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A quantitative stability result for the Prékopa–Leindler inequality for arbitrary measurable functions

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Abstract. We prove that if a triplet of functions satisfies almost-equality in the Prékopa–Leindler inequality, then these functions are close to a common log-concave function, up to multiplication and rescaling. Our result holds for general measurable functions in all dimensions, and provides a quantitative stability estimate with computable constants.

1. Introduction

1.1. Brunn–Minkowski and Prékopa–Leindler inequalities

Writing $|X|$ to denote Lebesgue measure of a measurable subset X of $\mathbb{R}^n$ (with $|\emptyset| = 0$), the Brunn–Minkowski–Lusternik inequality states that if $\alpha, \beta > 0$ and A, B, C are bounded measurable subsets of $\mathbb{R}^n$ with $\alpha A + \beta B \subset C$ (by convention, if one of the sets A or B is empty, then $\alpha A + \beta B := \emptyset$), then

$$|C|^{\frac{1}{n}} \geq \alpha |A|^{\frac{1}{n}} + \beta |B|^{\frac{1}{n}}. \tag{1.1}$$

Also, in the case when $|A| > 0$ and $|B| > 0$, equality holds if and only if there exist a convex body K (that is, a convex compact set with nonempty interior), constants $a, b > 0$, and vectors $x, y \in \mathbb{R}^n$, such that $\alpha a + \beta b = 1$, $\alpha x + \beta y = 0$, and

$$A \subset aK + x, \quad B \subset bK + y, \quad |(aK + x) \setminus A| = 0, \quad |(bK + y) \setminus B| = 0, \quad \text{and} \quad |K \Delta C| = 0,$$

where $K \Delta C$ stands for the symmetric difference between K and C. We note that even if A and B are Lebesgue measurable, the Minkowski linear combination $\alpha A + \beta B$ may not be measurable (while $\alpha A + \beta B$ is measurable if A and B are Borel). We refer to the monograph [44] for a detailed exposition on this beautiful topic.

The Prékopa–Leindler inequality is a functional generalization of the classical Brunn–Minkowski inequality. In order to state it precisely, we recall that a function $f : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is said to be log-concave if $f((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} f(y)^{\lambda}$ for all $x, y \in \mathbb{R}^n$ and $\lambda \in (0, 1)$; in other words, f is log-concave if it can be written as $f = e^{-\varphi}$ for some convex function $\varphi : \mathbb{R}^n \to (-\infty, \infty]$.

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Theorem 1.1 (Prékopa, Leindler; Dubuc). *Let $\lambda \in (0,1)$ and $f, g, h\colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be measurable functions such that*

$$h((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} g(y)^{\lambda} \quad \forall x, y \in \mathbb{R}^n. \tag{1.2}$$

Then

$$\int_{\mathbb{R}} h \geq \left(\int_{\mathbb{R}} f\right)^{1-\lambda} \left(\int_{\mathbb{R}} g\right)^{\lambda}.$$

Also, equality holds if and only if there exist $a > 0$, $w \in \mathbb{R}^n$, and a log-concave function $\tilde{h}$, such that $h = \tilde{h}$, $f = a^{-\lambda}\tilde{h}(\cdot - \lambda w)$, $g = a^{1-\lambda}\tilde{h}(\cdot + (1-\lambda)w)$ almost everywhere.

Note that, if f, g, h are the indicator functions of some sets A, B, C, then Theorem 1.1 corresponds exactly to the Brunn–Minkowski inequality.

The Prékopa–Leindler inequality, due to Prékopa [40] and Leindler [35] in one dimension, was generalized in Prékopa [41] and Borell [8] to any dimension (cf. Marsiglietti [37], Pivovarov, Rebollo Bueno [39], Brascamp, Lieb [10], Kolesnikov, Werner [34], Bobkov, Colesanti, Fragalà [7]). The case of equality is characterized by Dubuc [16]. Various applications are provided and surveyed in Gardner [29].

1.2. Stability questions

As discussed above, optimizers are known for both the Brunn–Minkowski and Prékopa–Leindler inequalities. However, in spite of knowing the equality cases for these inequalities, one might ask what geometric properties can be deduced if one knows that the equality is "almost" attained. This is what one usually refers to as *stability* estimates.

Recently, various important stability results about geometric and functional inequalities have been obtained. For example, Fusco, Maggi, Pratelli [28] proved an optimal stability version of the isoperimetric inequality. This result was extended to the anisotropic isoperimetric inequality and to the Brunn–Minkowski inequality for convex sets by Figalli, Maggi, Pratelli [23, 24] (for the latter problem, the current best estimate is due to Kolesnikov, Milman [33]). One can further mention, for instance, stronger versions of the functional Blaschke–Santaló inequality, provided by the work of Barthe, Böröczky, Fradelizi [5]; of the Borell–Brascamp–Lieb inequality, provided by Ghilli, Salani [30], Rossi, Salani [42,43], and Balogh, Kristály [3]; of the Sobolev inequality by Figalli, Zhang [26] (extending Bianchi, Egnell [6] and Figalli, Neumayer [25]), Nguyen [38], and Wang [47]; of the log-Sobolev inequality by Gozlan [31]; and of some related inequalities by Caglar, Werner [12], Cordero-Erausquin [15], and Kolesnikov, Kosov [32]. An "isomorphic" stability result for the Prékopa–Leindler inequality for log-concave functions in terms of the transportation distance has been obtained by Eldan [17, Lemma 5.2].

1.2.1. Stability for Brunn–Minkowski. For the specific case of the Brunn–Minkowski inequality (1.1), the stability question is rather delicate. The first contribution in the direction of stability was made by Freĭman [27], although indirectly, as a consequence of his celebrated $3k - 4$ theorem in dimension $n = 1$ (see also Christ [14]):

Theorem 1.2 (Freĭman). *Let $A, B, C \subset \mathbb{R}$ be bounded measurable sets satisfying $A + B \subset C$ and $|C| < |A| + |B| + \varepsilon$ for some $\varepsilon \leq \min\{|A|, |B|\}$. Then there exist intervals $I, J \subset \mathbb{R}$ such that $A \subset I$, $B \subset J$, $|I \setminus A| < \varepsilon$, and $|J \setminus B| < \varepsilon$.*

In the planar case, van Hintum, Spink, Tiba [46] have found the optimal stability version of (1.1).

Theorem 1.3 (Van Hintum, Spink, Tiba). *For $\tau \in (0, \frac{1}{2}]$ and $\lambda \in [\tau, 1-\tau]$, let A, B, C be bounded measurable subsets of $\mathbb{R}^2$ satisfying $(1-\lambda)A + \lambda B \subset C$ and*

$$\big||A| - 1\big| + \big||B| - 1\big| + \big||C| - 1\big| < \varepsilon$$

for some $\varepsilon \leq e^{-M(\tau)}$, with $M(\tau) > 0$ depending only on τ. Then there exists a convex body K, with $A \subset K + x$ and $B \subset K + y$ for some $x, y \in \mathbb{R}^2$, such that

$$|(K + x) \setminus A| + |(K + y) \setminus B| + |K \Delta C| < c\tau^{-\frac{1}{2}}\varepsilon^{\frac{1}{2}} \tag{1.3}$$

for an absolute constant $c > 0$.

We note that, for $n \geq 2$, in (1.3) one cannot have an estimate with better error term, in terms of the order of both τ and ε. In higher dimensions, the only available quantitative stability version of the Brunn–Minkowski inequality has been established by Figalli, Jerison [19].

Theorem 1.4 (Figalli, Jerison). *For $\tau \in (0, \frac{1}{2}]$ and $\lambda \in [\tau, 1-\tau]$, let A, B, C be bounded measurable subsets of $\mathbb{R}^n$, $n \geq 3$, with $(1-\lambda)A + \lambda B \subset C$ and*

$$\big||A| - 1\big| + \big||B| - 1\big| + \big||C| - 1\big| < \varepsilon$$

for some $\varepsilon < e^{-A_n(\tau)}$, with $A_n(\tau) := (2^{3^{n+2}} n^{3^n} |\log \tau|^{3^n})/\tau^{3^n}$. Then there exists a convex body K, with $A \subset K + x$ and $B \subset K + y$ for some $x, y \in \mathbb{R}^n$, such that

$$|(K + x) \setminus A| + |(K + y) \setminus B| + |K \Delta C| < \tau^{-N_n}\varepsilon^{\gamma_n(\tau)}, \tag{1.4}$$

where $\gamma_n(\tau) = \tau^{3^n}/(2^{3^{n+1}} n^{3^n} |\log \tau|^{3^n})$ and $N_n > 0$ depends only on n.

Remark 1.5. We list here some results for particular cases of Theorem 1.4.

- When $A = B$, van Hintum, Spink, Tiba [45] obtained the optimal stability version, where the error term in (1.4) is of the form $c_n\tau^{-\frac{1}{2}}\varepsilon^{\frac{1}{2}}$ with $c_n > 0$ depending only on n. Their result improves the previous contributions [18, 20, 21].

When at least one of the sets A or B is convex, several results have been obtained, as described below. However, it is important to observe that all these results measure stability by controlling the symmetric difference between A and a translate of B. This is weaker than the statement in Theorem 1.4, where one finds a convex set K that contains both A and B (up to a translation) with a control on the missing volume. Here are some important results.

- When either A or B is convex, an optimal stability estimate has been proved by Barchiesi, Julin [4]. This extends earlier results about the case when both A and B are convex [23, 24], or when either A or B is the unit ball [22].
- If A and B are convex and n is large, then Kolesnikov, Milman [33] provided an estimate on $|A\Delta(x+B)|$ with a bound of the form $cn^{2.75}\tau^{-\frac{1}{2}}\varepsilon^{\frac{1}{2}}$, for some absolute constant c. Actually, we note that the term $n^{2.75}$ can be improved to $n^{2.5+o(1)}$ by combining the general estimates of Kolesnikov, Milman [33, Section 12] with the bound $n^{o(1)}$ on the Cheeger constant of a convex body in isotropic position, which follows from Chen's work [13] on the Kannan–Lovasz–Simonovits conjecture.

1.2.2. Stability for Prékopa–Leindler. With respect to the Brunn–Minkowski inequality, before now much less was known about stability for the Prékopa–Leindler inequality, except for some results in the case of log-concave functions (see the discussion below). In this paper, we prove the first quantitative stability result for the Prékopa–Leindler inequality on arbitrary functions.

Theorem 1.6. *Given $\tau \in (0, \frac{1}{2}]$ and $\lambda \in [\tau, 1-\tau]$, let $f, g, h\colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be measurable functions such that $h((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} g(y)^{\lambda}$ for all $x, y \in \mathbb{R}^n$, and*

$$\int_{\mathbb{R}^n} h < (1+\varepsilon)\left(\int_{\mathbb{R}^n} f\right)^{1-\lambda}\left(\int_{\mathbb{R}^n} g\right)^{\lambda} \quad \text{for some } \varepsilon > 0. \tag{1.5}$$

There are a computable dimensional constant Θ_n and computable constants $Q_n(\tau)$ and $M_n(\tau)$ depending only on n and τ,[1] *such that the following holds: if $0 < \varepsilon < e^{-M_n(\tau)}$, then there exist $\tilde{h}$ log-concave and $w \in \mathbb{R}^n$ such that*

$$\int_{\mathbb{R}^n} |h - \tilde{h}| + \int_{\mathbb{R}^n} |a^{\lambda} f - \tilde{h}(\cdot + \lambda w)| + \int_{\mathbb{R}^n} |a^{\lambda-1} g - \tilde{h}(\cdot + (\lambda-1)w)| < \frac{\varepsilon^{Q_n(\tau)}}{\tau^{\Theta_n}} \int_{\mathbb{R}^n} h,$$

where $a = \int_{\mathbb{R}^n} g / \int_{\mathbb{R}^n} f$.

Remark 1.7. For f, g, h a priori assumed to be log-concave, Theorem 1.6 was established by Ball, Böröczky [2] and Böröczky, De [9] in the case $n = 1$ (in this case, $\varepsilon^{Q_n(\tau)}/\tau^{\Theta_n}$ in Theorem 1.6 can be essentially replaced by $(\varepsilon/\tau)^{\frac{1}{3}}$; see also Theorem 2.1), and by Böröczky, De [9] in the case $n \geq 2$ (in that case, $\varepsilon^{Q_n(\tau)}/\tau^{\Theta_n}$ in Theorem 1.6 can be replaced by $(\varepsilon/\tau)^{\frac{1}{19}}$). Further, we note that Bucur, Fragalà [11] proved another interesting stability version of the Prékopa–Leindler inequality for log-concave functions, bounding the distance of all one-dimensional projections.

Theorem 1.6 is probably quite far from the optimal version, which one could conjecture to provide a bound of the form $C(n,\tau)\varepsilon^{\frac{1}{2}}$. In this direction, even for $n = 1$, Example 1.8 below shows that the error term in Theorem 1.6 is at least $c\varepsilon^{\frac{1}{2}}$.

[1] At the end of the proof of Theorem 1.6 (see (5.34)), we indicate explicit values for the constants $M_n(\tau)$, $Q_n(\tau)$, Θ_n.

At first sight, this is perhaps surprising, because in the case of Freĭman's result (Theorem 1.2) the error is of order ε, which shows that the Brunn–Minkowski and Prékopa–Leindler inequalities exhibit different behaviors for $n = 1$. Nonetheless, our proof of Theorem 1.6 shows that the Prékopa–Leindler inequality in dimension n shares some – but not all – of the geometric aspects of the Brunn–Minkowski inequality in dimension $n + 1$, which explains, at least partially, the difference between the two exponents.

Another important difference between the stability version of the Prékopa–Leindler and the Brunn–Minkowski inequalities is shown by the following observation: when $A = B$, the convex set K in Theorem 1.4 coincides with the convex hull of A; on the other hand, for $f = g$, the function $\tilde{h}$ in Theorem 1.6 can be quite far from the log-concave hull of f (see Example 1.9 below). In other words, there is no direct geometric characterization of the function $\tilde{h}$ (see also Remark 1.10 below).

As mentioned above, the following example shows that the error term in Theorem 1.6 is at least $c\varepsilon^{\frac{1}{2}}$.

Example 1.8. There is an absolute constant $c \in (0, 1)$ such that the following holds. For any $\varepsilon \ll 1$, there exist log-concave probability densities f, g on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} \sup_{z=\frac{1}{2}x+\frac{1}{2}y} f(x)^{\frac{1}{2}} g(y)^{\frac{1}{2}}\, dz < 1 + \varepsilon,$$

while

$$\int_{\mathbb{R}} |g(x) - f(x + w)|\, dx \geq c\varepsilon^{\frac{1}{2}} \quad \text{for any } w \in \mathbb{R}.$$

Proof. We fix $f(x) = e^{-\pi x^2}$ and an odd C^2 function φ on $\mathbb{R}$ satisfying $\operatorname{supp}\varphi \subset [-1, 1]$ and $\max \varphi = 1$. Note that, since φ is odd, $\int_{\mathbb{R}} f\varphi = 0$.

Given $\eta \ll 1$ to be fixed later, we consider $g = (1 + \eta\varphi) f$ so that $\int_{\mathbb{R}} g = 1$. We note that there exists a constant $\tilde{c} \geq 2$ such that

$$|[\log(1 + \eta\varphi)]'| = \left|\eta \cdot \frac{\varphi'}{1 + \eta\varphi}\right| \leq \tilde{c}\eta, \tag{1.6}$$

$$|[\log(1 + \eta\varphi)]''| = \left|\eta \cdot \frac{\varphi''(1 + \eta\varphi) - \eta(\varphi')^2}{(1 + \eta\varphi)^2}\right| \leq \tilde{c}\eta \tag{1.7}$$

for any $\eta \in (0, \frac{1}{2})$. In particular, since $(\log f)'' = -2\pi$, it follows that g is log-concave provided $\eta \ll 1/\tilde{c}$.

Note now that, since $g(x) = f(x) = e^{-\pi x^2}$ for $|x| \geq 1$, there exists a constant $c_0 > 0$ such that

$$\begin{aligned} \int_{\mathbb{R}} |g(x) - f(x + w)|\, dx &\geq \int_1^{\infty} |e^{-\pi x^2} - e^{-\pi(x+w)^2}|\, dx \\ &\geq c_0 \min\{|w|, 1\}. \end{aligned} \tag{1.8}$$

On the other hand, we have

$$\int_{\mathbb{R}} |g(x) - f(x+w)|\,dx \geq \int_{\mathbb{R}} |g(x) - f(x)| - |f(x) - f(x+w)|\,dx$$
$$\geq \eta \int_{\mathbb{R}} f(x)|\varphi(x)|\,dx - \bar{c}|w|.$$

Hence, combining this last estimate with (1.8), we deduce the existence of a constant $c_1 > 0$ such that

$$\int_{\mathbb{R}} |g(x) - f(x+w)|\,dx \geq c_1\eta \quad \forall w \in \mathbb{R}. \tag{1.9}$$

Finally, we estimate $\int_{\mathbb{R}} h$ for $h(z) = \sup_{2z=x+y} \sqrt{f(x)g(y)}$. To this aim, consider the auxiliary function $\tilde{h}(z) = \sqrt{f(z)g(z)}$. Thanks to the Hölder inequality, this satisfies $\int_{\mathbb{R}} \tilde{h} \leq 1$.

Since f and g are log-concave and $g(x) = f(x)$ for $|x| \geq 1$, for any $z \in \mathbb{R}$, there exists a point $y_z \in \mathbb{R}$ such that $h(z) = \sqrt{f(2z - y_z)g(y_z)}$. Also, $y_z = z$ if $|z| \geq 1$, and $|y_z| \leq 1$ if $|z| \leq 1$.

We now observe that, for any $z \in \mathbb{R}$, the function $\psi_z(y) = \log\sqrt{f(2z-y)g(y)}$ satisfies $\psi_z(z) = \log\tilde{h}(z)$, $\psi_z(y_z) = \log h(z)$, and ψ_z has a maximum at y_z. Then, recalling (1.6), we have

$$0 = \psi_z'(y_z) = 2\pi(z - y_z) + \frac{1}{2}[\log(1+\eta\varphi)]'(y_z) \quad \Rightarrow \quad |z - y_z| \leq \tilde{c}\eta.$$

Hence, since $|\psi_z''|$ is bounded, a Taylor expansion yields (recall that $\psi_z'(y_z) = 0$)

$$\log\frac{h(z)}{\tilde{h}(z)} = \psi_z(y_z) - \psi_z(z) \leq c_2\eta^2 \quad \forall z \in \mathbb{R},$$

for some constant $c_2 > 1$, and we conclude that

$$\int_{\mathbb{R}} h \leq e^{c_2\eta^2}\int_{\mathbb{R}} \tilde{h} \leq e^{c_2\eta^2} < 1 + 2c_2\eta^2 \quad \text{for } \eta \ll 1.$$

Choosing $\eta := (2c_2)^{-\frac{1}{2}}\varepsilon^{\frac{1}{2}}$, (1.9) and the equation above prove the result. ∎

The next example shows that, even in the case $f = g$, the function $\tilde{h}$ provided by Theorem 1.6 *cannot* be chosen to be the log-concave hull of f (i.e., the smallest log-concave function above f).

Example 1.9. For any $\varepsilon > 0$ there exist $f, h\colon \mathbb{R} \to \mathbb{R}_{\geq 0}$ measurable functions such that $h(\frac{1}{2}x + \frac{1}{2}y) \geq f(x)^{\frac{1}{2}} f(y)^{\frac{1}{2}}$ for all $x, y \in \mathbb{R}^n$,

$$\int_{\mathbb{R}} h < (1+\varepsilon)\int_{\mathbb{R}} f,$$

but

$$\int_{\mathbb{R}} (F - f) \geq \frac{1}{2}\int_{\mathbb{R}} f,$$

where F denotes the log-concave hull of f.

Proof. Given $A \gg 1$, let f be defined as

$$f(x) = \begin{cases} e^{-x} & \text{on } [0,1] \cup [2A, 2A+1], \\ 0 & \text{otherwise,} \end{cases}$$

and set $h(z) := \sup_{z=\frac{1}{2}x+\frac{1}{2}y} f(x)^{\frac{1}{2}} f(y)^{\frac{1}{2}}$. Then

$$h(x) = \begin{cases} e^{-x} & \text{on } [0,1] \cup [A, A+1] \cup [2A, 2A+1], \\ 0 & \text{otherwise,} \end{cases}$$

and therefore

$$\int_{\mathbb{R}} h < (1+\varepsilon) \int_{\mathbb{R}} f$$

with $\varepsilon \simeq e^{-A} \ll 1$. On the other hand, the log-concave hull of f is given by

$$F(x) = \begin{cases} e^{-x} & \text{on } [0, 2A+1], \\ 0 & \text{otherwise.} \end{cases}$$

Hence, for $A \gg 1$,

$$\int_{\mathbb{R}} (F - f) = \int_1^{2A} e^{-x}\, dx = e^{-1} - e^{-2A} \geq \frac{1}{2}(1 - e^{-1}) = \frac{1}{2} \int_{\mathbb{R}} f,$$

as desired. ■

Remark 1.10. The argument used in Example 1.9 emphasizes a key difference between the Brunn–Minkowski inequality and the Prékopa–Leindler inequality: while in the Brunn–Minkowski inequality only arithmetic means of points are considered, in Prékopa–Leindler one considers points z that are the arithmetic mean of x and y, but then the value of $h(z)$ is obtained as a geometric mean of the values of $f(x)$ and $g(y)$. This key difference is the source of many new challenges when proving stability results for Prékopa–Leindler.

1.3. Outline of the proof of Theorem 1.6

We now sketch the structure of the proof of Theorem 1.6, which is split into four main steps. The first three steps deal with the one-dimensional case. Then, in Step 4, we exploit both the one-dimensional case and Theorem 1.4 to obtain the higher-dimensional result.

(1) We first deal with the case of symmetrically rearranged functions, and prove the result in this case. Note that, if f, g, h satisfy (1.2) and (1.5), then their rearrangements f^*, g^*, h^* also satisfy the same estimates.

(2) With the knowledge that the result holds for f^*, g^*, h^*, we deduce conditions on the distribution functions $t \mapsto \mathcal{H}^1(\{f > t\}), \mathcal{H}^1(\{g > t\})$. In particular, from (1.5)

applied to f, g, h, we use a stability version of the Brunn–Minkowski inequality in one dimension in order to prove that f and g are close to "bubble-shaped" functions (i.e., that are nondecreasing on an interval $(-\infty, a)$ and nonincreasing on $(a, +\infty)$).

Calling ϕ and ψ such bubble-shaped functions, we define

$$\lambda(z) = \sup_{(1-\lambda)x+\lambda y=z} \phi(x)^{1-\lambda}\psi(y)^{\lambda}.$$

This function is measurable (thanks to the fact that ϕ and ψ are bubble shaped), and an analysis similar to the proof of Proposition 2.6 shows that ϕ, ψ, λ satisfy both (1.2) and (1.5) (but for some smaller power of ε).

(3) Denote

$$\{x \in \mathbb{R}: \phi(x) > t\} = (a_f(t), b_f(t)), \quad \{x \in \mathbb{R}: \psi(x) > t\} = (a_g(t), b_g(t)).$$

Then we use the almost-optimality of ϕ, ψ, λ to prove that, on a large set, a four-point inequality (in the same spirit as [19, Lemma 3.6 and Remark 4.1]) is satisfied by the functions $\mathcal{B}_f(T) = b_f(e^T)$ and $\mathcal{B}_g(T) = b_g(e^T)$, and a "reversed" version of such a four-point inequality holds for $\mathcal{A}_f(T) = a_f(e^T)$ and $\mathcal{A}_g(T) = a_g(e^T)$.

As a consequence, we are able to prove that $\mathcal{A}_f$, $\mathcal{A}_g$ are both L^1-close to convex functions m_f, m_g on a large interval. Analogously, $\mathcal{B}_f$, $\mathcal{B}_g$ are L^1-close to concave functions n_f, n_g on the same large interval. Thanks to these facts, we show that there exist log-concave function $\tilde{\phi}$ and $\tilde{\psi}$ such that $\{\tilde{\phi} > t\} = (m_f(\log t), n_f(\log t))$ and $\{\tilde{\psi} > t\} = (m_g(\log t), n_g(\log t))$ on a large interval.

Finally, we translate the properties of $\mathcal{A}_f$, $\mathcal{A}_g$, $\mathcal{B}_f$, $\mathcal{B}_g$, m_f, m_g, n_f, n_g into a bound on $\|\phi - \tilde{\phi}\|_1$, which can thus be made small. By Proposition 2.6, we conclude the one-dimensional case of Theorem 1.6.

(4) In order to obtain the result in higher dimensions as well, we consider the hypographs of the logarithms of f, g, h. Denoting these sets by $\mathcal{S}_f$, $\mathcal{S}_g$, $\mathcal{S}_h$, respectively, we show that they satisfy the Brunn–Minkowski condition $\mathcal{S}_h \supset (1-\lambda)\mathcal{S}_f + \lambda\mathcal{S}_g$. In particular, due to the one-dimensional case, we can estimate how level sets of f, g, h are close to each other, in terms of volume. This enables us to use the main theorem in [19] on the sets $\mathcal{S}_f$, $\mathcal{S}_g$, $\mathcal{S}_h$, which in turn produces a natural algorithm to construct log-concave functions close to f, g, h.

