Uniform asymptotic stability for convection–reaction–diffusion equations in the inviscid limit towards Riemann shocks

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Abstract. The present contribution proves the asymptotic orbital stability of viscous regularizations of stable Riemann shocks of scalar balance laws, uniformly with respect to the viscosity/diffusion parameter ε. The uniformity is understood in the sense that all constants involved in the stability statements are uniform and that the corresponding multiscale ε-dependent topology reduces to the classical $W^{1,\infty}$-topology when restricted to functions supported away from the shock location. Main difficulties include that uniformity precludes any use of parabolic regularization to close regularity estimates, that the global-in-time analysis is also spatially multiscale due to the coexistence of nontrivial slow parts with fast shock-layer parts, that the limiting smooth spectral problem (in fast variables) has no spectral gap and that uniformity requires a very precise and unusual design of the phase shift encoding *orbital* stability. In particular, our analysis builds a phase that somehow interpolates between the hyperbolic shock location prescribed by the Rankine–Hugoniot conditions and the nonuniform shift arising merely from phasing out the nondecaying 0-mode, as in the classical stability analysis for fronts of reaction–diffusion equations.

1. Introduction

In the present contribution, we prove for the very first time an asymptotic stability result, uniform with respect to the viscosity parameter, for a viscous regularization of a discontinuous traveling wave of a hyperbolic equation. As detailed below, by this we mean that the discontinuous wave of interest is asymptotically stable for the inviscid equation and that our result provides, for parabolic approximations, uniformity in both decay rate and size of the basin of attraction. In turn, if one replaces asymptotic stability with bounded stability or simply drops the requirement that decay rates should be uniform, then the topic is covered by an extensive body of literature in the L^1-setting and has been completely renewed in the L^2-setting by an impressive series of work. Concerning the latter we point out [29, 35] as possible entering gates among the most closely related to the present contribution.

1.1. The original hyperbolic result

The purely inviscid result [11], which we extend to slightly viscous regimes, is itself quite recent. More generally, despite the fact that hyperbolic models are largely used for practical purposes and that for such models singularities such as shocks and characteristic points are ubiquitous, the analysis of nonlinear asymptotic stability of singular traveling waves of hyperbolic systems is still in its infancy. The state of the art is essentially reduced to a full classification of waves of scalar equations in any dimension [11, 12] (obtained using some significant insights about characteristic points from [25]) and the case study of a discontinuous wave without characteristic point for a system of two equations in dimension 1 [55, 58].

Let us stress that, in the foregoing, stability is understood in the sense of Lyapunov, that is, globally in time, and for a topology encoding piecewise smoothness. This is consistent with the fact that concerning stability in the sense of Hadamard, that is, short-time well-posedness, for piecewise-smooth topologies, a quite comprehensive (but not complete) theory is already available even for multidimensional systems; see [5, 38, 39, 43]. At this level of regularity, being a weak solution is characterized by a free-interface initial boundary value problem, composed of equations taken in the classical sense in zones of smoothness, and the Rankine–Hugoniot transmission conditions along the free interfaces of discontinuity.

As is well known, for hyperbolic equations, weak solutions are not unique and one needs to make an extra choice. The one we are interested in is the most classical one when the extra condition is to be obtained as a vanishing viscosity limit. For scalar equations, in any dimension, since the pioneering work of Kružkov [36] (see also [7, Chapters 4 and 6]), this is known to be sufficient to ensure uniqueness and to be characterized by the so-called entropy conditions, which at our level of smoothness are reduced to inequalities at the free interfaces of discontinuity. For systems, even in dimension 1, despite decisive breakthroughs achieved in [6], such questions are still the object of intensive research; see for instance [30]. The present contribution lies at the crossroads of these questions related to the basic definitions of the notion of solution for hyperbolic equations and the ongoing development of a robust general theory for the stability of traveling waves, for which we refer the reader to [20, 26, 32, 50, 51, 59]. From the former point of view, the present contribution may be thought of as a global-in-time scalar version of [16, 17, 49]. From the latter point of view, though of a very different technical nature, by many respects, it shares similar goals with other vanishing viscosity stability programs – see for instance [2, 21] – and the present contribution is thought of as being to [11] what [4] is to [3].

We focus on the most basic shock stability result of [11]. Consider a scalar balance law in dimension 1,

$$\partial_t u + \partial_x (f(u)) = g(u) \tag{1.1}$$

with traveling wave solutions $\mathbf{R} \times \mathbf{R} \to \mathbf{R}$, $(t, x) \mapsto \underline{u}(x - (\psi_0 + \sigma_0 t))$ with initial shock position $\psi_0 \in \mathbf{R}$, speed $\sigma_0 \in \mathbf{R}$ and wave profile $\underline{u}$ of Riemann shock type, that is,

$$\underline{u}(x) = \begin{cases} \underline{u}_{-\infty} & \text{if } x < 0, \\ \underline{u}_{+\infty} & \text{if } x > 0, \end{cases}$$

where $(\underline{u}_{-\infty}, \underline{u}_{+\infty}) \in \mathbf{R}^2$, $\underline{u}_{+\infty} \neq \underline{u}_{-\infty}$. The fact that this does define a weak solution is equivalent to

$$g(\underline{u}_{+\infty}) = 0, \quad g(\underline{u}_{-\infty}) = 0, \quad f(\underline{u}_{+\infty}) - f(\underline{u}_{-\infty}) = \sigma_0(\underline{u}_{+\infty} - \underline{u}_{-\infty}),$$

whereas a strict version of entropy conditions may be enforced in Oleinik's form

$$\begin{cases} \sigma_0 > f'(\underline{u}_{+\infty}), \\ \dfrac{f(\tau \underline{u}_{-\infty} + (1-\tau)\underline{u}_{+\infty}) - f(\underline{u}_{-\infty})}{\tau \underline{u}_{-\infty} + (1-\tau)\underline{u}_{+\infty} - \underline{u}_{-\infty}} \\ \qquad > \dfrac{f(\tau \underline{u}_{-\infty} + (1-\tau)\underline{u}_{+\infty}) - f(\underline{u}_{+\infty})}{\tau \underline{u}_{-\infty} + (1-\tau)\underline{u}_{+\infty} - \underline{u}_{+\infty}} \quad \forall \tau \in (0,1), \\ f'(\underline{u}_{-\infty}) > \sigma_0. \end{cases}$$

Requiring a strict version of entropy conditions ensures that they still hold for nearby functions and in particular they disappear at the linearized level. In the foregoing, and throughout the text, for the sake of simplicity, we assume that $f, g \in \mathcal{C}^\infty(\mathbf{R})$ though each result only requires a small amount of regularity.

The following statement is one of the alternative versions of [11, Theorem 2.2] described in [11, Remark 2.3].

