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On a reverse Kohler-Jobin inequality

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Abstract. In this paper, we consider the shape optimization problems for the quantities $\lambda(\Omega)T^q(\Omega)$, where Ω varies among open sets of $\mathbb{R}^d$ with a prescribed Lebesgue measure. While the characterization of the infimum is completely clear, the same does not happen for the maximization in the case $q > 1$. We prove that for q large enough a maximizing domain exists among quasi-open sets and that the ball is optimal among *nearly spherical domains*.

1. Introduction

In the present paper, we consider two well-known quantities that appear in the study of elliptic equations in the Euclidean space $\mathbb{R}^d$, $d \geq 2$. The first one is usually called *torsional rigidity*, and is defined, for every nonempty open set $\Omega \subset \mathbb{R}^d$ with finite Lebesgue measure (in the following, a *domain*), as

$$T(\Omega) = \int w_\Omega \, dx,$$

where w_Ω is the unique solution of the PDE

$$-\Delta u = 1 \quad \text{in } \Omega, \quad u \in H_0^1(\Omega).$$

Equivalently, we may define $T(\Omega)$ as

$$T(\Omega) = \max\left\{\frac{(\int u \, dx)^2}{\int |\nabla u|^2 \, dx} : u \in H_0^1(\Omega) \setminus \{0\}\right\}.$$

In the integrals above and in the following, we use the convention that integrals without the indicated domain are intended over the entire space $\mathbb{R}^d$. The quantity $T(\Omega)$ satisfies the scaling property

$$T(t\Omega) = t^{d+2} T(\Omega) \quad \text{for every } t > 0;$$

in addition, the maximum of $T(\Omega)$ among domains with prescribed measure is reached by the ball (*Saint-Venant's inequality*), which can be written in the scaling free formulation as

$$|\Omega|^{-(d+2)/d} \, T(\Omega) \leq |B|^{-(d+2)/d} \, T(B),$$

for every domain Ω and for every ball $B \subset \mathbb{R}^d$.

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The second quantity is the *first eigenvalue* $\lambda(\Omega)$ of the Dirichlet Laplacian, defined as the smallest λ such that the PDE

$$-\Delta u = \lambda u \quad \text{in } \Omega, \quad u \in H^1_0(\Omega),$$

admits a nonzero solution. Equivalently, $\lambda(\Omega)$ can be defined through the minimization of the Rayleigh quotient

$$\lambda(\Omega) = \min\left\{\frac{\int |\nabla u|^2\,dx}{\int u^2\,dx} : u \in H^1_0(\Omega) \setminus \{0\}\right\}.$$

The quantity $\lambda(\Omega)$ satisfies the scaling property

$$\lambda(t\Omega) = t^{-2}\,\lambda(\Omega) \qquad \text{for every } t > 0;$$

in addition, the minimum of $\lambda(\Omega)$ among domains with prescribed measure is reached by the ball (the *Faber–Krahn inequality*), which can be written in the scaling free formulation as

$$|\Omega|^{2/d}\,\lambda(\Omega) \ge |B|^{2/d}\,\lambda(B),$$

for every domain Ω and for every ball $B \subset \mathbb{R}^d$.

The study of relations between $T(\Omega)$ and $\lambda(\Omega)$ was performed in several papers (see for instance [1–4, 6, 12, 13, 18, 21–23]), where some important inequalities were established. In particular,

- the Kohler-Jobin inequality

  $$\lambda(\Omega)\,T^q(\Omega) \ge \lambda(B)\,T^q(B),$$

  valid for every $q \in [0, 2/(d+2)]$ and for every domain Ω, where B is any ball in $\mathbb{R}^d$ with $|B| = |\Omega|$;
- the Pólya inequality

  $$0 < \frac{\lambda(\Omega)\,T(\Omega)}{|\Omega|} < 1,$$

  valid for every domain Ω of $\mathbb{R}^d$.

In the present paper, we consider the scaling free shape functional

$$F_q(\Omega) = \frac{\lambda(\Omega)\,T^q(\Omega)}{|\Omega|^{\alpha_q}}, \quad \text{with} \quad \alpha_q = \frac{-2 + q(d+2)}{d},$$

and the two quantities

$$m_q = \inf\left\{F_q(\Omega) : \Omega \text{ domain}\right\} \quad \text{and} \quad M_q = \sup\left\{F_q(\Omega) : \Omega \text{ domain}\right\}.$$

While the situation for m_q is fully clear, and by the Kohler-Jobin inequality, together with the Saint-Venant inequality, we have

$$m_q = \begin{cases} F_q(B) & \text{if } q \le 2/(d+2), \\ 0 & \text{if } q > 2/(d+2), \end{cases}$$

the characterization of M_q is not yet complete. The results available up to now are the following (see [1] and [3]):

- $M_q = \infty$ for every $q < 1$;
- $M_q = 1$ when $q = 1$, with the upper bound 1 not reached by any domain Ω;
- $M_q < \infty$ for every $q > 1$.

We investigate here this last case. The maximal expectation would be having the following result (the reverse Kohler-Jobin inequality):

- for every $q > 1$, the supremum M_q is attained on an optimal domain Ω_q;
- for every $q > 1$, the free boundary $\partial\Omega_q$ is a smooth $d-1$ surface;
- there exists a threshold $q^* > 1$ such that for every $q \geq q^*$, the supremum M_q is attained by a ball[1].

We are unable to prove the results in the strong form above. We prove here the weaker results below, for which we need to extend the functional F_q to the set of capacitary measures (see Section 2):

- for every $q > 1$, the supremum M_q is reached on a capacitary measure μ_q (Theorem 4.3);
- there exists a threshold $q_0 > 1$ such that, for every $q \geq q_0$, the supremum M_q is reached by a quasi-open set Ω_q (Theorem 5.3);
- there exists another threshold q_1 such that, for every $q \geq q_1$, the ball is a maximizer for the shape functional F_q among *nearly spherical domains* (Theorem 6.2).

2. Capacitary measures

The concept of capacitary measure and the related properties shall be a very useful tool for our purposes. When dealing with sequences of PDEs of the form

$$-\Delta u = f \quad \text{in } \Omega_n, \quad u \in H_0^1(\Omega_n),$$

a natural question is to establish if the sequence $u_{n,f}$ of solutions, or a subsequence of it, converges in L^2 to some function u_f, and to determine in this case the PDE that the function u_f solves. Starting from the pioneering papers [14, 15], it is now well understood that the right framework to treat such a kind of questions is that of capacitary measures. Below we recall the main results and definitions following [10] and [24]. For further information we refer the reader to the monographs [8] and [20], and references therein.