The rest of the manuscript is organized as follows: In Section 2, we prove tail estimates that allow us to suitably truncate the functions under consideration, as well as to estimate the size of level sets. This allows us to perform a set of preliminary reductions of the one-dimensional problem. In Section 3 we prove Theorem 1.6 in the case when $n = 1$ and f, g, h are symmetrically decreasing, while in Section 4 we deal with the general one-dimensional case. Finally, in Section 5 we prove the theorem in arbitrary dimension.

Throughout the manuscript, we shall use the notation $\mathcal{H}^k$ for the k-dimensional Hausdorff measure of a set. Sometimes we shall use $c > 0$ to denote an absolute (computable) constant, whose exact value might change from one part of the paper to the next, and even from line to line. We shall also occasionally use a subscript, e.g. c_n, to indicate dependence of the constant on a dimensional parameter. Moreover, we write $a \lesssim b$ whenever a/b is bounded from above by an absolute and explicitly computable constant, and we shall use a subscript $a \lesssim_n b$ to emphasize the dependence of the bound on the dimension considered. Finally, we write $a \simeq b$ if both $a \lesssim b$ and $b \lesssim a$ hold.

2. Tail estimates in the case of almost-equality in the one-dimensional Prékopa–Leindler inequality

A useful tool for our study is the symmetric decreasing rearrangement. For a bounded function $\varphi: \mathbb{R} \to \mathbb{R}_{\geq 0}$ with $0 < \int_{\mathbb{R}} \varphi < \infty$, we define its symmetric decreasing rearrangement $\varphi^*: \mathbb{R} \to \mathbb{R}_{\geq 0}$ by

$$\varphi^*(t) = \inf\bigl\{\alpha: \mathcal{H}^1(\{\varphi \geq \alpha\}) \leq 2|t|\bigr\}.$$

In particular, φ^* is an even function that is monotone decreasing on $[0, \infty)$, $\varphi^*(0)$ is the essential supremum of φ, and

$$\mathcal{H}^1(\{\varphi \geq \alpha\}) = \mathcal{H}^1(\{\varphi^* \geq \alpha\})$$

for any $\alpha > 0$ with $\mathcal{H}^1(\{\varphi \geq \alpha\}) > 0$. In particular, the level sets $\{\varphi^* \geq \alpha\}$ are symmetric segments, and the layer-cake representation yields $\int_{\mathbb{R}} \varphi = \int_{\mathbb{R}} \varphi^*$.

Symmetric decreasing rearrangement works very well for the Prékopa–Leindler inequality. For $\lambda \in (0,1)$ and bounded functions $f, g, h: \mathbb{R} \to \mathbb{R}_{\geq 0}$ with positive integral, if $h((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} g(y)^{\lambda}$ for any $x, y \in \mathbb{R}$, then the one-dimensional Brunn–Minkowski inequality yields $h^*((1-\lambda)x + \lambda y) \geq f^*(x)^{1-\lambda} g^*(y)^{\lambda}$ for any $x, y \in \mathbb{R}$. Also, if φ is log-concave, then the same holds for φ^*.

The main goal of this section is to show that if we have almost-equality in the one-dimensional Prékopa–Leindler equality, then the functions f, g, h in (1.5) with positive integral satisfy similar tail estimates like log-concave functions (here $\varphi: \mathbb{R} \to \mathbb{R}_{\geq 0}$ has positive integral if $0 < \int \varphi < \infty$). First we review the related properties of log-concave functions. Let us recall the following estimate from [2, 9]:

Theorem 2.1 (Ball, Böröczky, De). *For $\tau \in (0, \frac{1}{2}]$ and $\lambda \in [\tau, 1-\tau]$, let $f, g, h: \mathbb{R} \to \mathbb{R}_{\geq 0}$ be log-concave functions with positive integral such that*

$$h((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} g(y)^{\lambda}$$

for all $x, y \in \mathbb{R}$, and

$$\int_{\mathbb{R}} h < (1+\varepsilon)\Bigl(\int_{\mathbb{R}} f\Bigr)^{1-\lambda}\Bigl(\int_{\mathbb{R}} g\Bigr)^{\lambda}$$

for some $\varepsilon \in (0,1)$. *Then there exists* $w \in \mathbb{R}$ *such that*

$$\int_{\mathbb{R}} |a^{\lambda} f - h(\cdot + \lambda w)| + \int_{\mathbb{R}} |a^{\lambda-1} g - h(\cdot + (\lambda - 1)w)| < c\Big(\frac{\varepsilon}{\tau}\Big)^{\frac{1}{3}} |\log \varepsilon|^{\frac{4}{3}} \int_{\mathbb{R}^n} h,$$

where $a = \int_{\mathbb{R}} g / \int_{\mathbb{R}} f$, *and* $c > 1$ *is an absolute constant.*

Next we prove some basic properties of log-concave functions. We observe that if φ is log-concave and $0 < \int_{\mathbb{R}} \varphi < \infty$, then the level sets are segments, φ is bounded, and its essential supremum coincides with its supremum $\|\varphi\|_\infty$.

Lemma 2.2. *Let* φ *be a log-concave function with* $0 < \int_{\mathbb{R}} \varphi < \infty$. *Then*

(i) $\mathcal{H}^1(\{\varphi > \|\varphi\|_\infty - s\}) \geq \frac{\|\varphi\|_1}{\|\varphi\|_\infty^2} s$ *provided* $0 < s < \|\varphi\|_\infty$;

(ii) $\mathcal{H}^1(\{\varphi > t\}) \leq \frac{2\|\varphi\|_1}{\|\varphi\|_\infty} |\log \frac{t}{\|\varphi\|_\infty}|$ *provided* $0 < t \leq \frac{1}{2}\|\varphi\|_\infty$;

(iii) $\int_{\{\varphi < t\}} \varphi \leq \frac{2\|\varphi\|_1}{\|\varphi\|_\infty} t$ *provided* $0 < t \leq \frac{1}{2}\|\varphi\|_\infty$.

Proof. Using symmetric decreasing rearrangement we can assume that φ is even. Also, by scaling, we may also suppose that $\varphi(0) = \|\varphi\|_\infty = \int_{\mathbb{R}} \varphi = 1$.

For (i), let $x_0 = \sup\{x\colon \varphi(x) > 1 - s\} = \frac{1}{2}\mathcal{H}^1(\{\varphi > 1 - s\})$, and choose $\gamma > 0$ such that $1 - s = e^{-\gamma x_0}$. It follows from the log-concavity and the evenness of φ that $\varphi(x) \leq 1$ if $|x| \leq |x_0|$, and $\varphi(x) \leq e^{-\gamma |x|}$ if $|x| \geq |x_0|$. Also, since $e^{-\gamma x_0} > 1 - \gamma x_0$ we get $\frac{1}{\gamma} < \frac{x_0}{s}$, thus

$$1 = \int_{\mathbb{R}} \varphi \leq 2x_0 + 2\int_{x_0}^{\infty} e^{-\gamma x}\, dx = 2x_0 + \frac{2e^{-\gamma x_0}}{\gamma} < 2x_0\Big(1 + \frac{1-s}{s}\Big) = \frac{2x_0}{s}.$$

For (ii) and (iii), let $x_1 = \sup\{x\colon \varphi(x) > t\} = \frac{1}{2}\mathcal{H}^1(\{\varphi > t\})$, and choose $\delta > 0$ such that $t = e^{-\delta x_1}$. It follows again by log-concavity and evenness that $\varphi(x) \geq e^{-\delta|x|}$ if $|x| \leq |x_1|$, and $\varphi(x) \leq e^{-\delta|x|}$ if $|x| \geq |x_1|$.

Then, on the one hand, we have

$$\frac{1}{2} \geq \int_0^{x_1} e^{-\delta x}\, dx = \frac{1 - e^{-\delta x_1}}{\delta} = \frac{1-t}{\delta} \geq \frac{1}{2\delta} = \frac{x_1}{2|\log t|}, \tag{2.1}$$

verifying (ii). On the other hand, using (2.1) we get

$$\int_{\{\varphi < t\}} \varphi \leq 2\int_{x_1}^{\infty} e^{-\delta x}\, dx = \frac{2e^{-\delta x_1}}{\delta} = \frac{2t x_1}{|\log t|} \leq 2t,$$

verifying (iii). ∎

Given $\varepsilon \in (0,1]$, $\tau \in (0, \frac{1}{2}]$, and $\lambda \in [\tau, 1 - \tau]$, we now consider measurable functions $f, g, h\colon \mathbb{R} \to \mathbb{R}_{\geq 0}$ with positive integral satisfying

$$h((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} g(y)^{\lambda} \quad \text{for } x, y \in \mathbb{R}, \tag{2.2}$$

$$\int_{\mathbb{R}} h < (1 + \varepsilon)\Big(\int_{\mathbb{R}} f\Big)^{1-\lambda}\Big(\int_{\mathbb{R}} g\Big)^{\lambda}. \tag{2.3}$$

For $t > 0$, we set

$$A_t = \{f \geq t\}, \quad B_t = \{g \geq t\}, \quad \text{and} \quad C_t = \{h \geq t\}, \tag{2.4}$$

so that

$$A_t = \bigcap_{0<s<t} A_s, \quad B_t = \bigcap_{0<s<t} B_s, \quad \text{and} \quad C_t = \bigcap_{0<s<t} C_s.$$

It follows from (2.2) that if $A_t, B_s \neq \emptyset$ for $t, s > 0$, then

$$(1-\lambda)A_t + \lambda B_s \subset C_{t^{1-\lambda}s^\lambda}. \tag{2.5}$$

Lemma 2.3. *Let f, g, h satisfy (2.2) and (2.3). Then f and g are bounded.*

Proof. For any $x_0 \in \mathbb{R}$ with $f(x_0) > 0$, we have

$$\begin{aligned} 2\Big(\int_{\mathbb{R}} f\Big)^{1-\lambda}\Big(\int_{\mathbb{R}} g\Big)^{\lambda} > \int_{\mathbb{R}} h &\geq \int_{\mathbb{R}} f(x_0)^{1-\lambda} g\Big(\frac{1}{\lambda}z - \frac{1-\lambda}{\lambda}x_0\Big)^\lambda \, dz \\ &= f(x_0)^{1-\lambda}\lambda \int_{\mathbb{R}} g^\lambda; \end{aligned}$$

therefore, f is bounded. Similarly, g is bounded as well. ∎

We use the following stability version of the inequality between the arithmetic and geometric means. It follows from Aldaz [1, Lemma 2.1] that if $a, b > 0$ and $\lambda \in [\tau, 1-\tau]$ for $\tau \in (0, \frac{1}{2}]$, then

$$(1-\lambda)a + \lambda b - a^{1-\lambda}b^\lambda \geq \tau(\sqrt{a} - \sqrt{b})^2. \tag{2.6}$$

According to Lemma 2.3, we can speak about $\|f\|_\infty$ and $\|g\|_\infty$.

Lemma 2.4. *Let f, g, h satisfy (2.2) and (2.3). If $\varepsilon < 2^{-6}\tau^3$, then*

$$\left| \frac{\|f\|_\infty}{\|g\|_\infty} \cdot \frac{\|g\|_1}{\|f\|_1} - 1 \right| \leq 4\tau^{-\frac{3}{2}}\varepsilon^{\frac{1}{2}}.$$

Proof. We may assume that $\int_{\mathbb{R}} f = \int_{\mathbb{R}} g = 1$.

We set $\theta = \|f\|_\infty / \|g\|_\infty$. Using the notation (2.4), it follows from (2.2) that if $0 < t < \|f\|_\infty^{1-\lambda}\|g\|_\infty^\lambda$, then

$$(1-\lambda)A_{\theta^\lambda t} + \lambda B_{\theta^{\lambda-1}t} \subset C_t.$$

We deduce from (2.5) and the one-dimensional Brunn–Minkowski inequality that

$$\begin{aligned} 1 + \varepsilon \geq \int_{\mathbb{R}} h &\geq \int_0^{\|f\|_\infty^{1-\lambda}\|g\|_\infty^\lambda} \mathcal{H}^1(C_t)\, dt \\ &\geq (1-\lambda)\int_0^{\|f\|_\infty^{1-\lambda}\|g\|_\infty^\lambda} \mathcal{H}^1(A_{\theta^\lambda t})\, dt + \lambda \int_0^{\|f\|_\infty^{1-\lambda}\|g\|_\infty^\lambda} \mathcal{H}^1(B_{\theta^{\lambda-1}t})\, dt \\ &= \frac{1-\lambda}{\theta^\lambda}\int_0^{\|f\|_\infty} \mathcal{H}^1(A_s)\, ds + \lambda\theta^{1-\lambda}\int_0^{\|g\|_\infty} \mathcal{H}^1(B_s)\, ds = \frac{1-\lambda}{\theta^\lambda} + \lambda\theta^{1-\lambda}. \end{aligned}$$

We conclude from (2.6) that

$$|\theta^{-\frac{\lambda}{2}} - \theta^{\frac{1-\lambda}{2}}| < \tau^{-\frac{1}{2}}\varepsilon^{\frac{1}{2}},$$

which in turn yields that

$$\tau^{-\frac{1}{2}}\varepsilon^{\frac{1}{2}} > e^{\frac{\tau|\log\theta|}{2}} - 1 > \frac{\tau|\log\theta|}{2}.$$

Since $|\log\theta| < 2\tau^{-\frac{3}{2}}\varepsilon^{\frac{1}{2}} \le \frac{1}{4}$ provided $\varepsilon \le \tau^3/64$, we have $|\theta - 1| < 4\tau^{-\frac{3}{2}}\varepsilon^{\frac{1}{2}}$. ∎

Lemma 2.5. *Let f, g, h satisfy (2.2) and (2.3). If $\varepsilon^{\frac{1}{2}} \le \eta < 1$, then*

$$\begin{aligned}\mathcal{H}^1\big(\{f \ge \eta\|f\|_\infty\}\big) &\lesssim \frac{\tau^{-\frac{5}{2}}\|f\|_1}{\|f\|_\infty}\cdot|\log\varepsilon|^{\frac{4}{\tau}},\\ \mathcal{H}^1\big(\{g \ge \eta\|g\|_\infty\}\big) &\lesssim \frac{\tau^{-\frac{5}{2}}\|g\|_1}{\|g\|_\infty}\cdot|\log\varepsilon|^{\frac{4}{\tau}},\end{aligned} \tag{2.7}$$

and

$$\int_{\{f<\eta\}} f \lesssim \tau^{-\frac{5}{2}}\|f\|_1\cdot\eta|\log\varepsilon|^{\frac{4}{\tau}}, \quad \int_{\{g<\eta\}} g \lesssim \tau^{-\frac{5}{2}}\|g\|_1\cdot\eta|\log\varepsilon|^{\frac{4}{\tau}}.$$

Proof. We may assume that $\|f\|_\infty = \|g\|_\infty = 1$ and $\min\{\int_{\mathbb{R}} f, \int_{\mathbb{R}} g\} = 1$, so that Lemma 2.4 yields

$$1 = \min\big\{\textstyle\int_{\mathbb{R}} f, \int_{\mathbb{R}} g\big\} \le \max\big\{\textstyle\int_{\mathbb{R}} f, \int_{\mathbb{R}} g\big\} \le 1 + 4\tau^{-\frac{3}{2}}\varepsilon^{\frac{1}{2}} < 2. \tag{2.8}$$

For $t > 0$, it follows from (2.5) that if $\varrho \in (0,1)$, then

$$C_{\varrho t} \supset \big((1-\lambda)A_{t^{1/(1-\lambda)}} + \lambda B_{\varrho^{1/\lambda}}\big) \cup \big((1-\lambda)A_{\varrho^{1/(1-\lambda)}} + \lambda B_{t^{1/\lambda}}\big), \tag{2.9}$$

thus the one-dimensional Brunn–Minkowski inequality yields that $\mathcal{H}^1(C_{\varrho t})$ is at least the arithmetic mean of $(1-\lambda)\mathcal{H}^1(A_{t^{1/(1-\lambda)}}) + \lambda\mathcal{H}^1(B_{\varrho^{1/\lambda}})$ and $(1-\lambda)\mathcal{H}^1(A_{\varrho^{1/(1-\lambda)}}) + \lambda\mathcal{H}^1(B_{t^{1/\lambda}})$, and hence letting ϱ tend to 1 implies

$$\mathcal{H}^1(C_t) \ge \frac{1}{2}\big[(1-\lambda)\mathcal{H}^1(A_{t^{1/(1-\lambda)}}) + \lambda\mathcal{H}^1(B_{t^{1/\lambda}})\big]. \tag{2.10}$$

In addition, $\mathcal{H}^1(C_t) - (1-\lambda)\mathcal{H}^1(A_t) - \lambda\mathcal{H}^1(B_t) \ge 0$ holds for any $t > 0$, thanks to (2.5) and the one-dimensional Brunn–Minkowski inequality.

Therefore, using the near optimality (2.3) for the Prékopa–Leindler inequality, (2.8), and (2.10), we deduce that for any $\alpha \in (0,1]$, we have

$$\begin{aligned}8\tau^{-\frac{3}{2}}\varepsilon^{\frac{1}{2}} &\ge \int_0^\alpha \big(\mathcal{H}^1(C_t) - (1-\lambda)\mathcal{H}^1(A_t) - \lambda\mathcal{H}^1(B_t)\big)\,dt\\ &\ge \int_0^\alpha \Big(\frac{1}{2}\big[(1-\lambda)\mathcal{H}^1(A_{t^{1/(1-\lambda)}}) + \lambda\mathcal{H}^1(B_{t^{1/\lambda}})\big]\\ &\qquad - (1-\lambda)\mathcal{H}^1(A_t) - \lambda\mathcal{H}^1(B_t)\Big)\,dt.\end{aligned} \tag{2.11}$$

We now define

$$\Gamma(\alpha) := \int_0^\alpha \big((1-\lambda)\mathcal{H}^1(A_t) + \lambda\mathcal{H}^1(B_t)\big)\, dt.$$

Note that Γ is an increasing function bounded by 2. Also, through a change of variables, it satisfies

$$\int_0^\alpha \big((1-\lambda)\mathcal{H}^1(A_{t^{1/s}}) + \lambda\mathcal{H}^1(B_{t^{1/s}})\big)\, dt \geq s\alpha^{1-\frac{1}{s}}\Gamma(\alpha^{\frac{1}{s}}) \quad \forall s \in (0,1).$$

Hence, assuming with no loss of generality that $\lambda \leq 1/2$, it follows from (2.11) that

$$8\tau^{-\frac{3}{2}}\varepsilon^{\frac{1}{2}} \geq \frac{1-\lambda}{2} \cdot \alpha^{-\frac{\lambda}{1-\lambda}}\Gamma(\alpha^{\frac{1}{1-\lambda}}) - \Gamma(\alpha). \tag{2.12}$$

As $1-\lambda \geq 1/2$, using the substitution $\beta = \alpha^{\frac{1}{1-\lambda}} \in (0,1)$, (2.12) leads to

$$\frac{\Gamma(\beta)}{\beta} \leq \frac{32\tau^{-\frac{3}{2}}\varepsilon^{\frac{1}{2}}}{\beta^{1-\lambda}} + 4\frac{\Gamma(\beta^{1-\lambda})}{\beta^{1-\lambda}},$$

and, by iteration,

$$\begin{aligned}\frac{\Gamma(\beta)}{\beta} &\leq 32\tau^{-\frac{3}{2}}\varepsilon^{\frac{1}{2}}\sum_{i=1}^{k}\frac{4^{i-1}}{\beta^{(1-\lambda)^i}} + 4^k\frac{\Gamma(\beta^{(1-\lambda)^k})}{\beta^{(1-\lambda)^k}} \\ &\leq c\Big(1+\tau^{-\frac{3}{2}}\frac{\varepsilon^{\frac{1}{2}}}{\beta^{1-\lambda}}\Big)\frac{4^k}{\beta^{(1-\lambda)^k}} \quad \forall k \geq 1.\end{aligned} \tag{2.13}$$

Hence, if $\varepsilon^{\frac{1}{2}} \leq \beta$, then (2.13) yields

$$\frac{\Gamma(\beta)}{\beta} \leq c\tau^{-\frac{3}{2}}\frac{4^k}{\beta^{(1-\lambda)^k}}.$$

Choosing $k \in [\frac{|\log|\log\beta||}{|\log(1-\lambda)|}, 2\frac{|\log|\log\beta||}{|\log(1-\lambda)|}]$ so that $\beta^{(1-\lambda)^k} \simeq 1$, then the bound above gives (recall that $\lambda \geq \tau$ and that $|\log(1-\tau)| \simeq \tau$)

$$\frac{\Gamma(\beta)}{\beta} \leq c\tau^{-\frac{3}{2}}4^{2\frac{|\log|\log\beta||}{\tau}} \leq c\tau^{-\frac{3}{2}}|\log\beta|^{\frac{4}{\tau}} \quad \forall \beta \in [\varepsilon^{\frac{1}{2}}, 1).$$

Since

$$\frac{\Gamma(\beta)}{\beta} \geq (1-\lambda)\mathcal{H}^1(A_\beta) + \lambda\mathcal{H}^1(B_\beta) \geq \tau(\mathcal{H}^1(A_t) + \mathcal{H}^1(B_t)),$$

this proves (2.7).

Finally, the layer-cake formula yields $\int_{\{f<\eta\}} f + \int_{\{g<\eta\}} g \leq \Gamma(\eta)/\tau$, and the monotonicity of A_t and B_t imply $\mathcal{H}^1(\{f \geq \eta\}) + \mathcal{H}^1(\{g \geq \eta\}) \leq \Gamma(\eta)/\eta$, completing the proof of Lemma 2.5. ∎

Proposition 2.6. *Let f, g, h satisfy (2.2) and (2.3) where $\tau \in (0, \frac{1}{2}]$ and $0 < \varepsilon < c\tau^3$ for a certain absolute constant $c \in (0, 2^{-6})$. For $\eta \geq \varepsilon$ with $\eta < 4c\tau^3$, we assume that there exist log-concave functions $\tilde{f}$, $\tilde{g}$ such that*

$$\|f - \tilde{f}\|_1 < \eta \|f\|_1 \quad \textit{and} \quad \|g - \tilde{g}\|_1 < \eta \|g\|_1.$$

Then, setting $a = \int_{\mathbb{R}} g / \int_{\mathbb{R}} f$, there exist a log-concave function $\tilde{h}$ and a constant $w \in \mathbb{R}$ such that

$$\int_{\mathbb{R}} |a^{\lambda} f(x) - \tilde{h}(x - \lambda w)|\, dx + \int_{\mathbb{R}} |a^{\lambda-1} g(x) - \tilde{h}(x + (1-\lambda) w)|\, dx \lesssim \tau^{-1} \eta^{\frac{1}{12}} |\log \varepsilon|^{\frac{4}{3}} \int_{\mathbb{R}} h,$$

$$\int_{\mathbb{R}} |h(x) - \tilde{h}(x)|\, dx \lesssim \tau^{-2} \eta^{\frac{1}{4}} |\log \varepsilon| \int_{\mathbb{R}} h.$$

Proof. We may assume that $\min\{\|f\|_\infty, \|g\|_\infty\} = 1$ and $\int_{\mathbb{R}} f = \int_{\mathbb{R}} g = 1$, so that Lemma 2.4 yields

$$1 = \min\{\|f\|_\infty, \|g\|_\infty\} \leq \max\{\|f\|_\infty, \|g\|_\infty\} \leq 1 + 4\tau^{-\frac{3}{2}} \varepsilon^{\frac{1}{2}} < 2. \tag{2.14}$$

It follows from the conditions $\|f - \tilde{f}\|_1 < \eta$ and $\|g - \tilde{g}\|_1 < \eta$ and $\eta < \frac{1}{2}$ that the approximating log-concave functions satisfy

$$\frac{1}{2} < \int_{\mathbb{R}} \tilde{f}, \int_{\mathbb{R}} \tilde{g} < 2. \tag{2.15}$$

The main idea of the proof is to show that, for a suitable log-concave function $\tilde{h}$, the log-concave functions $\tilde{f}_0 = \tilde{f} \chi_{\{\tilde{f} > \alpha\}}$ and $\tilde{g}_0 = \tilde{g} \chi_{\{\tilde{g} > \alpha\}}$ satisfy almost-equality in the Prékopa–Leindler inequality for some value $\alpha \geq \eta$; therefore, the stability version Theorem 2.1 of the Prékopa–Leindler inequality for log-concave functions implies that $\tilde{f}_0$ and $\tilde{g}_0$ can be expressed in terms of shifts and multiples of $\tilde{h}$.

