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Hölder estimate for a tug-of-war game with $1 < p < 2$ from Krylov–Safonov regularity theory

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Abstract. We propose a new version of the tug-of-war game and a corresponding dynamic programming principle related to the p-Laplacian with $1 < p < 2$. For this version, the asymptotic Hölder continuity of solutions can be directly derived from recent Krylov–Safonov type regularity results in the singular case. Moreover, existence of a measurable solution can be obtained without using boundary corrections. We also establish a comparison principle.

1. Introduction

In Section 7 of [3], we show that a solution to the dynamic programming principle

$$u(x) = \frac{\alpha}{2}\Big(\sup_{h\in B_1} u(x+\varepsilon h) + \inf_{h\in B_1} u(x+\varepsilon h)\Big) + \beta \fint_{B_1} u(x+\varepsilon h)\,dh + \varepsilon^2 f(x), \tag{1.1}$$

with $\beta = 1-\alpha \in (0,1]$ and $x \in \Omega \subset \mathbb{R}^N$, satisfies certain extremal inequalities, and thus has asymptotic Hölder regularity directly by the Krylov–Safonov theory developed in that paper. This dynamic programming principle corresponds to a version of the tug-of-war game with noise as explained in [19], and it is linked to the p-Laplacian when $p \geq 2$ with suitable choice of probabilities $\alpha = \alpha(N,p)$ and $\beta = \beta(N,p)$. Interested readers can consult the references [5, 12, 20, 22] for more information about the tug-of-war games.

In the case $1 < p < 2$, one usually considers a variant of the game known as the tug-of-war game with orthogonal noise as in [22], although there is also other recent variant covering this range and not using orthogonal noise but measures absolutely continuous with respect to the N-dimensional Lebesgue measure, see [13]. An inconvenience of the "orthogonal noise" approach of [22] is that the uniform part of the measure is supported in $(N-1)$-dimensional balls. Thus we do not expect that solutions to the corresponding dynamic programming principle satisfy the extremal inequalities required for the Krylov–Safonov type regularity estimates obtained in [2, 3].

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Second, there are some deep measurability issues related to the corresponding dynamic programming principles, and this introduces some difficulties in the existence and measurability proofs in the case $1 < p < 2$, as explained at the beginning of Section 3.2. As a matter of fact, in [8] and [4] a modification near the boundary was necessary in order to guarantee the measurability. See also [1], where a modification of the usual tug-of-war is made near the boundary.

For these reasons, and with the purpose of covering the case $1 < p < 2$, we propose a different variant of the tug-of-war game that can be described by a dynamic programming principle having a uniform part in a N-dimensional ball in (2.4) below. For this variant, no boundary modifications are needed in the existence proof (Theorem 3.7) because of better continuity properties that are addressed in Section 3. We also establish a comparison principle and thus uniqueness of solutions (Theorem 3.8). Moreover, the solutions to this dynamic programming principle are asymptotically Hölder continuous (Corollary 4.3) and converge, passing to a subsequence if necessary, to a solution of the normalized p-Laplace equation (Theorem 5.3).

2. Preliminaries

We denote by B_1 the open unit ball of $\mathbb{R}^N$ centered at the origin. For $|z| = 1$, we introduce the following notation:

$$\mathcal{I}_\varepsilon^z u(x) := \frac{1}{\gamma_{N,p}} \fint_{B_1} u(x+\varepsilon h)(z\cdot h)_+^{p-2}\,dh, \tag{2.1}$$

for each Borel measurable bounded function u, where $\gamma_{N,p}$ is the normalization constant

$$\gamma_{N,p} := \fint_{B_1} (z\cdot h)_+^{p-2}\,dh = \frac{1}{2}\fint_{B_1} |h_1|^{p-2}\,dh, \tag{2.2}$$

which is independent of the choice of $|z| = 1$. Here, we have used the following notation:

$$(t)_+^{p-2} = \begin{cases} t^{p-2} & \text{if } t > 0, \\ 0 & \text{if } t \le 0. \end{cases} \tag{2.3}$$

We remark that $t \mapsto (t)_+^{p-2}$ is a continuous function in $\mathbb{R}$ when $p > 2$, but it presents a discontinuity at $t = 0$ when $1 < p \le 2$. We observe, integrating for example over a cube containing B_1, that $\gamma_{N,p} < \infty$ for any $p > 1$. Later we compute the precise value of $\gamma_{N,p}$ in (A.4), but we immediately observe that $\gamma_{N,p} > 1/2$ if $1 < p < 2$, since

$$2\gamma_{N,p} = \fint_{B_1} |z\cdot h|^{p-2}\,dh \ge \fint_{B_1} |h|^{p-2}\,dh > 1.$$

Throughout the paper, $\Omega \subset \mathbb{R}^N$ denotes a bounded domain. For $\varepsilon > 0$, we define the ε-neighborhood of Ω as

$$\Omega_\varepsilon = \{x \in \mathbb{R}^N : \operatorname{dist}(x,\Omega) < \varepsilon\}.$$

Let $f:\Omega \to \mathbb{R}$ be a Borel measurable bounded function. We consider a dynamic programming principle (DPP)

$$u(x) = \frac{1}{2}\Big(\sup_{|z|=1} \mathcal{I}_\varepsilon^z u(x) + \inf_{|z|=1} \mathcal{I}_\varepsilon^z u(x)\Big) + \varepsilon^2 f(x) \tag{2.4}$$

for $x \in \Omega$, with prescribed Borel measurable bounded boundary values $g:\Omega_\varepsilon \setminus \Omega \to \mathbb{R}$. The parameter p above is linked to the p-Laplace operator as explained in Section 5.

Next, we recall the asymptotic Hölder continuity result derived in [3] and [2]. The results there apply to a quite general class of discrete stochastic processes with bounded and measurable increments and their expectations, or equivalently functions satisfying the corresponding dynamic programming principles. Moreover, the results actually hold for functions merely satisfying inequalities given in terms of extremal operators, which we recall below. This can be compared with the Hölder result for PDEs given in terms of Pucci operators (see for example [6] and [10, 11, 23]).

For $\Lambda \geq 1$, let $\mathcal{M}(B_\Lambda)$, as in those papers, denote the set of symmetric unit Radon measures with support in B_Λ, and let $\nu:\mathbb{R}^N \to \mathcal{M}(B_\Lambda)$ be such that

$$x \mapsto \int_{B_\Lambda} u(x+h)\, d\nu_x(h)$$

defines a Borel measurable function for every Borel measurable $u:\mathbb{R}^N \to \mathbb{R}$. By symmetric we mean that

$$\nu_x(E) = \nu_x(-E)$$

holds for every measurable set $E \subset \mathbb{R}^N$.

Definition 2.1 (Extremal operators). Let $u:\mathbb{R}^N \to \mathbb{R}$ be a Borel measurable bounded function. For $\beta = 1-\alpha \in (0,1]$, we define the extremal Pucci type operators

$$\begin{aligned}
\mathcal{L}_\varepsilon^+ u(x) &:= \frac{1}{2\varepsilon^2}\Big(\alpha \sup_{\nu \in \mathcal{M}(B_\Lambda)} \int_{B_\Lambda} \delta u(x,\varepsilon h)\, d\nu(h) + \beta \fint_{B_1} \delta u(x,\varepsilon h)\, dh\Big)\\
&= \frac{1}{2\varepsilon^2}\Big(\alpha \sup_{h \in B_\Lambda} \delta u(x,\varepsilon h) + \beta \fint_{B_1} \delta u(x,\varepsilon h)\, dh\Big)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{L}_\varepsilon^- u(x) &:= \frac{1}{2\varepsilon^2}\Big(\alpha \inf_{\nu \in \mathcal{M}(B_\Lambda)} \int_{B_\Lambda} \delta u(x,\varepsilon h)\, d\nu(h) + \beta \fint_{B_1} \delta u(x,\varepsilon h)\, dh\Big)\\
&= \frac{1}{2\varepsilon^2}\Big(\alpha \inf_{h \in B_\Lambda} \delta u(x,\varepsilon h) + \beta \fint_{B_1} \delta u(x,\varepsilon h)\, dh\Big),
\end{aligned}$$

where

$$\delta u(x,\varepsilon h) = u(x+\varepsilon h) + u(x-\varepsilon h) - 2u(x)$$

for every $h \in B_\Lambda$.

Naturally, also other domains of definition are possible instead of $\mathbb{R}^N$ above.

Theorem 2.2 (Asymptotic Hölder, [2, 3]). *There exists $\varepsilon_0 > 0$ such that if u satisfies $\mathcal{L}_\varepsilon^+ u \geq -\rho$ and $\mathcal{L}_\varepsilon^- u \leq \rho$ in B_R for some $1-\alpha = \beta > 0$, where $\varepsilon < \varepsilon_0 R$, there exist $C, \gamma > 0$ such that*

$$|u(x) - u(y)| \leq \frac{C}{R^\gamma}\Big(\sup_{B_R} |u| + R^2\rho\Big)\big(|x-y|^\gamma + \varepsilon^\gamma\big)$$

for every $x, y \in B_{R/2}$.

It is worth remarking that the constants C and γ are independent of ε, and depend exclusively on N, $\Lambda \geq 1$ and $\beta = 1-\alpha \in (0,1]$. Also a version of Harnack's inequality (see Theorem 5.5 in [2]) holds if the extremal inequalities are satisfied for some $\beta > 0$. For a different approach for regularity in the case of tug-of-war games, see [16] ($p > 2$) and [15] ($p > 1$).