Definition 2.1. We say that a nonnegative Borel regular measure μ, possibly taking the value ∞, is a capacitary measure if

$$\mu(E) = 0 \text{ whenever } E \text{ is a Borel set with } \mathrm{cap}(E) = 0,$$

being $\mathrm{cap}(E)$ the capacity

$$\mathrm{cap}(E) = \inf\left\{ \int_{\mathbb{R}^d} |\nabla u|^2 + u^2\,dx \, : \, u \in H_0^1(\mathbb{R}^d), \ u = 1 \text{ in a neighborhood of } E \right\}.$$

[1] After the submission of this paper, we have been informed that D. Bucur and coauthors were working on similar problems. In particular, by proving a new sharp stability inequality for the spectrum of the Dirichlet Laplacian, they are able to show that such a q^* does exist. Their results are now collected in the preprint [11].

A property $P(x)$ is said to hold quasi-everywhere (briefly, q.e.) if the set where $P(x)$ does not hold has zero capacity. A Borel set $\Omega \subset \mathbb{R}^d$ is said to be quasi-open if there exists a function $u \in H^1(\mathbb{R}^d)$ such that $\Omega = \{u > 0\}$ up to a set of capacity zero. A function $f\colon \mathbb{R}^d \to \mathbb{R}$ is said to be quasi-continuous if there is a sequence of open sets $\omega_n \subset \mathbb{R}^d$ such that $\lim_{n\to\infty} \operatorname{cap}(\omega_n) = 0$ and f is continuous when restricted to $\mathbb{R}^d \setminus \omega_n$. It is well known (see for instance [19]) that every Sobolev function has a quasi-continuous representative, and that two quasi-continuous representatives coincide quasi-everywhere. We then identify the space $H^1(\mathbb{R}^d)$ with the space of quasi-continuous representatives. We recall that a sequence $u_n \in H^1(\mathbb{R}^d)$ that converges in norm to some $u \in H^1(\mathbb{R}^d)$, converges quasi-everywhere (up to a subsequence) to u.

Given μ a capacitary measure, we denote by H^1_μ the following space:

$$H^1_\mu = H^1(\mathbb{R}^d) \cap L^2_\mu(\mathbb{R}^d) = \Big\{u \in H^1(\mathbb{R}^d) : \int u^2\, d\mu < \infty\Big\}.$$

The space H^1_μ is a Hilbert space when endowed with $\|u\|_{H^1_\mu} = \|u\|_{H^1(\mathbb{R}^d)} + \|u\|_{L^2_\mu(\mathbb{R}^d)}$, where the quantity $\|u\|_{L^2_\mu(\mathbb{R}^d)}$ is well defined, being Sobolev functions defined up to a set of zero capacity. We always identify two capacitary measures μ and ν for which

$$\int u^2\, d\mu = \int u^2\, d\nu, \quad \text{for every } u \in H^1(\mathbb{R}^d). \tag{2.1}$$

If instead (2.1) holds with "$\le$", we say that $\mu \le \nu$, and in this case we have $H^1_\nu \subseteq H^1_\mu$. We can associate to any open set (or more generally to any quasi-open set) $\Omega \subset \mathbb{R}^d$ the capacitary measure I_Ω defined as follows:

$$I_\Omega(E) := \begin{cases} 0 & \text{if } \operatorname{cap}(E \setminus \Omega) = 0,\\ \infty & \text{if } \operatorname{cap}(E \setminus \Omega) > 0.\end{cases}$$

Notice that, if $\mu = I_\Omega$ for some open set $\Omega \subset \mathbb{R}^d$, then $H^1_\mu = H^1_0(\Omega)$.

To extend the notion of torsional rigidity to a capacitary measure μ, we need to carefully deal with the fact that the embedding $H^1_\mu \hookrightarrow L^1(\mathbb{R}^d)$ can be noncompact, and even noncontinuous. Nevertheless, we can follow an approximation argument: for every $R > 0$, let w_R be the solution to the minimization problem

$$\min\Big\{\int |\nabla u|^2\, dx + \int u^2\, d\mu - \int u\, dx \ :\ u \in H^1_\mu \cap H^1_0(B_R)\Big\}.$$

The torsion function w_μ and the torsional rigidity $T(\mu)$ of the capacitary measure μ are defined as

$$w_\mu := \sup_{R>0} w_R \quad \text{and} \quad T(\mu) := \int w_\mu\, dx.$$

The Dirichlet eigenvalue of μ can be defined through the following Rayleigh-type quotient:

$$\lambda_1(\mu) = \inf_{u \in H^1_\mu \setminus\{0\}} \frac{\int |\nabla u|^2\, dx + \int u^2\, d\mu}{\int u^2\, dx}.$$

Clearly, if $\mu = I_\Omega$ for some domain $\Omega \subset \mathbb{R}^d$, we have $T(\mu) = T(\Omega)$ and $\lambda(\mu) = \lambda(\Omega)$ (we adopt this notation also if Ω is a quasi-open set). For a general capacitary measure μ, neither $\lambda(\mu)$ is necessarily attained by some function $u \in H^1_\mu$ nor $T(\mu)$ is necessarily finite. However, as shown in [9], it holds the following:

$$w_\mu \in L^1(\mathbb{R}^d) \iff T(\mu) < \infty \implies \lambda_1(\mu) \text{ is attained by some } u \in H^1_\mu.$$

For every capacitary measure μ with $T(\mu) < \infty$, we define the *set of finiteness* A_μ as the quasi-open set

$$A_\mu := \{w_\mu > 0\}.$$

In the case when $\mu = I_\Omega$, for some domain $\Omega \subset \mathbb{R}^d$, we have $A_\mu = \Omega$. The set of capacitary measures with finite torsion can be endowed with the following notion of distance.

