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A four-dimensional cousin of the Segre cubic

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Abstract. This note is devoted to a special Fano fourfold defined by a four-dimensional space of skew-symmetric forms in five variables. This fourfold appears to be closely related with the classical Segre cubic and its Cremona–Richmond configuration of planes. Among other exceptional properties, it is infinitesimally rigid and has Picard number six. We show how to construct it by blow-up and contraction, starting from a configuration of five planes in a four-dimensional quadric, compatibly with the symmetry group $\mathcal{S}_5$. From this construction, we are able to describe the Chow ring explicitly.

Dedicated to the memory of Laurent Gruson

1. Introduction

Fano threefolds were classified more than fourty years ago, after some fifty years of efforts. The classification of Fano fourfolds is still elusive and will probably remain so for a long time. There are many ways to construct such manifolds, and a systematic study was launched a few years ago, of those that can be constructed from vector bundles on products of Grassmannians and more general flag manifolds [6]; a sample has already appeared in [5]. In this database, there is a unique fourfold with maximal Picard number, equal to six: the study of this fourfold is the object of this note.

This study turned out to be related with interesting questions at the intersection of algebraic geometry with Lie theory. Consider two complex vector spaces V_4 and V_5, of dimensions four and five, respectively. The action of $\mathrm{GL}(V_4) \times \mathrm{GL}(V_5)$ on $V_4^\vee \otimes \wedge^2 V_5^\vee$ is known to be prehomogeneous, its open orbit being the complement of a degree 40 hypersurface, see [25], p. 98. It is in fact one of the most complicated prehomogeneous spaces, containing no less than 63 distinct orbits [10, 24]. An important literature has been devoted to this prehomogeneous space, including some in connections with quintic field extensions, in the spirit of Bhargava's work on higher reciprocity laws [7, 17, 18].

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The Fano fourfold X_4 we are interested in is defined by a generic element of the prehomogeneous space $V_4^\vee \otimes \wedge^2 V_5^\vee$. It has two natural projections to $G(2,4)$ and to the six-dimensional $G(3,5)$ that we describe in some detail in Section 4. In particular, we show it is a small resolution of a fourfold with ten singular points which appears to be a cousin, or a big brother of the Segre cubic primal; this small resolution contracts ten planes which can be seen as a special subcollection of the classical Cremona–Richmond configuration. We deduce the following.

Theorem. *Consider five general planes in one of the two families of projective planes in a smooth four-dimensional quadric Q_4. They intersect pairwise in ten points. Blow-up these ten points and then the strict transforms of the five planes. Then the strict transforms of the exceptional divisors of the first blow-up can be contracted to a smooth Fano fourfold, which is precisely X_4.*

Then we show that the automorphism group is $\mathrm{Aut}(X_4) = \mathcal{S}_5$, so that

$$\mathrm{Pic}(X_4)^{\mathcal{S}_5} \simeq \mathbb{Z}^2$$

is generated by the pull-back of the hyperplane classes by the two projections. This suggests to construct the tensor that defines X_4 by *reverse-engineering*, starting from the representation theory of $\mathcal{S}_5$; we show how this leads to a normal form from this tensor. We then use the previous constructions to describe the Chow ring of X_4 completely, including the action of $\mathcal{S}_5$. We also check that X_4, as expected, is infinitesimally rigid.

This study can be considered as a warm-up for the more mysterious case of $U_5^\vee \otimes \wedge^2 V_5^\vee$, directly related to E_8, which has infinitely many but well-described orbits for the action of $\mathrm{GL}(U_5) \times \mathrm{GL}(V_5)$ (see [16] for a first approach). Among other nice geometric objects, this representation will give rise to an interesting family of special Fano sixfolds.

2. Models

According to the classical Borel–Weil theorem, the representation $V_4^\vee \otimes \wedge^2 V_5^\vee$ can be interpreted as a space of global sections of an irreducible homogeneous vector bundle over a homogeneous space, and this in more than one way:

$$\begin{aligned} V_4^\vee \otimes \wedge^2 V_5^\vee &= \Gamma(G(2,V_4) \times \mathbb{P}(V_5), \mathcal{U}^\vee \boxtimes \mathcal{Q}^\vee(1)) \\ &= \Gamma(\mathbb{P}(V_4) \times \mathbb{P}(V_5^\vee), \mathcal{O}(1) \boxtimes \wedge^2 \mathcal{V}^\vee) \\ &= \Gamma(G(2,V_5), V_4^\vee \otimes \wedge^2 \mathcal{V}^\vee) \\ &= \Gamma(\mathbb{P}(V_4) \times \mathbb{P}(V_5), \mathcal{O}(1) \boxtimes \mathcal{Q}^\vee(1)) \\ &= \Gamma(G(2,V_4) \times G(3,V_5), \mathcal{U}^\vee \boxtimes \wedge^2 \mathcal{V}^\vee) \\ &= \Gamma(\mathbb{P}(V_4) \times G(3,V_5), \mathcal{O}(1) \boxtimes \wedge^2 \mathcal{V}^\vee) \\ &= \Gamma(\mathbb{P}(V_4) \times G(2,V_5), \mathcal{O}(1) \boxtimes \wedge^2 \mathcal{V}^\vee) \\ &= \Gamma(G(2,V_4) \times G(2,V_5), \mathcal{U}^\vee \boxtimes \wedge^2 \mathcal{V}^\vee). \end{aligned}$$

Here $\mathcal{U}$ and $\mathcal{V}$ denote tautological bundles on Grassmannians (with some abuse of notations, since we use the these symbols several times for distinct bundles on different Grassmannians); the general statement is that $\Gamma(G(k,V_d), \wedge^i \mathcal{U}^\vee) = \wedge^i V_d^\vee$ for $i \le k < d$.

The bundle $\mathcal{Q}$ on projective space is the tautological quotient bundle, and the slightly less familiar statement is that $\Gamma(\mathbb{P}(V_d), \mathcal{Q}^\vee(1)) = \wedge^2 V_d^\vee$: given a skew-symmetric bilinear form ω on V_d, for any nonzero vector v the linear form $\omega(v, \bullet)$ descends to the quotient of V_d by the line generated by v, hence the isomorphism.

As a consequence, consider a general element θ in $V_4^\vee \otimes \wedge^2 V_5^\vee$. Interpreting it as a global section of a vector bundle in these seven different ways, we obtain smooth subvarieties of codimensions equal to the ranks of the vector bundles in question, that we respectively denote as follows (the notation is such that X_d has dimension d):

$$\begin{array}{ll} X_0 \subset G(2, V_4) \times \mathbb{P}(V_5), & X_1 \subset \mathbb{P}(V_4) \times \mathbb{P}(V_5^\vee), \\ X_2 \subset G(2, V_5), & X_3 \subset \mathbb{P}(V_4) \times \mathbb{P}(V_5), \\ X_4 \subset G(2, V_4) \times G(3, V_5), & X_6 \subset \mathbb{P}(V_4) \times G(3, V_5), \\ X_8 \subset \mathbb{P}(V_4) \times G(2, V_5), & X_8' \subset G(2, V_4) \times G(2, V_5). \end{array}$$

Another obvious thing to do is to consider θ as a general morphism from V_4 to $\wedge^2 V_5^\vee$. The image of $\mathbb{P}(V_4)$ inside $\mathbb{P}(\wedge^2 V_5^\vee)$ is then a generic projective three-plane, that has to meet the Grassmannian $G(2, V_5^\vee)$ along a set Y_0 of five reduced points (the degree of the Grassmannian being equal to five). Correspondingly, we get a set P_0 of five points in $\mathbb{P}(V_4)$, and a set Π_0 of five planes in $\mathbb{P}(V_5)$, all in general position. Concretely, if we choose a basis $e_1, \dots, e_4$ of V_4, with dual basis $e_1^\vee, \dots, e_4^\vee$ of $V_4^\vee$, and decompose θ accordingly as

$$\theta = e_1^\vee \otimes \theta_1 + e_2^\vee \otimes \theta_2 + e_3^\vee \otimes \theta_3 + e_4^\vee \otimes \theta_4,$$

then the contraction $\theta(v) = v_1\theta_1 + v_2\theta_2 + v_3\theta_3 + v_4\theta_4$ has rank two when $[v]$ belongs to P_0; that is, it decomposes as $f_1^\vee \wedge f_2^\vee$ for two linear forms $f_1^\vee$ and $f_2^\vee$ whose kernels intersect along the corresponding plane in $\mathbb{P}(V_5)$. We will denote the five two-forms of rank two (defined up to scalars) obtained by contracting θ as $\omega_1, \dots, \omega_5$. It would be natural then to impose the normalization $\omega_1 + \cdots + \omega_5 = 0$, and decompose θ as

$$\theta = u_1^\vee \otimes \omega_1 + u_2^\vee \otimes \omega_2 + u_3^\vee \otimes \omega_3 + u_4^\vee \otimes \omega_4 + u_5^\vee \otimes \omega_5$$

for some linear forms $u_1^\vee, \dots, u_5^\vee$ such that $u_1^\vee + \cdots + u_5^\vee = 0$.

Notations. $P_0 = \{p_1, \dots, p_5\}$ is a set of five points in $\mathbb{P}(V_4)$, in natural bijection with the set $\{\omega_1, \dots, \omega_5\}$ of five decomposable two-forms in $\wedge^2 V_5^\vee$, that define five points in $G(2, V_5^\vee) \simeq G(3, V_5)$, hence five planes $P_1, \dots, P_5$ in $\mathbb{P}(V_5)$. They also define five planes $\pi_1, \dots, \pi_5$ in $G(2, V_4)$, where π_k is the set of planes in V_4 that contain p_k.

L_0 is the set of pairs of points in P_0. According to the previous identifications, it is in natural bijection with a set of ten lines in $\mathbb{P}(V_4)$, a set of ten points in $\mathbb{P}(V_5)$, and a set of ten points in $G(2, V_4)$.

3. Small dimensions

Most results in this section are classical. Our purpose is mainly to set up the scene for the main character, which will make its entry in the next section.

