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Local minimality of $\mathbb{R}^N$-valued and $\mathbb{S}^N$-valued Ginzburg–Landau vortex solutions in the unit ball B^N

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Abstract. We study the existence, uniqueness and minimality of critical points of the form $m_{\varepsilon,\eta}(x) = (f_{\varepsilon,\eta}(|x|)\frac{x}{|x|}, g_{\varepsilon,\eta}(|x|))$ of the functional $E_{\varepsilon,\eta}[m] = \int_{B^N} [\frac{1}{2}|\nabla m|^2 + \frac{1}{4\varepsilon^2}(1-|m|^2)^2 + \frac{1}{2\eta^2} m_{N+1}^2]\,dx$ for $m = (m_1, \dots, m_N, m_{N+1}) \in H^1(B^N, \mathbb{R}^{N+1})$ with $m(x) = (x, 0)$ on ∂B^N. We establish a necessary and sufficient condition on the dimension N and the parameters ε and η for the existence of an escaping vortex solution $(f_{\varepsilon,\eta}, g_{\varepsilon,\eta})$ with $g_{\varepsilon,\eta} > 0$. We also establish its uniqueness and local minimality. In particular, when $\eta = 0$, we prove the local minimality of the degree-one vortex solution for the Ginzburg–Landau (GL) energy for every $\varepsilon > 0$ and $N \geq 2$. Similarly, when $\varepsilon = 0$, we prove the local minimality of the degree-one escaping vortex solution to an $\mathbb{S}^N$-valued GL model in micromagnetics for all $\eta > 0$ and $2 \leq N \leq 6$.

Contents

1. Introduction

The minimality of the degree-one vortex solution for the Ginzburg–Landau system in the unit ball $B^N \subset \mathbb{R}^N$ in dimension $2 \leq N \leq 6$ is an important open question for which a rich literature is available. In dimension $N \geq 7$, this has been proved recently in a joint work of the authors with Slastikov and Zarnescu [28]. In this paper, we address the local minimality of this solution. Motivated by the theory of magnetic materials, we also consider the local minimality of a similar vortex structure taking values into the unit sphere $\mathbb{S}^N$.

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Our strategy is to treat the local minimality of the vortex solution for an extended model of which the previous two models are special limit cases.

We introduce first the Ginzburg–Landau (GL) functional

$$E_\varepsilon^{\mathrm{GL}}[u] = \int_{B^N} \Big[\frac{1}{2}|\nabla u|^2 + \frac{1}{2\varepsilon^2} W(1-|u|^2)\Big]\,dx,$$

where $\varepsilon > 0$, $W(t) = \frac{t^2}{2}$ and u belongs to the set

$$\mathcal{A}^{\mathrm{GL}} = \big\{u \in H^1(B^N, \mathbb{R}^N) : u(x) = x \text{ on } \partial B^N\big\}.$$

The functional $E_\varepsilon^{\mathrm{GL}}$ has a unique radially symmetric critical point of the form (see Definition A.1 and Lemma A.4)

$$u_\varepsilon(x) = f_\varepsilon(r) n(x) \in \mathcal{A}^{\mathrm{GL}}, \quad n(x) = \frac{x}{r}, \quad r = |x|, \tag{1.1}$$

where the profile f_ε is the unique solution to the ODE (see e.g. [22, 24])

$$f_\varepsilon'' + \frac{N-1}{r} f_\varepsilon' - \frac{N-1}{r^2} f_\varepsilon = -\frac{1}{\varepsilon^2} W'(1-f_\varepsilon^2) f_\varepsilon \quad \text{in } (0,1), \tag{1.2}$$

$$f_\varepsilon(1) = 1. \tag{1.3}$$

Note that $f_\varepsilon(0) = 0$ (see Lemma A.4). Here, a map $u_{\mathrm{crit}} \in \mathcal{A}^{\mathrm{GL}}$ is said to be a *bounded* critical point of $E_\varepsilon^{\mathrm{GL}}$ if $u_{\mathrm{crit}} \in L^\infty(B^N, \mathbb{R}^N)$ and $\langle DE_\varepsilon^{\mathrm{GL}}[u_{\mathrm{crit}}], \varphi\rangle := \frac{d}{dt}\big|_{t=0} E_\varepsilon^{\mathrm{GL}}[u_{\mathrm{crit}} + t\varphi] = 0$ for all $\varphi \in C_c^\infty(B^N \setminus \{0\}, \mathbb{R}^N)$ (which is dense in $H_0^1(B^N, \mathbb{R}^N)$), and is said to be a *radially symmetric* critical point of $E_\varepsilon^{\mathrm{GL}}$ if u_{crit} is radially symmetric[1] in the sense of Definition A.1 and $\langle DE_\varepsilon^{\mathrm{GL}}[u_{\mathrm{crit}}], \varphi\rangle = 0$ for all $\varphi \in C_c^\infty(B^N \setminus \{0\}, \mathbb{R}^N)$. By Lemma 2.7, radially symmetric critical points of $E_\varepsilon^{\mathrm{GL}}$ are bounded.

The map u_ε in (1.1), called the ($\mathbb{R}^N$*-valued*) *Ginzburg–Landau vortex solution* of topological degree one, can be considered a regularization of the singular harmonic map $n: B^N \to \mathbb{S}^{N-1}$ given by $n(x) = \frac{x}{|x|}$ for every $x \in B^N$, which is the unique minimizing $\mathbb{S}^{N-1}$-valued harmonic map for $N \geq 3$ within the boundary condition $n(x) = x$ on ∂B^N (see Brezis, Coron and Lieb [9] and Lin [34]). It is not hard to see that, when ε is sufficiently large, $E_\varepsilon^{\mathrm{GL}}$ is strictly convex and so u_ε is the unique bounded critical point of $E_\varepsilon^{\mathrm{GL}}$ in $\mathcal{A}^{\mathrm{GL}}$ for every $N \geq 2$ (see e.g. [7] or [29, Remark 3.3]). In dimension $N = 2$, Pacard and Rivière showed in [39] that, for small $\varepsilon > 0$, u_ε is the unique critical point of $E_\varepsilon^{\mathrm{GL}}$ in $\mathcal{A}^{\mathrm{GL}}$; however, whether u_ε is the unique minimizer of $E_\varepsilon^{\mathrm{GL}}$ for all $\varepsilon > 0$ remains an open question. In dimensions $N \geq 7$, it was shown in a recent work of Ignat, Nguyen, Slastikov and Zarnescu [28] that u_ε is the unique minimizer of $E_\varepsilon^{\mathrm{GL}}$ in $\mathcal{A}^{\mathrm{GL}}$ *for every* $\varepsilon > 0$. It is not known whether u_ε minimizes $E_\varepsilon^{\mathrm{GL}}$ in $\mathcal{A}^{\mathrm{GL}}$ in dimensions $3 \leq N \leq 6$ when ε is small.

[1] By Lemma A.2, radially symmetric maps in $H^1(B^N, \mathbb{R}^N)$ belong to $L^\infty_{\mathrm{loc}}(\bar{B}^N \setminus \{0\}, \mathbb{R}^N)$.

A different way to regularize the singular harmonic map n is to add an $(N+1)$st direction in the target space, while keeping the constraint of unit length, and to minimize

$$E_\eta^{\mathrm{MM}}[m] = \int_{B^N} \Big[\frac{1}{2}|\nabla m|^2 + \frac{1}{2\eta^2}\widetilde{W}(m_{N+1}^2)\Big]\,dx$$

where $\eta > 0$, $\widetilde{W}(t) = t$ and m belongs to

$$\mathcal{A}^{\mathrm{MM}} = \{m \in H^1(B^N, \mathbb{S}^N) : m(x) = (x, 0) \text{ on } \partial B^N\}.$$

This model comes from micromagnetics, where the order parameter m stands for the magnetization in ferromagnetic materials (see [18]),[2] and also the Oseen–Frank theory for nematic liquid crystals (see [5]). Considering radially symmetric critical points of E_η^{MM} in $\mathcal{A}^{\mathrm{MM}}$, one is led to (see Appendix A)

$$m_\eta(x) = (\tilde{f}_\eta(r)n(x), g_\eta(r)) \in \mathcal{A}^{\mathrm{MM}}, \tag{1.4}$$

where the radial profiles $\tilde{f}_\eta$ and g_η satisfy

$$\tilde{f}_\eta^2 + g_\eta^2 = 1 \quad \text{in } (0,1), \tag{1.5}$$

and the system of ODEs

$$\tilde{f}_\eta'' + \frac{N-1}{r}\tilde{f}_\eta' - \frac{N-1}{r^2}\tilde{f}_\eta = -\lambda(r)\tilde{f}_\eta \qquad \text{in } (0,1), \tag{1.6}$$

$$g_\eta'' + \frac{N-1}{r}g_\eta' = \frac{1}{\eta^2}\widetilde{W}'(g_\eta^2)g_\eta - \lambda(r)g_\eta \quad \text{in } (0,1), \tag{1.7}$$

$$\tilde{f}_\eta(1) = 1 \quad \text{and} \quad g_\eta(1) = 0, \tag{1.8}$$

where

$$\lambda(r) = (\tilde{f}_\eta')^2 + \frac{N-1}{r^2}\tilde{f}_\eta^2 + (g_\eta')^2 + \frac{1}{\eta^2}\widetilde{W}'(g_\eta^2)g_\eta^2 \tag{1.9}$$

is the Lagrange multiplier due to the unit length constraint in $\mathcal{A}^{\mathrm{MM}}$.

Remark 1.1. We will see in Lemma A.6 that solutions to (1.4)–(1.8) satisfy the dichotomy that either $\tilde{f}_\eta(0) = 0$ or $\tilde{f}_\eta(0) = 1$. Furthermore, in the latter case, it holds that $N \geq 3$ and $(\tilde{f}_\eta = 1, g_\eta = 0)$ in $(0,1)$, which corresponds to the *equator map*

$$\bar{m}(x) := (n(x), 0).$$

In dimension $N \geq 7$, $\bar{m}$ is the unique minimizing harmonic map from B^N into $\mathbb{S}^N$ in $\mathcal{A}^{\mathrm{MM}}$ (Jäger and Kaul [30]; see also [29, Example 1.6]), and so is the unique minimizer of E_η^{MM} in $\mathcal{A}^{\mathrm{MM}}$ *for every* $\eta > 0$.

[2] There is also a thin-film regime different from [18] where in the reduced micromagnetic model in dimension $N = 2$ (see e.g. [14, Section 4.5] or [23, Section 7]), after a rotation by $\frac{\pi}{2}$ in the first two components, the condition $\nabla \times (m_1, m_2) = 0$ is imposed in the space of admissible configurations in $\mathcal{A}^{\mathrm{MM}}$. Note that the vortex solution m_η in (1.4) satisfies the above curl-free condition and we will prove its local minimality in the larger class of H_0^1 perturbations (that are not necessarily curl-free in the in-plane components). A related model appears in the study of the cross-tie walls; see [4].

We will focus on "escaping" solutions $m_\eta(x) = (\tilde{f}_\eta(r)n(x), \pm g_\eta(r))$ satisfying $g_\eta > 0$ in $(0,1)$ which exist *only* in dimensions $2 \le N \le 6$ (see Theorem 2.6). More precisely, we will show in these dimensions that, *for every* $\eta > 0$, there exists a unique solution $(\tilde{f}_\eta, g_\eta)$ with $g_\eta > 0$ in $(0,1)$ of the system (1.5)–(1.8), and we call the two configurations $m_\eta = (\tilde{f}_\eta(r)n(x), \pm g_\eta(r)) \in \mathcal{A}^{\mathrm{MM}}$ the *escaping ($\mathbb{S}^N$-valued) Ginzburg–Landau vortex solutions*, or simply the *micromagnetic vortex solutions*. In addition, the micromagnetic vortex solutions m_η have lower energy than the equator map; in particular, the equator map is *no longer* a minimizer of E_η^{MM} in $\mathcal{A}^{\mathrm{MM}}$ (see Proposition 2.15). It is not known whether the micromagnetic vortex solutions m_η minimize E_η^{MM} in $\mathcal{A}^{\mathrm{MM}}$ in dimension $2 \le N \le 6$.

The goal of this paper is to study the local minimality of the vortex solutions u_ε and m_η with respect to $E_\varepsilon^{\mathrm{GL}}$ over the set $\mathcal{A}^{\mathrm{GL}}$ and E_η^{MM} over the set $\mathcal{A}^{\mathrm{MM}}$ respectively. We will in fact consider C^2 potentials $W\colon (-\infty, 1] \to [0,\infty)$ and $\widetilde{W}\colon [0,\infty) \to [0,\infty)$ more general than the ones described above. We make the following assumptions:

$$W(0) = 0, \quad W(t) \ge 0, \quad W''(t) \ge 0 \quad \text{in } (-\infty, 1] \setminus \{0\}, \tag{1.10}$$

$$\widetilde{W}(0) = 0, \quad \widetilde{W}(t) \ge 0, \quad \widetilde{W}''(t) \ge 0 \quad \text{in } (0,\infty). \tag{1.11}$$

We point out that (1.10) implies that $W'(0) = 0$ and $tW'(t) \ge 0$ in $(-\infty, 1] \setminus \{0\}$. Likewise, (1.11) implies that $\widetilde{W}'(0) \ge 0$ and $\widetilde{W}'(t) \ge 0$ in $(0,\infty)$. However, we allow the possibility that W or $\widetilde{W}$ is zero in a neighborhood of the origin. This leads to new difficulties as well as new behaviors of solutions; see for example Proposition B.1 (ii).

Under assumptions (1.10) and (1.11) for W and $\widetilde{W}$, we will prove the existence and uniqueness of the radial profiles f_ε and $(\tilde{f}_\eta, g_\eta)$ with $g_\eta > 0$ solving (1.1)–(1.3) and (1.4)–(1.8), respectively. See Theorems 2.1 and 2.6, where the global minimality of these solutions in the class of radial symmetric maps is also established. For these unique radial profiles, we will continue to refer to the maps $u_\varepsilon(x) = f_\varepsilon(|x|)n(x)$ and $m_\eta(x) = (\tilde{f}_\eta(r)n(x), g_\eta(r))$ as the $\mathbb{R}^N$-valued and $\mathbb{S}^N$-valued Ginzburg–Landau vortex solutions. Our main results concern the local minimizing property of these vortex solutions, in particular the positive-definiteness of the second variation at those solutions (see Section 3 for the definition).

Theorem 1.2. *Let $W \in C^2((-\infty, 1])$ satisfy (1.10). For $N \ge 2$ and every $\varepsilon > 0$, the $\mathbb{R}^N$-valued Ginzburg–Landau vortex solution $u_\varepsilon(x) = f_\varepsilon(r)n(x)$ is a local minimizer of $E_\varepsilon^{\mathrm{GL}}$ in $\mathcal{A}^{\mathrm{GL}}$ with a positive definite second variation.*

Theorem 1.3. *Let $\widetilde{W} \in C^2([0,\infty))$ satisfy (1.11). For $2 \le N \le 6$ and every $\eta > 0$, the escaping $\mathbb{S}^N$-valued Ginzburg–Landau vortex solution $m_\eta(x) = (\tilde{f}_\eta(r)n(x), g_\eta(r))$ with $g_\eta > 0$ is a local minimizer m_η of E_η^{MM} in $\mathcal{A}^{\mathrm{MM}}$ with a positive definite second variation. For $3 \le N \le 6$ and every $\eta > 0$, the equator map $\bar{m} = (n(x), 0)$ is an unstable critical point of E_η^{MM} in $\mathcal{A}^{\mathrm{MM}}$ and $E_\eta^{\mathrm{MM}}(m_\eta) < E_\eta^{\mathrm{MM}}(\bar{m})$.*

Remark 1.4. (a) In Theorem 1.3, we can replace (1.11) by

$$\widetilde{W} \in C^2([0,1]), \quad \widetilde{W}(0) = 0, \quad \widetilde{W}(t) \ge 0, \quad \widetilde{W}''(t) \ge 0 \quad \text{in } [0,1],$$

since any such function $\widetilde{W}$ can be extended to a function satisfying (1.11).

(b) In dimension $N = 2$, the equator map $\bar{m} \notin H^1(B^N, \mathbb{S}^N)$, so $\bar{m} \notin \mathcal{A}^{\mathrm{MM}}$. However, the second variation of E_η^{MM} at $\bar{m}$ can still be defined and it is negative in a certain direction compactly supported in $B^N \setminus \{0\}$, leading to the instability of $\bar{m}$ for $N = 2$ also (see (2.27)).

In the $\mathbb{R}^N$-valued Ginzburg–Landau case, when $N = 2$, Theorem 1.2 was proved by Mironescu [36] for $W(t) = \frac{t^2}{2}$. Also when $N = 2$, the non-negativity of the second variation was proved by Lieb and Loss [33] for potentials W which are strictly increasing and convex[3] in $[0, 1]$. In dimension $N \geq 7$, the global minimality of the vortex solution was proved by Ignat, Nguyen, Slastikov and Zarnescu [28, 29]. When the domain is $\mathbb{R}^N$ (instead of B^N), the local minimality of the entire vortex solution (in the sense of De Giorgi) was obtained in Mironescu [37] for $N = 2$, Millot and Pisante [35] for $N = 3$ and Pisante [40] for $N \geq 4$. For the stability of the entire vortex solution, see Ovchinnikov and Sigal [38], del Pino, Felmer and Kowalczyk [13] for $N = 2$ and Gustafson [19] for $N \geq 3$.

In the micromagnetic case, in dimension $N = 2$ and for $\widetilde{W}(t) = t$, Theorem 1.3 was proved by Hang and Lin [20]. For dimension $N \geq 7$, see Remark 1.1. See also Li and Melcher [32] for related stability analysis in the study of micromagnetics skyrmions.

More generally, we consider a family of extended energy functionals $E_{\varepsilon,\eta}$ depending on two positive parameters ε, η of which $E_\varepsilon^{\mathrm{GL}}$ and E_η^{MM} are limiting cases:

$$E_{\varepsilon,\eta}[m] = \int_{B^N} \Big[\frac{1}{2}|\nabla m|^2 + \frac{1}{2\varepsilon^2} W(1 - |m|^2) + \frac{1}{2\eta^2}\widetilde{W}(m_{N+1}^2)\Big]\, dx, \quad \varepsilon, \eta > 0,$$

where W and $\widetilde{W}$ satisfy (1.10)–(1.11) and m belongs to

$$\mathcal{A} = \big\{m \in H^1(B^N, \mathbb{R}^{N+1}) : m(x) = (x, 0) \text{ on } \partial B^N\big\}.$$

Under suitable conditions on $\widetilde{W}$ (e.g. $\widetilde{W}(t) > 0$ for $t > 0$), it can be shown that for a fixed $\varepsilon > 0$, minimizers of $E_{\varepsilon,\eta}$ in $\mathcal{A}$ converge in H^1 to minimizers of $E_\varepsilon^{\mathrm{GL}}$ in $\mathcal{A}^{\mathrm{GL}}$ as $\eta \to 0$. Likewise, under suitable conditions on W, for a fixed $\eta > 0$, minimizers of $E_{\varepsilon,\eta}$ in $\mathcal{A}$ converge in H^1 to minimizers of E_η^{MM} in $\mathcal{A}^{\mathrm{MM}}$ as $\varepsilon \to 0$. We hope that having a good understanding of critical points of $E_{\varepsilon,\eta}$ will lead to new insights on the open problem concerning the minimality of the vortex solutions u_ε and m_η.

We define a map $m_{\mathrm{crit}} \in \mathcal{A}$ to be a *bounded* critical point of $E_{\varepsilon,\eta}$ if $m_{\mathrm{crit}} \in L^\infty(B^N, \mathbb{R}^{N+1})$ and $\langle DE_{\varepsilon,\eta}[m_{\mathrm{crit}}], \varphi\rangle := \frac{d}{dt}\big|_{t=0} E_{\varepsilon,\eta}[m_{\mathrm{crit}} + t\varphi] = 0$ for all $\varphi \in C_c^\infty(B^N \setminus \{0\}, \mathbb{R}^{N+1})$, and to be a *radially symmetric* critical point of $E_{\varepsilon,\eta}$ if m_{crit} is radially symmetric in the sense of Definition A.1 and $\langle DE_{\varepsilon,\eta}[m_{\mathrm{crit}}], \varphi\rangle = 0$ for all $\varphi \in C_c^\infty(B^N \setminus \{0\}, \mathbb{R}^{N+1})$. By Lemma 2.7, radially symmetric critical points of $E_{\varepsilon,\eta}$ are bounded. Radially symmetric critical points of $E_{\varepsilon,\eta}$ in $\mathcal{A}$ take the form

$$(f_{\varepsilon,\eta}(r)n(x), g_{\varepsilon,\eta}(r)) \in \mathcal{A}, \tag{1.12}$$

[3] See Remark 3.5 for a related comment for $E_{\varepsilon,\eta}$.

where $(f_{\varepsilon,\eta}, g_{\varepsilon,\eta})$ satisfies the system of ODEs

$$f''_{\varepsilon,\eta} + \frac{N-1}{r} f'_{\varepsilon,\eta} - \frac{N-1}{r^2} f_{\varepsilon,\eta} = -\frac{1}{\varepsilon^2} W'(1 - f^2_{\varepsilon,\eta} - g^2_{\varepsilon,\eta}) f_{\varepsilon,\eta}, \tag{1.13}$$

$$g''_{\varepsilon,\eta} + \frac{N-1}{r} g'_{\varepsilon,\eta} = -\frac{1}{\varepsilon^2} W'(1 - f^2_{\varepsilon,\eta} - g^2_{\varepsilon,\eta}) g_{\varepsilon,\eta} + \frac{1}{\eta^2} \widetilde{W}'(g^2_{\varepsilon,\eta}) g_{\varepsilon,\eta}, \tag{1.14}$$

$$f_{\varepsilon,\eta}(1) = 1 \quad \text{and} \quad g_{\varepsilon,\eta}(1) = 0. \tag{1.15}$$

Note that the above implies $f_{\varepsilon,\eta}(0) = 0$ and $g'_{\varepsilon,\eta}(0) = 0$ (see Lemma A.5).

Of special interest to our discussion will be solutions to (1.12)–(1.15) satisfying the sign constraint $g_{\varepsilon,\eta} \geq 0$ in $(0, 1)$. It is easy to see by the strong maximum principle that either $g_{\varepsilon,\eta} \equiv 0$ or $g_{\varepsilon,\eta} > 0$ in $(0, 1)$. When $g_{\varepsilon,\eta} \equiv 0$, we obtain an η-independent solution given by $(f_\varepsilon, 0)$, where f_ε is the unique radial profile in (1.1)–(1.3). We will sometimes refer to $(f_\varepsilon, 0)$ as the *non-escaping solution* to (1.12)–(1.15) and

$$\bar{m}_\varepsilon(x) = (f_\varepsilon(r)n(x), 0)$$

as the *non-escaping (radially symmetric) critical point* of the extended energy functional $E_{\varepsilon,\eta}$ in $\mathcal{A}$. In contrast, we refer to solutions $(f_{\varepsilon,\eta}, g_{\varepsilon,\eta})$ of (1.12)–(1.15) with $g_{\varepsilon,\eta} > 0$ as *escaping solutions* and the corresponding maps

$$m_{\varepsilon,\eta}(x) = (f_{\varepsilon,\eta}(r)n(x), \pm g_{\varepsilon,\eta}(r))$$

as[4] *escaping (radially symmetric) critical points* of the extended energy functional $E_{\varepsilon,\eta}$ in $\mathcal{A}$. The escaping phenomenon[5] refers to the positivity of $g_{\varepsilon,\eta}$. We will prove that such escaping solutions satisfy $f_{\varepsilon,\eta} > 0$ in $(0, 1)$; see Proposition 2.10.

There exists a sufficiently large ε_* such that $E_{\varepsilon,\eta}$ is strictly convex for all $\varepsilon > \varepsilon_*$ and $\eta > 0$ and so $\bar{m}_\varepsilon$ is the unique critical point and hence the unique global minimizer of $E_{\varepsilon,\eta}$ in $\mathcal{A}$ if $N \geq 2$. In dimensions $N \geq 7$, it follows from [28, Theorem 2][6] (compare [29, Theorem 1.7]) that $\bar{m}_\varepsilon(x)$ is the unique global minimizer of $E_{\varepsilon,\eta}$ in $\mathcal{A}$ for every $\varepsilon > 0$. In dimension $2 \leq N \leq 6$ and for small $\varepsilon > 0$, it is not known whether a solution to (1.12)–(1.15) satisfying $g_{\varepsilon,\eta} \geq 0$ gives a global minimizer of $E_{\varepsilon,\eta}$ in $\mathcal{A}$. Our next theorem concerns the existence, uniqueness and local minimality of these solutions. See Figure 1.

Theorem 1.5. *Let $N \geq 2$, $W \in C^2((-\infty, 1])$ and $\widetilde{W} \in C^2([0, \infty))$ satisfy* (1.10) *and* (1.11).

(a) *There is at most one escaping critical point $m_{\varepsilon,\eta}(x) = (f_{\varepsilon,\eta}(r)n(x), g_{\varepsilon,\eta}(r))$ of $E_{\varepsilon,\eta}$ in $\mathcal{A}$ with $g_{\varepsilon,\eta} > 0$. Moreover, if such an escaping critical point exists, then it is a local minimizer of $E_{\varepsilon,\eta}$ in $\mathcal{A}$ with a positive definite second variation, and the non-escaping critical point $\bar{m}_\varepsilon(x) = (f_\varepsilon(r)n(x), 0)$ is unstable for $E_{\varepsilon,\eta}$.*

[4] When discussing escaping and non-escaping critical points, we will drop the term "radially symmetric" as here we only study radially symmetric critical points.

[5] For more about escaping phenomena in the context of harmonic maps, see e.g. [6].

[6] In [28], besides the convexity of W, it is assumed that W is positive away from 0; but it can be seen from the proof there that non-negativity $W \geq 0$ is sufficient, as in (1.10).

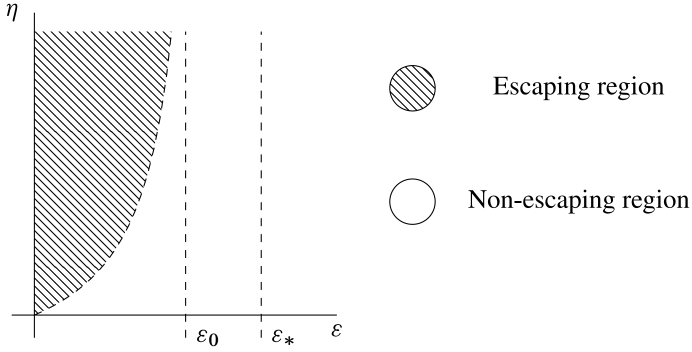

Figure 1. Radial critical points of the extended functional $E_{\varepsilon,\eta}$ when $W'(1) > 0$ and $\widetilde{W}'(0) > 0$. In the escaping region, there is a co-existence of non-escaping and escaping critical points. In the non-escaping region, only the non-escaping critical point exists.

(b) *An escaping critical point* $m_{\varepsilon,\eta}(x) = (f_{\varepsilon,\eta}(r)n(x), g_{\varepsilon,\eta}(r))$ *with* $g_{\varepsilon,\eta} > 0$ *exists if and only if* $2 \le N \le 6$, $W'(1) > 0$, $0 < \varepsilon < \varepsilon_0$ *and* $\eta > \eta_0(\varepsilon)$ *for some* $\varepsilon_0 \in (0,\infty)$ *and a continuous non-decreasing function*[7] $\eta_0\colon [0,\varepsilon_0) \to [0,\infty)$ *with* $\eta_0(0) = 0$.

(c) *In the absence of an escaping critical point* $m_{\varepsilon,\eta}(x) = (f_{\varepsilon,\eta}(r)n(x), g_{\varepsilon,\eta}(r))$ *with* $g_{\varepsilon,\eta} > 0$ *for* $E_{\varepsilon,\eta}$, *the non-escaping solution* $\overline{m}_\varepsilon(x) = (f_\varepsilon(r)n(x), 0)$ *is a local minimizer of* $E_{\varepsilon,\eta}$ *in* $\mathcal{A}$ *with a positive definite second variation unless* $2 \le N \le 6$, $W'(1) > 0$, $\widetilde{W}'(0) > 0$, $0 < \varepsilon < \varepsilon_0$ *and* $\eta = \eta_0(\varepsilon)$. *Moreover, in the latter case, the second variation of* $E_{\varepsilon,\eta}$ *at* $\overline{m}_\varepsilon$ *is non-negative semi-definite with a one-dimensional kernel generated by* $(0, q_\varepsilon) \in C^2(\overline{B}^N, \mathbb{R}^{N+1})$ *for some positive smooth function* $q_\varepsilon > 0$ *in* B^N *with* $q_\varepsilon = 0$ *on* ∂B^N.

A main part of our paper concerns the local minimality of vortex solutions. Let us explain our strategy for the Ginzburg–Landau model. We establish

$$E_\varepsilon^{\mathrm{GL}}[u_\varepsilon + v] \ge E_\varepsilon^{\mathrm{GL}}[u_\varepsilon] + c\|v\|_{H^1}^2 \quad \text{for } u_\varepsilon + v \in \mathcal{A}^{\mathrm{GL}},\ \|v\|_{H^1} < \delta,$$

for some small $c > 0$ and $\delta > 0$. This draws on a careful study of the second variation of $E_\varepsilon^{\mathrm{GL}}$ at u_ε based on a separation of variables and a Hardy decomposition technique [25]. To separate variables, we first decompose $v = sn + w$, where $w \cdot n = 0$, and then, for each $0 < r < 1$, we use the Helmholtz decomposition to write $w = \mathring{w} + \nabla\!\!\!/\,\psi$ on ∂B_r, where $\mathring{w}$ is a divergence-free vector field on ∂B_r and $\nabla\!\!\!/$ is the gradient operator. In the context of Ginzburg–Landau theory, our use of the Helmholtz decomposition appears new in dimension $N \ge 3$. The contribution of $\mathring{w}$ to the second variation is treated at once using the sharp Poincaré inequality in Appendix C and the Hardy decomposition technique.

[7]For more about ε_0 and η_0, see Lemma 2.3 (c) and Remark 2.5.

Finally, we decompose s and ψ into spherical harmonics and treat them using the Hardy decomposition technique again, with special choices of factoring functions.

An important point in proving our results resides in the analysis of the radial profiles f_ε, $(\tilde{f}_\eta, g_\eta)$ and $(f_{\varepsilon,\eta}, g_{\varepsilon,\eta})$ for general potentials W and $\widetilde{W}$ that goes beyond the existing (very rich) literature. For example, the choice of factoring functions in our use of the Hardy decomposition technique is based on the positivity and monotonicity of (a priori, nodal solutions) f_ε, $\tilde{f}_\eta$ and $f_{\varepsilon,\eta}$. The proof uses the moving plane method for cooperative systems [12, 16, 42]. A novel part of our argument is in the fact that cooperativity is obtained alongside the application of the moving plane method. Another issue is the uniqueness of the radial profiles, which is proved again using the Hardy decomposition technique which handles the non-linear part in the ODE. This analysis enables us to show the dichotomy of escaping vs. non-escaping critical points in the extended model introduced here for the first time.

The rest of the paper is organized as follows. In Section 2 we establish the existence and uniqueness of vortex radial profiles and discuss their minimality within radially symmetric configurations. In Section 3 we analyze their stability and give the proof of the main theorems. We also include four appendices on some miscellaneous results.

2. Existence and uniqueness of vortex radial profiles

We study existence and uniqueness properties of radially symmetric critical points of $E_\varepsilon^{\mathrm{GL}}$, E_η^{MM} and $E_{\varepsilon,\eta}$. We define the following reduced energy functionals relevant in the discussion of radially symmetric critical points in $\mathcal{A}^{\mathrm{GL}}$, $\mathcal{A}^{\mathrm{MM}}$ and $\mathcal{A}$ (see Appendix A):

- the reduced $\mathbb{R}^N$-valued Ginzburg–Landau functional

$$\begin{aligned} I_\varepsilon^{\mathrm{GL}}[f] &= \frac{1}{|\mathbb{S}^{N-1}|} E_\varepsilon^{\mathrm{GL}}[f(|x|)n(x)] \\ &= \frac{1}{2}\int_0^1 \Big[(f')^2 + \frac{N-1}{r^2} f^2 + \frac{1}{\varepsilon^2} W(1-f^2)\Big] r^{N-1}\, dr \end{aligned}$$

  where f belongs to $\mathcal{B}^{\mathrm{GL}} = \{f : r^{\frac{N-1}{2}} f',\ r^{\frac{N-3}{2}} f \in L^2(0,1),\ f(1) = 1\}$;

- the reduced $\mathbb{S}^N$-valued Ginzburg–Landau functional

$$\begin{aligned} I_\eta^{\mathrm{MM}}[f,g] &= \frac{1}{|\mathbb{S}^{N-1}|} E_\eta^{\mathrm{MM}}[(f(r)n(x), g(r))] \\ &= \frac{1}{2}\int_0^1 \Big[(f')^2 + (g')^2 + \frac{N-1}{r^2} f^2 + \frac{1}{\eta^2} \widetilde{W}(g^2)\Big] r^{N-1}\, dr, \end{aligned}$$

  where (f,g) belongs to $\mathcal{B}^{\mathrm{MM}} = \{(f,g) : r^{\frac{N-1}{2}} f',\ r^{\frac{N-3}{2}} f,\ r^{\frac{N-1}{2}} g',\ r^{\frac{N-1}{2}} g \in L^2(0,1),$ $f^2 + g^2 = 1,\ f(1) = 1,\ g(1) = 0\}$;

- the reduced extended functional

$$
\begin{aligned}
I_{\varepsilon,\eta}[f,g] &= \frac{1}{|\mathbb{S}^{N-1}|} E_{\varepsilon,\eta}[(f(r)n(x), g(r))] \\
&= \frac{1}{2}\int_0^1 \Big[(f')^2 + (g')^2 + \frac{N-1}{r^2} f^2 + \frac{1}{\varepsilon^2} W(1-f^2-g^2) \\
&\qquad + \frac{1}{\eta^2}\widetilde{W}(g^2)\Big] r^{N-1}\,dr,
\end{aligned}
$$

where (f,g) belongs to $\mathcal{B} = \{(f,g) : r^{\frac{N-1}{2}} f',\ r^{\frac{N-3}{2}} f,\ r^{\frac{N-1}{2}} g',\ r^{\frac{N-1}{2}} g \in L^2(0,1),$ $f(1) = 1,\ g(1) = 0\}$.