3. Existence of measurable solutions with $1 < p < \infty$

In this section, we prove existence and uniqueness of solutions to the DPP (2.4). In addition, when $1 < p < 2$, such DPP satisfies the requirements to get the asymptotic Hölder estimate from Theorem 2.2. The regularity result and the connection of such a DPP (as well as the corresponding tug-of-war game) to the p-Laplacian are addressed in Sections 4 and 5, respectively.

Remark 3.1. Observe that the operator $\mathcal{I}_\varepsilon^z$ defined in (2.1) is a linear average for each $|z| = 1$, in the sense that $\mathcal{I}_\varepsilon^z$ satisfies the following:

(i) stability: $\inf_{B_\varepsilon(x)} u \leq \mathcal{I}_\varepsilon^z u(x) \leq \sup_{B_\varepsilon(x)} u$;

(ii) monotonicity: $\mathcal{I}_\varepsilon^z u \leq \mathcal{I}_\varepsilon^z v$ for $u \leq v$;

(iii) linearity: $\mathcal{I}_\varepsilon^z(au + bv) = a\mathcal{I}_\varepsilon^z u + b\mathcal{I}_\varepsilon^z v$ for $a, b \in \mathbb{R}$.

3.1. Continuity estimates for $\mathcal{I}_\varepsilon^z u$

Given a Borel measurable bounded function $u\colon \Omega_\varepsilon \to \mathbb{R}$, we show that the function $\mathcal{I}_\varepsilon^z u(x)$ is continuous with respect to $x \in \overline{\Omega}$ and $|z| = 1$. In fact, we prove that $(x, z) \mapsto \mathcal{I}_\varepsilon^z u(x)$ is uniformly continuous. As a consequence of this, the function

$$x \longmapsto \frac{1}{2}\Big(\sup_{|z|=1} \mathcal{I}_\varepsilon^z u(x) + \inf_{|z|=1} \mathcal{I}_\varepsilon^z u(x)\Big)$$

is continuous in $\overline{\Omega}$, as shown in Lemma 3.4.

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^N$. For $u\colon \Omega_\varepsilon \to \mathbb{R}$ a Borel measurable bounded function and for $x \in \overline{\Omega}$, the function*

$$z \longmapsto \mathcal{I}_\varepsilon^z u(x)$$

is continuous on $|z| = 1$. Moreover, the family $\{z \mapsto \mathcal{I}_\varepsilon^z u(x) : x \in \overline{\Omega}\}$ is equicontinuous on $|z| = 1$.

Proof. For $|z| = |w| = 1$, we have

$$\left|\mathcal{I}_\varepsilon^z u(x) - \mathcal{I}_\varepsilon^w u(x)\right| \le \frac{\|u\|_\infty}{\gamma_{N,p}} \fint_{B_1} \left|(z \cdot h)_+^{p-2} - (w \cdot h)_+^{p-2}\right| dh$$

uniformly for every $x \in \overline{\Omega}$. We claim that the limit

$$\lim_{|z-w|\to 0} \fint_{B_1} \left|(z \cdot h)_+^{p-2} - (w \cdot h)_+^{p-2}\right| dh = 0 \tag{3.1}$$

holds for every $1 < p < \infty$. Observe that the limit in (3.1) is independent of x and u, and thus it holds uniformly for every $x \in \overline{\Omega}$; then, the (uniform) equicontinuity of $\mathcal{I}_\varepsilon^z u$ in Ω follows.

(i) *Case* $p = 2$. Since $(t)_+^0 = \chi_{(0,\infty)}(t)$, we have

$$\fint_{B_1} \left|(z \cdot h)_+^0 - (w \cdot h)_+^0\right| dh = \frac{|B_1 \cap (\{z \cdot h > 0\} \triangle \{w \cdot h > 0\})|}{|B_1|} \le C|z - w|$$

for some explicit constant $C > 0$ depending only on N, so (3.1) follows. Here, $\triangle$ stands for the symmetric difference $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

(ii) *Case* $p > 2$. We observe that the function $t \mapsto (t)_+^{p-2}$ is continuous in $\mathbb{R}$. In addition, given any $|z| = |w| = 1$, it holds that

$$|(z \cdot h)_+^{p-2} - (w \cdot h)_+^{p-2}| \le 1$$

for each $h \in B_1$. Then (3.1) follows by the dominated convergence theorem.

(iii) *Case* $1 < p < 2$. This case requires a bit care, since obtaining an integrable upper bound needed for the dominated convergence theorem is not as straightforward. To this end, we observe the inequality

$$|a - b| \le (a + b)\left(1 - \frac{\min\{a, b\}}{\max\{a, b\}}\right)$$

for every $a, b > 0$. Thus

$$|(z \cdot h)_+^{p-2} - (w \cdot h)_+^{p-2}| \le \left((z \cdot h)_+^{p-2} + (w \cdot h)_+^{p-2}\right)\left(1 - \frac{\min\{(z \cdot h)_+^{p-2}, (w \cdot h)_+^{p-2}\}}{\max\{(z \cdot h)_+^{p-2}, (w \cdot h)_+^{p-2}\}}\right).$$

In that way, applying Hölder's inequality with $q = \frac{1}{2}\frac{p-3}{p-2}$ and recalling the definition of $\gamma_{N,(p+1)/2}$, we can estimate

$$\begin{aligned}
&\fint_{B_1} |(z \cdot h)_+^{p-2} - (w \cdot h)_+^{p-2}|\, dh \\
&\qquad \le 2\gamma_{N,(p+1)/2}^{2\frac{p-2}{p-3}} \left(\fint_{B_1} \left(1 - \frac{\min\{(z \cdot h)_+^{p-2}, (w \cdot h)_+^{p-2}\}}{\max\{(z \cdot h)_+^{p-2}, (w \cdot h)_+^{p-2}\}}\right)^{-\frac{p-3}{p-1}} dh\right)^{-\frac{p-1}{p-3}}.
\end{aligned}$$

Observe that $(p + 1)/2 > 1$, and thus $\gamma_{N,(p+1)/2} < \infty$. Now, since $t \mapsto (t)_+^{p-2}$ is continuous in $(0, +\infty)$ (and we set other values identically to 0 in (2.3)), and so, for any given

$|z| = 1$ and for each $h \in B_1$ such that $z \cdot h > 0$, it holds that $(w \cdot h)_+^{p-2} \to (z \cdot h)_+^{p-2}$ as $w \to z$ with $|w| = 1$. Then, we see that the function in the integral on the right-hand side is bounded between 0 and 1 and converges to 0 as $w \to z$ for each $h \in B_1$, so the assumptions in the dominated convergence theorem are fulfilled, yielding that the right-hand side above converges to 0 as $w \to z$, and thus (3.1) follows. ∎

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^N$. For $|z| = 1$, the function*

$$x \longmapsto \mathcal{I}_\varepsilon^z u(x)$$

is continuous in $\overline{\Omega}$ for every Borel measurable bounded function $u\colon \Omega_\varepsilon \to \mathbb{R}$. Moreover, $\{\mathcal{I}_\varepsilon^z u : |z| = 1\}$ is equicontinuous in $\overline{\Omega}$.

Proof. Let $u\colon \Omega_\varepsilon \to \mathbb{R}$ be a Borel measurable bounded function defined in Ω_ε. Our aim is to show that

$$\lim_{x,y\in\overline{\Omega} \text{ s.t. } |x-y|\to 0} \left|\mathcal{I}_\varepsilon^z u(x) - \mathcal{I}_\varepsilon^z u(y)\right| = 0.$$

We can write

$$\begin{aligned}
\mathcal{I}_\varepsilon^z u(x) &= \frac{1}{\gamma_{N,p}\,|B_1|} \int_{\mathbb{R}^N} u(x+\varepsilon h)\, \chi_{B_1}(h)\, (z\cdot h)_+^{p-2}\, dh,\\
\mathcal{I}_\varepsilon^z u(y) &= \frac{1}{\gamma_{N,p}\,|B_1|} \int_{\mathbb{R}^N} u(x+\varepsilon h)\, \chi_{B_1(-\frac{x-y}{\varepsilon})}(h)\, (z\cdot (h+\tfrac{x-y}{\varepsilon}))_+^{p-2}\, dh,
\end{aligned}$$

so it follows immediately that

$$\begin{aligned}
&\left|\mathcal{I}_\varepsilon^z u(x) - \mathcal{I}_\varepsilon^z u(y)\right|\\
&\quad\le \frac{\|u\|_\infty}{\gamma_{N,p}\,|B_1|} \int_{\mathbb{R}^N} \left|\chi_{B_1}(h)(z\cdot h)_+^{p-2} - \chi_{B_1(-\frac{x-y}{\varepsilon})}(h)\, (z\cdot (h+\tfrac{x-y}{\varepsilon}))_+^{p-2}\right| dh.
\end{aligned}$$

We focus on the integral above. We can assume without loss of generality that $z = e_1$, otherwise we could perform a change of variables. In addition, and for the sake of simplicity, we denote $\xi = -(x-y)/\varepsilon$. Then the result follows from the following claim:

$$\lim_{\xi\to 0} \int_{\mathbb{R}^N} \left|\chi_{B_1}(h)\,(h_1)_+^{p-2} - \chi_{B_1(\xi)}(h)\,(h_1-\xi_1)_+^{p-2}\right| dh = 0. \tag{3.2}$$

To see this we need to distinguish two cases depending on the value of p.