Definition 2.2. Given two capacitary measures μ and ν such that $w_\mu, w_\nu \in L^1(\mathbb{R}^d)$, we define the γ-distance between them as

$$d_\gamma(\mu, \nu) = \|w_\mu - w_\nu\|_{L^1(\mathbb{R}^d)}.$$

We say that a sequence μ_n γ-converges to μ if $d_\gamma(\mu_n, \mu) \to 0$ as $n \to \infty$. When $I_{\Omega_n} \xrightarrow{\gamma} \mu$, we simply write $\Omega_n \xrightarrow{\gamma} \mu$.

We summarize the main properties of the γ-distance below:

- The space $(\{\mu : \mu \text{ capacitary measure with } w_\mu \in L^1(\mathbb{R}^d)\}, d_\gamma)$ is a complete metric space, and the set $\{I_\Omega : \Omega \subset \mathbb{R}^d \text{ open set with } w_\Omega \in L^1(\mathbb{R}^d)\}$ is a dense subset of it.
- The functionals $\mu \mapsto \lambda(\mu)$ and $\mu \mapsto T(\mu)$ are γ-continuous.
- The map $\mu \mapsto |A_\mu|$, or more generally integral functionals as $\int_{A_\mu} f(x)\,dx$ with $f \geq 0$ and measurable, are lower semicontinuous with respect to the γ-convergence.
- The γ-convergence of μ_n to μ implies the Γ-convergence in $L^2(\mathbb{R}^d)$ of the functionals $\|\cdot\|_{H^1_{\mu_n}} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ defined by

$$\|u\|_{H^1_{\mu_n}} = \begin{cases} \|u\|_{H^1(\mathbb{R}^d)} + \int u^2\,d\mu_n & \text{if } u \in H^1_{\mu_n}, \\ \infty & \text{if } u \notin H^1_{\mu_n}, \end{cases}$$

  to the functional $\|\cdot\|_{H^1_\mu} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$,

$$\|u\|_{H^1_\mu} = \begin{cases} \|u\|_{H^1(\mathbb{R}^d)} + \int u^2\,d\mu & \text{if } u \in H^1_\mu, \\ \infty & \text{if } u \notin H^1_\mu. \end{cases}$$

- For a given capacitary measures μ with finite torsion, we call resolvent of μ the linear compact and self-adjoint operator

$$R_\mu : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d), \quad R_\mu(f) = w_{\mu,f},$$

  where $w_{\mu,f}$ is the solution of the problem

$$w_{\mu,f} \in H^1_\mu, \quad -\Delta w_{\mu,f} + w_{\mu,f}\mu = f,$$

in the sense that

$$w_{\mu,f} \in H^1_\mu, \quad \int \nabla w_{\mu,f} \cdot \nabla \phi \, dx + \int w_{\mu,f} \phi \, d\mu = \int f \phi \, dx \quad \text{for every } \phi \in H^1_\mu.$$

The γ-convergence of μ_n to μ implies the norm convergence of R_{μ_n} to R_μ, i.e.,

$$\lim_{n\to\infty} \|R_{\mu_n} - R_\mu\|_{\mathcal{L}(L^2(\mathbb{R}^d),L^2(\mathbb{R}^d))} = 0.$$

- If μ_n is a sequence of capacitary measures whose set of finiteness have uniformly bounded measures $|A_{\mu_n}|$, then

$$\mu_n \xrightarrow{\gamma} \mu \iff \|R_{\mu_n} - R_\mu\|_{\mathcal{L}(L^2(\mathbb{R}^d),L^2(\mathbb{R}^d))} \to 0 \iff \|u\|_{H^1_{\mu_n}} \xrightarrow{\Gamma} \|u\|_{H^1_\mu} \text{ on } L^2(\mathbb{R}^d).$$

The classical concentration-compactness principle of P. L. Lions was extended to sequences of open sets in [7]. Notably, the following result holds.

Theorem 2.3. *Let Ω_n be a sequence of open sets (more generally, quasi-open sets) with uniformly bounded measures. Then there exists a subsequence (still denoted with the same indices n) such that one of the following situations occurs.*

- *(Compactness). There exists a sequence $x_n \subset \mathbb{R}^d$ such that the sequence of capacitary measures $x_n + \Omega_n$ γ-converges.*
- *(Vanishing). The sequence $R_{I_{\Omega_n}}$ converges in norm to 0. Moreover, we have that $\|w_{\Omega_n}\|_{L^\infty} \to 0$ and $\lambda(\Omega_n) \to \infty$, as $n \to \infty$.*
- *(Dichotomy). There exist two sequences of quasi-open sets $\Omega_n^1, \Omega_n^2 \subset \Omega_n$ such that*
  - $\operatorname{dist}(\Omega_n^1, \Omega_n^2) \to \infty$, *as $n \to \infty$;*
  - $d_\gamma(I_{\Omega_n}, I_{\Omega_n^1 \cup \Omega_n^2}) \to 0$, *as $n \to \infty$;*
  - $\liminf_{n\to\infty} T(\Omega_n^1) > 0$ *and* $\liminf_{n\to\infty} T(\Omega_n^2) > 0$.

The proof of the theorem above can be deduced by combining Theorem 2.2 of [7] and Theorem 3.5 of [10].

3. Relaxation of F_q

In this section, we characterize the relaxation of the functional F_q to the set of capacitary measures. We define the set $\mathcal{M}_{\text{ad}}$ of admissible capacitary measures as

$$\mathcal{M}_{\text{ad}} = \{\mu : \mu \text{ capacitary measure with } 0 < |A_\mu| < \infty\}.$$

For $\mu \in \mathcal{M}_{\text{ad}}$, we define the relaxed form of our functional F_q as

$$F_q(\mu) = \sup\Big\{\limsup_n F_q(\Omega_n) : \Omega_n \subset \mathbb{R}^d \text{ domain such that } \Omega_n \xrightarrow{\gamma} \mu\Big\},$$

so that

$$M_q = \sup\{F_q(\mu) : \mu \in \mathcal{M}_{\text{ad}}\}.$$

Lemma 3.1. *Let $\mu \in \mathcal{M}_{\text{ad}}$ and let Ω_n be a sequence of domains such that $\Omega_n \xrightarrow{\gamma} \mu$. If $|A_\mu| < \infty$, then $\Omega_n \cap A_\mu \xrightarrow{\gamma} \mu$.*

Proof. Being the sequence $\Omega_n \cap A_\mu$ of uniformly bounded measure, by the properties of γ-convergence seen above, we have to show that

$$\|u\|_{H^1_{\mu_n}} \xrightarrow{\Gamma} \|u\|_{H^1_\mu} \quad \text{on } L^2(\mathbb{R}^d),$$

where we set $\mu_n = I_{\Omega_n \cap A_\mu}$.

