb. Math. Grenzgeb. (3) 19, Springer, Berlin, 1990.

[4] Geiges, H.: *An introduction to contact topology*. Cambridge Stud. Adv. Math. 109, Cambridge University Press, Cambridge, 2008.

[5] Haefliger, A.: Variétés feuilletées. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)* **16** (1962), no. 4, 367–397.

[6] Herman, M.-R.: Inégalités "a priori" pour des tores lagrangiens invariants par des difféomorphismes symplectiques. *Inst. Hautes Études Sci. Publ. Math.* **70** (1989), 47–101.

[7] Katok, A. and Hasselblatt, B.: *Introduction to the modern theory of dynamical systems*. Encyclopedia Math. Appl. 54, Cambridge University Press, Cambridge, 1995.

[8] Lê, H. V. and Oh, Y.-G.: Deformations of coisotropic submanifolds in locally conformal symplectic manifolds. *Asian J. Math.* **20** (2016), no. 3, 553–596.

[9] Otiman, A. and Stanciu, M.: Darboux–Weinstein theorem for locally conformally symplectic manifolds. *J. Geom. Phys.* **111** (2017), 1–5.

[10] Ruelle, D.: Ergodic theory of differentiable dynamical systems. *Inst. Hautes Études Sci. Publ. Math.* **50** (1979), 27–58.

[11] Schwartzman, S.: Asymptotic cycles. *Ann. of Math. (2)* **66** (1957), no. 2, 270–284.

[12] Vaisman, I.: On locally conformal almost Kähler manifolds. *Israel J. Math.* **24** (1976), no. 3-4, 338–351.

[13] Weinstein, A.: *Lectures on symplectic manifolds*. Regional Conference Series in Mathematics 29, American Mathematical Society, Providence, RI, 1977.

[14] Wojtkowski, M. P. and Liverani, C.: Conformally symplectic dynamics and symmetry of the Lyapunov spectrum. *Comm. Math. Phys.* **194** (1998), no. 1, 47–60.

Received December 18, 2022; revised August 27, 2023.

Simon Allais
CNRS, IMJ-PRG, Université Paris Cité and Sorbonne Université
750013 Paris;
current affiliation: IRMA, UMR 7501, Université de Strasbourg
7 rue René-Descartes, 67084 Strasbourg, France;
simon.allais@math.unistra.fr

Marie-Claude Arnaud
CNRS, IMJ-PRG, Université Paris Cité and Sorbonne Université
Bâtiment Sophie Germain Case 7012, 75205 Paris, France;
marie-claude.arnaud@imj-prg.fr