(i) *Case* $p \ge 2$. The integrand in (3.2) converges to zero as $\xi \to 0$ for almost every $h \in \mathbb{R}^N$. Moreover, it is bounded by 2 and zero outside a bounded set. Then the claim follows by the dominated convergence theorem as $\xi \to 0$.

(ii) *Case* $1 < p < 2$. In order to apply the dominated convergence theorem when $1 < p < 2$, we observe similarly as in the proof of Lemma 3.2 that

$$\begin{aligned}
&\left|\chi_{B_1}(h)\,(h_1)_+^{p-2} - \chi_{B_1(\xi)}(h)\,(h_1-\xi_1)_+^{p-2}\right|\\
&\qquad\le \left(\chi_{B_1}(h)\,(h_1)_+^{p-2} + \chi_{B_1(\xi)}(h)\,(h_1-\xi_1)_+^{p-2}\right)\\
&\qquad\quad\cdot \left(1 - \frac{\min\{\chi_{B_1}(h)\,(h_1)_+^{p-2}, \chi_{B_1(\xi)}(h)\,(h_1-\xi_1)_+^{p-2}\}}{\max\{\chi_{B_1}(h)\,(h_1)_+^{p-2}, \chi_{B_1(\xi)}(h)\,(h_1-\xi_1)_+^{p-2}\}}\right).
\end{aligned}$$

In that way, applying Hölder's inequality with $q = \frac{1}{2}\frac{p-3}{p-2}$,

$$\int_{D_0} \left|\chi_{B_1}(h)\,(h_1)_+^{p-2} - \chi_{B_1(\xi)}(h)\,(h_1-\xi_1)_+^{p-2}\right| dh$$
$$\leq C\left(\int_{\mathbb{R}^N}\left(1 - \frac{\min\{\chi_{B_1}(h)\,(h_1)_+^{p-2}, \chi_{B_1(\xi)}(h)\,(h_1-\xi_1)_+^{p-2}\}}{\max\{\chi_{B_1}(h)\,(h_1)_+^{p-2}, \chi_{B_1(\xi)}(h)\,(h_1-\xi_1)_+^{p-2}\}}\right)^{-\frac{p-3}{p-1}} dh\right)^{-\frac{p-1}{p-3}}$$

for every small enough ξ, where $C > 0$ only depends on N and p. Now again, the right-hand side in the integral above is bounded between 0 and 1. Moreover, the integrand converges to 0 as $\xi \to 0$ for almost every $h \in \mathbb{R}^N$, so that the dominated convergence theorem implies (3.2). Moreover, since the estimates obtained in this proof hold uniformly for every $x \in \overline{\Omega}$ and $|z| = 1$, this implies the uniform equicontinuity of the family $x \mapsto \mathcal{I}_\varepsilon^z u(x)$ with respect to $|z| = 1$. ∎

As a direct consequence of the continuity estimate from the previous lemma, we get the following result.

Lemma 3.4. *Let $\Omega \subset \mathbb{R}^N$ and let $u\colon \Omega_\varepsilon \to \mathbb{R}$ be a Borel measurable bounded function. Then the function*

$$x \longmapsto \frac{1}{2}\Big(\sup_{|z|=1} \mathcal{I}_\varepsilon^z u(x) + \inf_{|z|=1} \mathcal{I}_\varepsilon^z u(x)\Big)$$

is continuous in $\overline{\Omega}$.

Proof. The result follows directly from the equicontinuity in $\overline{\Omega}$ of the set of functions $\{\mathcal{I}_\varepsilon^z u : |z| = 1\}$ (Lemma 3.3) and the elementary inequalities

$$\sup_{|z|=1} \mathcal{I}_\varepsilon^z u(x) - \sup_{|z|=1} \mathcal{I}_\varepsilon^z u(y) \leq \sup_{|z|=1} \big\{\mathcal{I}_\varepsilon^z u(x) - \mathcal{I}_\varepsilon^z u(y)\big\},$$
$$\inf_{|z|=1} \mathcal{I}_\varepsilon^z u(x) - \inf_{|z|=1} \mathcal{I}_\varepsilon^z u(y) \leq \sup_{|z|=1} \big\{\mathcal{I}_\varepsilon^z u(x) - \mathcal{I}_\varepsilon^z u(y)\big\}.$$ ∎

3.2. Existence and uniqueness

Next we show the existence of Borel measurable solutions to the DPP (2.4). We also establish a comparison principle and thus uniqueness of solutions.

We remark that, contrary to the existence proofs in [4, 8], no boundary correction is needed as in those references, since Lemma 3.3 guarantees that $u - \varepsilon^2 f$ is continuous in Ω, and the solutions to (2.4) are measurable. Also recall that measurability of operators containing sup and inf is not completely trivial, as shown by Example 2.4 in [17].

The proof of existence of solutions to the DPP (2.4) with prescribed values in

$$\Gamma_\varepsilon = \Omega_\varepsilon \setminus \Omega$$

is based on Perron's method.

For that, given Borel measurable bounded functions $f\colon \Omega \to \mathbb{R}$ and $g\colon \Gamma_\varepsilon \to \mathbb{R}$, we consider the family $\mathcal{S}_{f,g}$ of Borel measurable functions $u\colon \Omega_\varepsilon \to \mathbb{R}$ such that $u - \varepsilon^2 f$ is continuous in Ω and

$$\begin{cases} u \le \dfrac{1}{2}\Big(\sup\limits_{|z|=1} \mathcal{J}_\varepsilon^z u + \inf\limits_{|z|=1} \mathcal{J}_\varepsilon^z u\Big) + \varepsilon^2 f & \text{in } \Omega, \\ u \le g & \text{in } \Gamma_\varepsilon. \end{cases} \tag{3.3}$$

In the PDE literature, the corresponding class would be the class of subsolutions with suitable boundary conditions. In the following lemmas, we prove that $\mathcal{S}_{f,g}$ is non-empty and uniformly bounded.

Lemma 3.5. *Let f and g be Borel measurable bounded functions in Ω and Γ_ε, respectively. There exists a Borel measurable function $u\colon \Omega_\varepsilon \to \mathbb{R}$ such that $u - \varepsilon^2 f$ is continuous in Ω and u satisfies* (3.3) *with $u = g$ in Γ_ε.*

Proof. Let $C > 0$ be a constant to be fixed later, fix $R = \sup_{x\in\Omega_\varepsilon} |x|$, and define

$$u(x) = \begin{cases} C(|x|^2 - R^2) + \varepsilon^2 f(x) & \text{if } x \in \Omega, \\ g(x) & \text{if } x \in \Gamma_\varepsilon. \end{cases}$$

Then $u - \varepsilon^2 f$ is clearly continuous in Ω. To see that u satisfies (3.3), let

$$u_0(x) = C(|x|^2 - R^2) - \varepsilon^2 \|f\|_\infty - \|g\|_\infty$$

for every $x \in \Omega_\varepsilon$. Then $u_0 \le u$ in Ω_ε. By the linearity and the monotonicity of the operator $\mathcal{J}_\varepsilon^z$ (see Remark 3.1),

$$\begin{aligned} \frac{1}{2}\Big(\sup_{|z|=1} \mathcal{J}_\varepsilon^z u(x) + \inf_{|z|=1} \mathcal{J}_\varepsilon^z u(x)\Big) &\ge \frac{1}{2}\Big(\sup_{|z|=1} \mathcal{J}_\varepsilon^z u_0(x) + \inf_{|z|=1} \mathcal{J}_\varepsilon^z u_0(x)\Big) \\ &\ge \frac{1}{2} \inf_{|z|=1} \big\{\mathcal{J}_\varepsilon^z u_0(x) + \mathcal{J}_\varepsilon^{-z} u_0(x)\big\} \\ &= C \inf_{|z|=1} \Big\{\frac{1}{2\gamma_{N,p}} \fint_{B_1} |x + \varepsilon h|^2 |z\cdot h|^{p-2}\, dh\Big\} \\ &\quad - CR^2 - \varepsilon^2 \|f\|_\infty - \|g\|_\infty. \end{aligned}$$

By the symmetry properties and the identity (A.5), it turns out that

$$\frac{1}{2\gamma_{N,p}} \fint_{B_1} |x + \varepsilon h|^2 \, |z\cdot h|^{p-2}\, dh = |x|^2 + \varepsilon^2 \frac{N+p-2}{N+p} \ge |x|^2 + \frac{\varepsilon^2}{3}$$

holds for any $|z| = 1$, $N \ge 2$ and $p > 1$. Therefore,

$$\begin{aligned} \frac{1}{2}\Big(\sup_{|z|=1} \mathcal{J}_\varepsilon^z u(x) + \inf_{|z|=1} \mathcal{J}_\varepsilon^z u(x)\Big) &\ge C\Big(|x|^2 + \frac{\varepsilon^2}{3}\Big) - CR^2 - \varepsilon^2 \|f\|_\infty - \|g\|_\infty \\ &= u(x) - \varepsilon^2 f(x) + \Big(\frac{C\varepsilon^2}{3} - \varepsilon^2 \|f\|_\infty - \|g\|_\infty\Big). \end{aligned}$$

Then (3.3) follows for $C = 3(\|f\|_\infty + \varepsilon^{-2}\|g\|_\infty)$, since then the expression in the parenthesis right above equals zero. ■

Lemma 3.6. *Let f be a Borel measurable bounded function in Ω. If $u\colon \Omega_\varepsilon \to \mathbb{R}$ is a Borel measurable function satisfying (3.3) in Ω, then*

$$\sup_{\Omega} u \le C\varepsilon^2 \|f\|_\infty + \|g\|_\infty$$

for some constant $C > 0$ depending only on Ω and ε.