Proposition 3.1. *X_0 consists of* 10 *points of $G(2, V_4) \times \mathbb{P}(V_5)$, in natural bijection with L_0.*

Proof. By definition, a point (A_2, B_1) belongs to X_0 if and only if we can decompose θ in such a way that A_2 is cut out by the linear forms $e_3^\vee$ and $e_4^\vee$, and the skew-symmetric forms θ_1 and θ_2 have the same kernel $B_1 \subset V_5$. Otherwise said, θ_1 and θ_2 belong to $\wedge^2 B_1^\perp$. Since in the latter space decomposable tensors are parametrized by a quadric, we can make a change of basis in $A_2^\perp$ and suppose that θ_1 and θ_2 are indeed decomposable. Concretely, this means that we can write θ in the form

$$\theta = e_1^\vee \otimes f_1^\vee \wedge f_2^\vee + e_2^\vee \otimes f_3^\vee \wedge f_4^\vee + e_3^\vee \otimes \theta_3 + e_4^\vee \otimes \theta_4.$$

Then $[e_1]$ belongs to P_0, the associated plane in $\mathbb{P}(V_5)$ being $P_1 = \langle f_1, f_2\rangle^\perp$, and also $[e_2]$ belongs to P_0, the associated plane being $P_2 = \langle f_3, f_4\rangle^\perp$. In particular, $A_2 = \langle e_1, e_2\rangle$ and $B_1 = P_1 \cap P_2$, as claimed. ∎

Proposition 3.2. *X_1 is the union of five disjoint lines, in natural bijection with P_0.*

Proof. By definition, a point in X_1 is a pair (A_1, B_4) such that $\theta(v)$ vanishes on B_4 when v generates A_1. But then $\theta(v)$ must have rank two, of the form $f_1^\vee \wedge f_2^\vee$. In particular, A_1 must correspond to one of the five points of P_0, and the hyperplane B_4 can move in the pencil $\langle f_1^\vee, f_2^\vee\rangle$. ∎

Proposition 3.3. *$X_2 \subset G(2, V_5)$ is a del Pezzo surface of degree five.*

Proof. Obvious. ∎

Recall that the del Pezzo surface of degree five contains 10 lines. Since the embedding in $G(2, V_5)$ is anticanonical, this means in our setting that there exist ten flags $A_1 \subset A_3 \subset V_5$ such that $\theta(v, w) = 0$ for any $v \in A_1, w \in A_3$. It is easy to see that these ten flags are in natural bijection with L_0, the ten points $[A_1]$ in $\mathbb{P}(V_5)$ being exactly the intersections of the planes $P_1, \dots, P_5$.

The following statement is classical, see e.g. Chapter 9 in [13]. We include a proof as a warm-up.

Proposition 3.4. *The projection of X_3 to $\mathbb{P}(V_4)$ is the blow-up of the five points of P_0. The projection to $\mathbb{P}(V_5)$ is a small resolution of a Segre cubic primal C_3, ten lines being contracted to the ten singular points of C_3 defined by L_0.*

Proof. By definition, X_3 parametrizes the pairs $(A_1 = [v], B_1)$ such that B_1 is contained in the kernel of $\theta(v)$. Generically, this two-form has rank four and the kernel is one-dimensional, which implies that X_3 projects birationally to $\mathbb{P}(V_4)$. The projection has non-trivial fibers when the rank of $\theta(v)$ drops, that is, over one of the five points in P_0. Then the kernel has dimension three and the fiber is a projective plane, as it has to be.

Now we turn to the projection to $\mathbb{P}(V_5)$. By definition, the fibers are linear subspaces defined by the image of the morphism $Q(-1) \to V_4^\vee \otimes \mathcal{O}_{\mathbb{P}(V_5)}$ induced by θ. In particular, the fibers are non-trivial over the corresponding determinantal locus C_3, which is a cubic threefold since $\det(Q(-1)) = \mathcal{O}_{\mathbb{P}(V_5)}(-3)$. This threefold becomes singular exactly when the rank drops to two. If $w \in V_5$ generates B_1, this means that the morphism from V_4 to $V_5^\vee$ sending e_i to $\theta_i(w, \bullet)$ has rank two. So we may suppose after a change of basis that $\theta_1(w, \bullet) = \theta_2(w, \bullet) = 0$. In other words, θ_1 and θ_2 have the same kernel B_1, and after another change of basis if necessary, we have already seen that we can suppose they are decomposable. So they define two points in P_0, in such a way that B_1 is the point obtained as the intersection of the corresponding planes in $\mathbb{P}(V_5)$, while the line contracted to this point is the span of the corresponding points in $\mathbb{P}(V_4)$. ∎

As a result, C_3 is a cubic threefold with 10 nodes. In fact, C_3 is the image of the rational map from $\mathbb{P}(V_4)$ to $\mathbb{P}(V_5)$ sending $[v]$ to the kernel of the two-form $\theta(v)$, and essentially by definition this is a Segre cubic primal [11].

Reminder on the Segre cubic primal. It was Guido Castelnuovo who introduced the Segre cubic primal in 1888, as the union of lines in $\mathbb{P}^4$ parametrized by a quintic del Pezzo surface obtained as a linear section of the Grassmannian. It can also be described directly, in terms of homogeneous coordinates $x_0, \dots, x_5$ on $\mathbb{P}^5$, by the two equations

$$x_0 + \cdots + x_5 = 0 \quad \text{and} \quad x_0^3 + \cdots + x_5^3 = 0.$$

This presentation exhibits an $\mathcal{S}_6$ symmetry, and it is known that $\mathrm{Aut}(C_3) = \mathcal{S}_6$. Classically, the Segre primal contains 15 planes. (See 4.5 in Chapter X of [26] and Chapter 9 in [13] for much more information.)

The Segre cubic primal admits a classical modular interpretation, according to which $C_3 \simeq (\mathbb{P}^1)^6 /\!/ \mathrm{SL}_2$. Moreover, $\bar{M}_{0,6}$ is a resolution of its singularities (that just blows-up the singular points), and according to Kapranov, it can be constructed by blowing-up five general points in $\mathbb{P}^3$, plus the strict transforms of the ten lines that join them [19]. (Note also that $\bar{M}_{0,6}$ compactifies the moduli space of genus 2 curves.)

Note also that C_3 is known to be G-birationally rigid, and even G-birationally super-rigid, when $\mathcal{A}_5 \subset G \subset \mathcal{S}_6$ [1].

Blowing-up the ten singular points in C_3, we get ten exceptional divisors isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, each of which is contracted to $\mathbb{P}^1$ in X_3. According to [15], any of the rulings of $\mathbb{P}^1 \times \mathbb{P}^1$ can in fact be contracted, yielding $2^{10} = 1024$ small resolutions of the singularities of C_3, falling into 13 orbits of $\mathcal{S}_6$, including 6 for which the resolution is projective. Homological projective duality for the Segre cubic is discussed in [3].

On the planes in the Segre cubic. In coordinates, the 15 planes on the Segre cubic are given by three equations

$$x_a + x_b = x_c + x_d = x_e + x_f = 0,$$

for $(abcdef)$ a permutation of (123456); we denote such a plane by $(ab|cd|ef)$. Together with the 15 points in the hyperplane $x_0 + \cdots + x_5 = 0$ with four coordinates equal to zero, they form a $(15_3, 15_3)$ configuration classically known as the Cremona–Richmond configuration: each plane contains three of the 15 points, and each of those points belongs to three planes of the configuration. But beware that two planes may meet along a single point, or a projective line; the second possibility occurs when their symbols have a common pair.

Proposition 3.5. *There are exactly* 6 *collections of five planes among the fifteen planes in C_3, meeting pairwise along single points. These collections are exchanged transitively by the action of $\mathcal{S}_6$. Each one has for stabilizer a copy of $\mathcal{S}_5$, embedded in $\mathcal{S}_6$ in a non-standard way.*

To understand the last sentence, recall that $\mathcal{S}_6$ has the exceptional property that its outer automorphism group is non-trivial: there exists a unique outer automorphism, and a non-standard embedding of $\mathcal{S}_5$ in $\mathcal{S}_6$ is the composition of a standard embedding by such an outer automorphism. Note that this outer automorphism of $\mathcal{S}_6$ exchanges the two conjugacy classes consisting of transpositions on one hand, and products of three disjoint transpositions on the other hand; the former corresponds to points, the latter to planes in the Cremona–Richmond configuration, which is for this reason self-dual.

Proof. Suppose given a collection of five planes, any two of which meet at a single point. This means that each plane is represented by three pairs, none of which being shared with another plane. So we have a total amount of 15 distinct pairs; necessarily, all the 15 pairs of integers from 1 to 6 must appear exactly once.

Up to permutation, we can assume that one of our planes is (12|34|56). Then the plane containing (13) is either (13|25|46) or (13|26|45), and up to permuting 5 and 6, we can suppose it is the first one. Then the other planes are determined. For example, for the one containing (14), we must split (2356) into two pairs, and since (25) and (56) have already been used, the only possibility is (14|26|35). This also shows that we have three choices for the plane containing (12), then two choices for the plane containing (13), and then no more choices; this means there are exactly six possibilities. Explicitly, they are the following:

(12|34|56)	(12|34|56)	(12|35|46)	(12|35|46)	(12|36|45)	(12|36|45)
(13|25|46)	(13|26|45)	(13|24|56)	(13|26|45)	(13|25|46)	(13|24|56)
(14|26|35)	(14|25|36)	(14|25|36)	(14|23|56)	(14|23|56)	(14|26|35)
(15|24|36)	(15|23|46)	(15|26|34)	(15|24|36)	(15|26|34)	(15|23|46)
(16|23|45)	(16|24|35)	(16|23|45)	(16|25|34)	(16|24|35)	(16|25|34).

Let us denote these six configurations by $ABCDEF$. The action of $\mathcal{S}_6$ on them induces a morphism $\mathcal{S}_6 \to \mathcal{S}_6$, and a direct examination shows that it sends the transposition (12) to the permutation $(AB)(CD)(EF)$. So it has to correspond to the outer automorphism of $\mathcal{S}_6$, and our final claim follows. ∎

Question. Is there an interpretation in terms of the root system E_7? In fact, the Lie algebra $\mathfrak{e}_7$ admits a $\mathbb{Z}_3$-grading of the form

$$\mathfrak{e}_7 = \mathfrak{sl}_3 \times \mathfrak{sl}_6 \oplus (\mathbb{C}^3 \otimes \wedge^2\mathbb{C}^6) \oplus (\mathbb{C}^3 \otimes \wedge^2\mathbb{C}^6)^\vee,$$

and roots of $\mathfrak{e}_7$ defined by weights of $\mathbb{C}^3 \otimes \wedge^2\mathbb{C}^6$ can be interpreted as triples of pairs [21]. Note that roots of $\mathfrak{e}_7$ are classically connected with the 28 bitangents of a plane quartic.

4. The Fano fourfold

Recall that our main character $X_4 \subset G(2, V_4) \times G(3, V_5)$ is defined by θ a general element in $V_4^\vee \otimes \wedge^2 V_5^\vee$, considered as a general section of the vector bundle $\mathcal{U}^\vee \boxtimes \wedge^2 \mathcal{V}^\vee$. Here $\mathcal{U}$ denotes the tautological rank two bundle on $G(2, V_4)$, while $\mathcal{V}$ denotes the tautological rank three bundle on $G(3, V_5)$.

In this section, we describe the geometry of X_4 by blow-ups and contractions.

4.1. The main invariants

We start by computing the main numerical invariants of X_4, including its Hodge numbers.

Proposition 4.1. *X_4 is a rational Fano fourfold of index one.*
Its cohomology is pure, with $h^{1,1} = 6$ and $h^{2,2} = 17$.
Moreover, $h^0(-K_{X_4}) = 40$ and $K_{X_4}^4 = 172$.