Theorem 1 ([11]). *Let $(\sigma_0, \underline{u}_{-\infty}, \underline{u}_{+\infty}) \in \mathbf{R}^3$ define a strictly entropic Riemann shock of* (1.1) *in the above sense. Assume that it is spectrally stable in the sense that*

$$g'(\underline{u}_{+\infty}) < 0 \quad and \quad g'(\underline{u}_{-\infty}) < 0.$$

There exist $\delta > 0$ and $C > 0$ such that for any $\psi_0 \in \mathbf{R}$ and $v_0 \in \mathrm{BUC}^1(\mathbf{R}^*)$ satisfying

$$\|v_0\|_{W^{1,\infty}(\mathbf{R}^*)} \leq \delta,$$

there exists $\psi \in \mathcal{C}^2(\mathbf{R}^+)$ with initial data $\psi(0) = \psi_0$ such that the entropy solution to (1.1), *u, generated by the initial data $u(0, \cdot) = (\underline{u} + v_0)(\cdot + \psi_0)$, belongs to $\mathrm{BUC}^1(\mathbf{R}_+ \times \mathbf{R} \setminus \{(t, \psi(t)); t \geq 0\})$ and satisfies for any $t \geq 0$,*

$$\|u(t, \cdot - \psi(t)) - \underline{u}\|_{W^{1,\infty}(\mathbf{R}^*)} + |\psi'(t) - \sigma_0| \leq \|v_0\|_{W^{1,\infty}(\mathbf{R}^*)} C \mathrm{e}^{\max(\{g'(\underline{u}_{+\infty}), g'(\underline{u}_{-\infty})\})t},$$

and moreover there exists ψ_∞ such that

$$|\psi_\infty - \psi_0| \leq \|v_0\|_{L^\infty(\mathbf{R}^*)} C,$$

and for any $t \geq 0$,

$$|\psi(t) - \psi_\infty - t\sigma_0| \leq \|v_0\|_{L^\infty(\mathbf{R}^*)} C \mathrm{e}^{\max(\{g'(\underline{u}_{+\infty}), g'(\underline{u}_{-\infty})\})t}.$$

In the foregoing, we have used notation $\mathrm{BUC}^k(\Omega)$ to denote the set of $\mathscr{C}^k$ functions over Ω whose derivatives up to order k are bounded, and uniformly continuous on every connected component of Ω. In other words, $\mathrm{BUC}^k(\Omega)$ is the closure of $W^{\infty,\infty}(\Omega)$ for the $W^{k,\infty}(\Omega)$ topology. Working with BUC^k instead of $W^{k,\infty}$ allows us to use approximation by smooth functions, an argument ubiquitous in local well-posedness theories, without imposing vanishing at ∞.

Note that, expressed in classical stability terminology, the previous theorem provides asymptotic orbital stability with asymptotic phase. We stress however that the role of phase shifts is here deeper than in the classical stability analysis of smooth waves, since it is not only required to provide decay of suitable norms in large time but also to ensure that these norms are finite locally in time. In particular, here there is no freedom, even in finite time, in the definition of phase shifts that need to synchronize discontinuities to allow for comparisons in piecewise smooth topologies.

It is also instructive to consider the corresponding spectral problem. In a moving frame, linearizing from $u(t, x) = \underline{u}(x - (\psi_0 + \sigma_0 t) - \psi(t)) + v(t, x - (\psi_0 + \sigma_0 t) - \psi(t))$ gives a linear IBVP in (v, ψ):

$$
\begin{aligned}
&\left(\partial_t + (f'(\underline{u}_{+\infty}) - \sigma_0)\partial_x - g'(\underline{u}_{+\infty})\right)v(t, \cdot) = 0 \quad \text{on } \mathbb{R}_+^*, \\
&\left(\partial_t + (f'(\underline{u}_{-\infty}) - \sigma_0)\partial_x - g'(\underline{u}_{-\infty})\right)v(t, \cdot) = 0 \quad \text{on } \mathbb{R}_-^*, \\
&\psi'(t) - \left(\frac{f'(\underline{u}_{+\infty}) - \sigma_0}{\underline{u}_{+\infty} - \underline{u}_{-\infty}}v(t, 0^+) - \frac{f'(\underline{u}_{-\infty}) - \sigma_0}{\underline{u}_{+\infty} - \underline{u}_{-\infty}}v(t, 0^-)\right) = 0.
\end{aligned}
$$

The corresponding spectrum on $\mathrm{BUC}^1(\mathbf{R}^*) \times \mathbf{R}$ is

$$
\left\{\lambda; \, \Re(\lambda) \leq \max\left(\{g'(\underline{u}_{-\infty}); g'(\underline{u}_{+\infty})\}\right)\right\} \cup \{0\},
$$

and when $\max(\{g'(\underline{u}_{-\infty}); g'(\underline{u}_{+\infty})\}) < 0$, 0 has multiplicity 1 (in the sense provided by resolvent singularities) with eigenvector $(0, 1)$. This shows that Theorem 1 sharply reproduces linear behavior.

1.2. The vanishing viscosity problem

Since even the local-in-time notion of solution involves vanishing viscosity approximations, it is natural to wonder whether Theorem 1 may have a small-viscosity extension, or whether the local-in-time vanishing viscosity limits may be globalized in time about the stable Riemann shocks of Theorem 1. We answer such a question for the following parabolic approximation:

$$
\partial_t u + \partial_x(f(u)) = \varepsilon \partial_x^2 u + g(u). \tag{1.2}
$$

Note that solutions to (1.2) are smooth (not uniformly in ε) so that techniques based on free-interface IBVP formulations for (1.1) cannot easily be extended to the study of (1.2). In the reverse direction, to gain better control on smoothness of solutions to (1.2), it is

expedient to introduce fast variables

$$u(t, x) = \tilde{u} \underbrace{\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon} \right)}_{(\tilde{t}, \tilde{x})}$$

that turn (1.2) into

$$\partial_{\tilde{t}} \tilde{u} + \partial_{\tilde{x}} (f(\tilde{u})) = \partial_{\tilde{x}}^2 \tilde{u} + \varepsilon g(\tilde{u}). \tag{1.3}$$

We stress however that this is indeed in original variables (t, x) that we aim to prove a uniform result. In particular, a large part of the analysis is focused on distinctions between norms that get large and norms that get small when going from slow to fast variables. For a closely related discussion we refer the reader to [28, 30].

In order to carry out the extension, the first step is to elucidate the existence of traveling waves to (1.2) near $\underline{u}$. A preliminary observation in this direction is that the formal $\varepsilon \to 0$ limit of (1.3) does possess a smooth traveling-wave solution $(\tilde{t}, \tilde{x}) \mapsto \underline{U}_0(\tilde{x} - \sigma_0 \tilde{t})$ of speed σ_0 and profile $\underline{U}_0$ such that

$$\lim_{-\infty} \underline{U}_0 = \underline{u}_{-\infty}, \quad \lim_{+\infty} \underline{U}_0 = \underline{u}_{+\infty},$$

simply obtained by solving

$$\underline{U}_0(0) = \frac{\underline{u}_{-\infty} + \underline{u}_{+\infty}}{2}, \quad \underline{U}_0' = f(\underline{U}_0) - f(\underline{u}_{+\infty}) - \sigma_0 (\underline{U}_0 - \underline{u}_{+\infty}).$$

We recall that σ_0 is tuned to ensure $f(\underline{u}_{-\infty}) - f(\underline{u}_{+\infty}) - \sigma_0 (\underline{u}_{-\infty} - \underline{u}_{+\infty}) = 0$ and observe that the Oleinik's entropy conditions imply that $\underline{U}_0$ is strictly monotonic. This $\varepsilon = 0$ viscous profile is often called a viscous shock layer and plays the role of a short-time free-interface boundary layer. This simple limiting fast profile may be perturbed to yield profiles for (1.3), hence for (1.2). To state such a perturbation result with optimal spatial decay rates, we introduce, for $\varepsilon \geq 0$,

$$\theta_\varepsilon^r := \frac{1}{2} |f'(\underline{u}_{+\infty}) - \sigma_\varepsilon| + \frac{1}{2} \sqrt{(f'(\underline{u}_{+\infty}) - \sigma_\varepsilon)^2 + 4\varepsilon |g'(\underline{u}_{+\infty})|},$$

$$\theta_\varepsilon^\ell := \frac{1}{2} (f'(\underline{u}_{-\infty}) - \sigma_\varepsilon) + \frac{1}{2} \sqrt{(f'(\underline{u}_{-\infty}) - \sigma_\varepsilon)^2 + 4\varepsilon |g'(\underline{u}_{-\infty})|}.$$