The "Γ-liminf" inequality readily follows from the fact that $H^1_{\mu_n} = H^1_0(\Omega_n \cap A_\mu) \subseteq H^1_0(\Omega_n)$, and from the Γ-convergence of $\|\cdot\|_{H^1_0(\Omega_n)}$ to $\|\cdot\|_{H^1_\mu}$ in $L^2(\mathbb{R}^d)$.

To prove the "Γ-limsup" inequality, we can suppose without loss of generality that $u \in H^1_\mu$. Since $\Omega_n \xrightarrow{\gamma} \mu$, there exists a sequence $u_n \in H^1_0(\Omega_n)$ such that

$$u_n \to u \text{ strongly } L^2(\mathbb{R}^d) \quad \text{and} \quad \lim_{n\to\infty}\Big(\int |\nabla u_n|^2\,dx\Big) = \int |\nabla u|^2\,dx + \int |u|^2\,d\mu.$$

We denote respectively by u_n^+ and u_n^- the positive and negative part of u_n. Since we have

$$\int |\nabla(u_n^+ - u_n^-)|^2\,dx = \int |\nabla u_n^+|^2\,dx + \int |\nabla u_n^-|^2\,dx,$$

and $u_n = u_n^+ - u_n^-$, by possibly passing to a subsequence (still indexed by n), we can suppose that

$$\begin{aligned} &\limsup_{n\to\infty}\Big(\int |\nabla u_n^+|^2\,dx\Big) + \limsup_{n\to\infty}\Big(\int |\nabla u_n^-|^2\,dx\Big) \\ &\quad = \lim_{n\to\infty}\Big(\int |\nabla(u_n^+ - u_n^-)|^2\,dx\Big) = \int |\nabla u|^2\,dx + \int u^2\,d\mu. \end{aligned} \tag{3.1}$$

We define

$$v_n^+ = u_n^+ \wedge u^+ \in H^1(\mathbb{R}^d) \quad \text{and} \quad v_n^- = u_n^- \wedge u^- \in H^1(\mathbb{R}^d).$$

Since $u \in H^1_\mu$ and $u_n \in H^1_0(\Omega_n)$, we have $u = 0$ q.e. on A^c_μ and $u_n = 0$ q.e. on Ω^c_n. This implies that both v_n^+ and v_n^- vanish q.e. on $(\Omega_n \cap A_\mu)^c$, and consequently that $v_n^+, v_n^- \in H^1_0(\Omega_n \cap A_\mu)$. Moreover, it is easy to show that

$$v_n^+ - v_n^- \longrightarrow u \quad \text{strongly } L^2(\mathbb{R}^d).$$

Therefore the thesis is achieved if we show that

$$\limsup_{n\to\infty}\Big(\int |\nabla(v_n^+ - v_n^-)|^2\,dx\Big) \le \lim_{n\to\infty}\Big(\int |\nabla(u_n^+ - u_n^-)|^2\,dx\Big). \tag{3.2}$$

We have

$$\begin{aligned} \int |\nabla v_n^+|^2\,dx &= \int_{\{u_n^+ \le u^+\}} |\nabla u_n^+|^2\,dx + \int_{\{u_n^+ > u^+\}} |\nabla u^+|^2\,dx \\ &= \int |\nabla u_n^+|^2\,dx - \int \big(|\nabla u_n^+|^2 - |\nabla u|^2\big) 1_{\{u_n^+ > u^+\}}\,dx. \end{aligned} \tag{3.3}$$

By lower semicontinuity, we have

$$\liminf_n \int \big(|\nabla u_n^+|^2 - |\nabla u^+|^2\big) 1_{\{u_n^+ > u^+\}}\, dx \geq 0. \tag{3.4}$$

Indeed, to show the inequality above, it is enough to write

$$\begin{aligned}\int \big(|\nabla u_n^+|^2 - |\nabla u^+|^2\big)\, 1_{\{u_n^+ > u^+\}}\, dx &= \int \big(|\nabla u_n^+|^2 - |\nabla u^+|^2\big)\, 1_{\{u_n^+ \geq u^+\}}\, dx \\ &= \int |\nabla(u_n^+ \vee u^+)|^2 - |\nabla u^+|^2\, dx,\end{aligned}$$

and to notice that $u_n^+ \rightharpoonup u^+$ weakly in $H^1(\mathbb{R}^d)$ implies $u_n^+ \vee u^+ \rightharpoonup u^+$ weakly in $H^1(\mathbb{R}^d)$ and so, by lower semicontinuity,

$$\liminf_n \int |\nabla(u_n^+ \vee u^+)|^2 - |\nabla u^+|^2\, dx \geq 0.$$

Combining (3.3) and (3.4), we deduce that

$$\limsup_{n\to\infty} \Big(\int |\nabla v_n^+|^2\, dx\Big) \leq \limsup_{n\to\infty} \Big(\int |\nabla u_n^+|^2\, dx\Big). \tag{3.5}$$

Similarly we have

$$\limsup_{n\to\infty} \Big(\int |\nabla v_n^-|^2\, dx\Big) \leq \limsup_{n\to\infty} \Big(\int |\nabla u_n^-|^2\, dx\Big). \tag{3.6}$$

Combining (3.1), (3.5) and (3.6), we finally deduce (3.2), and this concludes the proof of the lemma. ■

Remark 3.2. By Lemma 3.1, for every measure $\mu \in \mathcal{M}_{\mathrm{ad}}$ there exists a sequence of quasi-open sets Ω_n (that can be taken open by a standard approximation procedure) such that $\Omega_n \xrightarrow{\gamma} \mu$ and for which

$$|\Omega_n| \to |A_\mu| \quad \text{as } n \to \infty.$$

To show this fact, we take $\Omega_n = A_\mu \cap O_n$, where O_n is any sequence of domains γ-converging to μ. We then have $|\Omega_n| \leq |A_\mu|$ and, by the lower semicontinuity of the Lebesgue measure with respect to γ-convergence (see the properties recalled in Section 2), we have $|A_\mu| \leq \liminf_n |\Omega_n|$. This in turns implies that the set

$$\{I_\Omega : \ \Omega \subset \mathbb{R}^d \text{ domain}\}$$

is γ-dense in $\mathcal{M}_{\mathrm{ad}}$. Furthermore, we can extend the Saint-Venant, Faber–Krahn and Pólya inequalities to any capacitary measure. That is,

$$|A_\mu|^{-(d+2)/d}\, T(\mu) \leq |B|^{-(d+2)/d}\, T(B), \quad |A_\mu|^{2/d}\, \lambda(\mu) \geq |B|^{2/d}\, \lambda(B), \tag{3.7}$$

and

$$0 < |A_\mu|^{-1}\, \lambda(\mu)\, T(\mu) < 1 \tag{3.8}$$

for every measure $\mu \in \mathcal{M}_{\mathrm{ad}}$ and every ball $B \subset \mathbb{R}^d$.