Proof. The first assertion is an immediate consequence of the adjunction formula:

$$\begin{aligned} K_{X_4} &= K_G \otimes \det(\mathcal{U}^\vee \boxtimes \wedge^2 \mathcal{V}^\vee)_{|X_4} \\ &= \mathcal{O}_G(-4,-5) \otimes \det(\mathcal{U}^\vee)^3 \otimes \det(\wedge^2 \mathcal{V}^\vee)^2_{|X_4} = \mathcal{O}_{X_4}(-1,-1), \end{aligned}$$

where for simplicity we let $G := G(2, V_4) \times G(3, V_5)$. The Hodge numbers and invariants can computed using exact sequences, along the lines explained in [5]. (They could also be deduced from the geometric descriptions that will follow.) Since $172 = 4 \times 43$ is not divisible by any fourth power, the index must be one. ■

Remark. There exist only very few examples, if we exclude products, of Fano fourfolds with Picard number six or more. See Section 6 in [8] for details.

Note that
$$h^0(-K_{X_4}) = 40 < \dim(\wedge^2 V_4 \otimes \wedge^3 V_5) = 60,$$
which means that X_4 is linearly degenerate inside $G = G(2, V_4) \times G(3, V_5)$. This can be checked by considering the twisted Koszul complex

$$0 \longrightarrow \wedge^6 E^\vee(1,1) \longrightarrow \cdots \longrightarrow E^\vee(1,1) \longrightarrow \mathcal{O}_G(1,1) \longrightarrow \mathcal{O}_{X_4}(1) \longrightarrow 0.$$

Indeed, $H^0(E^\vee(1,1)) \simeq V_4 \otimes V_5$ has dimension 20, while it can be checked that $H^0(\wedge^k E^\vee(1,1)) = 0$ for $k > 1$.

We will describe in some detail the two projections ρ_1 and ρ_2:

$$\begin{array}{ccccc} & & X_4 & & \\ & \overset{\rho_1}{\swarrow} & & \overset{\rho_2}{\searrow} & \\ G(2, V_4) & & & & G(3, V_5). \end{array}$$

We start with the second one.

4.2. The second projection and the Cremona–Richmond configuration

We start with the projection to $G(3, V_5)$, which is very similar to the resolution of singularities of the Segre cubic primal.

Proposition 4.2. *The projection of X_4 to $G(3, V_5)$ is a small resolution of a codimension two subvariety C_4 of degree* 12*, contracting ten planes to ten singular points in natural bijection with L_0.*

Proof. The fiber of $\rho_2\colon X_4 \longrightarrow G(3,V_5)$ over a point $[V]\in G(3,V_5)$ is defined by the morphism $\theta_V\colon \wedge^2 V \to V_4^\vee$ induced by θ. In particular, the fibers are non-trivial when the rank is at most two, which happens in codimension two. We conclude that the image C_4 of X_4 is a determinantal fourfold. Its structure sheaf admits a resolution by the Eagon–Northcott complex (see, e.g., (6.1.6) in [27])

$$0 \longrightarrow \mathcal{V}^\vee(-3) \longrightarrow V_4\otimes \mathcal{O}_{G(2,V_5)}(-2) \longrightarrow \mathcal{O}_{G(2,V_5)} \longrightarrow \mathcal{O}_{C_4} \longrightarrow 0, \tag{4.1}$$

where $\mathcal{V}$ denotes the rank three tautological bundle. We deduce in particular that the class of C_4 in the Chow ring of the Grassmannian $G(3,V_5)$ is $3\sigma_2+2\sigma_{11}$, so that its degree is $3\times 2+2\times 3=12$.

The rank of θ_V drops to one on the singular locus of C_4, which must have codimension 6, hence be a finite set, over which the fibers are projective lines. The fact that θ_V has a two dimensional kernel means that we can find a basis v_1,v_2,v_3 of V such that $\theta_i(v_1,v_2)=\theta_i(v_1,v_3)=0$ for all i. Completing with two vectors v_4,v_5 and taking the dual basis, we conclude that every θ_i belongs to the space of forms generated by $v_1^\vee\wedge v_4^\vee$, $v_1^\vee\wedge v_5^\vee$ and $\wedge^2(v_1^\perp)$. In particular, $\langle\theta_1,\theta_2,\theta_3,\theta_4\rangle$ has to meet $\wedge^2(v_1^\perp)$ in dimension at least two, which means that V defines a pair of planes π_p and π_q in P_0, whose intersection point is a line in V. Finally, V defines a hyperplane H_{pq} of V_4, and the corresponding fiber is the set $G(2,H_{pq})\simeq\mathbb{P}^2$ of planes in H_{pq}.

Conversely, such a pair of planes being given, we can decompose θ in an adapted basis as

$$\theta = e_1^\vee\otimes f_1^\vee\wedge f_2^\vee + e_2^\vee\otimes f_3^\vee\wedge f_4^\vee + e_3^\vee\otimes\theta_3 + e_4^\vee\otimes\theta_4,$$

and then the conditions $\theta_3(f_5,\bullet)=\theta_4(f_5,\bullet)=0$ define a 3-plane V containing f_5. This exactly means that the singular locus of C_4 consists in ten points, in natural bijection with L_0. ∎

Proposition 4.3. *Each singular point of C_4 defines a plane in the Segre cubic primal C_3. The five remaining planes are the projectivized kernels of the five singular form $\omega_1,\dots,\omega_5$.*

Proof. By definition, a point $[v]\in\mathbb{P}(V_5)$ belongs to C_3 when the four linear forms $\theta_i(v,\bullet)$ on V_5 are linearly dependent. In the proof above, we have seen that a singular point in C_4 corresponds to a three-plane $V=\langle v_1,v_2,v_3\rangle$ in V_5 with $\theta(v_1,v_2)=\theta(v_1,v_3)=0$. So for any $v\in V$, the linear forms $\theta_i(v,\bullet)$ vanish on v_1, and also on v by skew-symmetry. When v and v_1 are independent, the four linear forms $\theta_i(v,\bullet)$ thus belong to the three-dimensional space $\langle v,v_1\rangle^\perp\subset V_5^\vee$, so they must be linearly dependent. Hence $\mathbb{P}(V)\subset C_3$.

That the projectivized kernels $\mathbb{P}(K_j)$ of the five singular skew forms θ_j are contained in C_3 is obvious, since $\theta_j(v,\bullet)=0$ for $v\in K_j$ is a linear dependence relation between the $\theta_i(v,\bullet)$. ∎

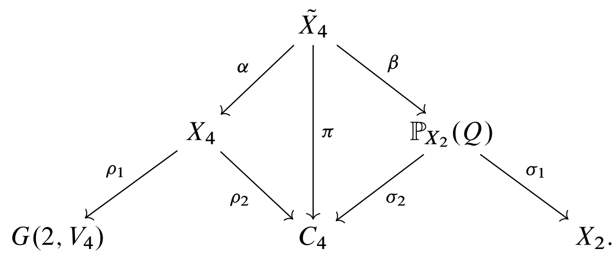
Note that we also have a special point $[v_1]$ in each of the ten planes $\mathbb{P}(V)$. Moreover, the five planes $\mathbb{P}(K_1),\dots,\mathbb{P}(K_5)$ meet pairwise at a single point. In particular, they provide one of the special subcollections of the Cremona–Richmond configuration described in Proposition 3.5.

Also observe that a form ω which is as above in the span of $v_1^\vee\wedge v_4^\vee$, $v_1^\vee\wedge v_5^\vee$ and $\wedge^2(v_1^\perp)$, but does not belong to $\wedge^2(v_1^\perp)$, can be written as $v_1^\vee\wedge w^\vee+\gamma$, with $\gamma\in\wedge^2(v_1^\perp)$ and $w^\vee$ a combination of $v_4^\vee$ and $v_5^\vee$. It has rank two when γ has rank (at most) two and $w^\vee \wedge \gamma = 0$, which means if $w^\vee \neq 0$ that γ is divisible by $w^\vee$. But then ω itself is divisible by $w^\vee$, and since $w^\vee$ is a combination of $v_4^\vee$ and $v_5^\vee$, this implies that $\omega(v_2, v_3) = 0$. In other words, the linear form that defines $H_{pq} \subset V_4$ vanishes at the point that corresponds to ω. This exactly means that

$$p_i \in H_{jk} \quad \text{for } i \neq j, k.$$

We thus get in $G(2, V_4)$ a collection of $5 + 10$ planes, such that each plane of the second type meets exactly three planes of the first type. Hence a configuration $(10_3, 5_6)$. The condition that (jk) be disjoint from (lm), so that the two hyperplanes meet in p_n, defines a copy of the Petersen graph.

Being a degeneracy locus of a morphism between vector bundles, C_4 admits two natural resolutions of singularities; X_4 is one of them. For the other one, we need to impose a rank one kernel in the source of the morphism $\wedge^2 V \to V_4^\vee$; note that a rank one subspace of $\wedge^2 V$ is always of the form $\wedge^2 W$ for $W \subset V$ a rank two subspace. But then the composition $\wedge^2 W \to V_4^\vee$ vanishes exactly when W defines a point in the del Pezzo surface $X_2 \subset G(2, V_5)$. Our second resolution of singularities is thus simply $\mathbb{P}_{X_2}(Q)$, the projectivisation of the quotient bundle of $G(2, V_5)$, restricted to X_2. The two resolutions are dominated by $\tilde{X}_4$, the set of triples $(U_2, V_3 \supset W_2)$ such that (U_2, V_3) belongs to X_4 and W_2 belongs to X_2. We get the following diagram:



Remark. Since ρ_1 and α are birational, this diagram induces a rational map from $G(2, V_4)$ to $X_2 \subset G(2, V_5)$: as we have just seen, the generic $U_2 \subset V_4$ is sent to the unique plane W_2 in V_5 such $W_2 \subset V_3$, where V_3 is generated by the kernels of the forms $\theta(v)$, $v \in U_2$. If $U_2 = \langle v_1, v_2 \rangle$, that we complete in a basis of V_4 by two vectors v_3 and v_4, this means that $W_2 = \langle a_1, a_2 \rangle$ is defined in V_3 by the two conditions $\theta(v_3)(a_1, a_2) = \theta(v_4)(a_1, a_2) = 0$. But then for any $[s_1, s_2] \in \mathbb{P}^1$, the condition that $\theta(v)(s_1 p_1 + s_2 p_2, \bullet) = 0$ reduces to only three scalar conditions on v, that can be realized by some nonzero vector. This implies that the line $\overline{p_1 p_2}$ is contained in C_3, and must be the residual line of the conic obtained by applying θ to the line $\mathbb{P}(U_2)$. As observed by one of the referees, it was already known to Castelnuovo that X_2 can be interpreted as one of the components of the Fano variety of lines on C_3. Then mapping a line in $\mathbb{P}(V_4)$ to the residual line of the corresponding conic as before defines a natural rational map from $G(2, V_4)$ to X_2, and our $\tilde{X}_4$ resolves the indeterminacies of this rational map.