Proposition 1. *Under the assumptions of Theorem* 1, *for any* $0 < \alpha^\ell < \theta_0^\ell$, $0 < \alpha^r < \theta_0^r$ *and* $k_0 \in \mathbf{N}^*$, *there exist* $\varepsilon_0 > 0$ *and* $C_0 > 0$ *such that there exist a unique* $(\sigma_\varepsilon, \underline{U}_\varepsilon)$, *with* $\underline{U}_\varepsilon \in \mathscr{C}^2(\mathbf{R})$,

$$\underline{U}_\varepsilon(0) = \frac{\underline{u}_{-\infty} + \underline{u}_{+\infty}}{2}, \quad (f(\underline{U}_\varepsilon) - \sigma_\varepsilon \underline{U}_\varepsilon)' = \underline{U}_\varepsilon'' + \varepsilon g(\underline{U}_\varepsilon)$$

and

$$|\sigma_\varepsilon - \sigma_0| + \|e^{\alpha^\ell |\cdot|}(\underline{U}_\varepsilon - \underline{U}_0)\|_{W^{1,\infty}(\mathbf{R}_-)} + \|e^{\alpha^r \cdot}(\underline{U}_\varepsilon - \underline{U}_0)\|_{W^{1,\infty}(\mathbf{R}_+)} \leq C_0 \varepsilon,$$

and, moreover, there also hold

$$\|e^{\theta_\varepsilon^\ell |\cdot|}(\underline{U}_\varepsilon - \underline{u}_{-\infty}) - e^{\theta_0^\ell |\cdot|}(\underline{U}_0 - \underline{u}_{-\infty})\|_{L^\infty(\mathbf{R}_-)} \le C_0\varepsilon,$$

$$\|e^{\theta_\varepsilon^r \cdot}(\underline{U}_\varepsilon - \underline{u}_{+\infty}) - e^{\theta_0^r \cdot}(\underline{U}_0 - \underline{u}_{+\infty})\|_{L^\infty(\mathbf{R}_+)} \le C_0\varepsilon,$$

$$\|e^{\theta_\varepsilon^\ell |\cdot|}\underline{U}_\varepsilon^{(k)} - e^{\theta_0^\ell |\cdot|}\underline{U}_0^{(k)}\|_{L^\infty(\mathbf{R}_-)} \le C_0\varepsilon, \quad 1 \le k \le k_0,$$

$$\|e^{\theta_\varepsilon^r \cdot}\underline{U}_\varepsilon^{(k)} - e^{\theta_0^r \cdot}\underline{U}_0^{(k)}\|_{L^\infty(\mathbf{R}_+)} \le C_0\varepsilon, \quad 1 \le k \le k_0.$$

Note that a traveling wave $(t, x) \mapsto \underline{u}_\varepsilon(x - (\psi_0 + \sigma_\varepsilon t))$ with $\psi_0 \in \mathbf{R}$ arbitrary, is obtained from $\underline{U}_\varepsilon$ through

$$\underline{u}_\varepsilon(x) := \underline{U}_\varepsilon\left(\frac{x}{\varepsilon}\right)$$

and that, uniformly in ε,

$$|\underline{u}_\varepsilon(x) - \underline{u}(x)| \lesssim e^{-\theta_\varepsilon^\ell \frac{|x|}{\varepsilon}}, \qquad x < 0,$$

$$|\underline{u}_\varepsilon(x) - \underline{u}(x)| \lesssim e^{-\theta_\varepsilon^r \frac{x}{\varepsilon}}, \qquad x > 0,$$

$$|\underline{u}_\varepsilon^{(k)}(x)| \lesssim \frac{1}{\varepsilon^k}e^{-\theta_\varepsilon^\ell \frac{|x|}{\varepsilon}}, \quad x < 0, \ k \ge 1,$$

$$|\underline{u}_\varepsilon^{(k)}(x)| \lesssim \frac{1}{\varepsilon^k}e^{-\theta_\varepsilon^r \frac{x}{\varepsilon}}, \quad x > 0, \ k \ge 1.$$

We prove Proposition 1 in Appendix A. The existence and uniqueness part with suboptimal spatial rates follows from a rather standard Lyapunov–Schmidt argument. We stress however that it is crucial for our linear and nonlinear stability analyses to gain control on $\underline{U}_\varepsilon'$ with sharp spatial decay rates. We obtain the claimed upgrade from suboptimal to optimal rates essentially as a corollary to the refined spectral analysis needed to carry out the nonlinear study. We point out that, despite the fact that the literature on the subject is quite extensive – see for instance [8,9,13,18,19] and references therein – we have not found there an existence result with the level of generality needed here, that is, including nonconvex fluxes and yielding optimal spatial decay rates.

With the existence of ε-versions of traveling waves in hand, the next natural question is whether these are spectrally stable. It is settled by standard arguments, as expounded in [32], combining direct computations of the essential spectrum with Sturm–Liouville theory. The latter uses, crucially, that $\underline{U}_\varepsilon'$ is monotonic, a consequence of the Oleinik entropy conditions. The upshot is that, in slow original variables, the spectrum of the linearization about $\underline{u}_\varepsilon$ in a co-moving frame, acting on $\mathrm{BUC}^1(\mathbf{R})$, is stable and exhibits a spectral gap between the simple eigenvalue 0 and the rest of the spectrum of size $\min(\{|g'(\underline{u}_{+\infty})|, |g'(\underline{u}_{-\infty})|\}) + \mathcal{O}(\varepsilon)$. Note that in fast variables the spectral gap is of size $\varepsilon \times \min(\{|g'(\underline{u}_{+\infty})|, |g'(\underline{u}_{-\infty})|\}) + \mathcal{O}(\varepsilon^2)$. Details of the latter are given in Section 2.

The real challenge is *uniform* nonlinear asymptotic stability. Indeed, if one removes the uniformity requirement, nonlinear stability follows from spectral stability by now well-known classical arguments as expounded in [20,32,50,51], and initially developed in, among others, [20,31,52,53,56,57]. Since the limit is singular, it is worth spelling out what we mean by uniform stability. There are two closely related parts in the requirement.

Explicitly, on initial data,

(1) the most obvious one is that the restriction on the sizes of allowed initial perturbations (encoded by the smallness of δ in Theorem 1) should be uniform with respect to ε, so that the lower bound on the size of the basin of attraction provided by the analysis is nontrivial in the limit $\varepsilon \to 0$;

(2) the second one is more intricate,[1] stating that the ε-dependent norms, say $\|\cdot\|_{(\varepsilon)}$, used to measure this smallness (in slow original variables) should be controlled by an ε-independent norm for functions supported away from the shock, so that in particular, for any $v \in \mathcal{C}_c^\infty(\mathbf{R})$ supported in $\mathbf{R}^*$, $\limsup_{\varepsilon \to 0} \|v\|_{(\varepsilon)} < +\infty$.

On the control of solutions arising from perturbations, we impose similar constraints but with upper bounds replacing lower bounds in the requirements. Constraints on the control of solutions ensure that the bounds provide a nontrivial control whereas constraints on the control of initial data ensure that nontrivial perturbations are allowed.