Proposition 3.3. *Let $\mu \in \mathcal{M}_{\rm ad}$. Then we have*

$$|A_\mu| = \inf\Big\{\liminf_n |\Omega_n| \,:\, \Omega_n \text{ domain, } \Omega_n \xrightarrow{\gamma} \mu\Big\}. \tag{3.9}$$

The quantity $|A_\mu|$ is then the relaxation, in the γ-convergence, of the Lebesgue measure $|\Omega|$. As a consequence, if $\alpha_q > 0$, we have

$$F_q(\mu) = \frac{\lambda(\mu)T^q(\mu)}{|A_\mu|^{\alpha_q}}. \tag{3.10}$$

Proof. The inequality $\le$ in (3.9) follows from the γ-lower semicontinuity of the map $\mu \mapsto |A_\mu|$ seen above. The opposite inequality follows at once by Remark 3.2. Since $T(\mu)$ and $\lambda(\mu)$ are γ-continuous, the proof of (3.10) is achieved by a similar argument. ∎

The scaling properties of the shape functionals $|\Omega|$, $\lambda(\Omega)$, $T(\Omega)$ and $F_q(\Omega)$ extend to their relaxations $|A_\mu|$, $\lambda(\mu)$, $T(\mu)$ and $F_q(\mu)$ in $\mathcal{M}_{\rm ad}$. More precisely, setting for $t > 0$,

$$\mu_t(E) = t^{d-2}\mu(E/t),$$

we have

$$|A_{\mu_t}| = t^d|A_\mu|, \quad \lambda(\mu_t) = t^{-2}\lambda(\mu), \quad T(\mu_t) = t^{d+2}T(\mu) \quad \text{and} \quad F_q(\mu_t) = F_q(\mu).$$

4. Existence of an optimal measure for $q > 1$

In [3], it is proved that the supremum $M_1 = 1$ is not attained in the class of domains. In the next proposition, we point out that the same occurs even in the class $\mathcal{M}_{\rm ad}$.

Proposition 4.1 (Nonexistence for $q = 1$ of an optimal measure). *The problem*

$$\sup\{F_1(\mu) : \ \mu \in \mathcal{M}_{\rm ad}\}$$

does not have a maximizer.

Proof. The proof follows at once by exploiting Theorem 1.1 in [3], which asserts that there exists a dimensional constant $c_d > 0$ for which

$$F_1(\Omega) \le 1 - \frac{c_d T(\Omega)}{|\Omega|^{1+2/d}}, \tag{4.1}$$

for every domain Ω. Then, for every $\mu \in \mathcal{M}_{\rm ad}$, by Remark 3.2 we can select a sequence $\Omega_n \xrightarrow{\gamma} A_\mu$ for which

$$F_1(\Omega_n) \to F_1(\mu), \quad T(\Omega_n) \to T(\mu), \quad |\Omega_n| \le |A_\mu| \quad \text{as } n \to \infty.$$

Thus, using (4.1) with $\Omega = \Omega_n$ and then passing to the limit as $n \to \infty$, we get that $F_1(\mu) < 1 = M_1$. ∎

To prove the main result of this section, we need the following elementary lemma.

Lemma 4.2. *Let $0 < c_1 < c_2 < \infty$ and $1 < \alpha_1 < \alpha_2 < \infty$. Then there exists $\beta < 1$ such that, for every $a, b, c, d \in (c_1, c_2)$, it holds*

$$\frac{(a+b)^{\alpha_1}}{(c+d)^{\alpha_2}} \le \beta \max\left\{\frac{a^{\alpha_1}}{c^{\alpha_2}}, \frac{b^{\alpha_1}}{d^{\alpha_2}}\right\}.$$

Proof. Letting $x = b/a$ and $y = d/c$, is enough to prove that

$$\frac{(1+x)^{\alpha_1}}{(1+y)^{\alpha_2}} \le \beta \max\left\{1, \frac{x^{\alpha_1}}{y^{\alpha_2}}\right\}.$$

Suppose that $x \le y$. Since $x \ge c_1/c_2$, it holds

$$(1+x)^{\alpha_1} = (1+x)^{\alpha_2}(1+x)^{\alpha_1-\alpha_2} \le (1+y)^{\alpha_2}\left(1+\frac{c_1}{c_2}\right)^{\alpha_1-\alpha_2}. \tag{4.2}$$

Similarly, if $x > y$, since $x \le c_2/c_1$, it holds

$$\left(1+\frac{1}{x}\right)^{\alpha_1} \le \left(1+\frac{1}{y}\right)^{\alpha_2}\left(1+\frac{1}{x}\right)^{\alpha_1-\alpha_2} \le \left(1+\frac{1}{y}\right)^{\alpha_2}\left(1+\frac{c_1}{c_2}\right)^{\alpha_1-\alpha_2}. \tag{4.3}$$

Eventually, we achieve the thesis by letting

$$\beta = \left(1+\frac{c_1}{c_2}\right)^{\alpha_1-\alpha_2}$$

and combining (4.2) and (4.3). ■

Theorem 4.3 (Existence for $q > 1$ of an optimal measure). *For every $q > 1$, there exists a measure $\mu^\star \in \mathcal{M}_{\text{ad}}$ such that*

$$F_q(\mu^\star) = \sup\{F_q(\mu) : \mu \in \mathcal{M}_{\text{ad}}\}.$$

Proof. We select a sequence $\mu_n \in \mathcal{M}_{\text{ad}}$ such that $F_q(\mu_n) \to M_q$, as $n \to \infty$. By density, we can suppose that $\mu_n = I_{\Omega_n}$, for some sequence of open sets Ω_n. Further, being F_q scaling free, we can also assume $|\Omega_n| = 1$. Hence, we can apply Theorem 2.3.