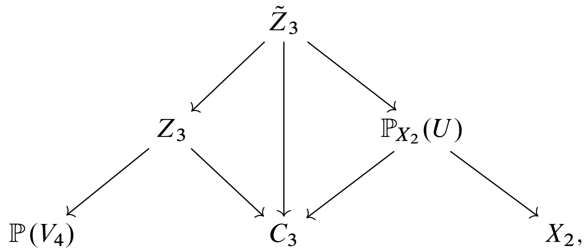
Proposition 4.4. *The morphism $\sigma_2 : \mathbb{P}_{X_2}(Q) \to C_4$ is a small resolution of singularities, contracting ten lines to the ten singular points of C_4. These ten lines are mapped by σ_1 to the ten lines in the del Pezzo surface X_2.*

The morphism β is the blow-up of the ten exceptional lines of σ_2, as well as α is the blow-up of the ten exceptional planes of ρ_2.

Finally, π is the blow-up of the ten singular points of C_4, its exceptional divisor being the disjoint union of ten copies of $\mathbb{P}^2 \times \mathbb{P}^1$.

Remark. Contrary to X_4, the fourfold $X_4' = \mathbb{P}_{X_2}(Q)$ is not Fano. Indeed, the canonical bundle of X_2 is $\det(Q^\vee) = \mathcal{O}_{X_2}(-1)$, and we deduce that the canonical bundle of X_4' is $\sigma_1^*\mathcal{O}_{X_2}(1) \otimes \sigma_2^*\mathcal{O}_{C_4}(-3)$. In particular, it has degree one on the lines contracted by σ_2.

Note also the striking similarity with the two main projective resolutions of the Segre cubic, which can be encapsulated in a similar diagram,



where Z_3 is the blow-up of $\mathbb{P}(V_4) = \mathbb{P}^3$ at five points. Two important differences: Z_3, contrary to X_4, is only weak Fano; Z_3 and $Z_3' = \mathbb{P}_{X_2}(U)$, contrary to X_4 and X_4', are related by flops and therefore derived-equivalent. Instead of that, we have:

Proposition 4.5. *The birational map $\sigma_2^{-1} \circ \rho_2 : X_4 \dashrightarrow X_4'$ is a flip.*

Proof. We have to check that the canonical bundle changes sign on the fibers of the projection to C_4, see Chapter 9 in [23]. On the one hand, since X_4 is Fano, K_{X_4} is certainly negative on the fibers of ρ_2. On the other hand, we have just seen that $K_{X_4'}$ has degree one on the lines contracted by σ_2. ∎

According to the Bondal–Orlov conjecture, there should therefore exist a fully faithful functor $D^b(X_4') \longrightarrow D^b(X_4)$ that would be interesting to describe explicitly.

4.3. Pencils of skew-forms and the first projection

In order to describe the projection to $G(2, V_4)$, we first note that a plane in V_4 defines through θ a pencil of skew-symmetric forms in five variables, and that such pencils have been classified. In fact, for a two dimensional vector space V_2, the action of $\mathrm{GL}(V_2) \times \mathrm{GL}(V_5)$ on $V_2^\vee \otimes \wedge^2 V_5^\vee$ has finitely many orbits, which are described in [20]. Let us only mention that there are exactly eight orbits: the open orbit $\mathcal{O}_7$, an orbit $\mathcal{O}_6$ of codimension two, and another $\mathcal{O}_5$ of codimension four, and then all the other orbits have bigger codimension.

The orbit $\mathcal{O}_5$ (or rather its closure) is characterized as consisting of tensors of rank at most four, in the sense that they belong to $V_2^\vee \otimes \wedge^2 V_4$ for some hyperplane $V_4 \subset V_5^\vee$. The orbit $\mathcal{O}_6$ (or rather its closure) is characterized as consisting of those pencils in $\wedge^2 V_5^\vee$ admitting a rank two element. So the open orbit $\mathcal{O}_7$ parametrizes pencils of forms of constant rank four. By Proposition 2 in [22], given such a pencil, one can find a basis of V_5 for which the two skew-forms

$$\varphi_1 = f_1^\vee \wedge f_3^\vee + f_2^\vee \wedge f_4^\vee \quad \text{and} \quad \varphi_2 = f_1^\vee \wedge f_4^\vee + f_2^\vee \wedge f_5^\vee$$

are generators. The projective line $\langle f_1^\vee, f_2^\vee \rangle$ is the *pivot* of the pencil. Now, observe that if a three-plane $V \subset V_5$ is isotropic with respect to any skew-form $s\varphi_1 + t\varphi_2$ of the pencil, it has to contain its kernel, which is generated by $s^2 f_3 - st f_4 + t^2 f_5$. So necessarily $V = \langle f_3, f_4, f_5 \rangle$, the orthogonal to the pivot.

Proposition 4.6. *The projection of X_4 to $G(2, V_4)$ is birational. The exceptional locus in $G(2, V_4)$ is the union of five planes, intersecting in the ten points of L_0, whose fibers are quadratic surfaces.*

Proof. The fiber of the projection $\rho_1 : X_4 \longrightarrow G(2, V_4)$ over the point $[U] \in G(2, V_4)$ is defined by the morphism $\theta_U : U \to \wedge^2 V_5^\vee$. This morphism is injective and we thus get a pencil of skew-symmetric forms. If this pencil is generic, which means that it has constant rank, then we have just seen that there is a unique three-plane in V_5 which is isotropic with respect to any skew-form in the pencil. This three-plane is the image of the induced map $\theta_U^{(2)} : S^2 U \to \wedge^4 V_5^\vee \simeq V_5$. In particular, ρ_1 is birational.

Special fibers will occur when the pencil $\mathrm{Im}(\theta_U)$ becomes special in some way. By the usual arguments for orbital degeneracy locus [4], each type of special pencil will appear along a locus whose codimension is equal to the codimension of the corresponding orbit inside the space of pencils. In particular, we only need to take into account the orbits of pencils of codimension smaller than five, which apart from the open one, are the orbits $\mathcal{O}_5$ and $\mathcal{O}_6$ we have described above.

Pencils in $\mathcal{O}_5$ contain two skew-forms of rank two. In our case, they must be two of the skew-forms $\omega_1, \dots, \omega_5$, say θ_1 and θ_2. Choose an adapted basis such that $\theta_1 = f_1^\vee \wedge f_2^\vee$ and $\theta_2 = f_3^\vee \wedge f_4^\vee$, so that

$$\theta_U = e_1^\vee \otimes f_1^\vee \wedge f_2^\vee + e_2^\vee \otimes f_3^\vee \wedge f_4^\vee.$$

It is straightforward to check that the three-planes that are isotropic with respect to any skew-form in the pencil are those generated by f_5, a vector in $\langle f_1, f_2 \rangle$, and a vector in $\langle f_3, f_4 \rangle$. We thus get for fiber a copy of $\mathbb{P}^1 \times \mathbb{P}^1$.

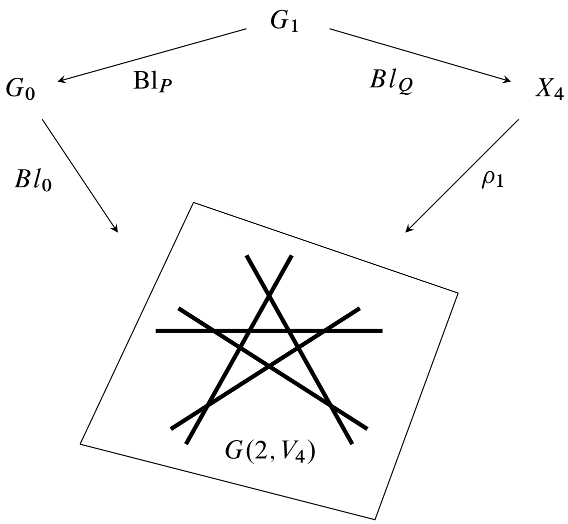
Finally, pencils in $\mathcal{O}_6$ contain exactly one skew-form of rank two, say θ_1. To describe the corresponding fiber, we must understand the 3-planes isotropic with respect to both the generic form θ_2 and the degenerate form $\theta_1 = f_1^\vee \wedge f_2^\vee$. Such a 3-plane must contain the kernel of θ_2; let us choose a generator f_5 and a hyperplane H_4 in V_5 not containing f_5. We may suppose that $f_2^\vee$ vanishes on f_5. The 3-planes we are looking for are in correspondence with the 2-planes $H = \langle h, h' \rangle$ in H_4 such that $\theta_2(h, h') = 0$ and $f_1^\vee(h) = f_1^\vee(h') = 0$. Such a 2-plane must be contained in the kernel K_3 of $f_1^\vee$, and it has to contain the kernel K_1 of the restriction of θ_2 to K_3. We finally get for fiber a pencil of planes. ∎

To summarize, the exceptional locus is the union of five planes $\pi_1, \dots, \pi_5$ in $G(2, V_4)$, where π_i parametrizes the planes in V_4 containing p_i. Their pre-images in X_4 are five divisors that we denote $F_1, \dots, F_5$. Any two among the five planes meet at a single point, over which the fiber of ρ_1 is a quadratic surface.

If U_2 does not belong to any of the five exceptional planes, we have seen that U_3 is the span of the kernels of the two-forms $\theta(v)$, for $v \in U_2$. Since this kernel can be computed as $\theta(v) \wedge \theta(v)$, there is a natural associated conic bundle over $G(2, V_4)$ minus the five exceptional planes. This also stresses the analogy with the construction of the Segre primal C_3 as the image of a rational map $\mathbb{P}(V_4) \dashrightarrow \mathbb{P}(V_5)$ defined by θ. Here we get C_4 as the image of a rational map $G(2, V_4) \dashrightarrow G(2, V_5)$ also defined by θ. We will put its equations in simple form in the next section.

4.4. Blow-up and contract

Proposition 4.6 suggests to construct X_4 by first blowing-up $G(2, V_4)$ along the 10 points of L_0, then the strict transforms of the 5 planes, which are Del Pezzo surfaces of degree five. The first blow-up $\mathrm{Bl}_0 : G_0 \longrightarrow G(2, V_4)$ gives 10 exceptional divisors $E^0_{ij} \simeq \mathbb{P}^3$ for $1 \leq i < j \leq 5$, each with a pair of skew lines ℓ_i and ℓ_j coming from the two planes π_i and π_j intersecting at p_{ij}. The second blow-up $\mathrm{Bl}_P : G_1 \longrightarrow G_0$ produces five other exceptional divisors $F^1_1, \dots, F^1_5$ that will be sent in X_4 to $F_1, \dots, F_5$. Moreover, the strict transform E^1_{ij} of E^0_{ij} is the blow-up of E^0_{ij} along $\ell_i \cup \ell_j$. Since the blow-up of $\mathbb{P}^3$ along two skew lines is the total space of $\mathbb{P}(\mathcal{O}(-1,0) \oplus \mathcal{O}(0,-1))$ over $\mathbb{P}^1 \times \mathbb{P}^1$, we deduce that the rational map to X_4 is a morphism. More precisely, it has to coincide with the blow-up $\mathrm{Bl}_Q : G_1 \longrightarrow X_4$ of the ten quadratic surfaces $S_{ij} = \rho_1^{-1}(p_{ij})$ in X_4. This explains in particular why the Picard number is equal to 6.