Note that $(f,g) \in \mathcal{B}$ if and only if $m(x) = (f(r)n(x), g(r)) \in H^1(B^N, \mathbb{R}^{N+1})$ with $m(x) = (x,0)$ on ∂B^N, in which case,

$$
\int_{B^N} |\nabla m|^2\,dx = |\mathbb{S}^{N-1}| \int_0^1 \Big[(f')^2 + (g')^2 + \frac{N-1}{r^2} f^2\Big] r^{N-1}\,dr.
$$

It is straightforward to check that bounded critical points of $I_\varepsilon^{\mathrm{GL}}$, I_η^{MM} and $I_{\varepsilon,\eta}$ correspond to bounded radially symmetric critical points of $E_\varepsilon^{\mathrm{GL}}$, E_η^{MM} and $E_{\varepsilon,\eta}$, respectively.[8]

The $\mathbb{R}^N$-valued Ginzburg–Landau model. We prove the following:

Theorem 2.1. *Let $N \geq 2$ and $W \in C^2((-\infty, 1])$ satisfy $W(0) = 0$ and $W \geq 0$. Then, for every $\varepsilon > 0$, (1.2)–(1.3) has a solution f_ε such that $\frac{f_\varepsilon}{r} \in C^2([0,1])$, $0 < f_\varepsilon < 1$ in $(0,1)$ and $f_\varepsilon(0) = 0$. If, in addition, W satisfies (1.10), then $f_\varepsilon' > 0$ in $(0,1]$ and f_ε is the unique solution to (1.1)–(1.3); in particular, f_ε is the unique minimizer of $I_\varepsilon^{\mathrm{GL}}$ in $\mathcal{B}^{\mathrm{GL}}$.*

Remark 2.2. The existence and uniqueness of the vortex radial profile for the $\mathbb{R}^N$-valued Ginzburg–Landau model has been studied by many authors. Closely related to our result above is a result in [28] which gives the uniqueness in dimensions $N \geq 7$. Earlier results in [2, 11, 15, 22, 24] are for all dimensions $N \geq 2$ but assume the inequality $W''(0) > 0$, while Theorem 2.1 above allows the case $W''(0) = 0$.

Let f_ε be the radial profile in Theorem 2.1. Note that $(f_\varepsilon, 0)$ is the non-escaping critical point for the extended functional $I_{\varepsilon,\eta}$ for any $\eta > 0$. For the existence of escaping solutions in the extended model, we give an estimate for the first eigenvalue $\ell(\varepsilon)$ of

$$
L_\varepsilon^{\mathrm{GL}} = -\Delta - \frac{1}{\varepsilon^2} W'(1 - f_\varepsilon^2) \tag{2.1}
$$

in B^N with respect to the zero Dirichlet boundary condition. Note that since the potential $\frac{1}{\varepsilon^2} W'(1-f_\varepsilon^2)$ is radially symmetric, any first eigenfunction of $L_\varepsilon^{\mathrm{GL}}$ is also radially symmetric. It is clear that, under (1.10), we have $\ell(\varepsilon) > -W'(1)\varepsilon^{-2}$ for every $\varepsilon > 0$.

---

[8] In this radially symmetric setting, when W and $\widetilde{W}$ satisfy (1.10) and (1.11), the boundedness assumption on critical points can be dropped, in view of Lemma 2.7.

Lemma 2.3. *Let $W \in C^2((-\infty, 1])$ satisfy (1.10). Then ℓ is a continuous function in ε with*

$$\varepsilon^2 \ell(\varepsilon) > \tilde{\varepsilon}^2 \ell(\tilde{\varepsilon}) \quad \text{for all } 0 < \tilde{\varepsilon} < \varepsilon < \infty, \tag{2.2}$$

and the following estimates hold:

(a) *If $W'(1) = 0$, then $W = 0$ in $(0, 1)$, $L_\varepsilon^{\mathrm{GL}} = -\Delta$ and*

$$\ell(\varepsilon) = \lambda_1(-\Delta) > 0 \quad \text{for all } \varepsilon > 0,$$

where $\lambda_1(-\Delta)$ is the first eigenvalue of the Laplacian on B^N with respect to the zero Dirichlet boundary value.

(b) *If $N \geq 7$,*

$$\ell(\varepsilon) \geq \frac{(N-2)^2}{4} - (N-1) > 0 \quad \text{for all } \varepsilon > 0.$$

(c) *If $2 \leq N \leq 6$ and $W'(1) > 0$, then there exists $\varepsilon_0 \in (0, \infty)$ such that $\ell(\varepsilon) < 0$ and increasing in $(0, \varepsilon_0)$, $\ell(\varepsilon_0) = 0$ and $\ell(\varepsilon) > 0$ in (ε_0, ∞). Furthermore, for some $\varepsilon_1 \in (0, \varepsilon_0)$ and $c_1 \in (0, W'(1))$,*

$$-\frac{W'(1)}{\varepsilon^2} < \ell(\varepsilon) \leq -\frac{c_1}{\varepsilon^2} \quad \text{for } \varepsilon \in (0, \varepsilon_1).$$

The extended model. We are now in position to give a necessary and sufficient condition for the existence of an escaping solution of (1.12)–(1.15). For an illustration see Figure 1.

Theorem 2.4. *Suppose $W \in C^2((-\infty, 1])$ and $\widetilde{W} \in C^2([0, \infty))$ satisfy (1.10) and (1.11).*

(a) *If $N \geq 7$ or $W'(1) = 0$, then for every $\varepsilon, \eta > 0$, (1.12)–(1.15) has no solution $(f_{\varepsilon,\eta}, g_{\varepsilon,\eta})$ that satisfies $g_{\varepsilon,\eta} > 0$ in $(0, 1)$. Moreover, the non-escaping solution $(f_\varepsilon, 0)$ is the unique minimizer of $I_{\varepsilon,\eta}$ in $\mathcal{B}$.*

(b) *Let $2 \leq N \leq 6$, $W'(1) > 0$, $\varepsilon_0 \in (0, \infty)$ be as in Lemma 2.3 and*

$$\eta_0(\varepsilon) = \sqrt{\frac{\widetilde{W}'(0)}{|\ell(\varepsilon)|}} \in [0, \infty) \quad \text{for } \varepsilon \in (0, \varepsilon_0).$$

(b1) *System (1.12)–(1.15) has an escaping solution $(f_{\varepsilon,\eta}, g_{\varepsilon,\eta})$ that satisfies $g_{\varepsilon,\eta} > 0$ in $(0, 1)$ if and only if $0 < \varepsilon < \varepsilon_0$ and $\eta > \eta_0(\varepsilon)$. In this case, it is the unique escaping solution of (1.12)–(1.15), $\frac{f_{\varepsilon,\eta}}{r}, g_{\varepsilon,\eta} \in C^2([0, 1])$, $f_{\varepsilon,\eta}^2 + g_{\varepsilon,\eta}^2 < 1$, $f_{\varepsilon,\eta} > 0$, $f'_{\varepsilon,\eta} > 0$, $g'_{\varepsilon,\eta} < 0$ in $(0, 1)$, and there are exactly two minimizers of $I_{\varepsilon,\eta}$ in $\mathcal{B}$ given by $(f_{\varepsilon,\eta}, \pm g_{\varepsilon,\eta})$.*

(b2) *If $\varepsilon \geq \varepsilon_0$ or $0 < \eta \leq \eta_0(\varepsilon)$, the non-escaping solution $(f_\varepsilon, 0)$ of (1.12)–(1.15) is the unique minimizer of $I_{\varepsilon,\eta}$ in $\mathcal{B}$. Otherwise (i.e. $0 < \varepsilon < \varepsilon_0$ and $\eta > \eta_0(\varepsilon)$), the non-escaping solution $(f_\varepsilon, 0)$ of (1.12)–(1.15) is an unstable critical point of $I_{\varepsilon,\eta}$ in $\mathcal{B}$.*

We note that if $2 \le N \le 6$, $W'(1) > 0$ and $\widetilde{W}'(0) = 0$, then $\eta_0(\varepsilon) = 0$ for all $\varepsilon \in (0, \varepsilon_0)$. In this case, the theorem asserts for all $\eta > 0$, an escaping solution of (1.12)–(1.15) exists if and only if $\varepsilon \in (0, \varepsilon_0)$.

Remark 2.5. By Lemma 2.3, when $2 \le N \le 6$, $W'(1) > 0$ and $\widetilde{W}'(0) > 0$, the function η_0 defined in Theorem 2.4 (b) belongs to $C([0, \varepsilon_0))$, $\frac{\eta_0(\varepsilon)}{\varepsilon}$ is increasing with respect to ε,

$$\lim_{\varepsilon \to \varepsilon_0} \eta_0(\varepsilon) = \infty, \quad \lim_{\varepsilon \to 0} \eta_0(0) = 0,$$

and, for some $C > 1$ and $\varepsilon_1 \in (0, \varepsilon_0)$, $\frac{\sqrt{\widetilde{W}'(0)}\varepsilon}{C} \le \eta_0(\varepsilon) \le C\sqrt{\widetilde{W}'(0)}\varepsilon$ for every $\varepsilon \in (0, \varepsilon_1)$.

Theorem 2.4 can be viewed as an extension of the results in [29] but within radial symmetry, relating the escaping phenomenon with the stability property of critical points.

The $\mathbb{S}^N$-valued Ginzburg–Landau model.

Theorem 2.6. *Suppose that $\widetilde{W} \in C^2([0, \infty))$ satisfies (1.11).*

(a) *If $N \ge 7$, then for every $\eta > 0$, system (1.4)–(1.8) has no escaping solution $(\tilde{f}_\eta, g_\eta)$ with $g_\eta > 0$ in $(0, 1)$.*

(b) *If $2 \le N \le 6$, then for any $\eta > 0$, (1.4)–(1.8) has a unique escaping solution $(\tilde{f}_\eta, g_\eta)$ with $g_\eta > 0$. Also, $(\tilde{f}_\eta, \pm g_\eta)$ are the only two minimizers of the functional I_η^{MM} in $\mathcal{B}^{\mathrm{MM}}$, $\frac{\tilde{f}_\eta}{r}, g_\eta \in C^2([0, 1])$, $\tilde{f}_\eta > 0$, $\tilde{f}_\eta' > 0$ and $g_\eta' < 0$ in $(0, 1)$. In addition, for $3 \le N \le 6$, the non-escaping solution $(1, 0)$ is an unstable critical point of I_η^{MM} in $\mathcal{B}^{\mathrm{MM}}$.*

Recall that, when $N \ge 7$, the non-escaping solution $(1, 0)$ is the unique minimizer of I_η^{MM} in $\mathcal{B}^{\mathrm{MM}}$ for every $\eta > 0$ (see Remark 1.1). Note that when $N = 2$, the non-escaping solution $(1, 0) \notin \mathcal{B}^{\mathrm{MM}}$; however, the second variation of I_η^{MM} at $(1, 0)$ can still be defined and it is negative in a certain direction with compact support in the interval $(0, 1)$, leading to the instability of the non-escaping solution $(1, 0)$ for $N = 2$ also (see (2.27)).

The rest of the section is organized as follows. In Section 2.1, for the extended model, we prove the monotonicity (see Proposition 2.9) and uniqueness (see Proposition 2.12) of escaping solutions (1.12)–(1.15), if they exist, together with the positivity of $f_{\varepsilon,\eta}$ in Proposition 2.10; we also prove the boundedness of arbitrary solutions to (1.12)–(1.15); see Lemma 2.7. In Section 2.2, for the $\mathbb{R}^N$-valued GL model, we give the proofs of Theorem 2.1 and Lemma 2.3. In Section 2.3 we give the proof of Theorem 2.4 for the extended model. Finally, Theorem 2.6 for the $\mathbb{S}^N$-valued GL model is proved in Section 2.4.

2.1. The extended model: Monotonicity and uniqueness

In this subsection we establish the monotonicity and uniqueness of escaping radially symmetric critical points of the extended functional $E_{\varepsilon,\eta}$, which correspond to escaping solutions $(f_{\varepsilon,\eta}, g_{\varepsilon,\eta})$ with $g_{\varepsilon,\eta} > 0$ of the ODE system (1.12)–(1.15). Furthermore, we show that $f_{\varepsilon,\eta} > 0$ and prove the minimality of this escaping solution with respect to radially symmetric competitors.

The next lemma shows that, under (1.10)–(1.11), all solutions to (1.12)–(1.15) are bounded in $(0,1)$. To dispel confusion, in this result, we do not assume a priori the boundedness or the non-negativity of $f_{\varepsilon,\eta}$ and $g_{\varepsilon,\eta}$.

Lemma 2.7. *Let $N \geq 2$, $\varepsilon > 0$, $\eta > 0$, $W \in C^2((-\infty,1])$ and $\widetilde{W} \in C^2([0,\infty))$ with (1.10)–(1.11). If $(f_{\varepsilon,\eta}, g_{\varepsilon,\eta})$ satisfies (1.12)–(1.15), then $f_{\varepsilon,\eta}^2 + g_{\varepsilon,\eta}^2 < 1$ in $(0,1)$ and the map $x \mapsto m_{\varepsilon,\eta}(x) = (f_{\varepsilon,\eta}(r)n(x), g_{\varepsilon,\eta}(r))$ is $C^2(\overline{B}^N)$. In particular, $f_{\varepsilon,\eta}(0) = 0$ and $g'_{\varepsilon,\eta}(0) = 0$.*

Proof. Note that $m_{\varepsilon,\eta} \in H^1(B^N)$ (as $(f_{\varepsilon,\eta}, g_{\varepsilon,\eta}) \in \mathcal{B}$) and, by (1.13)–(1.15),

$$\begin{aligned}
\Delta m_{\varepsilon,\eta} &= -\frac{1}{\varepsilon^2} W'(1 - |m_{\varepsilon,\eta}|^2) m_{\varepsilon,\eta} + \frac{1}{\eta^2} \widetilde{W}'(g_{\varepsilon,\eta}^2) g_{\varepsilon,\eta} e_{N+1} && \text{in } B^N \setminus \{0\}, \qquad (2.3)\\
m_{\varepsilon,\eta}(x) &= (n(x), 0) && \text{on } \partial B^N.
\end{aligned}$$

Let $M = f_{\varepsilon,\eta}^2 + g_{\varepsilon,\eta}^2$. Note that $M(1) = 1$ and

$$\begin{aligned}
&\frac{1}{2}\Big(M'' + \frac{N-1}{r} M'\Big)\\
&\quad = (f'_{\varepsilon,\eta})^2 + (g'_{\varepsilon,\eta})^2 + \frac{N-1}{r^2} f_{\varepsilon,\eta}^2 - \frac{1}{\varepsilon^2} W'(1-M)M + \frac{1}{\eta^2} \widetilde{W}'(g_{\varepsilon,\eta}^2) g_{\varepsilon,\eta}^2\\
&\quad \geq -\frac{1}{\varepsilon^2} W'(1-M)M.
\end{aligned}$$

In particular, the function $X = 1 - M$ satisfies

$$-X'' - \frac{N-1}{r} X' + 2a(r) X \geq 0, \qquad (2.4)$$

where $a\colon (0,1] \to [0,\infty)$ is given by

$$a(r) = \begin{cases} \dfrac{1}{\varepsilon^2} \dfrac{W'(1-M(r))}{1-M(r)} M(r) & \text{if } M(r) \neq 1,\\[2mm] \dfrac{1}{\varepsilon^2} W''(0) & \text{if } M(r) = 1. \end{cases} \qquad (2.5)$$

Note that (1.10) and the continuity of M in $(0,1]$ imply $a \geq 0$ and a is continuous on $(0,1]$. Now define

$$r_0 = \inf\big\{r \in (0,1] : M \leq 1 \text{ in } [r,1]\big\}.$$

The aim is to show that $r_0 = 0$.

Step 1: We show that if $r_0 > 0$, then $M > 1$ in $(0, r_0)$. Assume by contradiction that $M(r_1) \leq 1$ for some $r_1 \in (0, r_0)$. Multiplying (2.4) by $r^{N-1} X^-$ (where $X^\pm = \max\{0, \pm X\}$), noting that $X^-(1) = X^-(r_1) = 0$, and integrating over $[r_1, 1]$ give

$$\int_{r_1}^1 r^{N-1} [((X^-)')^2 + 2a(r)(X^-)^2]\, dr \leq 0.$$

This shows that $X^- = 0$ in $[r_1, 1]$, i.e. $X \geq 0$ and $M \leq 1$ in $[r_1, 1]$. By definition of r_0, this implies that $r_0 \leq r_1$, which contradicts the fact that $r_1 \in (0, r_0)$.

Step 2: We show that $f_{\varepsilon,\eta}^2 + g_{\varepsilon,\eta}^2 \le 1$ *in* $(0,1)$. Indeed, if $r_0 = 0$, this step is clear. Suppose that $r_0 > 0$. By Step 1, we have $M > 1$ and so $W'(1-M) \le 0$ in $(0, r_0)$. Returning to (1.13)–(1.14), as (1.11) implies $\widetilde{W}'(t) \ge \widetilde{W}'(0) \ge 0$ for $t \ge 0$, we have that the functions $f_{\varepsilon,\eta}$ and $g_{\varepsilon,\eta}$, considered as functions on the ball $B(0,r_0)$ in $\mathbb{R}^N$, satisfy

$$\Delta f_{\varepsilon,\eta} = c_1 f_{\varepsilon,\eta} \quad \text{and} \quad \Delta g_{\varepsilon,\eta} = c_2 g_{\varepsilon,\eta} \quad \text{in } B(0,r_0) \setminus \{0\},$$

where $c_1 = \frac{N-1}{r^2} - \frac{1}{\varepsilon^2} W'(1-M) \ge 0$ and $c_2 = -\frac{1}{\varepsilon^2} W'(1-M) + \frac{1}{\eta^2} \widetilde{W}'(g_{\varepsilon,\eta}^2) \ge 0$ in $(0, r_0)$. By Kato's inequality (see [31] or [8, Lemma A.1]), this implies

$$\Delta f_{\varepsilon,\eta}^{\pm} \ge 0 \quad \text{and} \quad \Delta g_{\varepsilon,\eta}^{\pm} \ge 0 \quad \text{in } B(0,r_0) \setminus \{0\}.$$

Since $f_{\varepsilon,\eta}, g_{\varepsilon,\eta} \in H^1(B(0,r_0))$, these hold in $B(0,r_0)$. By the maximum principle, $f_{\varepsilon,\eta}^{\pm} \le f_{\varepsilon,\eta}^{\pm}(r_0)$ and $g_{\varepsilon,\eta}^{\pm} \le g_{\varepsilon,\eta}^{\pm}(r_0)$ in $B(0,r_0)$. We deduce that $f_{\varepsilon,\eta}^2 + g_{\varepsilon,\eta}^2 \le M(r_0) \le 1$ in $(0,r_0)$. As $M = f_{\varepsilon,\eta}^2 + g_{\varepsilon,\eta}^2 \le 1$ in $[r_0, 1]$, the conclusion of Step 2 follows.

Step 3: Conclusion. By Step 2 and the fact that $m_\eta \in H^1(B^N)$, we deduce that (2.3) holds in the whole of B^N; then standard elliptic regularity theory yields that $m_{\varepsilon,\eta}$ and so X are C^2 in $\bar{B}^N$. In particular, $f_{\varepsilon,\eta}(0) = 0$ (as $f_{\varepsilon,\eta}(r)n(x) \in C^2(B^N)$) and $g_{\varepsilon,\eta}'(0) = 0$ (since $g_{\varepsilon,\eta}$ extends to an even C^2 function on $(-1,1)$). By Step 2, we know that $M \le 1$ in $(0,1)$. Moreover, since $f_{\varepsilon,\eta}(1) = 1$, we deduce that the inequality in (2.4) is strict near $r = 1$; in particular, X cannot be identically 0. Thus, the strong maximum principle applied to (2.4) yields $X > 0$ in $(0,1)$ i.e. $M < 1$ in $(0,1)$. ∎

By restricting attention to solutions with $g_{\varepsilon,\eta} \equiv 0$ (for any $\widetilde{W}$ satisfying (1.11) e.g. $\widetilde{W}(t) = t$), we immediately obtain the following corollary:

Corollary 2.8. *Let* $N \ge 2$ *and* $\varepsilon > 0$. *If* $W \in C^2((-\infty, 1])$ *satisfies* (1.10) *and* f_ε *satisfies* (1.1)–(1.3)*, then* $|f_\varepsilon| < 1$ *in* $(0,1)$ *and the map* $x \mapsto u_\varepsilon(x) = f_\varepsilon(r)n(x)$ *belongs to* $C^2(\bar{B}^N)$. *In particular,* $f_\varepsilon(0) = 0$.

Concerning the monotonicity of solutions of (1.12)–(1.15) satisfying $g_{\varepsilon,\eta} \ge 0$, we first prove it in Proposition 2.9 under an additional assumption that $f_{\varepsilon,\eta} \ge 0$. We then show in Proposition 2.10 that this additional non-negativity assumption on $f_{\varepsilon,\eta}$ can be removed.

Proposition 2.9. *Let* $W \in C^2((-\infty,1])$ *and* $\widetilde{W} \in C^2([0,\infty))$ *satisfy* (1.10) *and* (1.11)*, and* $(f_{\varepsilon,\eta}, g_{\varepsilon,\eta})$ *satisfy* (1.12)–(1.15) *with* $f_{\varepsilon,\eta} \ge 0$, $g_{\varepsilon,\eta} \ge 0$ *in* $(0,1)$. *Then* $f_{\varepsilon,\eta}' > 0$, $(\frac{f_{\varepsilon,\eta}}{r})' \le 0$ *and either* $g_{\varepsilon,\eta}' < 0$ *or* $g_{\varepsilon,\eta} = 0$ *in* $(0,1]$.

Proof of Proposition 2.9. To simplify notation, we write (f,g) for $(f_{\varepsilon,\eta}, g_{\varepsilon,\eta})$. By Lemma 2.7, we know that $f^2 + g^2 < 1$ in $(0,1)$, $f(0) = 0$ and $g'(0) = 0$. By the strong maximum principle applied to (1.13) for $f \ge 0$ in $(0,1)$, we get $f > 0$ in $(0,1)$ (as $f = 0$ in $(0,1)$ would contradict the boundary condition $f(1) = 1$ in (1.15)). By the strong maximum principle applied to (1.14) (as a PDE in B^N for $g \ge 0$) we get $g > 0$ in $[0,1)$ or $g = 0$ in $(0,1)$.

Case 1: $g > 0$ in $[0,1)$. For $a, b \in [0,1]$, let

$$A(a,b) = -\frac{1}{\varepsilon^2} W'(1-a^2-b^2)a, \quad B(a,b) = -\frac{1}{\varepsilon^2} W'(1-a^2-b^2)b + \frac{1}{\eta^2}\widetilde{W}'(b^2)b.$$

Then (1.13) and (1.14) can be rewritten as

$$\Delta f - \frac{N-1}{r^2} f = f'' + \frac{N-1}{r} f' - \frac{N-1}{r^2} f = A(f,g) \quad \text{in } (0,1), \tag{2.6}$$

$$\Delta g = g'' + \frac{N-1}{r} g' = B(f,g) \quad \text{in } (0,1). \tag{2.7}$$

The convexity assumption on W in (1.10) yields

$$\partial_b A(a,b) = \partial_a B(a,b) \geq 0 \quad \text{for all } a, b \in [0,1].$$

These inequalities give system (2.6)–(2.7) a cooperative structure; see e.g. [12, 16, 42]. In order to prove the monotonicity of f and g, we follow the ideas based on a moving plane argument in the proof of [27, Theorem 1.6]. See also [1] for a similar argument in the context of phase segregation in Bose–Einstein condensates. For $0 < s < 1$, define

$$f_s(r) = f(2s-r) \quad \text{and} \quad g_s(r) = g(2s-r) \quad \text{for } \max(0, 2s-1) < r < s.$$

By (1.15) and (1.10) (in particular, $W'(0) = 0$), we have $A(f(1), g(1)) = B(f(1), g(1)) = 0$ and recall that $0 < f < 1 = f(1)$ and $g > 0 = g(1)$ in $(0,1)$. As $\partial_b A(a,b) \geq 0$, we deduce that the function $\hat{f} = f - f(1)$ satisfies

$$\begin{aligned}\Delta \hat{f} - \frac{N-1}{r^2}\hat{f} &= \frac{N-1}{r^2} f(1) + A(f,g) - A(f(1), g(1)) \\ &\geq A(f,g) - A(f(1), g) = c(r)\hat{f}\end{aligned}$$

for some continuous function $c \in C[0,1]$. As $\hat{f}(1) = 0$ and $\hat{f} < 0$ in $(0,1)$, we deduce from the Hopf lemma (see e.g. [17, Lemma 3.4]) that $f'(1) > 0$. Likewise, we can show that $g'(1) < 0$. Consequently, there is some small $\delta > 0$ such that $f_s > f$ and $g_s < g$ in $\max(0, 2s-1) < r < s$ for any $s \in (1-\delta, 1)$. We define

$$\underline{s} = \inf\big\{s \in (0,1) : f_t > f \text{ and } g_t < g \text{ in } (\max(0, 2t-1), t) \text{ for all } t \in (s,1)\big\}.$$

It follows that $\underline{s} \in [0, 1-\delta]$.

Claim: *$\underline{s} = 0$, $f' > 0$ and $g' < 0$ in $(0,1]$.*

Proof of claim. Assume by contradiction that $\underline{s} > 0$. Then

(a) $f' \geq 0$ and $g' \leq 0$ in $(\underline{s}, 1)$,

(b) and $f_{\underline{s}} \geq f > 0$ and $g_{\underline{s}} \leq g$ in $\max(0, 2\underline{s}-1) < r < \underline{s}$.

Combined with the monotonicity of $A(a,\cdot)$ and $B(\cdot,b)$, it follows for every $s \in [\underline{s},1)$ and every $r \in (\max(0,2s-1),s)$,

$$\begin{aligned}\Delta f_s(r) - \frac{N-1}{r^2} f_s(r) &= f''(2s-r) - \frac{N-1}{r} f'(2s-r) - \frac{N-1}{r^2} f(2s-r)\\ &\le A(f(2s-r), g(2s-r))\\ &= A(f_s(r), g_s(r)) \le A(f_s(r), g(r)), \qquad (2.8)\\ \Delta g_s(r) &\ge B(f_s(r), g_s(r)) \ge B(f(r), g_s(r)), \qquad (2.9)\end{aligned}$$

and equality in all the inequalities (2.8) (resp. in (2.9)) for some $s \in [\underline{s},1)$ implies that, for every $r \in (\max(0,2s-1),s)$,

$$f'(2s-r) = 0 \quad (\text{resp. } g'(2s-r) = 0). \tag{2.10}$$

Combining (2.8) and (2.9) with (2.6) and (2.7), we have for all $s \in [\underline{s},1)$,

$$\begin{aligned}\Delta(f_s - f)(r) - \frac{N-1}{r^2}(f_s - f) &\le A(f_s, g) - A(f,g) = (f_s - f)c_1(r),\\ \Delta(g_s - g)(r) &\ge B(f, g_s) - B(f,g) = (g_s - g)c_2(r),\end{aligned}$$

with c_1, c_2 being two continuous functions on $[\max(0,2s-1),s]$ and equality in the above inequalities again implies (2.10).

By the definition of $\underline{s}$, $f_s > f$ and $g_s < g$ in $(\max(0,2s-1),s)$ for $s \in (\underline{s},1)$. By the Hopf lemma, applied to the above differential inequalities, we have $f_s'(s) < f'(s)$ and $g_s'(s) > g'(s)$, i.e. $f'(s) > 0$ and $g'(s) < 0$ for $s \in (\underline{s},1)$. We now show that these assertions continue to hold with $s = \underline{s}$, i.e.

Fact 1. $f_{\underline{s}} > f$ and $g_{\underline{s}} < g$ in $\max(0,2\underline{s}-1) < r < \underline{s}$.

Fact 2. $f' > 0$ and $g' < 0$ in $[\underline{s},1)$.

Indeed, since $f' > 0$ and $g' < 0$ in $(\underline{s},1)$, (2.10) does not hold and so the above differential inequalities for $f_{\underline{s}} - f$ and $g_{\underline{s}} - g$ are strict in $(\max(0,2\underline{s}-1),\underline{s})$. Since $f_{\underline{s}} - f \ge 0$ and $g_{\underline{s}} - g \le 0$ in $(\max(0,2\underline{s}-1),\underline{s})$, the strong maximum principle applied to those differential inequalities gives Fact 1. By the Hopf lemma, we then have $f_{\underline{s}}'(\underline{s}) < f'(\underline{s})$ and $g_{\underline{s}}'(\underline{s}) > g'(\underline{s})$, i.e. $f'(\underline{s}) > 0$ and $g'(\underline{s}) < 0$, and Fact 2 follows.

Conclusion. We now show that Facts 1 and 2 contradict the minimality of $\underline{s}$. Indeed, observe first that $(f_{\underline{s}} - f)(\max(0,2\underline{s}-1)) > 0$ since

$$\begin{aligned}f_{\underline{s}}(\max(0,2\underline{s}-1)) = 1 > f(\max(0,2\underline{s}-1)) &\quad \text{when } \frac{1}{2} \le \underline{s} < 1,\\ f_{\underline{s}}(\max(0,2\underline{s}-1)) > 0 = f(\max(0,2\underline{s}-1)) &\quad \text{when } \underline{s} < \frac{1}{2}.\end{aligned}$$

Likewise, we have $(g_{\underline{s}} - g)(\max(0, 2\underline{s} - 1)) < 0$ since

$$g_{\underline{s}}(\max(0, 2\underline{s} - 1)) = 0 < g(\max(0, 2\underline{s} - 1)) \qquad \text{when } \frac{1}{2} \le \underline{s} < 1,$$
$$g'_{\underline{s}}(\max(0, 2\underline{s} - 1)) = -g'(2\underline{s}) > 0 = g'(0) = g'(\max(0, 2\underline{s} - 1)) \quad \text{when } \underline{s} < \frac{1}{2}$$

(in the latter case, this is combined with $g_{\underline{s}} < g$ on $(0, \underline{s})$ by Fact 1). Thus, thanks to Facts 1 and 2, we deduce by continuity the existence of a small $\tilde{\delta} > 0$ such that, for every $s \in (\underline{s} - \tilde{\delta}, \underline{s}]$, $f_s > f$ and $g_s < g$ in $\max(0, 2s - 1) < r < s$, contradicting the minimality of $\underline{s}$. Thus, $\underline{s} = 0$. Also, by Fact 2, $f' > 0$ and $g' < 0$ in $(0, 1]$. The claim is proved. ∎

Case 2: $g = 0$ in $(0, 1)$. . The above argument applies to solutions $f \ge 0$ of (1.1)–(1.3), where equation (2.8) is replaced by $\Delta f_s(r) - \frac{N-1}{r^2} f_s(r) \le A(f_s(r), 0)$, yielding $f' > 0$. (Note that the assumption $W'' \ge 0$ is no longer needed in this case, though the condition $W'(0) = 0$ is used.)

Proof of $(\frac{f}{r})' \le 0$ in $(0, 1)$. Indeed, by Lemma A.5, we know that $v := \frac{f}{r} \in C^2([0, 1])$. To prove that v is decreasing, we follow the argument in [24, Proposition 2.2]: by (1.10) we have $W' \ge 0$ in $(0, 1)$ so that

$$(r^{N+1} v'(r))' = -\frac{r^{N+1}}{\varepsilon^2} W'(1 - f^2 - g^2) v(r) \le 0, \quad r \in (0, 1).$$

This implies that $r^{N+1} v'(r)$ is a non-increasing C^1 function in $[0, 1]$. Then, since $\lim_{r \to 0} r^{N+1} v'(r) = 0$ (as $v \in C^1([0, 1])$), we have $v'(r) \le 0$ in $[0, 1]$. ∎

Next, we prove the positivity of $f_{\varepsilon,\eta}$ when $g_{\varepsilon,\eta} \ge 0$. When $g_{\varepsilon,\eta} \equiv 0$, the result was obtained in [22, 25] under a slightly different condition on W.

Proposition 2.10. *Suppose $W \in C^2((-\infty, 1])$ and $\tilde{W} \in C^2([0, \infty))$ satisfy (1.10) and (1.11), and $(f_{\varepsilon,\eta}, g_{\varepsilon,\eta})$ satisfies (1.12)–(1.15) with $g_{\varepsilon,\eta} \ge 0$ in $(0, 1)$. Then $f_{\varepsilon,\eta} > 0$ in $(0, 1)$.*

Proof. As in the proof of the previous proposition, we drop the indices ε and η, so that in the following we denote by f and g the solution considered in (1.12)–(1.15). Suppose by contradiction that f changes sign in $(0, 1)$. Let $r_1 \in (0, 1)$ be such that $f(r_1) = 0$ and $f > 0$ in $(r_1, 1]$. Applying the Hopf lemma to (1.13) in $(r_1, 1)$, we have $f'(r_1) > 0$. In particular, $f < 0$ in some small interval $(r_1 - \delta, r_1)$. Note that $(|f|, g)$ satisfies distributionally

$$\Delta|f| - \frac{N-1}{r^2}|f| = A(|f|, g) \quad \text{in } (r_1, 1),$$
$$\Delta|f| - \frac{N-1}{r^2}|f| \ge A(|f|, g) \quad \text{in } (0, 1),$$
$$\Delta g'' + \frac{N-1}{r} g' = B(|f|, g) \quad \text{in } (0, 1),$$

where A and B are as in the proof of Proposition 2.9. Consequently, we can apply the proof of Proposition 2.9 to the pair $(|f|, g)$ to obtain

$$(|f|)_s \geq |f| \quad \text{and} \quad g_s \leq g \quad \text{in } \max(0, 2s-1) < r < s \text{ for all } r_1 \leq s < 1,$$

where $(|f|)_s(r) = |f|(2s-r)$ and $g_s(r) = g(2s-r)$. Observe also that, by definition, both $|f|$ and $(|f|)_{r_1}$ have the same first left-derivative at r_1; thus, we deduce by the Hopf lemma that $(|f|)_{r_1} \equiv |f|$ and $f'(2r_1 - r) = 0$ in $\max(0, 2r_1 - 1) < r < r_1$ (see (2.10)). The latter identity is impossible, since $f'(r_1) > 0$. We conclude that $f \geq 0$ in $(0,1)$. The positivity of f follows by the strong maximum principle applied to (1.13) (as $f(1) = 1$). ∎

Applying Propositions 2.9 and 2.10 to the solution $(f_\varepsilon, 0)$, we obtain the following corollary:

Corollary 2.11. *Suppose $W \in C^2((-\infty, 1])$ satisfies (1.10), and f_ε satisfies (1.1)–(1.3). Then $f_\varepsilon > 0$, $f_\varepsilon' > 0$ and $(\frac{f_\varepsilon}{r})' \leq 0$ in $(0, 1]$.*

Finally, we prove the uniqueness of escaping solutions of (1.12)–(1.15).