As a first step, we claim that

$$\big|\|\tilde{f}\|_\infty - \|f\|_\infty\big| \leq 32\tau^{-\frac{3}{2}} \eta^{\frac{1}{2}} \quad \text{and} \quad \big|\|\tilde{g}\|_\infty - \|g\|_\infty\big| \leq 32\tau^{-\frac{3}{2}} \eta^{\frac{1}{2}}. \tag{2.16}$$

As the roles of f and g are symmetric, we only prove the statement about f.

First, we assume that $\|\tilde{f}\|_\infty > \|f\|_\infty$, hence $\|f\|_\infty = \|\tilde{f}\|_\infty - \alpha$ for some $\alpha > 0$. In this case, Lemma 2.2 (i) and (2.15) imply that $\mathcal{H}^1(\{\tilde{f} > \|\tilde{f}\|_\infty - s\}) \geq \frac{s}{2} \|\tilde{f}\|_\infty^{-2}$ for $s \in (0, \alpha)$; thus the layer-cake representation gives

$$\eta \geq \int_{\|f\|_\infty}^{\|\tilde{f}\|_\infty} \mathcal{H}^1(\{\tilde{f} > t\})\, dt > \frac{\alpha^2}{4\|\tilde{f}\|_\infty^2}.$$

Therefore, $\|f\|_\infty = \|\tilde{f}\|_\infty - \alpha \geq \|\tilde{f}\|_\infty (1 - 2\sqrt{\eta})$, and we deduce that

$$\|\tilde{f}\|_\infty - \|f\|_\infty \leq \|f\|_\infty \big[(1 - 2\sqrt{\eta})^{-1} - 1\big] < 8\eta^{\frac{1}{2}}.$$

Next we assume that $\|\tilde{f}\|_\infty < \|f\|_\infty$. We consider the function

$$f_1 = f \cdot \chi_{\{f \le \|\tilde{f}\|_\infty\}} + \|\tilde{f}\|_\infty \cdot \chi_{\{f > \|\tilde{f}\|_\infty\}},$$

which satisfies

$$1 \le \left(\int_{\mathbb{R}} f_1\right)^{-1} \le \left(\int_{\mathbb{R}} f - \int_{\mathbb{R}} |f - \tilde{f}|\right)^{-1} < 1 + 2\eta.$$

As $f_1 \le f$, we have $h((1-\lambda)x + \lambda y) \ge f_1(x)^{1-\lambda} g(y)^\lambda$ for any $x, y \in \mathbb{R}$ where

$$\int_{\mathbb{R}} h \le (1+\varepsilon)\left(\int_{\mathbb{R}} f\right)^{1-\lambda}\left(\int_{\mathbb{R}} g\right)^{\lambda} \le (1+4\eta)\left(\int_{\mathbb{R}} f_1\right)^{1-\lambda}\left(\int_{\mathbb{R}} g\right)^{\lambda}.$$

We deduce from Lemma 2.4 applied to f and g on the one hand, and to f_1 and g on the other hand that

$$\frac{\|f\|_\infty}{\|\tilde{f}\|_\infty} = \frac{\|f\|_\infty}{\|g\|_\infty} \cdot \frac{\|g\|_\infty}{\|f_1\|_\infty} \le (1 + 4\tau^{-\frac{3}{2}}\varepsilon^{\frac{1}{2}}) \cdot (1 + 4\tau^{-\frac{3}{2}}\eta^{\frac{1}{2}})(1+4\eta) < 1 + 16\tau^{-\frac{3}{2}}\eta^{\frac{1}{2}}.$$

Recalling (2.14), this proves claim (2.16). In turn, combining (2.14) and (2.16) leads to

$$\frac{1}{2} < \|f\|_\infty, \|g\|_\infty, \|\tilde{f}\|_\infty, \|\tilde{g}\|_\infty < 2. \tag{2.17}$$

For any $r > 0$, we define

$$A_r = \{f > r\}, \quad \tilde{A}_r = \{\tilde{f} > r\}, \quad B_r = \{g > r\}, \quad \tilde{B}_r = \{\tilde{g} > r\}.$$

According to the layer-cake representation (representing $\|\varphi - \psi\|_1$ for nonnegative $\varphi, \psi \in L_1(\mathbb{R})$ as the area of the symmetric difference of the parts between the graphs and the first axis),

$$\int_0^\infty \mathcal{H}^1(A_r \Delta \tilde{A}_r)\, dr = \|f - \tilde{f}\|_1 \le \eta,$$
$$\int_0^\infty \mathcal{H}^1(B_r \Delta \tilde{B}_r)\, dr = \|g - \tilde{g}\|_1 \le \eta.$$

In particular, the set $S \subset (0, \infty)$ defined by the property

$$\mathcal{H}^1(A_r \Delta \tilde{A}_r) + \mathcal{H}^1(B_r \Delta \tilde{B}_r) \le \eta^{\frac{1}{2}} \quad \text{for } r \in S \tag{2.18}$$

satisfies that

$$\mathcal{H}^1((0,\infty) \setminus S) < 4\eta^{\frac{1}{2}}. \tag{2.19}$$

It follows from (2.18) that if $r, s \in S$ and $x \in \mathbb{R}$, then $\mathcal{H}^1((1-\lambda)A_r \Delta (1-\lambda)\tilde{A}_r) \le (1-\lambda)\eta^{\frac{1}{2}}$ and $\mathcal{H}^1((x - \lambda B_s)\Delta(x - \lambda\tilde{B}_s)) \le \lambda\eta^{\frac{1}{2}}$; therefore,

$$\left|\mathcal{H}^1((1-\lambda)A_r \cap (x - \lambda B_s)) - \mathcal{H}^1((1-\lambda)\tilde{A}_r \cap (x - \lambda\tilde{B}_s))\right| \le \eta^{\frac{1}{2}}. \tag{2.20}$$

Consider

$$r_0 = \|\tilde f\|_\infty - 32\tau^{-1}\eta^{\frac14} \quad \text{and} \quad s_0 = \|\tilde g\|_\infty - 32\tau^{-1}\eta^{\frac14}. \tag{2.21}$$

Using (2.15) and (2.17), we deduce from Lemma 2.2 (i) that

$$\mathcal{H}^1(\tilde A_{r_0}), \mathcal{H}^1(\tilde B_{s_0}) \geq 4\tau^{-1}\eta^{\frac14}.$$

Possibly after shifting f and $\tilde f$ together on the one hand, and g and $\tilde g$ together on the other hand, we may assume that zero is the common midpoint of the segments $\tilde A_{r_0}$ and $\tilde B_{s_0}$. In particular, setting

$$\mathrm{cl}\,\tilde A_r = [a_1(r), a_2(r)] \ \text{ and } \ \mathrm{cl}\,\tilde B_s = [b_1(s), b_2(s)] \quad \text{for } 0 < r < \|\tilde f\|_\infty, 0 < s < \|\tilde g\|_\infty,$$

using that $a_1(r), b_1(r)$ are monotone increasing and $a_2(r), b_2(r)$ are monotone decreasing provided $0 < r < \min\{\|\tilde f\|_\infty, \|\tilde g\|_\infty\}$, we have

$$a_2(r), b_2(s) \geq 2\tau^{-1}\eta^{\frac14} \ \text{ and } \ a_1(r), b_1(s) \leq -2\tau^{-1}\eta^{\frac14} \quad \text{for } r \in (0, r_0], s \in (0, s_0].$$

We deduce that if $r \in S \cap (0, r_0)$, $s \in S \cap (0, s_0)$ and

$$x \in (1 + 2\eta^{\frac14})^{-1}((1-\lambda)\tilde A_r + (\lambda \tilde B_s)) \subset (1 - \eta^{\frac14})((1-\lambda)\tilde A_r + (\lambda \tilde B_s)),$$

then $(1-\lambda)a_i(r), \lambda b_i(s) \geq 2\eta^{\frac14}$ for $i = 1, 2$, and $x - \lambda\tilde B_s = [x - \lambda b_2(s), x + \lambda b_1(s)]$ satisfies $x - \lambda b_2(s) \leq (1-\lambda)a_2(r) - \eta^{\frac14}\lambda b_2(s)$ and $x + \lambda b_1(s) \geq -(1-\lambda)a_1(r) + \eta^{\frac14}\lambda b_1(s)$; therefore,

$$\mathcal{H}^1((1-\lambda)\tilde A_r \cap (x - \lambda \tilde B_s)) \geq 2\eta^{\frac12}.$$

In turn, (2.20) yields that if $x \in (1 + 2\eta^{\frac14})^{-1}((1-\lambda)\tilde A_r + (\lambda \tilde B_s))$, then

$$x \in (1-\lambda)A_r + (\lambda B_s).$$

In other words, if $r \in S \cap (0, r_0)$ and $s \in S \cap (0, s_0)$, then

$$(1-\lambda)\tilde A_r + \lambda \tilde B_s \subset (1 + 2\eta^{\frac14})((1-\lambda)A_r + \lambda B_s) \subset (1 + 2\eta^{\frac14})\{h > r^{1-\lambda}s^\lambda\}. \tag{2.22}$$

On the other hand, for any $r \in (\eta^{\frac14}, \|\tilde f\|_\infty)$ and $s \in (\eta^{\frac14}, \|\tilde g\|_\infty)$, (2.19) and the definitions of r_0, s_0 yield the existence of some $\tilde r \in S \cap (0, \min\{r, r_0\})$ and $\tilde s \in S \cap (0, \min\{s, s_0\})$ with

$$\tilde r \geq r - \theta(r) \quad \text{and} \quad \tilde s \geq s - \theta(s),$$

where $\theta(t) = 2^6\tau^{-1}\eta^{\frac14}$ if $t \geq \frac12$, and $\theta(t) = 4\eta^{\frac12}$ if $t \in (0, \frac12)$. In particular,

$$\tilde r \geq (1 - 2^7\tau^{-1}\eta^{\frac14})r \ \text{ and } \ \tilde s \geq (1 - 2^7\tau^{-1}\eta^{\frac14})s \quad \text{for } r, s \geq \eta^{\frac14},$$

thus setting $t = r^{1-\lambda}s^\lambda$, we have

$$\tilde r^{1-\lambda}\tilde s^\lambda \geq (1 - 2^7\tau^{-1}\eta^{\frac14})t \geq t - 2^8\tau^{-1}\eta^{\frac14}.$$

Therefore, if we define

$$\alpha = 2^8 \tau^{-1} \eta^{\frac{1}{4}},$$

then, for any $r \in (\alpha, \|\tilde{f}\|_\infty)$ and $s \in (\alpha, \|\tilde{g}\|_\infty)$, we deduce from (2.22) that $t = r^{1-\lambda} s^\lambda$ satisfies

$$\begin{aligned}(1-\lambda)\tilde{A}_r + \lambda \tilde{B}_s \subset (1-\lambda)\tilde{A}_{\tilde{r}} + \lambda \tilde{B}_{\tilde{s}} &\subset (1+2\eta^{\frac{1}{4}})\big\{h > \tilde{r}^{1-\lambda}\tilde{s}^{\lambda}\big\} \\ &\subset (1+2\eta^{\frac{1}{4}})\big\{h > t - \alpha\big\}. \end{aligned} \tag{2.23}$$

Next we replace $\tilde{f}$ by $\tilde{f}_0 = \tilde{f}\chi_{\{\tilde{f}>\alpha\}}$ and $\tilde{g}$ by $\tilde{g}_0 = \tilde{g}\chi_{\{\tilde{g}>\alpha\}}$. Then Lemma 2.2, (2.15), and $\frac{1}{2} < \|\tilde{f}\|_\infty, \|\tilde{g}\|_\infty < 2$ (cf. (2.16)), yield

$$\|\tilde{f} - \tilde{f}_0\|_1 + \|\tilde{g} - \tilde{g}_0\|_1 \le 32\alpha, \tag{2.24}$$

$$\mathcal{H}^1(\operatorname{supp} \tilde{f}_0) + \mathcal{H}^1(\operatorname{supp} \tilde{g}_0) \le 32|\log \alpha|. \tag{2.25}$$

In particular, we deduce from (2.24) that

$$\|f - \tilde{f}_0\|_1 + \|g - \tilde{g}_0\|_1 \le 2^6\alpha, \tag{2.26}$$

hence

$$\int_{\mathbb{R}} \tilde{f}_0, \int_{\mathbb{R}} \tilde{g}_0 \ge 1 - 2^6 \cdot \alpha. \tag{2.27}$$

Consider now the log-concave function $\tilde{h}$ defined as

$$\tilde{h}(z) = \sup_{z=(1-\lambda)x+\lambda y} \tilde{f}_0(x)^{1-\lambda}\tilde{g}_0(y)^{\lambda},$$

which satisfies $\tilde{h}(z) \ge \alpha$ for any $z \in \operatorname{int}\operatorname{supp}\tilde{h}$ and

$$\mathcal{H}^1(\operatorname{supp}\tilde{h}) \le 32|\log \alpha| \tag{2.28}$$

(see (2.25)). According to (2.27) and the Prékopa–Leindler inequality, we have

$$\int_{\mathbb{R}} \tilde{h} \ge 1 - 2^6\alpha. \tag{2.29}$$

It follows from the definition of $\tilde{h}$ and (2.23) that, for any $t > \alpha$, we have

$$\{\tilde{h} > t\} = \bigcup_{t=r^{1-\lambda}s^{\lambda}} ((1-\lambda)\tilde{A}_r + \lambda\tilde{B}_s) \subset (1+2\eta^{\frac{1}{4}})\{h > t - \alpha\}. \tag{2.30}$$

To relate $\tilde{h}$ to f and g, we deduce from (2.27) and (2.30) that

$$\begin{aligned}\int_{\mathbb{R}} \tilde{h} = \int_\alpha^\infty \mathcal{H}^1(\{\tilde{h} > t\})\,dt &\le (1+2\eta^{\frac{1}{4}})\int_\alpha^\infty \mathcal{H}^1(\{h > t-\alpha\})\,dt \\ &= (1+2\eta^{\frac{1}{4}})\int_{\mathbb{R}} h < 1 + 4\eta^{\frac{1}{4}} \\ &\le (1+2^9\alpha)\Big(\int_{\mathbb{R}} \tilde{f}_0\Big)^{1-\lambda}\Big(\int_{\mathbb{R}} \tilde{g}_0\Big)^{\lambda}. \end{aligned} \tag{2.31}$$

Recalling that $\alpha = 2^8\tau^{-1}\eta^{\frac{1}{4}}$, thanks to Theorem 2.1 there exists $w \in \mathbb{R}$ such that

$$\int_{\mathbb{R}^n} |a_0^\lambda \tilde{f}_0 - \tilde{h}(\cdot + \lambda w)| + \int_{\mathbb{R}^n} |a_0^{\lambda-1}\tilde{g}_0 - \tilde{h}(\cdot + (\lambda-1)w)| \lesssim \tau^{-\frac{2}{3}}\eta^{\frac{1}{12}}|\log\alpha|^{\frac{4}{3}} \int_{\mathbb{R}^n} \tilde{h},$$

where $a_0 = \int_{\mathbb{R}^n}\tilde{g}_0/\int_{\mathbb{R}^n}\tilde{f}_0$. Also, by (2.27) and the conditions $\int_{\mathbb{R}}\tilde{f}, \int_{\mathbb{R}}\tilde{g} \le 1+\eta$, it holds that

$$1 - 2^{14}\tau^{-1}\eta^{\frac{1}{4}} \le \int_{\mathbb{R}} \tilde{f}_0, \int_{\mathbb{R}} \tilde{g}_0 \le 1+\eta,$$

In particular, $|a_0 - 1| \lesssim \tau^{-1}\eta^{\frac{1}{4}}$; therefore,

$$\int_{\mathbb{R}^n} |\tilde{f}_0 - \tilde{h}(\cdot + \lambda w)| + \int_{\mathbb{R}^n} |\tilde{g}_0 - \tilde{h}(\cdot + (\lambda-1)w)| \lesssim \tau^{-\frac{2}{3}}\eta^{\frac{1}{12}}|\log\alpha|^{\frac{4}{3}} \int_{\mathbb{R}^n} \tilde{h}.$$

Recalling (2.26), this proves the first bound in the statement of Proposition 2.6.

To relate $\tilde{h}$ to h, consider the auxiliary function

$$\tilde{h}_0(x) = \begin{cases} \tilde{h}((1+2\eta^{\frac{1}{4}})x) - \alpha & \text{if } x \in \operatorname{int}\operatorname{supp}\tilde{h}, \\ 0 & \text{otherwise,} \end{cases}$$

so that, if $t > \alpha$, then

$$\{\tilde{h} > t\} = (1+2\eta^{\frac{1}{4}})\{\tilde{h}_0 > t - \alpha\}. \tag{2.32}$$

Comparing (2.32) and (2.30), it follows that $\tilde{h}_0 \le h$. In addition, (2.29) implies that

$$1 - 2^7\alpha < (1+2\eta^{\frac{1}{4}})^{-1}\int_{\mathbb{R}} \tilde{h} = \int_{\mathbb{R}} \tilde{h}_0 \le \int_{\mathbb{R}} h < 1+\varepsilon,$$

therefore

$$\|h - \tilde{h}_0\|_1 < 2^8\alpha. \tag{2.33}$$

Next we claim that

$$\tilde{h}((1+2\eta^{\frac{1}{4}})x) < \tilde{h}(x) + 2^7\tau^{-2}\eta^{\frac{1}{4}} \quad \text{for any } x \in \operatorname{supp}\tilde{h}. \tag{2.34}$$

We observe that $t_0 = r_0^{1-\lambda}s_0^\lambda \ge 1 - 2^6\tau^{-\frac{3}{2}}\eta^{\frac{1}{4}}$ according to (2.14), (2.16), and (2.21). Since $\tilde{f}$ and $\tilde{g}$ were translated to ensure $\tilde{f}_0(0) \ge r_0$ and $\tilde{g}_0(0) \ge s_0$, we deduce that $\tilde{h}(0) \ge t_0$. Using that $\tilde{h}$ is log-concave, we deduce that if $\tilde{h}(x) \le t_0$, then $\tilde{h}((1+2\eta^{\frac{1}{4}})x) \le \tilde{h}(x)$. On the other hand, if $\tilde{h}(x) > t_0$ then (2.34) follows from $\|\tilde{h}\|_\infty \le 1 + 32\tau^{-\frac{3}{2}}\eta^{\frac{1}{2}}$ (see (2.14) and (2.16)) and the bound $t_0 \ge 1 - 2^6\tau^{-\frac{3}{2}}\eta^{\frac{1}{4}}$.

Thanks to (2.34), since $\alpha \le 2^7\tau^{-2}\eta^{\frac{1}{4}}$ we get

$$\begin{aligned} \|\tilde{h} - \tilde{h}_0\|_1 &= \int_{\operatorname{supp}\tilde{h}} \big|\tilde{h}(x) - \tilde{h}((1+2\eta^{\frac{1}{4}})x) + \alpha\big|\,dx \\ &= \int_{\operatorname{supp}\tilde{h}} \big|\tilde{h}(x) + 2^7\tau^{-2}\eta^{\frac{1}{4}} - \tilde{h}((1+2\eta^{\frac{1}{4}})x) + (\alpha - 2^7\tau^{-2}\eta^{\frac{1}{4}})\big|\,dx \\ &\le \int_{\operatorname{supp}\tilde{h}} \tilde{h}(x) + 2^7\tau^{-2}\eta^{\frac{1}{4}} - \tilde{h}((1+2\eta^{\frac{1}{4}})x)\,dx + \int_{\operatorname{supp}\tilde{h}} 2^7\tau^{-2}\eta^{\frac{1}{4}}\,dx \\ &= \left(1 - \frac{1}{1+2\eta^{\frac{1}{4}}}\right)\int_{\operatorname{supp}\tilde{h}} \tilde{h}(x)\,dx + 2\cdot\mathcal{H}^1(\operatorname{supp}\tilde{h})\cdot 2^7\tau^{-2}\eta^{\frac{1}{4}}. \end{aligned}$$

Since $\int_{\mathbb{R}} \tilde{h} < 2$ and $\mathcal{H}^1(\operatorname{supp} \tilde{h}) \le 32|\log \alpha|$ (see (2.31) and (2.28)), we conclude that $\|\tilde{h} - \tilde{h}_0\|_1 < 2^{14}\tau^{-2}\eta^{\frac{1}{4}}|\log \alpha|$. Combining this estimate with (2.33) implies that $\|h - \tilde{h}\|_1 < 2^{15}\tau^{-2}\eta^{\frac{1}{4}}|\log \alpha|$. As $\alpha = 2^8\tau^{-1}\eta^{\frac{1}{4}}$, we have $|\log \alpha| \lesssim \max\{|\log \tau|, |\log \varepsilon|\} \lesssim |\log \varepsilon|$. Plugging this into the statements above, we obtain the original claim, which finishes the proof. ∎

3. The case of symmetric-rearranged functions

For this part and for the remainder of the paper, we assume that all the reductions and results from Section 2 hold.

As noticed at the beginning of the previous section, the *symmetric decreasing rearrangements* of functions f, g, h satisfying (1.2) and (1.5), denoted by f^*, g^*, h^*, also satisfy (1.2) and (1.5) with the same constant, as rearrangements preserve L^p-norms. By changing these functions on a zero-measure set, we may suppose that their level sets are all open. The main result of this section Theorem 3.2 lays out the foundation for the analysis in the following ones. But first we state a lemma that is used in the proof of Theorem 3.2 and also later in the paper.

Lemma 3.1. *Let $f, g, h\colon \mathbb{R} \to \mathbb{R}_{\ge 0}$ satisfy (1.2) and (1.5) for $0 < \varepsilon < 2^{-6}\tau^3$, $\|f\|_1 = \|g\|_1 = 1$, $\min\{\|f\|_\infty, \|g\|_\infty\} = 1$, and let $A_t = \{f \ge t\}$, $B_t = \{g \ge t\}$, $C_t = \{h \ge t\}$ be their level sets. Then*

(i) $\displaystyle\int_{\mathbb{R}_+} \big|\mathcal{H}^1(C_t) - (1-\lambda)\mathcal{H}^1(A_t) - \lambda\mathcal{H}^1(B_t)\big|\, dt \le 9\tau^{-\frac{3}{2}}\varepsilon^{\frac{1}{2}}$;

(ii) *there exists a measurable set $F \subset \mathbb{R}_+$ such that $\mathcal{H}^1(\mathbb{R}_+ \setminus F) \le 9\varepsilon^{\frac{1}{4}}$ and*

$$\big|\mathcal{H}^1(C_t) - (1-\lambda)\mathcal{H}^1(A_t) - \lambda\mathcal{H}^1(B_t)\big| \le \tau^{-\frac{3}{2}}\varepsilon^{\frac{1}{4}} \quad \forall t \in F.$$

Proof. We may assume that $\min\{\|f\|_\infty, \|g\|_\infty\} = \|f\|_\infty = 1$, and hence Lemma 2.4 yields that

$$1 \le \|g\|_\infty \le 1 + 4\tau^{-\frac{3}{2}}\varepsilon^{\frac{1}{2}}.$$

Let $S_1 = \{t \ge 0;\, \mathcal{H}^1(C_t) \ge (1-\lambda)\mathcal{H}^1(A_t) + \lambda\mathcal{H}^1(B_t)\}$. By the reductions made, we know that $S_1 \supseteq (0,1)$ as $A_t \ne \emptyset$ and $B_t \ne \emptyset$ if $0 < t < 1 = \|f\|_\infty \le \|g\|_\infty$, and $S_1 \supseteq (1 + 4\tau^{-\frac{3}{2}}, \infty)$ as $A_t = B_t = \emptyset$ if $t > 1 + 4\tau^{-\frac{3}{2}} \ge \|g\|_\infty \ge \|f\|_\infty$. If $t \in S_2$ for $S_2 = \mathbb{R}_+ \setminus S_1$, then $t \ge 1$ and $\int_{\mathbb{R}} f = \int_{\mathbb{R}} g \le \int_{\mathbb{R}} h \le 1 + \varepsilon$ yield $\mathcal{H}^1(A_t), \mathcal{H}^1(B_t), \mathcal{H}^1(C_t) \le 1 + \varepsilon$; therefore,

$$|\mathcal{H}^1(C_t) - (1-\lambda)\mathcal{H}^1(A_t) - \lambda\mathcal{H}^1(B_t)| \le 1 + \varepsilon < 2 \quad \forall t \in S_2.$$

Thus,

$$\int_{S_2} |\mathcal{H}^1(C_t) - (1-\lambda)\mathcal{H}^1(A_t) - \lambda\mathcal{H}^1(B_t)|\, dt \le \int_1^{1+4\tau^{-\frac{3}{2}}\varepsilon^{\frac{1}{2}}} 2\, dt = 8\tau^{-\frac{3}{2}}\varepsilon^{\frac{1}{2}}.$$

By the fact that the integral $\int_{\mathbb{R}_+}(\mathcal{H}^1(C_t)-(1-\lambda)\mathcal{H}^1(A_t)-\lambda\mathcal{H}^1(B_t))\,dt \le \varepsilon$, we obtain

$$\int_{\mathbb{R}_+} |\mathcal{H}^1(C_t)-(1-\lambda)\mathcal{H}^1(A_t)-\lambda\mathcal{H}^1(B_t)|\,dt \le 9\tau^{-\frac{3}{2}}\varepsilon^{\frac{1}{2}}.$$

By using Chebyshev's inequality, we obtain that the set of $t \ge 0$ where the integrand is larger than $\tau^{-\frac{3}{2}}\varepsilon^{\frac{1}{4}}$ has measure at most $9\varepsilon^{\frac{1}{4}}$, which finishes the proof of Lemma 3.1. ■

Theorem 3.2. *There is an absolute constant $c > 0$ such that the following holds. Suppose $f, g, h\colon \mathbb{R} \to \mathbb{R}_{\ge 0}$ satisfy (1.2) and (1.5) for $0 < \varepsilon < ce^{-1000|\log\tau|^4/\tau^4}$. Then there exist even log-concave functions $\tilde{f}$, $\tilde{g}$ such that*

$$\|f^* - \tilde{f}\|_1 + \|g^* - \tilde{g}\|_1 \lesssim \tau^{-\omega}\varepsilon^{\frac{\tau}{2^{21}|\log\tau|}},$$

where ω is an absolute constant given by $\omega = 6 + \frac{3\omega_0}{2}$, with ω_0 as in Lemma 3.3.