Proof. We start by extending the function u as $\|g\|_\infty$ outside Ω_ε. For each $x \in \Omega$, let

$$S_x = \{h \in B_1 : 1/2 \le |h| < 1 \ \text{ and } \ x \cdot h \ge 0\}.$$

Then we define the constant

$$\vartheta = \vartheta(N,p) = \frac{1}{2\gamma_{N,p}|B_1|} \int_{S_x} |z \cdot h|^{p-2}\, dh,$$

which is independent of $x \in \Omega$ and $|z| = 1$. Indeed, since

$$\int_{S_x \cap \{z\cdot h < 0\}} |z \cdot h|^{p-2}\, dh = \int_{S_{-x} \cap \{z\cdot h > 0\}} |z \cdot h|^{p-2}\, dh,$$

we can write

$$\begin{aligned}\int_{S_x} |z \cdot h|^{p-2}\, dh &= \int_{S_x \cap \{z\cdot h > 0\}} |z \cdot h|^{p-2}\, dh + \int_{S_x \cap \{z\cdot h < 0\}} |z \cdot h|^{p-2}\, dh \\ &= \int_{S_x \cap \{z\cdot h > 0\}} |z \cdot h|^{p-2}\, dh + \int_{S_{-x} \cap \{z\cdot h > 0\}} |z \cdot h|^{p-2}\, dh.\end{aligned}$$

Using this and the fact that $S_x \cup S_{-x} = B_1 \setminus B_{1/2}$, we get

$$\int_{S_x} |z \cdot h|^{p-2}\, dh = \int_{B_1 \setminus B_{1/2}} (z \cdot h)_+^{p-2}\, dh = \gamma_{N,p}|B_1|\Big(1 - \frac{1}{2^{N+p-2}}\Big).$$

In the last equality, we used the definition of $\gamma_{N,p}$ and a change of variables as

$$\begin{aligned}\int_{B_1 \setminus B_{1/2}} (z \cdot h)_+^{p-2}\, dh &= \int_{B_1} (z \cdot h)_+^{p-2}\, dh - \int_{B_{1/2}} (z \cdot h)_+^{p-2}\, dh \\ &= \int_{B_1} (z \cdot h)_+^{p-2}\, dh - 2^{-(N+p-2)} \int_{B_1} (z \cdot h)_+^{p-2}\, dh.\end{aligned}$$

Thus we obtain

$$\vartheta = \frac{1}{2} - \frac{1}{2^{N+p-1}} \in \Big(\frac{1}{4}, \frac{1}{2}\Big) \quad \text{for any } N \ge 2 \text{ and } 1 < p < \infty.$$

Let $x \in \Omega$. By Lemma 3.2, there exists $|z_0| = 1$ maximizing $\mathcal{I}_\varepsilon^z u(x)$ among all $|z| = 1$. Then

$$\begin{aligned}u(x) - \varepsilon^2 f(x) &\le \frac{1}{2}\Big(\sup_{|z|=1} \mathcal{I}_\varepsilon^z u(x) + \inf_{|z|=1} \mathcal{I}_\varepsilon^z u(x)\Big) \le \frac{1}{2}\big(\mathcal{I}_\varepsilon^{z_0} u(x) + \mathcal{I}_\varepsilon^{-z_0} u(x)\big) \\ &= \frac{1}{2\gamma_{N,p}} \fint_{B_1} u(x+\varepsilon h)|z_0 \cdot h|^{p-2}\, dh \le \vartheta \sup_{h \in B_1 \cap S_x} \{u(x+\varepsilon h)\} + (1-\vartheta) \sup_{\mathbb{R}^N} u.\end{aligned}$$

For each $k \in \mathbb{N}$, let $V_k = \mathbb{R}^N \setminus B_{\sqrt{k}\varepsilon/2}$. Since

$$|x + \varepsilon h|^2 \geq |x|^2 + \frac{\varepsilon^2}{4}$$

for every $h \in B_1 \cap S_x$, it turns out that $x + \varepsilon h \in V_{k+1}$ for every $h \in B_1 \cap S_x$ and $x \in V_k$. Therefore,

$$\sup_{V_k} u \leq \vartheta \sup_{V_{k+1}} u + (1-\vartheta) \sup_{\mathbb{R}^N} u + \varepsilon^2 \|f\|_\infty.$$

Iterating this inequality starting from $V_0 = \mathbb{R}^N$, we obtain

$$\sup_{\mathbb{R}^N} u \leq \vartheta^k \sup_{V_k} u + \Big(\sum_{j=0}^{k-1} \vartheta^j\Big) \Big((1-\vartheta) \sup_{\mathbb{R}^N} u + \varepsilon^2 \|f\|_\infty\Big),$$

and rearranging terms,

$$\sup_{\mathbb{R}^N} u \leq \sup_{V_k} u + \frac{1-\vartheta^k}{\vartheta^k(1-\vartheta)} \varepsilon^2 \|f\|_\infty.$$

Since Ω is bounded, choosing large enough $k_0 = k_0(\varepsilon, \Omega)$ we ensure that $\Omega \subset B_{\sqrt{k_0}\varepsilon/2}$, and thus $V_{k_0} \subset \mathbb{R}^N \setminus \Omega$. Thus there is necessarily a step $k \leq k_0$ so that $\sup_{V_k} u \leq \sup_{\mathbb{R}^N \setminus \Omega} u \leq \|g\|_\infty$. Using also that $\vartheta \in (1/4, 1/2)$, we get

$$\sup_{\Omega} u \leq \sup_{\mathbb{R}^N} u \leq \|g\|_\infty + 2^{2k+1} \varepsilon^2 \|f\|_\infty. \qquad \blacksquare$$

Now we have the necessary lemmas to work out the existence through Perron's method. The idea is to take the pointwise supremum of functions in $\mathcal{S}_{f,g}$, the family of Borel measurable functions u with $u - \varepsilon^2 f \in C(\Omega)$ satisfying (3.3) (this would be subsolutions in corresponding PDE context), and to show that this is the desired solution. Here we also utilize the continuity of $u - \varepsilon^2 f$ in Ω, so that the supremum of functions in the uncountable set $\mathcal{S}_{f,g}$ is measurable. Indeed, otherwise to the best of our knowledge, Perron's method does not work as such (unless $p = \infty$ [14]), but one needs to construct a countable sequence of functions as in [17] to guarantee the measurability.

Theorem 3.7. *Let f and g be Borel measurable bounded functions in Ω and Γ_ε, respectively. There exists a Borel measurable function $u\colon \Omega_\varepsilon \to \mathbb{R}$ satisfying*

$$\begin{cases} u = \dfrac{1}{2}\Big(\sup\limits_{|z|=1} \mathcal{I}_\varepsilon^z u + \inf\limits_{|z|=1} \mathcal{I}_\varepsilon^z u\Big) + \varepsilon^2 f & \textit{in } \Omega, \\ u = g & \textit{in } \Gamma_\varepsilon. \end{cases} \tag{3.4}$$

Proof. In view of Lemmas 3.5 and 3.6, the set $\mathcal{S}_{f,g}$ is non-empty and uniformly bounded. Thus, we can define $\bar{u}$ as the pointwise supremum of functions in $\mathcal{S}_{f,g}$, that is,

$$\bar{u}(x) = \sup_{u \in \mathcal{S}_{f,g}} u(x)$$

for each $x \in \Omega_\varepsilon$. The boundedness of $\bar{u}$ is immediate. Moreover, $\bar{u}$ is Borel measurable. Indeed, since $\bar{u} - \varepsilon^2 f$ can be expressed as the pointwise supremum of continuous functions $u - \varepsilon^2 f$ with $u \in \mathcal{S}_{f,g}$, it turns out that $\bar{u} - \varepsilon^2 f$ is lower semicontinuous in Ω, and thus measurable, so the measurability of $\bar{u}$ follows.