Proposition 2.12. *Let $N \geq 2$ and suppose that $W \in C^2((-\infty, 1])$ and $\widetilde{W} \in C^2([0,\infty))$ satisfy (1.10) and (1.11). Then, for every $\varepsilon > 0$ and $\eta > 0$, system (1.12)–(1.15) has at most one escaping solution $(f_{\varepsilon,\eta}, g_{\varepsilon,\eta})$ with $g_{\varepsilon,\eta} > 0$ in $(0,1)$. Furthermore, when it exists, $(f_{\varepsilon,\eta}, \pm g_{\varepsilon,\eta})$ are the only two minimizers of $I_{\varepsilon,\eta}$ over the set $\mathcal{B}$; in particular, $I_{\varepsilon,\eta}[f_\varepsilon, 0] > I_{\varepsilon,\eta}[f_{\varepsilon,\eta}, g_{\varepsilon,\eta}]$, where f_ε is the radial profile satisfying (1.2)–(1.3).*

Proof. We use ideas from [28, 29]. Suppose that $(f_{\varepsilon,\eta}, g_{\varepsilon,\eta})$ solves (1.12)–(1.15) and $g_{\varepsilon,\eta} > 0$ in $(0,1)$. By Proposition 2.10, $f_{\varepsilon,\eta} > 0$ in $(0,1)$. For $(f,g) \in \mathcal{B}$, we write $(f,g) = (f_{\varepsilon,\eta}, g_{\varepsilon,\eta}) + (s,q)$ and $V(x) = (s(r)n(x), q(r)) \in H_0^1(B^N, \mathbb{R}^{N+1})$. By the convexity of W, $\widetilde{W}$ and (1.13)–(1.14),

$$\begin{aligned}
&I_{\varepsilon,\eta}[f,g] - I_{\varepsilon,\eta}[f_{\varepsilon,\eta}, g_{\varepsilon,\eta}] \\
&\quad\geq \frac{1}{2}\int_0^1 \Big\{2f_{\varepsilon,\eta}' s' + (s')^2 + 2g_{\varepsilon,\eta}' q' + (q')^2 + \frac{N-1}{r^2}(2f_{\varepsilon,\eta}s + s^2) \\
&\qquad\qquad - \frac{1}{\varepsilon^2}W'(1 - f_{\varepsilon,\eta}^2 - g_{\varepsilon,\eta}^2)[2(f_{\varepsilon,\eta}s + g_{\varepsilon,\eta}q) + s^2 + q^2] \\
&\qquad\qquad + \frac{1}{\eta^2}\widetilde{W}'(g_{\varepsilon,\eta}^2)(2g_{\varepsilon,\eta}q + q^2)\Big\} r^{N-1}\,dr \\
&\quad= \frac{1}{2}\int_0^1 \Big\{(s')^2 + (q')^2 + \frac{N-1}{r^2}s^2 \\
&\qquad\qquad - \frac{1}{\varepsilon^2}W'(1 - f_{\varepsilon,\eta}^2 - g_{\varepsilon,\eta}^2)(s^2 + q^2) + \frac{1}{\eta^2}\widetilde{W}'(g_{\varepsilon,\eta}^2)q^2\Big\} r^{N-1}\,dr
\end{aligned}$$

$$= \frac{1}{2|\mathbb{S}^{N-1}|}\int_{B^N}\left\{|\nabla V|^2 - \frac{1}{\varepsilon^2}W'(1-f_{\varepsilon,\eta}^2-g_{\varepsilon,\eta}^2)|V|^2 + \frac{1}{\eta^2}\widetilde{W}'(g_{\varepsilon,\eta}^2)V_{N+1}^2\right\}dx$$
$$=: \frac{F_{\varepsilon,\eta}[V]}{2|\mathbb{S}^{N-1}|}.$$

Claim: *For every $V(x) = (s(r)n(x), q(r)) \in H_0^1(B^N, \mathbb{R}^{N+1})$, it holds that*

$$F_{\varepsilon,\eta}[V] \geq \int_{B^N}\left\{f_{\varepsilon,\eta}^2(|x|)\left|\left(\frac{s}{f_{\varepsilon,\eta}}\right)'(|x|)\right|^2 + g_{\varepsilon,\eta}^2(|x|)\left|\left(\frac{q}{g_{\varepsilon,\eta}}\right)'(|x|)\right|^2\right\}dx,$$

and as a consequence, $(f_{\varepsilon,\eta}, g_{\varepsilon,\eta})$ minimizes $I_{\varepsilon,\eta}$ in $\mathcal{B}$.

Proof of claim. Since $F_{\varepsilon,\eta}$ is continuous in $H_0^1(B^N, \mathbb{R}^{N+1})$ (because $W'(1-f_{\varepsilon,\eta}^2 - g_{\varepsilon,\eta}^2)$, $\widetilde{W}'(g_{\varepsilon,\eta}^2) \in L^\infty(B^N)$ by Lemma 2.7), by standard density results and Fatou's lemma, it suffices to show the claim for $V = (s(r)n, q(r)) \in C_c^\infty(B^N \setminus \{0\}, \mathbb{R}^{N+1})$. For that, we will apply [24, Lemma A.1] for

$$\begin{cases} L := -\Delta - \frac{1}{\varepsilon^2}W'(1-f_{\varepsilon,\eta}^2-g_{\varepsilon,\eta}^2), \\ T := -\Delta - \frac{1}{\varepsilon^2}W'(1-f_{\varepsilon,\eta}^2-g_{\varepsilon,\eta}^2) + \frac{1}{\eta^2}\widetilde{W}'(g_{\varepsilon,\eta}^2). \end{cases} \tag{2.11}$$

Indeed, writing $V = (s(r)n, q(r)) = (V_1, \ldots, V_N, V_{N+1}) \in C_c^\infty(B^N \setminus \{0\}, \mathbb{R}^{N+1})$ and decomposing $V_j = f_{\varepsilon,\eta}\widehat{V}_j$ with $\widehat{V}_j = \frac{V_j}{f_{\varepsilon,\eta}}$ for $j = 1, \ldots, N$ and $V_{N+1} = g_{\varepsilon,\eta}\widehat{V}_{N+1}$ with $\widehat{V}_{N+1} = \frac{q}{g_{\varepsilon,\eta}}$,

$$\begin{aligned} F_{\varepsilon,\eta}[V] &= \sum_{j=1}^N \int_{B^N} LV_j \cdot V_j\, dx + \int_{B^N} TV_{N+1}\cdot V_{N+1}\, dx \\ &= \sum_{j=1}^N \int_{B^N}\Big\{f_{\varepsilon,\eta}^2|\nabla \widehat{V}_j|^2 + \widehat{V}_j^2 Lf_{\varepsilon,\eta}\cdot f_{\varepsilon,\eta} \\ &\qquad + g_{\varepsilon,\eta}^2|\nabla\widehat{V}_{N+1}|^2 + \widehat{V}_{N+1}^2 Tg_{\varepsilon,\eta}\cdot g_{\varepsilon,\eta}\Big\}dx \\ &= \int_{B^N}\left\{f_{\varepsilon,\eta}^2(|x|)\left|\nabla\left(\frac{s(r)}{f_{\varepsilon,\eta}(r)}n(x)\right)\right|^2 - \frac{N-1}{r^2}s^2 + g_{\varepsilon,\eta}^2(|x|)\left|\left(\frac{q}{g_{\varepsilon,\eta}}\right)'(|x|)\right|^2\right\}dx \\ &= \int_{B^N}\left\{f_{\varepsilon,\eta}^2(|x|)\left|\left(\frac{s}{f_{\varepsilon,\eta}}\right)'(|x|)\right|^2 + g_{\varepsilon,\eta}^2(|x|)\left|\left(\frac{q}{g_{\varepsilon,\eta}}\right)'(|x|)\right|^2\right\}dx, \end{aligned} \tag{2.12}$$

because $Lf_{\varepsilon,\eta} = -\frac{N-1}{r^2}f_{\varepsilon,\eta}$, $Tg_{\varepsilon,\eta} = 0$ (by (1.13)–(1.14)) and

$$(\widehat{V}_1, \ldots, \widehat{V}_N) = \frac{s(r)}{f_{\varepsilon,\eta}(r)}n(x)$$

with $|\nabla n|^2 = \frac{N-1}{r^2}$. Hence, the claim is proved. ∎

Step 1: We prove that $\{(f_{\varepsilon,\eta}, \pm g_{\varepsilon,\eta})\}$ *is the set of minimizers of* $I_{\varepsilon,\eta}$ *in* $\mathcal{B}$. Indeed, we have seen that $(f_{\varepsilon,\eta}, \pm g_{\varepsilon,\eta})$ minimizes $I_{\varepsilon,\eta}$ in $\mathcal{B}$. Suppose $(\tilde{f}_{\varepsilon,\eta}, \tilde{g}_{\varepsilon,\eta})$ also minimizes $I_{\varepsilon,\eta}$ in $\mathcal{B}$, in particular, $I_{\varepsilon,\eta}[f_{\varepsilon,\eta}, g_{\varepsilon,\eta}] = I_{\varepsilon,\eta}[\tilde{f}_{\varepsilon,\eta}, \tilde{g}_{\varepsilon,\eta}]$ so that, for $V = ((\tilde{f}_{\varepsilon,\eta} - f_{\varepsilon,\eta})n(x), \tilde{g}_{\varepsilon,\eta} - g_{\varepsilon,\eta})$, one has $F[V] = 0$ leading to

$$\frac{\tilde{f}_{\varepsilon,\eta} - f_{\varepsilon,\eta}}{f_{\varepsilon,\eta}} \quad \text{and} \quad \frac{\tilde{g}_{\varepsilon,\eta} - g_{\varepsilon,\eta}}{g_{\varepsilon,\eta}} \quad \text{are constant in } (0, 1).$$

This together with $\tilde{f}_{\varepsilon,\eta}(1) - f_{\varepsilon,\eta}(1) = 0$ gives $\tilde{f}_{\varepsilon,\eta} \equiv f_{\varepsilon,\eta}$ and $\tilde{g}_{\varepsilon,\eta} \equiv a g_{\varepsilon,\eta}$ in $(0, 1)$ for some $a \in \mathbb{R}$. Since $g_{\varepsilon,\eta} > 0$, this implies that $\tilde{g}_{\varepsilon,\eta}$ has a fixed sign. Furthermore, either $a = 0$ (so $\tilde{g}_{\varepsilon,\eta} \equiv 0$), or $|\tilde{g}_{\varepsilon,\eta}| > 0$ in $(0, 1)$ in which case, we can interchange $g_{\varepsilon,\eta}$ and $\pm\tilde{g}_{\varepsilon,\eta}$ if necessary (note that $(\tilde{f}_{\varepsilon,\eta}, -\tilde{g}_{\varepsilon,\eta})$ also minimizes $I_{\varepsilon,\eta}$ in $\mathcal{B}$), so that we may always assume that $0 \le a \le 1$.

To finish the proof, we prove that $a = 1$, i.e. $\tilde{g}_{\varepsilon,\eta} \equiv g_{\varepsilon,\eta}$ in $(0, 1)$. Assume by contradiction that $0 \le a < 1$. We will show that

$$W'(1 - f_{\varepsilon,\eta}^2 - g_{\varepsilon,\eta}^2) \equiv 0 \quad \text{in } (0, 1). \tag{2.13}$$

Once this is done, we deduce from (1.14) that $-\Delta g_{\varepsilon,\eta} + \frac{1}{\eta^2}\widetilde{W}'(g_{\varepsilon,\eta}^2) g_{\varepsilon,\eta} = 0$ in B^N. Since $\widetilde{W}' \ge \widetilde{W}'(0) \ge 0$ in $[0, \infty)$ (by (1.11)) and $g_{\varepsilon,\eta} = 0$ on ∂B^N, we deduce that $g_{\varepsilon,\eta} = 0$ in B^N, which gives a contradiction to the assumption $g_{\varepsilon,\eta} > 0$ in B^N, and completes the proof.

Let us now prove (2.13). Returning to (1.13), we see that

$$W'(1 - f_{\varepsilon,\eta}^2 - g_{\varepsilon,\eta}^2) \equiv W'(1 - f_{\varepsilon,\eta}^2 - a^2 g_{\varepsilon,\eta}^2) \quad \text{in } [0, 1]. \tag{2.14}$$

Therefore, to prove (2.13), it suffices to show that $W'(t) = 0$ for every $0 \le t \le \max_{[0,1]}(1 - f_{\varepsilon,\eta}^2 - a^2 g_{\varepsilon,\eta}^2) =: \tau$. For that, we have $f_{\varepsilon,\eta}^2 + a^2 g_{\varepsilon,\eta}^2 < f_{\varepsilon,\eta}^2 + g_{\varepsilon,\eta}^2 < 1$ in $(0, 1)$ by Lemma 2.7, and hence $\tau > 0$. Note that the range of $1 - f_{\varepsilon,\eta}^2 - a^2 g_{\varepsilon,\eta}^2$ over $[0, 1]$ is $[0, \tau]$ because of (1.15). Set $t_0 = \inf\{t > 0 : W'(s) = W'(\tau) \text{ for all } s \in [t, \tau]\}$. We show that $t_0 = 0$. For that, let $r_0 \in [0, 1]$ be such that $1 - f_{\varepsilon,\eta}^2(r_0) - a^2 g_{\varepsilon,\eta}^2(r_0) = t_0$. By the continuity of W' and (2.14), we deduce for $t_1 := 1 - f_{\varepsilon,\eta}^2(r_0) - g_{\varepsilon,\eta}^2(r_0) \le t_0$ that $W'(t_1) = W'(t_0) = W'(\tau)$. As W' is non-decreasing (because W is convex), we deduce that $W'(s) = W'(\tau)$ for every $s \in [t_1, \tau]$. By the minimality of t_0, it means that $t_1 = t_0$, i.e. $g_{\varepsilon,\eta}^2(r_0) = 0$. Since $g_{\varepsilon,\eta} > 0$ in $[0, 1)$ (which is a consequence of the strong maximum principle applied to (1.14), considered as a PDE on B^N), this yields $r_0 = 1$, i.e. $t_0 = 0$. It follows that $W' \equiv W'(0) = 0$ on $[0, \tau]$ as desired (where we use that 0 is a minimum point of W by the assumption (1.10)).

Step 2: We prove the uniqueness of escaping solutions of (1.12)–(1.15). Indeed, assume that $(\check{f}_{\varepsilon,\eta}, \check{g}_{\varepsilon,\eta})$ is also a solution to (1.12)–(1.15) with $\check{g}_{\varepsilon,\eta} > 0$ in $(0, 1)$. Then the claim yields that both $(f_{\varepsilon,\eta}, g_{\varepsilon,\eta})$ and $(\check{f}_{\varepsilon,\eta}, \check{g}_{\varepsilon,\eta})$ minimize $I_{\varepsilon,\eta}$ in $\mathcal{B}$. By Step 1, we have $f_{\varepsilon,\eta} \equiv \check{f}_{\varepsilon,\eta}$ and $g_{\varepsilon,\eta} \equiv \check{g}_{\varepsilon,\eta}$ as desired. The proof is complete. ∎

2.2. The $\mathbb{R}^N$-valued model: Existence and uniqueness

We prove existence and uniqueness of the radial profile and its minimality for $I_\varepsilon^{\mathrm{GL}}$ as stated in Theorem 2.1. Then we prove Lemma 2.3.

Proof of Theorem 2.1. Let f_ε be a minimizer of the reduced energy functional $I_\varepsilon^{\mathrm{GL}}$ in $\mathcal{B}^{\mathrm{GL}}$. (It is easy to see that such a minimizer exists.) Since $I_\varepsilon^{\mathrm{GL}}[f] \geq I_\varepsilon^{\mathrm{GL}}[\min\{|f|, 1\}]$, we may also assume that $0 \leq f_\varepsilon \leq 1$. In addition, we have that f_ε satisfies (1.2), $f_\varepsilon(1) = 1$ and $f_\varepsilon \in C^2((0, 1])$. Noting also that the constant functions 0 and 1 are a solution and a super-solution to (1.2) respectively (since $W'(0) = 0$), the strong maximum principle implies that $0 < f_\varepsilon < 1$ in $(0, 1)$. By Lemma A.4, $f_\varepsilon/r \in C^2([0, 1])$ and $f_\varepsilon(0) = 0$.

If (1.10) holds, then by Corollary 2.11 we have $f_\varepsilon' > 0$ in $(0, 1]$. Also, the same argument as in the proof of Proposition 2.12 applies, also giving the uniqueness of f_ε as solution of (1.2)–(1.3), in particular, as the unique minimizer of $I_\varepsilon^{\mathrm{GL}}$ over $\mathcal{B}^{\mathrm{GL}}$. We omit the details. ∎

We next prove estimates for $\ell(\varepsilon)$.

Proof of Lemma 2.3. Note that by the definition of the first eigenvalue for $L_\varepsilon^{\mathrm{GL}}$ and standard elliptic regularity, ℓ depends continuously on ε. Let us prove (2.2) for $0 < \tilde{\varepsilon} < \varepsilon < \infty$. We have, for all $\varphi \in H_0^1(B^N)$,

$$\int_{B^N} \Big[|\nabla \varphi|^2 - \frac{1}{\tilde{\varepsilon}^2} W'(1 - f_{\tilde{\varepsilon}}^2)\varphi^2\Big]\, dx \geq \ell(\tilde{\varepsilon}) \int_{B^N} \varphi^2\, dx.$$

By rescaling, we deduce for all $\psi \in H_0^1(B(0, 1/\tilde{\varepsilon}))$ that

$$\int_{B(0,1/\tilde{\varepsilon})} \big[|\nabla \psi|^2 - W'(1 - f_{\tilde{\varepsilon}}^2(\tilde{\varepsilon}|x|))\psi^2\big] \geq \tilde{\varepsilon}^2 \ell(\tilde{\varepsilon}) \int_{B(0,1/\tilde{\varepsilon})} \psi^2\, dx.$$

As $B(0, 1/\varepsilon) \subset B(0, 1/\tilde{\varepsilon})$, by the strict monotonicity of the first eigenvalue with respect to domains (due to the positivity of the first eigenfunctions), we have, for all $0 \not\equiv \psi \in H_0^1(B(0, 1/\varepsilon))$,

$$\int_{B(0,1/\varepsilon)} \big[|\nabla \psi|^2 - W'(1 - f_{\tilde{\varepsilon}}^2(\tilde{\varepsilon}|x|))\psi^2\big] > \tilde{\varepsilon}^2 \ell(\tilde{\varepsilon}) \int_{B(0,1/\varepsilon)} \psi^2\, dx.$$

Now using the inequality $1 \geq f_\varepsilon(\varepsilon|x|) \geq f_{\tilde{\varepsilon}}(\tilde{\varepsilon}|x|) \geq 0$ for $|x| < 1/\varepsilon$ (see Proposition B.1 (i)) and the monotonicity of W', we deduce that

$$\int_{B(0,1/\varepsilon)} \big[|\nabla \psi|^2 - W'(1 - f_\varepsilon^2(\varepsilon|x|))\psi^2\big] > \tilde{\varepsilon}^2 \ell(\tilde{\varepsilon}) \int_{B(0,1/\varepsilon)} \psi^2\, dx$$

for all $0 \not\equiv \psi \in H_0^1(B(0, 1/\varepsilon))$. Rescaling once again we get

$$\int_{B^N} \Big[|\nabla \varphi|^2 - \frac{1}{\varepsilon^2} W'(1 - f_\varepsilon^2)\varphi^2\Big] > \frac{\tilde{\varepsilon}^2 \ell(\tilde{\varepsilon})}{\varepsilon^2} \int_{B^N} \varphi^2\, dx$$

for all $0 \not\equiv \varphi \in H_0^1(B^N)$, which is equivalent to (2.2).

Assertion (a) is clear because if $W'(1) = 0$, then (1.10) implies that $W = 0$ in $(0, 1)$. Assertion (b) for $N \geq 7$ is a consequence of the inequality

$$\int_{B^N} L_\varepsilon^{\mathrm{GL}} v \cdot v \, dx \geq \Big(\frac{(N-2)^2}{4} - (N-1)\Big) \int_{B^N} \frac{v^2}{r^2} \, dx \quad \text{for all } v \in H_0^1(B^N),$$

which was proved in Step 4 of the proof of [28, Theorem 2].

We next prove assertion (c) for $2 \leq N \leq 6$ and $W'(1) > 0$. We have seen that $\ell(\varepsilon) > -W'(1)\varepsilon^{-2}$. We prove the rest in two steps.

Step 1: We show that there exist $\varepsilon_1 > 0$ and $c_1 > 0$ such that $\ell(\varepsilon) \leq -\frac{c_1}{\varepsilon^2}$ for $\varepsilon \in (0, \varepsilon_1)$, by exhibiting a function $0 \not\equiv q = q_\varepsilon(r) \in \mathrm{Lip}_c((0, 1))$ with

$$\int_{B^N} L_\varepsilon^{\mathrm{GL}} q \cdot q \, dx \leq -\frac{c_1}{\varepsilon^2} \int_{B^N} q^2 \, dx.$$

(Note that by the lower bound of $\ell(\varepsilon)$, it is clear that $c_1 < W'(1)$.)

Note that, by [25, Lemma A.1], for every positive function $\varphi \in C_{\mathrm{loc}}^{1,1}((0, 1))$, we have the following identity for every $q = f_\varepsilon \varphi \tilde{q} \in \mathrm{Lip}_c(B^N \setminus \{0\})$:

$$\int_{B^N} L_\varepsilon^{\mathrm{GL}} q \cdot q \, dx = \int_{B^N} \varphi^2 \Big\{ f_\varepsilon^2 |\nabla \tilde{q}|^2 + \frac{L_\varepsilon^{\mathrm{GL}}(\varphi f_\varepsilon) f_\varepsilon}{\varphi} \tilde{q}^2 \Big\} \, dx. \tag{2.15}$$

We choose[9] $\varphi = r^{-\frac{N-2}{2}} \in C^\infty((0, 1))$, and note that, by (1.2),

$$L_\varepsilon^{\mathrm{GL}}(\varphi f_\varepsilon) f_\varepsilon = \frac{(N^2 - 8N + 8) f_\varepsilon^2 \varphi}{4r^2} - 2 f_\varepsilon f_\varepsilon' \varphi' \quad \text{in } (0, 1).$$

The idea now is to exploit the negativity of $N^2 - 8N + 8$ for $2 \leq N \leq 6$ to reach the desired conclusion. Let $t_0 = \sup\{0 \leq t < 1 : W(t) = 0\}$. By Proposition B.1 (ii), for every small $\delta > 0$, there exists $C_\delta > 0$ such that for every $a > C_\delta$ we can find $\varepsilon_1 = \varepsilon_1(\delta, a)$ for which

$$1 - t_0 - \delta \leq f_\varepsilon^2 \leq 1 - t_0 \quad \text{in } [C_\delta \varepsilon, a\varepsilon] \text{ for all } \varepsilon \in (0, \varepsilon_1). \tag{2.16}$$

The contribution of the term $-2 f_\varepsilon f_\varepsilon' \varphi'$ in the above expression of $L_\varepsilon^{\mathrm{GL}}(\varphi f_\varepsilon) f_\varepsilon$ to the right-hand side of (2.15) is handled as follows. (Note that if $N = 2$, then $\varphi' = 0$, so that term vanishes and the reader can proceed directly to estimate (2.17) below.) We impose that $\tilde{q} = \tilde{q}(r)$ is supported in $[C_\delta \varepsilon, a\varepsilon]$. Then integration by parts combined with (2.16) and $(r^{N-1}(\varphi^2)')' = 0$ for $r \in (0, 1)$ yields, by Cauchy–Schwarz,

$$\begin{aligned} -2 \int_0^1 r^{N-1} \tilde{q}^2 f_\varepsilon f_\varepsilon' \varphi \varphi' \, dr &= \frac{1}{2} \int_0^1 r^{N-1} \tilde{q}^2 (1 - t_0 - f_\varepsilon^2)' (\varphi^2)' \, dr \\ &= -\int_0^1 r^{N-1} \tilde{q} \tilde{q}' (1 - t_0 - f_\varepsilon^2) (\varphi^2)' \, dr \\ &\leq \delta \int_0^1 (\tilde{q}')^2 r \, dr + \frac{(N-2)^2}{4} \delta \int_0^1 \frac{\tilde{q}^2}{r} \, dr. \end{aligned}$$

[9] See [28, inequality (6)] for an explanation of this choice of φ.

Since $2 \le N \le 6$ implies $\widetilde{N} := N^2 - 8N + 8 < 0$, using (2.16), we deduce

$$\int_{B^N} \Big[L_\varepsilon^{\mathrm{GL}} q \cdot q + \frac{c_1 q^2}{\varepsilon^2} \Big] dx \le |\mathbb{S}^{N-1}| \int_0^1 r \Big\{ (1 - t_0 + \delta)(\tilde{q}')^2 + \frac{1}{r^2} \Big[\frac{\widetilde{N}(1 - t_0 - \delta) + (N-2)^2 \delta}{4} + \frac{c_1 r^2}{\varepsilon^2} \Big] \tilde{q}^2 \Big\} dr. \quad (2.17)$$

We now specify $\tilde{q} \in \mathrm{Lip}_c((0,1))$ by setting $\tilde{q}(r) = \sin(\frac{\pi}{\ln(a/C_\delta)} \ln \frac{r}{C_\delta \varepsilon})$ for $r \in (C_\delta \varepsilon, a\varepsilon)$ and $\tilde{q}(r) = 0$ otherwise. Note that $\widetilde{N}(1 - t_0 - \delta) + (N-2)^2 \delta = \widetilde{N}(1 - t_0) + c\delta$ for $c = 4N - 4 > 0$. Inserting into (2.17), we get

$$\int_{B^N} \Big[L_\varepsilon^{\mathrm{GL}} q \cdot q + \frac{c_1 q^2}{\varepsilon^2} \Big] dx \le \frac{|\mathbb{S}^{N-1}| \ln \frac{a}{C_\delta}}{2} \Big(\Big(\frac{\pi}{\ln \frac{a}{C_\delta}} \Big)^2 (1 - t_0 + \delta) + \frac{\widetilde{N}(1 - t_0) + c\delta}{4} + c_1 a^2 \Big). \quad (2.18)$$

Recalling $\widetilde{N} = N^2 - 8N + 8 < 0$ for $2 \le N \le 6$, we can choose $\delta > 0$ small, $a = a_\delta > 0$ large and then $c_1 = c_1(\delta) > 0$ small such that the right-hand side of (2.18) is negative for $\varepsilon < \varepsilon_1(\delta)$, yielding Step 1.

Step 2: We prove that there exists $\varepsilon_0 > 0$ such that $\ell(\varepsilon) < 0$ and increasing in $(0, \varepsilon_0)$, $\ell(\varepsilon_0) = 0$ and $\ell(\varepsilon) > 0$ for $\varepsilon > \varepsilon_0$. Let $I = \{\varepsilon \in (0, \infty) : \ell(\varepsilon) < 0\}$. It is clear that $\ell(\varepsilon) > 0$ for large ε and so I is bounded. By Step 1, I contains $(0, \varepsilon_1)$. Let

$$\varepsilon_0 = \sup\{\tilde{\varepsilon} : \ell(\varepsilon) < 0 \quad \text{for } \varepsilon \in (0, \tilde{\varepsilon})\} \in (\varepsilon_1, \infty).$$

By the continuity of ℓ, we must have $\ell(\varepsilon_0) = 0$. Then (2.2) yields the monotonicity of ℓ in $(0, \varepsilon_0)$ and also, $\ell(\varepsilon) > 0$ for $\varepsilon > \varepsilon_0$. Step 2 is proved. ■

2.3. The extended model: Existence

The aim is to prove Theorem 2.4 for the extended model.

Proof of Theorem 2.4.

Proof of (a) *when $N \ge 7$.* By [28, Theorem 2],[10] when $N \ge 7$, $\overline{m}_\varepsilon(x) = (f_\varepsilon(|x|) n(x), 0)$ is the unique minimizer for $E_{\varepsilon,\infty} \colon \mathcal{A} \subset H^1(B^N, \mathbb{R}^{N+1}) \to [0, \infty]$, i.e.

$$E_{\varepsilon,\infty}[m] = \int_{B^N} \Big[\frac{1}{2} |\nabla m|^2 + \frac{1}{2\varepsilon^2} W(1 - |m|^2) \Big] dx, \quad \varepsilon > 0.$$

As $\widetilde{W} \ge 0$, it follows that for every $\varepsilon, \eta > 0$, $\overline{m}_\varepsilon$ is the unique minimizer of $E_{\varepsilon,\eta}$ in $\mathcal{A}$ and so $(f_\varepsilon, 0)$ is the unique minimizer of $I_{\varepsilon,\eta}$ in $\mathcal{B}$. This together with Proposition 2.12 implies that (1.12)–(1.15) has no escaping solution.

[10] [28, Theorem 1] assumes strict convexity of W, but its proof uses only (1.10).

Proof of (a) *when* $W'(1) = 0$. When $W'(1) = 0$, we have by (1.10) that $W = 0$ in $[0, 1]$. In particular, $E_{\varepsilon,\infty}$ is exactly the Dirichlet energy (and hence convex) when restricting to the set $\{m \in \mathcal{A} : |m| \le 1 \text{ a.e.}\}$. This together with the fact that for $m \in \mathcal{A}$,

$$E_{\varepsilon,\infty}[m] \ge E_{\varepsilon,\infty}[m^\sharp] \quad \text{where } m^\sharp(x) = \begin{cases} m(x) & \text{if } |m| \le 1, \\ \dfrac{m(x)}{|m(x)|} & \text{if } |m(x)| > 1, \end{cases}$$

implies that the unique minimizer of $E_{\varepsilon,\infty}$ is the map $Y(x) = (x, 0)$ (i.e. the unique $H^1(B^N, \mathbb{R}^{N+1})$ harmonic map with boundary value $(x, 0)$). Also, if $W \equiv 0$ in $[0, 1]$, then $f_\varepsilon(r) = r$ solves (1.2)–(1.3), so by Theorem 2.1, f_ε is the unique solution of (1.2)–(1.3). Thus, $\bar{m}_\varepsilon = (f_\varepsilon n(x), 0) = Y$. Thus, $\bar{m}_\varepsilon$ is the unique minimizer of $E_{\varepsilon,\infty}$ and hence of $E_{\varepsilon,\eta}$ (since $\widetilde{W} \ge \widetilde{W}(0)$) in $\mathcal{A}$; in particular, $(f_\varepsilon, 0)$ is the unique minimizer of $I_{\varepsilon,\eta}$ over $\mathcal{B}$. By Proposition 2.12, we conclude that (1.12)–(1.15) has no escaping solution.

Proof of (b). First, we focus on the existence of escaping solutions of (1.12)–(1.15) when $2 \le N \le 6$ and $W'(1) > 0$. It is easy to see that $I_{\varepsilon,\eta}$ admits a minimizer $(f_{\varepsilon,\eta}, g_{\varepsilon,\eta}) \in \mathcal{B}$. Since $(f_{\varepsilon,\eta}, g_{\varepsilon,\eta}) \in \mathcal{B}$, $(f_{\varepsilon,\eta}, g_{\varepsilon,\eta}) \in C((0, 1])$. It follows that $(f_{\varepsilon,\eta}, g_{\varepsilon,\eta})$ satisfies (1.13)–(1.15) in the weak sense, and so $(f_{\varepsilon,\eta}, g_{\varepsilon,\eta}) \in C^2((0, 1])$.

Since $(|f_{\varepsilon,\eta}|, |g_{\varepsilon,\eta}|)$ is also a minimizer of $I_{\varepsilon,\eta}$ in $\mathcal{B}$, the above argument also shows that $(|f_{\varepsilon,\eta}|, |g_{\varepsilon,\eta}|) \in C^2((0, 1])$ satisfies (1.13)–(1.15). Since $|f_{\varepsilon,\eta}|, |g_{\varepsilon,\eta}| \ge 0$ and $f_{\varepsilon,\eta}(1) = 1$, by the strong maximum principle, we have that $|f_{\varepsilon,\eta}| > 0$ in $(0, 1)$, and either $|g_{\varepsilon,\eta}| > 0$ in $(0, 1)$ or $g_{\varepsilon,\eta} \equiv 0$ in $(0, 1)$. It follows that $f_{\varepsilon,\eta} > 0$ in $(0, 1)$, and either $g_{\varepsilon,\eta} > 0$ in $(0, 1)$ or $g_{\varepsilon,\eta} < 0$ in $(0, 1)$ or $g_{\varepsilon,\eta} \equiv 0$ in $(0, 1)$. Clearly, when $g_{\varepsilon,\eta} \equiv 0$, $f_{\varepsilon,\eta}$ is equal to the radial profile f_ε obtained in Theorem 2.1. By considering $(f_{\varepsilon,\eta}, -g_{\varepsilon,\eta})$ instead of $(f_{\varepsilon,\eta}, g_{\varepsilon,\eta})$ if necessary, we assume in the sequel that $g_{\varepsilon,\eta} \ge 0$.

Claim: $g_{\varepsilon,\eta} > 0$ *if and only if* $(\varepsilon, \eta) \in A := \{(\varepsilon, \eta) : 0 < \varepsilon < \varepsilon_0,\ \eta > \eta_0(\varepsilon)\}$.

Proof of claim. Define

$$Q_{\varepsilon,\eta}[\alpha, \beta] = \int_{B^N} \Big[L_\varepsilon^{\mathrm{GL}} \alpha \cdot \alpha + L_\varepsilon^{\mathrm{GL}} \beta \cdot \beta + \frac{N-1}{r^2} \alpha^2 \\ + \frac{2}{\varepsilon^2} W''(1 - f_\varepsilon^2) f_\varepsilon^2 \alpha^2 + \frac{1}{\eta^2} \widetilde{W}'(0) \beta^2 \Big] \, dx,$$

for (α, β) belonging to the Hilbert space $\mathcal{H} = \{(\alpha, \beta) : (f_\varepsilon + \alpha, \beta) \in \mathcal{B}\}$ with the norm $\|(\alpha, \beta)\|_{\mathcal{H}} := \|(\alpha n, \beta)\|_{H^1(B^N, \mathbb{R}^{N+1})}$. This can be considered as the second variation of $I_{\varepsilon,\eta}$ at $(f_\varepsilon, 0)$; see equation (3.1) in Section 3.1. The C^2 regularity of W together with (1.10), $\widetilde{W}'(0) \ge 0$ and the boundedness of f_ε yield a constant $c_1 > 0$ (independent of ε and η) such that

$$Q_{\varepsilon,\eta}[\alpha, \beta] \ge \|(\alpha, \beta)\|_{\mathcal{H}}^2 - \frac{c_1}{\varepsilon^2} \|(\alpha, \beta)\|_{L^2(B^N)}^2 \quad \text{for all } (\alpha, \beta) \in \mathcal{H}. \tag{2.19}$$

($\Leftarrow$). If $(\varepsilon, \eta) \in A$, then $\frac{\widetilde{W}'(0)}{\eta^2} < -\ell(\varepsilon)$. Taking $\beta \in H_0^1(B^N)$ to be any first eigenfunction of $L_\varepsilon^{\mathrm{GL}}$, which is radially symmetric, we have $r^{\frac{N-1}{2}} \beta', r^{\frac{N-1}{2}} \beta \in L^2(0, 1)$, $\beta(1) = 0$ and $Q_{\varepsilon,\eta}[0, \beta] < 0$. This implies that $(f_\varepsilon, 0)$ is not minimizing $I_{\varepsilon,\eta}$ in $\mathcal{B}$, and thus $g_{\varepsilon,\eta} > 0$.