Let $F^1 = F^1_1 + \cdots + F^1_5$, and let E^1 be the sum of the ten divisors E^1_{ij} in G_1. Let H_1 and H_2 be the pull-backs of the hyperplane classes by ρ_1 and ρ_2, that we denote in the same way on X_4 and on G_1. From the identity

$$K_{G_1} = -4H_1 + 3E^1 + F^1 = K_{X_4} + E^1 = -H_1 - H_2 + E^1,$$

we deduce the relation $3H_1 = H_2 + 2E^1 + F^1$.

The exceptional locus of ρ_2 defines a collection of 10 planes in X_4, contracted to the ten singular points of C_4, and that we can identify with their isomorphic images in $G(2, V_4)$. Recall that in this Grassmannian we have the five planes $\pi_1, \dots, \pi_5$.

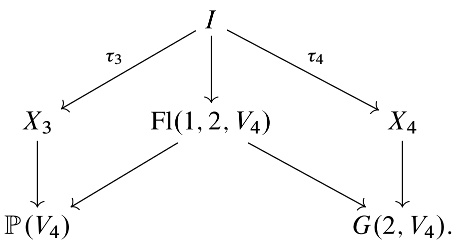
Proposition 4.7. *The resulting collection of* $10 + 5$ *planes in* $G(2, V_4)$ *is in natural correspondence with the Cremona–Richmond configuration.*

4.5. Incidences with the Segre cubic

Now we relate the two varieties X_3 and X_4 by considering the incidence correspondence

$$I = \{(A_1, B_1), (U_2, U_3) \in X_3 \times X_4,\ A_1 \subset U_2,\ B_1 \subset U_3\}.$$

Recall that, by definition, B_1 is (contained in) the kernel of $\theta(v)$ for $v \in A_1$, while the general U_3 is the linear span of the kernels of the two forms $\theta(u)$ for $u \in U_2$; this kernel depends quadratically on u since it is given by $\theta(u) \wedge \theta(u)$. So I is birationally equivalent to the flag variety $\mathrm{Fl}(1, 2, V_4)$, and we have a commutative diagram:



We leave the proof of the next statement to the interested reader.

Proposition 4.8. *The morphism* τ_4 *is a conic fibration, while* τ_3 *is a fibration in del Pezzo surfaces of degree four.*

4.6. Projective duality

We have seen that X_4 is birationally equivalent to the projective bundle $\mathbb{P}_{X_2}(Q)$ over the del Pezzo surface X_2. Since $Q^\vee$ has no section, we would rather write it as $\mathbb{P} = \mathbb{P}_{X_2}(\wedge^2 Q^\vee)$, in which case the relative tautological bundle $\mathcal{O}_{\mathbb{P}}(-1)$ sends $\mathbb{P}$ to $\mathbb{P}(\wedge^2 V_5^\vee) \simeq \mathbb{P}(\wedge^3 V_5)$, the image being $C_4 \subset G(3, V_5)$. We are then in the context of homological projective duality for projective bundles, according to which $\mathbb{P} \to \mathbb{P}(\wedge^2 V_5^\vee)$ is dual to $\mathbb{P}^* \to \mathbb{P}(\wedge^2 V_5)$, with $\mathbb{P}^*$ the projective bundle $\mathbb{P}_{X_2}(W \wedge V_5)$, where W denotes the rank to tautological bundle.

Proposition 4.9. *The image of* $\mathbb{P}^* \to \mathbb{P}(\wedge^2 V_5)$ *is an octic hypersurface in* $\mathbb{P}(\wedge^2 V_5)$*, containing the Grassmannian* $G(2, V_5)$ *in its singular locus.*

Proof. First consider the full projective bundle $\mathbb{P}_{G(2,V_5)}(W \wedge V_5)$ and its projection to $\mathbb{P}(\wedge^2 V_5)$. The generic fiber is a smooth three-dimensional quadric Q_3 (while the special fibers, that occur over $G(2, V_5)$, are codimension two Schubert cycles). When we restrict to X_2, we cut the fibers by linear spaces of codimension four. Generically, they meet the span of the fiber at one point; in codimension one, this point will be on the fiber itself. This implies that $\mathbb{P}^* \to \mathbb{P}(\wedge^2 V_5)$ is birational onto its image, which must be a hypersurface. As usual, we compute the degree of this hypersurface as

$$\int_{\mathbb{P}^*} \mathcal{O}_{\mathbb{P}^*}(1)^8 = \int_{X_2} s_2(W \wedge V_5) = \int_{G(2,V_5)} (2\sigma_2 + \sigma_{11})\sigma_1^4 = 8.$$

Here we used exact sequences to compute the Segre class

$$s(W \wedge V_5) = c(Q)^5 c(S^2 U),$$

with $c(Q) = 1 + \sigma_1 + \sigma_2 + \sigma_3$ and $c(U) = 1 - \sigma_1 + \sigma_{11}$.

Over a point W^0 of the Grassmannian, the fiber of $\mathbb{P}_{G(2,V_5)}(W \wedge V_5) \to \mathbb{P}(\wedge^2 V_5)$ is the Schubert cycle of planes W meeting W^0 along at least a line. It is desingularized by a $\mathbb{P}^3$-bundle over $\mathbb{P}(W^0)$. If we fix a line $L \subset W^0$, there exists a plane $W \supset L$ in X_2 if and only if the four linear forms $\theta(L, \bullet)$ on V_5/L are linearly dependent. This defines a section of $\wedge^4(Q^\vee(1)) = \mathcal{O}(3)$ over $\mathbb{P}^1$, and we conclude that the general fiber of $\mathbb{P}_{X_2}(W \wedge V_5) \to \mathbb{P}(\wedge^2 V_5)$ over $G(2, V_5)$ consists in three points. Since this morphism is birational onto its image, Zariski's main theorem implies that $G(2, V_5)$ is contained in the singular locus. ■

5. Symmetries

The symmetries of the Segre cubic primal must be reflected in X_4. In this section, we describe the symmetries of X_4 in some detail. In particular, we will prove:

Proposition 5.1. *The generic stabilizer of the action of* $\mathrm{PGL}(V_4) \times \mathrm{PGL}(V_5)$ *on* $\mathbb{P}(V_4^\vee \otimes \wedge^2 V_5^\vee)$ *is the symmetric group* $\mathcal{S}_5$.

What is classically known, as we mentioned in the introduction, is that the action of $\mathrm{PGL}(V_4) \times \mathrm{PGL}(V_5)$ on $V_4^\vee \otimes \wedge^2 V_5^\vee$ is prehomogeneous. The representative of the open orbit given in [24] is

$$\theta = e_1^\vee \otimes (f_{25} - f_{34}) + e_2^\vee \otimes (f_{15} - f_{24}) + e_3^\vee \otimes (f_{23} - f_{14}) + e_4^\vee \otimes (f_{45} - f_{12}),$$

with the notation $f_{ij} = f_i^\vee \wedge f_j^\vee$. The corresponding points in $\mathbb{P}(V_4)$ and rank two forms are easy to identify; we get

$$\begin{aligned}
p_1 &= e_2 + ie_4, & \omega_1 &= (f_1 + if_4) \wedge (f_2 + if_5),\\
p_2 &= e_2 - ie_4, & \omega_2 &= (f_1 - if_4) \wedge (f_2 - if_5),\\
p_3 &= e_1 + e_3 + e_4, & \omega_3 &= (f_2 + f_4) \wedge (f_1 + f_3 + f_5),\\
p_4 &= e_1 + je_3 + j^2 e_4, & \omega_4 &= (f_2 + j^2 f_4) \wedge (f_1 + j^2 f_3 + j f_5),\\
p_5 &= e_1 + j^2 e_3 + je_4, & \omega_5 &= (f_2 + jf_4) \wedge (f_1 + jf_3 + j^2 f_5).
\end{aligned}$$

Here j and i are primitive fourth and third roots of unity. Each pair ω_p, ω_q defines two planes in $V_5^\vee$ whose common orthogonal is a line $[e_{pq}]$. Then the planes of the Cremona–Richmond configuration are obtained as follows: P_{pq} is generated by the three points e_{ij},

e_{jk} and e_{ik} for ijk distinct from pq; and P_p is generated by the four points e_{ip} for $i \neq p$. Explicitly, the ten vectors e_{pq} can be chosen as follows:

$$\begin{array}{ll} e_{12} = (0, 0, 1, 0, 0), & e_{24} = (i, -j^2, -2ij, 1, ij^2), \\ e_{13} = (1, -i, -2, i, 1), & e_{25} = (i, -j, -2ij^2, 1, ij), \\ e_{14} = (-i, -j^2, 2ij, 1, -ij^2), & e_{34} = (1, 0, j^2, 0, j), \\ e_{15} = (-i, -j, 2ij^2, 1, -ij), & e_{35} = (1, 0, j, 0, j^2), \\ e_{23} = (1, i, -2, -i, 1), & e_{45} = (1, 0, 1, 0, 1). \end{array}$$

Each ω_i defines a plane π_i in $V_5^\vee$, from which we can deduce a collection of hyperplanes $\pi_{ij} = \pi_i + \pi_j$ and points $p_{ijk} = \pi_i \cap (\pi_j + \pi_k)$.

Proposition 5.2. *For any permutation i, j, k, l, m of $1, \dots, 5$, $p_{ijk} = p_{ilm}$.*

Proof. Explicit check. ∎

We have no convincing explanation of this coincidence, but as a consequence, we do not get thirty but only fifteen points in $\mathbb{P}(V_5^\vee)$. Obviously, p_{ijk} belongs to π_i, hence to any of the four hyperplanes $\pi_{il}, l \neq i$. Conversely, π_{ij} contains the three points p_{iab} plus the three points p_{jcd}.

Proposition 5.3. *The fifteen points π_{ijk} and the ten hyperplanes π_{ij} in $\mathbb{P}(V_4)$ form a configuration $(15_4, 10_6)$.*

We thus recover the abstract configuration classically defined by the Segre primal. In particular, the fifteen points π_{ijk} should be in natural correspondence with planes in the Segre primal.

Automorphisms in $\mathrm{PGL}(V_4) \times \mathrm{PGL}(V_5)$ that fix $\langle\theta\rangle$ are in bijective correspondence with elements of $\mathrm{PGL}(V_5)$ fixing the four-plane generated by the ω_i's. Automatically, such an automorphism will preserve the set of five planes $\pi_1, \dots, \pi_5$, hence the collection of the thirty points p_{ijk}.