By Lemma 3.5, there exists at least one function u in $\mathcal{S}_{f,g}$ such that $u = g$ in Γ_ε, so $\bar{u}$ agrees with g in Γ_ε. On the other hand, since $\bar{u} \geq u$ for every $u \in \mathcal{S}_{f,g}$, then

$$u - \varepsilon^2 f \leq \frac{1}{2}\Big(\sup_{|z|=1} \mathcal{I}_\varepsilon^z u + \inf_{|z|=1} \mathcal{I}_\varepsilon^z u \Big) \leq \frac{1}{2}\Big(\sup_{|z|=1} \mathcal{I}_\varepsilon^z \bar{u} + \inf_{|z|=1} \mathcal{I}_\varepsilon^z \bar{u} \Big)$$

in Ω. Taking the pointwise supremum in $\mathcal{S}_{f,g}$, we have that

$$\bar{u} - \varepsilon^2 f \leq \frac{1}{2}\Big(\sup_{|z|=1} \mathcal{I}_\varepsilon^z \bar{u} + \inf_{|z|=1} \mathcal{I}_\varepsilon^z \bar{u} \Big). \tag{3.5}$$

Hence $\bar{u}$ is a Borel measurable bounded subsolution to (3.3) with $\bar{u} = g$ in Γ_ε. Next we show that $\bar{u} - \varepsilon^2 f$ is indeed continuous in Ω. For this, let $\widetilde{u} \colon \Omega_\varepsilon \to \mathbb{R}$ be the Borel measurable function defined by

$$\widetilde{u} = \begin{cases} \dfrac{1}{2}\Big(\sup\limits_{|z|=1} \mathcal{I}_\varepsilon^z \bar{u} + \inf\limits_{|z|=1} \mathcal{I}_\varepsilon^z \bar{u} \Big) + \varepsilon^2 f & \text{in } \Omega, \\ g & \text{in } \Gamma_\varepsilon. \end{cases}$$

Then $\bar{u} \leq \widetilde{u}$ in Ω by (3.5), so $\widetilde{u}$ is a subsolution to (3.3), since the right-hand side above can be estimated from above by $\frac{1}{2}\big(\sup_{|z|=1} \mathcal{I}_\varepsilon^z \widetilde{u} + \inf_{|z|=1} \mathcal{I}_\varepsilon^z \widetilde{u} \big) + \varepsilon^2 f$. Observe also that $\widetilde{u} - \varepsilon^2 f$ is continuous in Ω by Lemma 3.3, so $\widetilde{u} \in \mathcal{S}_{f,g}$. Thus $\widetilde{u} \leq \bar{u}$, and in consequence, $\bar{u} = \widetilde{u} \in \mathcal{S}_{f,g}$. Moreover,

$$\bar{u} = \frac{1}{2}\Big(\sup_{|z|=1} \mathcal{I}_\varepsilon^z \bar{u} + \inf_{|z|=1} \mathcal{I}_\varepsilon^z \bar{u} \Big) + \varepsilon^2 f$$

in Ω, and the proof is finished. ∎

The uniqueness of solutions to (3.4) is directly deduced from the following comparison principle.

Theorem 3.8. *Let f be a Borel measurable bounded function in Ω, and let $u, v : \Omega_\varepsilon \to \mathbb{R}$ be two Borel measurable solutions to the DPP (2.4) in Ω such that $u \leq v$ in Γ_ε. Then $u \leq v$ in Ω.*

Proof. For simplicity, we define $w = u - v$. Then w is continuous in Ω by Lemma 3.4 and $w \leq 0$ in Γ_ε. Furthermore, w is uniformly continuous in Ω, and thus we can define $\widetilde{w} \colon \Omega_\varepsilon \to \mathbb{R}$ by

$$\widetilde{w}(x) = \begin{cases} \lim\limits_{\Omega \ni y \to x} w(y) & \text{if } x \in \partial\Omega, \\ w(x) & \text{otherwise,} \end{cases}$$

so that $\widetilde{w} \in C(\overline{\Omega})$. Let us suppose thriving for a contradiction that

$$M = \sup_{\Omega_\varepsilon} w = \max_{\overline{\Omega}} \widetilde{w} > 0,$$

where the fact that $w \leq 0$ in Γ_ε is used above.

By continuity, the set $A = \{x \in \overline{\Omega} : \widetilde{w}(x) = M\}$ is non-empty and closed. Indeed, since $\overline{\Omega}$ is bounded, then A is compact. For any fixed $y \in \Omega$, by Lemma 3.2 there exist $|z_1| = |z_2| = 1$ such that

$$\mathcal{I}_\varepsilon^{z_1} u(y) = \sup_{|z|=1} \mathcal{I}_\varepsilon^z u(y) \quad \text{and} \quad \mathcal{I}_\varepsilon^{z_2} v(y) = \inf_{|z|=1} \mathcal{I}_\varepsilon^z v(y).$$

Then

$$\begin{aligned} w(y) &= u(y) - v(y) \\ &= \frac{1}{2}\Big(\sup_{|z|=1} \mathcal{I}_\varepsilon^z u(y) + \inf_{|z|=1} \mathcal{I}_\varepsilon^z u(y)\Big) - \frac{1}{2}\Big(\sup_{|z|=1} \mathcal{I}_\varepsilon^z v(y) + \inf_{|z|=1} \mathcal{I}_\varepsilon^z v(y)\Big) \\ &\leq \frac{1}{2}\left(\mathcal{I}_\varepsilon^{z_1} u(y) + \mathcal{I}_\varepsilon^{z_2} u(y)\right) - \frac{1}{2}\left(\mathcal{I}_\varepsilon^{z_1} v(y) + \mathcal{I}_\varepsilon^{z_2} v(y)\right) \\ &= \frac{1}{2}\left(\mathcal{I}_\varepsilon^{z_1} w(y) + \mathcal{I}_\varepsilon^{z_2} w(y)\right) \leq \sup_{|z|=1} \mathcal{I}_\varepsilon^z w(y). \end{aligned}$$

Using this, for any $x \in A$,

$$M = \widetilde{w}(x) = \lim_{\Omega \ni y \to x} w(y) \leq \lim_{\Omega \ni y \to x} \Big(\sup_{|z|=1} \mathcal{I}_\varepsilon^z w(y)\Big) = \sup_{|z|=1} \mathcal{I}_\varepsilon^z w(x) \leq M,$$

where the continuity of $x \mapsto \sup_{|z|=1} \mathcal{I}_\varepsilon^z w(x)$ by Lemma 3.3 has been used in the last equality. That is, $\sup_{|z|=1} \mathcal{I}_\varepsilon^z w(x) = M$, and again by Lemma 3.2, there exists $|z_0| = 1$ such that

$$\frac{1}{\gamma_{N,p}} \fint_{B_1} w(x + \varepsilon h)(z_0 \cdot h)_+^{p-2}\, dh = M.$$

By the definition of $\gamma_{N,p}$ and the fact that $w \leq M$, this implies that $w(x + \varepsilon h) = M$ for a.e. $|h| < 1$ such that $z_0 \cdot h > 0$. Then $x + \varepsilon h \in A \subset \overline{\Omega}$ for a.e. $|h| < 1$ such that $z_0 \cdot h > 0$. Moreover, by the continuity of $\widetilde{w}$ in $\overline{\Omega}$, it turns out that $x + \varepsilon h \in A$ for every $|h| \leq 1$ with $z_0 \cdot h \geq 0$. In particular, taking any $|h| = 1$ such that $z_0 \cdot h = 0$, we have that $x \pm \varepsilon h \in A$, so $x = \frac{1}{2}(x + \varepsilon h) + \frac{1}{2}(x - \varepsilon h)$. That is, any point $x \in A$ is the midpoint between two different points $x_1, x_2 \in A$. The contradiction follows by choosing $x \in A$ to be an extremal point of A, i.e., a point which cannot be written as a convex combination $\lambda x_1 + (1-\lambda)x_2$ of points $x_1, x_2 \in A$ with $\lambda \in (0,1)$ (take for instance any point $x \in A$ maximizing the Euclidean norm among all points in A). Then $M \leq 0$, and the proof is finished. ∎

4. Regularity for the tug-of-war game with $1 < p < 2$

The above DPP can also be stochastically interpreted. It is related to a two-player zero-sum game played in a bounded domain $\Omega \subset \mathbb{R}^N$. When the players are at $x \in \Omega$, they toss a fair coin and the winner of the toss may choose $z \in \mathbb{R}^N$, $|z| = 1$, so the next point is chosen according to the probability measure

$$A \mapsto \frac{1}{\gamma_{N,p}} \frac{1}{|B_\varepsilon(x)|} \int_{A \cap B_\varepsilon(x)} \left(z \cdot \frac{h-x}{\varepsilon}\right)_+^{p-2} dh.$$

Then the players play a new round starting from the current position. When the game exits the domain and the first point outside the domain is denoted by x_τ, Player II pays Player I the amount given by $F(x_\tau)$, where $F\colon \mathbb{R}^N \setminus \Omega \to \mathbb{R}$ is a given payoff function. Since Player I gets the payoff at the end, she tries (heuristically speaking) to maximize the outcome, and since Player II has to pay it, he tries to minimize it. This is a variant of a so called tug-of-war game considered for example in [19, 21, 22]. As explained in more detail in those references, u denotes the value of the game, i.e., the expected payoff of the game when players are optimizing over their strategies. Then for this u, the DPP holds and it can be heuristically interpreted by considering one round of the game and summing up the different outcomes (either Player I or Player II wins the toss) with the corresponding probabilities.

Next we show that if u is a solution to the DPP (2.4), then it satisfies the extremal inequalities (when $1 < p < 2$ and also $p = 2$) needed in order to apply the Hölder result in Theorem 2.2. However, in the case $2 < p < \infty$, the DPP (2.4) does not have any Pucci bounds, as we explain later in Remark 4.2.

Proposition 4.1. *Let $1 < p < 2$ and let u be a bounded Borel measurable function satisfying*

$$u(x) = \frac{1}{2}\Big(\sup_{|z|=1} \mathcal{I}_\varepsilon^z u(x) + \inf_{|z|=1} \mathcal{I}_\varepsilon^z u(x)\Big) + \varepsilon^2 f(x).$$

Then $\mathcal{L}_\varepsilon^+ u + f \ge 0$ and $\mathcal{L}_\varepsilon^- u + f \le 0$ for some $1-\alpha = \beta > 0$ depending on N and p, where $\mathcal{L}_\varepsilon^+$ and $\mathcal{L}_\varepsilon^-$ are the extremal operators as in Definition 2.1.