It may be intuitive that the stronger the norm is, the larger the size of the basin of attraction is, since a qualitatively better control is offered by the topology. In the present case, the discussion is on the amount of localization encoded by the norm since, though this is somewhat hidden, time decay is controlled by initial spatial localization (as opposed to cases where regularity drives decay, as for instance in [2–4]). To offer a quantitative insight, let us use as in [21] a simple ODE as a toy model to predict the size constraints. Consider the stability of $y \equiv 0$ for $y' = -\tau y + \rho y^2$, where $\tau > 0$ encodes the size of the spectral gap and $\rho > 0$ measures the size of nonlinear forcing. For such an equation, a ball of radius r_0 and center 0 is uniformly attracted to 0 provided that $r_0 < \tau/\rho$. Now, if one considers (1.2) directly in $\mathrm{BUC}^1(\mathbf{R})$ (or any reasonable unweighted topology) and forgets about issues related to phase definitions and possible regularity losses, the spectral gap offered by a linearization about $\underline{u}_\varepsilon$ is of order 1, whereas the forcing by nonlinear terms is of order ε^{-1} (since this is the size of $\underline{u}_\varepsilon'$), hence the rough prediction of a basin of size $\mathcal{O}(\varepsilon)$. Yet, working with weights such as $\mathrm{e}^{-\theta \varepsilon^{-\alpha}|x|}$, for some sufficiently small $\theta > 0$ and some $0 < \alpha \leq 1$, moves the spectrum to increase the size of the gap to the order $\varepsilon^{-\alpha}$, yielding the expectation of an $\mathcal{O}(\varepsilon^{1-\alpha})$ basin. Note that the choice $\alpha = 1$ would provide a uniform size and is consistent with the size of viscous shock layers, but it would force initial perturbations to be located in an $\mathcal{O}(\varepsilon)$ spatial neighborhood of the shock location.

The foregoing simple discussion predicts quite accurately[2] what could be obtained by applying the most classical parabolic strategy to the problem at hand. The failure of the classical strategy may also be read on the deeply related, but not equivalent, fact that it uses the phase only to pull out the contribution of nonlinear terms through the spectrally nondecaying 0-mode. This is inconsistent with the stronger role of the phase for the hyperbolic problem, all the spectrum contributing to the phase in the latter case. A completely different approach is needed.

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[1]But our result satisfies a much simpler and stronger version of the requirement.
[2]Actually it is even a bit optimistic for the unweighted and $\alpha < 1$ cases.

Additional strong signs of the very challenging nature of the uniform stability problem may also be gathered from examination of the viscous layer stability problem, that is, the stability of $(\tilde{t}, \tilde{x}) \mapsto \underline{U}_0(\tilde{x} - \sigma_0\tilde{t})$ as a solution to (1.3) with $\varepsilon = 0$. The problem has been extensively studied; see for instance [14, 15, 22, 23, 27, 34, 37] for a few key contributions and [59] for a thorough account. The spectrum of the linearization includes essential spectrum touching the imaginary axis at 0, which is still an eigenvalue, so that the decay is not exponential but algebraic and requires a trade-off, localization against decay, as for the heat equation. This is a consequence of the conservative nature of the equation, but the conservative structure may also be used to tame some of the apparent difficulties. To give one concrete example, one may remove the embedded eigenvalue 0 from the essential spectrum by using the classical antiderivative trick, dating back at least to [42], either directly at the nonlinear level under the restriction of zero-mean perturbations (as in [15] or in [42] for a system case) or only to facilitate the linear analysis as in [22, 23]. In fast variables, turning on $\varepsilon > 0$ moves the essential to the left, creating an $\mathcal{O}(\varepsilon)$ spectral gap, but breaks the conservative structure, thus rendering almost impossible, and at least quite inconvenient, the use of classical conservative tools. Our stability analysis requires a description as detailed as those of [22, 23] and, without the antiderivative trick at hand, this involves the full machinery of [60, 61]. Roughly speaking, one of the main outcomes of our detailed spectral analysis, expressed in fast variables, is that the ε-proximity of essential spectrum and 0-eigenvalue induces that the essential spectrum has an impact of size $1/\varepsilon$ on the linear time evolution, but that at leading order the algebraic structure of the essential-spectrum contribution is such that it may be absorbed in a suitably designed phase modulation. Note that this is consistent with the fact that, in fast variables, variations in shock positions are expected to be of size $1/\varepsilon$ and with the fact that, in slow variables, the phase is involved in the resolution of all the hyperbolic spectral problems, not only the 0-mode.

To summarize and extend the discussion so far, we may hope

(1) to overcome the discrepancy between the Rankine–Hugoniot prescription of the phase and the pure 0-mode modulation, and to phase out the hidden singularity caused by the proximity of essential spectrum and 0 eigenvalue, by carefully identifying the most singular contribution of the essential spectrum as phase variations and including this in a carefully designed phase;

(2) to guarantee uniform nonlinear decay estimates provided we can ensure that, in slow variables, nonlinear terms of size $1/\varepsilon$ also come with a spectral-gap-enhancing factor $e^{-\theta|x|/\varepsilon}$ (for some $\theta > 0$).

The latter expectation is motivated by the fact that it is indeed the case for terms forced by $\underline{u}'_\varepsilon$ but we need to prove that it is so also for stiff terms caused by the derivatives of the perturbation itself. Concerning the latter, we stress that even if one starts with a very gentle perturbation supported away from the shock, the nonlinear coupling instantaneously creates stiff parts of shock-layer type in the perturbation, thus making it effectively multi-scale.

There remains a somewhat hidden issue, that we have not discussed so far. Throughout the foregoing discussion, we have acted as if we could use Duhamel principle based on a straightforward linearization, as in classical semilinear parabolic problems. Yet, here, closing nonlinear estimates in regularity by using parabolic regularization either explicitly through gains of derivatives or indirectly through $L^q \to L^p$, $q < p$, mapping properties, effectively induces losses in power of ε in an already ε-critical problem, thus is completely forbidden. Instead, we estimate

- the variation in shock position ψ, the shape variation v and the restriction of its derivative $\partial_x v$ to an $\mathcal{O}(\varepsilon)$ neighborhood of the shock location through the Duhamel formula and linear decay estimates;

- the remaining part of $\partial_x v$ by a suitably modified Goodman-type hyperbolic energy estimate.

The latter energy estimate is similar in spirit to those in [14, 48, 58], but the hard part of its design is precisely in going from a classical hyperbolic estimate that would work in the complement of an $\mathcal{O}(1)$ neighborhood of the shock location to a finely tuned estimate covering the complement of an $\mathcal{O}(\varepsilon)$ neighborhood, since this is required for the combination with a lossless parabolic regularization argument. Moreover, there are two more twists in the argument: on one hand we need the estimate to include weights encoding the multiscale nature of $\partial_x v$; on the other hand, for the sake of sharpness, to remain at the $\mathcal{C}^1$ level of regularity, we actually apply the energy estimates on a suitable nonlinear version of $\partial_x v$ so that they yield L^∞ bounds for $\partial_x v$.

The arguments sketched above, appropriately worked out, provide the main result of the present paper. To state such results, we introduce multiscale weights and corresponding norms: for $k \in \mathbf{N}$, $\varepsilon > 0$ and $\theta \geq 0$,

$$\omega_{k,\varepsilon,\theta}(x) := \frac{1}{1 + \frac{1}{\varepsilon^k}e^{-\theta\frac{|x|}{\varepsilon}}}, \quad \|v\|_{W^{k,\infty}_{\varepsilon,\theta}(\mathbf{R})} = \sum_{j=0}^{k} \|\omega_{j,\varepsilon,\theta}\partial_x^j v\|_{L^\infty(\mathbf{R})}.$$

Note that

(1) each norm $\|\cdot\|_{W^{k,\infty}_{\varepsilon,\theta}(\mathbf{R})}$ is equivalent to any standard norm on $\|\cdot\|_{W^{k,\infty}(\mathbf{R})}$ but non-uniformly in ε and that the uniformity is restored if one restricts it to functions supported in the complement of a fixed neighborhood of the origin;

(2) the norm $\|\cdot\|_{W^{0,\infty}_{\varepsilon,\theta}(\mathbf{R})}$ is uniformly equivalent to $\|\cdot\|_{L^\infty(\mathbf{R})}$;

(3) if $\theta < \min(\{\theta_0^\ell, \theta_0^r\})$ then $\|\underline{u}_\varepsilon - \underline{u}\|_{W^{k,\infty}_{\varepsilon,\theta}(\mathbf{R})}$ is bounded uniformly with respect to ε.