Here and henceforth, given a family of sets $\{S_\alpha\}$, we shall use the notation $\bigcup_\alpha^* S_\alpha$ to denote the union $\bigcup_{\alpha : S_\alpha \ne \emptyset} S_\alpha$.

Proof of Theorem 3.2. First, we may suppose without loss of generality that $\|f\|_1 = \|g\|_1 = 1$, and that $\min\{\|f\|_\infty, \|g\|_\infty\} = \|f\|_\infty = 1$. These assumptions, together with Lemma 2.4, imply that

$$0 \le \|g\|_\infty - 1 \le 4\tau^{-\frac{3}{2}}\varepsilon^{\frac{1}{2}}.$$

Consider, thus, the functions $a, b, c\colon \mathbb{R} \to \mathbb{R}_+$ defined to satisfy, for any $R \in \mathbb{R}$,

$$\{f^* > e^R\} = (-a(R), a(R)) =: \mathcal{A}_R,$$
$$\{g^* > e^R\} = (-b(R), b(R)) =: \mathcal{B}_R,$$
$$\{h^* > e^R\} = (-c(R), c(R)) =: \mathcal{C}_R.$$

By (1.2) applied to h^*, we have (remember, $\bigcup_\alpha^* S_\alpha = \bigcup_{\alpha : S_\alpha \ne \emptyset} S_\alpha$ for any sets S_α)

$$\mathcal{C}_T \supseteq \bigcup_{(1-\lambda)R+\lambda S = T}^{*} \{(1-\lambda)\mathcal{A}_R + \lambda\mathcal{B}_S\}. \tag{3.1}$$

Thus, as $\int f^* = \int g^* = 1$, by a change of variables $t = e^T$, we have

$$\varepsilon \ge \int_{-\infty}^{\infty} \big(\mathcal{H}^1(\mathcal{C}_T) - ((1-\lambda)\mathcal{H}^1(\mathcal{A}_T) + \lambda\mathcal{H}^1(\mathcal{B}_T))\big)e^T\,dT.$$

Notice that the map $T \mapsto \mathcal{H}^1(\mathcal{C}_T) - (1-\lambda)\mathcal{H}^1(\mathcal{A}_T) - \lambda\mathcal{H}^1(\mathcal{B}_T)$ is, by (3.1) and the Brunn–Minkowski inequality, nonnegative for all $T \in \mathbb{R}$ for which $\mathcal{A}_T, \mathcal{B}_T \ne \emptyset$. We observe that

$$\mathcal{A}_T = A_{e^T}, \quad \mathcal{B}_T = B_{e^T}, \quad \mathcal{C}_T = C_{e^T}.$$

Let F be the set constructed in Lemma 3.1 (ii). In particular, Lemma 3.1 yields that if $\mathcal{A}_R, \mathcal{B}_S \neq \emptyset, (1-\lambda)R + \lambda S = T$, and $e^T = t \in F$, we have

$$(1-\lambda)a(R) + \lambda b(S) \leq ((1-\lambda)a + \lambda b)(T) + \tau^{-\frac{3}{2}}\varepsilon^{\frac{1}{4}}. \tag{3.2}$$

Fix thus $M = \theta \log(1/\varepsilon)$, with $\theta > 0$ small to be chosen later. Denote $F_M = F \cap [e^{-M}, e^M]$. With this definition, we have that the set

$$\log(F_M) = \{T \in \mathbb{R} \colon e^T \in F_M\}$$

has large measure within $[-M, M]$. Indeed, recalling that $\mathcal{H}^1(\mathbb{R}_+ \setminus F) \leq \varepsilon^{\frac{1}{4}}$,

$$\begin{aligned}\int_{\mathbb{R}} \chi_{[-M,M]\setminus\log(F_M)}(T)\,dT &\leq e^M \int_{\mathbb{R}} \chi_{[-M,M]\setminus\log(F_M)}(T) e^T\,dT \\ &= \varepsilon^{-\theta}\mathcal{H}^1([e^{-M}, e^M] \setminus F) \leq \varepsilon^{\frac{1}{4}-\theta}.\end{aligned} \tag{3.3}$$

Thus, if $\theta < 1/8$, then $\mathcal{H}^1([-M, M] \setminus \log(F_M)) \leq \varepsilon^{\frac{1}{8}}$.

Therefore, if $T_1, T_2 \in \log(F_M)$, and additionally

$$T_{1,2} = \frac{1}{2-\lambda}T_1 + \frac{1-\lambda}{2-\lambda}T_2 \in \log(F_M), \quad T_{2,1} = \frac{1}{2-\lambda}T_2 + \frac{1-\lambda}{2-\lambda}T_1 \in \log(F_M),$$

then the reduction in [19, Remark 4.1] and inequality (3.2) show that the following *four-point inequalities* hold:

$$\begin{aligned} a(T_1) + a(T_2) &\leq a(T_{1,2}) + a(T_{2,1}) + \frac{2}{\lambda}\tau^{-\frac{3}{2}}\varepsilon^{\frac{1}{4}}, \\ b(T_1) + b(T_2) &\leq b(T_{1,2}) + b(T_{2,1}) + \frac{2}{\lambda}\tau^{-\frac{3}{2}}\varepsilon^{\frac{1}{4}}. \end{aligned}$$

Inspired by this, we recall the statement of [19, Lemma 3.6] in the one-dimensional case:

Lemma 3.3 ([19, Lemma 3.6]). *Let $G \subset \mathbb{R}$ be a measurable subset and $\psi \colon G \to \mathbb{R}$ be a function, such that the following properties hold:*

(1) *The four-point inequality*

$$\psi(T_1) + \psi(T_2) \leq \psi(T_{1,2}) + \psi(T_{2,1}) + \sigma \tag{3.4}$$

holds, whenever $T_1, T_2, T_{1,2}, T_{2,1} \in G$.

(2) *The convex hull $\mathrm{co}(G) = \Omega$ satisfies $\mathcal{H}^1(\Omega \setminus G) \leq \zeta$.*

(3) *There is $r \in (1/2, 2)$ with $[-r, r] = \Omega$.*

(4) *The inequalities $-\kappa \leq \psi(T) \leq \kappa$ hold for all $T \in G$ for some $\kappa \geq 1$.*

(5) *There is $H \subset \mathbb{R}$ such that*

$$\int_H \mathcal{H}^1(\mathrm{co}(\{\psi > s\}) \setminus \{\psi > s\})\,ds + \int_{\mathbb{R}\setminus H} \mathcal{H}^1(\{\psi > s\}) \leq \zeta. \tag{3.5}$$

Then there exist a concave function $\Psi\colon \Omega \to [-2\kappa, 2\kappa]$ *and an absolute constant* $c > 0$ *such that*

$$\int_G |\Psi(T) - \psi(T)|\, dT \le c\kappa\tau^{-\omega_0}(\sigma + \zeta)^{\alpha_\tau},$$

where we let $\alpha_\tau = \frac{\tau}{16|\log\tau|}$, *and* $\omega_0 > 0$ *is an absolute constant.*

We are almost ready to apply Lemma 3.3: we change variables and set $\tilde{a}(T') = a(MT')$.

If $T_1', T_2', T_{1,2}', T_{2,1}' \in \log(F_M)/M$ and $\lambda \in [\tau, 1-\tau]$, then the four-point inequality (3.4) holds for $\tilde{a}$, with $\sigma = \frac{2\varepsilon^{1/4}}{\tau^{5/2}}$. Moreover, the properties of $\log(F_M)$ (see (3.3)) imply

$$\mathcal{H}^1(\mathrm{co}(\log(F_M)/M) \setminus (\log(F_M)/M)) \le \varepsilon^{\frac{1}{8}}.$$

From that, we see that $\widetilde{\Omega}_M := \mathrm{co}(\log(F_M)/M)$ is an interval that differs by at most $\varepsilon^{\frac{1}{8}}$ from the interval $[-1,1]$, and thus can be written as $T_0 + I$, with $I = [-r, r]$ and $|r-1| \le 2\varepsilon^{\frac{1}{8}}$, and $T_0 \in \mathbb{R}$ with $|T_0| \le \varepsilon^{\frac{1}{8}}$.

Defining the function $\tilde{a}'(T'') = \tilde{a}(T' + T_0)$ preserves conditions (1), (2), (4), and (5), in Lemma 3.3. In addition, now condition (3) is also fulfilled. Furthermore, by Lemma 2.5, we have that $\tilde{a}'$ is bounded in absolute value by $\kappa = \frac{c}{\tau^4}|\log\varepsilon|^{4/\tau}$, with c an absolute constant.

Finally, as the function a is nonincreasing on $\mathbb{R}$, the level sets of $\tilde{a}'$ are all intervals. Hence we may take H to be the support of $\tilde{a}'$ in (3.5) and $\zeta = 4\varepsilon^{\frac{1}{8}}$.

Therefore, by Lemma 3.3, there is a concave function $\tilde{\mathfrak{a}}'\colon \widetilde{\Omega}_M' := \widetilde{\Omega}_M - T_0 \to [-2\kappa, 2\kappa]$ such that

$$\int_{\log(F_M)/M - T_0} |\tilde{\mathfrak{a}}'(T) - \tilde{a}'(T)|\, dT \le \kappa\tau^{-\omega_0} \cdot \frac{\varepsilon^{\frac{\alpha_\tau}{8}}}{\tau^{5\alpha_\tau/2}}.$$

Thus, the function $\tilde{\mathfrak{a}}(T) = \tilde{\mathfrak{a}}'(T - T_0)$ satisfies

$$\int_{\log(F_M)/M} |\tilde{\mathfrak{a}}(T) - \tilde{a}(T)|\, dT \lesssim |\log\varepsilon|^{\frac{4}{\tau}} \frac{\varepsilon^{\frac{\alpha_\tau}{8}}}{\tau^{4+\omega_0}}.$$

This follows from the definition of κ and the fact that $\tau^{\alpha_\tau} = e^{-\tau/16}$, which is bounded from below and above whenever $\tau \in (0, 1/2]$. Changing variables $T = T'/M$ above yields that $\mathfrak{a}(T) = \tilde{\mathfrak{a}}(T/M)$ satisfies (recall that $M = \theta\log(1/\varepsilon)$)

$$\int_{\log(F_M)} |\mathfrak{a}(T') - a(T')|\, dT' \lesssim |\log\varepsilon|^{1+\frac{4}{\tau}} \frac{\varepsilon^{\frac{\alpha_\tau}{8}}}{\tau^{4+\omega_0}}. \tag{3.6}$$

We observe that, if we denote by $\Omega_M = M\widetilde{\Omega}_M$ the domain of definition of $\mathfrak{a}$, then it follows from the considerations above that $\mathcal{H}^1([-M, M] \setminus \Omega_M) \lesssim |\log\varepsilon|\varepsilon^{\frac{1}{8}}$.

Notice that the process above can be adapted verbatim to b, and we find a concave function $\mathfrak{b}\colon \Omega_M \to [-2\kappa, 2\kappa]$ such that

$$\int_{\log(F_M)} |\mathfrak{b}(T') - b(T')|\, dT' \lesssim |\log\varepsilon|^{1+\frac{4}{\tau}} \frac{\varepsilon^{\frac{\alpha_\tau}{8}}}{\tau^{4+\omega_0}}. \tag{3.7}$$

For brevity, let $\omega_1 := 4 + \omega_0$. We must now ensure that $\mathfrak{a}$, $\mathfrak{b}$ satisfy the requirements of distribution functions. Indeed, in the case that $\mathfrak{a}$, $\mathfrak{b}$ are both nonincreasing on the subinterval $I_M = [-3M/4, 3M/4] \subset \Omega_M$, we do not change them.

On the other hand, if either $\mathfrak{a}$ or $\mathfrak{b}$ are not nonincreasing on such a large interval, we use Chebyshev's inequality in conjunction with (3.6) and (3.7).

This implies that there is a set $\mathcal{F} \subset \log(F_M)$ such that $\mathcal{H}^1(\log(F_M) \setminus \mathcal{F}) \leq \tau^{-\frac{\omega_1}{2}} \varepsilon^{\frac{\alpha_\tau}{32}}$, and

$$|\mathfrak{b}(T) - b(T)| + |\mathfrak{a}(T) - a(T)| \lesssim \tau^{-\frac{\omega_1}{2}} \varepsilon^{\frac{\alpha_\tau}{32}} \quad \forall T \in \mathcal{F}.$$

Changing $\mathfrak{a}$, $\mathfrak{b}$ on a zero measure set, we may suppose that both are lower semicontinuous. Suppose then, without loss of generality, that $\mathfrak{a}$ attains its maximum at a point $T_0 \in I_M$.

As $\mathcal{H}^1(\Omega_M \setminus \mathcal{F}) \lesssim \tau^{-\frac{\omega_1}{2}} \varepsilon^{\frac{\alpha_\tau}{32}}$, there is a point $T_1 \in \mathcal{F}$ such that

$$|T_0 - T_1| \lesssim \tau^{-\frac{\omega_1}{2}} \varepsilon^{\frac{\alpha_\tau}{32}}.$$

Analogously, there is a point $T_2 \in \mathcal{F}$ such that $|T_2 + M| \lesssim \tau^{-\frac{\omega_1}{2}} \varepsilon^{\frac{\alpha_\tau}{32}}$, thus,

$$\begin{aligned}\mathfrak{a}(T_0) - \mathfrak{a}(T_2) &\leq |\mathfrak{a}(T_2) - a(T_2)| + a(T_1) - a(T_2) + |a(T_1) - \mathfrak{a}(T_1)| + |\mathfrak{a}(T_1) - \mathfrak{a}(T_0)| \\ &\leq c\tau^{-\frac{\omega_1}{2}} \varepsilon^{\frac{\alpha_\tau}{32}} + |\mathfrak{a}(T_1) - \mathfrak{a}(T_0)|. \end{aligned} \tag{3.8}$$

On the other hand, by concavity,

$$\mathfrak{a}(T_1) \geq \gamma \mathfrak{a}(T_0) + (1-\gamma)\mathfrak{a}(T_2), \quad \text{with } \gamma \in (0,1) \text{ such that } \gamma T_0 + (1-\gamma) T_2 = T_1.$$

It follows from the manner we have chosen T_0, T_1, T_2 that

$$\tau^{-\frac{\omega_1}{2}} \varepsilon^{\frac{\alpha_\tau}{32}} \gtrsim |T_1 - T_0| = (1-\gamma)|T_0 - T_2| \geq \Big(\frac{M}{4} - c\tau^{-\frac{\omega_1}{2}} \varepsilon^{\frac{\alpha_\tau}{32}}\Big)(1-\gamma).$$

Thus, if $\varepsilon > 0$ is sufficiently small, we have

$$\gamma \geq 1 - 10\tau^{-\frac{\omega_1}{2}} \varepsilon^{\frac{\alpha_\tau}{64}}.$$

Also, by boundedness of $\mathfrak{a}$, we have

$$|\mathfrak{a}(T_1) - \mathfrak{a}(T_0)| \lesssim |\log \varepsilon|^{\frac{4}{\tau}} \tau^{-\frac{3\omega_1}{2}} \varepsilon^{\frac{\alpha_\tau}{64}}. \tag{3.9}$$

Combining (3.9) and (3.8) implies

$$\mathfrak{a}(T_0) \leq \mathfrak{a}(T_2) + c|\log \varepsilon|^{\frac{4}{\tau}} \tau^{-\frac{3\omega_1}{2}} \varepsilon^{\frac{\alpha_\tau}{64}},$$

where $c > 0$ is an absolute constant, and so, by monotonicity,

$$\mathfrak{a}(T_0) \leq \mathfrak{a}(T) + c|\log \varepsilon|^{\frac{4}{\tau}} \tau^{-\frac{3\omega_1}{2}} \varepsilon^{\frac{\alpha_\tau}{64}} \quad \forall T \in I_M T < T_0. \tag{3.10}$$

We thus define

$$\tilde{\mathfrak{a}}(T) = \begin{cases} \mathfrak{a}(T) & \text{if } T \in I_M, T \geq T_0, \\ \mathfrak{a}(T_0) & \text{if } T \in I_M, T < T_0. \end{cases}$$

This new function, besides being concave, is also nonincreasing on I_M, and, by (3.6) and (3.10),

$$\int_{\log(F_M)\cap I_M} |\tilde{\mathrm{a}}(T)-a(T)|\,dT \lesssim |\log\varepsilon|^{1+\frac{4}{\tau}}\tau^{-\frac{3\omega_1}{2}}\varepsilon^{\frac{\alpha_\tau}{64}}.$$

As both a, $\tilde{\mathrm{a}}$ are bounded by $c|\log\varepsilon|^{\frac{4}{\tau}}/\tau^4$ on I_M and $\mathcal{H}^1(I_M\setminus\log(F_M))\le\varepsilon^{\frac{1}{8}}$, we conclude moreover that

$$\int_{I_M} |\tilde{\mathrm{a}}(T)-a(T)|\,dT \lesssim |\log\varepsilon|^{1+\frac{4}{\tau}}\tau^{-\frac{3\omega_1}{2}}\varepsilon^{\frac{\alpha_\tau}{64}}.$$

By symmetry, the same method can be applied to the function b. Given the two resulting concave functions $\tilde{\mathrm{a}}$, $\tilde{\mathrm{b}}$, they define an almost-everywhere unique pair $\tilde{f}$, $\tilde{g}$ of functions such that

$$\{x\in\mathbb{R}\colon \tilde{f}(x)>t\}=(-\tilde{\mathrm{a}}(\log t),\tilde{\mathrm{a}}(\log t)),\quad \{x\in\mathbb{R}\colon \tilde{g}(x)>t\}=(-\tilde{\mathrm{b}}(\log t),\tilde{\mathrm{b}}(\log t)),$$

whenever $\log t\in I_M$ (that is, $t\in(\varepsilon^{\frac{3\theta}{4}},\varepsilon^{-\frac{3\theta}{4}})$),

$$\mathrm{supp}(\tilde{f})=\bigcup_{t\in(\varepsilon^{\frac{3\theta}{4}},\varepsilon^{-\frac{3\theta}{4}})}(-\tilde{\mathrm{a}}(\log t),\tilde{\mathrm{a}}(\log t)),\quad \mathrm{supp}(\tilde{g})=\bigcup_{t\in(\varepsilon^{\frac{3\theta}{4}},\varepsilon^{-\frac{3\theta}{4}})}(-\tilde{\mathrm{b}}(\log t),\tilde{\mathrm{b}}(\log t)),$$

and $\{x\in\mathbb{R}\colon\tilde{f}(x)>t\}=\{x\in\mathbb{R}\colon\tilde{g}(x)>s\}=\emptyset$ for $t,s>\varepsilon^{-\frac{3\theta}{4}}$ or whenever $\tilde{\mathrm{a}}(\log t)=0=\tilde{\mathrm{b}}(\log s)$.

We claim that these functions are log-concave. Indeed, if $\tilde{f}(x_1)>s_1$ and $\tilde{f}(x_2)>s_2$ with $s_1,s_2\in(\varepsilon^{\frac{3\theta}{4}},\varepsilon^{-\frac{3\theta}{4}})$ then

$$x_1\in(-\tilde{\mathrm{a}}(\log s_1),\tilde{\mathrm{a}}(\log s_1)),\quad x_2\in(-\tilde{\mathrm{a}}(\log s_2),\tilde{\mathrm{a}}(\log s_2)).$$

By concavity, for any $t\in(0,1)$,

$$\begin{aligned}tx_1+(1-t)x_2&\in(-t\tilde{\mathrm{a}}(\log s_1)-(1-t)\tilde{\mathrm{a}}(\log s_2),t\tilde{\mathrm{a}}(\log s_1)+(1-t)\tilde{\mathrm{a}}(\log s_2))\\&\subseteq(-\tilde{\mathrm{a}}(\log(s_1^ts_2^{1-t})),\tilde{\mathrm{a}}(\log(s_1^ts_2^{1-t}))).\end{aligned}$$

Thus $\tilde{f}(tx_1+(1-t)x_2)>s_1^ts_2^{1-t}$, which concludes in this case.

The case $\max\{s_1,s_2\}>\varepsilon^{-\frac{3\theta}{4}}$ or $\tilde{\mathrm{a}}(\max\{\log s_1,\log s_2\})=0$ is trivial by definition. Also, if $s_1\in(0,\varepsilon^{\frac{3\theta}{4}})$, then $x_1\in(-\tilde{\mathrm{a}}(\log t_0),\tilde{\mathrm{a}}(\log t_0))$, for $t_0\in(\varepsilon^{\frac{3\theta}{4}},\varepsilon^{-\frac{3\theta}{4}})$, and thus we reduce to the previous one. By symmetry, the same holds for $\tilde{g}$, and the claim is proved.

Finally, it remains to prove that $\|f-\tilde{f}\|_1+\|g-\tilde{g}\|_1$ is small. By the layer-cake representation, choosing $\theta=\alpha_\tau/100$ we have

$$\begin{aligned}\|f-\tilde{f}\|_1&=\int_0^\infty\mathcal{H}^1(\{f>t\}\Delta\{\tilde{f}>t\})\,dt=\int_{\mathbb{R}}|a(T)-\tilde{\mathrm{a}}(T)|e^T\,dT\\&\le\int_0^{\varepsilon^{\frac{3\theta}{4}}}\left(\mathcal{H}^1(\{f>t\})+\mathcal{H}^1(\{\tilde{f}>t\})\right)dt+\varepsilon^{-\frac{3\theta}{4}}\int_{I_M}|a(T)-\tilde{\mathrm{a}}(T)|\,dT\\&\lesssim\frac{\varepsilon^{\frac{3\theta}{4}}|\log\varepsilon|^{\frac{4}{\tau}}}{\tau^4}+|\log\varepsilon|^{1+\frac{4}{\tau}}\varepsilon^{\frac{\alpha_\tau}{64}-\frac{3\theta}{4}}\tau^{-\frac{3\omega_1}{2}}\lesssim\varepsilon^{\frac{\alpha_\tau}{128}}|\log\varepsilon|^{1+\frac{4}{\tau}}\tau^{-\frac{3\omega_1}{2}},\end{aligned}$$

where we used $\|f\|_\infty, \|g\|_\infty \leq 2$ and Lemma 2.5. Naturally, all such considerations hold in the exact same manner for $g, \tilde{g}$.

We now notice that, if $\varepsilon > 0$ satisfies the smallness condition as in the statement of the result, then we may bound

$$|\log \varepsilon|^{1+\frac{4}{\tau}} \varepsilon^{\frac{\alpha\tau}{128}} \leq \varepsilon^{\frac{\alpha\tau}{256}}.$$

By Proposition 2.6, this is enough to conclude the case of symmetrically decreasing functions. As we do not need an explicit estimate on the distance between h and a log-concave function, we omit the final bound one could obtain using that proposition, limiting ourselves thus to the statement of Theorem 3.2. ■

4. The general case

We now turn to the general case, assuming the results in the previous subsection. We shall prove the following result:

Theorem 4.1. *There is an explicitly computable constant $c_0 > 0$ such that the following holds. For $\tau \in (0, \frac{1}{2}]$ and $\lambda \in [\tau, 1-\tau]$, if $f, g, h: \mathbb{R} \to \mathbb{R}_{\geq 0}$ are measurable functions for which (1.2) and (1.5) hold, with $0 < \varepsilon < c_0 e^{-M(\tau)}$, then there exist a log-concave function $\tilde{h}$ and $w \in \mathbb{R}$ such that*

$$\int_{\mathbb{R}} |h - \tilde{h}| + \int_{\mathbb{R}} |a^\lambda f - \tilde{h}(\cdot + \lambda w)| + \int_{\mathbb{R}} |a^{\lambda-1} g - \tilde{h}(\cdot + (\lambda - 1)w)| < c_0 \frac{\varepsilon^{Q(\tau)}}{\tau^\omega} \int_{\mathbb{R}} h,$$

where $\omega = \frac{5}{2} + \frac{\omega_0}{8}$, with ω_0 being the exponent of τ in Lemma 3.3, $M(\tau) = 10^{40}(\omega_0 + 4)\frac{|\log(\tau)|^4}{\tau^4}$, and $Q(\tau) = \frac{\tau^4}{2^{100}|\log \tau|^4}$.