In order to show that any permutation of the five planes can be lifted to $\mathrm{PGL}(V_5)$, it is enough to lift two generators of $\mathcal{S}_5$, say a transposition and a complete cycle. By sending f_i to $\varepsilon_i f_i$, with $\varepsilon_i = 1$ for i odd and $\varepsilon_i = -1$ for i even, we exchange π_1 and π_2 and let the three other planes be fixed. So let us turn to a maximal cycle. We claim that the cycle $(12345) \in \mathcal{S}_5$ can be lifted to the transformation of $\mathrm{GL}(V_5)$ given by

$$\begin{array}{lll} f_1 & \mapsto & \frac{j}{3} f_1 - 2ijf_2 + \frac{j}{3} f_3 - ijf_4 + \frac{4j}{3} f_5, \\ f_2 & \mapsto & -\frac{2i}{3} f_1 - f_2 + \frac{i}{3} f_3 + \frac{i}{3} f_5, \\ f_3 & \mapsto & \frac{4j^2}{3} f_1 + 4ij^2 f_2 - \frac{2j^2}{3} f_3 - 4ij^2 f_4 + \frac{4j^2}{3} f_5, \\ f_4 & \mapsto & -\frac{ij}{3} f_1 - \frac{ij}{3} f_3 - jf_4 + \frac{2ij}{3} f_5, \\ f_5 & \mapsto & \frac{4}{3} f_1 + if_2 + \frac{1}{3} f_3 + 2if_4 + \frac{1}{3} f_5. \end{array}$$

Corollary 5.4. *The automorphism group of the Fano fourfold X_4 is $\mathrm{Aut}(X_4) = \mathcal{S}_5$.*

Proof. An automorphism of X_4 is induced by a linear transformation in $\mathrm{PGL}(V_4) \times \mathrm{PGL}(V_5)$ preserving θ. Considered as a homomorphism from V_4 to $\wedge^2 V_5^\vee$, θ defines a codimension four linear section of $G(2, V_5)$, that is a degree five del Pezzo surface S_5.

This implies that $\mathrm{Stab}(\theta)$ embeds into $\mathrm{Aut}(\mathcal{S}_5)$, which is well known to be $\mathcal{S}_5$. Since we know by the previous computations that $\mathrm{Stab}(\theta)$ contains $\mathcal{S}_5$, we are done. ∎

Once we identify $\mathcal{S}_5$ with the stabilizer of θ in $\mathrm{SL}(V_4) \times \mathrm{SL}(V_5)$, we get actions of $\mathcal{S}_5$ on V_4 and V_5, clearly irreducible. Up to the sign representation there is a unique irreducible representation of $\mathcal{S}_5$ of dimension 4, and a unique one of dimension 5. The complex (4.1) shows that C_4 is cut out by a family of quadrics on $G(2, V_5)$ parametrized by V_4, hence a $\mathcal{S}_5$-invariant copy of V_4 inside the Schur module $S_{22}V_5^\vee = H^0(G(2,V_5), \mathcal{O}_{G(2,V_5)}(2))$. We will show later on that this copy is unique. (This point of view from finite group representation theory is typically used in [14]. Something with the same flavour has been done in [2] for the quintic del Pezzo surface.)

We use the character table of $\mathcal{S}_5$ (see for example [12]) to compute some plethysm and tensor product representations. Recall that $\mathcal{S}_5$ has irreducible representations, of dimensions 1, 1, 4, 4, 5, 5 and 6, that we denote by U_1 U_1^-, U_4, U_4^-, U_5 and U_5^-, U_6. All these representations are self-dual, being defined over the real numbers. Concretely, U_1 is the trivial representation, U_1^- is the sign representation, U_4 is the natural representation, $U_4^- = U_4 \otimes U_1^-$ and $U_6 = \wedge^2 U_4$. One computes that

$$S^2U_4 = U_5 \oplus U_4 \oplus U_1 \quad \text{and} \quad \wedge^2 U_5 = U_4^- \oplus U_6.$$

The last decomposition implies, in particular, that $(U_4^-)^\vee \otimes \wedge^2 U_5^\vee$ contains a unique $\mathcal{S}_5$-invariant tensor $\theta_{\mathcal{S}_5}$, up to scalars.

At this point, it could therefore be reasonable to reverse the whole process and start from the representation theory of $\mathcal{S}_5$. One should be able to check directly that $\theta_{\mathcal{S}_5}$ is generic, and then we should get $\theta_{\mathcal{S}_5}$-invariant descriptions of all the objects we have been studying.

Note that $S^2U_4 = U_5 \oplus U_4 \oplus U_1$ allows to construct U_5 from U_4, as the space of quadrics which are apolar to the obvious invariant cubic. In coordinates $x_1, \dots, x_5$ permuted by $\mathcal{S}_5$, the representation U_4 is the hyperplane $x_1 + \cdots + x_5 = 0$, the invariant cubic is $x_1^3 + \cdots + x_5^3$ and the apolar quadrics are of the form $\sum_{i \neq j} a_{ij} x_i x_j$, with

$$a_{ij} = a_{ji} \quad \text{for all } i \neq j, \qquad \sum_{i \neq k} a_{ik} = 0 \quad \text{for all } k.$$

We get ten indeterminates and five independent relations, consistently with the fact that these quadrics should span a copy of V_5.

Inside the space V_5 of apolar quadrics to the invariant cubic, note that we have $q_{ij,kl} = (x_i - x_j)(x_k - x_l)$ for i, j, k, l distinct integers. These quadrics are subject to the Plücker type relations $q_{ij,kl} - q_{ik,jl} + q_{il,jk} = 0$. This suggests to define the following elements of $\wedge^2 V_5$:

$$\begin{aligned}
Q_1 &= q_{23,45} \wedge q_{24,35},\\
Q_2 &= q_{13,45} \wedge q_{14,53},\\
Q_3 &= q_{12,45} \wedge q_{14,25},\\
Q_4 &= q_{12,35} \wedge q_{13,52},\\
Q_5 &= q_{12,34} \wedge q_{13,24}.
\end{aligned}$$

Obviously, for any permutation $\sigma \in \mathcal{S}_5$ one must have $\sigma(Q_i) = \pm Q_{\sigma(i)}$. We also let, for a pair $i \neq j$ with complement p, q, r in $1, \dots, 5$,

$$Q_{i,j} = q_{ip,qr} \wedge q_{jp,qr} + q_{iq,rp} \wedge q_{jq,rp} + q_{ir,pq} \wedge q_{jr,pq}.$$

Proposition 5.5. *The action of $\mathcal{S}_5$ on $\langle Q_1, \dots, Q_5 \rangle$ gives a copy of the representation U_4^- in $\wedge^2 U_5$. Similarly, the action of $\mathcal{S}_5$ on $\langle Q_{i,j}, 1 \leq i < j \leq 5 \rangle$ gives a copy of the representation U_6.*

What have we gained in doing all that? First, we get a better, more symmetric normal form for the generic θ than that of Ozeki, as

$$\theta_{\mathcal{S}_5} = e_1 \otimes Q_1 + e_2 \otimes Q_2 + e_3 \otimes Q_3 + e_4 \otimes Q_4 + e_5 \otimes Q_5,$$

with $e_1 + \cdots + e_5 = 0$.

Also, we can make explicit the quadratic equations of $\mathcal{C}_4$. A character computation yields:

Lemma 5.6. *The multiplicity of U_4^- inside $S^2(\wedge^2 U_5)$ is equal to one.*

So the space of quadratic equations we are looking for is uniquely defined in terms of the $\mathcal{S}_5$-action. Moreover, recall that $\wedge^2 U_5 = U_4^- \oplus U_6$. Another character computation shows that the copy of U_4^- that we are looking for inside $S^2(\wedge^2 U_5)$ is in fact contained inside $U_4^- \otimes U_6 = U_4^- \otimes \wedge^2(U_4^-) \subset U_4^- \otimes \mathrm{End}(U_4^-)$ (recall that U_4^- is self-dual). So there is an obvious map to U_4^-, and dually, this says that the space of quadrics we are looking for is generated by the five quadrics

$$CQ_i = \sum_{j \neq i} Q_{i,j} Q_j, \quad \text{for } 1 \leq i \leq 5.$$

Remark. Since $\mathrm{Aut}(C_3) = \mathcal{S}_6$, certain automorphisms of the Segre primal do not lift to X_4. Would it be possible that $\mathcal{S}_6$ act on X_4 by birational transformations?

6. The Chow ring of X_4

In this section, we completely determine the Chow ring of X_4, with its structure of $\mathcal{S}_5$-module. Let us start with the Picard group.

From the relation $3H_1 = H_2 + 2E + F$ that we found on G_1, we compute that

$$H_1^4 = 2, \quad H_1^3 H_2 = 6, \quad H_1^2 H_2^2 = 13, \quad H_1 H_2^3 = 14 \quad \text{and} \quad H_2^4 = 12.$$

The Picard group is generated by H_1, H_2 and the five components of F, which are permuted by $\mathcal{S}_5$. We deduce:

Proposition 6.1. *The Chow ring of X_4 is generated by $A^*(G)$ and the five divisors $F_1, \dots, F_5$. As a representation of $\mathcal{S}_5$, the Picard group decomposes as*

$$\mathrm{Pic}(X_4) \otimes_{\mathbb{Z}} \mathbb{C} = 2U_0 \oplus U_4.$$

We know by Proposition 4.1 that the middle dimensional Chow group $A^2(X_4)$ has dimension 17, and we expect that the invariant part has dimension four, with two classes coming from $G(2, V_4)$ and two other classes from $G(2, V_5)$. We will show that are all come (at least over $\mathbb{Q}$) from products of divisor classes.

We compute the multiplicative structure of the Chow ring by embedding it in the Chow ring of G_1, that we shall now describe. First, the Chow ring of G_0 is generated by the Chow ring of $G = G(2, V_4)$ and the ten exceptional divisors E^0_{pq} of the blow-up $b_0 = \mathrm{Bl}_0 \colon G_0 \longrightarrow G(2, V_4)$, such that

$$(E^0_{pq})^4 = -1, \quad E^0_{pq}E^0_{p'q'} = 0 \text{ for } \{p, q\} \neq \{p', q'\}, \quad E^0_{pq}.b_0^*C = 0$$

for any class $C \in A^*(G)$ of positive degree. After this first blow-up, the five planes $\pi_1, \dots, \pi_5$ give five disjoint surfaces $\Sigma_1, \dots, \Sigma_5$, each one being a plane blow-up in five points, that is a del Pezzo surface of degree 5. We denote the four exceptional lines in Σ_p by ℓ^q_p, whose image in G is the point π_{pq}, for $q \neq p$.