Theorem 2. *Enforce the assumptions and notation of Theorem 1 and Proposition 1.*

There exists $\theta_0 > 0$ such that for any $0 < \theta \leq \theta_0$, there exist $\varepsilon_0 > 0$, $\delta > 0$ and $C > 0$ such that for any $0 < \varepsilon \leq \varepsilon_0$, any $\psi_0 \in \mathbf{R}$ and any $v_0 \in \mathrm{BUC}^1(\mathbf{R})$ satisfying

$$\|v_0\|_{W^{1,\infty}_{\varepsilon,\theta}(\mathbf{R})} \leq \delta,$$

there exists $\psi \in \mathcal{C}^1(\mathbf{R}^+)$ with initial data $\psi(0) = \psi_0$ such that the strong[3] solution to (1.2), u, generated by the initial data $u(0, \cdot) = (\underline{u}_\varepsilon + v_0)(\cdot + \psi_0)$, is global in time and satisfies for any $t \geq 0$,

$$\|u(t, \cdot - \psi(t)) - \underline{u}_\varepsilon\|_{W^{1,\infty}_{\varepsilon,\theta}(\mathbf{R})} + |\psi'(t) - \sigma_\varepsilon|$$
$$\leq \|v_0\|_{W^{1,\infty}_{\varepsilon,\theta}(\mathbf{R})} C e^{\max(\{g'(\underline{u}_{+\infty}), g'(\underline{u}_{-\infty})\})t},$$

and moreover there exists ψ_∞ such that

$$|\psi_\infty - \psi_0| \leq \|v_0\|_{W^{1,\infty}_{\varepsilon,\theta}(\mathbf{R})} C,$$

and for any $t \geq 0$,

$$|\psi(t) - \psi_\infty - t\sigma_\varepsilon| \leq \|v_0\|_{W^{1,\infty}_{\varepsilon,\theta}(\mathbf{R})} C e^{\max(\{g'(\underline{u}_{+\infty}), g'(\underline{u}_{-\infty})\})t}.$$

Among the many variations and extensions of Theorem 1 provided in [11], the simplest one to extend to a uniform small viscosity result is [11, Proposition 2.5], which proves that the exponential time decay also holds for higher-order derivatives without further restriction on the sizes of perturbations. It does not require any new insight besides those used to prove Theorem 2 and we leave it aside only to cut unnecessary technicalities.

Likewise, one may obtain in an even more direct way, that is, up to immaterial changes, exponential damping of norms encoding further slow spatial localization. To give an explicit example, let us extend the notation $W^{k,\infty}_{\varepsilon,\theta}$, L^∞, into $W^{k,\infty}_{\varepsilon,\theta,\theta'}$, $L^\infty_{\theta'}$, according to weights

$$\omega_{k,\varepsilon,\theta,\theta'}(x) := \frac{1}{e^{-\theta'|x|} + \frac{1}{\varepsilon^k} e^{-\theta \frac{|x|}{\varepsilon}}},$$
$$\omega_{\theta'}(x) := e^{\theta'|x|},$$

with $\theta' \geq 0$ arbitrary. One may prove for instance that for any $\theta' \geq 0$ there exist $C_{\theta'}$ and $\varepsilon_{\theta'} > 0$ such that, under the sole further restrictions $0 < \varepsilon \leq \varepsilon_{\theta'}$ and $e^{\theta'|\cdot|} v_0 \in L^\infty(\mathbf{R})$, there holds

$$\|u(t, \cdot - \psi(t)) - \underline{u}_\varepsilon\|_{L^\infty_{\theta'}(\mathbf{R})} \leq \|v_0\|_{L^\infty_{\theta'}(\mathbf{R})} C_\theta e^{\max(\{g'(\underline{u}_{+\infty}), g'(\underline{u}_{-\infty})\})t}.$$

One point in considering these weighted topologies is that, when $\theta' > 0$, $L^\infty_{\theta'}$ is continuously embedded in $L^1 \cap L^\infty$, so that an estimate on $\|u(t, \cdot - \psi(t)) - \underline{u}\|_{L^p}$ is provided by the combination of the foregoing bound with the already known bound

$$\|\underline{u}_\varepsilon - \underline{u}\|_{L^p(\mathbf{R})} \lesssim \varepsilon^{\frac{1}{p}}.$$

---

[3]We ensure $u \in \mathrm{BUC}^0(\mathbf{R}_+; \mathrm{BUC}^1(\mathbf{R})) \cap \mathcal{C}^\infty(\mathbf{R}_+^*; \mathrm{BUC}^\infty(\mathbf{R}))$.

1.3. Outline and perspectives

The most natural nontrivial extensions of Theorems 1/2 that we have chosen to leave for future work concern on one hand the parabolic regularization by quasilinear terms and on the other hand planar Riemann shocks in higher spatial dimensions (see [11, Theorem 3.4] for the hyperbolic case). We expect many parts of the present analysis to be directly relevant in quasilinear or multidimensional cases, but we also believe that their treatments would require sufficiently many new arguments to deserve a separate treatment.

In the multidimensional case, even the outcome is expected to be significantly different. In this direction, let us point out that the hyperbolic spectral problem is critical in the stronger sense that the spectrum includes the whole imaginary axis, instead of having an intersection with the imaginary axis reduced to $\{0\}$. This may be tracked back to the fact that the linearized Rankine–Hugoniot equation takes the form of a transport equation in transverse variables for the phase. Consistently, as proved in [11, Theorem 3.4], for the hyperbolic problem, perturbing a planar shock may lead asymptotically in large time to another nonplanar Riemann shock sharing the same constant states. This may still be interpreted as a space-modulated asymptotic stability result, in the sense coined in [26] and thoroughly discussed in [12, 45–47]. A similar phenomenon is analyzed for scalar *conservation* laws in [54].

Concerning the quasilinear case, the main new difficulty is expected to arise from the fact that, to close the argument, one needs to prove that the L^∞ decay of $\varepsilon\partial_x^2 v$, where v still denotes the shape variation, is at least as good as that of $\partial_x v$. A priori, outside the shock layer this leaves the freedom to pick some initial typical size $\varepsilon^{-\eta_0}$, $\eta_0 \in [0, 1]$, for $\partial_x^2 v$ and to try to propagate it. Indeed, roughly speaking, in the complement of an $\mathcal{O}(\varepsilon)$ neighborhood of the shock location, this L^∞ propagation stems from arguments similar to the ones sketched above for $\partial_x v$. The key difference is that now one cannot complete it with a bound obtained through the Duhamel formula since this would involve an L^∞ bound on $\partial_x^3 v$. Thus the quasilinear study seems to require being able to close an estimate for $\partial_x^2 v$ entirely with energy-type arguments, a highly nontrivial task.

In another direction, we expect that the study of waves with characteristic points, as arising in the full classification obtained in [12] for scalar balance laws, should not only involve some new patches here and there but follow very different routes and thus will require significantly new insights even at a general abstract level. As a strong token of this expectation, we point out that regularity is expected to play a paramount role there since, at the hyperbolic level, the regularity class chosen deeply modifies the spectrum when a characteristic point is present in the wave profile; see [12, 25].