($\Rightarrow$). For the converse, we suppose by contradiction that there exists $(\varepsilon, \eta) \in B = (0,\infty)^2 \setminus A$ with $g_{\varepsilon,\eta} > 0$. By (2.15) with the choice $\varphi = 1$ and by (1.2),

$$\int_{B^N} L_\varepsilon^{\mathrm{GL}} \alpha \cdot \alpha \, dx = \int_{B^N} \Big\{ f_\varepsilon^2 \Big| \nabla \Big(\frac{\alpha}{f_\varepsilon} \Big) \Big|^2 - \frac{N-1}{r^2} \alpha^2 \Big\} \, dx \quad \text{for } \alpha \in C_c^\infty(0,1).$$

By a density argument in $H_0^1(B^N)$ using Fatou's lemma, we deduce by (1.10),

$$Q_{\varepsilon,\eta}[\alpha,\beta] \geq \int_{B^N} \Big\{ f_\varepsilon^2 \Big| \nabla \Big(\frac{\alpha}{f_\varepsilon} \Big) \Big|^2 + \Big(\ell(\varepsilon) + \frac{\widetilde{W}'(0)}{\eta^2} \Big) \beta^2 \Big\} \, dx \quad \text{for } (\alpha,\beta) \in \mathcal{H}.$$

In view of Lemma 2.3, we thus have that $Q_{\varepsilon,\eta}$ is positive definite over $\mathcal{H}$ for $(\varepsilon,\eta) \in \mathring{B} = (0,\infty)^2 \setminus \bar{A}$ where $\ell(\varepsilon) + \frac{\widetilde{W}'(0)}{\eta^2} > 0$. More precisely, there exists a constant $c > 0$ (depending on ε and η) such that $Q_{\varepsilon,\eta}[\alpha,\beta] \geq c \|(\alpha,\beta)\|^2_{L^2(B^N)}$ for every $(\alpha,\beta) \in \mathcal{H}$. This follows by the above inequality for $Q_{\varepsilon,\eta}[\alpha,\beta]$ combined with the following estimate based on the Hardy inequality in $\mathbb{R}^{N+2}$ using $r \leq f_\varepsilon(r) \leq 1$ for every $r \in (0,1)$ (see Corollary 2.11):

$$\begin{aligned} \int_0^1 r^{N-1} f_\varepsilon^2 (h')^2 \, dr &\geq \int_0^1 r^{N+1} (h')^2 \, dr \geq \frac{N^2}{4} \int_0^1 r^{N-1} h^2 \, dr \\ &\geq \frac{N^2}{4} \int_0^1 r^{N-1} f_\varepsilon^2 h^2 \, dr, \end{aligned} \tag{2.20}$$

where h plays the role of $\frac{\alpha}{f_\varepsilon}$. Thus, by (2.19), for $(\varepsilon,\eta) \in \mathring{B}$, there exists a constant $\tilde{c} > 0$ (depending on ε and η) such that

$$Q_{\varepsilon,\eta}[\alpha,\beta] \geq \tilde{c} \|(\alpha,\beta)\|^2_{\mathcal{H}} \quad \text{for all } (\alpha,\beta) \in \mathcal{H}. \tag{2.21}$$

Fact. *If $(\varepsilon,\eta) \in \mathring{B}$, then $(f_\varepsilon, 0)$ is a local minimizer of $I_{\varepsilon,\eta}$.* For $(\alpha,\beta) \in \mathcal{H}$,

$$|\mathbb{S}^{N-1}| (I_{\varepsilon,\eta}[f_\varepsilon + \alpha, \beta] - I_{\varepsilon,\eta}[f_\varepsilon, 0]) - \frac{1}{2} Q_{\varepsilon,\eta}[\alpha,\beta] \overset{(1.2)}{=} \int_{B^N} h(x, \alpha(r)n, \beta(r)) \, dx,$$

$$\begin{aligned} h(x,V) = \frac{1}{2\varepsilon^2} \big\{ & W(1 - |f_\varepsilon(r)n + V_\parallel|^2 - V_{N+1}^2) - W(1 - f_\varepsilon(r)^2) \\ & + W'(1 - f_\varepsilon(r)^2)(2 f_\varepsilon(r) n \cdot V_\parallel + |V|^2) \\ & - 2W''(1 - f_\varepsilon(r)^2) f_\varepsilon(r)^2 (n \cdot V_\parallel)^2 \big\} \\ + \frac{1}{2\eta^2} \big\{ & \widetilde{W}(V_{N+1}^2) - \widetilde{W}(0) - \widetilde{W}'(0) V_{N+1}^2 \big\}, \qquad r = |x|, \\ & \qquad V = (V_\parallel, V_{N+1}) \in \mathbb{R}^{N+1}. \end{aligned}$$

We have $h \in C^0(\bar{B}^N, C^2(\mathbb{R}^{N+1}))$ (since $W, \widetilde{W} \in C^2$ and $f_\varepsilon n \in C^2(\bar{B}^N)$ by Lemma A.4), $h(x,0) = 0$, $\nabla_V h(x,0) = 0$, $\nabla_V^2 h(x,0) = 0$ (thus, (D.1) holds true in Proposition D.1) and h satisfies the growth assumption (D.2) in Proposition D.1 for $p = 2$ (due to the convexity of W and $\widetilde{W}$); therefore, Proposition D.1 applies and yields some small radius $\tilde{r} > 0$ such that

$$\int_{B^N} h(x, \alpha(r)n, \beta(r)) \, dx \geq -\frac{\tilde{c}}{4} \|(\alpha,\beta)\|^2_{\mathcal{H}} \quad \text{for } \|(\alpha,\beta)\|_{\mathcal{H}} < \tilde{r}.$$

Combined with (2.21), the local minimality of $(f_\varepsilon, 0)$ follows.

End of proof of the claim. As the constructed minimizer $(f_{\varepsilon,\eta}, g_{\varepsilon,\eta})$ of $I_{\varepsilon,\eta}$ satisfies $g_{\varepsilon,\eta} > 0$, Fact 2.3 combined with Lemma 2.13 below yields $(\varepsilon,\eta) \in B \setminus \mathring{B}$ and, for all $(\tilde{\varepsilon},\tilde{\eta}) \in \mathring{B}$, $(f_{\tilde{\varepsilon}}, 0)$ is the unique minimizer for $I_{\tilde{\varepsilon},\tilde{\eta}}$ in $\mathcal{B}$. Thanks to the latter, by considering a sequence $\{(\tilde{\varepsilon}_j, \tilde{\eta}_j)\} \subset \mathring{B}$ which converges to (ε,η), since $f_{\tilde{\varepsilon}_j}$ converges to f_ε in $H^1(B^N)$, Fatou's lemma implies that $(f_\varepsilon, 0)$ is a minimizer for $I_{\varepsilon,\eta}$ in $\mathcal{B}$, which contradicts the fact that $(f_{\varepsilon,\eta}, \pm g_{\varepsilon,\eta})$ are the only two minimizers of $I_{\varepsilon,\eta}$ in $\mathcal{B}$ (see Proposition 2.12). ■

Proof of (b1). By the claim, an escaping solution of (1.12)–(1.15) exists if and only if $0 < \varepsilon < \varepsilon_0$ and $\eta > \eta_0(\varepsilon)$. In this case, the uniqueness of an escaping solution and the classification of minimizers of $I_{\varepsilon,\eta}$ are obtained in Proposition 2.12, Lemma 2.7 yields $f_{\varepsilon,\eta}^2 + g_{\varepsilon,\eta}^2 < 1$, the regularity of $(f_{\varepsilon,\eta}, g_{\varepsilon,\eta})$ follows from Lemma A.5, while the positivity of $f_{\varepsilon,\eta}$ and monotonicity of $f_{\varepsilon,\eta}$ and $g_{\varepsilon,\eta}$ are given by Propositions 2.10 and 2.9.

Proof of (b2). The fact that the non-escaping solution $(f_\varepsilon, 0)$ is an unstable critical point (and hence not a minimizer) of $I_{\varepsilon,\eta}$ in $\mathcal{B}$ when $0 < \varepsilon < \varepsilon_0$ and $\eta > \eta_0(\varepsilon)$ was obtained in the proof of the ($\Leftarrow$) part of the claim. The fact that the non-escaping solution $(f_\varepsilon, 0)$ is the unique minimizer of $I_{\varepsilon,\eta}$ in $\mathcal{B}$ when $\varepsilon \geq \varepsilon_0$ or $0 < \eta \leq \eta_0(\varepsilon)$ follows from the claim. ■

It remains to prove the following lemma used above:

Lemma 2.13. *Let $N \geq 2$, $\varepsilon, \eta > 0$, and suppose that $W \in C^2((-\infty, 1])$ and $\widetilde{W} \in C^2([0,\infty))$ satisfy* (1.10) *and* (1.11). *If $I_{\varepsilon,\eta}$ admits an escaping critical point $(f_{\varepsilon,\eta}, g_{\varepsilon,\eta})$ in $\mathcal{B}$ with $g_{\varepsilon,\eta} > 0$ in $(0,1)$, then the non-escaping critical point $(f_\varepsilon, 0)$ is not a local minimizer of $I_{\varepsilon,\eta}$. As a consequence, if the non-escaping critical point $(f_\varepsilon, 0)$ is a local minimizer of $I_{\varepsilon,\eta}$, then $(f_\varepsilon, 0)$ is the unique* global *minimizer of $I_{\varepsilon,\eta}$ in $\mathcal{B}$ and $I_{\varepsilon,\eta}$ does not admit any escaping critical point $(f_{\varepsilon,\eta}, g_{\varepsilon,\eta})$ in $\mathcal{B}$ with $g_{\varepsilon,\eta} > 0$ in $(0,1)$.*

Proof. By Proposition 2.12, $(f_{\varepsilon,\eta}, \pm g_{\varepsilon,\eta})$ are the only two minimizers of $I_{\varepsilon,\eta}$ in $\mathcal{B}$. In particular, $I_{\varepsilon,\eta}[f_{\varepsilon,\eta}, g_{\varepsilon,\eta}] < I_{\varepsilon,\eta}[f_\varepsilon, 0]$. Suppose by contradiction that $(f_\varepsilon, 0)$ is a local minimizer of $I_{\varepsilon,\eta}$. We use some ideas from [3,27]: we show, by means of a mountain-pass theorem, the existence of a second escaping critical point $(\hat{f}, \hat{g})$ of $I_{\varepsilon,\eta}$ with $\hat{g} > 0$, which would lead to a contradiction with Proposition 2.12. Along the way, care is given due to the fact that $I_{\varepsilon,\eta}$ is not always finite in $\mathcal{B}$. To avoid this problem, let $V, \widetilde{V} \in C^2(\mathbb{R})$ be bounded non-negative functions such that $V|_{[0,1]} = W|_{[0,1]}$, $\widetilde{V}|_{[0,1]} = \widetilde{W}|_{[0,1]}$ and define $J\colon \mathcal{H} \to \mathbb{R}$ by

$$J[\alpha,\beta] = \frac{1}{2}\int_0^1 \Big[((f_\varepsilon+\alpha)')^2 + (\beta')^2 + \frac{N-1}{r^2}(f_\varepsilon+\alpha)^2 + \frac{1}{\varepsilon^2}V(1-(f_\varepsilon+\alpha)^2-\beta^2) + \frac{1}{\eta^2}\widetilde{V}(\beta^2)\Big] r^{N-1}\,dr.$$

Let $\mathcal{M} := \{(\alpha,\beta) \in \mathcal{H} : f_\varepsilon + \alpha \geq 0,\ \beta \geq 0 \text{ and } (f_\varepsilon+\alpha)^2 + \beta^2 \leq 1 \text{ in } (0,1)\}$. Then $J \in C^1(\mathcal{H})$, $\mathcal{M}$ is a closed convex subset of $\mathcal{H}$, $J[\alpha,\beta] = I_{\varepsilon,\eta}[f_\varepsilon+\alpha,\beta]$ for $(\alpha,\beta) \in \mathcal{M}$, and $(0,0)$ and $(f_{\varepsilon,\eta} - f_\varepsilon, g_{\varepsilon,\eta})$ are two relative minima of J in $\mathcal{M}$ with $J(f_{\varepsilon,\eta} - f_\varepsilon, g_{\varepsilon,\eta}) <$

$J(0,0)$. We proceed to check that J satisfies the Palais–Smale condition on $\mathcal{M}$ (see e.g. [43, Theorem II.12.8]): if $\{(\alpha_j, \beta_j)\} \subset \mathcal{M}$ is such that $\{J[\alpha_j, \beta_j]\}$ is bounded and

$$G[\alpha_j, \beta_j] := \sup_{\substack{(\alpha_j-\varphi, \beta_j-\psi)\in\mathcal{M} \\ \|(\varphi,\psi)\|_{\mathcal{H}}\le 1}} \langle DJ[\alpha_j, \beta_j], (\varphi, \psi)\rangle \to 0, \tag{2.22}$$

then $\{(\alpha_j, \beta_j)\}$ is relatively compact in $\mathcal{H}$. Indeed, since $\{J[\alpha_j, \beta_j]\}$ is bounded, $\{(\alpha_j, \beta_j)\}$ is bounded in $\mathcal{H}$. Thus, we assume that (α_j, β_j) converges to $(\alpha_*, \beta_*) \in \mathcal{M}$ weakly in $\mathcal{H}$, strongly in $L^2(B^N)$ and a.e. in $(0,1)$.

We may use

$$(\varphi, \psi) = t(\alpha_j - \alpha_*, \beta_j - \beta_*) = t((f_\varepsilon + \alpha_j) - (f_\varepsilon + \alpha_*), \beta_j - \beta_*)$$

for a small $t > 0$ (which is independent of j) in (2.22), since $(\alpha_j - \varphi, \beta_j - \psi)$ is a convex combination of $(\alpha_j, \beta_j), (\alpha_*, \beta_*) \in \mathcal{M}$ and $\mathcal{M}$ is convex. It gives

$$\begin{aligned}
0 &\ge \limsup_{j\to\infty} \langle DJ[\alpha_j, \beta_j], (\alpha_j - \alpha_*, \beta_j - \beta_*)\rangle \\
&= \limsup_{j\to\infty} \int_0^1 \Big[(f_\varepsilon + \alpha_j)'(\alpha_j - \alpha_*)' + \beta_j'(\beta_j - \beta_*)' + \frac{N-1}{r^2}(f_\varepsilon + \alpha_j)(\alpha_j - \alpha_*) \\
&\qquad - \frac{1}{\varepsilon^2} W'(1 - (f_\varepsilon + \alpha_j)^2 - \beta_j^2)[(f_\varepsilon + \alpha_j)(\alpha_j - \alpha_*) + \beta_j(\beta_j - \beta_*)] \\
&\qquad + \frac{1}{\eta^2} \widetilde{W}'(\beta_j^2)\beta_j(\beta_j - \beta_*)\Big] r^{N-1}\, dr.
\end{aligned}$$

Using the strong convergence of (α_j, β_j) to (α_*, β_*) in $L^2(B^N)$ and the boundedness of (α_j, β_j) in $L^\infty(B^N)$, the last two lines above converge to 0 as $j \to \infty$. Then writing $\alpha_j - \alpha_* = (f_\varepsilon + \alpha_j) - (f_\varepsilon + \alpha_*)$, by the weak convergence of (α_j, β_j) in $\mathcal{H}$, we get

$$\begin{aligned}
0 \ge \limsup_{j\to\infty} &\int_0^1 \Big[((f_\varepsilon + \alpha_j)')^2 + (\beta_j')^2 + \frac{N-1}{r^2}(f_\varepsilon + \alpha_j)^2\Big] r^{N-1}\, dr \\
&- \int_0^1 \Big[((f_\varepsilon + \alpha_*)')^2 + (\beta_*')^2 + \frac{N-1}{r^2}(f_\varepsilon + \alpha_*)^2\Big] r^{N-1}\, dr.
\end{aligned}$$

Thus, $\|((f_\varepsilon + \alpha_j)n, \beta_j)\|_{H^1(B^N, \mathbb{R}^{N+1})}$ converges to $\|((f_\varepsilon + \alpha_*)n, \beta_*)\|_{H^1(B^N, \mathbb{R}^{N+1})}$ and so $((f_\varepsilon + \alpha_j)n, \beta_j)$ converges strongly in $H^1(B^N, \mathbb{R}^{N+1})$ to $((f_\varepsilon + \alpha_*)n, \beta_*)$. This means also that (α_j, β_j) converges strongly in $\mathcal{H}$ to (α_*, β_*), giving the desired Palais–Smale property for J.

Applying the mountain pass theorem (see e.g. [43, Theorem II.12.8]), J has a mountain-pass critical point $(\hat\alpha_{\varepsilon,\eta}, \hat\beta_{\varepsilon,\eta}) \in \mathcal{M}$ relative to $\mathcal{M}$, i.e.

$$\sup_{\substack{(\hat\alpha_{\varepsilon,\eta}-\varphi, \hat\beta_{\varepsilon,\eta}-\psi)\in\mathcal{M} \\ \|(\varphi,\psi)\|_{\mathcal{H}}\le 1}} \langle DJ[\hat\alpha_{\varepsilon,\eta}, \hat\beta_{\varepsilon,\eta}], (\varphi, \psi)\rangle = 0. \tag{2.23}$$

In addition, $(\hat{\alpha}_{\varepsilon,\eta}, \hat{\beta}_{\varepsilon,\eta})$ is not a local minimizer of J relative to $\mathcal{M}$. For ease of exposition, we write $\hat{f} = f_\varepsilon + \hat{\alpha}_{\varepsilon,\eta}$ and $\hat{g} = \hat{\beta}_{\varepsilon,\eta}$. Then (2.23) means

$$0 = \sup\Big\{\int_0^1 r^{N-1}\Big[\hat{f}'\varphi' + \hat{g}'\psi' + \frac{N-1}{r^2}\hat{f}\varphi - \frac{1}{\varepsilon^2}W'(1-\hat{f}^2-\hat{g}^2)(\hat{f}\varphi+\hat{g}\psi) + \frac{1}{\eta^2}\widetilde{W}'(\hat{g}^2)\hat{g}\psi\Big]\,dr: \|(\varphi,\psi)\|_{\mathcal{H}} \leq 1, \hat{f}-\varphi \geq 0, \hat{g}-\psi \geq 0, (\hat{f}-\varphi)^2 + (\hat{g}-\psi)^2 \leq 1\Big\}. \quad (2.24)$$

To proceed, we show that $\hat{f}^2 + \hat{g}^2 < 1$ in $(0,1)$, $\hat{f} > 0$ in $(0,1)$ and either $\hat{g} \equiv 0$ in $(0,1)$ or $\hat{g} > 0$ in $(0,1)$, so that we have in fact that $(\hat{f},\hat{g})$ is either a non-escaping solution $(f_\varepsilon, 0)$ or an escaping solution of (1.12)–(1.15). Once this is proved, by Theorem 2.1 and Proposition 2.12, we then have that $(\hat{f},\hat{g})$ must be identical to either $(f_\varepsilon, 0)$ or $(f_{\varepsilon,\eta}, g_{\varepsilon,\eta})$, which contradicts the fact that $(\hat{\alpha}_{\varepsilon,\eta}, \hat{\beta}_{\varepsilon,\eta})$ is not a local minimizer of J relative to $\mathcal{M}$.

Indeed, if $(\varphi,\psi) = t\zeta(\hat{f},\hat{g})$ in (2.24), where $\zeta \in C_c^\infty(0,1)$ is non-negative and $t \geq 0$ is small enough that $0 \leq 1 - t\zeta \leq 1$ in $(0,1)$, then

$$-\frac{1}{2}\Big[(\hat{f}^2)'' + \frac{N-1}{r}(\hat{f}^2)'\Big] - \frac{1}{2}\Big[(\hat{g}^2)'' + \frac{N-1}{r}(\hat{g}^2)'\Big] + (\hat{f}')^2 + (\hat{g}')^2 + \frac{N-1}{r^2}\hat{f}^2 - \frac{1}{\varepsilon^2}W'(1-\hat{f}^2-\hat{g}^2)(\hat{f}^2+\hat{g}^2) + \frac{1}{\eta^2}\widetilde{W}'(\hat{g}^2)\hat{g}^2 \leq 0 \quad \text{in } (0,1)$$

in the sense of a distribution. It follows that the function $\widehat{X} = 1 - \hat{f}^2 - \hat{g}^2$, considered as a radially symmetric function in B^N, satisfies

$$-\widehat{X}'' - \frac{N-1}{r}\widehat{X}' + 2a(r)\widehat{X} \geq \frac{2(N-1)}{r^2}\hat{f}^2 \geq 0 \quad \text{in } (0,1),$$

where the continuous function $a\colon (0,1] \to [0,\infty)$ is given in (2.5). Since $\widehat{X} \geq 0$, we deduce from the strong maximum principle that either $\widehat{X} \equiv 0$ in $(0,1)$ or $\widehat{X} > 0$ in $(0,1)$. The case $\widehat{X} \equiv 0$ is impossible since it would imply, in view of the above differential inequality, that $\hat{f} \equiv 0$, contradicting that $\hat{f}(1) = 1$. We thus have $\widehat{X} > 0$ and $\hat{f}^2 + \hat{g}^2 < 1$ in $(0,1)$.

As $\hat{f}^2 + \hat{g}^2 < 1$ in $(0,1)$, we may use $(\varphi,\psi) = (-t\zeta, 0)$ in (2.24), where $\zeta \in C_c^\infty(0,1)$ is non-negative and $t \geq 0$ is sufficiently small so that $(\hat{f} + t\zeta)^2 + \hat{g}^2 < 1$ in $(0,1)$ to get $\hat{f}'' + \frac{N-1}{r}\hat{f}' - b(r)\hat{f} \leq 0$ in $(0,1)$ with $b(r) := \frac{N-1}{r^2} - \frac{1}{\varepsilon^2}W'(1-\hat{f}^2-\hat{g}^2) \in L^\infty_{\mathrm{loc}}((0,1])$. Since $\hat{f} \geq 0$ and $\hat{f}(1) = 1$, we have by the strong maximum principle that $\hat{f} > 0$ in $(0,1)$.

Likewise, we use $(\varphi,\psi) = (0, -t\zeta)$ in (2.24), where $\zeta \in C_c^\infty(0,1)$ is non-negative and $t \geq 0$ is sufficiently small so that $\hat{f}^2 + (\hat{g} + t\zeta)^2 < 1$ in $(0,1)$ to get $\hat{g}'' + \frac{N-1}{r}\hat{g}' - c(r)\hat{g} \leq 0$ in $(0,1)$ with $c(r) := -\frac{1}{\varepsilon^2}W'(1-\hat{f}^2-\hat{g}^2) + \frac{1}{\eta^2}\widehat{W}'(\hat{g}^2)$. Since $\hat{g} \geq 0$, we have by the strong maximum principle that either $\hat{g} \equiv 0$ in $(0,1)$ or $\hat{g} > 0$ in $(0,1)$. As explained earlier, this together with the previous shown facts that $\hat{f}^2 + \hat{g}^2 < 1$ and $\hat{f} > 0$ in $(0,1)$ shows that the statement that $(\varepsilon,\eta) \in \mathring{B}$ amounts to a contradiction.

Finally, we explain the stated consequence: by the proof of Theorem 2.4 b), any minimizer $(f_{\varepsilon,\eta}, g_{\varepsilon,\eta})$ of $I_{\varepsilon,\eta}$ in $\mathcal{B}$ satisfies $|g_{\varepsilon,\eta}| > 0$ or $g_{\varepsilon,\eta} \equiv 0$. As we have just proved that escaping critical points of $I_{\varepsilon,\eta}$ cannot exist whenever $(f_\varepsilon, 0)$ is a local minimizer of $I_{\varepsilon,\eta}$, we conclude that every minimizer satisfies $g_{\varepsilon,\eta} \equiv 0$, i.e. it is given by $(f_\varepsilon, 0)$. ■

2.4. The $\mathbb{S}^N$-valued model: Existence, monotonicity and uniqueness

We start with positivity of $\tilde{f}_\eta$ and the monotonicity for an escaping solution $(\tilde{f}_\eta, g_\eta)$ of (1.4)–(1.8) with $g_\eta > 0$. Next we prove Theorem 2.6.

Proposition 2.14. *Suppose* $\widetilde{W} \in C^2([0,\infty))$ *satisfies* (1.11)*, and* $(\tilde{f}_\eta, g_\eta)$ *satisfies* (1.4)–(1.8) *with* $g_\eta > 0$ *in* $(0,1)$*. Then* $\tilde{f}_\eta > 0$, $\tilde{f}_\eta' > 0$, $g_\eta' < 0$ *and* $(\frac{\tilde{f}_\eta}{r})' \le 0$ *in* $(0,1]$.

Proof. We adapt the strategy in the proofs of Propositions 2.9 and 2.10. By Lemma A.6, $(\tilde{f}_\eta, g_\eta) \in C^2([0,1], \mathbb{S}^1)$ and $f(0) = 0$. Recalling also that $g_\eta > 0$, we may thus write $\tilde{f}_\eta = \sin\theta$, $g_\eta = \cos\theta$ in $[0,1]$, where the lifting $\theta\colon [0,1] \to [-\pi/2, \pi/2]$ is C^2, $\theta(0) = 0$ and $\theta(1) = \pi/2$. Then θ satisfies

$$\theta'' + \frac{N-1}{r}\theta' = \frac{N-1}{r^2}\sin\theta\cos\theta - \frac{1}{\eta^2}\widetilde{W}'(\cos^2\theta)\sin\theta\cos\theta =: P(r,\theta) \qquad (2.25)$$

in $(0,1)$. Since $\theta(1) = \pi/2$, $\theta \le \pi/2$ in $(0,1)$ and $\pi/2$ is a constant solution of (2.25), the maximum principle and the Hopf lemma applied to (2.25) yield $\theta < \pi/2$ in $(0,1)$ and $\theta'(1) > 0$.

Let $r_1 \in [0,1)$ be such that $\theta(r_1) = 0$ and $\theta > 0$ in $(r_1, 1]$. Observe that, if $r_1 > 0$, then by applying the Hopf lemma to (2.25) in $(r_1, 1)$, we have $\theta'(r_1) > 0$. In particular, $\theta < 0$ in a small interval $(r_1 - \delta, r_1)$ when $r_1 > 0$.

Note that, since $P(r,\theta)$ is odd in θ, $|\theta|$ satisfies in the sense of distributions,

$$|\theta|'' + \frac{N-1}{r}|\theta|' = P(r,|\theta|) \quad \text{in } (r_1, 1),$$
$$|\theta|'' + \frac{N-1}{r}|\theta|' \ge P(r,|\theta|) \quad \text{in } (0, 1).$$

Since P is non-increasing in r, we apply the proof of Proposition 2.9 to get

$$(|\theta|)_s \ge |\theta| \quad \text{in } \max(0, 2s-1) < r < s \text{ for all } r_1 \le s < 1,$$

where $(|\theta|)_s(r) = |\theta|(2s - r)$. As in the proof of Proposition 2.10, the Hopf lemma then implies that $r_1 = 0$, i.e. $\theta > 0$ in $(0,1)$, and so the above gives

$$\theta_s \ge \theta \quad \text{in } \max(0, 2s-1) < r < s \text{ for all } 0 < s < 1.$$

In addition, we have that $\theta' > 0$ in $(0,1]$ (see Fact 2 in the proof of Proposition 2.9). In particular, $0 = \theta(0) < \theta < \theta(1) = \pi/2$ in $(0,1)$.

Returning to $(\tilde f_\eta, g_\eta)$, we have shown that $\tilde f_\eta > 0$, $\tilde f_\eta' > 0$ and $g_\eta' < 0$ in $(0,1]$. The statement $(\frac{\tilde f_\eta}{r})' \le 0$ in $(0,1]$ is obtained in the same way as in the last part of the proof of Proposition 2.9 using the following equivalent form of (1.6):

$$\Big(r^{N+1}\Big(\frac{\tilde f_\eta}{r}\Big)'\Big)' = -r^{N+1}\lambda(r)\frac{\tilde f_\eta}{r} \le 0 \quad \text{for } r \in (0,1). \qquad \blacksquare$$

Next we prove the uniqueness of escaping solutions of (1.4)–(1.8).

Proposition 2.15. *Let $N \ge 2$, $\eta > 0$ and $\widetilde W \in C^2([0,\infty))$ with (1.11). Then system (1.4)–(1.8) has at most one escaping solution $(\tilde f_\eta, g_\eta)$ with $g_\eta > 0$ in $(0,1)$. Furthermore, when it exists, then $(\tilde f_\eta, \pm g_\eta)$ are the only two minimizers of the functional I_η^{MM} in $\mathcal{B}^{\mathrm{MM}}$.*

Proof. By Proposition 2.14, we have $\tilde f_\eta > 0$ in $(0,1)$ for any escaping $(\tilde f_\eta, g_\eta)$ with $g_\eta > 0$ in $(0,1)$ of system (1.4)–(1.8). To prove the uniqueness, we follow a similar argument to the proof of Proposition 2.12, adapted to the new target space $\mathbb{S}^N$. Indeed, denoting $m_\eta = (\tilde f_\eta(r)n(x), g_\eta(r)) \in H^1(B^N, \mathbb{S}^N)$ for a solution $(\tilde f_\eta, g_\eta)$ in $(0,1)$ of system (1.4)–(1.8) with $g_\eta > 0$ in $(0,1)$, take an arbitrary radial configuration $m = (f(r)n(x), g(r)) \in H^1(B^N, \mathbb{S}^N)$ with $m = (n, 0)$ on ∂B^N. Setting $V = m - m_\eta = (s(r)n, q(r)) \in H_0^1(B^N, \mathbb{R}^{N+1})$, the constraints $|m| = |m_\eta| = 1$ yield $\tilde f_\eta s + g_\eta q = m_\eta \cdot V = -\frac12 |V|^2$ in B^N. By the convexity of $\widetilde W$ and (1.6)–(1.7), we have

$$\begin{aligned}
&I_\eta^{\mathrm{MM}}[f,g] - I_\eta^{\mathrm{MM}}[\tilde f_\eta, g_\eta]\\
&\quad\ge \frac12 \int_0^1 r^{N-1}\Big\{2\tilde f_\eta' s' + (s')^2 + 2g_\eta' q' + (q')^2\\
&\qquad\qquad + \frac{N-1}{r^2}(2\tilde f_\eta s + s^2) + \frac{1}{\eta^2}\widetilde W'(g_\eta^2)(2g_\eta q + q^2)\Big\}\,dr\\
&\quad= \frac12 \int_0^1 r^{N-1}\Big\{(s')^2 + (q')^2 + \frac{N-1}{r^2}s^2 + \frac{1}{\eta^2}\widetilde W'(g_\eta^2)q^2 + 2\lambda(\tilde f_\eta s + g_\eta q)\Big\}\,dr\\
&\quad= \frac{1}{2|\mathbb{S}^{N-1}|}\int_{B^N}\Big\{|\nabla V|^2 + \frac{1}{\eta^2}\widetilde W'(g_\eta^2)V_{N+1}^2 - \lambda|V|^2\Big\}\,dx\\
&\quad=: \frac{1}{2|\mathbb{S}^{N-1}|}F_\eta^{\mathrm{MM}}[V].
\end{aligned}$$

Claim: *For every $V(x) = (s(r)n(x), q(r)) \in H_0^1(B^N, \mathbb{R}^{N+1})$, it holds that*

$$F_\eta^{\mathrm{MM}}[V] \ge \int_{B^N}\Big\{\tilde f_\eta^2(|x|)\Big|\nabla\Big(\frac{s}{\tilde f_\eta}\Big)(|x|)\Big|^2 + g_\eta^2(|x|)\Big|\nabla\Big(\frac{q}{g_\eta}\Big)(|x|)\Big|^2\Big\}\,dx.$$

Proof of claim. Since F_η^{MM} is continuous in $H_0^1(B^N, \mathbb{R}^{N+1})$ (because $\lambda, \widetilde W'(g_\eta^2) \in L^\infty(B^N)$ by Lemma A.6), by standard density results and Fatou's lemma, it suffices to show the claim for $V = (s(r)n, q(r)) \in C_c^\infty(B^N \setminus \{0\}, \mathbb{R}^{N+1})$. For that, we apply [25, Lemma A.1] to

$$\tilde L := -\Delta - \lambda(r) \quad\text{and}\quad \widetilde T := -\Delta + \frac{1}{\eta^2}\widetilde W'(g_{\varepsilon,\eta}^2) - \lambda(r).$$

Writing

$$V = (s(r)n, q(r)) = (V_1, \dots, V_N, V_{N+1}) \in C_c^\infty(B^N \setminus \{0\}, \mathbb{R}^{N+1}),$$
$$V_j = \tilde{f}_\eta \widehat{V}_j \quad \text{with } \widehat{V}_j = \frac{V_j}{\tilde{f}_\eta} \text{ for } j = 1, \dots, N,$$
$$V_{N+1} = g_\eta \widehat{V}_{N+1} \quad \text{with } \widehat{V}_{N+1} = \frac{q}{g_\eta},$$

we have

$$\begin{aligned} F_\eta^{\mathrm{MM}}[V] &= \sum_{j=1}^N \int_{B^N} \tilde{L} V_j \cdot V_j \, dx + \int_{B^N} \widetilde{T} V_{N+1} \cdot V_{N+1} \, dx \\ &= \sum_{j=1}^N \int_{B^N} \{\tilde{f}_\eta^2 |\nabla \widehat{V}_j|^2 + \widehat{V}_j^2 \tilde{L} \tilde{f}_\eta \cdot \tilde{f}_\eta\} \, dx \\ &\quad + \int_{B^N} \{g_\eta^2 |\nabla \widehat{V}_{N+1}|^2 + \widehat{V}_{N+1}^2 \widetilde{T} g_\eta \cdot g_\eta\} \, dx \\ &= \int_{B^N} \Big\{\tilde{f}_\eta^2 \Big|\nabla\Big(\frac{s(r)}{\tilde{f}_\eta(r)} n(x)\Big)\Big|^2 - \frac{N-1}{r^2} s^2 + g_\eta^2(|x|)\Big|\nabla\Big(\frac{q}{g_\eta}\Big)(|x|)\Big|^2\Big\} \, dx \\ &= \int_{B^N} \Big\{\tilde{f}_\eta^2(|x|)\Big|\nabla\Big(\frac{s}{\tilde{f}_\eta}\Big)(|x|)\Big|^2 + g_\eta^2(|x|)\Big|\nabla\Big(\frac{q}{g_\eta}\Big)(|x|)\Big|^2\Big\} \, dx, \end{aligned}$$

because $\tilde{L}\tilde{f}_\eta = -\frac{N-1}{r^2}\tilde{f}_\eta$, $\widetilde{T} g_\eta = 0$ (by (1.6)–(1.7)) and $(\widehat{V}_1, \dots, \widehat{V}_N) = \frac{s(r)}{\tilde{f}_\eta(r)} n(x)$ with $|\nabla n|^2 = \frac{N-1}{r^2}$. Hence, the claim is proved. ∎

Consequently, $(\tilde{f}_\eta, \pm g_\eta)$ minimizes I_η^{MM} in $\mathcal{B}^{\mathrm{MM}}$. If $(\hat{f}_\eta, \hat{g}_\eta)$ also minimizes I_η^{MM} in $\mathcal{B}^{\mathrm{MM}}$, the argument in Step 1 of the proof of Proposition 2.12 gives $(\hat{f}_\eta - \tilde{f}_\eta)/\tilde{f}_\eta$ and $(\hat{g}_\eta - g_\eta)/g_\eta$ are constant in $(0, 1)$. This together with $\hat{f}_\eta(1) - \tilde{f}_\eta(1) = 0$ gives $\hat{f}_\eta \equiv \tilde{f}_\eta$ and $\hat{g}_\eta \equiv a g_\eta$ in $(0, 1)$ for some constant $a \in \mathbb{R}$. Since $\tilde{f}_\eta^2 + g_\eta^2 = 1 = \hat{f}_\eta^2 + \hat{g}_\eta^2$ we deduce that $\hat{g}_\eta \equiv \pm g_\eta$ in $(0, 1)$. This proves that $(\tilde{f}_\eta, \pm g_\eta)$ are the only two minimizers of I_η^{MM} in $\mathcal{B}^{\mathrm{MM}}$. Lastly, if $(\check{f}_\eta, \check{g}_\eta)$ is also a solution to (1.4)–(1.8) with $\check{g}_\eta > 0$ in $(0, 1)$, then the claim yields that $(\check{f}_\eta, \check{g}_\eta)$ also minimizes I_η^{MM} in $\mathcal{B}^{\mathrm{MM}}$, and by the above, $\check{f}_\eta \equiv \tilde{f}_\eta$ and $\check{g}_\eta \equiv g_\eta$ in $(0, 1)$. The proof is complete. ∎

Proof of Theorem 2.6. Recall that for $N \geq 7$, since $\widetilde{W} \geq 0$, the equator map $\overline{m}(x) = (n(x), 0)$ is the unique minimizer of E_η^{MM} in $\mathcal{A}$ for every $\eta > 0$ (see Remark 1.1). Thus, by (1.11) and Proposition 2.15, escaping solutions of (1.4)–(1.8) do not exist for any $\eta > 0$.