As pointed out in the introduction, in order to prove such a result we shall break the proof into several steps.

Step 1: Finding better behaving functions $\bar{f}, \bar{g}, \bar{h}$ (cf. (4.3)) that satisfy (1.2) and (1.5) with a possibly smaller power of ε. Once more, we assume that the reductions made in Sections 2 and 3 hold. That is, we have $\|f\|_1 = \|g\|_1 = 1$, $\min\{\|f\|_\infty, \|g\|_\infty\} = \|f\|_\infty = 1$. Lemma 2.4 then yields that

$$\|g\|_\infty \in (1, 1 + 4\tau^{-\frac{3}{2}}\varepsilon^{\frac{1}{2}}).$$

Also, as $\|f\|_1 = \|g\|_1 = 1$, using notation from Lemma 2.5,

$$\varepsilon > \int_0^\infty \big(\mathcal{H}^1(C_t) - (1-\lambda)\mathcal{H}^1(A_t) - \lambda \mathcal{H}^1(B_t)\big)\, dt \geq 0.$$

Thus Lemma 3.1 implies

$$\tau^{-\frac{3}{2}}\varepsilon^{\frac{1}{2}} \gtrsim \int_0^\infty \big|\mathcal{H}^1(C_t) - (1-\lambda)\mathcal{H}^1(A_t) - \lambda \mathcal{H}^1(B_t)\big|\, dt.$$

Let F be the set constructed in Lemma 3.1 (ii). Moreover, if $t < 1 - c\tau^{-\frac{3}{2}}\varepsilon^{\frac{1}{2}}$, then we know that $C_t \supset (1-\lambda)A_t + \lambda B_t$. Thus, Lemma 3.1 and the Brunn–Minkowski inequality yield

$$0 \leq \mathcal{H}^1(C_t) - (1-\lambda)\mathcal{H}^1(A_t) - \lambda\mathcal{H}^1(B_t) \lesssim \tau^{-\frac{3}{2}}\varepsilon^{\frac{1}{4}} \quad \forall t \in F \cap (0, 1 - c\tau^{-\frac{3}{2}}\varepsilon^{\frac{1}{2}}). \tag{4.1}$$

We need one more preliminary result in order to move on with our construction.

Lemma 4.2. *Let $f, g, h \colon \mathbb{R} \to \mathbb{R}_{\geq 0}$ satisfy (1.2) and (1.5) for $0 < \varepsilon < 2^{-6}\tau^3$, $\|f\|_1 = \|g\|_1 = 1$, $\min\{\|f\|_\infty, \|g\|_\infty\} = 1$, and let $A_t = \{f \geq t\}$, $B_t = \{g \geq t\}$, $C_t = \{h \geq t\}$ be their level sets. Then there exists a measurable set $F' \subset \mathbb{R}_+$ such that*

1. *$\mathcal{H}^1(\mathbb{R}_+ \setminus F') \lesssim \varepsilon^\delta$, whenever $\delta < \alpha_\tau/2048$;*
2. *$|\mathcal{H}^1(C_t) - (1-\lambda)\mathcal{H}^1(A_t) - \lambda\mathcal{H}^1(B_t)| \lesssim \tau^{-\frac{3}{2}}\varepsilon^{\frac{1}{4}}$ for all $t \in F'$;*
3. *$\min\{\mathcal{H}^1(A_t), \mathcal{H}^1(B_t)\} \geq \varepsilon^\delta$ for all $t \in (0, 1 + c\tau^{-\frac{3}{2}}\varepsilon^{\frac{1}{2}}) \cap F'$, $\delta \leq \alpha_\tau/2048$,*

where, as before, we let $\alpha_\tau = \frac{\tau}{16|\log\tau|}$.

Proof. By the considerations in Section 3, we know that there are log-concave functions $\tilde{f}^*, \tilde{g}^*$ such that

$$\|f^* - \tilde{f}^*\|_1 + \|g^* - \tilde{g}^*\|_1 \lesssim \tau^{-\frac{3\omega_1}{2}}\varepsilon^{\frac{\alpha_\tau}{256}},$$

where f^*, g^* denote the symmetric decreasing rearrangements of f, g, respectively. By the reductions in the proof of Proposition 2.6, we may suppose that (2.16) holds for the functions $\tilde{f}^*, \tilde{g}^*$. In particular, applying it in conjunction with Lemma 2.2 to these functions, we conclude that

$$\mathcal{H}^1\big(\{t > 0 \colon \mathcal{H}^1(\{\tilde{f}^* > t\}) \leq \varepsilon^\delta\}\big) \lesssim \varepsilon^\delta,$$

for all $\delta > 0$. By writing

$$\|f^* - \tilde{f}^*\|_1 = \int_0^\infty \mathcal{H}^1\big(\{f^* > t\}\Delta\{\tilde{f}^* > t\}\big)\,dt \lesssim \tau^{-\frac{3\omega_1}{2}}\varepsilon^{\frac{\alpha_\tau}{256}}$$

and using the argument with Chebyshev's inequality that we have employed extensively throughout this manuscript, we obtain

$$\mathcal{H}^1\big(\{t > 0 \colon \mathcal{H}^1(\{f^* > t\}) \leq \varepsilon^\delta\}\big) \lesssim \varepsilon^\delta$$

for all $\delta \in (0, \frac{\alpha_\tau}{1024})$, and $\varepsilon > 0$ sufficiently small (independently of $\tau > 0$). Thus, by equimeasurability of the rearrangement,

$$\mathcal{H}^1\big(\{t > 0 \colon \mathcal{H}^1(\{f > t\}) \leq \varepsilon^\delta\}\big) \lesssim \varepsilon^\delta$$

for all $\delta < \alpha_\tau/1024$. In particular, we see that

$$\mathcal{H}^1(A_t) > \varepsilon^{\frac{\alpha_\tau}{2048}},$$

whenever $t \in F' \subseteq F \cap (0, 1 - c\tau^{-\frac{3}{2}}\varepsilon^{\frac{1}{2}})$, where $\mathcal{H}^1(F \setminus F') \leq \varepsilon^{\frac{\alpha_\tau}{2048}}$. The same holds for g, and thus we may still denote by F' the set where the above properties hold for both f and g. By the considerations above, the set F' thus defined satisfies the assertions in Lemma 4.2, and we are done. ■

We now wish to employ Freĭman's theorem in order to conclude that the convex hull of the level sets A_t, B_t are not too far off from A_t, B_t themselves. To that extent, notice that, for $\varepsilon \leq \tau^4 \ll 1$,

$$\min\{\mathcal{H}^1(A_t), \mathcal{H}^1(B_t)\} > \varepsilon^{\frac{\alpha_\tau}{2048}} \gg \tau^{-\frac{3}{2}}\varepsilon^{\frac{1}{4}} \quad \forall t \in F',$$

Thus, thanks to (4.1), we can apply Freĭman's theorem. This yields that

$$\mathcal{H}^1(\mathrm{co}(A_t) \setminus A_t) + \mathcal{H}^1(\mathrm{co}(B_t) \setminus B_t) \lesssim \tau^{-\frac{3}{2}}\varepsilon^{\frac{1}{4}} \tag{4.2}$$

for all $t \in F'$. Notice also that, since the sets $\{A_t\}_{t>0}$ are nested, the same property holds for their convex hulls $\{\mathrm{co}(A_t)\}_{t>0}$.

With this in mind, we set

$$\mathrm{co}(A_t) = (a_f^1(t), b_f^1(t)), \quad \mathrm{co}(B_t) = (a_g^1(t), b_g^1(t)).$$

The main idea is to slightly change the functions $a_f^1, a_g^1, b_f^1, b_g^1$, in order to construct two functions $\bar{f}$, $\bar{g}$ close to f, g respectively, and whose level sets are intervals coinciding with $\mathrm{co}(A_t)$, $\mathrm{co}(B_t)$ for the vast majority of levels $t > \varepsilon^\theta$, where $\theta > 0$ will be a small constant to be chosen later.

By redefining on a set of zero measure, we may assume that the functions a_f^1, a_g^1, b_f^1, b_g^1 are all right continuous. Then we define

$$b_f(t) = \sup_{t'>t, t' \in F'} b_f^1(t'), \qquad b_g(t) = \sup_{t'>t, t' \in F'} b_g^1(t'),$$
$$a_f(t) = \inf_{t'>t, t' \in F'} a_f^1(t'), \qquad a_g(t) = \inf_{t'>t, t' \in F'} a_g^1(t').$$

The functions a_f, a_g, b_f, b_g defined in such a way are all, by definition, monotone. Moreover, modifying on a zero-measure set, we may suppose them to be right continuous as well.

Now let $\theta > 0$ be a fixed parameter, whose exact value we shall determine later. We define

$$(\bar{a}_f, \bar{b}_f) = (a_f(\varepsilon^\theta), b_f(\varepsilon^\theta)).$$

As $\mathcal{H}^1((0, 1 - c\tau^{-\frac{3}{2}}\varepsilon^{\frac{1}{2}}) \setminus F') \leq \varepsilon^{\frac{\alpha_\tau}{2048}}$, as long as we choose $\theta < \alpha_\tau/2^{12}$ we may always find a point $t_0 \in F'$ so that $\frac{1}{100}\varepsilon^\theta < t_0 < \varepsilon^\theta$. Thus, for all $t \geq \varepsilon^\theta$, (4.2) yields

$$(b_f(t) - a_f(t)) \leq (b_f(t_0) - a_f(t_0)) \leq \mathcal{H}^1(A_{t_0}) + c\tau^{-\frac{3}{2}}\varepsilon^{\frac{1}{4}} \lesssim \tau^{-4}|\log \varepsilon|^{\frac{4}{\tau}},$$

where we used Lemma 2.5 in the last inequality. We then build the function $\bar{f}$ supported in $(\bar{a}_f, \bar{b}_f)$, for $x \le a_f(1)$, as

$$\bar{f}(x) = \sup\{t : a_f(t) < x\}.$$

We further define it to be 1 in the interval $(a_f(1), b_f(1))$, and for $x \ge b_f(1)$ we let

$$\bar{f}(x) = \sup\{t : x < b_f(t)\}.$$

An entirely analogous construction yields the function $\bar{g}$. Notice now that, for $s \in (0,1)$,

$$\begin{aligned}\{x \in \mathbb{R} : \bar{f}(x) > s\} &= \big\{x \in \mathbb{R} : \exists\, t > s \text{ so that either } a_f(t) < x \text{ and } x \le a_f(1) \\ &\qquad\qquad \text{or } b_f(t) > x \ge b_f(1)\big\} \cup (a_f(1), b_f(1)) \\ &= \bigcup_{t>s} (a_f(t), b_f(t)) \\ &= \Big(\inf_{t>s} a_f(t), \sup_{t>s} b_f(t)\Big) = (a_f(s), b_f(s)). \qquad (4.3)\end{aligned}$$

Notice that we used the hypothesis of right continuity of a_f, b_f in order to obtain the last equality above. Thus, we have

$$\bar{A}_t =: \{\bar{f} > t\} = \mathrm{co}(A_t) \quad \forall t \in F'.$$

This allows us to estimate

$$\begin{aligned}\int_{\mathbb{R}} |\bar{f}(x) - f(x)|\, dx &= \int_0^\infty \mathcal{H}^1(A_t \Delta \bar{A}_t)\, dt \\ &\le \int_0^{\varepsilon^\theta} (\mathcal{H}^1(A_t) + \mathcal{H}^1(\bar{A}_{t_0}))\, dt + \int_{(\varepsilon^\theta,1)\cap F'} \mathcal{H}^1(\mathrm{co}(A_t) \setminus A_t)\, dt \\ &\quad + \int_{(\varepsilon^\theta,1)\setminus F'} (\mathcal{H}^1(A_t) + \mathcal{H}^1(\bar{A}_t))\, dt \\ &\lesssim \tau^{-4} \varepsilon^\theta |\log \varepsilon|^{\frac{4}{\tau}}, \qquad (4.4)\end{aligned}$$

where we used (4.2), $\theta < \alpha_\tau / 2^{12}$, and once more Lemma 2.5. The same conclusion holds in an entirely analogous way for $\|g - \bar{g}\|_1$.

We now build a function $\bar{h}$ so that (1.2) and (1.5) are satisfied. In fact, we take the most natural choice

$$\bar{h}(z) = \sup_{(1-\lambda)x + \lambda y = z} \bar{f}(x)^{1-\lambda} \bar{g}(y)^\lambda.$$

The level sets $\bar{C}_t = \{x \in \mathbb{R} : \bar{h}(x) > t\}$ satisfy, by definition,

$$\bar{C}_t = \bigcup^{*}_{r^{1-\lambda}s^\lambda = t} ((1-\lambda)\bar{A}_r + \lambda \bar{B}_s).$$

As the level sets of $\bar{f}$, $\bar{g}$ are intervals, the function $\bar{h}$ is measurable. It remains to verify that we have a control of the form

$$\int_{\mathbb{R}} \bar{h} \le 1 + c(\tau)\varepsilon^{\gamma}$$

for some $\gamma > 0$ and some function $c(\tau) > 0$. The strategy here is similar to the proof of Proposition 2.6.

First, we may choose $\theta = \alpha_\tau/2^{13}$ in (4.4), so that we obtain

$$\|f - \bar{f}\|_1 = \int_0^\infty \mathcal{H}^1\big(\{f > t\}\Delta\{\bar{f} > t\}\big)\, dt \lesssim \tau^{-4}\varepsilon^{\frac{\alpha_\tau}{2^{13}}}|\log\varepsilon|^{\frac{4}{\tau}} \tag{4.5}$$

(with the same estimate holding for $g, \bar{g}$) and then use Chebyshev's inequality in order to conclude that

$$\mathcal{H}^1\big(\{t > 0 \colon \mathcal{H}^1(\{\bar{f} > t\}) \le \varepsilon^\delta\}\big) \lesssim \varepsilon^\delta \tag{4.6}$$

for all $\delta < \alpha_\tau/2^{15}$. Then we fix $\gamma_0 < \alpha_\tau/2^{15}$ and define $\bar{S} \subset (0, +\infty)$ to be the largest measurable subset of $(0, +\infty)$ satisfying

(1) $\min\{\mathcal{H}^1(\{\bar{f} > t\}), \mathcal{H}^1(\{\bar{g} > t\})\} > \varepsilon^{\gamma_0}$ for all $t \in \bar{S} \cap (0, 1 + c\tau^{-4}\varepsilon^{\frac{1}{2}})$;

(2) $\mathcal{H}^1(\{f > t\}\Delta\{\bar{f} > t\}) + \mathcal{H}^1(\{g > t\}\Delta\{\bar{g} > t\}) \lesssim \varepsilon^{\frac{\alpha_\tau}{2^{15}}}$ for all $t \in \bar{S}$.

By (4.5) and (4.6), we have $\mathcal{H}^1(\mathbb{R}_+ \setminus \bar{S}) \lesssim \tau^{-4}\varepsilon^{\gamma_0}$. Thus, for some absolute constant $c > 0$, there is an element $r_0 \in (1 - c\tau^{-4}\varepsilon^{\gamma_0}, 1 + c\tau^{-4}\varepsilon^{\gamma_0}) \cap \bar{S}$. Fix this element until the end of the proof.

Note that transformations of the form

$$(f, g, h) \mapsto (f(\cdot - x_0), g(\cdot + x_0), h),$$
$$(f, g, h) \mapsto (f(\cdot - x_0), g(\cdot - x_0), h(\cdot - x_0))$$

preserve (1.2) and (1.5) with the same constant. Also, they leave the set $\bar{S}$ defined above unaltered. Hence, with no loss of generality, we may suppose that the barycenters of $\{\bar{f} > r_0\}$ and $\{\bar{g} > r_0\}$ both coincide with the origin. Assume this additional fact until the end of the proof as well.

Now we employ the same strategy as in the final part of the proof of Proposition 2.6. Fix $t > \varepsilon^{\frac{\tau\gamma_0}{2}}$. It is not hard to see that the set $\{\bar{h} > t\}$ splits as

$$\bar{C}_t = \bigcup^{*}_{\substack{r^{1-\lambda}s^\lambda = t \\ r,s\in\bar{S} \\ r_0 > r,s > \varepsilon^{\gamma_0}}} ((1-\lambda)\bar{A}_r + \lambda\bar{B}_s) \cup \bigcup^{*}_{\substack{r^{1-\lambda}s^\lambda = t \\ r,s\in\bar{S} \\ \text{either } r > r_0 \text{ or } s > r_0}} ((1-\lambda)\bar{A}_r + \lambda\bar{B}_s)$$
$$\cup \bigcup^{*}_{\substack{r^{1-\lambda}s^\lambda = t \\ \text{either } r \notin \bar{S} \text{ or } s \notin \bar{S}}} ((1-\lambda)\bar{A}_r + \lambda\bar{B}_s) =: \bar{C}_t^1 \cup \bar{C}_t^2 \cup \bar{C}_t^3.$$

Case 1*: Analysis of* $\bar{C}_t^1$. By Young's convolution inequality and the definition of $\bar{S}$, we have

$$\begin{aligned}
&\|\chi_{(1-\lambda)A_r} * \chi_{\lambda B_s} - \chi_{(1-\lambda)\bar{A}_r} * \chi_{\lambda \bar{B}_s}\|_\infty \\
&\quad \le \|\chi_{(1-\lambda)A_r} - \chi_{(1-\lambda)\bar{A}_r}\|_1 + \|\chi_{\lambda B_s} - \chi_{\lambda \bar{B}_s}\|_1 \\
&\quad \lesssim \varepsilon^{\frac{\alpha_\tau}{2^{15}}} \quad \forall r, s \in \bar{S}.
\end{aligned} \tag{4.7}$$

On the other hand, by the definition of $\bar{S}$ and the fact that we are analyzing $\bar{C}_t^1$, we have

$$\min\big\{(1-\lambda)\mathcal{H}^1(\bar{A}_r), \lambda \mathcal{H}^1(\bar{B}_s)\big\} \ge \tau \varepsilon^{\gamma_0}.$$

We thus have the convolution estimate

$$\chi_{(1-\lambda)\bar{A}_r} * \chi_{\lambda \bar{B}_s}(x) > 3\varepsilon^{2\gamma_0} \tag{4.8}$$

whenever

$$x \in \big((1-\lambda)a_f(r) + \lambda a_g(s) + 3\varepsilon^{2\gamma_0}, (1-\lambda)b_f(r) + \lambda b_g(s) - 3\varepsilon^{2\gamma_0}\big).$$

Since $(1-\lambda)a_f(r) + \lambda a_g(s) \le -\varepsilon^{\gamma_0}$, $(1-\lambda)b_f(r) + \lambda b_g(s) \ge \varepsilon^{\gamma_0}$, and $r, s \in (\varepsilon^{\gamma_0}, r_0)$, due to the fact that the barycenters of $\bar{A}_{r_0}$ and $\bar{B}_{r_0}$ coincide with the origin, we have that the set

$$\big((1-\lambda)a_f(r) + \lambda a_g(s) + 3\varepsilon^{2\gamma_0}, (1-\lambda)b_f(r) + \lambda b_g(s) - 3\varepsilon^{2\gamma_0}\big)$$

contains $(1-\varepsilon^{\frac{\gamma_0}{4}})((1-\lambda)\bar{A}_r + \lambda \bar{B}_s)$ whenever $\gamma_0 < \alpha_\tau/2^{15}$.

On the other hand, (4.7) and (4.8) imply that

$$x \in \operatorname{supp}(\chi_{(1-\lambda)A_r} * \chi_{\lambda B_s}) = (1-\lambda)A_r + \lambda B_s.$$

Thus,

$$(1-\lambda)\bar{A}_r + \lambda \bar{B}_s \subset \frac{1}{1-\varepsilon^{\frac{\gamma_0}{4}}}((1-\lambda)A_r + \lambda B_s) \subset \frac{1}{1-\varepsilon^{\frac{\gamma_0}{4}}}\{h > t\},$$

hence

$$\bar{C}_t^1 \subset \frac{1}{1-\varepsilon^{\frac{\gamma_0}{4}}} C_t.$$

Case 2*: Analysis of* $\bar{C}_t^2 \cup \bar{C}_t^3$. Recall that, by assumption, $t > \varepsilon^{\frac{\tau\gamma_0}{2}}$. Hence, since $\|\bar{f}\|_\infty$, $\|\bar{g}\|_\infty \le 2$, we readily obtain

$$r, s \gtrsim \varepsilon^{\frac{\gamma_0}{2}}.$$

Since $\mathcal{H}^1(\mathbb{R}_+ \setminus \bar{S}) \le \varepsilon^{\gamma_0}$, there exist $r', s' \in \bar{S}$, with $r', s' \in (\varepsilon^{\gamma_0}, r_0)$, such that $|r - r'| + |s - s'| \le \varepsilon^{\gamma_0}$ and $r > r'$, $s > s'$. Therefore,

$$\begin{aligned}
(1-\lambda)\bar{A}_r + \lambda \bar{B}_s \subset (1-\lambda)\bar{A}_{r'} + \lambda \bar{B}_{s'} &\subset \frac{1}{1-\varepsilon^{\frac{\gamma_0}{4}}}\big\{h > (r')^{1-\lambda}(s')^{\lambda}\big\} \\
&\subset \frac{1}{1-\varepsilon^{\frac{\gamma_0}{4}}}\big\{h > t - \varepsilon^{\tau\gamma_0}\big\},
\end{aligned}$$

which implies

$$\bar{C}_t \subseteq \frac{1}{1-\varepsilon^{\frac{\gamma_0}{4}}}\{h > t - \varepsilon^{\tau\gamma_0}\} \quad \forall t > \varepsilon^{\frac{\tau\gamma_0}{2}}.$$

Moreover, since $\mathrm{supp}(\bar{h}) \subset (1-\lambda)\,\mathrm{supp}(\bar{f}) + \lambda\,\mathrm{supp}(\bar{g})$ and all sets involved are intervals, $\mathcal{H}^1(\mathrm{supp}(\bar{h})) \lesssim \tau^{-4}|\log\varepsilon|^{\frac{4}{\tau}}$. Thus,

$$\begin{aligned}\int_{\mathbb{R}} \bar{h} &= \int_0^\infty \mathcal{H}^1(\{\bar{h} > t\})\,dt \\ &\le \int_0^{\frac{1}{2}\varepsilon^{\frac{\tau\gamma_0}{2}}} \mathcal{H}^1(\mathrm{supp}(\bar{h}))\,dt + \frac{1}{1-\varepsilon^{\frac{\gamma_0}{4}}}\int_{\frac{1}{2}\varepsilon^{\frac{\tau\gamma_0}{2}}}^\infty \mathcal{H}^1(\{h>t\})\,dt \\ &\le 1 + \frac{c}{\tau^4}\varepsilon^{\frac{\tau\gamma_0}{2}}|\log\varepsilon|^{\frac{4}{\tau}},\end{aligned}$$

for some absolute constant $c > 0$. This concludes Step 1, as long as we take $\gamma \in (0, \frac{\tau\gamma_0}{2})$ and $c(\tau) = \tau^{-4}$.

Step 2: The functions a_f, a_g, b_f, b_g are suitably close to satisfying four-point inequalities. We now use similar methods to those employed in Section 3 in order to conclude that the functions we constructed are close to being concave.

Indeed, for notational simplicity, we reset our construction from the beginning, additionally assuming the reductions and conclusions of Step 1 to hold. In other words, we assume that f, g, h satisfy (1.2) and (1.5), and moreover the level sets of f, g are intervals. We further assume that $\|f\|_\infty = 1$, $\int_{\mathbb{R}} f = \int_{\mathbb{R}} g = 1$, as in Section 2.