The second blow-up $b_1 = \mathrm{Bl}_P \colon G_1 \longrightarrow G_0$ is the blow-up of these five surfaces. Recall that we denoted by F^1_p the five exceptional divisors, and by E^1_{pq} the strict transforms of the divisors E^0_{pq} in G_0. Since $F^1_p = \mathbb{P}(N_p)$, for N_p the normal bundle of Σ_p inside G_0, we need to describe this normal bundle. Recall that when one blows up one point in a smooth variety X, creating an exceptional divisor E inside the blow-up $Y \xrightarrow{\pi} X$, the tangent exact sequence is $0 \to TY \to \pi^*TX \to i_*TE \to 0$, where $i \colon E \to Y$ denotes the inclusion. Since the normal bundle of π_p inside the Grassmannian G is the quotient bundle $\mathcal{Q}$, we get the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T\Sigma_p & \longrightarrow & TG_{0|\Sigma_p} & \longrightarrow & N_p & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & b_0^*T\pi_p & \longrightarrow & b_0^*TG_{|\pi_p} & \longrightarrow & b_0^*\mathcal{Q} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \oplus_{q\neq p}T\ell^q_p & \longrightarrow & \oplus_{q\neq p}TE^0_{pq} & \longrightarrow & \oplus_{q\neq p}N^q_p & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

Here we denoted by N^q_p the normal bundle to $\ell^q_p \simeq \mathbb{P}^1$ inside $E^0_{pq} \simeq \mathbb{P}^3$, which is just $\mathcal{O}_{\ell^q_p}(1) \oplus \mathcal{O}_{\ell^q_p}(1)$. We deduce the Segre class

$$s(N_p) = s(b_0^*Q) \prod_{q\neq p} c(\mathcal{O}_{\ell^q_p}(1))^2 \in A^*(\Sigma_p).$$

One the one hand, the Segre class $s(\mathcal{Q})$ equals the Chern class of the tautological bundle on G, that is, $s(\mathcal{Q}) = 1 - H_1 + \sigma_{11}$, and the Schubert class σ_{11} restricts to zero on π_p. On the other hand, on the del Pezzo surface Σ_p we have $\mathcal{O}_{\ell_p^q}(1) = \mathcal{O}(-\ell_p^q)_{|\ell_p^q}$, from which we get the Segre class $s(\mathcal{O}_{\ell_p^q}(1)) = 1 + \ell_p^q + 2(\ell_p^q)^2$. Finally,

$$s(N_p) = 1 - H_1 + 2\sum_{q\neq p} \ell_p^q + 2\sum_{q\neq p}(\ell_p^q)^2.$$

We can deduce several intersection numbers, since for any class C_{3-k} of degree $3-k$ on G_0, we have the classical formulas

$$(F_p^1)^{k+1} b_1^* C_{3-k} = \int_{F_p^1} (F_p^1)^k b_1^* C_{3-k} = (-1)^k \int_{\Sigma_p} s_{k-1}(N_p) C_{3-k}.$$

Lemma 6.2.

$$(F_p^1)^4 = 8, \quad (F_p^1)^3 H_1 = -1, \quad (F_p^1)^2 H_1^2 = -1 \quad \textit{and} \quad F_p^1 H_1^3 = 0.$$

Note also that F_p^1 does not meet E_{rq}^1 for $r, q \neq p$, but it meets E_{pq}^1 transversally along the surface $S_p^q = b_1^{-1}(\ell_p^q)$. Therefore,

$$\mathcal{O}_{G_1}(E^1_{pq|F_p^1}) = \mathcal{O}_{F_p^1}(S_p^q) = b_1^* \mathcal{O}_{\Sigma_p}(\ell_p^q).$$

Applying the previous formula to $C_{3-k} = (E_{pq}^0)^{3-k}$, we get:

Lemma 6.3. *$F_p^1 E_{rq}^1 = 0$ if $r, q \neq p$, but*

$$(F_p^1)^3 E_{pq}^1 = -2, \quad (F_p^1)^2 (E_{pq}^1)^2 = 1 \quad \textit{and} \quad F_p^1 (E_{pq}^1)^3 = 0.$$

On the other hand, E_{pq}^0 gets blown-up along the two-skew lines ℓ_p^q and ℓ_q^p, and its strict transform E_{pq}^1 is contracted to the quadratic surface $\ell_p^q \times \ell_q^p$ in X_4. This surface is also the intersection of F_p and F_q in X_4, in particular it is contained in F_p. We deduce, denoting Bl_Q by c, that

$$c^* F_p = F_p^1 + \sum_{q\neq p} E_{pq}^1.$$

Summing up over p, we get the relation $c^* F = F^1 + 2E^1$, where $F = F_1 + \cdots + F_5$.

Corollary 6.4. *$C_4 \subset G(3, V_5)$ is the image of $G = G(2, V_4)$ by the linear system $|I_\pi(3H_1)|$ of cubics vanishing along the union π of the five planes $\pi_1, \ldots, \pi_5$.*

This is similar to the classical statement that C_3 is the image of $\mathbb{P}^3$ by the linear system of quadrics passing through five points in general position, see Proposition 9.4.15 in [13]. As a referee points out, this somehow reveals the mystery of C_4.

We now have enough information to describe the intersection product on X_4.

Proposition 6.5. *The nonzero intersection numbers among $H_1, F_1, \ldots, F_5$ are the following: for $1 \leq p \neq q \leq 5$,*

$$F_p^4 = 12, \quad F_p^3 F_q = -2, \quad F_p^2 F_q^2 = 1, \quad F_p^3 H_1 = -1, \quad F_p^2 H_1^2 = -1 \quad \textit{and} \quad H_1^4 = 2.$$

Moreover, we always have $H_1 F_p F_q = 0$ for $p \neq q$ and $F_p F_q F_r = 0$ for $p \neq q \neq r \neq p$.

Proof. The values of $F_p^3 H_1$ and $F_p^2 H_1^2$ can be computed directly by restricting to a general hyperplane or a general codimension two section of G; then we avoid the points π_{qr} and we are reduced to compute the self-intersection of the exceptional divisor for the blow-up of a line in a three-dimensional quadric, or a point in a surface. Then we can deduce the value of F_p^4 by computing the self-intersection of $H_2 = 3H_1 - F$, which we know is equal to

$$12 = 81\,H_1^2 - 108\,H_1^3 F + 54\,H_1^2 F^2 - 12\,H_1 F^3 + F^4 = 162 - 270 + 60 + F^4.$$

This gives $F^4 = F_1^4 + \cdots + F_5^4 = 60$, hence $F_p^4 = 12$. (But note that this is not equal to $(c^* F_p)^4 = -4$, as a consequence of the fact that F_p contains four of the quadratic surfaces blown-up by c.)

The other intersection numbers can be computed by pulling-back by c and using Lemma 6.2. ■

Proposition 6.6. *The square map $S^2 A_1(X_4) \longrightarrow A^2(X_4)$ is surjective. As a consequence, the $\mathcal{S}_5$-module structure of $A^2(X_4)$ is*

$$A^2(X_4) = 4U_0 \oplus 2U_4 \oplus U_5.$$

Proof. The decomposition of the $\mathcal{S}_5$-module $S^2 A^1(X_4)$ is $4U_0 \oplus 3U_4 \oplus U_5$, the sum of three isotypical components, and the kernel of the square map must decompose accordingly.

First consider the four invariant classes H_1^2, $H_1 F$, $F^{(2)}$ and $F^{(11)}$, where

$$F^{(2)} = \sum_p F_p^2 \quad \text{and} \quad F^{(11)} = \sum_{p<q} F_p F_q.$$

Suppose that there is a relation $aH_1^2 + bH_1 F + cF^{(2)} + dF^{(11)} = 0$. Multiplying successively by H_1^2, $H_1 F_p$, F_p^2 and $F_p F_q$, and using the results of Proposition 6.5, we deduce that $2a - 5c = 0, b + c = 0, a + b + 16c - 8d = 0, 4c - d = 0$, hence $a = b = c = d = 0$.

Now consider the possibility that U_5 be contained in the kernel of the square map. We claim that U_5 is embedded inside $S^2 A^1(X_4)$ as the space of linear combinations $\sum_{p\neq q} a_{pq} F_p F_q$ with $a_{pq} = a_{qp}$ and $\sum_r a_{pr} = 0$ for all p, q. Indeed, this defines an invariant five-dimensional subspace of $S^2 A^1(X_4)$, not containing any invariant class, so it must be U_5. A typical element is

$$3F_p F_q - (F_p + F_q) \sum_{r\neq p,q} F_r + \sum_{s,t\neq p,q} F_s F_t.$$

If this was zero in $A^2(X_4)$, multiplying by $F_p F_q$ would imply that the intersection number $F_p^2 F_q^2 = 0$, which is not the case.

We can conclude that the kernel of the square map must be contained in the isotypical component $3U_4$ of $S^2 A^1(X_4)$, which is generated by the three copies of U_4 respectively obtained as the linear combinations

$$\sum_p a_p H_1 F_p, \quad \sum_p a_p F F_p \quad \text{and} \quad \sum_p a_p F_p^2,$$

for $\sum_p a_p = 0$. A copy of U_4 in the kernel corresponds to a relation of the form

$$uH_1F_p + vFF_p + wF_p^2 = I, \quad \text{for all } p,$$

for I an invariant class. Since I is invariant, multiplying by H_1F_p and H_1F_q must then give the same intersection number, which gives the relation $-u - v - w = 0$. Similarly, multiplying by F_p^2 or F_q^2 must give the same result, that is $-u + 4v + 12w = -v + w$. Finally, multiplying by F_pF_q or F_qF_r with q, r distinct from p must also give the same result, that is, $-v - 2w = 0$. These three equations are linearly dependent and reduce to $u = w$ and $v = -2w$, which proves that there is a unique copy of U_4 in the kernel of the square map. This concludes the proof. ■

Threefolds

Consider the two families of divisors in X_4 given by sections of H_1 and H_2, respectively. Since a general hyperplane section in $G(2, V_4)$ will avoid the ten points π_{pq}, the first ones are just blow-ups of five disjoint lines in a smooth three-dimensional quadric. For the same reason, the second ones, say Z_3, are isomorphic with their images in $G(3, V_5) \cap H_2$, which are codimension two degeneraci loci defined by the condition that the morphism $\wedge^2 V \to V_4^\vee$ has rank exactly two. Its image is then the pull-back from $G(2, V_4)$ of the dual quotient bundle $\mathcal{Q}^\vee$. In particular, we get an exact sequence

$$0 \to \mathcal{O}(1, -2) \to \rho_2^*(\wedge^2 V) \to \rho_1^*\mathcal{Q}^\vee \to 0$$

on Z_3. This shows in particular that $\mathcal{O}(-1, 2)$, the restriction of $2H_2 - H_1 = 5H_1 - 2F$, is generated by sections on Z_3. The image in Z_3 is the closure of the planes $U \subset V_4$ such that the image of $S^2U \to \wedge^2 V_5 \simeq V_5$ is isotropic with respect to some three-form on V_5. This defines a section of $\wedge^3(S^2U)^\vee = \det(U^\vee)^3$, so that the image of Z_3 in $G(2, V_4)$ is a singular cubic hypersurface.

K3 surfaces

By taking sections of $H_1 \oplus H_2$ in X_4, we get a family of smooth K3 surfaces S in X_4. We denote by $h_1, h_2, f_1, \dots, f_5$ the restriction to S of the divisors $H_1, H_2, F_1, \dots, F_5$.