Proof. Let

$$1-\alpha = \beta = \frac{1}{2\gamma_{N,p}} \in (0,1)$$

and, for $|z| = 1$, consider the measure defined as

$$\nu(E) = \frac{1}{|B_1|}\int_{B_1\cap E} \frac{|z\cdot h|^{p-2}-1}{2\gamma_{N,p}-1}\,dh$$

for every Borel measurable set. Then $\nu \in \mathcal{M}(B_1)$. Indeed, by the definition of $\gamma_{N,p}$ and the fact that $\gamma_{N,p} > 1/2$ for $1 < p < 2$, we have that ν is a positive measure such that $\nu(B_1) = 1$. Therefore,

$$\begin{aligned}\alpha\int_{B_\Lambda} u(x+\varepsilon h)\,d\nu(h) + \beta \fint_{B_1} u(x+\varepsilon h)\,dh &= \frac{1}{2\gamma_{N,p}}\fint_{B_1} u(x+\varepsilon h)|z\cdot h|^{p-2}\,dh\\ &= \frac{1}{2}\big(\mathcal{I}_\varepsilon^z u(x) + \mathcal{I}_\varepsilon^{-z}u(x)\big),\end{aligned}$$

so

$$\mathcal{L}_\varepsilon^- u(x) \le \frac{\mathcal{I}_\varepsilon^z u(x) + \mathcal{I}_\varepsilon^{-z}u(x) - 2u(x)}{2\varepsilon^2} \le \mathcal{L}_\varepsilon^+ u(x)$$

for every $|z| = 1$. Now, if u is a solution to the DPP, then

$$\begin{aligned}-f(x) &= \frac{1}{2\varepsilon^2}\Big(\sup_{|z|=1}\mathcal{I}_\varepsilon^z u(x) + \inf_{|z|=1}\mathcal{I}_\varepsilon^z u(x) - 2u(x)\Big)\\ &\le \sup_{|z|=1}\left\{\frac{\mathcal{I}_\varepsilon^z u(x) + \mathcal{I}_\varepsilon^{-z}u(x) - 2u(x)}{2\varepsilon^2}\right\} \le \mathcal{L}_\varepsilon^+ u(x),\end{aligned}$$

and similarly for $\mathcal{L}_\varepsilon^- u(x) \le -f(x)$. ∎

Remark 4.2. The extremal inequalities do not hold for $2 < p < \infty$. Indeed, the map

$$E \mapsto \frac{1}{2\gamma_{N,p}|B_1|} \int_{B_1 \cap E} |z \cdot h|^{p-2}\, dh$$

defines a probability measure in B_1 which is absolutely continuous with respect the Lebesgue measure, and whose density function vanishes as h approaches an orthogonal direction to z when $p > 2$. Thus, it is not possible to decompose the measure as a convex combination of the uniform probability measure on B_1 and any probability measure ν, which is an essential step in the proof of the Hölder estimate in [3] and [2].

By Proposition 4.1, a solution u to the DPP (2.4) satisfies the conditions of Theorem 2.2. Thus it immediately follows that u is asymptotically Hölder continuous.

Corollary 4.3. *There exists $\varepsilon_0 > 0$ such that if u is a solution to the DPP* (2.4) *in B_R, where $\varepsilon < \varepsilon_0 R$, there exist $C, \gamma > 0$ (independent of ε) such that*

$$|u(x) - u(y)| \leq \frac{C}{R^\gamma}\Big(\sup_{B_R}|u| + R^2 \sup_{B_R}|f|\Big)\big(|x-y|^\gamma + \varepsilon^\gamma\big)$$

for every $x, y \in B_{R/2}$.

5. A connection to the p-Laplacian

In this section, we consider a connection of solutions to the DPP (2.4) to the viscosity solutions to

$$\Delta_p^{\mathrm{N}} u = -f, \tag{5.1}$$

where we now assume $f \in C(\overline{\Omega})$. Here $\Delta_p^{\mathrm{N}} u$ stands for the normalized p-Laplacian, which is the non-divergence form operator

$$\Delta_p^{\mathrm{N}} u = \Delta u + (p-2)\,\frac{\langle D^2 u \nabla u, \nabla u\rangle}{|\nabla u|^2}.$$

In Section 7 of [3], it was already pointed out that in the case $2 \leq p < \infty$ this follows (up to a multiplicative constant) for the dynamic programming principle describing the usual tug-of-war game (1.1), so here the main interest lies in the case $1 < p < 2$.

First, to establish the connection to the p-Laplace equation, we need to derive asymptotic expansions related to the DPP (2.4) for C^2-functions. The expansion below holds for the full range $1 < p < \infty$.

Proposition 5.1. *Let $u \in C^2(\Omega)$. If $1 < p < \infty$, then*

$$\begin{aligned} \mathcal{I}_\varepsilon^z u(x) = {} & u(x) + \varepsilon\,\frac{\gamma_{N,p+1}}{\gamma_{N,p}}\,\nabla u(x)\cdot z \\ & + \frac{\varepsilon^2}{2(N+p)}\big[\Delta u(x) + (p-2)\langle D^2 u(x) z, z\rangle\big] + o(\varepsilon^2). \end{aligned} \tag{5.2}$$

In particular, if $\nabla u(x) \neq 0$ *and* $z^* = \nabla u(x)/|\nabla u(x)|$, *then*

$$\lim_{\varepsilon \to 0} \frac{\mathcal{I}_\varepsilon^{z^*} u(x) + \mathcal{I}_\varepsilon^{-z^*} u(x) - 2u(x)}{2\varepsilon^2} = \frac{1}{2(N+p)} \Delta_p^{\mathrm{N}} u(x).$$

Proof. For the sake of simplicity, we use the notation for the tensor product of (column) vectors in $\mathbb{R}^N$, $v \otimes w = vw^\intercal$, which allows to write $\langle Mv, v\rangle = \mathrm{Tr}\{Mv \otimes v\}$. Using the second order Taylor's expansion of u, we obtain

$$\begin{aligned} \frac{\mathcal{I}_\varepsilon^{z} u(x) - u(x)}{\varepsilon} &= \frac{1}{\gamma_{N,p}} \fint_{B_1} \Big(\nabla u(x) \cdot h + \frac{\varepsilon}{2} \,\mathrm{Tr}\left\{ D^2 u(x) h \otimes h \right\} + o(\varepsilon) \Big) (z \cdot h)_+^{p-2}\, dh \\ &= \nabla u(x) \cdot \Big(\frac{1}{\gamma_{N,p}} \fint_{B_1} h (z \cdot h)_+^{p-2}\, dh \Big) \\ &\quad + \frac{\varepsilon}{2} \,\mathrm{Tr} \Big\{ D^2 u(x) \Big(\frac{1}{\gamma_{N,p}} \fint_{B_1} h \otimes h (z \cdot h)_+^{p-2}\, dh \Big) \Big\} + o(\varepsilon). \end{aligned} \tag{5.3}$$

In order to compute the first integral in the right-hand side of the previous identity, let R be any orthogonal transformation such that $Re_1 = z$, i.e., $e_1 = R^\intercal z$. Then a change of variables $Rw = h$ yields

$$\fint_{B_1} h (z \cdot h)_+^{p-2}\, dh = R \fint_{B_1} w (w_1)_+^{p-2}\, dw$$

using

$$z \cdot Rw = z^\intercal Rw = (R^\intercal z)^\intercal w = e_1^\intercal w = w_1.$$

Going back to the original notation, and observing by symmetry that

$$\fint_{B_1} h_i (h_1)_+^{p-2}\, dh = \begin{cases} \fint_{B_1} (h_1)_+^{p-1}\, dh = \gamma_{N,p+1} & \text{if } i = 1, \\ 0 & \text{if } i \neq 1, \end{cases}$$

we get

$$\fint_{B_1} h (z \cdot h)_+^{p-2}\, dh = R \fint_{B_1} h (h_1)_+^{p-2}\, dh = \gamma_{N,p+1} Re_1 = \gamma_{N,p+1}\, z.$$

Next we repeat the change of variables in the second integral on the right-hand side of (5.3) to get

$$\frac{1}{\gamma_{N,p}} \fint_{B_1} h \otimes h (z \cdot h)_+^{p-2}\, dh = R \Big(\frac{1}{\gamma_{N,p}} \fint_{B_1} h \otimes h (h_1)_+^{p-2}\, dh \Big) R^\intercal.$$

Observe that the integral in the parenthesis above is a diagonal matrix. Indeed, for $i \neq j$, by symmetry,

$$\frac{1}{\gamma_{N,p}} \fint_{B_1} h_i h_j (h_1)_+^{p-2}\, dh = 0.$$

In order to compute the diagonal elements, we utilize the explicit values of the normalization constants from Lemma A.2. Then

$$\frac{1}{\gamma_{N,p}}\fint_{B_1} h_i^2 (h_1)_+^{p-2}\,dh = \frac{1}{2\gamma_{N,p}}\fint_{B_1} h_i^2 |h_1|^{p-2}\,dh = \begin{cases} \frac{p-1}{N+p}, & i=1,\\ \frac{1}{N+p}, & i=2,\dots,n.\end{cases}$$

Combining, we get

$$\begin{aligned}\frac{1}{\gamma_{N,p}}\fint_{B_1} h\otimes h (z\cdot h)_+^{p-2}\,dh &= R\Big(\frac{1}{N+p}(I-e_1\otimes e_1)+\frac{p-1}{N+p}e_1\otimes e_1\Big)R^{\mathsf{T}}\\ &= \frac{1}{N+p}\,(I+(p-2)z\otimes z).\end{aligned}$$

The proof is concluded after replacing these integrals in the expansion for $\mathcal{I}_\varepsilon^z u(x)$. ■

Next we show that the solutions to the DPP (2.4) converge uniformly as $\varepsilon\to 0$ to a viscosity solution of

$$\Delta_p^{\mathrm{N}} u = -2(N+p)f.$$

But before, we recall the definition of viscosity solutions for the convenience of the reader. Below $\lambda_{\max}(D^2\phi(x_0))$ and $\lambda_{\min}(D^2\phi(x_0))$ refer to the largest and smallest eigenvalues, respectively, of $D^2\phi(x_0)$. This definition is equivalent to the standard way of defining viscosity solutions through convex envelopes. Different definitions of viscosity solutions in this context are analyzed for example in Section 2 of [9].