Now Lemma 3.1 yields that there is a set $F \subset (0, +\infty)$ such that $\mathcal{H}^1(\mathbb{R}_+ \setminus F) \lesssim \varepsilon^{\frac{1}{4}}$, and moreover

$$\left|\mathcal{H}^1(C_t) - (1-\lambda)\mathcal{H}^1(A_t) - \lambda\mathcal{H}^1(B_t)\right| \lesssim \tau^{-\frac{3}{2}}\varepsilon^{\frac{1}{4}} \quad \forall t \in F.$$

We may now invoke the set F' constructed in Lemma 4.2. With this in hand, we define the set $\mathcal{F}'_M := \log(F') \cap [-M, M]$, $M = \theta\log(1/\varepsilon)$ ($\theta < \delta/2$ to be chosen later). From this definition and a change of variables we see that $\mathcal{H}^1([-M, M] \setminus \mathcal{F}'_M) \lesssim \varepsilon^{\frac{\delta}{2}}$, and $\mathcal{F}'_M$ is such that the sets

$$\begin{aligned}\mathcal{A}_R &= A_{e^R} = (\mathbf{a}_f(R), \mathbf{b}_f(R)), \\ \mathcal{B}_S &= B_{e^S} = (\mathbf{a}_g(S), \mathbf{b}_g(S)), \\ \mathcal{C}_T &= C_{e^T} = (\mathbf{a}_h(T), \mathbf{b}_h(T))\end{aligned}$$

satisfy

$$\left|\mathcal{H}^1(\mathcal{C}_T) - (1-\lambda)\mathcal{H}^1(\mathcal{A}_T) - \lambda\mathcal{H}^1(\mathcal{B}_T)\right| \lesssim \tau^{-\frac{3}{2}}\varepsilon^{\frac{1}{4}} \quad \forall T \in \mathcal{F}'_M \tag{4.9}$$

and

$$\min\{\mathcal{H}^1(\mathcal{A}_T), \mathcal{H}^1(\mathcal{B}_T)\} \ge \varepsilon^\delta \quad \forall T \in (-\infty, \log(1 + c\tau^{-\frac{3}{2}}\varepsilon^{\frac{1}{2}})) \cap \mathcal{F}'_M. \tag{4.10}$$

We claim that, for $R, S, T \in \mathcal{F}'_M$ such that $\mathcal{A}_R, \mathcal{B}_S \neq \emptyset$, $(1-\lambda)R + \lambda S = T$,

$$\begin{aligned}&(1-\lambda)\mathcal{A}_R + \lambda\mathcal{B}_S\\ &\subset \left((1-\lambda)\mathbf{a}_f(T) + \lambda\mathbf{a}_g(T) - \frac{1}{1000}\varepsilon^\delta, (1-\lambda)\mathbf{a}_f(T) + \lambda\mathbf{a}_g(T) + \frac{1}{1000}\varepsilon^\delta\right).\end{aligned} \tag{4.11}$$

Indeed, if this is not the case, then, by (4.10) and the Brunn–Minkowski inequality,

$$\mathcal{H}^1((1-\lambda)\mathcal{A}_R + \lambda\mathcal{B}_S) \geq \varepsilon^\delta,$$

and thus, as all sets involved are intervals,

$$\mathcal{H}^1\big(((1-\lambda)\mathcal{A}_R + \lambda\mathcal{B}_S) \setminus ((1-\lambda)\mathcal{A}_T + \lambda\mathcal{B}_T)\big) \geq \frac{1}{1000}\varepsilon^\delta.$$

This implies, on the other hand, that

$$\mathcal{H}^1\big(\mathcal{C}_T \setminus ((1-\lambda)\mathcal{A}_T + \lambda\mathcal{B}_T)\big) \geq \frac{1}{1000}\varepsilon^\delta,$$

which, together with (4.9) and the one-dimensional Brunn–Minkowski inequality, contradicts the definition of $\mathcal{F}'_M$, as long as we take $\varepsilon \ll \tau^3$. Thus, whenever $R, S, T \in \mathcal{F}'_M$, $(1-\lambda)R + \lambda S = T$, $\mathcal{A}_R, \mathcal{B}_S \neq \emptyset$, we have

$$\begin{aligned}(1-\lambda)\mathbf{a}_f(R) + \lambda\mathbf{a}_g(S) &\geq (1-\lambda)\mathbf{a}_f(T) + \lambda\mathbf{a}_g(T) - \frac{1}{1000}\varepsilon^\delta,\\ (1-\lambda)\mathbf{b}_f(R) + \lambda\mathbf{b}_g(S) &\leq (1-\lambda)\mathbf{b}_f(T) + \lambda\mathbf{b}_g(T) + \frac{1}{1000}\varepsilon^\delta,\end{aligned} \tag{4.12}$$

which proves (4.11).

As indicated in Section 3, we can apply [19, Remark 4.1] to translate the three-point inequalities presented in (4.12) into the following *four-point inequalities*:

$$\begin{aligned}\mathbf{a}_f(T_1) + \mathbf{a}_f(T_2) &\geq \mathbf{a}_f(T_{1,2}) + \mathbf{a}_f(T_{2,1}) - \frac{1}{\lambda}\varepsilon^\delta,\\ \mathbf{a}_g(T_1) + \mathbf{a}_g(T_2) &\geq \mathbf{a}_g(T_{1,2}) + \mathbf{a}_g(T_{2,1}) - \frac{1}{\lambda}\varepsilon^\delta,\end{aligned} \tag{4.13}$$

$$\begin{aligned}\mathbf{b}_f(T_1) + \mathbf{b}_f(T_2) &\leq \mathbf{b}_f(T_{1,2}) + \mathbf{b}_f(T_{2,1}) + \frac{1}{\lambda}\varepsilon^\delta,\\ \mathbf{b}_g(T_1) + \mathbf{b}_g(T_2) &\leq \mathbf{b}_g(T_{1,2}) + \mathbf{b}_g(T_{2,1}) + \frac{1}{\lambda}\varepsilon^\delta,\end{aligned} \tag{4.14}$$

whenever

$$T_1, T_2 \in \mathcal{F}'_M, \quad T_{1,2} = \frac{1}{2-\lambda}T_1 + \frac{1-\lambda}{2-\lambda}T_2 \in \mathcal{F}'_M, \quad T_{2,1} = \frac{1}{2-\lambda}T_2 + \frac{1-\lambda}{2-\lambda}T_1 \in \mathcal{F}'_M.$$

This concludes this step, as the functions a_f, a_g, b_f, b_g are close to $\mathbf{a}_f, \mathbf{a}_g, \mathbf{b}_f, \mathbf{b}_g$, which themselves satisfy the four-point inequalities.

Step 3: Constructing the log-concave approximations. We now apply Lemma 3.3 to the functions $\mathbf{a}_f, \mathbf{a}_g, \mathbf{b}_f, \mathbf{b}_g$.

Indeed, fixing a level $r_0 > 1 - c\varepsilon^\delta$ with $\min\{\mathcal{H}^1(\{f > r_0\}), \mathcal{H}^1(\{g > r_0\})\} \geq \varepsilon^\delta$, we may suppose that the barycenters of the intervals $\{f > r_0\}$, $\{g > r_0\}$ coincide with the origin; the existence of such a level follows once again by the definition and properties of the set $\mathcal{F}'_M$.

After this reduction, the definition of $\mathcal{F}'_M$ and Lemma 2.5 ensure that the additional hypothesis

$$|\mathbf{a}_f(T)| + |\mathbf{b}_f(T)| + |\mathbf{a}_g(T)| + |\mathbf{b}_g(T)| \lesssim \tau^{-4} |\log \varepsilon|^{\frac{4}{\tau}}$$

holds on a subset $\mathfrak{F} \subset \mathcal{F}'_M$ so that $\mathcal{H}^1(\mathcal{F}'_M \setminus \mathfrak{F}) \lesssim \varepsilon^\delta$. We thus replace $\mathcal{F}'_M$ by $\mathfrak{F}$, and henceforth still denote it by $\mathcal{F}'_M$. Notice also that, in such a set, one has $\mathbf{a}_f, \mathbf{a}_g$ nonpositive and $\mathbf{b}_f, \mathbf{b}_g$ nonnegative.

At the present point, one notices that all other prerequisites for Lemma 3.3 are satisfied; thus we may apply it to $\mathbf{b}_f, \mathbf{b}_g$, and to $-\mathbf{a}_f, -\mathbf{a}_g$ (thanks to (4.13) and (4.14)).

Applying Lemma 3.3 and arguing as in Section 3, we find functions $\mathfrak{b}_f, \mathfrak{b}_g, \mathfrak{a}_f, \mathfrak{a}_g$, defined on an interval Ω_M satisfying $\mathcal{H}^1((-M, M) \setminus \Omega_M) \lesssim \varepsilon^{\frac{\delta}{2}}$, such that

$$\begin{aligned} \int_{\mathcal{F}'_M} |\mathfrak{b}_f(T) - \mathbf{b}_f(T)|\, dT + \int_{\mathcal{F}'_M} |\mathfrak{a}_f(T) - \mathbf{a}_f(T)|\, dT &\lesssim \frac{|\log \varepsilon|^{\frac{4}{\tau}}}{\tau^{\omega_1}} \varepsilon^{\frac{\delta\alpha_\tau}{2}}, \\ \int_{\mathcal{F}'_M} |\mathfrak{b}_g(T) - \mathbf{b}_g(T)|\, dT + \int_{\mathcal{F}'_M} |\mathfrak{a}_g(T) - \mathbf{a}_g(T)|\, dT &\lesssim \frac{|\log \varepsilon|^{\frac{4}{\tau}}}{\tau^{\omega_1}} \varepsilon^{\frac{\delta\alpha_\tau}{2}}. \end{aligned} \tag{4.15}$$

Moreover, $\mathfrak{b}_f, \mathfrak{b}_g$ are *concave*, $\mathfrak{a}_f, \mathfrak{a}_g$ are *convex*, and they are all bounded in absolute value by $c\tau^{-4}|\log \varepsilon|^{\frac{4}{\tau}}$.

Again, the considerations in Section 3 applied almost verbatim to $\mathbf{b}_f, \mathbf{b}_g, -\mathbf{a}_f, -\mathbf{a}_g$ imply that, by potentially decreasing the power of ε in the left-hand side of (4.15), we may suppose that $\mathfrak{a}_f, \mathfrak{a}_g, \mathfrak{b}_f, \mathfrak{b}_g$ are all *monotone* on a smaller interval $I_M = (-3M/4, 3M/4)$, and thus, as $\mathbf{a}_f, \mathbf{a}_g, \mathbf{b}_f, \mathbf{b}_g$ are themselves bounded by $c\tau^{-4}|\log \varepsilon|^{\frac{4}{\tau}}$,

$$\begin{aligned} \int_{I_M} |\mathfrak{a}_f(T) - \mathbf{a}_f(T)|\, dT + \int_{I_M} |\mathfrak{b}_f(T) - \mathbf{b}_f(T)|\, dT &\lesssim \frac{|\log \varepsilon|^{1+\frac{4}{\tau}}}{\tau^{\frac{3\omega_1}{2}}} \varepsilon^{\frac{\delta\alpha_\tau}{16}}, \\ \int_{I_M} |\mathfrak{a}_g(T) - \mathbf{a}_g(T)|\, dT + \int_{I_M} |\mathfrak{b}_g(T) - \mathbf{b}_g(T)|\, dT &\lesssim \frac{|\log \varepsilon|^{1+\frac{4}{\tau}}}{\tau^{\frac{3\omega_1}{2}}} \varepsilon^{\frac{\delta\alpha_\tau}{16}}. \end{aligned}$$

Similarly to before, we pick the unique pair $\tilde{f}, \tilde{g}$ of functions such that

$$\begin{aligned} \left\{x \in \mathbb{R} \colon \tilde{f}(x) > t\right\} &= (\mathfrak{a}_f(\log t), \mathfrak{b}_f(\log t)), \\ \left\{x \in \mathbb{R} \colon \tilde{g}(x) > t\right\} &= (\mathfrak{a}_g(\log t), \mathfrak{b}_g(\log t)), \end{aligned}$$

whenever $\log t \in I_M$ (that is, $t \in (\varepsilon^{\frac{3\theta}{4}}, \varepsilon^{-\frac{3\theta}{4}})$),

$$\operatorname{supp}(\tilde{f}) = \bigcup_{t \in (\varepsilon^{\frac{3\theta}{4}}, \varepsilon^{-\frac{3\theta}{4}})} (\mathfrak{a}_f(\log t), \mathfrak{b}_f(\log t)),$$

$$\operatorname{supp}(\tilde{g}) = \bigcup_{t \in (\varepsilon^{\frac{3\theta}{4}}, \varepsilon^{-\frac{3\theta}{4}})} (\mathfrak{a}_g(\log t), \mathfrak{b}_g(\log t)),$$

and $\{x \in \mathbb{R} \colon \tilde{f}(x) > t\} = \{x \in \mathbb{R} \colon \tilde{g}(x) > s\} = \emptyset$ for $t, s > \varepsilon^{-\frac{3\theta}{4}}$ or whenever $\mathfrak{a}_f(\log t) = \mathfrak{b}_f(\log t) = 0 = \mathfrak{a}_g(\log s) = \mathfrak{b}_g(\log s)$.

It follows from the convexity of $\mathfrak{a}_f$, $\mathfrak{a}_g$, the concavity of $\mathfrak{b}_f$, $\mathfrak{b}_g$, and the argument in Section 3 that these functions are log-concave.

Step 4: Conclusion. We can finally conclude the proof. Assume, as in previous sections, that $\|f\|_1 = \|g\|_1 = 1$ and $\min\{\|f\|_\infty, \|g\|_\infty\} = \|f\|_\infty = 1$. Moreover, we assume that Steps 1, 2, 3 hold. Thus, using the functions $\tilde{f}$, $\tilde{g}$ and the way we built them, we are led to estimate

$$\begin{aligned}
\|f - \tilde{f}\|_1 &= \int_0^\infty \mathcal{H}^1(\{f > t\} \Delta \{\tilde{f} > t\})\, dt \\
&\le \int_{I_M} |\mathbf{a}_f(T) - \mathfrak{a}_f(T)| e^T\, dT + \int_{I_M} |\mathbf{b}_f(T) - \mathfrak{b}_f(T)| e^T\, dT \\
&\quad + \int_0^{\varepsilon^\theta} \mathcal{H}^1(\{f > t\})\, dt \\
&\le \varepsilon^{-\frac{3\theta}{4}} \left(\int_{I_M} |\mathbf{a}_f(T) - \mathfrak{a}_f(T)|\, dT + \int_{I_M} |\mathbf{b}_f(T) - \mathfrak{b}_f(T)|\, dT \right) \\
&\quad + \frac{c}{\tau^4} \varepsilon^\theta |\log \varepsilon|^{\frac{4}{\tau}} \\
&\lesssim |\log \varepsilon|^{1 + \frac{4}{\tau}} \tau^{-\frac{3\omega_1}{2}} \varepsilon^{\frac{\delta \alpha_\tau}{32}} \lesssim \tau^{-\frac{3\omega_1}{2}} \varepsilon^{\frac{\delta \alpha_\tau}{64}},
\end{aligned}$$

by choosing $\theta = \frac{4}{3} \frac{\delta \alpha_\tau}{32}$ and using $\varepsilon \ll e^{-10^{10} \frac{|\log \tau|^4}{\tau^4}}$. Note that, in this computation, we assumed that f and g fulfill the requirements in Steps 1–3. In doing so, we lose powers of ε along the way. More precisely, combining estimates from Section 3 and Steps 1–3, we have the following requirements:

(1) We must not incorporate any further power from Section 3, as it has only been used in the reduction to the case of functions whose level sets are intervals.

(2) In Steps 1–3, we must substitute $\varepsilon \mapsto \frac{c}{\tau^4} \varepsilon^{\frac{\tau \alpha_\tau}{2048}}$, by the reduction made in Step 1.

Thus, we conclude that if the functions f, g, h satisfy (1.2) and (1.5), then there are log-concave functions $\tilde{f}$, $\tilde{g}$ such that

$$\|f - \tilde{f}\|_1 + \|g - \tilde{g}\|_1 \le c \tau^{-\frac{3\omega_1}{2}} \varepsilon^{\frac{\tau \alpha_\tau^3}{2^{30}}} =: c \tau^{-\frac{3\omega_1}{2}} \varepsilon^{Q_0(\tau)}.$$

We are now in a position to use Proposition 2.6. We choose $\eta = c\tau^{\frac{-3\omega_1}{2}}\varepsilon^{Q_0(\tau)}$. The condition $\eta < c'\tau^3$ for some $c' \in (0,1)$ becomes

$$\varepsilon \leq ce^{-M(\tau)}, \tag{4.16}$$

where we define $M(\tau) = 10^{40}\omega_1\frac{|\log(\tau)|^4}{\tau^4}$, and $c > 0$ is an absolute constant. Under that condition, notice that all the smallness conditions in the proof above are also fulfilled.

Hence, thanks to Proposition 2.6 and the smallness condition (4.16), there exists a log-concave function $\tilde{h}$ such that, for f, g, h satisfying (1.2) and (1.5), if we let $a = \|g\|_1/\|f\|_1$, then there is $w \in \mathbb{R}$ for which

$$\begin{aligned}
\int_{\mathbb{R}} |a^\lambda f(x) - \tilde{h}(x - \lambda w)|\,dx &\lesssim \tau^{-\omega_2}\varepsilon^{\frac{Q_0(\tau)}{32}}\int_{\mathbb{R}} h,\\
\int_{\mathbb{R}} |a^{\lambda-1} g(x) - \tilde{h}(x + (1-\lambda)w)| &\lesssim \tau^{-\omega_2}\varepsilon^{\frac{Q_0(\tau)}{32}}\int_{\mathbb{R}} h,\\
\int_{\mathbb{R}} |h(x) - \tilde{h}(x)|\,dx &\lesssim \tau^{-\omega_2}\varepsilon^{\frac{Q_0(\tau)}{8}}\int_{\mathbb{R}} h.
\end{aligned}$$

Here, we have let $\omega_2 = \frac{\omega_1}{8} + 2$. Thus, noting the choices of $Q(\tau)$, $M(\tau)$ in the statement of Theorem 4.1, we notice that this finishes the proof of that result, and thus also the proof of Theorem 1.6 in dimension $n = 1$.

5. The high-dimensional case

With the one-dimensional case already resolved in the previous section, we now employ a recent strategy by the first author and De [9] in order to reduce the higher-dimensional version to the one-dimensional one, with the aid of the stability version of the Brunn–Minkowski inequality proved by the second author and Jerison [19]. Indeed, we note that the main result in one dimension implies the following result:

Corollary 5.1. *Let $F, G, H : \mathbb{R}_+ \to \mathbb{R}_+$ be measurable functions such that*

$$H(r^{1-\lambda}s^\lambda) \geq F(r)^{1-\lambda}G(s)^\lambda \quad \forall r, s \geq 0, \tag{5.1}$$

where $\lambda \in [\tau, 1-\tau]$ for some $\tau \in (0, 1/2]$. Suppose that

$$\int_{\mathbb{R}_+} H \leq (1+\varepsilon)\left(\int_{\mathbb{R}_+} F\right)^{1-\lambda}\left(\int_{\mathbb{R}_+} G\right)^\lambda \tag{5.2}$$

holds for $0 < \varepsilon < e^{-M(\tau)}$. Then there are constant $a, b > 0$, with $a/b = \|F\|_1/\|G\|_1$, such that

$$\int_{\mathbb{R}_+} |a^{-\lambda}F(b^{-\lambda}t) - H(t)|\,dt + \int_{\mathbb{R}_+} |a^{(1-\lambda)}G(b^{(1-\lambda)}t) - H(t)|\,dt \lesssim \tau^{-\omega}\varepsilon^{Q(\tau)}\int_{\mathbb{R}_+} H.$$

Here, ω and $Q(\tau)$ are the same as in Theorem 4.1.

Proof. We change variables and define $f(x) = F(e^x)e^x$, $g(x) = G(e^x)e^x$, $h(x) = H(e^x)e^x$. These functions satisfy (1.2), and, as

$$\int_{\mathbb{R}} f = \int_{\mathbb{R}_+} F, \quad \int_{\mathbb{R}} g = \int_{\mathbb{R}_+} G, \quad \int_{\mathbb{R}} h = \int_{\mathbb{R}_+} H,$$

they also satisfy (1.5). By the result in Section 4, there is a constant $\eta \in \mathbb{R}$ such that

$$\int_{\mathbb{R}} \left| f(x) - (\|f\|_1/\|g\|_1)^{\lambda} h(x + \lambda\eta) \right| dx \lesssim \tau^{-\omega} \varepsilon^{Q(\tau)} \|f\|_1,$$
$$\int_{\mathbb{R}} \left| g(x) - (\|g\|_1 \|f\|_1)^{1-\lambda} h(x + (\lambda - 1)\eta) \right| dx \lesssim \tau^{-\omega} \varepsilon^{Q(\tau)} \|g\|_1,$$

for $Q(\tau)$ as in the statement of Theorem 4.1. Changing variables back, we obtain

$$\int_{\mathbb{R}} \left| F(t) - e^{\lambda\eta} (\|F\|_1/\|G\|_1)^{\lambda} H(te^{\lambda\eta}) \right| dt \lesssim \tau^{-\omega} \varepsilon^{Q(\tau)} \|F\|_1,$$
$$\int_{\mathbb{R}} \left| G(t) - e^{(\lambda-1)\eta} (\|G\|_1 \|F\|_1)^{1-\lambda} H(te^{(\lambda-1)\eta}) \right| dt \lesssim \tau^{-\omega} \varepsilon^{Q(\tau)} \|G\|_1,$$

which implies that

$$\int_{\mathbb{R}} \left| e^{-\lambda\eta} (\|G\|_1/\|F\|_1)^{\lambda} F(e^{-\lambda\eta} s) - H(s) \right| dt \lesssim \tau^{-\omega} \varepsilon^{Q(\tau)} \|F\|_1^{1-\lambda} \|G\|_1^{\lambda},$$
$$\int_{\mathbb{R}} \left| e^{(1-\lambda)\eta} (\|F\|_1/\|G\|_1)^{1-\lambda} G(e^{(1-\lambda)\eta} s) - H(s) \right| dt \lesssim \tau^{-\omega} \varepsilon^{Q(\tau)} \|F\|_1^{1-\lambda} \|G\|_1^{\lambda}.$$

Taking $a = \frac{e^{\eta}\|F\|_1}{\|G\|_1}$, $b = e^{\eta}$ and using the Prékopa–Leindler inequality on the right-hand side of the last expression implies the result. ∎

Let $f, g, h: \mathbb{R}^n \to \mathbb{R}_+$ satisfy the n-dimensional version of (1.2). We use Corollary 5.1 for the triple F, G, H defined by

$$\mathcal{H}^n\big(\{x \in \mathbb{R}^n : f(x) > t\}\big) = F(t),$$
$$\mathcal{H}^n\big(\{x \in \mathbb{R}^n : g(x) > t\}\big) = G(t),$$
$$\mathcal{H}^n\big(\{x \in \mathbb{R}^n : h(x) > t\}\big) = H(t).$$

By (1.2) and the n-dimensional Brunn–Minkowski inequality, we have

$$H(r^{1-\lambda} s^{\lambda}) \geq ((1 - \lambda) F(r)^{1/n} + \lambda G(s)^{1/n})^n,$$

whenever $F(s), G(r) > 0$. Thus, using the weighted inequality between arithmetic and geometric means, we get condition (5.1) for $F(s), G(r) > 0$. Whenever one of them is zero, (5.1) holds trivially, and thus we have verified (5.1). By the layer-cake representation, (5.2) follows at once from (1.5).

As conditions are verified, we are in position to use the following result:

Lemma 5.2. *If $\varepsilon \in (0, e^{-M_n(\tau)})$, and $f, g, h\colon \mathbb{R}^n \to \mathbb{R}_+$ satisfy (1.2), (1.5) and $\int_{\mathbb{R}^n} f = \int_{\mathbb{R}^n} g = 1$, then there is a dimensional constant $c_n > 0$ such that*

$$\int_0^\infty |F(t) - H(t)|\, dt + \int_0^\infty |G(t) - H(t)|\, dt \le c_n \tau^{-\frac{\omega}{2}-1} \varepsilon^{\frac{Q(\tau)}{2}}.$$

Proof. In analogy with the notation employed in Sections 2, 3, and 4, in what follows, we let

$$\{x \in \mathbb{R}^n : f(x) > t\} = A_t,$$
$$\{x \in \mathbb{R}^n : g(x) > t\} = B_t,$$
$$\{x \in \mathbb{R}^n : h(x) > t\} = C_t$$

denote the level sets of f, g, h, respectively. Since $\|f\|_1 = \|g\|_1 = 1$, $\int_0^\infty H = \int_{\mathbb{R}^n} h \le 1 + \varepsilon$, it follows from Corollary 5.1 that there exists some $b > 0$ such that

$$\int_0^\infty |b^\lambda F(b^\lambda t) - H(t)|\, dt + \int_0^\infty |b^{-(1-\lambda)} G(b^{-(1-\lambda)} t) - H(t)|\, dt \le a(\tau, \varepsilon),$$

where we denote $a(\tau, \varepsilon) = c\tau^{-\omega} e^{Q(\tau)}$. We may assume, without loss of generality, that $b \ge 1$.