Proposition 6.7. *The intersection numbers of these divisors in S are*

$$h_1^2 = 6, \quad h_2^2 = 14, \quad h_1h_2 = 13, \quad h_1f_i = 1, \quad h_2f_i = 5 \quad \textit{and} \quad f_if_j = -2\delta_{ij}.$$

Proof. This is an immediate consequence of the computations above, since for two divisors A and B on X restricting to a and b on S, we have $ab = ABH_1H_2$. ■

An obvious consequence is that $h_1, f_1, \dots, f_5$ are linearly independent. Moreover, the curves $C_i = F_i \cap S$ are (-2)-curves on S, mapping to lines on $G(2, V_4)$ and to rational quintics in $G(2, V_5)$. The divisor $5h_1 - h_2 = 2h_1 + f$ should contract these five (-2)-curves to the five singular points of a surface $\bar{S}$. Note that this is a divisor of degree 34, so $\bar{S}$ could be a degeneration of a smooth K3 surface of genus 18. Mukai described the generic such K3 surface as the zero locus in $OG(3, 9)$ of five sections of the rank two spinor bundle. What is the connection? Note that we have a family of surfaces of dimension $5 + 9 = 14 = 19 - 5$, which is coherent with the expectation that imposing 5 nodes on a K3 surface of genus 18 should give five independent conditions.

7. The Igusa quartic and the Coble fourfold

Given a linear form h on V_5, there is an associated quadratic form Q_h on V_4:

$$Q_h(v) = h \wedge \theta(v) \wedge \theta(v) \in \wedge^5 V_5^\vee \simeq \mathbb{C}.$$

Proposition 7.1. *The quartic* $\det(Q_h) = 0$ *is the Igusa quartic in* $\mathbb{P}^4 = \mathbb{P}(V_5^\vee)$.

Proof. Recall that the generic point of the Segre cubic $C_3 \subset \mathbb{P}(V_5)$ is the kernel of one of the two-forms $\theta(v)$, and that we can get this kernel as the line generated by $\theta(v) \wedge \theta(v) \in \wedge^4 V_5^\vee \simeq V_5$. At this generic point, the affine tangent space to the Segre cubic is therefore the hyperplane of V_5 generated by the vectors of the form $\theta(v) \wedge \theta(w) \in \wedge^4 V_5^\vee \simeq V_5$. This hyperplane is defined by a linear form $h_v \in V_5^\vee$ that vanishes on these vectors, which exactly means that $h_v \wedge \theta(v) \wedge \theta(w) = 0$ for any $w \in V_5$. In other words, $Q_{h_v}(\theta(v), \theta(w)) = 0$ for all $w \in V_5$, which means that $\theta(v)$ belongs to the kernel of the quadratic form Q_{h_v}. In particular, the latter is degenerate.

We have thus proved that the generic point of the projective dual variety of the Segre cubic is contained in the quartic hypersurface $\det(Q_h) = 0$. But this projective dual is well known to be the Igusa quartic in $\mathbb{P}(V_5^\vee)$, and these two quartics have to coincide. ∎

This yields a simple determinantal representation of the Igusa quartic. Using Ozeki's representative, we get

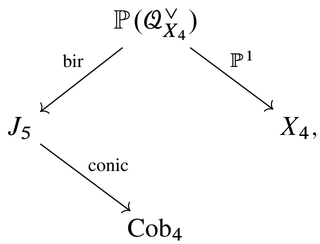
$$Q_h = \begin{pmatrix} 2h_1 & -h_2 & -h_3 & -h_5 \\ -h_2 & 2h_3 & -h_4 & 0 \\ -h_3 & -h_4 & 2h_5 & -h_1 \\ -h_5 & 0 & -h_1 & 2h_3 \end{pmatrix},$$

whose determinant is readily computed to be

$$-\det(Q_h) = 4h_3^4 + 4h_3^2(3h_1h_5 - h_2h_4) - 4h_3(h_1^3 + h_5^3 + h_1h_4^2 + h_2^2h_5) + (h_1h_2 - h_4h_5)^2.$$

One can consider inside $\mathbb{P}(V_5^\vee) \times G(2, V_4)$ the locus J_5 of pairs $([h], U)$ such that U is isotropic with respect to Q_h. Recall that $OG_Q(2,4) = \mathbb{P}^1 \cup \mathbb{P}^1$ is the disjoint union of two smooth conics when Q is non-degenerate. When Q is a quadratic form of corank one on V_4, the corresponding orthogonal Grassmannian $OG_Q(2, V_4)$ is a single conic (while if Q has corank two, $OG_Q(2, V_4)$ is the union of two planes meeting at one point, defined by the kernel). This means that the Stein factorization of the projection of J_5 to $\mathbb{P}(V_5^\vee)$ is $J_5 \to \mathrm{Cob}_4 \to \mathbb{P}(V_5^\vee)$, where Cob_4 is the double cover of $\mathbb{P}(V_5)$ branched over the Igusa quartic: that is, the *Coble fourfold* [9].

On the other hand, denote by $\mathcal{Q}_{X_4}$ the pull-back to X_4 of the rank two quotient bundle on $G(3, V_5)$. The $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{Q}_{X_4})$ over X_4 has a natural map to $G(2, V_4) \times \mathbb{P}(V_5^\vee)$, and we claim that its image is precisely J_5. Indeed, a generic element $(U, V \subset H)$ of $G(2, V_4) \times Fl(3, 4, V_5)$ belongs to $\mathbb{P}(\mathcal{Q}_{X_4}^\vee)$ when V, hence H, contains the kernels of all the two-forms $\theta(v)$, $v \in U$. But if h is a linear form defining H, the condition that H contains the kernel of $\theta(v)$ exactly means that $h \wedge \theta(v) \wedge \theta(v) = 0$, hence our claim. Moreover, the projection map $\mathbb{P}(\mathcal{Q}_{X_4}^\vee) \to J_5$ is birational, being clearly bijective outside the five special planes in $G(2, V_4)$. So we get a diagram



where the south-west arrow is a conic bundle, at least generically. But the picture does not seem to recover the small resolutions of Cob_4 described in [9].

8. Local rigidity

Since $V_4^\vee \otimes \wedge^2 V_5^\vee$ is prehomogeneous, we expect that X_4 has strong rigidity properties. What we can prove is the following statement.

Proposition 8.1. *X_4 is locally rigid.*

Proof. Local rigidity is equivalent to the vanishing of $H^1(TX_4)$. In order to check this, as usual we rely on the normal exact sequence, which yields an exact sequence of cohomology groups

$$H^0(TG_{|X_4}) \longrightarrow H^0(E_{|X_4}) \longrightarrow H^1(TX_4) \longrightarrow H^1(TG_{|X_4}).$$

So local rigidity will follow from the following statements, to be proved separately:

(1) $H^1(TG_{|X_4}) = 0$;

(2) $H^0(E_{|X_4}) = V_4^\vee \otimes \wedge^2 V_5^\vee / \langle\theta\rangle$;

(3) $H^0(TG_{|X_4}) \longrightarrow H^0(E_{|X_4})$ is surjective.

The third statement follows from the fact that $\mathbb{P}(V_4^\vee \otimes \wedge^2 V_5^\vee)$ is prehomogeneous under $\mathrm{PGL}(V_4) \times \mathrm{PGL}(V_5)$, more precisely from the fact that the orbit of $[\theta]$ is open, since this implies that the image of the natural differential $\mathfrak{sl}_4 \times \mathfrak{sl}_5 = H^0(TG) \longrightarrow V_4^\vee \otimes \wedge^2 V_5^\vee / \langle\theta\rangle$ sending X to $X(\theta)$ mod θ is surjective; since this morphism can also be defined by restricting first to X_4 and then composing with the morphism we are interested in, the latter must also be surjective.

In order to prove the second statement, we twist by E the Koszul complex resolving the structure sheaf of X_4. By standard cohomological arguments, it is enough to check that $H^0(G, \mathrm{End}_0(E)) = 0$ and $H^i(G, E \otimes \wedge^{i+1} E^\vee) = 0$ for any $i > 0$.

For the first claim, observe that

$$\mathrm{End}_0(E) = \mathrm{End}_0(U) \oplus \mathrm{End}_0(V) \oplus \mathrm{End}_0(U) \otimes \mathrm{End}_0(V)$$

is in fact acyclic.

For the second claim, check that $E \otimes \wedge^{i+1} E^\vee$ is also acyclic for any $i > 0$. Similarly, in order to prove the first statement we need to check that $H^{i+1}(G, TG \otimes \wedge^i E^\vee) = 0$ for any $i \geq 0$, which is again a straightforward application of Bott's theorem. ■

The question remains open, whether X_4 is also globally rigid, which would be remarkable for a Fano fourfold with such a big Picard number. The first thing to be checked is whether X_4 remains smooth when we degenerate θ to the codimension one orbit. If the answer were yes, we would get a similar situation to the case of codimension two linear sections of the spinor tenfold (which has Picard number one, though).

Another question one may ask is whether the quotient bundle restricted to X_4 is rigid. In other words, is the morphism to $G(3, V_5)$ uniquely defined?

9. Higher dimensions

Let us briefly describe the higher dimensional models.

Proposition 9.1. *X_6 is a rational Fano sixfold of index one and Picard rank two.*

The projection of X_6 to $G(3, V_5)$ is birational, with non-trivial fibers isomorphic to $\mathbb{P}^1$ over the smooth locus of C_4, and to $\mathbb{P}^2$ over its ten singular points.

The projection to $\mathbb{P}(V_4)$ is a Q_3-bundle outside P_0, with five four-dimensional fibers over P_0.

From this description and that of X_4, we deduce that in the Grothendieck ring of varieties, one has the relation $[X_6] + L^3[Y_0] = [G(3, V_5)] + L[X_4]$. This yields the Poincaré polynomial of X_6,

$$P_{X_6}(t) = 1 + 2t + 8t^2 + 9t^3 + 8t^4 + 2t^5 + t^6.$$

Proposition 9.2. *X_8 is a Fano eightfold of pseudo-index three, while X_8' is Fano eigthfold of index three.*

Proposition 9.3. *The projections of X_8 and X_8' to $G(2, V_5)$ are dual $\mathbb{P}^2$-fibrations over the complement of a del Pezzo surface of degree five, the exceptional fibers being isomorphic to $\mathbb{P}(V_4)$ and $G(2, V_4)$ respectively.*

We can readily deduce that X_8 and X_8' have pure cohomology, with Poincaré polynomials

$$P_{X_8}(t) = 1 + 2t + 4t^2 + 6t^3 + 11t^4 + 6t^5 + 4t^6 + 2t^7 + t^8$$

and

$$P_{X_8'}(t) = 1 + 2t + 5t^2 + 11t^3 + 13t^4 + 11t^5 + 5t^6 + 2t^7 + t^8.$$

Of course, X_6, X_8, X_8' inherit the same symmetries as X_4. E. Fatighenti and F. Tanturri checked the necessary vanishing conditions to establish, as for X_4, that they are also infinitesimally rigid.

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