Suppose now that $2 \leq N \leq 6$ and fix $\eta > 0$. The uniqueness of the escaping solution $(\tilde{f}_\eta, g_\eta)$ of (1.4)–(1.8) with $g_\eta > 0$, together with its minimality, monotonicity and positivity, was proved in Propositions 2.14 and 2.15 and its regularity follows from Lemma A.6 in Appendix A. It remains to prove the existence[11] of an escaping solution of (1.4)–(1.8) for $2 \leq N \leq 6$ and the instability of the non-escaping solution $(1, 0)$ for $3 \leq N \leq 6$.

[11]For the existence of an escaping solution, it suffices to assume $\widetilde{W} \in C^2([0, \infty))$ instead of (1.11).

Proof of the instability of $(1,0)$ *when* $3 \le N \le 6$. We show the second variation of I_η^{MM} in $\mathcal{B}^{\mathrm{MM}}$ at $(1,0)$ is not non-negative semi-definite, i.e. there exists $q \in \mathrm{Lip}_c(0,1)$ such that

$$Q_\eta^{\mathrm{MM}}[0,q] = \frac{d^2}{dt^2}\Big|_{t=0} I_\eta^{\mathrm{MM}}\Big(\frac{(1,tq)}{\sqrt{1+t^2q^2}}\Big)$$
$$= \int_0^1 \Big[(q')^2 - \frac{N-1}{r^2}q^2 + \frac{\widetilde{W}'(0)}{\eta^2}q^2\Big]r^{N-1}\,dr < 0. \tag{2.26}$$

To this end, we adapt the computation in Step 1 of the proof of Lemma 2.3 (c). Writing $q = \varphi\tilde{q}$ with $\varphi = r^{-\frac{N-2}{2}}$ and applying [24, Lemma A.1] (for the Laplace operator), we have for $\widetilde{N} = N^2 - 8N + 8 < 0$,

$$Q_\eta^{\mathrm{MM}}[0,q] = \int_0^1 \Big\{(\tilde{q}')^2 + \frac{1}{r^2}\Big[\frac{\widetilde{N}}{4} + \frac{\widetilde{W}'(0)r^2}{\eta^2}\Big]\tilde{q}^2\Big\}r\,dr.$$

For $0 < b < a < 1$ to be fixed, let $\tilde{q}(r) = \sin(\frac{\pi}{\ln\frac{a}{b}}\ln\frac{r}{b})$ for $r \in (b,a)$ and $\tilde{q}(r) = 0$ otherwise. We have

$$Q_\eta^{\mathrm{MM}}[0,q] \le \frac{1}{2}\ln\frac{a}{b}\Big\{\Big(\frac{\pi}{\ln\frac{a}{b}}\Big)^2 + \frac{\widetilde{N}}{4} + \frac{\widetilde{W}'(0)a^2}{\eta^2}\Big\}.$$

Noting that $\widetilde{N} < 0$ for $3 \le N \le 6$, we can select $0 \ll b \ll a \ll \eta$ such that the above quantity is negative.

Proof of the existence of an escaping solution. Minimizing I_η^{MM} in $\mathcal{B}^{\mathrm{MM}}$, we obtain a minimizer $(\tilde{f}_\eta, g_\eta) \in \mathcal{B}^{\mathrm{MM}}$. Replacing $(\tilde{f}_\eta, g_\eta)$ by $(|\tilde{f}_\eta|, |g_\eta|)$ if necessary, we have $\tilde{f}_\eta \ge 0$ and $g_\eta \ge 0$. It is readily seen that $(\tilde{f}_\eta, g_\eta)$ satisfies (1.4)–(1.8). By (1.6), the fact that $\tilde{f}_\eta(1) = 1$ and the strong maximum principle, $\tilde{f}_\eta > 0$ in $(0,1)$. By (1.7) and the strong maximum principle, either $g_\eta > 0$ or $g_\eta \equiv 0$ in $(0,1)$. The case $g_\eta \equiv 0$ cannot hold since it would imply $\tilde{f}_\eta \equiv 1$ in $(0,1)$ (since $\tilde{f}_\eta^2 + g_\eta^2 = 1$, $\tilde{f}_\eta(1) = 1$ and $\tilde{f}_\eta \in C((0,1]))$ and $N \ge 3$ (since $r^{\frac{N-3}{2}}\tilde{f}_\eta \in L^2(0,1)$), which contradicts the instability statement established above. ∎

Remark 2.16. For $N = 2$, if we define the second variation of I_η^{MM} at $(1,0)$ (in $\mathcal{B}^{\mathrm{MM}}$) along directions $(0,q)$ compactly supported in $(0,1)$ by (2.26), then the same proof as above yields a perturbation $q \in \mathrm{Lip}_c(0,1)$ such that

$$Q_\eta^{\mathrm{MM}}[0,q] < 0. \tag{2.27}$$

Remark 2.17. One can also prove Theorem 2.6 by considering the limit as $\varepsilon \to 0$ of the escaping (minimizing) solutions $(f_{\varepsilon,\eta} > 0, g_{\varepsilon,\eta} > 0)$ obtained in Theorem 2.4 for a fixed $\eta > 0$ with $W(t) = t^2$. The strong limit $(\tilde{f}_\eta, g_\eta)$ of $\{(f_{\varepsilon,\eta}, g_{\varepsilon,\eta})\}_{\varepsilon\to 0}$ in $\mathcal{B}$ is indeed escaping because the non-escaping solution $(1,0)$ (which corresponds to the equator map $\overline{m}(x) = (n(x),0))$ is unstable for I_η^{MM}. We omit the standard proof.

3. Stability analysis of vortex solutions

3.1. An orthogonal decomposition for the second variation

Assume that $N \geq 2$ and $W \in C^2((-\infty, 1])$ and $\widetilde{W} \in C^2([0,\infty))$. Let $m_{\varepsilon,\eta} = (f_{\varepsilon,\eta} n, g_{\varepsilon,\eta})$ be any (bounded) radially symmetric critical point of $E_{\varepsilon,\eta}$ in $\mathcal{A}$, and define the second variation $Q_{\varepsilon,\eta}\colon H_0^1(B^N, \mathbb{R}^{N+1}) \to \mathbb{R}$ of $E_{\varepsilon,\eta}$ at $m_{\varepsilon,\eta}$ as follows. Under our assumptions on W and $\widetilde{W}$, $E_{\varepsilon,\eta}$ may take an infinite value in any neighborhood of $m_{\varepsilon,\eta}$. To bypass this technical matter, we first define the second variation $Q_{\varepsilon,\eta}[V]$ along a direction $V = (v,q) \in C_c^\infty(B^N \setminus \{0\}, \mathbb{R}^N) \times C_c^\infty(B^N \setminus \{0\}, \mathbb{R}) \cong C_c^\infty(B^N \setminus \{0\}, \mathbb{R}^{N+1})$ by

$$\begin{aligned} Q_{\varepsilon,\eta}[V] &= \frac{d^2}{dt^2}\Big|_{t=0} E_{\varepsilon,\eta}[m_{\varepsilon,\eta} + tV] \\ &= \int_{B^N} \Big[|\nabla v|^2 + |\nabla q|^2 - \frac{1}{\varepsilon^2} W'(1 - f_{\varepsilon,\eta}^2 - g_{\varepsilon,\eta}^2)(|v|^2 + q^2) + \frac{1}{\eta^2}\widetilde{W}'(g_{\varepsilon,\eta}^2) q^2 \\ &\qquad + \frac{2}{\varepsilon^2} W''(1 - f_{\varepsilon,\eta}^2 - g_{\varepsilon,\eta}^2)(f_{\varepsilon,\eta} n \cdot v + g_{\varepsilon,\eta} q)^2 \\ &\qquad + \frac{2}{\eta^2} \widetilde{W}''(g_{\varepsilon,\eta}^2) g_{\varepsilon,\eta}^2 q^2 \Big]\, dx, \end{aligned} \tag{3.1}$$

and extend this definition to $V \in H_0^1(B^N, \mathbb{R}^{N+1})$ by density using the fact that the right-hand side of (3.1) is continuous $H_0^1(B^N, \mathbb{R}^{N+1})$ (because $f_{\varepsilon,\eta}, g_{\varepsilon,\eta} \in L^\infty(B^N)$ and W and $\widetilde{W}$ are twice continuously differentiable). We will see that this definition is appropriate for our proof of the local minimality of the escaping critical points.

In the sequel $A : B$ denotes the Frobenius scalar product of matrices. Writing $v = sn + w$, where $w \cdot n = 0$, with $s \in C_c^\infty(B^N \setminus \{0\}, \mathbb{R})$ and $w \in C_c^\infty(B^N \setminus \{0\}, \mathbb{R}^N)$, we compute

$$|\nabla v|^2 = |\nabla s|^2 + \frac{N-1}{r^2} s^2 + |\nabla w|^2 + 2\nabla(sn)\colon \nabla w,$$

$$\begin{aligned} \int_{B^N} \nabla(sn)\colon \nabla w\, dx &= -\int_{B^N} \Delta(sn) \cdot w\, dx = -2 \int_{B^N} \nabla s \cdot ((\nabla n)^t w)\, dx \\ &= -\int_{B^N} \frac{2}{r}(w \cdot \nabla) s\, dx, \end{aligned}$$

where we used $w \cdot \partial_k n = \frac{w_k}{r}$ for $1 \leq k \leq N$ because $w \cdot n = 0$. It follows that

$$\begin{aligned} Q_{\varepsilon,\eta}[V] = \int_{B^N} \Big[&|\nabla s|^2 + \frac{N-1}{r^2} s^2 + |\nabla w|^2 - \frac{4}{r}(w \cdot \nabla) s + |\nabla q|^2 \\ &- \frac{1}{\varepsilon^2} W'(1 - f_{\varepsilon,\eta}^2 - g_{\varepsilon,\eta}^2)(s^2 + |w|^2 + q^2) + \frac{1}{\eta^2} \widetilde{W}'(g_{\varepsilon,\eta}^2) q^2 \\ &+ \frac{2}{\varepsilon^2} W''(1 - f_{\varepsilon,\eta}^2 - g_{\varepsilon,\eta}^2)(f_{\varepsilon,\eta} s + g_{\varepsilon,\eta} q)^2 + \frac{2}{\eta^2} \widetilde{W}''(g_{\varepsilon,\eta}^2) g_{\varepsilon,\eta}^2 q^2 \Big]\, dx. \end{aligned}$$

We identify $x = (r, \theta)$, where $r = |x| \geq 0$ and $\theta = \frac{x}{|x|} \in \mathbb{S}^{N-1}$. Let $\not{D}$ denote the covariant derivative of the standard metric g_{round} on the unit sphere $\mathbb{S}^{N-1}$ and $d\sigma$ denote the surface measure on $\mathbb{S}^{N-1}$. For a tangent vector field w on $\mathbb{S}^{N-1}$ (i.e. $w \cdot n = 0$), one computes

$$|\nabla w|^2 = |\partial_r w|^2 + \frac{1}{r^2}(|w|^2 + |\not{D} w|^2), \tag{3.2}$$

$$\begin{aligned} Q_{\varepsilon,\eta}[V] = \int_0^1 \int_{\mathbb{S}^{N-1}} r^{N-1} \Big\{ &(\partial_r s)^2 + \frac{1}{r^2}|\not{D} s|^2 + \frac{N-1}{r^2} s^2 + |\partial_r w|^2 \\ &+ \frac{1}{r^2}|\not{D} w|^2 + \frac{1}{r^2}|w|^2 - \frac{4}{r^2}(w \cdot \not{D})s + (\partial_r q)^2 + \frac{1}{r^2}|\not{D} q|^2 \\ &- \frac{1}{\varepsilon^2} W'(1 - f_{\varepsilon,\eta}^2 - g_{\varepsilon,\eta}^2)(s^2 + |w|^2 + q^2) + \frac{1}{\eta^2}\widetilde{W}'(g_{\varepsilon,\eta}^2) q^2 \\ &+ \frac{2}{\varepsilon^2} W''(1 - f_{\varepsilon,\eta}^2 - g_{\varepsilon,\eta}^2)(f_{\varepsilon,\eta} s + g_{\varepsilon,\eta} q)^2 \\ &+ \frac{2}{\eta^2}\widetilde{W}''(g_{\varepsilon,\eta}^2) g_{\varepsilon,\eta}^2 q^2 \Big\}\, d\sigma\, dr. \end{aligned} \tag{3.3}$$

We start with an orthogonal decomposition for $Q_{\varepsilon,\eta}$. Let $\lambda_0 = 0 < \lambda_1 \le \lambda_2 \le \cdots \to \infty$ be the eigenvalues of the Laplacian $-\not{\Delta}$ on $\mathbb{S}^{N-1}$, and let $\zeta_0, \zeta_1, \ldots$ be a corresponding orthonormal eigenbasis of $L^2(\mathbb{S}^{N-1})$. In particular, $\lambda_k = N-1$ for $k = 1, \ldots, N$, $\lambda_k \ge 2N$ for $k \ge N+1$, and the first $N+1$ eigenfunctions can be taken as $\zeta_0(\theta) = \frac{1}{\sqrt{|\mathbb{S}^{N-1}|}}$, $\zeta_k(\theta) = \sqrt{\frac{N}{|\mathbb{S}^{N-1}|}}\theta_k$ for $1 \le k \le N$. Moreover, $\int_{\mathbb{S}^{N-1}} \zeta_k\, d\sigma = 0$ for all $k \ge 1$.

Proposition 3.1. *Let $N \ge 2$, $W \in C^2((-\infty, 1])$, $\widetilde{W} \in C^2([0,\infty))$, $m_{\varepsilon,\eta} = (f_{\varepsilon,\eta} n, g_{\varepsilon,\eta})$ be a radially symmetric critical point of $E_{\varepsilon,\eta}$ in $\mathcal{A}$ and $Q_{\varepsilon,\eta}$ be the second variation of $E_{\varepsilon,\eta}$ at $m_{\varepsilon,\eta}$ defined by (3.1). Suppose that $V = (v = sn + w, q) \in C_c^\infty(B^N \setminus \{0\}, \mathbb{R}^{N+1})$ with $w \cdot n = 0$. For $r \in (0,1]$, let*

- *$w(r,\cdot) = \mathring{w}(r,\cdot) + \not{D}\psi(r,\cdot)$ be the Helmholtz decomposition of $w(r,\cdot)$ as a tangent vector field on $\mathbb{S}^{N-1}$, where $\mathring{w} \in C_c^\infty(B^N \setminus \{0\}, \mathbb{R}^N)$, $\psi \in C_c^\infty(B^N \setminus \{0\}, \mathbb{R})$, $\not{D} \cdot \mathring{w}(r,\cdot) = 0$ and $\int_{\mathbb{S}^{N-1}} \psi(r,\theta)\, d\sigma = 0$, where we use the convention that $\mathring{w} = 0$ when $N = 2$;*
- *the expansions of $s(r,\theta)$, $\psi(r,\theta)$ and $q(r,\theta)$ in the basis $\{\zeta_i\}_{i=0}^\infty$ be*

$$s(r,\cdot) = \sum_{i=0}^\infty s_i(r)\zeta_i, \quad \psi(r,\cdot) = \sum_{i=0}^\infty \psi_i(r)\zeta_i, \quad q(r,\cdot) = \sum_{i=0}^\infty q_i(r)\zeta_i, \tag{3.4}$$

with $s_i, \psi_i, q_i \in C_c^\infty((0,1))$ for every $i \ge 0$.[12] *Then $\mathring{V} := (\mathring{w}, 0)$, $V_i := (s_i\zeta_i n + \psi_i \not{D}\zeta_i, q_i\zeta_i) \in C_c^\infty(B^N \setminus \{0\}, \mathbb{R}^{N+1})$ for $i \ge 0$ and*

$$Q_{\varepsilon,\eta}[V] = Q_{\varepsilon,\eta}[\mathring{V}] + \sum_{i=0}^\infty Q_{\varepsilon,\eta}[V_i]. \tag{3.5}$$

[12] Note that $\psi_0 = 0$ since $\psi(r,\cdot)$ as well as ζ_i has zero average on $\mathbb{S}^{N-1}$ for $i \ge 1$.

For a related decomposition see [13, 19, 36] (in the context of the Ginzburg–Landau energy), [20, 32] (in the context of micromagnetics), [25, 27] (for the Landau–de Gennes energy).

Proof of Proposition 3.1. Observe that for a tangent vector field w (i.e. $w \cdot n = 0$),

$$\int_{\mathbb{S}^{N-1}} (w \cdot \not{D})s \, d\sigma = -\int_{\mathbb{S}^{N-1}} \not{D} \cdot ws \, d\sigma. \tag{3.6}$$

Hence, in the coupling term $(w \cdot \not{D})s$ between s and w in the expression for $Q_{\varepsilon,\eta}[V]$ in (3.3), the divergence-free part of the tangent vector field w does not contribute. If $w = \mathring{w} + \not{D}\psi$ is the Helmholtz decomposition of w with $\not{D} \cdot \mathring{w} = 0$ and $\int_{\mathbb{S}^{N-1}} \psi \, d\sigma = 0$, then, by (3.6),

$$\begin{aligned}
\int_{\mathbb{S}^{N-1}} |w|^2 \, d\sigma &= \int_{\mathbb{S}^{N-1}} |\mathring{w}|^2 \, d\sigma + \int_{\mathbb{S}^{N-1}} |w - \mathring{w}|^2 \, d\sigma, \\
\int_{\mathbb{S}^{N-1}} |\partial_r w|^2 \, d\sigma &= \int_{\mathbb{S}^{N-1}} |\partial_r \mathring{w}|^2 \, d\sigma + \int_{\mathbb{S}^{N-1}} |\partial_r (w - \mathring{w})|^2 \, d\sigma, \\
\int_{\mathbb{S}^{N-1}} (w \cdot \not{D})s \, d\sigma &= \int_{\mathbb{S}^{N-1}} ((w - \mathring{w}) \cdot \not{D})s \, d\sigma, \\
\int_{\mathbb{S}^{N-1}} |\not{D} w|^2 \, d\sigma &= \int_{\mathbb{S}^{N-1}} |\not{D} \mathring{w}|^2 \, d\sigma + \int_{\mathbb{S}^{N-1}} |\not{D}(w - \mathring{w})|^2 \, d\sigma \\
&= \int_{\mathbb{S}^{N-1}} |\not{D} \mathring{w}|^2 \, d\sigma + \int_{\mathbb{S}^{N-1}} [(\not{\Delta}\psi)^2 - (N-2)|\not{D}\psi|^2] \, d\sigma,
\end{aligned}$$

where we used the Bochner identity (see e.g. [41, Ch. I, Proposition 2.2])

$$\int_{\mathbb{S}^{N-1}} |\not{D}^2 \psi|^2 \, d\sigma = \int_{\mathbb{S}^{N-1}} [(\not{\Delta}\psi)^2 - (N-2)|\not{D}\psi|^2] \, d\sigma,$$

where $\not{D}^2\psi$ and $\not{\Delta}\psi$ are the covariant Hessian and Laplacian of ψ. Summing and using (3.4), the Dirichlet part in $Q_{\varepsilon,\eta}[V]$ in (3.3) becomes

$$\begin{aligned}
\mathrm{Dir} := \int_{\mathbb{S}^{N-1}} r^{N-1} \Big\{ &(\partial_r s)^2 + (\partial_r q)^2 + |\partial_r w|^2 \\
&+ \frac{(N-1)s^2 + |\not{D}s|^2 + |\not{D}q|^2 + |\not{D}w|^2 + |w|^2 - 4(w \cdot \not{D})s}{r^2} \Big\} \, d\sigma \\
= \int_{\mathbb{S}^{N-1}} r^{N-1} \Big\{ &|\partial_r \mathring{w}|^2 + \frac{1}{r^2}|\not{D}\mathring{w}|^2 + \frac{1}{r^2}|\mathring{w}|^2 + (\partial_r s)^2 + \frac{1}{r^2}|\not{D}s|^2 + \frac{N-1}{r^2}s^2 \\
&+ (\partial_r q)^2 + \frac{1}{r^2}|\not{D}q|^2 + |\partial_r \not{D}\psi|^2 + \frac{1}{r^2}(\not{\Delta}\psi)^2 - \frac{N-3}{r^2}|\not{D}\psi|^2 \\
&- \frac{4}{r^2}\not{D}\psi \cdot \not{D}s \Big\} \, d\sigma
\end{aligned}$$

$$= \int_{\mathbb{S}^{N-1}} r^{N-1}\Big\{|\partial_r \mathring{w}|^2 + \frac{1}{r^2}|\not{D}\mathring{w}|^2 + \frac{1}{r^2}|\mathring{w}|^2\Big\}\,d\sigma$$
$$+ \sum_{i=0}^{\infty} r^{N-1}\Big\{(s_i')^2 + \frac{\lambda_i + N - 1}{r^2}s_i^2 + (q_i')^2 + \frac{\lambda_i}{r^2}q_i^2 + \lambda_i(\psi_i')^2$$
$$+ \frac{\lambda_i(\lambda_i - N + 3)}{r^2}\psi_i^2 - \frac{4\lambda_i}{r^2}\psi_i s_i\Big\}.$$

Noting that, as $\lambda_i(\lambda_i + N - 1)(\lambda_i - N + 3) - 4\lambda_i^2 = \lambda_i(\lambda_i + N - 3)(\lambda_i - N + 1) \geq 0$ for $\lambda_i \geq N - 1$, which holds for $i \geq 1$, we have

$$\frac{\lambda_i + N - 1}{r^2}x^2 + \frac{\lambda_i(\lambda_i - N + 3)}{r^2}y^2 - \frac{4\lambda_i}{r^2}xy \geq 0 \quad \text{for all } i \geq 1.$$

Recall also that $\psi_0 \equiv 0$ and $\lambda_0 = 0$. Hence, all the summands on the right-hand side of the identity above are non-negative. Hence, by the Fubini–Tonelli theorem, we obtain the following formula for $Q_{\varepsilon,\eta}[V]$ in (3.3):

$$Q_{\varepsilon,\eta}[V] = \int_0^1 \mathrm{Dir}\,dr$$
$$+ \int_0^1\int_{\mathbb{S}^{N-1}} r^{N-1}\Big\{-\frac{1}{\varepsilon^2}W'(1 - f_{\varepsilon,\eta}^2 - g_{\varepsilon,\eta}^2)(s^2 + |w|^2 + q^2) + \frac{1}{\eta^2}\widetilde{W}'(g_{\varepsilon,\eta}^2)q^2$$
$$+ \frac{2}{\varepsilon^2}W''(1 - f_{\varepsilon,\eta}^2 - g_{\varepsilon,\eta}^2)(f_{\varepsilon,\eta}s + g_{\varepsilon,\eta}q)^2$$
$$+ \frac{2}{\eta^2}\widetilde{W}''(g_{\varepsilon,\eta}^2)g_{\varepsilon,\eta}^2q^2\Big\}\,d\sigma\,dr$$
$$= Q_{\varepsilon,\eta}[\mathring{V}] + \sum_{i=0}^{\infty} Q_{\varepsilon,\eta}[V_i],$$

because the same computation as for the Dirichlet energy Dir yields

$$\|\nabla\mathring{V}\|^2_{L^2(B^N,\mathbb{R}^{N+1})} = \int_0^1\int_{\mathbb{S}^{N-1}} r^{N-1}\Big\{|\partial_r\mathring{w}|^2 + \frac{1}{r^2}|\not{D}\mathring{w}|^2 + \frac{1}{r^2}|\mathring{w}|^2\Big\}\,d\sigma\,dr < \infty,$$
$$\|\nabla V_i\|^2_{L^2(B^N,\mathbb{R}^{N+1})} = \int_0^1 r^{N-1}\Big\{(s_i')^2 + \frac{\lambda_i + N - 1}{r^2}s_i^2 + (q_i')^2 + \frac{\lambda_i}{r^2}q_i^2$$
$$+ \lambda_i(\psi_i')^2 + \frac{\lambda_i(\lambda_i - N + 3)}{r^2}\psi_i^2 - \frac{4\lambda_i}{r^2}\psi_i s_i\Big\}\,dr < \infty,$$

which finally gives the expressions of $Q_{\varepsilon,\eta}[\mathring{V}]$ and $Q_{\varepsilon,\eta}[V_i]$ used above:

$$Q_{\varepsilon,\eta}[\mathring{V}] = \int_0^1\int_{\mathbb{S}^{N-1}} r^{N-1}\Big\{|\partial_r\mathring{w}|^2 + \frac{|\not{D}\mathring{w}|^2 + |\mathring{w}|^2}{r^2}$$
$$- \frac{W'(1 - f_{\varepsilon,\eta}^2 - g_{\varepsilon,\eta}^2)|\mathring{w}|^2}{\varepsilon^2}\Big\}\,d\sigma\,dr,$$

$$
\begin{aligned}
Q_{\varepsilon,\eta}[V_i] = \int_0^1 r^{N-1}\Big\{ &(s_i')^2 + \frac{\lambda_i + N - 1}{r^2} s_i^2 + \lambda_i (\psi_i')^2 + \frac{\lambda_i(\lambda_i - N + 3)}{r^2}\psi_i^2 \\
&- \frac{4\lambda_i \psi_i s_i}{r^2} + (q_i')^2 + \frac{\lambda_i}{r^2} q_i^2 \\
&- \frac{1}{\varepsilon^2} W'(1 - f_{\varepsilon,\eta}^2 - g_{\varepsilon,\eta}^2)(s_i^2 + \lambda_i \psi_i^2 + q_i^2) + \frac{1}{\eta^2}\widetilde{W}'(g_{\varepsilon,\eta}^2) q_i^2 \\
&+ \frac{2}{\varepsilon^2} W''(1 - f_{\varepsilon,\eta}^2 - g_{\varepsilon,\eta}^2)(f_{\varepsilon,\eta} s_i + g_{\varepsilon,\eta} q_i)^2 \\
&+ \frac{2}{\eta^2}\widetilde{W}''(g_{\varepsilon,\eta}^2) g_{\varepsilon,\eta}^2 q_i^2 \Big\}\, dr. \qquad (3.7)
\end{aligned}
$$

Thus, (3.5) holds. ∎

Strategy of the proof of stability/instability. The aim is to study the positivity of the terms in the decomposition of $Q_{\varepsilon,\eta}[V]$ in (3.5). For that, we will use the Hardy decomposition [25, Lemma A.1] for the two operators L and T defined in (2.11) (as in the proof of Proposition 2.12). By equations (1.13)–(1.14), one easily computes for $\alpha \in \mathbb{R}$,

$$
\begin{cases}
L(r^\alpha f_{\varepsilon,\eta}) = -2\alpha r^{\alpha-1} f_{\varepsilon,\eta}' - (\alpha(\alpha + N - 2) + N - 1) r^{\alpha-2} f_{\varepsilon,\eta}, \\
L(f_{\varepsilon,\eta}') = -\dfrac{2(N-1)}{r^2} f_{\varepsilon,\eta}' + \dfrac{2(N-1)}{r^3} f_{\varepsilon,\eta} \\
\qquad - \dfrac{2}{\varepsilon^2} W''(1 - f_{\varepsilon,\eta}^2 - g_{\varepsilon,\eta}^2)(f_{\varepsilon,\eta}^2 f_{\varepsilon,\eta}' + f_{\varepsilon,\eta} g_{\varepsilon,\eta} g_{\varepsilon,\eta}'), \\
T g_{\varepsilon,\eta} = 0, \\
T g_{\varepsilon,\eta}' = -\dfrac{N-1}{r^2} g_{\varepsilon,\eta}' - \dfrac{2}{\varepsilon^2} W''(1 - f_{\varepsilon,\eta}^2 - g_{\varepsilon,\eta}^2)(g_{\varepsilon,\eta} f_{\varepsilon,\eta} f_{\varepsilon,\eta}' + g_{\varepsilon,\eta}^2 g_{\varepsilon,\eta}') \\
\qquad - \dfrac{2}{\eta^2}\widetilde{W}''(g_{\varepsilon,\eta}^2) g_{\varepsilon,\eta}^2 g_{\varepsilon,\eta}',
\end{cases} \qquad (3.8)
$$

paying attention to the differences in the cases $g_{\varepsilon,\eta} > 0$ and $g_{\varepsilon,\eta} \equiv 0$.

Stability in direction $\mathring{V} = (\mathring{w}, 0)$.

Lemma 3.2. *Suppose $N \geq 3$ and $W \in C^2((-\infty, 1])$ and $\widetilde{W} \in C^2([0,\infty))$ satisfy (1.10) and (1.11). Let $m_{\varepsilon,\eta} = (f_{\varepsilon,\eta} n, g_{\varepsilon,\eta})$ be a radially symmetric critical point of $E_{\varepsilon,\eta}$ in $\mathcal{A}$ with $g_{\varepsilon,\eta} \geq 0$ in $(0,1)$, and let $Q_{\varepsilon,\eta}$ be the second variation of $E_{\varepsilon,\eta}$ at $m_{\varepsilon,\eta}$ defined by (3.1). Then there exists a constant $C > 0$ independent of $\varepsilon, \eta > 0$ such that for every $\mathring{w} \in C_c^\infty(B^N \setminus \{0\}, \mathbb{R}^N)$ with $\mathring{w} \cdot n = 0$ and $\not{D} \cdot \mathring{w} = 0$,*

$$
Q_{\varepsilon,\eta}[(\mathring{w}, 0)] \geq C \int_{B^N} |\mathring{w}|^2\, dx.
$$

In the lemma above, $m_{\varepsilon,\eta}$ can be either an escaping solution with $g_{\varepsilon,\eta} > 0$ or a non-escaping solution with $g_{\varepsilon,\eta} \equiv 0$. Also, for $N = 2$, this inequality is obvious since $\mathring{w} = 0$.

Proof of Lemma 3.2. Note that $f_{\varepsilon,\eta} > 0$ by Proposition 2.10. Let $\alpha \in \mathbb{R}$ to be chosen later (see (3.10) at the end of the proof). We factor $\mathring{w} = r^\alpha f_{\varepsilon,\eta}\widehat{w}$ with $\widehat{w} = (\widehat{w}_1, \dots, \widehat{w}_N) \in C_c^\infty(B^N \setminus \{0\}, \mathbb{R}^N)$ and we apply [25, Lemma A.1] for the operator L in (2.11):

$$
\begin{aligned}
Q_{\varepsilon,\eta}[(\mathring{w}, 0)] &= \int_{B^N} \sum_{j=1}^N L\mathring{w}_j \cdot \mathring{w}_j \, dx \\
&= \sum_{j=1}^N \int_{B^N} \{r^{2\alpha} f_{\varepsilon,\eta}^2 |\nabla \widehat{w}_j|^2 + \widehat{w}_j^2 L(r^\alpha f_{\varepsilon,\eta}) \cdot (r^\alpha f_{\varepsilon,\eta})\} \, dx \\
&= \int_0^1 \int_{\mathbb{S}^{N-1}} r^{2\alpha+N-1} f_{\varepsilon,\eta}^2 \Big\{ |\partial_r \widehat{w}|^2 - \frac{2\alpha f_{\varepsilon,\eta}'}{r f_{\varepsilon,\eta}} |\widehat{w}|^2 - \frac{(\alpha+1)(\alpha+N-3)}{r^2} |\widehat{w}|^2 \\
&\qquad + \frac{1}{r^2}(|\not{D}\widehat{w}|^2 - |\widehat{w}|^2) \Big\} \, d\sigma \, dr,
\end{aligned}
\tag{3.9}
$$

because of (3.2) for the tangent vector field $\widehat{w}$ and (3.8). By the Poincaré inequality for a divergence-free vector field on the unit sphere (see Lemma C.1),

$$
\int_{\mathbb{S}^{N-1}} |\not{D}\widehat{w}|^2 \, d\sigma \geq (N-2) \int_{\mathbb{S}^{N-1}} |\widehat{w}|^2 \, d\sigma.
$$

We then choose $\alpha \in (-(N-2), 0)$ yielding

$$
\alpha < 0 \quad \text{and} \quad (\alpha+1)(\alpha+N-3) < N-3. \tag{3.10}
$$

Since $f_{\varepsilon,\eta}' > 0$ (see Proposition 2.9) and $\frac{1}{r^2} > 1$ in $(0,1)$, it follows that $Q_{\varepsilon,\eta}[(\mathring{w}, 0)] \geq C\|\mathring{w}\|_{L^2}^2$ for a constant $C > 0$ independent of $\varepsilon, \eta > 0$. ∎

3.2. The extended model: Stability of the escaping vortex solution

Stability for the zero-mode V_0. Recall that $\lambda_0 = 0$ and ζ_0 is a non-zero constant that satisfies $\|\zeta_0\|_{L^2(\mathbb{S}^{N-1})} = 1$, in particular, $\not{D}\zeta_0 = 0$; thus, the zero-mode in (3.5) is given by $V_0 = (s\zeta_0 n, q\zeta_0)$ for two functions $s, q \in C_c^\infty(0,1)$.