Definition 5.2. Let $\Omega\subset\mathbb{R}^N$ be a bounded domain and let $1<p<\infty$. A lower semicontinuous function u is a viscosity supersolution of (5.1) if for all $x_0\in\Omega$ and $\phi\in C^2(\Omega)$ such that $u-\phi$ attains a local minimum at x_0, one has

$$\begin{cases} \Delta_p^{\mathrm{N}}\phi(x_0)\le -f(x_0) & \text{if } \nabla\phi(x_0)\ne 0,\\ \Delta\phi(x_0)+(p-2)\,\lambda_{\max}(D^2\phi(x_0))\le -f(x_0) & \text{if } \nabla\phi(x_0)=0 \text{ and } p\ge 2,\\ \Delta\phi(x_0)+(p-2)\,\lambda_{\min}(D^2\phi(x_0))\le -f(x_0) & \text{if } \nabla\phi(x_0)=0 \text{ and } 1<p<2.\end{cases}$$

An upper semicontinuous function u is a viscosity subsolution of (5.1) if $-u$ is a supersolution. We say that u is a viscosity solution of (5.1) in Ω if it is both a viscosity sub- and supersolution.

Theorem 5.3. *Let $1<p<2$ and let $\{u_\varepsilon\}$ be a family of uniformly bounded Borel measurable solutions to the DPP (2.4). Then there are a subsequence and a Hölder continuous function u such that*

$$u_\varepsilon\to u \quad \textit{locally uniformly.}$$

Moreover, u is a viscosity solution to $\Delta_p^{\mathrm{N}} u=-2(N+p)f$.

Proof. First we can use the asymptotic Arzelà–Ascoli theorem (see Lemma 4.2 in [19]) in connection to Theorem 2.2 to find a locally uniformly converging subsequence to a Hölder continuous function. Then it remains to verify that the limit is a viscosity solution

to the p-Laplace equation. For $\phi \in C^2(\Omega)$, fix $x \in \Omega$. By Lemma 3.2, there exists $|z_1^\varepsilon| = 1$ such that
$$\mathcal{I}_\varepsilon^{z_1^\varepsilon}\phi(x) = \inf_{|z|=1} \mathcal{I}_\varepsilon^z\phi(x).$$

By Proposition 5.1,

$$\begin{aligned}
(5.4)\qquad &\frac{\sup_{|z|=1} \mathcal{I}_\varepsilon^z\phi(x) + \inf_{|z|=1} \mathcal{I}_\varepsilon^z\phi(x) - 2\phi(x)}{2\varepsilon^2}\\
&\quad \geq \frac{\mathcal{I}_\varepsilon^{z_1^\varepsilon}\phi(x) + \mathcal{I}_\varepsilon^{-z_1^\varepsilon}\phi(x) - 2\phi(x)}{2\varepsilon^2}\\
&\quad = \frac{1}{2(N+p)}\big[\Delta\phi(x) + (p-2)\operatorname{Tr}\{D^2\phi(x)z_1^\varepsilon \otimes z_1^\varepsilon\}\big] + \frac{o(\varepsilon^2)}{\varepsilon^2}.
\end{aligned}$$

Let u be the Hölder continuous limit obtained as a uniform limit of the solutions to the DPP. Choose a point $x_0 \in \Omega$ and a C^2-function ϕ defined in a neighborhood of x_0 touching u at x_0 from below. By the uniform convergence, there exists a sequence x_ε converging to x_0 such that $u_\varepsilon - \phi$ has a minimum at x_ε (see Section 10.1.1 in [7]) up to an error $\eta_\varepsilon > 0$, that is, there exists x_ε such that

$$u_\varepsilon(y) - \phi(y) \geq u_\varepsilon(x_\varepsilon) - \phi(x_\varepsilon) - \eta_\varepsilon$$

at the vicinity of x_ε. The arbitrary error η_ε is due to the fact that u_ε may be discontinuous and we might not attain the infimum. Moreover, by adding a constant, we may assume that $\phi(x_\varepsilon) = u_\varepsilon(x_\varepsilon)$, so that ϕ approximately touches u_ε from below. Recalling the fact that u_ε is a solution to the DPP (2.4) and that $\mathcal{I}_\varepsilon^z$ is monotone and linear (see Remark 3.1), we have that
$$\mathcal{I}_\varepsilon^z u_\varepsilon(x_\varepsilon) \geq \mathcal{I}_\varepsilon^z\phi(x_\varepsilon) + u_\varepsilon(x_\varepsilon) - \phi(x_\varepsilon) - \eta_\varepsilon.$$
Thus, by choosing $\eta_\varepsilon = o(\varepsilon^2)$, we obtain

$$\frac{o(\varepsilon^2)}{\varepsilon^2} \geq \frac{\sup_{|z|=1} \mathcal{I}_\varepsilon^z\phi(x_\varepsilon) + \inf_{|z|=1} \mathcal{I}_\varepsilon^z\phi(x_\varepsilon) - 2\phi(x_\varepsilon) + 2\varepsilon^2 f(x_\varepsilon)}{2\varepsilon^2}.$$

Using (5.4) at x_ε and combining this with the previous estimate, we obtain

$$(5.5)\qquad -f(x_\varepsilon) + \frac{o(\varepsilon^2)}{\varepsilon^2} \geq \frac{1}{2(N+p)}\big[\Delta\phi(x_\varepsilon) + (p-2)\operatorname{Tr}\{D^2\phi(x_\varepsilon)z_1^\varepsilon \otimes z_1^\varepsilon\}\big].$$

Let us assume first that $\nabla\phi(x_0) \neq 0$. By (5.2), we see that

$$\lim_{\varepsilon\to 0} z_1^\varepsilon = -\frac{\nabla\phi(x_0)}{|\nabla\phi(x_0)|},$$

and thus we end up with

$$-f(x_0) \geq \frac{1}{2(N+p)}\,\Delta_p^{\mathrm{N}}\phi(x_0).$$

Finally, we consider the case $\nabla\phi(x_0) = 0$. Similarly as above, (5.5) follows. Even if we now have no information on the convergence of z_1^ε, since

$$\lambda_{\min}(M) \le \operatorname{Tr}\{M z \otimes z\} \le \lambda_{\max}(M)$$

for every $|z| = 1$, we still can deduce

$$\begin{cases} \Delta\phi(x_0) + (p-2)\,\lambda_{\min}(D^2\phi(x_0)) \le -2(N+p) f(x_0) & \text{if } p \ge 2, \\ \Delta\phi(x_0) + (p-2)\,\lambda_{\max}(D^2\phi(x_0)) \le -2(N+p) f(x_0) & \text{if } 1 < p < 2. \end{cases}$$

Thus we have shown that u is a viscosity supersolution to the p-Laplace equation. Similarly, starting with sup instead of inf, we can show that u is a subsolution, and thus a solution. ∎

Remark 5.4. For $1 < p < \infty$, $u \in C^2(\Omega)$ and $x \in \Omega$ such that $\nabla u(x) \neq 0$, we could also show that

$$\frac{1}{2}\Big(\sup_{|z|=1} \mathcal{I}_\varepsilon^z u(x) + \inf_{|z|=1} \mathcal{I}_\varepsilon^z u(x)\Big) = u(x) + \frac{\varepsilon^2}{2(N+p)}\,\Delta_p^{\mathrm{N}} u(x) + o(\varepsilon^2).$$

Indeed, by working carefully through the estimates similarly as in Lemmas 2.1 and 2.2 of [22], we could show that

$$\frac{1}{2}\Big(\sup_{|z|=1} \mathcal{I}_\varepsilon^z u(x) + \inf_{|z|=1} \mathcal{I}_\varepsilon^z u(x)\Big) = \frac{\mathcal{I}_\varepsilon^{z^*} u(x) + \mathcal{I}_\varepsilon^{-z^*} u(x)}{2} + o(\varepsilon^2)$$

for $z^* = \nabla u(x)/|\nabla u(x)|$. By Proposition 5.1, we have

$$\frac{\mathcal{I}_\varepsilon^{z^*} u(x) + \mathcal{I}_\varepsilon^{-z^*} u(x)}{2} + o(\varepsilon^2) = u(x) + \frac{\varepsilon^2}{2(N+p)}\,\Delta_p^{\mathrm{N}} u(x) + o(\varepsilon^2),$$

and combining these estimates we obtain the desired estimate. This would give an alternative way to write down the proof that the limit is a p-harmonic function. Reading this expansion in a viscosity sense gives a different characterization of p-harmonic functions as in [18] and [9] by the same proof as above.