For $t > 0$, let

$$\tilde{A}_t = b^{\frac{\lambda}{n}} A_{b^\lambda t} \qquad \text{if } \tilde{A}_t \neq \emptyset,$$
$$\tilde{B}_t = b^{\frac{-(1-\lambda)}{n}} B_{b^{-(1-\lambda)} t} \quad \text{if } \tilde{B}_t \neq \emptyset.$$

These sets satisfy $|\tilde{A}_t| = b^\lambda F(b^\lambda t)$, $|\tilde{B}_t| = b^{-(1-\lambda)} G(b^{-(1-\lambda)} t)$, and

$$\int_0^\infty \big||\tilde{A}_t| - H(t)\big|\, dt + \int_0^\infty \big||\tilde{B}_t| - H(t)\big|\, dt \le a(\tau, \varepsilon). \tag{5.3}$$

In addition, we also know from the Prékopa–Leindler condition that

$$(1-\lambda) b^{\frac{-\lambda}{n}} \tilde{A}_t + \lambda b^{\frac{1-\lambda}{n}} \tilde{B}_t \subset C_t.$$

We proceed to divide the positive line $[0, \infty)$ into two sets, where the measures of $\tilde{A}_t$, $\tilde{B}_t$ are either both close to that of $H(t)$, or otherwise. Indeed, we write $[0, +\infty) = I \cup J$, where $t \in I$ if $\frac{3}{4} H(t) < |\tilde{A}_t| < \frac{5}{4} H(t)$ and $\frac{3}{4} H(t) < |\tilde{B}_t| < \frac{5}{4} H(t)$, and $t \in J$ otherwise. For J, since $\varepsilon < e^{-M_n(\tau)}$, (5.3) yields

$$\int_J H(t)\, dt \le 4 \int_J \left(\big||\tilde{A}_t| - H(t)\big| + \big||\tilde{B}_t| - H(t)\big|\right) dt \le 8a(\tau, \varepsilon) < \frac{1}{2}. \tag{5.4}$$

Turning to I, it follows from the Prékopa–Leindler inequality and (5.4) that

$$\int_I H(t)\, dt \ge 1 - \int_J H(t)\, dt > \frac{1}{2}. \tag{5.5}$$

For $t \in I$, we define $\alpha(t) = |\tilde{A}_t|/H(t)$ and $\beta(t) = |\tilde{B}_t|/H(t)$, and hence $\frac{3}{4} < \alpha(t), \beta(t) < \frac{5}{4}$, and (5.3) implies

$$\int_0^\infty H(t) \cdot \big(|\alpha(t) - 1| + |\beta(t) - 1|\big)\, dt \leq 2a(\tau, \varepsilon). \tag{5.6}$$

We then proceed by estimating, by the Brunn–Minkowski inequality,

$$\begin{aligned} H(t) &\geq \Big((1-\lambda)|A_{b^\lambda t}|^{\frac{1}{n}} + \lambda |B_{b^{\lambda-1} t}|^{\frac{1}{n}}\Big)^n = \Big((1-\lambda) b^{\frac{-\lambda}{n}} |\tilde{A}_t|^{\frac{1}{n}} + \lambda b^{\frac{1-\lambda}{n}} |\tilde{B}_t|^{\frac{1}{n}}\Big)^n \\ &= |\tilde{A}_t|^{1-\lambda} \cdot |\tilde{B}_t|^{\lambda} \left((1-\lambda) b^{-\frac{\lambda}{n}} \frac{|\tilde{A}_t|^{\frac{\lambda}{n}}}{|\tilde{B}_t|^{\frac{\lambda}{n}}} + \lambda b^{\frac{1-\lambda}{n}} \frac{|\tilde{B}_t|^{\frac{1-\lambda}{n}}}{|\tilde{A}_t|^{\frac{1-\lambda}{n}}}\right)^n \\ &= H(t) \cdot \alpha(t)^{1-\lambda} \cdot \beta(t)^{\lambda} \Big((1-\lambda)\gamma^{\frac{\lambda}{n}} + \lambda \gamma^{-\frac{1-\lambda}{n}}\Big)^n, \end{aligned} \tag{5.7}$$

where we let $\gamma = \frac{|\tilde{A}_t|}{b|\tilde{B}_t|}$. Then (2.6) yields

$$(1-\lambda)\gamma^{\frac{\lambda}{n}} + \lambda \gamma^{-\frac{1-\lambda}{n}} \geq 1 + \tau(\gamma^{\frac{\lambda}{2n}} - \gamma^{-\frac{1-\lambda}{2n}})^2 \geq 1 + \tau(\gamma^{\frac{1}{4n}} - \gamma^{-\frac{1}{4n}})^2.$$

We now note that for $s \geq 1$, we have

$$s^{\frac{1}{4n}} - s^{-\frac{1}{4n}} = s^{-\frac{1}{4n}}(s^{\frac{1}{2n}} - 1) \geq s^{-\frac{1}{4n}} \cdot \frac{s^{\frac{1}{2n}-1}}{2n}(s-1) \geq \frac{1}{2n}\Big(s - \frac{1}{s}\Big),$$

and thus (5.7) implies

$$H(t) \geq H(t) \cdot \alpha(t)^{1-\lambda} \cdot \beta(t)^{\lambda} \Big(1 + \frac{\tau}{4n}(\gamma - \gamma^{-1})^2\Big).$$

We claim that if $t \in I$, then

$$\begin{aligned} &\alpha(t)^{1-\lambda} \cdot \beta(t)^{\lambda} \Big(1 + \frac{\tau}{4n}(\gamma - \gamma^{-1})^2\Big) \\ &\qquad \geq 1 - 2|\alpha(t) - 1| - 2|\beta(t) - 1| + \tau \frac{(\sqrt{b} - 1)^2}{8n \cdot b}. \end{aligned} \tag{5.8}$$

Since $\alpha(t)^{1-\lambda} \cdot \beta(t)^{\lambda} \geq 1 - |\alpha(t) - 1| - |\beta(t) - 1|$, (5.8) readily holds if $|\alpha(t) - 1| + |\beta(t) - 1| \geq \frac{(\sqrt{b}-1)^2}{16n \cdot b}$. Therefore we may assume that

$$|\alpha(t) - 1| + |\beta(t) - 1| \leq \frac{(\sqrt{b} - 1)^2}{16n \cdot b} < \frac{1}{2}, \tag{5.9}$$

which condition in turn yields that

$$\frac{b\beta(t)}{\alpha(t)} \geq \frac{b\big(1 - \frac{(\sqrt{b}-1)^2}{16n^2 \cdot b}\big)}{1 + \frac{(\sqrt{b}-1)^2}{16n \cdot b}} \geq b\Big(1 - 2 \cdot \frac{(\sqrt{b} - 1)^2}{32n \cdot b}\Big) \geq b\Big(1 - \frac{\sqrt{b} - 1}{\sqrt{b}}\Big) = \sqrt{b}. \tag{5.10}$$

We deduce, first applying (5.9), and then (5.10) and the fact that $\gamma = \frac{\alpha(t)}{b\beta(t)}$, that

$$\begin{aligned}
&\alpha(t)^{1-\lambda} \cdot \beta(t)^{\lambda}\Big(1 + \frac{\tau}{4n}(\gamma - \gamma^{-1})^2\Big) \\
&\quad\geq (1 - |\alpha(t) - 1| - |\beta(t) - 1|)\Big(1 + \frac{\tau}{4n}(\gamma - \gamma^{-1})^2\Big) \\
&\quad\geq 1 - |\alpha(t) - 1| - |\beta(t) - 1| + \frac{\tau}{8n}(\gamma - \gamma^{-1})^2 \\
&\quad\geq 1 - |\alpha(t) - 1| - |\beta(t) - 1| + \frac{\tau}{8n}\Big(\sqrt{b} - \frac{1}{\sqrt{b}}\Big)^2,
\end{aligned}$$

proving (5.8), under assumption (5.9) as well.

It follows first from (5.5), then from (5.7) and (5.8), and finally from (5.6) that

$$\begin{aligned}
\frac{(\sqrt{b}-1)^2}{16n \cdot b} \leq \int_I H(t) \cdot \frac{(\sqrt{b}-1)^2}{8n \cdot b}\, dt &\leq \frac{1}{\tau}\int_I H(t) \cdot (2|\alpha(t) - 1| + 2|\beta(t) - 1|)\, dt \\
&\leq \frac{4a(\tau, \varepsilon)}{\tau}.
\end{aligned}$$

Since $\varepsilon < e^{-M_n(\tau)}$, we deduce that $b < 2$; therefore, one easily deduces that

$$b \leq 1 + 50n^{\frac{1}{2}}\tau^{-\frac{1}{2}}a(\tau, \varepsilon)^{\frac{1}{2}}.$$

Next we claim that

$$\int_0^{\infty} \big||A_t| - |\tilde{A}_t|\big|\, dt + \int_0^{\infty} \big||B_t| - |\tilde{B}_t|\big|\, dt \leq 200n^{\frac{1}{2}}\tau^{-\frac{1}{2}}a(\tau, \varepsilon)^{\frac{1}{2}}. \tag{5.11}$$

Since $|A_{b^{\lambda}t}| \leq |A_t|$, we have

$$\begin{aligned}
\int_0^{\infty} \big||A_t| - |\tilde{A}_t|\big|\, dt &= \int_0^{\infty} \big||A_t| - b^{\lambda}|A_{b^{\lambda}t}|\big|\, dt \\
&\leq \int_0^{\infty} \big||A_t| - b^{\lambda}|A_t|\big|\, dt + b^{\lambda}\int_0^{\infty} \big||A_t| - |A_{b^{\lambda}t}|\big|\, dt \\
&= (b^{\lambda} - 1) + b^{\lambda}\int_0^{\infty} (|A_t| - |A_{b^{\lambda}t}|)\, dt \\
&= 2(b^{\lambda} - 1) \leq 100\lambda 2^{\lambda-1}n^{\frac{1}{2}}\tau^{-\frac{1}{2}}a(\tau, \varepsilon)^{\frac{1}{2}} \leq 100n^{\frac{1}{2}}\tau^{-\frac{1}{2}}a(\tau, \varepsilon)^{\frac{1}{2}}.
\end{aligned}$$

Similarly, $|B_t| \leq |B_{b^{\lambda-1}t}|$, and hence

$$\begin{aligned}
\int_0^{\infty} \big||B_t| - |\tilde{B}_t|\big|\, dt &= \int_0^{\infty} \big||B_t| - b^{\lambda-1}|B_{b^{\lambda-1}t}|\big|\, dt \\
&\leq \int_0^{\infty} \big||B_t| - b^{\lambda-1}|B_t|\big|\, dt + b^{\lambda-1}\int_0^{\infty} \big||B_t| - |B_{b^{\lambda-1}t}|\big|\, dt \\
&= (1 - b^{\lambda-1}) + b^{\lambda-1}\int_0^{\infty} (|B_{b^{\lambda-1}t}| - |B_t|)\, dt \\
&= 2(1 - b^{\lambda-1}) \leq 100n^{\frac{1}{2}}\tau^{-\frac{1}{2}}a(\tau, \varepsilon)^{\frac{1}{2}},
\end{aligned}$$

proving (5.11). We conclude the proof by combining (5.3) and (5.11). ∎

As a by-product of Lemma 5.2, notice that, by setting $\min(\|f\|_\infty, \|g\|_\infty) = \|f\|_\infty = 2$, then

$$\tau^{-\frac{1}{2}} a(\tau,\varepsilon)^{\frac{1}{2}} \gtrsim_n \int_2^{\max \|g\|_\infty, \|h\|_\infty} (G(t) + H(t))\, dt.$$

In particular, we know that

$$C_t \supset (1-\lambda)A_t + \lambda B_t \tag{5.12}$$

whenever $t \in (0,2)$. We claim, before proceeding with the proof, that under such conditions,

$$\|g\|_\infty \le \frac{2e \cdot 3^{n+1}}{\tau^{n+1}}. \tag{5.13}$$

Indeed, if $y_0 \in \mathbb{R}^n$ is fixed, we have

$$C_t \supset (1-\lambda)A_{t^{1/(1-\lambda)}/g(y_0)^{\lambda/(1-\lambda)}} + \lambda y_0.$$

In particular,

$$\begin{aligned}
\int_0^t F(s)\, ds &= \frac{1}{1-\lambda} \int_0^{t^{1-\lambda} g(y_0)^\lambda} F\left(\frac{r^{1/(1-\lambda)}}{g(y_0)^{\lambda/(1-\lambda)}}\right)\left(\frac{r}{g(y_0)}\right)^{\lambda/(1-\lambda)} dr \\
&\le \frac{1}{1-\lambda}\left(\frac{t}{g(y_0)}\right)^\lambda \int_0^{t^{1-\lambda} g(y_0)^\lambda} F\left(\frac{r^{1/(1-\lambda)}}{g(y_0)^{\lambda/(1-\lambda)}}\right) dr \\
&\le \frac{1}{(1-\lambda)^{n+1}}\left(\frac{t}{g(y_0)}\right)^\lambda \int_0^{t^{1-\lambda} g(y_0)^\lambda} H(r)\, dr.
\end{aligned}$$

Therefore, by picking $t = 2$ and using that $\int H \le 1 + \varepsilon$, $\int_0^2 F(s)\, ds = 1$,

$$g(y_0) \le \frac{2 \cdot (1+\varepsilon)^{1/\lambda}}{(1-\lambda)^{(n+1)/\lambda}}.$$

A quick analysis shows that, for $\lambda \in (0,1)$, the inequality

$$(1-\lambda)^{1/\lambda} \ge \frac{1}{3}(1-\lambda)$$

holds. If $\varepsilon < \tau$, then the numerator is at most $2e$, and thus, as y_0 was arbitrary above, we conclude the claim. Now using (5.12), we get

$$H(t) \ge \left((1-\lambda)F(t)^{1/n} + \lambda G(t)^{1/n}\right)^n \ge \frac{F(t)+G(t)}{2} - \frac{|F(t)-G(t)|}{2} \quad \forall t \in (0,2).$$

Notice also that, by Lemma 5.2,

$$\int_0^\infty |F(t) - G(t)|\, dt \lesssim_n \tau^{-\frac{1}{2}} a(\tau,\varepsilon)^{\frac{1}{2}}.$$

Thus, by these considerations and the almost-optimality of f, g, h for the Prékopa–Leindler inequality, we obtain

$$c_n \tau^{-\frac{1}{2}} a(\tau,\varepsilon)^{\frac{1}{2}} \geq \int_0^\alpha \Big(H(t) - \frac{F(t)+G(t)}{2} + \frac{|F(t)-G(t)|}{2} \Big)\, dt \quad \forall \alpha \geq 0. \tag{5.14}$$

On the other hand, notice that (2.9) implies, together with a limiting argument and the Brunn–Minkowski inequality,

$$H(t) \geq \max\big\{ \big(\lambda G(t^{\frac{1}{\lambda}})^{1/n} + (1-\lambda) F(1)^{1/n}\big)^n, \big((1-\lambda) F(t^{\frac{1}{1-\lambda}})^{1/n} + \lambda G(1)^{1/n}\big)^n \big\},$$

for all $t \in (0,2)$ so that $H(t) > 0$. Thus, (5.14) implies

$$c_n \tau^{-\frac{1}{2}} a(\tau,\varepsilon)^{\frac{1}{2}} \geq \int_0^\alpha \Big(\frac{1}{2}\big((1-\lambda)^n F(t^{\frac{1}{1-\lambda}}) + \lambda^n G(t^{\frac{1}{\lambda}})\big) - \frac{F(t)+G(t)}{2} \Big)\, dt. \tag{5.15}$$

We thus let, in analogy with Lemma 2.5,

$$\Gamma(\alpha) = \int_0^\alpha ((1-\lambda)^n F(t) + \lambda^n G(t))\, dt.$$

Again in analogy with Lemma 2.5, we may suppose without loss of generality that $\lambda \leq 1/2$. Then (5.15) implies

$$\frac{1-\lambda}{2} \Gamma(\alpha^{\frac{1}{1-\lambda}}) \alpha^{-\frac{\lambda}{1-\lambda}} \leq c_n \tau^{-\frac{1}{2}} a(\tau,\varepsilon)^{\frac{1}{2}} + \frac{\Gamma(\alpha)}{2\tau^n}.$$

As in the proof of Lemma 2.5, we let $\beta = \alpha^{\frac{1}{1-\lambda}}$. We thus have

$$\frac{\Gamma(\beta)}{\beta} \leq 2 c_n \tau^{-\frac{3}{2}} a(\tau,\varepsilon)^{\frac{1}{2}} \cdot \frac{1}{\beta^{1-\lambda}} + \frac{1}{\tau^{n+1}} \frac{\Gamma(\beta^{1-\lambda})}{\beta^{1-\lambda}},$$

and therefore

$$\frac{\Gamma(\beta)}{\beta} \leq \Big(2 c_n \tau^{-\frac{3}{2}} a(\tau,\varepsilon)^{\frac{1}{2}} \sum_{i=1}^k \frac{(1/\tau^{n+1})^{i-1}}{\beta^{(1-\lambda)^i}} \Big) + (1/\tau^{n+1})^k \frac{\Gamma(\beta^{(1-\lambda)^k})}{\beta^{(1-\lambda)^k}}.$$

We now select $k \in \mathbb{N}$ to be the first natural number such that $\beta^{(1-\lambda)^k} > e^{-1}$. This implies that

$$\Gamma(\beta) \lesssim (1/\tau^{n+1})^k \Big(1 + c_n \frac{\sqrt{a(\tau,\varepsilon)}}{\beta^{1-\lambda} \tau^{\frac{3}{2}}} \Big) \beta.$$

If $\beta > \varepsilon^{\frac{Q(\tau)}{2}}$, then the estimate above yields

$$\Gamma(\beta) \leq c_n \tau^{-\frac{\omega+3}{2}} \beta |\log(\beta)|^{\frac{4(n+3)|\log \tau|}{\tau}}.$$

In particular, one concludes directly from the definition of Γ that

$$F(\beta) + G(\beta) \leq c_n \tau^{-\frac{\omega+3+n}{2}} |\log \varepsilon|^{\frac{4(n+3)|\log \tau|}{\tau}} \quad \forall \beta > \varepsilon^{\frac{Q(\tau)}{2}}. \tag{5.16}$$

We are now ready to give the proof of Theorem 1.6 in dimensions $n \geq 2$. For that, we use the shorthand $\rho_n(\tau) = \frac{4(n+10)|\log \tau|}{\tau}$.

Proof of Theorem 1.6, $n \geq 2$. Let $\theta > 0$ be small, to be chosen later. Define the (truncated) log-hypographs of f, g, h as

$$\mathcal{S}_f = \{(x,T) \in \mathbb{R}^{n+1} : x \in \{f > \varepsilon^\theta\},\ \varepsilon^\theta \leq e^T < f(x)\},$$
$$\mathcal{S}_g = \{(x,T) \in \mathbb{R}^{n+1} : x \in \{g > \varepsilon^\theta\},\ \varepsilon^\theta \leq e^T < g(x)\},$$
$$\mathcal{S}_h = \{(x,T) \in \mathbb{R}^{n+1} : x \in \{h > \varepsilon^\theta\},\ \varepsilon^\theta \leq e^T < h(x)\}.$$

We first claim that the measures of the first two such sets are well controlled. Indeed, it follows directly from the definition of such sets and (5.16) that, for $\theta < Q(\tau)/4$,

$$c_n \theta \tau^{-\frac{\omega+3+n}{2}} |\log \varepsilon|^{\rho_n(\tau)} \geq \theta |\log \varepsilon| \cdot \mathcal{H}^n(\{f > \varepsilon^\theta\}) \geq \mathcal{H}^{n+1}(\mathcal{S}_f). \tag{5.17}$$

On the other hand, by a change of variables and the normalization chosen for f, one obtains

$$\mathcal{H}^{n+1}(\mathcal{S}_f) = \int_{\theta \log \varepsilon}^{\log \|f\|_\infty} F(e^s)\, ds > \frac{1}{2}. \tag{5.18}$$

The same estimates together with (5.13) show that

$$c_n \theta \tau^{-\frac{\omega+3+n}{2}} |\log \varepsilon|^{\rho_n(\tau)} \geq \mathcal{H}^{n+1}(\mathcal{S}_g) > \frac{\tau^{(n+1)}}{2e \cdot 3^{n+1}} \tag{5.19}$$

holds as well. Employing Lemma 5.2, we obtain

$$\begin{aligned}
&|\mathcal{H}^{n+1}(\mathcal{S}_f) - \mathcal{H}^{n+1}(\mathcal{S}_h)| + |\mathcal{H}^{n+1}(\mathcal{S}_g) - \mathcal{H}^{n+1}(\mathcal{S}_h)| \\
&\quad \leq \int_{\theta \log \varepsilon}^{\infty} \big(|F(e^s) - H(e^s)| + |G(e^s) - H(e^s)|\big)\, ds \\
&\quad \leq \varepsilon^{-\theta} \left(\int_0^\infty \big(|F(t) - H(t)| + |G(t) - H(t)|\big)\, ds \right) \\
&\quad \leq c_n \tau^{-\frac{\omega+3}{2}} \varepsilon^{\frac{Q(\tau)}{2} - \theta} =: \tau^n \cdot \delta(\varepsilon, \tau, \theta).
\end{aligned} \tag{5.20}$$

We denote, until the end of the proof, $\delta = \delta(\varepsilon, \tau, \theta)$ for shortness. By (1.2), we have

$$(1-\lambda)\mathcal{S}_f + \lambda \mathcal{S}_g \subset \mathcal{S}_h. \tag{5.21}$$

In particular, (5.20), (5.21), and the fact that $\mathcal{H}^{n+1}(\mathcal{S}_f) > 1/2$ imply the following control on the measure of $\mathcal{S}_h$:

$$2c_n \tau^{-\frac{\omega+3+n}{2}} |\log \varepsilon|^{\rho_n(\tau)} \geq \mathcal{H}^{n+1}(\mathcal{S}_h) \geq \frac{\tau^n}{2}.$$

We are in position to use Theorem 1.4. That result states that, under the conditions satisfied by the sets $\mathcal{S}_f$, $\mathcal{S}_g$, and $\mathcal{S}_h$ in (5.17), (5.18), (5.19), (5.20), and (5.21), for $\delta < e^{-A_n(\tau)}$, the sets $\mathcal{S}_f$, $\mathcal{S}_g$ are both close (in quantitative terms of $\delta = \delta(\varepsilon, \tau, \theta)$) to their convex hulls. Here we let $A_n(\tau) = 2^{3^{n+2}} n^{3^n} |\log \tau|^{3^n} / \tau^{3^n}$, in accordance with [19, Theorem 1.3].