The rest of the paper is organized as follows. We have decided to shift the derivation of wave profile asymptotics, proving Proposition 1, to Appendix A, because we believe that the backbone of the paper is stability and we provide it mostly for completeness' sake. The next section contains a detailed examination of the required spectral preliminaries. The following one explains how to use these to obtain a practical representation of the linearized time evolution. Though we mostly follow there the arguments in [60], with

some twists here and there, we provide a detailed exposition for two distinct reasons. The first one is that we need to track in constructions which parts are ε-uniform and which parts are not, a crucial point in our analysis. The second one is that most of the papers of the field requiring a detailed analysis, as we do, are either extremely long [60] or cut into a few long pieces [40, 41] and we want to save the reader from back-and-forth consultations of the literature. This makes our analysis essentially self-contained (up to basic knowledge of spectral analysis) and we believe that it could serve as a gentle introduction to the latter massive literature. Note however, that, to keep the paper to a reasonable size, we only expound the bare minimum required by our analysis. After these two preliminary sections, we enter into the technical core of the paper, with first a section devoted to detailed linear estimates, including the identification of the most-singular parts of the time evolution as phase variations, and then a section devoted to nonlinear analysis, including adapted nonlinear maximum principles proved through energy estimates and the proof of Theorem 2.

2. Spectral analysis

We investigate stability for traveling waves introduced in Proposition 1. We have chosen to carry out all our proofs within co-moving fast variables. Explicitly, we introduce new unknowns and variables through[4]

$$u(t, x) = \tilde{u} \underbrace{\left(\frac{t}{\varepsilon}, \frac{x - \sigma_\varepsilon t}{\varepsilon} \right)}_{(\tilde{t}, \tilde{x})}.$$

However, since we never go back to the original slow variables, we drop tildes on fast quantities from now on. One reason to opt for the fast variables is that it provides a simpler reading of size dependencies on ε.

Therefore, our starting point is

$$\partial_t u + \partial_x (f(u) - \sigma_\varepsilon u) = \partial_x^2 u + \varepsilon g(u), \tag{2.1}$$

about the stationary solution $\underline{U}_\varepsilon$. Accordingly we consider the operator

$$\mathcal{L}_\varepsilon := -\partial_x \big((f'(\underline{U}_\varepsilon) - \sigma_\varepsilon) \cdot \big) + \partial_x^2 + \varepsilon g'(\underline{U}_\varepsilon)$$

on $\mathrm{BUC}^0(\mathbf{R})$ with domain $\mathrm{BUC}^2(\mathbf{R})$.

Though the elements we provide are sufficient to reconstruct the classical theory, the reader may benefit from consulting [32] for background on spectral analysis specialized to nonlinear wave stability. In particular, we shall make extensive implicit use of the characterizations of essential spectrum in terms of endstates of wave profiles and of the spectrum

---

[4]Note the slight co-moving inconsistency with the introduction.

at the right-hand side[5] of the essential spectrum[6] in terms of zeros of Evans functions. The reader is referred to [10, 33] for less specialized, basic background on spectral theory.

The backbone of the theory is the interpretation of spectral properties of one-dimensional differential operators in terms of spatial dynamics and a key part of the corresponding studies is the investigation of exponential dichotomies. It starts with the identification between the eigenvalue equation

$$(\lambda - \mathscr{L}_\varepsilon)v = 0$$

and the system of ODEs

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{V}(x) = \mathbf{A}_\varepsilon(\lambda, x)\mathbf{V}(x)$$

for the vector[7] $\mathbf{V} = (v, \partial_x v - (f'(\underline{U}_\varepsilon) - \sigma_\varepsilon)v)$, where

$$\mathbf{A}_\varepsilon(\lambda, x) := \begin{pmatrix} f'(\underline{U}_\varepsilon) - \sigma_\varepsilon & 1 \\ \lambda - \varepsilon g'(\underline{U}_\varepsilon) & 0 \end{pmatrix}.$$

For later use, we shall denote by $\mathbf{\Phi}_\varepsilon^\lambda(x, y)$ the corresponding solution operators, mapping a datum at point y to a value at point x.

The essential spectrum is characterized in terms of matrices $\mathbf{A}_\varepsilon^r(\lambda) := \mathbf{A}^\varepsilon(\lambda; \underline{u}_{+\infty})$ and $\mathbf{A}_\varepsilon^\ell(\lambda) := \mathbf{A}^\varepsilon(\lambda; \underline{u}_{-\infty})$ with

$$\mathbf{A}^\varepsilon(\lambda; u) := \begin{pmatrix} f'(u) - \sigma_\varepsilon & 1 \\ \lambda - \varepsilon g'(u) & 0 \end{pmatrix}.$$

The eigenvalues of $\mathbf{A}^\varepsilon(\lambda; u)$ are given by

$$\mu_\pm^\varepsilon(\lambda; u) := \frac{f'(u) - \sigma_\varepsilon}{2} \pm \sqrt{\frac{(f'(u) - \sigma_\varepsilon)^2}{4} + \lambda - \varepsilon g'(u)}$$

and are distinct when $\lambda \neq \varepsilon g'(u) - \frac{1}{4}(f'(u) - \sigma_\varepsilon)^2$. In this case, the matrix may be diagonalized as

$$\mathbf{A}^\varepsilon(\lambda; u) = \begin{pmatrix} \mathbf{R}_+^\varepsilon(\lambda; u) & \mathbf{R}_-^\varepsilon(\lambda; u) \end{pmatrix} \begin{pmatrix} \mu_+^\varepsilon(\lambda; u) & 0 \\ 0 & \mu_-^\varepsilon(\lambda; u) \end{pmatrix} \begin{pmatrix} \mathbf{L}_+^\varepsilon(\lambda; u) \\ \mathbf{L}_-^\varepsilon(\lambda; u) \end{pmatrix}$$

with

$$\mathbf{R}_\pm^\varepsilon(\lambda; u) := \begin{pmatrix} 1 \\ -\mu_\mp^\varepsilon(\lambda; u) \end{pmatrix}, \quad \mathbf{L}_\pm^\varepsilon(\lambda; u) := \frac{(\pm\mu_\pm^\varepsilon(\lambda; u) \quad \pm 1)}{\mu_+^\varepsilon(\lambda; u) - \mu_-^\varepsilon(\lambda; u)}.$$

---

[5]We picture the complex plane with the real axis pointing to the right and the imaginary axis pointing to the top.

[6]There are (at least) two reasonable definitions of essential spectrum, either through failure to satisfy the Fredholm property or through failure to satisfy the Fredholm property with zero index. In the context of semigroup generators, both definitions provide the same right-hand boundary, thus the conventional choice is immaterial to stability issues.

[7]The use of flux variables is not necessary but it simplifies a few computations here and there.

The eigenvalues $\mu_{\pm}^{\varepsilon}(\lambda;u)$ have distinct real parts when λ does not belong to

$$\mathcal{D}_{\varepsilon}(u) := \varepsilon g'(u) - \tfrac{1}{4}(f'(u) - \sigma_{\varepsilon})^2 + \mathbf{R}^-.$$

All our spectral studies will take place far from the half-lines $\mathcal{D}_{\varepsilon}(\underline{u}_{+\infty}) \cup \mathcal{D}_{\varepsilon}(\underline{u}_{-\infty})$, which correspond to the set termed *absolute spectrum* in [32].

From now on, throughout the text, we shall use $\sqrt{\cdot}$ to denote the determination of the square root on $\mathbf{C} \setminus \mathbf{R}^-$ with positive real part.

2.1. Conjugation to constant coefficients

Our starting point is a conjugation of spectral problems to a piecewise constant coefficient spectral problem. This is mostly relevant in compact zones of the spectral plane and in the literature by Kevin Zumbrun and his collaborators this is known as a *gap lemma* – since a gap or in other words an exponential dichotomy is the key assumption; see for instance [44, Lemma 2.6] for a version relevant for the present analysis. Since we need to ensure uniformity in ε, for the case at hand we provide both a statement and a proof.