If dichotomy occurs, then there exist two sequences of quasi-open sets $\Omega_n^1, \Omega_n^2 \subset \Omega_n$ such that

$$\Omega_n^1 \cap \Omega_n^2 = \emptyset, \quad d_\gamma(I_{\Omega_n}, I_{\Omega_n^1 \cup \Omega_n^2}) \to 0 \quad \text{as } n \to \infty.$$

Taking into account the Saint-Venant inequality and the fact that $|\Omega_n| = 1$, there exist constants $c_1, c_2 > 0$, which depend only on the dimension, such that

$$c_1 < \inf_n |T(\Omega_n^i)| \le \sup_n |T(\Omega_n^i)| < c_2, \quad c_1 < \inf_n |\Omega_n^i| \le \sup_n |\Omega_n^i| < c_2, \text{ for } i = 1, 2.$$

Since λ is decreasing with respect to set inclusion, we have

$$\lambda(\Omega_n) \le \min\{\lambda(\Omega_n^1), \lambda(\Omega_n^2)\}. \tag{4.4}$$

Lemma 4.2 together with (4.4) gives

$$\frac{\lambda(\Omega_n)(T(\Omega_n^1\cup\Omega_n^2))^q}{|\Omega_n|^{\alpha_q}}\le\frac{\lambda(\Omega_n)\big(T(\Omega_n^1)+T(\Omega_n^2)\big)^q}{(|\Omega_n^1|+|\Omega_n^2|)^{\alpha_q}}\le\beta\max_{i=1,2}\frac{\lambda(\Omega_n^i)T^q(\Omega_n^i)}{|\Omega_n^i|^{\alpha_q}}.$$

By taking the limit as $n\to\infty$ in the latter inequality, we obtain the contradiction

$$\sup_{\mu\in\mathcal{M}_{\mathrm{ad}}}F_q(\mu)<\sup_{\mu\in\mathcal{M}_{\mathrm{ad}}}F_q(\mu),$$

and hence dichotomy cannot occur. Now, the maximality condition on the sequence Ω_n, together with Pólya's inequality, give that for n large enough,

$$\lambda(B)\,\frac{T^q(B)}{|B|^{\alpha_q}}\le\lambda(\Omega_n)T^q(\Omega_n)=\lambda(\Omega_n)T(\Omega_n)\cdot T^{q-1}(\Omega_n)\le T^{q-1}(\Omega_n),\tag{4.5}$$

where B is any ball of $\mathbb{R}^d$. In particular, it cannot be $\lim_{n\to\infty}T(\Omega_n)=0$, and this rules out the vanishing case.

Therefore, compactness holds and there exist a capacitary measure $\mu^\star$ and a sequence $x_n\in\mathbb{R}^d$ such that $I_{x_n+\Omega_n}$ γ-converges to $\mu^\star$.

By (4.5), we deduce that $T(\mu^\star)>0$, which by (3.7) implies $|A_{\mu^\star}|>0$, and hence that $\mu^\star$ belongs to $\mathcal{M}_{\mathrm{ad}}$. Clearly, the measure $\mu^\star$ maximizes the functional F_q on $\mathcal{M}_{\mathrm{ad}}$, and this concludes the proof. ■

5. Optimal measures are quasi-open sets for large q

We are now interested to prove that, when q is large enough, optimal measures μ coming from Theorem 4.3 can be represented as quasi-open sets. We begin by recalling the following result, see [17] and Proposition 3.83 in [24].

Theorem 5.1. *Let μ be a capacitary measure with finite torsion. Then the eigenfunctions $u\in L^2(\mathbb{R}^d)$ of the operator $-\Delta+\mu$ with unitary L^2 norm are in $L^\infty(\mathbb{R}^d)$, and satisfy*

$$\|u\|_\infty\le e^{1/(8\pi)}\lambda(\mu)^{d/4}.$$

We also use the following lemma.

Lemma 5.2. *For every $q>1$, let $\mu_q\in\mathcal{M}_{\mathrm{ad}}$ be a maximal measure for the functional F_q, such that $|A_{\mu_q}|=1$. Then*

$$\liminf_{q\to\infty}T(\mu_q)>0.$$

Proof. Let q_n be a diverging sequence and let $B\subset\mathbb{R}^d$ be a ball of unitary measure. By the density of domains among capacitary measures, we can select a sequence $\Omega_n\subset\mathbb{R}^d$ of open sets such that $|\Omega_n|=1$ for every n, and

$$|F_{q_n}(\Omega_n)-F_{q_n}(\mu_{q_n})|=o(T^{q_n}(B))\quad\text{as } n\to\infty.\tag{5.1}$$

Notice that $T^{q_n}(B) \to 0$ as $n \to \infty$. Then we can apply Theorem 2.3 to the sequence Ω_n. Dichotomy can be ruled out by the same argument as that of the proof of Theorem 4.3 once noticed that (3.8) implies

$$F_{q_n}^{1/q_n}(\mu_{q_n}) \le T^{(q_n-1)/q_n}(B) \to T(B) \quad \text{as } n \to \infty.$$

The vanishing case can be excluded too by following again the proof of Theorem 4.3. Indeed, for n large enough, Pólya's inequality and (5.1) imply

$$T^{q_n-1}(\Omega_n) \ge F_{q_n}(\Omega_n) \ge F_{q_n}(\mu_{q_n}) - |F_{q_n}(\Omega_n) - F_{q_n}(\mu_{q_n})| \ge F_{q_n}(B) + o(T^{q_n}(B)).$$

Hence we deduce

$$\liminf_{n\to\infty} T^{(1-1/q_n)}(\Omega_n) > 0,$$

which implies that it cannot be $T(\Omega_n) \to 0$, as $n \to \infty$. Therefore, compactness holds true and the sequence Ω_n has a subsequence (still denoted by the same indices) that γ-converges to some $\mu \in \mathcal{M}_{\mathrm{ad}}$ up to translations.