In more effective terms, Theorem 1.4 implies that there exist an absolute constant $c_n > 0$ and an exponent $\gamma_n(\tau) = \tau^{3^n}/2^{3^{n+1}} n^{3^n} |\log \tau|^{3^n}$ such that the following holds. Denote the closure of the convex hulls of $\mathcal{S}_f, \mathcal{S}_g, \mathcal{S}_h$ by S_f, S_g, S_h respectively. There are $\widetilde{w} = (w, \varrho) \in \mathbb{R}^{n+1}$, and a convex set $\mathbb{S}_h \supset S_h$ with

$$\begin{aligned}
&\mathbb{S}_h \supset (\mathcal{S}_f - \widetilde{w}) \cup (\mathcal{S}_g + \widetilde{w}),\\
&\mathcal{H}^{n+1}(S_h \setminus \mathcal{S}_h) + \mathcal{H}^{n+1}(S_f \setminus \mathcal{S}_f) + \mathcal{H}^{n+1}(S_g \setminus \mathcal{S}_g)\\
&\qquad \le c_n \tau^{-N_n - \frac{\omega+3+n}{2}} |\log \varepsilon|^{\rho_n(\tau)} \delta^{\gamma_n(\tau)},\\
&\mathcal{H}^{n+1}(\mathbb{S}_h \setminus \mathcal{S}_h) + \mathcal{H}^{n+1}(\mathbb{S}_h \setminus (\mathcal{S}_f - \widetilde{w})) + \mathcal{H}^{n+1}(\mathbb{S}_h \setminus (\mathcal{S}_g + \widetilde{w}))\\
&\qquad \le c_n \tau^{-N_n - \frac{\omega+3+n}{2}} |\log \varepsilon|^{\rho_n(\tau)} \delta^{\gamma_n(\tau)}.
\end{aligned} \tag{5.22}$$

We thus use the shorthand $N_n' = N_n + \frac{\omega+3+n}{2}$. Now (5.22) readily implies that

$$\mathcal{H}^{n+1}(\mathbb{S}_h \setminus S_h) \le 2c_n \tau^{-N_n'} |\log \varepsilon|^{\rho_n(\tau)} \delta^{\gamma_n(\tau)},$$

and thus

$$\mathcal{H}^{n+1}(S_h \Delta (S_f - \widetilde{w})) + \mathcal{H}^{n+1}(S_h \Delta (S_g - \widetilde{w})) \le 6c_n \tau^{-N_n'} |\log \varepsilon|^{\rho_n(\tau)} \delta^{\gamma_n(\tau)}. \tag{5.23}$$

We now employ the analysis of [9, Lemma 6.1]. Explicitly, suppose first that $\widetilde{w} = (w, \varrho)$, $\varrho > 0$. We let

$$S_f^{\varrho} = \{(x, T) \in S_f : \theta \log \varepsilon \le T \le \theta \log \varepsilon + \varrho\}.$$

By the fact that

$$\mathcal{H}^{n+1}(S_f + (0, \varrho)) = \mathcal{H}^{n+1}(S_f) = \mathcal{H}^{n+1}(S_f \cap (S_f + (0, \varrho))) + \mathcal{H}^{n+1}(S_f^{\varrho}),$$

it follows that $\mathcal{H}^{n+1}(S_f \Delta (S_f + (0, \varrho))) = 2\mathcal{H}^{n+1}(S_f^{\varrho})$. But we also have that $S_f^{\omega} \subset S_f \setminus (S_h + \widetilde{w})$, which, by (5.22) and (5.23), implies that

$$\mathcal{H}^{n+1}(S_f^{\varrho}) \le 6c_n \tau^{-N_n'} |\log \varepsilon|^{\rho_n(\tau)} \delta^{\gamma_n(\tau)}.$$

Thus, by the triangle inequality,

$$\begin{aligned}
\mathcal{H}^{n+1}(S_f \Delta (S_h + (w, 0))) &\le 2\mathcal{H}^{n+1}(S_f^{\varrho}) + \mathcal{H}^{n+1}(S_f \Delta (S_h + \widetilde{w}))\\
&\le 18 c_n \tau^{-N_n'} |\log \varepsilon|^{\rho_n(\tau)} \delta^{\gamma_n(\tau)}.
\end{aligned}$$

A similar argument works in the case $\varrho < 0$, if one considers $S_h^{|\varrho|}$ instead of S_f^{ϱ}. In the end, this allows one to conclude that the $w \in \mathbb{R}^n$ from before satisfies that

$$\mathcal{H}^{n+1}(S_h \Delta (S_f - w)) + \mathcal{H}^{n+1}(S_h \Delta (S_g + w)) \le 72 c_n \tau^{-N_n'} |\log \varepsilon|^{\rho_n(\tau)} \delta^{\gamma_n(\tau)}. \tag{5.24}$$

We now note that, as $\{f > \varepsilon^{\theta}\} \times \{T = \theta \log \varepsilon\} \subset \mathcal{S}_f$, then

$$S_f \supset \mathrm{co}(\{f > \varepsilon^{\theta}\}) \times \{T = \theta \log \varepsilon\}.$$

We associate to each $x \in \mathrm{co}(\{f > \varepsilon^\theta\})$ the function

$$T_f(x) = \sup\{T \in \mathbb{R} : (x, T) \in S_f\}.$$

Clearly, this satisfies $T_f(x) \geq \theta \log \varepsilon$ for all $x \in \mathrm{co}(\{f > \varepsilon^\theta\})$. We claim that this function is, moreover, concave. Indeed, if $(x, T_1), (y, T_2) \in S_f$, by convexity of that set we get

$$(tx + (1-t)y, tT_1 + (1-t)T_2) \in S_f.$$

Thus,

$$\begin{aligned} T_f(tx + (1-t)y) &= \sup\{T \in \mathbb{R} : (tx + (1-t)y, T) \in S_f\} \\ &\geq t \sup\{T \in \mathbb{R} : (x, T_1) \in S_f\} + (1-t)\sup\{T \in \mathbb{R} : (y, T_2) \in S_f\} \\ &= tT_f(x) + (1-t)T_f(y) \quad \forall t \in (0,1). \end{aligned}$$

By definition of S_f, it also follows that $T_f(x) \geq \log f(x)$ for all $x \in \mathrm{co}(\{f > \varepsilon^\theta\})$. Let

$$\tilde{f}(x) = \begin{cases} e^{T_f(x)} & \text{if } x \in \mathrm{co}(\{f > \varepsilon^\theta\}), \\ 0 & \text{otherwise.} \end{cases}$$

Now notice that (x, r) belongs to the interior of S_f if and only if $T_f(x) > r > \theta \log \varepsilon$ and x belongs to the interior of $\mathrm{co}(\{f > \varepsilon^\theta\})$. Writing $A(r) = \{(x, T) \in A, T = r\}$ for horizontal slices of a set $A \subset \mathbb{R}^{n+1}$, we compute, by Fubini,

$$\begin{aligned} \mathcal{H}^{n+1}(S_f \setminus \mathcal{S}_f) &= \int_{-\infty}^{\infty} \mathcal{H}^n(S_f(r) \setminus \mathcal{S}_f(r))\, dr \\ &= \int_{\theta \log \varepsilon}^{\log 2} \mathcal{H}^n(\{\log \tilde{f} > r\} \setminus \{\log f > r\})\, dr \\ &= \int_{\varepsilon^\theta}^{2} \mathcal{H}^n(\{\tilde{f} > s\} \Delta \{f > s\}) \frac{ds}{s} \\ &\geq \frac{1}{2} \int_{\varepsilon^\theta}^{2} \mathcal{H}^n(\{\tilde{f} > s\} \Delta \{f > s\})\, ds. \end{aligned} \tag{5.25}$$

By Chebyshev's inequality and (5.22), there is

$$\begin{aligned} &s_0 \in (\varepsilon^\theta, \varepsilon^\theta + c_n \tau^{-\frac{N_n'}{2}} \delta^{\frac{\gamma_n(\tau)}{2}}) \\ &\text{so that } \mathcal{H}^n(\{\tilde{f} > s_0\} \Delta \{f > s_0\}) \leq \tau^{-\frac{N_n'}{2}} |\log \varepsilon|^{\rho_n(\tau)} \delta^{\frac{\gamma_n(\tau)}{2}}. \end{aligned}$$

Recalling the definition of δ, one notices that, if $\frac{Q(\tau)}{4} > \theta$, and $\varepsilon < (c_n)^{-1} e^{\frac{2^{10} N_n \log(\tau)}{\gamma_n(\tau) Q(\tau)}}$ we may take $s_0 \in (\varepsilon^\theta, 2\varepsilon^\theta)$ so that

$$\mathcal{H}^n(\{\tilde{f} > s_0\} \Delta \{f > s_0\}) \lesssim \tau^{-N_n'/2} |\log \varepsilon|^{\rho_n(\tau)} \varepsilon^{\frac{\gamma_n(\tau) Q(\tau)}{8}}. \tag{5.26}$$

Then define the function $\tilde{f}_1$ to be zero whenever $\tilde{f} \le s_0$, and equal to $\tilde{f}$ otherwise. This new function is again log-concave.

We claim that this new function is still sufficiently close to f. Indeed, by gathering (5.25), (5.26), and (5.16), we have

$$\begin{aligned}
\|\tilde{f}_1 - f\|_1 &= \int_0^2 \mathcal{H}^n(\{\tilde{f}_1 > t\}\Delta\{f > t\})\,dt \\
&\le \int_0^{s_0} \big(\mathcal{H}^n(\{\tilde{f}_1 > s_0\}) + \mathcal{H}^n(\{f > t\})\big)\,dt \\
&\quad + \int_{s_0}^2 \mathcal{H}^n(\{\tilde{f}_1 > t\}\Delta\{f > t\})\,dt \\
&\le c_n \tau^{-\frac{\omega+3+n}{2}} \varepsilon^{\theta} |\log\varepsilon|^{\rho_n(\tau)} + \int_{s_0}^2 \mathcal{H}^n(\{\tilde{f} > t\}\Delta\{f > t\})\,dt \\
&\le c_n \tau^{-\frac{\omega+3+n}{2}} \varepsilon^{\theta} |\log\varepsilon|^{\rho_n(\tau)} + 2\mathcal{H}^{n+1}(S_f \setminus \mathcal{S}_f) \\
&\lesssim_n \tau^{-N_n'} \varepsilon^{\frac{\gamma_n(\tau)Q(\tau)}{16}} |\log\varepsilon|^{\rho_n(\tau)},
\end{aligned} \tag{5.27}$$

where we chose $\theta = \frac{\gamma_n(\tau)Q(\tau)}{16}$. Fix this value, and thus the value of δ, for the rest of the proof. Such an inequality is evidently not restrictive to f, and the same argument yields that there is a log-concave function $\tilde{g}_1$ so that

$$\|\tilde{g}_1 - g\|_1 \lesssim_n \tau^{-N_n'-(n+1)} \varepsilon^{\frac{\gamma_n(\tau)Q(\tau)}{16}} |\log\varepsilon|^{\rho_n(\tau)}. \tag{5.28}$$

In order to conclude, we only need to prove that both $\tilde{f}_1$, $\tilde{g}_1$ are sufficiently close, after a translation, to a log-concave function $\tilde{h}_1$. In order to prove that, one only needs to construct the function $\tilde{h}$ in entire analogy with what we did for $\tilde{f}$, $\tilde{g}$; that is, we let

$$T_h(x) = \sup\{T \in \mathbb{R} \colon (x, T) \in S_h\}.$$

One readily verifies that this new function is, again, concave, and that the function

$$\tilde{h}(x) = \begin{cases} e^{T_h(x)} & \text{if } x \in \mathrm{co}(\{h > e^{\theta}\}), \\ 0 & \text{otherwise}, \end{cases}$$

is log-concave. Using (5.24) together with an argument similar to (5.27) implies that

$$\begin{aligned}
&\mathcal{H}^{n+1}(S_h \Delta (S_f - w)) + \mathcal{H}^{n+1}(S_h \Delta (S_g + w)) \\
&\quad \ge \int_0^{\|\tilde{h}_1\|_\infty} \Big(\mathcal{H}^n\big(\{\tilde{h} > s\}\Delta\{\tilde{f}(\cdot + w) > s\}\big) \\
&\qquad\qquad + \mathcal{H}^n\big(\{\tilde{h} > s\}\Delta\{\tilde{g}(\cdot - w) > s\}\big)\Big)\,\frac{ds}{s}.
\end{aligned} \tag{5.29}$$

Notice now that $\|\tilde{f}_1\|_\infty = \|f\|_\infty$, $\|\tilde{g}_1\|_\infty = \|g\|_\infty$, by construction. The idea is then to truncate from below at height $\{\tilde{h} > s_0\}$ and from above at height $\varrho := \max(\|\tilde{f}_1\|_\infty, \|\tilde{g}_1\|_\infty)$

in order to generate a new function, which is again log-concave by construction. Denote this new function by $\tilde{h}_1$. Moreover, by (5.29) in conjunction with (5.13), we have

$$\begin{aligned} 2e \cdot 3^{n+1} \tau^{-n-1} c_n \tau^{-N_n'} |\log \varepsilon|^{\rho_n(\tau)} \delta^{\gamma_n(\tau)} \\ \geq \int_{s_0}^{\varrho} \Big(\mathcal{H}^n \big(\{\tilde{h}_1 > s\} \Delta \{\tilde{f}_1(\cdot + w) > s\} \big) + \mathcal{H}^n \big(\{\tilde{h}_1 > s\} \Delta \{\tilde{g}_1(\cdot - w) > s\} \big) \Big) \, ds \\ = \int_{\mathbb{R}^n} \big(|\tilde{h}_1(x) - \tilde{f}_1(x+w)| + |\tilde{h}_1(x) - \tilde{g}_1(x-w)| \big) \, dx. \end{aligned} \tag{5.30}$$

Combining (5.27), (5.28), and (5.30) implies that

$$\|\tilde{h}_1(\cdot - w) - f\|_1 + \|\tilde{h}_1(\cdot + w) - g\|_1 \lesssim_n \tau^{-N_n'-n-1} |\log \varepsilon|^{\rho_n(\tau)} \varepsilon^{\frac{\gamma_n(\tau)Q(\tau)}{16}}.$$

Finally, in order to prove that h is close to $\tilde{h}_1$, we estimate

$$\begin{aligned} \int_{\mathbb{R}^n} |h(x) - \tilde{h}_1(x)| \, dx &= \int_0^{s_0} \mathcal{H}^n(\{h > s\}) \, ds \\ &\quad + \int_{s_0}^{\varrho} \mathcal{H}^n(\{h > s\} \Delta \{\tilde{h} > s\}) \, ds + \int_{\varrho}^{\infty} \mathcal{H}^n(\{h > s\}) \, ds \\ &\leq c_n \tau^{-\frac{\omega+3+n}{2}} \varepsilon^{Q(\tau)\gamma_n(\tau)/16} |\log \varepsilon|^{\rho_n(\tau)} \\ &\quad + \int_{s_0}^{\varrho} \mathcal{H}^n(\{h > s\} \Delta \{\tilde{h}_1 > s\}) \, ds \\ &\quad + c_n \tau^{-\omega/2} \varepsilon^{\frac{Q(\tau)}{2}}, \end{aligned} \tag{5.31}$$

where we used both (5.16) and Lemma 5.2 in the last line. In order to deal with the middle term, we remark that an argument entirely analogous to that of (5.25) implies that

$$\mathcal{H}^n(S_h \setminus \mathcal{S}_h) \geq \frac{1}{\varrho} \int_{s_0}^{\varrho} \mathcal{H}^n(\{h > s\} \Delta \{\tilde{h} > s\}) \, ds,$$

which on the other hand implies

$$\int_{s_0}^{\varrho} \mathcal{H}^n(\{h > s\} \Delta \{\tilde{h}_1 > s\}) \, ds \lesssim_n \tau^{-n-1} \tau^{-N_n'} \varepsilon^{\gamma_n Q(\tau)/16} |\log \varepsilon|^{\rho_n(\tau)}. \tag{5.32}$$

Inserting (5.32) into (5.31) implies

$$\|h - \tilde{h}_1\|_1 \lesssim_n \tau^{-N_n'-(n+1)} \varepsilon^{\frac{\gamma_n(\tau)Q(\tau)}{16}} |\log \varepsilon|^{\rho_n(\tau)}. \tag{5.33}$$

Finally, in order to arrive at the statement of Theorem 1.6, we notice that the expression on the right-hand side of (5.33) may be bounded by $c_n \tau^{-N_n'-n-1} \varepsilon^{\frac{\gamma_n(\tau)Q(\tau)}{32}}$, as long as

$$\varepsilon < e^{-c_n \frac{|\log \tau| \rho_n(\tau)^2}{Q_n(\tau)^2}} \quad \text{for } c_n \gg 1 \text{ a sufficiently large absolute constant.}$$

An inspection of the constants needed for the proof above allows us to conclude that Theorem 1.6 holds with $\Sigma_n = N_n + \frac{\omega+3+n}{2} + (n+1)$, as $\tau^{\gamma_n(\tau)}$ is bounded by an explicitly computable absolute constant $\widetilde{C}_n$ whenever $\tau \in [0,1]$. We also conclude that we may take $Q_n(\tau) = \frac{Q(\tau)\gamma_n(\tau)}{16}$, and the result holds whenever $\varepsilon < c_n e^{-M_n(\tau)}$, where $c_n > 0$ is an explicitly computable absolute constant, and one may take

$$M_n(\tau) = c_n |\log(\tau)| \max\left\{ \frac{A_n(\tau)}{Q(\tau)}, \frac{\rho_n(\tau)^2}{Q_n(\tau)^2} \right\}, \tag{5.34}$$

for $c_n > 0$ a sufficiently large absolute constant, depending only on the dimension $n \geq 2$. This finishes the proof of the higher-dimensional case, and thus also of Theorem 1.6. ∎

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References

[1] J. M. Aldaz, A stability version of Hölder’s inequality. *J. Math. Anal. Appl.* **343** (2008), no. 2, 842–852 Zbl 1138.26308 MR 2401540

[2] K. M. Ball and K. J. Böröczky, Stability of the Prékopa–Leindler inequality. *Mathematika* **56** (2010), no. 2, 339–356 Zbl 1205.39024 MR 2678033

[3] Z. M. Balogh and A. Kristály, Equality in Borell–Brascamp–Lieb inequalities on curved spaces. *Adv. Math.* **339** (2018), 453–494 Zbl 1402.49035 MR 3866904

[4] M. Barchiesi and V. Julin, Robustness of the Gaussian concentration inequality and the Brunn–Minkowski inequality. *Calc. Var. Partial Differential Equations* **56** (2017), no. 3, article no. 80 Zbl 1378.60042 MR 3646982

[5] F. Barthe, K. J. Böröczky, and M. Fradelizi, Stability of the functional forms of the Blaschke–Santaló inequality. *Monatsh. Math.* **173** (2014), no. 2, 135–159 Zbl 1288.52003 MR 3154168

[6] G. Bianchi and H. Egnell, A note on the Sobolev inequality. *J. Funct. Anal.* **100** (1991), no. 1, 18–24 Zbl 0755.46014 MR 1124290

[7] S. G. Bobkov, A. Colesanti, and I. Fragalà, Quermassintegrals of quasi-concave functions and generalized Prékopa–Leindler inequalities. *Manuscripta Math.* **143** (2014), no. 1-2, 131–169 Zbl 1290.26019 MR 3147446

[8] C. Borell, Convex set functions in d-space. *Period. Math. Hungar.* **6** (1975), no. 2, 111–136 Zbl 0307.28009 MR 404559

[9] K. J. Böröczky and A. De, Stability of the Prékopa–Leindler inequality for log-concave functions. *Adv. Math.* **386** (2021), article no. 107810 Zbl 1472.26004 MR 4266751

[10] H. J. Brascamp and E. H. Lieb, On extensions of the Brunn–Minkowski and Prékopa–Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation. *J. Functional Analysis* **22** (1976), no. 4, 366–389 Zbl 0334.26009 MR 0450480

[11] D. Bucur and I. Fragalà, Lower bounds for the Prékopa–Leindler deficit by some distances modulo translations. *J. Convex Anal.* **21** (2014), no. 1, 289–305 Zbl 1326.26032 MR 3235317

[12] U. Caglar and E. M. Werner, Stability results for some geometric inequalities and their functional versions. In *Convexity and concentration*, pp. 541–564, IMA Vol. Math. Appl. 161, Springer, New York, 2017 Zbl 1375.52007 MR 3837282

[13] Y. Chen, An almost constant lower bound of the isoperimetric coefficient in the KLS conjecture. *Geom. Funct. Anal.* **31** (2021), no. 1, 34–61 Zbl 1495.52003 MR 4244847

[14] M. Christ, An approximate inverse Riesz–Sobolev inequality. 2011, arXiv:1112.3715

[15] D. Cordero-Erausquin, Transport inequalities for log-concave measures, quantitative forms, and applications. *Canad. J. Math.* **69** (2017), no. 3, 481–501 Zbl 1388.60057 MR 3679684

[16] S. Dubuc, Critères de convexité et inégalités intégrales. *Ann. Inst. Fourier (Grenoble)* **27** (1977), no. 1, 135–165 Zbl 0331.26008 MR 444863

[17] R. Eldan, Thin shell implies spectral gap up to polylog via a stochastic localization scheme. *Geom. Funct. Anal.* **23** (2013), no. 2, 532–569 Zbl 1277.52013 MR 3053755

[18] A. Figalli and D. Jerison, Quantitative stability for sumsets in $\mathbb{R}^n$. *J. Eur. Math. Soc. (JEMS)* **17** (2015), no. 5, 1079–1106 Zbl 1325.49052 MR 3346689

[19] A. Figalli and D. Jerison, Quantitative stability for the Brunn–Minkowski inequality. *Adv. Math.* **314** (2017), 1–47 Zbl 1380.52010 MR 3658711

[20] A. Figalli and D. Jerison, Quantitative stability of the Brunn–Minkowski inequality for sets of equal volume. *Chinese Ann. Math. Ser. B* **38** (2017), no. 2, 393–412 Zbl 1369.49065 MR 3615496

[21] A. Figalli and D. Jerison, A sharp Freiman type estimate for semisums in two and three dimensional Euclidean spaces. *Ann. Sci. Éc. Norm. Supér. (4)* **54** (2021), no. 1, 235–257 Zbl 1482.11139 MR 4245865

[22] A. Figalli, F. Maggi, and C. Mooney, The sharp quantitative Euclidean concentration inequality. *Camb. J. Math.* **6** (2018), no. 1, 59–87 Zbl 1385.39005 MR 3786098

[23] A. Figalli, F. Maggi, and A. Pratelli, A refined Brunn–Minkowski inequality for convex sets. *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **26** (2009), no. 6, 2511–2519 Zbl 1192.52015 MR 2569906

[24] A. Figalli, F. Maggi, and A. Pratelli, A mass transportation approach to quantitative isoperimetric inequalities. *Invent. Math.* **182** (2010), no. 1, 167–211 Zbl 1196.49033 MR 2672283

[25] A. Figalli and R. Neumayer, Gradient stability for the Sobolev inequality: The case $p \geq 2$. *J. Eur. Math. Soc. (JEMS)* **21** (2019), no. 2, 319–354 Zbl 1417.46023 MR 3896203

[26] A. Figalli and Y. R.-Y. Zhang, Sharp gradient stability for the Sobolev inequality. *Duke Math. J.* **171** (2022), no. 12, 2407–2459 Zbl 1504.46040 MR 4484209

[27] G. A. Freĭman, The addition of finite sets. I. *Izv. Vysš. Učebn. Zaved. Matematika* **1959** (1959), no. 6 (13), 202–213 Zbl 0096.25904 MR 0126388

[28] N. Fusco, F. Maggi, and A. Pratelli, The sharp quantitative isoperimetric inequality. *Ann. of Math. (2)* **168** (2008), no. 3, 941–980 Zbl 1187.52009 MR 2456887

[29] R. J. Gardner, The Brunn–Minkowski inequality. *Bull. Amer. Math. Soc. (N.S.)* **39** (2002), no. 3, 355–405 Zbl 1019.26008 MR 1898210

[30] D. Ghilli and P. Salani, Quantitative Borell–Brascamp–Lieb inequalities for power concave functions. *J. Convex Anal.* **24** (2017), no. 3, 857–888 Zbl 1385.49004 MR 3684805

[31] N. Gozlan, The deficit in the Gaussian log-Sobolev inequality and inverse Santaló inequalities. *Int. Math. Res. Not. IMRN* (2022), no. 17, 13396–13446 Zbl 07582346 MR 4475270

[32] A. V. Kolesnikov and E. D. Kosov, Moment measures and stability for Gaussian inequalities. *Theory Stoch. Process.* **22** (2017), no. 2, 47–61 Zbl 1413.28025 MR 3843524

[33] A. V. Kolesnikov and E. Milman, Local L^p-Brunn–Minkowski inequalities for $p < 1$. *Mem. Amer. Math. Soc.* **277** (2022), no. 1360 Zbl 1502.52002 MR 4438690

[34] A. V. Kolesnikov and E. M. Werner, Blaschke–Santaló inequality for many functions and geodesic barycenters of measures. *Adv. Math.* **396** (2022), article no. 108110 Zbl 1482.52004 MR 4370472

[35] L. Leindler, On a certain converse of Hölder's inequality. II. *Acta Sci. Math. (Szeged)* **33** (1972), no. 3-4, 217–223 Zbl 0245.26011 MR 2199372

[36] G. V. Livshyts, On a conjectural symmetric version of Ehrhard's inequality. 2022, arXiv:2103.11433

[37] A. Marsiglietti, Borell's generalized Prékopa–Leindler inequality: A simple proof. *J. Convex Anal.* **24** (2017), no. 3, 807–817 Zbl 1387.28005 MR 3684803

[38] V. H. Nguyen, New approach to the affine Pólya–Szegö principle and the stability version of the affine Sobolev inequality. *Adv. Math.* **302** (2016), 1080–1110 Zbl 1355.26018 MR 3545947

[39] P. Pivovarov and J. Rebollo Bueno, A stochastic Prékopa–Leindler inequality for log-concave functions. *Commun. Contemp. Math.* **23** (2021), no. 2, article no. 2050019 Zbl 1461.52006 MR 4201030

[40] A. Prékopa, Logarithmic concave measures with application to stochastic programming. *Acta Sci. Math. (Szeged)* **32** (1971), 301–316 Zbl 0235.90044 MR 315079

[41] A. Prékopa, On logarithmic concave measures and functions. *Acta Sci. Math. (Szeged)* **34** (1973), 335–343 Zbl 0264.90038 MR 404557

[42] A. Rossi and P. Salani, Stability for Borell–Brascamp–Lieb inequalities. In *Geometric aspects of functional analysis*, pp. 339–363, Lecture Notes in Math. 2169, Springer, Cham, 2017 Zbl 1366.60060 MR 3645132

[43] A. Rossi and P. Salani, Stability for a strengthened Borell–Brascamp–Lieb inequality. *Appl. Anal.* **98** (2019), no. 10, 1773–1784 Zbl 1428.26052 MR 3977216

[44] R. Schneider, *Convex bodies: The Brunn–Minkowski theory*. Expanded edn., Encycl. Math. Appl. 151, Cambridge University Press, Cambridge, 2014 Zbl 1287.52001 MR 3155183

[45] P. van Hintum, H. Spink, and M. Tiba, Sharp stability of Brunn–Minkowski for homothetic regions. *J. Eur. Math. Soc. (JEMS)* **24** (2022), no. 12, 4207–4223 Zbl 1501.52007 MR 4493623

[46] P. van Hintum, H. Spink, and M. Tiba, Sharp stability of Brunn–Minkowski for homothetic regions. *J. Eur. Math. Soc. (JEMS)* **24** (2022), no. 12, 4207–4223 Zbl 1501.52007 MR 4493623

[47] T. Wang, The affine Pólya–Szegö principle: Equality cases and stability. *J. Funct. Anal.* **265** (2013), no. 8, 1728–1748 Zbl 1291.46035 MR 3079233

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