Proposition 2. *Let K be a compact subset of $\mathbf{C} \setminus \mathcal{D}_0(\underline{u}_{+\infty})$. There exist positive constants $(\varepsilon_0, C, \theta)$ such that there exists[8] a smooth map*

$$P^r : [0, \varepsilon_0] \times K \times \mathbf{R} \mapsto \mathrm{GL}_2(\mathbf{C}), \quad (\varepsilon, \lambda, x) \mapsto P_{\varepsilon}^r(\lambda, x)$$

locally uniformly analytic in λ on a neighborhood of K and such that, for any $(\varepsilon, \lambda, x) \in [0, \varepsilon_0] \times K \times [0, +\infty)$,

$$\|P_{\varepsilon}^r(\lambda, x) - \mathrm{I}_2\| \le C\mathrm{e}^{-\theta|x|}, \quad \|(P_{\varepsilon}^r(\lambda, x))^{-1} - \mathrm{I}_2\| \le C\mathrm{e}^{-\theta|x|},$$

and, for any $(\varepsilon, \lambda, x, y) \in [0, \varepsilon_0] \times K \times (\mathbf{R}_+)^2$,

$$\Phi_{\varepsilon}^{\lambda}(x, y) = P_{\varepsilon}^r(\lambda, x)\mathrm{e}^{(x-y)\mathbf{A}_{\varepsilon}^r(\lambda)}(P_{\varepsilon}^r(\lambda, y))^{-1}.$$

The same argument applies to the conjugation on $(-\infty, 0]$ with the flow of $\mathbf{A}_{\varepsilon}^{\ell}(\lambda)$ and defines a conjugation map denoted P^{ℓ} from now on.

Proof of Proposition 2. The proof is essentially a quantitative "cheap" gap lemma – conjugating only one trajectory instead of solution operators – but applied in $\mathcal{M}_2(\mathbf{C})$ instead of $\mathbf{C}^2$.

Let us first observe that it is sufficient to define P^r on $[0, \varepsilon_0] \times K \times [x_0, +\infty)$ for some suitably large x_0. Indeed, then one may extend P^r by

$$P_{\varepsilon}^r(\lambda, x) := \Phi_{\varepsilon}^{\lambda}(x, x_0)P_{\varepsilon}^r(\lambda, x_0)\mathrm{e}^{(x_0-x)\mathbf{A}_{\varepsilon}^r(\lambda)},$$

---

[8] As follows from the proof, $P_{\varepsilon}^r(\lambda, \cdot)$ is defined as soon as $\lambda \notin \mathcal{D}_{\varepsilon}(\underline{u}_{+\infty}) \cup \mathcal{D}_{\varepsilon}(\underline{u}_{-\infty})$.

and bounds are extended by a continuity-compactness argument. Likewise, the uniformity in ε is simply derived from a continuity-compactness argument since the construction below is continuous at the limit $\varepsilon = 0$. Note moreover that in the large-x regime, the bound on $(P_\varepsilon^r(\lambda, x))^{-1}$ may be derived from the bound on $P_\varepsilon^r(\lambda, x)$ by using properties of the inverse map.

The requirements on $P_\varepsilon^r(\lambda, \cdot)$ are equivalent to the fact that it converges exponentially fast to I_2 at $+\infty$ (uniformly in ε) and that it satisfies for any x,

$$\frac{d}{dx} P_\varepsilon^r(\lambda, x) = \mathscr{A}_\varepsilon^r(\lambda)(P_\varepsilon^r(\lambda, x)) + (\mathbf{A}_\varepsilon(\lambda, x) - \mathbf{A}_\varepsilon^r(\lambda)) P_\varepsilon^r(\lambda, x),$$

where $\mathscr{A}_\varepsilon^r(\lambda) := \mathscr{A}^\varepsilon(\lambda; \underline{u}_{+\infty})$ is a linear operator on $\mathcal{M}_2(\mathbf{C})$ defined through

$$\mathscr{A}^\varepsilon(\lambda; u)(P) := [\mathbf{A}^\varepsilon(\lambda; u), P] = \mathbf{A}^\varepsilon(\lambda; u)P - P\mathbf{A}^\varepsilon(\lambda; u).$$

When $\lambda \notin \mathcal{D}_\varepsilon(u)$, $\mathscr{A}^\varepsilon(\lambda; u)$ admits

$$\big(\mathbf{R}_+^\varepsilon(\lambda; u)\mathbf{L}_-^\varepsilon(\lambda; u), \ \mathbf{R}_-^\varepsilon(\lambda; u)\mathbf{L}_+^\varepsilon(\lambda; u), \ \mathbf{R}_+^\varepsilon(\lambda; u)\mathbf{L}_+^\varepsilon(\lambda; u), \ \mathbf{R}_-^\varepsilon(\lambda; u)\mathbf{L}_-^\varepsilon(\lambda; u)\big)$$

as a basis of eigenvectors corresponding to eigenvalues

$$\big(\mu_+^\varepsilon(\lambda; u) - \mu_-^\varepsilon(\lambda; u), \ -(\mu_+^\varepsilon(\lambda; u) - \mu_-^\varepsilon(\lambda; u)), \ 0, \ 0\big).$$

Note that I_2 always lies in the kernel of $\mathscr{A}^\varepsilon(\lambda; u)$. We denote by $\Pi_u^\varepsilon(\lambda; u)$, $\Pi_s^\varepsilon(\lambda; u)$, $\Pi_0^\varepsilon(\lambda; u)$ the corresponding spectral projections respectively on the unstable space, the stable space and the kernel of $\mathscr{A}^\varepsilon(\lambda; u)$, and for further later study we point out that they are given as

$$\Pi_u^\varepsilon(\lambda; u)(P) = \mathbf{L}_+^\varepsilon(\lambda; u) P \mathbf{R}_-^\varepsilon(\lambda; u)\mathbf{R}_+^\varepsilon(\lambda; u)\mathbf{L}_-^\varepsilon(\lambda; u),$$

$$\Pi_s^\varepsilon(\lambda; u)(P) = \mathbf{L}_-^\varepsilon(\lambda; u) P \mathbf{R}_+^\varepsilon(\lambda; u)\mathbf{R}_-^\varepsilon(\lambda; u)\mathbf{L}_+^\varepsilon(\lambda; u),$$

$$\Pi_0^\varepsilon(\lambda; u)(P) = \mathbf{L}_+^\varepsilon(\lambda; u) P \mathbf{R}_+^\varepsilon(\lambda; u)\mathbf{R}_+^\varepsilon(\lambda; u)\mathbf{L}_+^\varepsilon(\lambda; u)$$
$$+ \mathbf{L}_-^\varepsilon(\lambda; u) P \mathbf{R}_-^\varepsilon(\lambda; u)\mathbf{R}_-^\varepsilon(\lambda; u)\mathbf{L}_-^\varepsilon(\lambda; u).$$

Then the result follows when θ is sufficiently small and x_0 is sufficiently large from a use of the implicit function theorem on

$$P_\varepsilon^r(\lambda, x) = I_2 - \int_x^{+\infty} \Pi_0^\varepsilon(\lambda; \underline{u}_{+\infty})\big((\mathbf{A}_\varepsilon(\lambda, y) - \mathbf{A}_\varepsilon^r(\lambda))P_\varepsilon^r(\lambda, y)\big) \, dy$$

$$- \int_x^{+\infty} e^{-(y-x)(\mu_+^\varepsilon(\lambda; \underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda; \underline{u}_{+\infty}))} \Pi_u^\varepsilon(\lambda; \underline{u}_{+\infty})\big((\mathbf{A}_\varepsilon(\lambda, y) - \mathbf{A}_\varepsilon^r(\lambda))P_\varepsilon^r(\lambda, y)\big) \, dy$$

$$+ \int_{x_0}^x e^{-(x-y)(\mu_+^\varepsilon(\lambda; \underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda; \underline{u}_{+\infty}))} \Pi_s^\varepsilon(\lambda; \underline{u}_{+\infty})\big((\mathbf{A}_\varepsilon(\lambda, y) - \mathbf{A}_\varepsilon^r(\lambda))P_\varepsilon^r(\lambda, y)\big) \, dy$$

with norm control on matrix-valued maps through

$$\sup_{x \geq x_0} e^{\theta|x|} \|P(x) - I_2\|. \qquad \blacksquare$$

Remark 1. The properties of the foregoing proposition do not determine P^r uniquely. The normalizing choice made in the proof is $\Pi_s^\varepsilon(\lambda; \underline{u}_{+\infty})(P_\varepsilon^r(\lambda, x_0)) = 0_2$ but we could have replaced 0_2 with any analytic choice of an element of the stable space of $\mathcal{A}^\varepsilon(\lambda; \underline{u}_{+\infty})$.