By the maximality of μ_{q_n}, it holds

$$F_{q_n}^{1/q_n}(B) \le F_{q_n}^{1/q_n}(\mu_{q_n}) = T(\Omega_n)(\lambda(\Omega_n) + o(1))^{1/q_n},$$

and we deduce, passing to the limit as $n \to \infty$,

$$T(B) \le T(\mu) = \lim_{n\to\infty} T(\Omega_n).$$

Since the sequence q_n was arbitrary, we obtain the conclusion. ∎

Theorem 5.3. *Let $\mu \in \mathcal{M}_{\mathrm{ad}}$ be an optimal measure for F_q with $q > 1$. There exists $q_0 > 1$ such that for $q > q_0$ we have $\mu = I_{A_\mu}$. In particular, the optimal measure can be represented by a quasi-open set.*

Proof. Since F_q is scaling free, we can suppose that $|A_\mu| = 1$. Let $\varepsilon > 0$ be a small parameter, and let μ_ε be the capacitary measure defined by

$$\mu_\varepsilon(E) = (1-\varepsilon)\mu(E).$$

Being $A_\mu = A_{\mu_\varepsilon}$, we have $\mu_\varepsilon \in \mathcal{M}_{\mathrm{ad}}$. We assume by contradiction that $\mu \ne I_{A_\mu}$ (notice that this implies $\mu_\varepsilon \ne \mu$). For the sake of brevity, we denote respectively by w and w_ε the torsion functions of μ and μ_ε. It is easy to verify that, as $\varepsilon \to 0$,

$$\|\cdot\|_{H^1_{\mu_\varepsilon}} \xrightarrow{\Gamma} \|\cdot\|_{H^1_\mu}, \quad \text{on } L^2(\mathbb{R}^d),$$

and therefore we have $\mu_\varepsilon \xrightarrow{\gamma} \mu$ and $w_\varepsilon \to w$ in $L^1(\mathbb{R}^d)$, as $\varepsilon \to 0$. Let us denote by $t(\varepsilon)$, $l(\varepsilon)$ and $f_q(\varepsilon)$ the real functions

$$\varepsilon \mapsto t(\varepsilon) = T(\mu_\varepsilon), \quad \varepsilon \mapsto l(\varepsilon) = \lambda(\mu_\varepsilon) \quad \text{and} \quad \varepsilon \mapsto f_q(\varepsilon) = F_q(\mu_\varepsilon),$$

and by $t'_+(0)$, $l'_+(0)$ and $(f_q)'_+(0)$ the limits for $\varepsilon \to 0$ of the respective different quotients.

By writing $w_\varepsilon = w + \varepsilon\xi_\varepsilon$ for some $\xi_\varepsilon \in L^1(\mathbb{R}^d)$ and using the fact that w and w_ε, respectively, weakly solve the PDEs

$$-\Delta w + w\mu = 1$$

and

$$-\Delta w_\varepsilon + w_\varepsilon \mu_\varepsilon = 1, \tag{5.2}$$

we deduce that ξ_ε weakly solves the PDE

$$-\Delta \xi_\varepsilon + \xi_\varepsilon \mu_\varepsilon = w\mu. \tag{5.3}$$

This allows us to compute the derivative

$$t'_+(0) = \lim_{\varepsilon\to 0}\Big(\int \xi_\varepsilon\, dx\Big) = \lim_{\varepsilon\to 0}\Big(\int \nabla w_\varepsilon \nabla \xi_\varepsilon dx + \int w_\varepsilon \xi_\varepsilon\, d\mu_\varepsilon\Big) = \lim_{\varepsilon\to 0}\Big(\int w w_\varepsilon\, d\mu\Big),$$

where we test (5.2) with ξ_ε and we use (5.3) tested with w_ε. Since as $\varepsilon \to 0$, $w_\varepsilon \to w$ in $L^1(\mathbb{R}^d)$, we obtain

$$t'_+(0) = \int w^2\, d\mu. \tag{5.4}$$

We can treat with a similar argument the eigenvalue. Let u and u_ε be the first eigenfunctions (with unitary L^2 norm), respectively, of the operators $-\Delta + \mu_\varepsilon$ and $-\Delta + \mu$, and let $v_\varepsilon \in L^2(\mathbb{R}^d)$ be such that $u_\varepsilon = u + \varepsilon v_\varepsilon$. Since

$$-\Delta u + u\mu = \lambda(\mu) u \quad \text{and} \quad -\Delta u_\varepsilon + u_\varepsilon \mu_\varepsilon = \lambda(\mu_\varepsilon) u_\varepsilon,$$

we have

$$-\Delta v_\varepsilon + v_\varepsilon \mu - u\mu - \varepsilon v_\varepsilon \mu = \Big(\frac{\lambda(\mu_\varepsilon) - \lambda(\mu)}{\varepsilon}\Big) u + \lambda(\mu_\varepsilon) v_\varepsilon.$$

By testing the PDE above with $u \in H^1_\mu$, and since $\int u^2\, dx = 1$, we obtain

$$\begin{aligned}\Big(\frac{\lambda(\mu_\varepsilon) - \lambda(\mu)}{\varepsilon}\Big) &= \int \nabla v_\varepsilon \nabla u\, dx + \int v_\varepsilon u\, d\mu - \int u^2\, d\mu \\ &\quad - \varepsilon \int v_\varepsilon u\, d\mu - \lambda(\mu_\varepsilon) \int v_\varepsilon u\, dx.\end{aligned}$$

By taking the limit as $\varepsilon \to 0$ and exploiting the fact that $u_\varepsilon \to u$ weakly in H^1_μ and $\lambda(\mu_\varepsilon) \to \lambda(\mu)$, we get

$$l'_+(0) = -\int u^2\, d\mu. \tag{5.5}$$

By combining (5.4) and (5.5), we get

$$(f_q)'_+(0) = l'_+(0) T^q(\mu) + q\lambda(\mu) T^{q-1}(\mu) t'_+(0) = F_q(\mu) \int \Big(-\frac{u^2}{\lambda(\mu)} + q\frac{w^2}{T(\mu)}\Big)\, d\mu.$$

Now, the optimality condition on μ implies $(f_q)'_+(0) \le 0$, and hence that

$$\int \Big(\frac{u^2}{\lambda(\mu)} - q\,\frac{w^2}{T(\mu)} \Big)\, d\mu \ge 0. \tag{5.6}$$