Lemma 3.3. *Let $N \geq 2$, $W \in C^2((-\infty, 1])$ and $\widetilde{W} \in C^2([0,\infty))$. Let $m_{\varepsilon,\eta} = (f_{\varepsilon,\eta} n, g_{\varepsilon,\eta})$ be a bounded radially symmetric critical point of $E_{\varepsilon,\eta}$ in $\mathcal{A}$ and let $Q_{\varepsilon,\eta}$ be the second variation of $E_{\varepsilon,\eta}$ at $m_{\varepsilon,\eta}$ defined by (3.1). If $f_{\varepsilon,\eta} > 0$ and $g_{\varepsilon,\eta} > 0$ in $(0,1)$, then for $(s,q) \in C_c^\infty(0,1)$,*

$$
\begin{aligned}
Q_{\varepsilon,\eta}[(s\zeta_0 n, q\zeta_0)] = \int_0^1 r^{N-1} \Big\{ & f_{\varepsilon,\eta}^2 \Big| \Big(\frac{s}{f_{\varepsilon,\eta}} \Big)' \Big|^2 + g_{\varepsilon,\eta}^2 \Big| \Big(\frac{q}{g_{\varepsilon,\eta}} \Big)' \Big|^2 \\
& + \frac{2}{\varepsilon^2} W''(1 - f_{\varepsilon,\eta}^2 - g_{\varepsilon,\eta}^2)(f_{\varepsilon,\eta} s + g_{\varepsilon,\eta} q)^2 \\
& + \frac{2}{\eta^2} \widetilde{W}''(g_{\varepsilon,\eta}^2) g_{\varepsilon,\eta}^2 q^2 \Big\} \, dr.
\end{aligned}
$$

Proof. Recalling the operators L and T defined in (2.11), by (3.7),

$$Q_{\varepsilon,\eta}[(s\zeta_0 n, q\zeta_0)] = \frac{1}{|\mathbb{S}^{N-1}|}\int_{B^N}\Big\{Ls\cdot s + \frac{N-1}{|x|^2}s^2 + Tq\cdot q + \frac{2}{\varepsilon^2}W''(1-f_{\varepsilon,\eta}^2 - g_{\varepsilon,\eta}^2)(f_{\varepsilon,\eta}s + g_{\varepsilon,\eta}q)^2 + \frac{2}{\eta^2}\widetilde{W}''(g_{\varepsilon,\eta}^2)g_{\varepsilon,\eta}^2 q^2\Big\}\,dx.$$

We factor $s = f_{\varepsilon,\eta}\hat{s}$ and $q = g_{\varepsilon,\eta}\hat{q}$ and (3.8) combined with [25, Lemma A.1] yields the conclusion. (For details, see (2.12).) ∎

Stability for the modes V_i, $i \geq 1$.

Lemma 3.4. *Assume $N \geq 2$ and $W \in C^2((-\infty,1])$ and $\widetilde{W} \in C^2([0,\infty))$ satisfy (1.10) and (1.11). Let $m_{\varepsilon,\eta} = (f_{\varepsilon,\eta}n, g_{\varepsilon,\eta})$ be a radially symmetric critical point of $E_{\varepsilon,\eta}$ in $\mathcal{A}$ and let $Q_{\varepsilon,\eta}$ be the second variation of $E_{\varepsilon,\eta}$ at $m_{\varepsilon,\eta}$ defined by (3.1). Suppose that $g_{\varepsilon,\eta} > 0$ in $(0,1)$. If $s, \psi, q \in C_c^\infty(0,1)$ then, for $i \geq 1$ and $V_i = (s\zeta_i n + \psi \not{D}\zeta_i, q\zeta_i)$,*

$$Q_{\varepsilon,\eta}[V_i] \geq \int_0^1 r^{N-1}\Big\{(f_{\varepsilon,\eta}')^2\Big|\Big(\frac{s}{f_{\varepsilon,\eta}'}\Big)'\Big|^2 + \frac{\lambda_i}{r^2}f_{\varepsilon,\eta}^2\Big|\Big(\frac{r\psi}{f_{\varepsilon,\eta}}\Big)'\Big|^2 + (g_{\varepsilon,\eta}')^2\Big|\Big(\frac{q}{g_{\varepsilon,\eta}'}\Big)'\Big|^2 + \frac{2}{r^3}f_{\varepsilon,\eta}f_{\varepsilon,\eta}'\Big(\frac{\sqrt{N-1}s}{f_{\varepsilon,\eta}'} - \frac{\sqrt{\lambda_i}r\psi}{f_{\varepsilon,\eta}}\Big)^2\Big\}\,dr \geq 0.$$

Moreover, there exists a constant $C > 0$ independent of $\varepsilon, \eta > 0$ such that

$$Q_{\varepsilon,\eta}[V_i] \geq C\|V_i\|_{L^2(B^N)}^2 \quad \textit{for every } i \geq N+1.$$

Proof. By Proposition 2.10, $f_{\varepsilon,\eta} > 0$ in $(0,1)$. By Proposition 2.9 we have that $f_{\varepsilon,\eta}' > 0$ and $g_{\varepsilon,\eta}' < 0$ in $(0,1)$. We factor $s = f_{\varepsilon,\eta}'\hat{s}$, $\psi = \frac{f_{\varepsilon,\eta}}{r}\hat{\psi}$ and $q = g_{\varepsilon,\eta}'\hat{q}$. Recalling the operators L and T defined in (2.11), by (3.7),

$$Q_{\varepsilon,\eta}[V_i] = \frac{1}{|\mathbb{S}^{N-1}|}\int_{B^N}\Big\{Ls\cdot s + \lambda_i L\psi\cdot\psi + Tq\cdot q + \frac{\lambda_i + N - 1}{r^2}s^2 + \frac{\lambda_i(\lambda_i - N + 3)}{r^2}\psi^2 - \frac{4\lambda_i}{r^2}s\psi + \frac{\lambda_i}{r^2}q^2 + \frac{2}{\varepsilon^2}W''(1-f_{\varepsilon,\eta}^2 - g_{\varepsilon,\eta}^2)(f_{\varepsilon,\eta}s + g_{\varepsilon,\eta}q)^2 + \frac{2}{\eta^2}\widetilde{W}''(g_{\varepsilon,\eta}^2)g_{\varepsilon,\eta}^2q^2\Big\}\,dx$$

$$= \int_0^1 r^{N-1}\Big\{(f'_{\varepsilon,\eta})^2(\hat{s}')^2 + \frac{\lambda_i-(N-1)}{r^2}(f'_{\varepsilon,\eta})^2\hat{s}^2 + \frac{2(N-1)}{r^3} f_{\varepsilon,\eta} f'_{\varepsilon,\eta}\hat{s}^2 + \frac{\lambda_i}{r^2} f^2_{\varepsilon,\eta}(\hat{\psi}')^2 + \frac{2\lambda_i}{r^3} f_{\varepsilon,\eta} f'_{\varepsilon,\eta}\hat{\psi}^2 + \frac{\lambda_i(\lambda_i-(N-1))}{r^4} f^2_{\varepsilon,\eta}\hat{\psi}^2 - \frac{4\lambda_i}{r^3} f'_{\varepsilon,\eta} f_{\varepsilon,\eta}\hat{s}\hat{\psi} + \frac{\lambda_i-(N-1)}{r^2}(g'_{\varepsilon,\eta})^2\hat{q}^2 + (g'_{\varepsilon,\eta})^2(\hat{q}')^2 - \frac{2}{\varepsilon^2}W''(1-f^2_{\varepsilon,\eta}-g^2_{\varepsilon,\eta}) f_{\varepsilon,\eta} f'_{\varepsilon,\eta} g_{\varepsilon,\eta} g'_{\varepsilon,\eta}(\hat{s}-\hat{q})^2\Big\}\,dr, \quad (3.11)$$

where we used [25, Lemma A.1] and (3.8). As $f_{\varepsilon,\eta} > 0$, $f'_{\varepsilon,\eta} > 0$ and $\lambda_i \geq N-1$ for $i \geq 1$,

$$\frac{\lambda_i-(N-1)}{r^2}(f'_{\varepsilon,\eta})^2\hat{s}^2 + \frac{\lambda_i(\lambda_i-(N-1))}{r^4} f^2_{\varepsilon,\eta}\hat{\psi}^2 \geq \frac{2\sqrt{\lambda_i}(\lambda_i-(N-1))}{r^3} f_{\varepsilon,\eta} f'_{\varepsilon,\eta}|\hat{s}\hat{\psi}|.$$

For $i \geq 1$,

$$4\sqrt{\lambda_i(N-1)} + 2\sqrt{\lambda_i}(\lambda_i-(N-1)) - 4\lambda_i = 2\sqrt{\lambda_i}[(\sqrt{\lambda_i}-1)^2 - (\sqrt{N-1}-1)^2] \geq 0,$$

so

$$\frac{2\sqrt{\lambda_i}(\lambda_i-(N-1))}{r^3} f_{\varepsilon,\eta} f'_{\varepsilon,\eta}|\hat{s}\hat{\psi}| \geq \frac{4\lambda_i - 4\sqrt{\lambda_i(N-1)}}{r^3} f_{\varepsilon,\eta} f'_{\varepsilon,\eta}\hat{s}\hat{\psi}.$$

Putting these inequalities in (3.11), we conclude

$$Q_{\varepsilon,\eta}[V_i] \geq \int_0^1 r^{N-1}\Big\{(f'_{\varepsilon,\eta})^2(\hat{s}')^2 + \frac{\lambda_i}{r^2} f^2_{\varepsilon,\eta}(\hat{\psi}')^2 + (g'_{\varepsilon,\eta})^2(\hat{q}')^2 + \frac{2}{r^3} f_{\varepsilon,\eta} f'_{\varepsilon,\eta}(\sqrt{N-1}\hat{s} - \sqrt{\lambda_i}\hat{\psi})^2\Big\}\,dr.$$

This proves the first assertion.

Consider the second assertion concerning the case $i \geq N+1$. We prove a uniform L^2 lower bound by a different Hardy decomposition using $\lambda_i \geq 2N$. Indeed, we factor $s = f_{\varepsilon,\eta}\tilde{s}$, $\psi = f_{\varepsilon,\eta}\tilde{\psi}$ and $q = g_{\varepsilon,\eta}\tilde{q}$ and we compute using [25, Lemma A.1] and (3.8):

$$Q_{\varepsilon,\eta}[V_i] = \frac{1}{|\mathbb{S}^{N-1}|}\int_{B^N}\Big\{f^2_{\varepsilon,\eta}|\nabla\tilde{s}|^2 + \tilde{s}^2 L f_{\varepsilon,\eta}\cdot f_{\varepsilon,\eta} + \lambda_i(f^2_{\varepsilon,\eta}|\nabla\tilde{\psi}|^2 + \tilde{\psi}^2 L f_{\varepsilon,\eta}\cdot f_{\varepsilon,\eta}) + g^2_{\varepsilon,\eta}|\nabla\tilde{q}|^2 + \frac{\lambda_i+N-1}{r^2}s^2 + \frac{\lambda_i(\lambda_i-N+3)}{r^2}\psi^2 - \frac{4\lambda_i}{r^2}s\psi + \frac{\lambda_i}{r^2}q^2 + \frac{2}{\varepsilon^2}W''(1-f^2_{\varepsilon,\eta}-g^2_{\varepsilon,\eta})(f_{\varepsilon,\eta}s + g_{\varepsilon,\eta}q)^2 + \frac{2}{\eta^2}\widetilde{W}''(g^2_{\varepsilon,\eta})g^2_{\varepsilon,\eta}q^2\Big\}\,dx$$

$$= \int_0^1 r^{N-1}\Big\{f_{\varepsilon,\eta}^2(\tilde{s}')^2 + \frac{\lambda_i}{r^2}s^2 + \lambda_i f_{\varepsilon,\eta}^2(\tilde{\psi}')^2 + \frac{\lambda_i(\lambda_i - 2N + 4)}{r^2}\psi^2 + g_{\varepsilon,\eta}^2(\tilde{q}')^2 + \frac{\lambda_i}{r^2}q^2 - \frac{4\lambda_i}{r^2}s\psi + \frac{2}{\varepsilon^2}W''(1 - f_{\varepsilon,\eta}^2 - g_{\varepsilon,\eta}^2)(f_{\varepsilon,\eta}s + g_{\varepsilon,\eta}q)^2 + \frac{2}{\eta^2}\tilde{W}''(g_{\varepsilon,\eta}^2)g_{\varepsilon,\eta}^2 q^2\Big\}\,dr$$

$$\geq \int_0^1 r^{N-1}\Big\{f_{\varepsilon,\eta}^2(\tilde{s}')^2 + \lambda_i f_{\varepsilon,\eta}^2(\tilde{\psi}')^2 + \frac{\lambda_i}{r^2}(s - 2\psi)^2 + \frac{\lambda_i}{r^2}q^2\Big\}\,dr, \tag{3.12}$$

where we used (1.10) and $\lambda_i \geq 2N$ for $i \geq N+1$. Finally, the L^2 lower bound (uniform in $\varepsilon, \eta > 0$) follows by the Hardy inequality in $\mathbb{R}^{N+2}$ using $r \leq f_{\varepsilon,\eta}(r) \leq 1$ (as in (2.20)):

$$\int_0^1 r^{N-1} f_{\varepsilon,\eta}^2(h')^2\,dr \geq \int_0^1 r^{N+1}(h')^2\,dr \geq \frac{N^2}{4}\int_0^1 r^{N-1}h^2\,dr \geq \frac{N^2}{4}\int_0^1 r^{N-1} f_{\varepsilon,\eta}^2 h^2\,dr, \tag{3.13}$$

where h stands for either $\tilde{s}$ or $\tilde{\psi}$. ■

We are in position to give the following proofs:

Proof of Theorem 1.5 (a) *and* (b). By Theorem 2.4, we only need to prove that, when an escaping critical point $m_{\varepsilon,\eta}(x) = (f_{\varepsilon,\eta}(r)n(x), g_{\varepsilon,\eta}(r))$ with $g_{\varepsilon,\eta} > 0$ exists, the second variation $Q_{\varepsilon,\eta}$ of $E_{\varepsilon,\eta}$ at $m_{\varepsilon,\eta}$ is positive definite, and that $m_{\varepsilon,\eta}$ is a local minimizer of $E_{\varepsilon,\eta}$.

Proof of the positive-definiteness of $Q_{\varepsilon,\eta}$. Fix some $V \in C_c^\infty(B^N \setminus \{0\}, \mathbb{R}^{N+1})$ and define $\mathring{V} = (\mathring{w}, 0)$, $V_i = (s_i\zeta_i n + \psi_i \not{D}\zeta_i, q_i\zeta_i)$ as in Proposition 3.1. By the orthogonal decomposition (3.5), Lemmas 3.2, 3.3 and 3.4, we have

$$Q_{\varepsilon,\eta}[V] \geq C\Big\|V - \sum_{i=0}^N V_i\Big\|_{L^2(B^N)}^2 + Q_0 + \sum_{i=1}^N Q_i, \tag{3.14}$$

where

$$Q_0 = \int_0^1 r^{N-1}\Big\{f_{\varepsilon,\eta}^2\Big|\Big(\frac{s_0}{f_{\varepsilon,\eta}}\Big)'\Big|^2 + g_{\varepsilon,\eta}^2\Big|\Big(\frac{q_0}{g_{\varepsilon,\eta}}\Big)'\Big|^2 + \frac{2}{\varepsilon^2}W''(1 - f_{\varepsilon,\eta}^2 - g_{\varepsilon,\eta}^2)(f_{\varepsilon,\eta}s_0 + g_{\varepsilon,\eta}q_0)^2 + \frac{2}{\eta^2}\tilde{W}''(g_{\varepsilon,\eta}^2)g_{\varepsilon,\eta}^2 q_0^2\Big\}\,dr,$$

$$Q_i = \int_0^1 r^{N-1}\Big\{(f'_{\varepsilon,\eta})^2\Big|\Big(\frac{s_i}{f'_{\varepsilon,\eta}}\Big)'\Big|^2 + \frac{N-1}{r^2} f_{\varepsilon,\eta}^2\Big|\Big(\frac{r\psi_i}{f_{\varepsilon,\eta}}\Big)'\Big|^2 + (g'_{\varepsilon,\eta})^2\Big|\Big(\frac{q_i}{g'_{\varepsilon,\eta}}\Big)'\Big|^2 + \frac{2(N-1)}{r^3} f_{\varepsilon,\eta} f'_{\varepsilon,\eta}\Big(\frac{s_i}{f'_{\varepsilon,\eta}} - \frac{r\psi_i}{f_{\varepsilon,\eta}}\Big)^2\Big\}\,dr, \quad 1 \le i \le N.$$

By the density of $C_c^\infty(B^N \setminus \{0\}, \mathbb{R}^{N+1})$ in $H_0^1(B^N, \mathbb{R}^{N+1})$ and Fatou's lemma, the above inequality holds for all $V \in H_0^1(B^N, \mathbb{R}^{N+1})$, proving that $Q_{\varepsilon,\eta}$ is non-negative semi-definite.

Suppose next that $Q_{\varepsilon,\eta}[V] = 0$ for some non-trivial $V \in H_0^1(B^N, \mathbb{R}^{N+1})$. The above inequality implies that $V = \sum_{i=0}^N V_i$, $s_0 = c_0 f_{\varepsilon,\eta}$, $q_0 = \tilde{c}_0 g_{\varepsilon,\eta}$, $s_i = c_i f'_{\varepsilon,\eta}$, $\psi_i = \frac{\hat{c}_i}{r} f_{\varepsilon,\eta}$, $q_i = \tilde{c}_i g'_{\varepsilon,\eta}$ in $(0,1)$ for $1 \le i \le N$ and some constants $c_i, \tilde{c}_i, \hat{c}_i$. As $V_i = 0$ on ∂B^N and $f_{\varepsilon,\eta}(1), f'_{\varepsilon,\eta}(1), g'_{\varepsilon,\eta}(1) \neq 0$, we deduce that $V = V_0 = (0, q_0\zeta_0)$.

Suppose by contradiction that $\tilde{c}_0 \neq 0$. Then q_0 has no zeros inside $(0,1)$, therefore $W''(1 - f_{\varepsilon,\eta}^2 - g_{\varepsilon,\eta}^2) \equiv \widetilde{W}''(g_{\varepsilon,\eta}^2) \equiv 0$ in $(0,1)$. It follows that W' is constant in $[\min(1 - f_{\varepsilon,\eta}^2 - g_{\varepsilon,\eta}^2), \max(1 - f_{\varepsilon,\eta}^2 - g_{\varepsilon,\eta}^2)] =: [0,\tau]$ and hence $W' = 0$ in $[0,\tau]$ since $W'(0) = 0$ (by (1.10)). Recalling (1.14), we thus have that $-\Delta g_{\varepsilon,\eta} + \frac{1}{\eta^2}\widetilde{W}'(g_{\varepsilon,\eta}^2) g_{\varepsilon,\eta} = 0$ in B^N. Since $\widetilde{W}' \ge \widetilde{W}'(0) \ge 0$ in $[0,\infty)$ (by (1.11)) and $g_{\varepsilon,\eta} = 0$ on ∂B^N, we deduce that $g_{\varepsilon,\eta} = 0$ in B^N, which gives a contradiction to the assumption $g_{\varepsilon,\eta} > 0$ in B^N. Thus, $\tilde{c}_0 = 0$, leading to $q_0 = 0$ and $V = 0$. This proves that $Q_{\varepsilon,\eta}$ is positive definite.

By (3.1), the convexity of W and $\widetilde{W}$, the fact that $\widetilde{W}' \ge 0$ and the boundedness of $(f_{\varepsilon,\eta}, g_{\varepsilon,\eta})$, we have for some constant $C_1 = C_1(\varepsilon) > 0$ that

$$Q_{\varepsilon,\eta}[V] \ge \|\nabla V\|^2_{L^2(B^N)} - C_1\|V\|^2_{L^2(B^N)} \quad \text{for all } V \in H_0^1(B^N, \mathbb{R}^{N+1}).$$

This together with the weak lower semi-continuity of $Q_{\varepsilon,\eta}$ in $H_0^1(B^N, \mathbb{R}^{N+1})$ implies that $\min\{Q_{\varepsilon,\eta}[V] : V \in H_0^1(B^N, \mathbb{R}^{N+1}), \|V\|_{L^2(B^N)} = 1\}$ is achieved and positive (as $Q_{\varepsilon,\eta}$ is positive definite), yielding, for $C_2 = C_2(\varepsilon,\eta) > 0$,

$$Q_{\varepsilon,\eta}[V] \ge \frac{1}{C_2}\|V\|^2_{L^2(B^N)} \quad \text{for all } V \in H_0^1(B^N, \mathbb{R}^{N+1}).$$

The above two inequalities imply for $C_3 = C_3(\varepsilon,\eta) = 1 + C_2(C_1 + 1)$ that

$$Q_{\varepsilon,\eta}[V] \ge \frac{1}{C_3}\|V\|^2_{H^1(B^N)} \quad \text{for all } V \in H_0^1(B^N, \mathbb{R}^{N+1}).$$

Proof of the local minimality of $m_{\varepsilon,\eta}$. We note a subtlety in this step due to the fact that $E_{\varepsilon,\eta}$ may not be finite in an H_0^1 neighborhood of $m_{\varepsilon,\eta}$ as we make no growth assumption for W and $\widetilde{W}$. Since $m_{\varepsilon,\eta}$ is a critical point for $E_{\varepsilon,\eta}$ in $\mathcal{A}$, we have, for $V = (v,q) \in H_0^1(B^N, \mathbb{R}^{N+1})$,

$$E_{\varepsilon,\eta}[m_{\varepsilon,\eta} + V] - E_{\varepsilon,\eta}[m_{\varepsilon,\eta}] - \frac{1}{2}Q_{\varepsilon,\eta}[V] = \int_{B^N} h(x, V(x))\,dx,$$

$$h(x,y) = \frac{1}{2\varepsilon^2}\{W(1-|m_{\varepsilon,\eta}(x)+y|^2) - W(1-|m_{\varepsilon,\eta}(x)|^2)$$
$$+ W'(1-|m_{\varepsilon,\eta}(x)|^2)(2m_{\varepsilon,\eta}(x)\cdot y + |y|^2)$$
$$- 2W''(1-|m_{\varepsilon,\eta}(x)|^2)(m_{\varepsilon,\eta}(x)\cdot y)^2\}$$
$$+ \frac{1}{2\eta^2}\{\widetilde{W}\big((g_{\varepsilon,\eta}(x)+y_{N+1})^2\big) - \widetilde{W}(g^2_{\varepsilon,\eta}(x))$$
$$- \widetilde{W}'(g^2_{\varepsilon,\eta}(x))(2g_{\varepsilon,\eta}(x)y_{N+1} + y^2_{N+1})$$
$$- 2\widetilde{W}''(g^2_{\varepsilon,\eta}(x))g^2_{\varepsilon,\eta}(x)y^2_{N+1}\}.$$

Note that $h \in C^0(\overline{B}^N, C^2(\mathbb{R}^{N+1}))$, $h(x,0) = 0$, $\nabla_y h(x,0) = 0$, $\nabla^2_y h(x,0) = 0$ (thus, (D.1) holds true in Proposition D.1) and, due to the convexity of W and $\widetilde{W}$, h satisfies the growth assumptions in Proposition D.1 for $p = 2$, namely

$$h(x,y) \geq -\frac{1}{\varepsilon^2}W''(1-|m_{\varepsilon,\eta}(x)|^2)(m_{\varepsilon,\eta}(x)\cdot y)^2 - \frac{1}{\eta^2}\widetilde{W}''(g^2_{\varepsilon,\eta}(x))g^2_{\varepsilon,\eta}(x)y^2_{N+1}$$
$$\geq -C(\varepsilon,\eta)|y|^2$$

for every $x \in B^N$ and $y \in \mathbb{R}^{N+1}$ and a constant $C(\varepsilon,\eta) > 0$. Combining Proposition D.1 with the positive-definiteness of $Q_{\varepsilon,\eta}$ yields for some $\delta > 0$ and $\widetilde{C} > 0$ (depending on ε and η),

$$E_{\varepsilon,\eta}[m_{\varepsilon,\eta} + V] \geq E_{\varepsilon,\eta}[m_{\varepsilon,\eta}] + \widetilde{C}\|V\|^2_{H^1(B^N)}$$

for all $V \in H^1_0(B^N, \mathbb{R}^{N+1})$ with $\|V\|_{H^1(B^N)} < \delta$. ■

Remark 3.5. The above result can be used to obtain the local minimality of any escaping radially symmetric critical point $m_{\varepsilon,\eta} = (f_{\varepsilon,\eta}n, g_{\varepsilon,\eta})$ of $E_{\varepsilon,\eta}$ with $g_{\varepsilon,\eta} > 0$ and $f^2_{\varepsilon,\eta} + g^2_{\varepsilon,\eta} \leq 1$ under a slightly weaker assumption that $W \in C^2([0,1])$, $\widetilde{W} \in C^2([0,1])$ and

$$W(0) = 0, \quad W(t) \geq 0 \text{ in } (-\infty, 1], \quad W''(t) \geq 0 \text{ in } [0,1], \tag{3.15}$$
$$\widetilde{W}(0) = 0, \quad \widetilde{W}(t) \geq 0, \quad \widetilde{W}''(t) \geq 0 \text{ in } [0,1], \quad \widetilde{W}(t) \geq \widetilde{W}(1) \text{ in } [1,\infty). \tag{3.16}$$

In the Ginzburg–Landau context, similar conditions appeared in [33].

Indeed, for $m \in \mathcal{A}$, define the truncation $Tm \in \mathcal{A}$ of m by

$$Tm(x) = \begin{cases} m(x) & \text{if } |m(x)| \leq 1, \\ \dfrac{m(x)}{|m(x)|} & \text{if } |m(x)| > 1. \end{cases}$$

Observe that, by (3.15)–(3.16), $E_{\varepsilon,\eta}[m] \geq E_{\varepsilon,\eta}[Tm]$ for $m \in \mathcal{A}$. On the other hand, by applying Theorem 1.5 to a pair of potentials satisfying (1.10)–(1.11), which agree with $(W, \widetilde{W})$ in $[0,1]$ (e.g. by using suitable quadratic polynomials outside $[0,1]$), we obtain that there exist $\delta > 0$ and $C > 0$ such that $E_{\varepsilon,\eta}[Tm] \geq E_{\varepsilon,\eta}[m_{\varepsilon,\eta}] + \frac{1}{C}\|Tm - m_{\varepsilon,\eta}\|_{H^1(B^N,\mathbb{R}^{N+1})}$

whenever $m \in \mathcal{A}$ and $\|Tm - m_{\varepsilon,\eta}\|_{H^1(B^N,\mathbb{R}^{N+1})} \le \delta$. Therefore, to prove the local minimality of $m_{\varepsilon,\eta}$, it suffices to show that the truncation map is continuous at $m_{\varepsilon,\eta}$, i.e. if $m_j \to m_{\varepsilon,\eta}$ in $H^1(B^N, \mathbb{R}^{N+1})$, then $Tm_j \to m_{\varepsilon,\eta}$ in $H^1(B^N, \mathbb{R}^{N+1})$. Indeed, observe that, for $a, b \in \mathbb{R}^N$ with $|a| \ge 1$, $|b| \le 1$,

$$|a-b|^2 = \Big(|a| - \frac{b \cdot a}{|a|}\Big)^2 + \Big|b - \frac{b \cdot a}{|a|^2} a\Big|^2 \ge \Big(1 - \frac{b \cdot a}{|a|}\Big)^2 + \Big|b - \frac{b \cdot a}{|a|^2} a\Big|^2 = \Big|\frac{a}{|a|} - b\Big|^2.$$

This implies that

$$\|m_j - m_{\varepsilon,\eta}\|^2_{L^2(B^N,\mathbb{R}^{N+1})} \ge \|Tm_j - m_{\varepsilon,\eta}\|^2_{L^2(B^N,\mathbb{R}^{N+1})},$$

and so $Tm_j \to m_{\varepsilon,\eta}$ in $L^2(B^N, \mathbb{R}^{N+1})$. Since $\|Tm_j\|_{H^1(B^N)} \le \|m_j\|_{H^1(B^N)}$, $\{Tm_j\}$ has an H^1-weakly convergent subsequence $\{T_{m_{j_k}}\}$, whose weak limit must be $m_{\varepsilon,\eta}$ (in view of the strong L^2 convergence of Tm_j), and

$$\|\nabla m_{\varepsilon,\eta}\|_{L^2(B^N,\mathbb{R}^{N+1})} \le \liminf_{k\to\infty} \|\nabla T m_{j_k}\|_{L^2(B^N,\mathbb{R}^{N+1})}.$$

On the other hand, by construction,

$$\|\nabla T m_j\|_{L^2(B^N,\mathbb{R}^{N+1})} \le \|\nabla m_j\|_{L^2(B^N,\mathbb{R}^{N+1})} \to \|\nabla m_{\varepsilon,\eta}\|_{L^2(B^N,\mathbb{R}^{N+1})}.$$

We thus have that $\|\nabla T m_{j_k}\|_{L^2(B^N,\mathbb{R}^{N+1})} \to \|\nabla m_{\varepsilon,\eta}\|_{L^2(B^N,\mathbb{R}^{N+1})}$ and so $Tm_{j_k} \to m_{\varepsilon,\eta}$ in $H^1(B^N, \mathbb{R}^{N+1})$. Applying the above argument to any subsequence of $\{Tm_j\}$, we get $Tm_j \to m_{\varepsilon,\eta}$ in $H^1(B^N, \mathbb{R}^{N+1})$ as desired.

3.3. The $\mathbb{R}^N$-valued model: Stability of the vortex solution

Let $N \ge 2$ and $W \in C^2((-\infty, 1])$ satisfy (1.10). Let $u_\varepsilon = f_\varepsilon n$ be the radially symmetric critical point of $E_\varepsilon^{\mathrm{GL}}$ in $\mathcal{A}^{\mathrm{GL}}$ obtained in Theorem 2.1, and let $Q_\varepsilon^{\mathrm{GL}}$ be the second variation of $E_\varepsilon^{\mathrm{GL}}$ at $u_\varepsilon = f_\varepsilon n$,

$$Q_\varepsilon^{\mathrm{GL}}[v] := \int_{B^N} \Big[|\nabla v|^2 - \frac{1}{\varepsilon^2} W'(1 - f_\varepsilon^2)|v|^2 + \frac{2}{\varepsilon^2} W''(1 - f_\varepsilon^2) f_\varepsilon^2 (n \cdot v)^2\Big]\, dx \quad (3.17)$$

for every $v \in H_0^1(B^N, \mathbb{R}^N)$.