A. Some useful integrals

In this appendix, we record some useful integrals that, no doubt, are known to the experts but are hard to find in the literature.

Lemma A.1. *Let $\alpha_1, \dots, \alpha_N > -1$. Then*

$$\fint_{B_1} |h_1|^{\alpha_1} \cdots |h_N|^{\alpha_N}\, dh_1 \cdots dh_N = \frac{1}{\pi^{N/2}} \cdot \frac{\Gamma\big(\frac{N}{2}+1\big)\,\Gamma\big(\frac{\alpha_1+1}{2}\big)\cdots\Gamma\big(\frac{\alpha_N+1}{2}\big)}{\Gamma\big(\frac{N+\alpha_1+\cdots+\alpha_N+2}{2}\big)}. \tag{A.1}$$

Proof. For convenience, we denote by B_1^N the N-dimensional unit ball centered at the origin. We decompose the integral over B_1^N by integrating in the first place with respect to $t = h_N$, that is,

$$\begin{aligned}\int_{B_1^N} &|h_1|^{\alpha_1} \cdots |h_N|^{\alpha_N}\, dh_1 \cdots dh_N \\ &= \int_{-1}^{1} |t|^{\alpha_N} \Big(\int_{\sqrt{1-t^2}\, B_1^{N-1}} |h_1|^{\alpha_1} \cdots |h_{N-1}|^{\alpha_{N-1}}\, dh_1 \cdots dh_{N-1} \Big)\, dt.\end{aligned}$$

Then the change of variables $\sqrt{1-t^2}\,(w_1, \ldots, w_{N-1}) = (h_1, \ldots, h_{N-1})$ and returning to the original notation gives

$$\begin{aligned}&\int_{B_1^N} |h_1|^{\alpha_1} \cdots |h_N|^{\alpha_N}\, dh_1 \cdots dh_N \\ &= \int_{-1}^{1} |t|^{\alpha_N} (1-t^2)^{(N+\alpha_1+\cdots+\alpha_{N-1}-1)/2}\, dt \cdot \int_{B_1^{N-1}} |h_1|^{\alpha_1} \cdots |h_{N-1}|^{\alpha_{N-1}}\, dh_1 \cdots dh_{N-1}.\end{aligned}$$

Next we focus on the first integral in the right-hand side. Using the symmetry properties and performing a change of variables, we get

$$\begin{aligned}\int_{-1}^{1} &|t|^{\alpha_N} (1-t^2)^{(N+\alpha_1+\cdots+\alpha_{N-1}-1)/2}\, dt = 2 \int_0^1 t^{\alpha_N}\, (1-t^2)^{(N+\alpha_1+\cdots+\alpha_{N-1}-1)/2}\, dt \\ &= \int_0^1 t^{(\alpha_N-1)/2}\, (1-t)^{(N+\alpha_1+\cdots+\alpha_{N-1}-1)/2}\, dt = \frac{\Gamma\big(\frac{\alpha_N+1}{2}\big)\, \Gamma\big(\frac{N+\alpha_1+\cdots+\alpha_{N-1}+1}{2}\big)}{\Gamma\big(\frac{N+\alpha_1+\cdots+\alpha_{N-1}+\alpha_N+2}{2}\big)},\end{aligned}$$

where for the last equality we have recalled the well-known formula arising in connection to the β-function

$$\int_0^1 t^{x-1}(1-t)^{y-1}\, dt = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$$

for $x, y > 0$. In the same way, we in general obtain

$$\begin{aligned}\int_{B_1^{N-k}} &|h_1|^{\alpha_1} \cdots |h_{N-k}|^{\alpha_{N-k}}\, dh_1 \cdots dh_{N-k} \\ &= \int_{-1}^{1} |t|^{\alpha_{N-k}}\, (1-t^2)^{(\alpha_1+\cdots+\alpha_{N-k-1}+N-k-1)/2}\, dt \\ &\quad \cdot \int_{B_1^{N-k-1}} |h_1|^{\alpha_1} \cdots |h_{N-k-1}|^{\alpha_{N-k-1}}\, dh_1 \cdots dh_{N-k-1} \\ &= \int_0^1 |t|^{(\alpha_{N-k}+1)/2-1}\, (1-t)^{(\alpha_1+\cdots+\alpha_{N-k-1}+N-k+1)/2-1}\, dt \\ &\quad \cdot \int_{B_1^{N-k-1}} |h_1|^{\alpha_1} \cdots |h_{N-k-1}|^{\alpha_{N-k-1}}\, dh_1 \cdots dh_{N-k-1}.\end{aligned}$$

Iterating the above formula, and dividing out the repeating terms, we get

$$\int_{B_1^N} |h_1|^{\alpha_1} \cdots |h_N|^{\alpha_N} \, dh_1 \cdots dh_N$$
$$= \frac{1}{\Gamma\big(\frac{\alpha_1+\cdots+\alpha_N+N+2}{2}\big)} \Gamma\big(\tfrac{\alpha_N+1}{2}\big) \cdots \Gamma\big(\tfrac{\alpha_2+1}{2}\big) \Gamma\big(\tfrac{\alpha_1+3}{2}\big) \int_{-1}^{1} |h_1|^{\alpha_1} \, dh_1.$$

Then using the definition of the Gamma function and integration by parts, we get

$$\Gamma\big(\tfrac{\alpha_1+3}{2}\big) \int_{-1}^{1} |h_1|^{\alpha_1} \, dh_1 = \Gamma\big(\tfrac{\alpha_1+1}{2} + 1\big) \frac{2}{\alpha_1 + 1} = \Gamma\big(\tfrac{\alpha_1+1}{2}\big).$$

Finally, the result follows by recalling that

$$|B_1^N| = \frac{\pi^{N/2}}{\Gamma(N/2+1)}. \qquad \blacksquare$$

The next lemma follows as a special case of the previous result. This lemma is the one we actually use in the proofs.

Lemma A.2. *Let* $1 < p < \infty$*. Then*

$$\frac{1}{2\gamma_{N,p}} \fint_{B_1} |h_1|^p \, dh = \frac{p-1}{N+p} \tag{A.2}$$

and

$$\frac{1}{2\gamma_{N,p}} \fint_{B_1} |h_1|^{p-2} |h_2|^2 \, dh = \frac{1}{N+p}. \tag{A.3}$$

Proof. First we recall the definition of $\gamma_{N,p}$ in (2.2) and use (A.1) to obtain

$$\gamma_{N,p} = \fint_{B_1} (z \cdot h)_+^{p-2} \, dh = \frac{1}{2} \fint_{B_1} |h_1|^{p-2} \, dh = \frac{1}{2\sqrt{\pi}} \cdot \frac{\Gamma\big(\frac{N}{2}+1\big)\Gamma\big(\frac{p-1}{2}\big)}{\Gamma\big(\frac{N+p}{2}\big)}. \tag{A.4}$$

Since $\Gamma(s+1) = s\,\Gamma(s)$ and $\Gamma(1/2) = \sqrt{\pi}$, applying the identity (A.1) with $\alpha_1 = p$ and $\alpha_2 = \cdots = \alpha_N = 0$, we have

$$\fint_{B_1} |h_1|^p \, dh = \frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma(\frac{N}{2}+1)\Gamma(\frac{p+1}{2})}{\Gamma(\frac{N+p+2}{2})} = \frac{p-1}{N+p} \cdot \frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma(\frac{N}{2}+1)\Gamma(\frac{p-1}{2})}{\Gamma(\frac{N+p}{2})},$$

and (A.2) follows by combining the previous formulas. Similarly, since $\Gamma(3/2) = \sqrt{\pi}/2$,

$$\fint_{B_1} |h_1|^{p-2} |h_2|^2 \, dh = \frac{1}{\pi} \cdot \frac{\Gamma\big(\frac{N}{2}+1\big)\Gamma\big(\frac{p-1}{2}\big)\Gamma\big(\frac{3}{2}\big)}{\Gamma\big(\frac{N+p+2}{2}\big)}$$
$$= \frac{1}{N+p} \cdot \frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma\big(\frac{N}{2}+1\big)\Gamma\big(\frac{p-1}{2}\big)}{\Gamma\big(\frac{N+p}{2}\big)},$$

and (A.3) follows. $\blacksquare$

Corollary A.3. *Let* $1 < p < \infty$ *and* $|z| = 1$. *Then*

$$\frac{1}{2\gamma_{N,p}} \fint_{B_1} |h|^2 |z \cdot h|^{p-2} \, dh = \frac{N+p-2}{N+p}. \tag{A.5}$$

Proof. By symmetry, we can take $z = e_1$, so $z \cdot h = h_1$. Then

$$|h|^2 |h_1|^{p-2} = |h_1|^p + |h_1|^{p-2} |h_2|^2 + \cdots + |h_1|^{p-2} |h_N|^2,$$

so

$$\fint_{B_1} |h|^2 |z \cdot h|^{p-2} \, dh = \fint_{B_1} |h_1|^p \, dh + (N-1) \fint_{B_1} |h_1|^{p-2} \, |h_2| \, dh,$$

and thus (A.5) follows from (A.2) and (A.3). ∎

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