We claim that

$$\frac{u^2}{\lambda(\mu)} - q\,\frac{w^2}{T(\mu)} < 0 \quad \text{q.e. on } \mathbb{R}^d \tag{5.7}$$

for q large enough. Indeed, by an application of Theorem 5.1 together with a comparison principle, we have

$$u \le e^{1/(8\pi)}\,\lambda^{d/4+1}(\mu)\, w \quad \text{q.e. on } \mathbb{R}^d,$$

and so by the Pólya inequality,

$$u^2 \le e^{1/(4\pi)}\,\lambda^{d/2}(\mu)\,\frac{\lambda(\mu)}{T(\mu)}\, w^2 \quad \text{q.e. on } \mathbb{R}^d.$$

This implies that

$$\frac{u^2}{\lambda(\mu)} - q\,\frac{w^2}{T(\mu)} \le \frac{w^2}{T(\mu)} \big(e^{1/(4\pi)}\,\lambda^{d/2}(\mu) - q \big) \quad \text{q.e. on } \mathbb{R}^d.$$

Therefore, for every q such that

$$\sup_{\nu \text{ optimal}} \; e^{1/(4\pi)}\,\lambda^{d/2}(\nu) < q,$$

the inequality (5.7) is satisfied. Notice that the supremum in the inequality above is finite as a consequence of Lemma 5.2 combined again with Pólya's inequality.

To conclude, it is now enough to notice that (5.7) contradicts (5.6). ■

6. Optimality for nearly spherical domains

In the following, we consider the classes $\mathcal{S}_{\delta,\gamma}$ of *nearly spherical domains*. Let B_1 be the unitary ball of $\mathbb{R}^d$. A domain Ω such that

$$|\Omega| = |B_1| \quad \text{and} \quad \int_\Omega x\, dx = 0$$

belongs to the class $\mathcal{S}_{\delta,\gamma}$ if there exists $\phi \in C^{2,\gamma}(\partial B_1)$ with $\|\phi\|_{L^\infty(\partial B_1)} \le 1/2$ and such that

$$\partial\Omega = \{ x \in \mathbb{R}^d : x = (1+\phi(y))y,\ y \in \partial B_1 \} \quad \text{and} \quad \|\phi\|_{C^{2,\gamma}(\partial B_1)} \le \delta.$$

We recall the following result.

Theorem 6.1. *Let $\gamma \in (0,1)$. There exists $\delta = \delta(d,\gamma) > 0$ such that, if $\Omega \in \mathcal{S}_{\delta,\gamma}$, then*

$$T(B_1) - T(\Omega) \geq C_1 \|\phi\|^2_{H^{1/2}(\partial B_1)} \quad \text{and} \quad \lambda(\Omega) - \lambda(B_1) \leq C_2 \|\phi\|^2_{H^{1/2}(\partial B_1)}$$

for suitable constants C_1 and C_2 depending only on the dimension d.

Proof. The inequality for the torsional rigidity follows from Theorem 3.3 in [5]. The inequality for the eigenvalue follows by combining Theorem 1.4 and Lemma 2.8 of [16]. ■

Theorem 6.2. *Let $\gamma \in (0,1)$. There exist $\delta > 0$ and $q_1 > 1$ such that, for every $q \geq q_1$ and every $\Omega \in \mathcal{S}_{\gamma,\delta}$, it holds*

$$\lambda(B_1)T^q(B_1) \geq \lambda(\Omega)T^q(\Omega).$$

Proof. For every domain Ω we have

$$\lambda(B_1)T^q(B_1) - \lambda(\Omega)T^q(\Omega) = \lambda(B_1)(T^q(B_1) - T^q(\Omega)) + T^q(\Omega)(\lambda(B_1) - \lambda(\Omega)),$$

which, by the elementary inequality

$$x^q - y^q \geq q\, y^{q-1}(x-y), \quad \text{for every } x, y \geq 0,\ q > 1,$$

implies

$$\begin{aligned} \lambda(B_1)T^q(B_1) - \lambda(\Omega)T^q(\Omega) \\ \geq T^{q-1}(\Omega)\left[q\,\lambda(B_1)(T(B_1) - T(\Omega)) - T(\Omega)(\lambda(\Omega) - \lambda(B_1))\right]. \end{aligned} \tag{6.1}$$

Let δ be the constant determined by Theorem 6.1 and assume $\Omega \in \mathcal{S}_{\gamma,\delta}$. Since $2^{-1}B_1 \subset \Omega \subset 2B_1$, we get

$$2^{-(2+d)}T(B_1) \leq T(\Omega) \leq 2^{2+d}T(B_1).$$

Combining Theorem 6.1 and inequality (6.1) we get

$$\begin{aligned} \lambda(B_1)T^q(B_1) - \lambda(\Omega)T^q(\Omega) \\ \geq (2^{-(2+d)}T(B_1))^{q-1}(qC_1 - 2^{2+d}C_2T(B_1))\|\phi\|^2_{H^{1/2}(\partial B_1)}. \end{aligned}$$

Hence, if q is such that

$$q \geq 2^{d+2}\frac{C_2}{C_1}T(B_1),$$

we obtain

$$\lambda(B_1)T^q(B_1) \geq \lambda(\Omega)T^q(\Omega),$$

and this concludes the proof. ■

Remark 6.3. Although for large q we expect the ball to be optimal for the functional F_q, it is easy to see that this does not occur when q approaches 1. Indeed, if the ball maximizes F_q for every $q > 1$, passing to the limit as $q \to 1$, this would happen also for $q = 1$, which is not true, even in the class of convex domains. To see this, it is enough to notice that

$$F_1(B_1) = \frac{\lambda(B_1)}{d(d+2)} \leq \frac{d+4}{2(d+2)},$$

where the last inequality follows simply by taking $u(x) = 1 - |x|^2$ as a test function for $\lambda(B_1)$. On the other hand, taking as Ω_ε the thin slab $(0,1)^{d-1} \times (0,\varepsilon)$ gives

$$\lim_{\varepsilon \to 0} F_1(\Omega_\varepsilon) = \frac{\pi^2}{12},$$

and

$$\frac{\pi^2}{12} > \frac{d+4}{2(d+2)} \quad \text{for every } d \geq 2.$$

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