Proof of Theorem 1.2. We will only prove the positive-definiteness of $Q_\varepsilon^{\mathrm{GL}}$ in $C_c^\infty(B^N \setminus \{0\}, \mathbb{R}^N)$. As in the proof of Theorem 1.5 (a), the estimate we obtain (see (3.18) below) implies that $Q_\varepsilon^{\mathrm{GL}}[v] \ge C\|v\|^2_{H^1(B^N)}$ for $v \in H_0^1(B^N, \mathbb{R}^N)$ and that u_ε is a local minimizer of $E_\varepsilon^{\mathrm{GL}}$ in $\mathcal{A}^{\mathrm{GL}}$, more precisely, for some constants $\delta > 0$ and $\widetilde{C} > 0$ (depending on ε),

$$E_\varepsilon^{\mathrm{GL}}[u_\varepsilon + v] \ge E_\varepsilon^{\mathrm{GL}}[u_\varepsilon] + \widetilde{C}\|v\|^2_{H^1(B^N)} \quad \text{for all } v \in H_0^1(B^N, \mathbb{R}^N),\ \|v\|_{H^1(B^N)} < \delta.$$

Take an arbitrary $v \in C_c^\infty(B^N \setminus \{0\}, \mathbb{R}^N)$. We use the decomposition in Proposition 3.1 in the orthonormal basis $(\zeta_i)_{i\ge 0}$ of $L^2(\mathbb{S}^{N-1})$. We write $v = sn + \mathring{w} + \not{D}\psi$ with

$s \in C_c^\infty(B^N \setminus \{0\})$, $\mathring{w} \in C_c^\infty(B^N \setminus \{0\}, \mathbb{R}^N)$ being a tangent vector field (i.e. $\mathring{w} \cdot n = 0$) with $\not{D} \cdot \mathring{w}(r, \cdot) = 0$ on $\mathbb{S}^{N-1}$ and $\psi \in C_c^\infty(B^N \setminus \{0\}, \mathbb{R})$ satisfying $\int_{\mathbb{S}^{N-1}} \psi(r, \theta)\, d\sigma = 0$, and decompose

$$s(r, \theta) = \sum_{i=0}^{\infty} s_i(r)\zeta_i(\theta), \quad \psi(r, \theta) = \sum_{i=0}^{\infty} \psi_i(r)\zeta_i(\theta),$$

with $s_i, \psi_i \in C_c^\infty((0, 1))$ for all $i \geq 0$ and all $r \in (0, 1]$. We will prove

$$\begin{aligned} Q_\varepsilon^{\mathrm{GL}}[v] &\geq C \Big\| v - \sum_{i=1}^{N} v_i \Big\|_{L^2(B^N)}^2 \\ &\quad + \sum_{i=1}^{N} \int_0^1 r^{N-1} \Big\{ (f_\varepsilon')^2 \Big| \Big(\frac{s_i}{f_\varepsilon'} \Big)' \Big|^2 + \frac{2(N-1)}{r^3} f_\varepsilon f_\varepsilon' \Big(\frac{s_i}{f_\varepsilon'} - \frac{r\psi_i}{f_\varepsilon} \Big)^2 \\ &\qquad + \frac{N-1}{r^2} f_\varepsilon^2 \Big| \Big(\frac{r\psi_i}{f_\varepsilon} \Big)' \Big|^2 \Big\} \, dr, \end{aligned} \tag{3.18}$$

where $v_i = s_i \zeta_i n + \psi_i \not{D} \zeta_i \in C_c^\infty(B^N \setminus \{0\}, \mathbb{R}^N)$, $i \geq 0$. By Proposition 3.1,

$$Q_\varepsilon^{\mathrm{GL}}[v] = Q_\varepsilon^{\mathrm{GL}}[\mathring{w}] + \sum_{i=0}^{\infty} Q_\varepsilon^{\mathrm{GL}}[v_i].$$

First, Lemma 3.2 yields a constant $C > 0$ independent of ε such that

$$Q_\varepsilon^{\mathrm{GL}}[\mathring{w}] \geq C \|\mathring{w}\|_{L^2}^2$$

for every tangent vector field $\mathring{w} \in C_c^\infty(B^N \setminus \{0\}, \mathbb{R}^N)$ with $\not{D} \cdot \mathring{w}(r, \cdot) = 0$. Second, for the zero mode $v_0 = s_0 \zeta_0 n$, the proof of Lemma 3.3 yields

$$\begin{aligned} Q_\varepsilon^{\mathrm{GL}}[s_0\zeta_0 n] &= \int_0^1 r^{N-1} \Big\{ f_\varepsilon^2 \Big| \Big(\frac{s_0}{f_\varepsilon} \Big)' \Big|^2 + \frac{2}{\varepsilon^2} W''(1 - f_\varepsilon^2) f_\varepsilon^2 s_0^2 \Big\} \, dr \\ &\geq \int_0^1 r^{N+1} \Big| \Big(\frac{s_0}{f_\varepsilon} \Big)' \Big|^2 \, dr \geq \frac{N^2}{4} \int_0^1 r^{N-1} s_0^2 \, dr = \frac{N^2}{4} \|v_0\|_{L^2}^2, \end{aligned}$$

where we used $r \leq f_\varepsilon \leq 1$ in $(0, 1)$ and the Hardy inequality in $\mathbb{R}^{N+2}$ (as in (2.20)). Third, for the modes $v_i = s_i \zeta_i n + \psi_i \not{D} \zeta_i$ for $1 \leq i \leq N$ (so that $\lambda_i = N - 1$), we factor $s_i = f_\varepsilon' \hat{s}_i$ and $\psi_i = \frac{f_\varepsilon}{r} \hat{\psi}_i$, and the computation in the proof of Lemma 3.4 yields

$$\begin{aligned} Q_\varepsilon^{\mathrm{GL}}[v_i] &= \int_0^1 r^{N-1} \Big\{ (f_\varepsilon')^2 (\hat{s}_i')^2 + \frac{2(N-1)}{r^3} f_\varepsilon f_\varepsilon' (\hat{s}_i - \hat{\psi}_i)^2 + \frac{N-1}{r^2} f_\varepsilon^2 (\hat{\psi}_i')^2 \Big\} \, dr \\ &\geq 0. \end{aligned}$$

Finally, for the modes $v_i = s_i \zeta_i n + \psi_i \not{D} \zeta_i$ for $i \geq N + 1$, we factor $s_i = f_\varepsilon \tilde{s}$ and $\psi_i = f_\varepsilon \tilde{\psi}_i$; by the computation in the proof of Lemma 3.4 (see (3.12)),

$$Q_\varepsilon^{\mathrm{GL}}[v_i] \geq C \|v_i\|_{L^2(B^N)}^2 \quad \text{for every } i \geq N + 1,$$

for some $C > 0$ independent of ε and i. These estimates yield (3.18). ■

3.4. The extended model: (In)stability of the non-escaping solution

Let $N \geq 2$, $W \in C^2((-\infty, 1])$ and $\widetilde{W} \in C^2([0, \infty))$ satisfy (1.10) and (1.11). Let $\bar{m}_\varepsilon = (f_\varepsilon n, 0)$ be the in-plane radially symmetric critical point of $E_{\varepsilon,\eta}$ in $\mathcal{A}$, where f_ε is given by Theorem 2.1. Let $\bar{Q}_{\varepsilon,\eta}$ be the second variation of $E_{\varepsilon,\eta}$ at $\bar{m}_\varepsilon$: for $V = (v, q) \in H_0^1(B^N, \mathbb{R}^N) \times H_0^1(B^N, \mathbb{R}) \cong H_0^1(B^N, \mathbb{R}^{N+1})$,

$$\begin{aligned}\bar{Q}_{\varepsilon,\eta}[V] &= Q_\varepsilon^{\mathrm{GL}}[v] + \bar{Q}_{\varepsilon,\eta}[(0,q)],\\ \bar{Q}_{\varepsilon,\eta}[(0,q)] &= \int_{B^N}\Big[|\nabla q|^2 - \frac{1}{\varepsilon^2}W'(1-f_\varepsilon^2)q^2 + \frac{1}{\eta^2}\widetilde{W}'(0)q^2\Big]\,dx\\ &= \int_{B^N}\Big[L_\varepsilon^{\mathrm{GL}} q\cdot q + \frac{1}{\eta^2}\widetilde{W}'(0)q^2\Big]\,dx,\end{aligned}$$

where $Q_\varepsilon^{\mathrm{GL}}$ is the second variation at the critical point $u_\varepsilon = f_\varepsilon n$ of the Ginzburg–Landau energy $E_\varepsilon^{\mathrm{GL}}$ given in (3.17) and $L_\varepsilon^{\mathrm{GL}}$ is defined by (2.1).

Proof of Theorem 1.5 (c). We will only discuss the positive-definiteness of $\bar{Q}_{\varepsilon,\eta}$. As in the proof of Theorem 1.5 (a), in the case when $\bar{Q}_{\varepsilon,\eta}$ is positive definite, we have that $\bar{Q}_{\varepsilon,\eta}[V] \geq C\|V\|^2_{H^1(B^N)}$ for $V \in H_0^1(B^N, \mathbb{R}^{N+1})$ and that $\bar{m}_\varepsilon$ is a local minimizer of $E_{\varepsilon,\eta}$ in $\mathcal{A}$: for some $\delta > 0$ and $\widetilde{C} > 0$ (depending on ε and η) and all $V \in H_0^1(B^N, \mathbb{R}^{N+1})$ with $\|V\|_{H^1(B^N)} < \delta$,

$$E_{\varepsilon,\eta}[\bar{m}_\varepsilon + V] \geq E_{\varepsilon,\eta}[\bar{m}_\varepsilon] + \widetilde{C}\|V\|^2_{H^1(B^N)}.$$

By Theorem 1.2, $Q_\varepsilon^{\mathrm{GL}}$ is positive definite. Therefore, $\bar{Q}_{\varepsilon,\eta}$ is positive definite if and only if $\bar{Q}_{\varepsilon,\eta}[(0,\cdot)]$ is positive definite, i.e. $\ell(\varepsilon) + \frac{1}{\eta^2}\widetilde{W}'(0) > 0$, where $\ell(\varepsilon)$ is the first eigenvalue of $L_\varepsilon^{\mathrm{GL}}$ on B^N with zero Dirichlet boundary value. Recalling that we are assuming that (1.12)–(1.15) has no escaping solutions, we deduce from Theorem 2.4 (a) and (b), Lemma 2.3 and the fact that $\widetilde{W}'(0) \geq 0$ that the above inequality fails if and only if $2 \leq N \leq 6$, $W'(1) > 0$, $\widetilde{W}'(0) > 0$, $0 < \varepsilon < \varepsilon_0$ and $\eta = \eta_0(\varepsilon)$. In this case, $\ell(\varepsilon) + \frac{1}{\eta^2}\widetilde{W}'(0) = 0$, $\bar{Q}_{\varepsilon,\eta}$ is non-negative semi-definite with the one-dimensional kernel $\{(0,q) : q \in H_0^1(B^N), L_\varepsilon^{\mathrm{GL}} q = \ell(\varepsilon)q\}$ generated by $(0, q_\varepsilon)$ for any first eigenfunction q_ε of $L_\varepsilon^{\mathrm{GL}}$. ■

3.5. The $\mathbb{S}^N$-valued model: Stability of the escaping vortex solution

Assume that $N \geq 2$ and $\widetilde{W} \in C^2([0, \infty))$. Let $m_\eta = (\tilde{f}_\eta n, g_\eta)$ be the escaping radially symmetric critical point of E_η^{MM} in $\mathcal{A}^{\mathrm{MM}}$ with $\tilde{f}_\eta > 0$ and $g_\eta > 0$, and let Q_η^{MM} be the second variation of E_η^{MM} at m_η: For $V = (v, q) \in H_0^1(B^N, \mathbb{R}^N) \times H_0^1(B^N, \mathbb{R}) \cong H_0^1(B^N, \mathbb{R}^{N+1})$ with $V \cdot m_\eta = 0$,

$$\begin{aligned}Q_\eta^{\mathrm{MM}}[V] &= \frac{d^2}{dt^2}\Big|_{t=0} E_\eta^{\mathrm{MM}}\Big[\frac{m_\eta + tV}{|m_\eta + tV|}\Big]\\ &= \int_{B^N}\Big[|\nabla V|^2 - \lambda(r)|V|^2 + \frac{1}{\eta^2}\widetilde{W}'(g_\eta^2)q^2 + \frac{2}{\eta^2}\widetilde{W}''(g_\eta^2)g_{\varepsilon,\eta}^2 q^2\Big]\,dx,\end{aligned}$$

where $\lambda \in C^1([0, 1])$ is given by (1.9), so Q_η^{MM} is continuous in $H_0^1(B^N, \mathbb{R}^{N+1})$.

Proof of Theorem 1.3. By the instability of the equator map proved in Theorem 2.6 (b), we only need to prove the stability and local minimality of the escaping solution m_η.

Proof of the positive-definiteness of Q_η^{MM}. Let $W(t) = t^2$ and let ε_0 and $\eta_0 \in C^0([0, \varepsilon_0))$ be as in Theorem 2.4; they are well defined as $W'(1) > 0$. If $\widetilde{W}'(0) > 0$, then η_0 is increasing and $\lim_{\varepsilon\to\varepsilon_0} \eta_0(\varepsilon) = \infty$ (see Remark 2.5), so η_0 has an increasing inverse $\eta_0^{-1}\colon [0,\infty) \to [0, \varepsilon_0)$. If $\widetilde{W}'(0) = 0$, then $\eta_0(\varepsilon) = 0$ for all $\varepsilon \in (0, \varepsilon_0)$ and by abuse of notation, we set $\eta_0^{-1}(\eta) = \varepsilon_0$ for every $\eta > 0$. In both cases, by Theorem 2.4, for $0 < \varepsilon < \eta_0^{-1}(\eta)$, (1.12)–(1.15) has an escaping solution $(f_{\varepsilon,\eta}, g_{\varepsilon,\eta})$ with $f_{\varepsilon,\eta} > 0$ and $g_{\varepsilon,\eta} > 0$. By Remark 2.17, $(f_{\varepsilon,\eta}, g_{\varepsilon,\eta}) \to (\tilde{f}_\eta, g_\eta)$ in $\mathcal{B}$ as $\varepsilon \to 0$, and so uniformly on compact subsets of $(0, 1]$.

We will prove the positive-definiteness of Q_η^{MM} from the positive-definiteness of the second variation $Q_{\varepsilon,\eta}$ of the escaping critical point $m_{\varepsilon,\eta} = (f_{\varepsilon,\eta} n, g_{\varepsilon,\eta})$ of $E_{\varepsilon,\eta}$ (established in Theorem 1.5 (a)). Fix some $V = (v, q) \in C_c^\infty(B^N \setminus \{0\}, \mathbb{R}^{N+1})$ with $V \cdot m_\eta = 0$ in B^N. We write $v = sn + \mathring{w} + \not{D}\psi$ with $s \in C_c^\infty(B^N \setminus \{0\})$, $\mathring{w} \in C_c^\infty(B^N \setminus \{0\}, \mathbb{R}^N)$ being a tangent vector field (i.e. $\mathring{w} \cdot n = 0$) having vanishing covariant divergence $\not{D} \cdot \mathring{w}(r, \cdot) = 0$ on $\mathbb{S}^{N-1}$ and $\psi \in C_c^\infty(B^N \setminus \{0\}, \mathbb{R})$ satisfying $\int_{\mathbb{S}^{N-1}} \psi(r, \theta)\, d\sigma = 0$.

For $0 < \varepsilon < \eta_0^{-1}(\eta)$, define

$$q_\varepsilon = q - \frac{f_{\varepsilon,\eta} - \tilde{f}_\eta}{g_{\varepsilon,\eta}} s - \frac{g_{\varepsilon,\eta} - g_\eta}{g_{\varepsilon,\eta}} q$$

and $V_\varepsilon = (v, q_\varepsilon) \in C_c^\infty(B^N \setminus \{0\}, \mathbb{R}^{N+1})$. Then $\operatorname{supp} V_\varepsilon \subset \operatorname{supp} V \subset B^N \setminus \{0\}$, and $V_\varepsilon \to V$ in $C^0(\bar{B}^N) \cap H^1(B^N)$ as $\varepsilon \to 0$ and $V_\varepsilon \cdot m_{\varepsilon,\eta} = 0$ in B^N. We decompose

$$s(r,\theta) = \sum_{i=0}^\infty s_i(r)\zeta_i(\theta), \quad \psi(r,\theta) = \sum_{i=0}^\infty \psi_i(r)\zeta_i(\theta),$$
$$q(r,\theta) = \sum_{i=0}^\infty q_i(r)\zeta_i(\theta), \quad q_\varepsilon(r,\theta) = \sum_{i=0}^\infty q_{\varepsilon,i}(r)\zeta_i(\theta),$$

define $\mathring{V} = (\mathring{w}, 0)$, $V_i = (s_i\zeta_i n + \psi_i \not{D}\zeta_i, q_i\zeta_i)$ and $V_{\varepsilon,i} = (s_i\zeta_i n + \psi_i \not{D}\zeta_i, q_{\varepsilon,i}\zeta_i)$ as in Proposition 3.1. Note that $V_{\varepsilon,i} \to V_i$ in $C^0(\bar{B}^N) \cap H^1(B^N)$ as $\varepsilon \to 0$ for every $i \geq 0$, $0 = V \cdot m_\eta = s\tilde{f}_\eta + q g_\eta = \sum_{i=0}^\infty (s_i \tilde{f}_\eta + q_i g_\eta)\zeta_i$ and so $s_i \tilde{f}_\eta + q_i g_\eta = 0$ for all $i \geq 0$. By the positivity inequality (3.14), we have

$$\begin{aligned} Q_{\varepsilon,\eta}[V_\varepsilon] \geq C \Big\| V_\varepsilon - \sum_{i=0}^N V_{\varepsilon,i} \Big\|_{L^2(B^N)}^2 &+ \int_0^1 r^{N-1} f_{\varepsilon,\eta}^2 \Big| \Big(\frac{s_0}{f_{\varepsilon,\eta}}\Big)' \Big|^2 dr \\ + (N-1)\sum_{i=1}^N \int_0^1 r^{N-3} \Big\{ f_{\varepsilon,\eta}^2 \Big| \Big(\frac{r\psi_i}{f_{\varepsilon,\eta}}\Big)' \Big|^2 \\ &+ \frac{2}{r} f_{\varepsilon,\eta} f_{\varepsilon,\eta}' \Big(\frac{s_i}{f_{\varepsilon,\eta}'} - \frac{r\psi_i}{f_{\varepsilon,\eta}}\Big)^2 \Big\}\, dr. \end{aligned} \tag{3.19}$$

Claim: $Q_{\varepsilon,\eta}[V_\varepsilon] \to Q_\eta^{\mathrm{MM}}[V]$ *as* $\varepsilon \to 0$.

Indeed, as $f_{\varepsilon,\eta}$ converges to $\tilde{f}_\eta$ in $H^1_{\mathrm{loc}}(0,1)$, we have for any open set K compactly supported in $B^N \setminus \{0\}$ and $(\varphi_\varepsilon) \subset H^1_0(K)$ converging in H^1 to $\varphi \in H^1_0(K)$, by multiplying from (1.13) and (1.6) with $\varphi_\varepsilon/f_{\varepsilon,\eta}$ and $\varphi/\tilde{f}_\eta$ respectively,

$$\lim_{\varepsilon\to 0}\int_{B^N} \frac{1}{\varepsilon^2} W'(1-f_{\varepsilon,\eta}^2 - g_{\varepsilon,\eta}^2)\varphi_\varepsilon\,dx = \int_{B^N} \lambda(r)\varphi\,dx.$$

Recalling the expressions of $Q_{\varepsilon,\eta}[V_\varepsilon]$ and $Q_\eta^{\mathrm{MM}}[V]$, together with the fact that $sf_{\varepsilon,\eta} + q_\varepsilon g_{\varepsilon,\eta} = V_\varepsilon \cdot m_{\varepsilon,\eta} = 0$, $\operatorname{supp} V_\varepsilon \subset \operatorname{supp} V \subset B^N \setminus \{0\}$, and $|V_\varepsilon|^2 \to |V|^2$ in $H^1_0(\operatorname{supp} V)$, the claim is readily seen from the above identity.

Passing $\varepsilon \to 0$ in (3.19) using the claim on the left-hand side and Fatou's lemma on the right-hand side, we obtain

$$\begin{aligned} Q_\eta^{\mathrm{MM}}[V] \geq C\Big\|V - \sum_{i=0}^N V_i\Big\|^2_{L^2(B^N)} + \int_0^1 r^{N-1}\tilde{f}_\eta^2\Big|\Big(\frac{s_0}{\tilde{f}_\eta}\Big)'\Big|^2\,dr \\ + (N-1)\sum_{i=1}^N \int_0^1 r^{N-3}\Big\{\tilde{f}_\eta^2\Big|\Big(\frac{r\psi_i}{\tilde{f}_\eta}\Big)'\Big|^2 \\ + \frac{2}{r}\tilde{f}_\eta\tilde{f}_\eta'\Big(\frac{s_i}{\tilde{f}_\eta'} - \frac{r\psi_i}{\tilde{f}_\eta}\Big)^2\Big\}\,dr \end{aligned} \tag{3.20}$$

for any $V \in C_0^\infty(B^N \setminus \{0\}, \mathbb{R}^{N+1})$ satisfying $V \cdot m_\eta = 0$ in B^N.

Suppose next that $V \in H^1_0(B^N, \mathbb{R}^{N+1})$ with $V \cdot m_\eta = 0$ in B^N. Pick a sequence $\{V_j\} \subset C_c^\infty(B^N \setminus \{0\}, \mathbb{R}^{N+1})$ which converges in $H^1(B^N, \mathbb{R}^{N+1})$ to V. Let $\tilde{V}_j = V_j - (V_j \cdot m_\eta)m_\eta \in C_c^\infty(B^N \setminus \{0\}, \mathbb{R}^{N+1})$. Then $\{\tilde{V}_j\}$ also converges in $H^1(B^N, \mathbb{R}^{N+1})$ to V. Applying (3.20) to $\tilde{V}_j$ (since $\tilde{V}_j \cdot m_\eta = 0$), and sending $j \to \infty$ (using the continuity of Q_η^{MM} on the left-hand side and Fatou's lemma on the right-hand side), we see that (3.20) holds for $V \in H^1_0(B^N, \mathbb{R}^{N+1})$ satisfying $V \cdot m_\eta = 0$ in B^N. Moreover, if $Q_\eta^{\mathrm{MM}}[V] = 0$, then $V = \sum_{i=0}^N V_i$, and

$$\frac{s_0}{\tilde{f}_\eta},\ \frac{r\psi_i}{\tilde{f}_\eta} \text{ are constant} \quad\text{and}\quad \frac{s_i}{\tilde{f}_\eta'} - \frac{r\psi_i}{\tilde{f}_\eta} = 0 \ \text{ for } 1 \leq i \leq N.$$

Recalling also that $s_i\tilde{f}_\eta + q_i g_\eta = 0$ in $(0,1)$ and $s_i(1) = \psi_i(1) = 0$ for all $i \geq 0$, we deduce that $V \equiv 0$, i.e. the positive-definiteness of Q_η^{MM}.

Proof of the local minimality of m_η. We relate the functional E_η^{MM} in a neighborhood of m_η to the second variation Q_η^{MM}, notwithstanding the fact that $H^1(B^N, \mathbb{S}^N)$ is not a manifold.

Consider a map $m_\eta + V \in \mathcal{A}^{\mathrm{MM}}$, and write $V = (v, q)$ and $\tilde{V} := V - (V \cdot m_\eta)m_\eta = (\tilde{v}, \tilde{q})$ so that $V, \tilde{V} \in H^1_0(B^N, \mathbb{R}^{N+1})$ and $\tilde{V} \cdot m_\eta = 0$. By the Euler–Lagrange equation for m_η (as a critical point for E_η^{MM} in $\mathcal{A}^{\mathrm{MM}}$) and $V \cdot m_\eta = -\frac{1}{2}|V|^2$ (since $|m_\eta + V|^2 = |m_\eta|^2 = 1$),

$$\begin{aligned} E_\eta^{\mathrm{MM}}[m_\eta + V] - E_\eta^{\mathrm{MM}}[m_\eta] - \frac{1}{2} Q_\eta^{\mathrm{MM}}[\widetilde{V}] &= \frac{1}{2}\int_{B^N} \Big\{(|\nabla V|^2 - |\nabla \widetilde{V}|^2) - \lambda(r)(|V|^2 - |\widetilde{V}|^2) \\ &\qquad + \frac{1}{\eta^2}\big[\widetilde{W}'(g_\eta^2) + 2\widetilde{W}''(g_\eta^2)g_\eta^2\big](q^2 - \tilde{q}^2)\Big\}\, dx \\ &\quad + \int_{B^N} h(x, V(x))\, dx, \\ h(x,y) = \frac{1}{2\eta^2}\big\{\widetilde{W}((g_\eta(x) + y_{N+1})^2) - \widetilde{W}(g_\eta^2(x)) & \\ - \widetilde{W}'(g_\eta^2(x))(2g_\eta(x)y_{N+1} + y_{N+1}^2) & \\ - 2\widetilde{W}''(g_\eta^2(x))g_\eta^2(x)y_{N+1}^2\big\}. & \end{aligned} \tag{3.21}$$

As in the proof of Theorem 1.5 (a), the positive-definiteness of Q_η^{MM} implies that there is a constant $c > 0$ depending only on η, W and $\widetilde{W}$ such that

$$Q_\eta^{\mathrm{MM}}[\widetilde{V}] \geq c\|\nabla \widetilde{V}\|^2_{L^2(B^N)} \quad \text{for every } \widetilde{V} \in H_0^1(B^N, \mathbb{R}^{N+1}) \text{ with } \widetilde{V} \cdot m_\eta = 0.$$

Since $h \in C^0(\overline{B}^N, C^2(\mathbb{R}^{N+1}))$, $h(x,0) = 0$, $\nabla_y h(x,0) = 0$, $\nabla_y^2 h(x,0) = 0$ and h satisfies the growth assumptions in Proposition D.1 for $p = 2$ (due to the convexity of $\widetilde{W}$), by Proposition D.1, for any $a > 0$, there exists $\delta > 0$ such that

$$\int_{B^N} h(x, V(x))\, dx \geq -a\|\nabla V\|^2_{L^2(B^N)} \quad \text{if } V \in H_0^1(B^N, \mathbb{R}^{N+1}),\ \|V\|_{H^1(B^N)} \leq \delta.$$

Let us consider the first integral on the right-hand side of (3.21). We start with the term $|V|^2 - |\widetilde{V}|^2$, using the facts that $V \cdot m_\eta = -\frac{1}{2}|V|^2$, $|V|^2 - |\widetilde{V}|^2 = |V \cdot m_\eta|^2 = \frac{1}{4}|V|^4$. Likewise, since $|q| \leq |V|$, $|\tilde{q}| \leq |\widetilde{V}| \leq |V|$, $0 < g_\eta \leq 1$ and $q - \tilde{q} = (V \cdot m_\eta)g_\eta$,

$$|q^2 - \tilde{q}^2| = |q - \tilde{q}|\,|q + \tilde{q}| \leq 2|V \cdot m_\eta|\,|V| = |V|^3.$$

Next, the term $|\nabla V|^2 - |\nabla \widetilde{V}|^2$ is estimated using that $\nabla(V - \widetilde{V}) = \nabla((V \cdot m_\eta)m_\eta) = -\frac{1}{2}\nabla(|V|^2 m_\eta)$ and $-m_\eta \cdot \partial_j \widetilde{V} = \partial_j m_\eta \cdot \widetilde{V}$ for $1 \leq j \leq N$,

$$\begin{aligned} |\nabla V|^2 - |\nabla \widetilde{V}|^2 &= |\nabla(V - \widetilde{V})|^2 + 2\nabla(V - \widetilde{V}) : \nabla \widetilde{V} \\ &= |\nabla(V - \widetilde{V})|^2 - \nabla(|V|^2 m_\eta) : \nabla \widetilde{V} \\ &= |\nabla(V - \widetilde{V})|^2 + \sum_{j=1}^N \partial_j(|V|^2)\widetilde{V} \cdot \partial_j m_\eta - |V|^2 \nabla m_\eta : \nabla \widetilde{V} \\ &\geq |\nabla(V - \widetilde{V})|^2 - C|V|^2(|\nabla V| + |V|^2) \end{aligned}$$

for some $C = C(\|\nabla m_\eta\|_{C^1(\bar{B}^N)})$, where we have used $|V| \le |m_\eta + V| + |m_\eta| = 2$ and $|\nabla \widetilde{V}| = |\nabla V + \frac{1}{2}\nabla(|V|^2 m_\eta)| \le C(|\nabla V| + |V|^2)$.

Putting things together in (3.21) with $a = \frac{1}{8}\min(c, 1)$, by the Cauchy–Schwarz and triangle inequalities, we get for all $m_\eta + V \in \mathcal{A}^{\mathrm{MM}}$ with $\|V\|_{H^1(B^N)} \le \delta$ that

$$\begin{aligned} &E_\eta^{\mathrm{MM}}[m_\eta + V] - E_\eta^{\mathrm{MM}}[m_{\varepsilon,\eta}] \\ &\quad \ge \frac{c}{2}\|\nabla \widetilde{V}\|^2_{L^2(B^N)} + \frac{1}{2}\|\nabla(V - \widetilde{V})\|^2_{L^2(B^N)} - a\|\nabla V\|^2_{L^2(B^N)} \\ &\qquad - C(\|\nabla V\|_{L^2(B^N)}\|V\|^2_{L^4(B^N)} + \|V\|^4_{L^4(B^N)} + \|V\|^3_{L^3(B^N)}) \\ &\quad \ge \frac{\min(c,1)}{8}\|\nabla V\|^2_{L^2(B^N)} - \widetilde{C}(\|V\|^4_{L^4(B^N)} + \|V\|^3_{L^3(B^N)}). \end{aligned}$$

Note also that, since $|V| \le 2$ and by the Sobolev embedding theorem for $V \in H_0^1(B^N)$, we have for any fixed $2 < p < \min(3, \frac{2N}{N-2})$ that

$$\|V\|^4_{L^4(B^N)} + \|V\|^3_{L^3(B^N)} \le C_p\|V\|^p_{L^p(B^N)} \le C_{N,p}\|\nabla V\|^p_{L^2(B^N)}.$$

By the last two estimates, for small $\delta > 0$, we obtain for some $\widehat{C} > 0$,

$$E_\eta^{\mathrm{MM}}[m_\eta + V] \ge E_\eta^{\mathrm{MM}}[m_\eta] + \widehat{C}\|\nabla V\|^2_{L^2(B^N)} \quad \text{if } m_\eta + V \in \mathcal{A}^{\mathrm{MM}},\ \|V\|_{H^1(B^N)} < \delta,$$

yielding the desired local minimality of m_η for E_η^{MM} in $\mathcal{A}^{\mathrm{MM}}$. ∎

A. Radially symmetric vector-valued maps

In the sequel, let $\mathrm{SO}(N)$ denote the group of $N \times N$ special orthogonal matrices, equipped with the Haar measure. Naturally, $\mathrm{SO}(N) \times B^N$ is equipped with the product measure.

Definition A.1. Let $N \ge 2$ and $k \ge 0$. A measurable map $m: B^N \to \mathbb{R}^{N+k}$ is said to be $\mathrm{SO}(N)$-equivariant, or simply radially symmetric, if

$$m(Rx) = \widetilde{R}m(x) \quad \text{for almost all } (R, x) \in \mathrm{SO}(N) \times B^N,$$

where $\widetilde{R} = \begin{pmatrix} R & 0_{N\times k} \\ 0_{k\times N} & I_{k\times k}\end{pmatrix} \in \mathrm{SO}(N+k)$, and $0_{i\times j}$ and $I_{k\times k}$ denote respectively the $i \times j$ zero matrix and the $k \times k$ identity matrix.

Lemma A.2. *Let $N \ge 2$, $k \ge 0$ and $m \in L^1_{\mathrm{loc}}(B^N, \mathbb{R}^{N+k})$.*

(a) *If $N \ge 3$, then m is radially symmetric if and only if there exist functions f, $g_1, \dots, g_k \in L^1_{\mathrm{loc}}(0,1)$ such that $m(x) = (f(|x|)\frac{x}{|x|}, g_1(|x|), \dots, g_k(|x|))$ for almost all $x \in B^N$.*

(b) *If $N = 2$, then m is radially symmetric if and only if there exist functions f_1, f_2, $g_1, \dots, g_k \in L^1_{\mathrm{loc}}(0,1)$ such that*

$$m(x) = (f_1(|x|)\frac{x}{|x|} + f_2(|x|)\frac{x^\perp}{|x|}, g_1(|x|), \dots, g_k(|x|))$$

for almost all $x = (x_1, x_2) \in B^2$, where $x^\perp = (-x_2, x_1)$.

Proof. It is clear that if m has the stated form, then m is radially symmetric. For the converse, suppose that m is radially symmetric. Let us make an observation on mollifications of a radially symmetric map. Let (ϱ_ε) be a sequence of smooth radially symmetric mollifiers (i.e. $\varrho_\varepsilon(x) = \varrho_\varepsilon(|x|)$) satisfying $\operatorname{supp}\varrho_\varepsilon \subset (-\varepsilon, \varepsilon)$ and let $m_\varepsilon = m * \varrho_\varepsilon$ in $B_{1-\varepsilon}$, where B_r is the ball centered at zero of radius $r > 0$.

We claim that m_ε is radially symmetric in $B_{1-\varepsilon}$. Indeed, by Fubini's theorem, for almost all $R \in \mathrm{SO}(N)$, we have

$$m(Rx) = \widetilde{R}m(x) \quad \text{for almost all } x \in B^N.$$

Therefore, for almost all $R \in \mathrm{SO}(N)$ and for all $0 < |x| < 1 - \varepsilon$,

$$\begin{aligned} m_\varepsilon(Rx) &= \int_{B^N} m(y)\varrho_\varepsilon(Rx - y)\, dy = \int_{B^N} m(Rz)\varrho_\varepsilon(Rx - Rz)\, dz \\ &= \int_{B^N} \widetilde{R}m(z)\varrho_\varepsilon(x - z)\, dz = \widetilde{R}m_\varepsilon(x), \end{aligned}$$

which proves the claim. Thus, it suffices to consider continuous m in our proof. In this case,

$$m(Rx) = \widetilde{R}m(x) \quad \text{for all } (R, x) \in \mathrm{SO}(N) \times B^N. \tag{A.1}$$

Clearly (A.1) implies that, for $1 \le j \le k$ and $x \in B^N$, $m_{N+j}(Rx) = m_{N+j}(x)$ for all $R \in \mathrm{SO}(N)$ and so $m_{N+j}(x) = g_j(|x|)$ for some $g_j \in C(0, 1)$. We thus assume without loss of generality that $k = 0$, i.e. $m\colon B^N \to \mathbb{R}^N$.

Let $e_N = (0, \dots, 0, 1)$. For $r \in (0, 1)$, we write $m(re_N) = (a(r), b(r))$, where $a(r) \in \mathbb{R}^{N-1}$ and $b(r) \in \mathbb{R}$. Since m is continuous, $a, b \in C((0, 1))$.

Case (a)*: $N \ge 3$.* Taking R of the form $R = \begin{pmatrix} S & 0_{(n-1)\times 1} \\ 0_{1\times(n-1)} & 1 \end{pmatrix}$, where $S \in \mathrm{SO}(N - 1)$, we obtain from (A.1) that $a(r) = Sa(r)$ for all $S \in \mathrm{SO}(N - 1)$. As $N \ge 3$, there exists $S(r) \in \mathrm{SO}(N - 1)$ so that $S(r)a(r) = -a(r)$ and so the above implies that $a(r) = 0$. In particular, $m(re_N) = b(r)e_N$ for every $r \in (0, 1)$. Now if $|x| = r \in (0, 1)$, we select $R \in \mathrm{SO}(N)$ such that $R(re_N) = x$ and obtain from (A.1) that

$$m(x) = m(R(re_N)) = Rm(re_N) = b(r)Re_N = b(r)\frac{x}{r}.$$

The conclusion follows with $f(r) = b(r)$.

Case (b)*: $N = 2$.* In this case, $a(r)$ is a scalar so that $m(re_2) = -a(r)e_2^\perp + b(r)e_2$. Now if $x = (r\cos\varphi, r\sin\varphi)$ for some $r > 0$ and $\varphi \in [0, 2\pi)$, setting $R_\varphi := \begin{pmatrix} \sin\varphi & \cos\varphi \\ -\cos\varphi & \sin\varphi \end{pmatrix} \in \mathrm{SO}(2)$, then we have $R_\varphi(re_2) = x$ and $R_\varphi(re_2^\perp) = x^\perp$. By (A.1),

$$m(x) = m(R_\varphi(re_2)) = R_\varphi m(re_2) = -a(r)R_\varphi e_2^\perp + b(r)R_\varphi e_2 = -a(r)\frac{x^\perp}{r} + b(r)\frac{x}{r}.$$

The conclusion follows with $f_1(r) = b(r)$ and $f_2(r) = -a(r)$. ∎

Remark A.3. In a similar fashion to Definition A.1, one can also define $O(N)$-equivariant maps. It is easy to see from the above lemma that, for $N \geq 3$ and $k \geq 0$, $\mathrm{SO}(N)$-equivariant maps are $O(N)$-equivariant. For $N = 2$ and $k \geq 0$, $m \in L^1_{\mathrm{loc}}(B^2; \mathbb{R}^{2+k})$ is $O(2)$-equivariant if and only if there exist functions $f, g_1, \dots, g_k \in L^1_{\mathrm{loc}}(0,1)$ such that

$$m(x) = \Big(f(|x|)\frac{x}{|x|}, g_1(|x|), \dots, g_k(|x|) \Big) \quad \text{for almost all } x \in B^2.$$

This is because the map $x \mapsto f_2(|x|)\frac{x^\perp}{|x|}$ is $O(2)$-invariant if and only if $f_2 = 0$, because $(Rx)^\perp = -R(x^\perp)$ with R being the reflection about the x_1-axis, i.e. $R(x_1, x_2) = (x_1, -x_2)$.

Lemma A.4. *Suppose $N \geq 2$, $\varepsilon > 0$ and $W \in C^2((-\infty, 1])$. If m is a bounded[13] radially symmetric critical point of $E^{\mathrm{GL}}_\varepsilon$ in $\mathcal{A}^{\mathrm{GL}}$, then $m \in C^2(\bar{B}^N)$ and takes the form $m(x) = f(|x|)\frac{x}{|x|}$ for some $f \in C^2([0,1])$ with $\frac{f}{r} \in C^2([0,1])$. In particular, $f(0) = 0$ and m is $O(N)$-equivariant.*

Lemma A.5. *Suppose $N \geq 2$, $\varepsilon, \eta > 0$, $W \in C^2((-\infty, 1])$ and $\widetilde{W} \in C^2([0, \infty))$. If m is a bounded[14] radially symmetric critical point of $E_{\varepsilon,\eta}$ in $\mathcal{A}$, then $m \in C^2(\bar{B}^N)$ and takes the form $m(x) = (f(|x|)\frac{x}{|x|}, g(|x|))$ for some $f, g \in C^2([0,1])$ with $\frac{f}{r} \in C^2([0,1])$. In particular, $f(0) = 0$, $g'(0) = 0$ and m is $O(N)$-equivariant.*

We will only give the proof of the latter result. The proof of the other one requires minor modifications and is omitted.

Proof of Lemma A.5. As m is bounded, it satisfies

$$\begin{cases} -\Delta m - \frac{1}{\varepsilon^2} W'(1 - |m|^2)m + \frac{1}{\eta^2}\widetilde{W}'(m_{N+1}^2) m_{N+1} e_{N+1} = 0 & \text{in } B^N \setminus \{0\}, \\ m(x) = x & \text{on } \partial B^N. \end{cases} \tag{A.2}$$

Due to $m \in H^1 \cap L^\infty(B^N)$ (in particular, $W'(1 - |m|^2), \widetilde{W}'(m_{N+1}^2) \in L^\infty(B^N)$), it follows that (A.2) holds in all of B^N, and, by elliptic regularity theory, $m \in C^2(\bar{B}^N)$. On the other hand, using Lemma A.2 and the regularity of m, we write

$$m(x) = \begin{cases} \Big(f_1(|x|)\dfrac{x}{|x|} + f_2(|x|)\dfrac{x^\perp}{|x|}, g(|x|) \Big) & \text{if } N = 2, \\ \Big(f_1(|x|)\dfrac{x}{|x|}, g(|x|) \Big) & \text{if } N \geq 3, \end{cases} \tag{A.3}$$

where $f_1, f_2 \in C^2 \cap L^\infty((0,1])$, $g \in C^2([0,1])$, $f_1(0) = f_2(0) = 0$, $g'(0) = 0$.

To conclude, we show that $\frac{f_1}{r} \in C^2([0,1])$ and, when $N = 2$, $f_2 = 0$ in $(0,1)$. For the last claim, we use ideas from the proof of [26, Proposition 2.3]. From (A.2), we have

$$\nabla \cdot (-m_2 \nabla m_1 + m_1 \nabla m_2) = (m_1, m_2)^\perp \cdot \Delta(m_1, m_2) = 0 \quad \text{in } B^2.$$

[13] If W satisfies (1.10), then the boundedness of m is a consequence of Corollary 2.8.

[14] If W and $\widetilde{W}$ satisfy (1.10)-(1.11), then the boundedness of m follows from Lemma 2.7.

Integrating over balls B_r of radius $r \in (0,1)$, the Gauss formula yields

$$\int_{\partial B_r} (m_1, m_2)^{\perp} \cdot \partial_r (m_1, m_2)\, dS = \int_{\partial B_r} (-m_2 \partial_r m_1 + m_1 \partial_r m_2)\, dS = 0. \tag{A.4}$$

Using (A.3) in (A.4), we obtain

$$-f_1' f_2 + f_2' f_1 = 0 \quad \text{in } (0,1). \tag{A.5}$$

Since $f_1(1) = 1$, we have that $f_1 > 0$ in some interval $(r_1, 1)$ with $0 \le r_1 < 1$. Dividing (A.5) by f_1^2 in $(r_1, 1)$, we get $(f_2/f_1)' = 0$, and using the fact that $f_2(1) = 0$, we have $f_2 = 0$ in $(r_1, 1)$. In particular, $f_2'(1) = 0$. Now, by (A.2),

$$f_2'' + \frac{N-1}{r} f_2' + c(r) f_2 = 0 \quad \text{in } (0,1), \tag{A.6}$$

where $c(r) := -\frac{N-1}{r^2} + \frac{1}{\varepsilon^2} W'(1 - f_1^2 - f_2^2 - g^2)$ belongs to $C^1((0,1])$. Since $f_2(1) = f_2'(1) = 0$, standard uniqueness results for ODEs imply that $f_2 = 0$ in $(0,1)$ as desired.

Let us show next that $\frac{f_1}{r} \in C^2([0,1])$ for any $N \ge 2$. By (A.2) and (A.3),

$$f_1'' + \frac{N-1}{r} f_1' + \Big(-\frac{N-1}{r^2} + \frac{1}{\varepsilon^2} W'(1 - f_1^2 - g^2)\Big) f_1 = 0 \quad \text{in } (0,1).$$

Setting $v = \frac{f_1}{r}$ and

$$d = \frac{1}{\varepsilon^2} W'(1 - f_1^2 - g^2) = \frac{1}{\varepsilon^2} W'(1 - |m|^2) \in C^1([0,1])$$

(as $m \in C^2(\bar{B}^N)$), we then have

$$v'' + \frac{N+1}{r} v' + d(r) v(r) = 0 \quad \text{in } (0,1).$$

Considering v as a radially symmetric function on the $(N+2)$-dimensional ball B^{N+2}, we have that v satisfies $\Delta v + dv = 0$ in $B^{N+2} \setminus \{0\}$. On the other hand, since $m \in H^1(B^N)$, we have $r^{\frac{N-1}{2}} f_1', r^{\frac{N-3}{2}} f_1 \in L^2(0,1)$ and so $v \in H^1(B^{N+2})$. It follows that $\Delta v + dv = 0$ in B^{N+2} and since $d \in C^1([0,1])$, we deduce that $v \in C^2(B^{N+2})$. The conclusion follows. ∎

Lemma A.6. *Suppose $N \ge 2$, $\eta > 0$ and $\tilde{W} \in C^2([0,1])$. If m is a radially symmetric critical point of E_η^{MM} in $\mathcal{A}^{\mathrm{MM}}$, then m takes the form*

$$m(x) = \Big(f(|x|)\frac{x}{|x|}, g(|x|)\Big), \tag{A.7}$$

with $f, g \in C^2_{\mathrm{loc}}((0,1])$, $f^2 + g^2 = 1$ and $r^{\frac{N-1}{2}}(|f'| + |g'|) + r^{\frac{N-3}{2}}|f| \in L^2(0,1)$. In particular, m is $O(N)$-equivariant. Furthermore, either $\frac{f}{r}, g \in C^2([0,1])$ or both $(f,g) \equiv (1,0)$ and $N \ge 3$, where in the former case one also has that $m \in C^2(\bar{B}^N)$, $f(0) = 0$ and $g'(0) = 0$.

Proof. We adapt the proof of Lemma A.5. Without loss of generality, we may assume that $\widetilde{W}(0) = 0$. As a critical point of E_η^{MM} in $\mathcal{A}^{\mathrm{MM}}$, m satisfies

$$\begin{cases} -\Delta m - \lambda(x)m + \dfrac{1}{\eta^2}\widetilde{W}'(m_{N+1}^2)m_{N+1}e_{N+1} = 0 & \text{in } B^N, \\ m(x) = x & \text{on } \partial B^N, \end{cases} \tag{A.8}$$

where $\lambda = |\nabla m|^2 + \frac{1}{\eta^2}\widetilde{W}'(m_{N+1}^2)m_{N+1}^2 \in L^1(B^N)$. By Lemma A.2, m takes the form (A.3). In particular, $\lambda = \lambda(r) \in L^1_{\mathrm{loc}}((0,1])$, which together with (A.8) (recast as ODEs for f_1, f_2, g) implies that $f_1'', f_2'', g'' \in L^1_{\mathrm{loc}}((0,1])$, where f_2 is absent when $N \geq 3$. This in turn implies that $\lambda \in C^0((0,1])$ and then again, by regularity theory, $f_1, f_2, g \in C^2((0,1])$ (and hence $m \in C^2(\bar{B}^N \setminus \{0\})$. Next, as in the proof of Lemma A.5, when $N = 2$, we prove that (A.4)–(A.5) hold here also, yielding $f_2 = 0$ in $(0,1)$. We have thus shown that m has the form (A.7), where $f^2 + g^2 = 1$, $r^{\frac{N-1}{2}}(|f'| + |g'|) + r^{\frac{N-3}{2}}|f| \in L^2(0,1)$, and $f, g \in C^2((0,1])$.

Step 1: We prove $f, g \in C([0,1])$. We distinguish the cases $N = 2$ and $N \geq 3$.

Case 1: $N = 2$. It is known that the continuity of m in $\bar{B}^2$ can be proved using Wente's lemma (see e.g. Hélein [21] or Carbou [10, Theorem 1]). However, in this ODE setting, the continuity of f (and hence of g) in $[0,1]$ is a consequence of the fact that $r^{\frac{1}{2}}|f'| + r^{-\frac{1}{2}}|f| \in L^2(0,1)$,

$$\begin{aligned} |f^2(r_1) - f^2(r_2)| &\leq 2\int_{r_2}^{r_1} |f'(r)|\,|f(r)|\,dr \\ &\leq \int_{r_2}^{r_1} \Big(r|f'(r)|^2 + \frac{1}{r}|f(r)|^2\Big)\,dr \xrightarrow{r_1,r_2\to 0} 0. \end{aligned}$$

Also, since $r^{-\frac{1}{2}}|f| \in L^2(0,1)$, we get $f(0) = 0$. It follows that $m \in C(\bar{B}^2)$.

Case 2: $N \geq 3$. As $f, g \in C^2((0,1])$ and $f^2 + g^2 = 1$, we can find a lifting $\theta \in C^2((0,1])$ such that $r^{\frac{N-1}{2}}|\theta'| \in L^2(0,1)$, $f = \sin\theta$, $g = \cos\theta$ in $(0,1]$ and $\theta(1) = \pi/2$. (To prepare for Steps 2 and 3 later on, we note that the existence of such a lifting θ also holds for $N = 2$ where we have in addition to the above that $\theta \in C([0,1])$, $r^{-1/2}\sin\theta \in L^2(0,1)$ and $\theta(0) \in \pi\mathbb{Z}$.)

A direct computation using (A.8) gives

$$\theta'' + \frac{N-1}{r}\theta' - \frac{N-1}{r^2}\sin\theta\cos\theta + \frac{1}{\eta^2}\widetilde{W}'(\cos^2\theta)\sin\theta\cos\theta = 0 \quad \text{in } (0,1). \tag{A.9}$$

Set $F(r) = [(N-1) - \frac{1}{\eta^2}r^2\widetilde{W}'(\cos^2\theta(r))]\sin\theta(r)\cos\theta(r) \in L^\infty(0,1)$ so that (A.9) is equivalent to $(r^{N-1}\theta')' = F(r)r^{N-3}$. Thus, for a constant c,

$$\theta'(r) = \frac{c}{r^{N-1}} + \frac{1}{r^{N-1}}\int_0^r F(s)s^{N-3}\,ds = \frac{c}{r^{N-1}} + O\Big(\frac{1}{r}\Big) \quad \text{as } r \to 0.$$

Using that $r^{\frac{N-1}{2}}|\theta'| \in L^2(0,1)$, we deduce that $c = 0$ and

$$\theta'(r) = \frac{1}{r^{N-1}} \int_0^r F(s) s^{N-3}\, ds. \tag{A.10}$$

It follows that, for some positive constant C independent of r,

$$|\theta'(r)| \le \frac{C}{r} \quad \text{and} \quad |\theta(r)| \le C(1 + |\log r|) \quad \text{in } (0,1). \tag{A.11}$$

Claim: *We prove that $\theta \in C([0,1])$ and $\theta(0) = \frac{k\pi}{2}$ for some $k \in \mathbb{Z}$.*

Proof of claim. Indeed, let

$$P(r) = r^2(\theta')^2 + (N-1)\cos^2\theta - \frac{r^2}{\eta^2}\widetilde{W}(\cos^2\theta).$$

By (A.11), $P \in L^\infty(0,1)$. Multiplying (A.9) by $2r^2\theta'$, we see that

$$P'(r) = -2(N-2)r(\theta')^2 - \frac{2r}{\eta^2}\widetilde{W}(\cos^2\theta). \tag{A.12}$$

In particular, the function $\widetilde{P}(r) := P(r) + \int_0^r \frac{2s}{\eta^2}\widetilde{W}(\cos^2\theta(s))\, ds$ satisfies $\widetilde{P} \in L^\infty(0,1)$ and $\widetilde{P}'(r) = -2(N-2)r(\theta')^2 \le 0$. It follows that $r(\theta')^2 = \frac{1}{2(N-2)}|\widetilde{P}'| \in L^1(0,1)$ and $\widetilde{P}, P \in W^{1,1}(0,1) \subset C([0,1])$. By (A.10) and integrating by parts,

$$\theta'(r) = \frac{F(r)}{(N-2)r} - \frac{1}{(N-2)r^{N-1}} \int_0^r F'(s) s^{N-2}\, ds.$$

Since $|F'(r)| \le C(|\theta'(r)| + r)$ for every $r \in (0,1)$, we obtain

$$\begin{aligned}|F(r)| &= \left|(N-2)r\theta'(r) + \frac{1}{r^{N-2}} \int_0^r F'(s)s^{N-2}\, ds\right| \\ &\le Cr^2 + Cr|\theta'(r)| + \frac{C}{r^{N-2}} \int_0^r |\theta'(s)| s^{N-2}\, ds.\end{aligned}$$

Noting that, by the Cauchy–Schwarz inequality,

$$\int_0^r |\theta'(s)| s^{N-2}\, ds \le Cr^{N-2}\left(\int_0^r s|\theta'(s)|^2\, ds\right)^{1/2},$$

we deduce from the above bound for $|F|$ that

$$\begin{aligned}\int_0^r |F(s)| s^{N-3}\, ds &\le Cr^N + C\int_0^r |\theta'(s)| s^{N-2}\, ds + C\int_0^r \int_0^s |\theta'(t)| t^{N-2}\, dt \,\frac{ds}{s} \\ &\le Cr^N + Cr^{N-2}\left(\int_0^r s|\theta'(s)|^2\, ds\right)^{1/2}.\end{aligned}$$

Returning to (A.10), since $r|\theta'(r)|^2 \in L^1(0,1)$, we have

$$r|\theta'(r)| \le Cr^2 + C\left(\int_0^r s|\theta'(s)|^2\,ds\right)^{1/2} \to 0 \quad \text{as } r \to 0.$$

Recalling the expression of P and its continuity, we deduce that $\cos^2\theta$ and hence θ belong to $C([0,1])$. By (A.10) and the continuity of F, $r\theta'(r) = \frac{1}{N-2}F(0) + o(1)$ for small $r > 0$. Hence, $F(0) = 0$, i.e. $\theta(0) = \frac{k\pi}{2}$ for some $k \in \mathbb{Z}$. The claim and Step 1 are now completed. ∎

Step 2: We prove that if k is odd, then $(f,g) \equiv (1,0)$ and $N \ge 3$. When k is odd, $f(0) \ne 0$. We saw in Step 1 that this is possible only if $N \ge 3$.

In the absence of $\widetilde{W}$ (i.e. for the harmonic map problem), the assertion that $(f,g) \equiv (1,0)$ can be dealt with as in [30]: (A.12) implies that $P' \le 0$, which leads to $0 = P(0) \ge P(r) \ge P(1) = (\theta'(1))^2 \ge 0$. Thus $\theta'(1) = 0$; since $\theta(1) = \frac{\pi}{2}$, uniqueness results for second-order ODEs give that $\theta \equiv \frac{\pi}{2}$.

To account for the presence of $\widetilde{W}$ in (A.9), we argue as follows. By (A.12), $P'(r) \le 2ar$ for some constant $a > 0$. Since $r\theta'(r) \to 0$ as $r \to 0$ and k is odd, $P(r) \to 0$ as $r \to 0$. Hence $P(r) \le ar^2$. By (A.12), we have $(r^{-2}P)' \le 0$ and since $\cos\theta(1) = 0$, $\widetilde{W}(0) = 0$,

$$P(r) \ge P(1)r^2 = (\theta'(1))^2 r^2 \ge 0 \quad \text{in } (0,1). \tag{A.13}$$

Also by (A.12), we have $(r^{-1}P)' \le -\frac{(N-1)}{r^2}\cos^2\theta - \frac{1}{\eta^2}\widetilde{W}(\cos^2\theta)$. Using the fact that $\cos\theta(0) = 0$, $\widetilde{W}(0) = 0$ and $\widetilde{W} \in C^1$, in particular, $|\widetilde{W}(t)| \le \tilde{c}t$ for $t \in [0,1]$, we thus have that $(r^{-1}P)' \le 0$ in some interval $(0, r_0)$. By $r^{-1}P(r) \to 0$ as $r \to 0$ (as $0 \le P(r) \le ar^2$),

$$P(r) \le 0 \quad \text{in } (0, r_0) \tag{A.14}$$

and so, $P \equiv 0$ in $(0, r_0)$. Putting together (A.13) and (A.14), $\theta'(1) = 0$. By uniqueness results for ODEs, we then have $\theta \equiv \frac{\pi}{2}$, i.e. $(f,g) \equiv (1,0)$.

Step 3: We prove that if $\theta(0) \in \pi\mathbb{Z}$ and $N \ge 2$, then $\frac{f}{r}, g \in C^2([0,1])$. Since $\theta(0) \in \pi\mathbb{Z}$, $F(r) = (N-1)d(r)(\theta(r) - \theta(0))$, where $d(r) = 1 + O(r^2 + |\theta(r) - \theta(0)|^2)$ as $r \to 0$. We can then recast (A.9) in the form

$$L(\theta - \theta(0)) := (\theta - \theta(0))'' + \frac{N-1}{r}(\theta - \theta(0))' - \frac{(N-1)d(r)}{r^2}(\theta - \theta(0)) = 0.$$

It is easy to check that, for $\delta \in (0,1)$, there exists $r_\delta > 0$ such that

$$L(r^{-(N-1)+\delta}) < 0 \quad \text{and} \quad L(r^{1-\delta}) < 0 \quad \text{in } (0, r_\delta).$$

Thus, by the maximum principle (see e.g. [24, Lemma B.1]), we have

$$\frac{|\theta(r_\delta) - \theta(0)|}{r_\delta^{1-\delta}} r^{1-\delta} \pm (\theta(r) - \theta(0)) \ge 0 \quad \text{in } (0, r_\delta).$$

This shows that $r^{-(1-\delta)}|\theta - \theta(0)| \in L^\infty(0,1)$ for all $\delta \in (0,1)$.

Taking $\delta = 1/2$ above, we have $d(r) = 1 + O(r)$. Then, for some large $A > 0$ and small $r_0 > 0$, we have

$$L(r - Ar^2) < 0 \quad \text{and} \quad r - Ar^2 > 0 \quad \text{in } (0, r_0).$$

Again, by the maximum principle, we then have

$$\frac{|\theta(r_0) - \theta(0)|}{r_0 - Ar_0^2}(r - Ar^2) \pm (\theta(r) - \theta(0)) \geq 0 \quad \text{in } (0, r_0).$$

Thus, $r^{-1}(\theta - \theta(0)) \in L^\infty(0,1)$. This yields $F(r) = O(r)$ and by (A.10),

$$\theta' \in L^\infty(0,1).$$

Since $f(0) = \sin\theta(0) = 0$, we get $\frac{f}{r} \in L^\infty(0,1)$. Returning to m, as $|\nabla m|^2 = (\theta')^2 + \frac{(N-1)f^2}{r^2}$, we see that $m \in C^{0,1}(B^N)$ and $\lambda \in L^\infty(B^N)$ (given in (A.8)), and by bootstrapping (A.8), $m \in C^2(B^N)$ and $\lambda \in C^1(B^N)$. By the same argument as in Lemma A.5, it follows that $\frac{f}{r}, g \in C^2([0,1])$, $f(0) = 0$ and $g'(0) = 0$ as desired. ■

B. Properties of the $\mathbb{R}^N$-valued vortex radial profile

Proposition B.1. *Suppose that $N \geq 2$, $W \in C^2((-\infty, 1])$ satisfies (1.10) and let $f_\varepsilon\colon [0,1] \to [0,1]$ be given by Theorem 2.1 and $f_\varepsilon^{-1}\colon [0,1] \to [0,1]$ its inverse. Then,*

(i) *$f_\varepsilon(\varepsilon r) \geq f_{\tilde\varepsilon}(\tilde\varepsilon r)$ for $0 < r < 1/\varepsilon$ and $0 < \tilde\varepsilon \leq \varepsilon$;*

(ii) *if $W'(1) > 0$ and $t_0 := \sup\{0 \leq t < 1 : W(t) = 0\}$, then $t_0 < 1$,*

$$\lim_{\varepsilon \to 0} \frac{f_\varepsilon^{-1}(\sqrt{1 - t_0})}{\varepsilon} = \infty,$$

and, for every $\delta \in (0, 1 - t_0)$,

$$\lim_{\varepsilon \to 0} \frac{f_\varepsilon^{-1}(\sqrt{1 - t_0 - \delta})}{\varepsilon} \in (0, \infty).$$

In particular, for every $a > 0$, there exists $\varepsilon_a > 0$ such that

$$f_\varepsilon^2 \leq 1 - t_0 \quad \text{in } [0, a\varepsilon] \text{ for every } \varepsilon \in (0, \varepsilon_a],$$

and, for every $\delta \in (0, 1 - t_0)$, there exists $C_\delta > 0$ such that

$$1 - t_0 - \delta \leq f_\varepsilon^2 \quad \text{in } [C_\delta \varepsilon, 1] \text{ for every } \varepsilon \in (0, 1/C_\delta].$$

Proof. For $\varepsilon > 0$, define

$$\hat f_\varepsilon(r) = \begin{cases} f_\varepsilon(\varepsilon r) & \text{if } r \in (0, 1/\varepsilon), \\ 1 & \text{if } r \in (1/\varepsilon, \infty). \end{cases}$$

Note that

$$\hat{f}_\varepsilon'' + \frac{N-1}{r}\hat{f}_\varepsilon' - \frac{N-1}{r^2}\hat{f}_\varepsilon = -W'(1-\hat{f}_\varepsilon^2)\hat{f}_\varepsilon \quad \text{in } (0,1/\varepsilon)$$

and $\hat{v}_\varepsilon := \frac{\hat{f}_\varepsilon}{r}$ considered as a radially symmetric function in $\mathbb{R}^{N+2}$ satisfies

$$\Delta \hat{v}_\varepsilon = -W'(1-\hat{f}_\varepsilon^2)\hat{v}_\varepsilon \le 0 \quad \text{in } B(0,1/\varepsilon). \tag{B.1}$$

As in Proposition 2.9, $\hat{v}_\varepsilon$ is non-increasing in $(0,1/\varepsilon)$ and so in $(0,\infty)$.

Proof of (i). This is equivalent to proving that $\hat{f}_\varepsilon \ge \hat{f}_{\tilde\varepsilon}$ for $0 < \tilde\varepsilon \le \varepsilon$. This is a direct consequence of the comparison principle[15] [24, Proposition 3.5] and the fact that $\hat{f}_\varepsilon'(0) = \hat{v}_\varepsilon(0) > 0$ (since $\frac{\hat{f}_\varepsilon}{r} = \hat{v}_\varepsilon$ is non-increasing), $\hat{f}_\varepsilon(1/\tilde\varepsilon) = \hat{f}_{\tilde\varepsilon}(1/\tilde\varepsilon) = 1$, and

$$\hat{f}_{\tilde\varepsilon}'' + \frac{N-1}{r}\hat{f}_{\tilde\varepsilon}' - \frac{N-1}{r^2}\hat{f}_{\tilde\varepsilon} = -W'(1-\hat{f}_{\tilde\varepsilon}^2)\hat{f}_{\tilde\varepsilon} \quad \text{in } (0,1/\tilde\varepsilon),$$
$$\hat{f}_\varepsilon'' + \frac{N-1}{r}\hat{f}_\varepsilon' - \frac{N-1}{r^2}\hat{f}_\varepsilon \le -W'(1-\hat{f}_\varepsilon^2)\hat{f}_\varepsilon \quad \text{in } (0,1/\tilde\varepsilon).$$

Proof of (ii). By (1.10), $t_0 < 1$, $W > 0$ and $W' > 0$ in $(t_0, 1]$. We prove

$$\lim_{\varepsilon\to 0} \hat{f}_\varepsilon^{-1}(\sqrt{1-t_0}) = \infty \quad \text{and} \quad \lim_{\varepsilon\to 0} \hat{f}_\varepsilon^{-1}(\sqrt{1-t_0-\delta}) \in (0,\infty). \tag{B.2}$$

By (i), $\{\hat{f}_\varepsilon\}$ is non-increasing as $\varepsilon \to 0$ and hence converges pointwise to some limit function $\hat{f}_*$. In particular, $\hat{f}_*(0) = 0$, $0 \le \hat{f}_* \le 1$ in $(0,\infty)$, $\hat{f}_*$ is continuous at 0, and, by the monotonicity of $\hat{f}_\varepsilon$, $\hat{f}_*$ is non-decreasing. By the equation of $\hat{f}_\varepsilon$ and the bound $0 \le \hat{f}_\varepsilon \le 1$, for every compact interval $[1/C, C] \subset (0,\infty)$, the family $\{\hat{f}_\varepsilon\}_{0<\varepsilon<1/C}$ is bounded in $C^3([1/C,C])$. By the Arzelà–Ascoli theorem, it follows that $\hat{f}_* \in C^2((0,\infty))$, $\hat{f}_\varepsilon$ converges to $\hat{f}_*$ in $C^2_{\text{loc}}((0,\infty))$ as $\varepsilon \to 0$ and

$$\hat{f}_*'' + \frac{N-1}{r}\hat{f}_*' - \frac{N-1}{r^2}\hat{f}_* = -W'(1-\hat{f}_*^2)\hat{f}_* \quad \text{in } (0,\infty).$$

Since $W' > 0$ in $(t_0, 1]$, one can argue as in Step 3 of the proof of [24, Proposition 2.4] to show that $W'(1-\hat{f}_*(\infty)^2)\hat{f}_*(\infty) = 0$, which implies that $\hat{f}_*(\infty) \in \{0\} \cup [\sqrt{1-t_0}, 1]$. Moreover, using again that $W' > 0$ in $(t_0, 1]$, we can argue as in Steps 4 and 5 of the proof of [24, Proposition 2.4] to show that $\hat{f}_* \not\equiv 0$ and so $\hat{f}_*(\infty) \in [\sqrt{1-t_0}, 1]$. Differentiating the equation for $\hat{f}_*$ and applying the strong maximum principle, we have that $\hat{f}_*' > 0$ in $(0,\infty)$.

Claim: $\hat{f}_*(\infty) = \sqrt{1-t_0}$.

[15] Though the comparison principle [24, Proposition 3.5] was stated with the assumption that $W' > 0$ in $(0,1)$ and $W''(0) > 0$, it is straightforward to see that it remains valid under the weaker condition that $W' \ge 0$ in $(0,1)$. Alternatively, one can first apply [24, Proposition 3.5] for the unique radial profiles corresponding to the strictly convex potentials $t \mapsto W(t) + \delta t^2$ with $\delta > 0$ and then send $\delta \to 0$.

Once this claim is proved, since $\{\hat{f}_\varepsilon^{-1}\}$ is non-decreasing as $\varepsilon \to 0$, the desired estimate (B.2) follows.

Proof of the claim. Indeed, suppose by contradiction that this does not hold, i.e. $\hat{f}_*(\infty) > \sqrt{1-t_0}$. Then we can select $r_0 \in (0,\infty)$ so that $\hat{f}_*(r_0) = \sqrt{1-t_0}$, $\hat{f}_* \in [\sqrt{1-t_0}, 1]$ and so $W'(1-\hat{f}_*^2) = 0$ in $[r_0,\infty)$. It follows that $\hat{f}_*'' + \frac{N-1}{r}\hat{f}_*' - \frac{N-1}{r^2}\hat{f}_* = 0$ in $[r_0,\infty)$ and so

$$\hat{f}_*(r) = c_1 r + c_2 r^{1-N} \quad \text{in } [r_0,\infty) \text{ for some constants } c_1, c_2.$$

Since $\hat{f}_*$ is bounded, we must have $c_1 = 0$, which implies that $\hat{f}_*(\infty) = 0$, which gives a contradiction. The claim is proved. ∎

C. A sharp Poincaré inequality for solenoidal vector fields

Lemma C.1. *Suppose $N \geq 3$ and let $\not{D}$ and $d\sigma$ denote the covariant derivative and the volume form on the standard sphere $\mathbb{S}^{N-1}$. For every smooth divergence-free vector field v on $\mathbb{S}^{N-1}$, i.e. $\not{D} \cdot v = 0$ on $\mathbb{S}^{N-1}$, one has*

$$\int_{\mathbb{S}^{N-1}} |\not{D} v|^2\, d\sigma = (N-2)\int_{\mathbb{S}^{N-1}} |v|^2\, d\sigma + 2\int_{\mathbb{S}^{N-1}} |\operatorname{Sym}(\not{D} v)|^2\, d\sigma.$$

In particular,

$$\int_{\mathbb{S}^{N-1}} |\not{D} v|^2\, d\sigma \geq (N-2)\int_{\mathbb{S}^{N-1}} |v|^2\, d\sigma,$$

and equality holds if and only if v is a Killing field, i.e. $\operatorname{Sym}(\not{D} v) = 0$.

Proof. In the following, we raise and lower indices using the standard metric g on the round sphere, i.e. $\not{D}^i = g^{ij}\not{D}_j$, $v_i = g_{ij}v^j$, etc. Also, repeated upper-lower indices are summed from 1 to $N-1$. As the commutator $[\not{D}^j, \not{D}_i]v_j = \operatorname{Ric}_{ki} v^k$, integration by parts yields

$$\begin{aligned}
\int_{\mathbb{S}^{N-1}} \not{D}_i v_j \not{D}^j v^i\, d\sigma &= -\int_{\mathbb{S}^{N-1}} \not{D}^j \not{D}_i v_j v^i\, d\sigma \\
&= -\int_{\mathbb{S}^{N-1}} (\not{D}_i \underbrace{\not{D}^j v_j}_{=0} + \underbrace{\operatorname{Ric}_{ki}}_{=(N-2)g_{ki}} v^k) v^i\, d\sigma \\
&= -(N-2)\int_{\mathbb{S}^{N-1}} |v|^2\, d\sigma.
\end{aligned}$$

It follows that

$$\begin{aligned}
4\int_{\mathbb{S}^{N-1}} |\operatorname{Sym}(\not{D} v)|^2\, d\sigma &= \int_{\mathbb{S}^{N-1}} |\not{D}_i v_j + \not{D}_j v_i|^2\, d\sigma \\
&= 2\int_{\mathbb{S}^{N-1}} \left[|\not{D} v|^2 - (N-2)|v|^2\right] d\sigma,
\end{aligned}$$

which clearly gives the assertion. ∎

D. Miscellaneous

Proposition D.1. *Suppose $N \geq 2$, $M \geq 1$, and $2 \leq p < \infty$ if $N = 2$ and $2 \leq p \leq \frac{2N}{N-2}$ if $N \geq 3$. Let Ω be a bounded smooth open subset of $\mathbb{R}^N$ and $h \in C^0(\Omega \times \mathbb{R}^M)$ satisfies*

$$\lim_{\substack{|y|\to 0\\ y\neq 0}} \sup_{x\in\Omega} \frac{|h(x,y)|}{|y|^2} = 0 \tag{D.1}$$

and, for some $C > 0$,

$$h(x,y) \geq -C|y|^2(|y|^{p-2}+1) \quad \text{for all } x \in \Omega,\ y \in \mathbb{R}^M. \tag{D.2}$$

Then

$$\liminf_{\substack{\|v\|_{H^1(\Omega,\mathbb{R}^M)}\to 0\\ v\neq 0, v\in H_0^1(\Omega,\mathbb{R}^M)}} \frac{\int_\Omega h(x,v(x))\,dx}{\|v\|^2_{H^1(\Omega,\mathbb{R}^M)}} \geq 0.$$

Note that by the Sobolev embedding theorem and the lower bound of h, the integral $\int_\Omega h(x,v(x))\,dx \in \mathbb{R} \cup \{+\infty\}$ makes sense for $v \in H_0^1(\Omega,\mathbb{R}^M)$.

Proof of Proposition D.1. Suppose by contradiction that the conclusion fails. Then there exist $t_j \to 0^+$ and $v_j \in H_0^1(\Omega,\mathbb{R}^M)$ with $\|v_j\|_{H^1} = 1$ such that, for some $\varepsilon > 0$ independent of j,

$$\int_\Omega \frac{1}{t_j^2} h(x,t_j v_j(x))\,dx \leq -\varepsilon < 0. \tag{D.3}$$

Without loss of generality, we may assume that v_j converges weakly in H^1 and a.e. in Ω to some $v \in H_0^1(\Omega,\mathbb{R}^M)$. Fix some small $\delta > 0$. By Egorov's theorem, we can select a measurable set $A \subset \Omega$ such that v_j converges uniformly to v in A and $|\Omega \setminus A| \leq \delta/2$. Also, since $v \in L^2(\Omega)$, then for large $K = K(\delta) \geq 1$, we can select a measurable set $B \subset A$ such that $|v| \leq K$ in B and $|A \setminus B| \leq \delta/2$. Thus, $|v_j| \leq 2K$ in B for all large j. By (D.1),

$$\lim_{j\to\infty} \int_B \frac{1}{t_j^2} |h(x,t_j v_j(x))|\,dx = 0.$$

Let $q = \frac{2N}{N-2}$ if $N \geq 3$ and q be arbitrary in (p,∞) if $N = 2$. Using the bound $h(x,y) \geq -C|y|^2(|y|^{p-2}+1)$, Hölder's inequality, the Sobolev embedding theorem for $\|v_j\|_{H^1} = 1$ and the fact that $|\Omega \setminus B| \leq \delta$, we have for some constant $C' > 0$ (independent of δ) that

$$\begin{aligned}\int_{\Omega\setminus B} \frac{1}{t_j^2} h(x,t_j v_j(x))\,dx &\geq -C \int_{\Omega\setminus B} (t_j^{p-2}|v_j|^p + |v_j|^2)\,dx\\ &\geq -C'(t_j^{p-2}\delta^{1-\frac{p}{q}} + \delta^{1-\frac{2}{q}}).\end{aligned}$$

Putting together the last two estimates, we get

$$\liminf_{j\to\infty} \int_\Omega \frac{1}{t_j^2} h(x,t_j v_j(x))\,dx \geq -C' \limsup_{j\to\infty} (t_j^{p-2}\delta^{1-\frac{p}{q}} + \delta^{1-\frac{2}{q}}).$$

Clearly, when δ is sufficiently small, this gives a contradiction to (D.3). ∎

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