Remark 2. The proposition is sufficient to prove classical results about the determination of the essential spectrum from endstates spectra.

We now investigate the possible failure of uniformity in the regime of large spectral parameters. In the literature by Kevin Zumbrun and his collaborators, similar purposes are achieved through comparison of unstable manifolds with their frozen-coefficient approximations by a type of lemma termed there a *tracking lemma*; see for instance [1, 24]. The rationale is that to large frequencies, smooth coefficients seem almost constant and thus may be treated in some adiabatic way, a fact ubiquitous in high-frequency/semiclassical analysis.

We follow here a different path and rather effectively build a conjugation as in the foregoing gap lemma. The first step is a suitable scaling to ensure some form of uniformity in the large-x contraction argument of the proof of Proposition 2. The second step is a high-frequency approximate diagonalization combined with an explicit solving of the leading-order part of the system, ensuring that in the large-frequency regime the latter construction could actually be carried out with $x_0 = 0$.

Proposition 3. *There exist positive constants* $(\varepsilon_0, C, \theta, \delta)$ *such that setting*

$$\Omega_\delta := \left\{ \lambda; \ \Re(\sqrt{\lambda}) \geq \tfrac{1}{\delta} \right\},$$

there exists a smooth map

$$P^{r,\mathrm{HF}} \colon [0, \varepsilon_0] \times \Omega_\delta \times \mathbf{R} \mapsto \mathrm{GL}_2(\mathbf{C}), \quad (\varepsilon, \lambda, x) \mapsto P_\varepsilon^{r,\mathrm{HF}}(\lambda, x),$$

locally uniformly analytic in λ on a neighborhood of Ω_δ and such that, for any $(\varepsilon, \lambda, x) \in [0, \varepsilon_0] \times \Omega_\delta \times [0, +\infty)$,

$$\left\| \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} \end{pmatrix} P_\varepsilon^{r,\mathrm{HF}}(\lambda, x) \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix} - \mathrm{e}^{-\frac{1}{2} \int_x^{+\infty} (f'(\underline{U}_\varepsilon(y)) - f'(\underline{u}_{+\infty})) \, \mathrm{d}y} \mathrm{I}_2 \right\| \leq C \frac{\mathrm{e}^{-\theta|x|}}{\Re(\sqrt{\lambda})},$$

$$\left\| \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} \end{pmatrix} (P_\varepsilon^{r,\mathrm{HF}}(\lambda, x))^{-1} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix} - \mathrm{e}^{\frac{1}{2} \int_x^{+\infty} (f'(\underline{U}_\varepsilon(y)) - f'(\underline{u}_{+\infty})) \, \mathrm{d}y} \mathrm{I}_2 \right\| \leq C \frac{\mathrm{e}^{-\theta|x|}}{\Re(\sqrt{\lambda})},$$

and, for any $(\varepsilon, \lambda, x, y) \in [0, \varepsilon_0] \times \Omega_\delta \times \mathbf{R}^2$,

$$\Phi_\varepsilon^\lambda(x, y) = P_\varepsilon^{r,\mathrm{HF}}(\lambda, x) \mathrm{e}^{(x-y)\mathbf{A}_\varepsilon^r(\lambda)} (P_\varepsilon^{r,\mathrm{HF}}(\lambda, y))^{-1}.$$

We point out that for our main purposes we do not need to identify explicitly the leading-order part of the conjugation.

As for Proposition 2 the same argument applies to the conjugation on $(-\infty, 0]$ with the flow of $\mathbf{A}_\varepsilon^\ell(\lambda)$ and defines a conjugation map denoted $P^{\ell,\mathrm{HF}}$ from now on.

Proof of Proposition 3. As a preliminary remark, we note that the conditions $\Re(\sqrt{\lambda}) \gg 1$ and $\Re(\mu_+^\varepsilon(\lambda; \underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda; \underline{u}_{+\infty})) \gg 1$ are equivalent, with uniform control of one by the other and vice versa.

Scaling P_ε^r to Q_ε^r defined as

$$
Q_\varepsilon^r(\lambda, \cdot) := \begin{pmatrix} 1 & 0 \\ 0 & \mu_+^\varepsilon(\lambda; \underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda; \underline{u}_{+\infty}) \end{pmatrix}^{-1}
$$
$$
\times P_\varepsilon^r(\lambda, \cdot) \begin{pmatrix} 1 & 0 \\ 0 & \mu_+^\varepsilon(\lambda; \underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda; \underline{u}_{+\infty}) \end{pmatrix}
$$

removes high-frequency singularities by replacing $\mathbf{R}_\pm^\varepsilon(\lambda; \underline{u}_{+\infty})$ and $\mathbf{L}_\pm^\varepsilon(\lambda; \underline{u}_{+\infty})$ with

$$
\begin{pmatrix} 1 \\ -\dfrac{\mu_\mp^\varepsilon(\lambda; \underline{u}_{+\infty})}{\mu_+^\varepsilon(\lambda; \underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda; \underline{u}_{+\infty})} \end{pmatrix}, \quad \begin{pmatrix} \dfrac{\pm\mu_\pm^\varepsilon(\lambda; \underline{u}_{+\infty})}{\mu_+^\varepsilon(\lambda; \underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda; \underline{u}_{+\infty})} & \pm 1 \end{pmatrix},
$$

whereas the only other effect is the replacement of $\mathbf{A}_\varepsilon(\lambda, y) - \mathbf{A}_\varepsilon^r(\lambda)$ with

$$
\begin{pmatrix} \dfrac{f'(\underline{U}_\varepsilon) - f'(\underline{u}_{+\infty})}{g'(\underline{U}_\varepsilon) - g'(\underline{u}_{+\infty})} & 0 \\ -\varepsilon \dfrac{}{\mu_+^\varepsilon(\lambda; \underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda; \underline{u}_{+\infty})} & 0 \end{pmatrix}.
$$

At this stage, let us choose coordinates to identify $\mathcal{M}_2(\mathbf{C})$ with $\mathbf{C}^4$ in such a way that $\mathbf{C}^2 \times \{0\}^2$, $\{0\} \times \mathbf{C} \times \{0\}$, $\{0\}^3 \times \mathbf{C}$, $(1, 0, 0, 0)$ and $(0, 1, 0, 0)$ correspond respectively – after scaling and choice of coordinates – to the kernel of $\mathcal{A}_\varepsilon^r(\lambda)$, its unstable space, its stable space, I_2 and

$$
\frac{1}{\mu_+^\varepsilon(\lambda; \underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda; \underline{u}_{+\infty})} A_{\varepsilon,\mathrm{HF}}^r.
$$

Then the problem to be solved takes the form

$$
\frac{\mathrm{d}}{\mathrm{d}x}\boldsymbol{\alpha}(x) = \mathcal{B}_0(x)\boldsymbol{\alpha}(x) + \mathcal{O}(\mathrm{e}^{-\theta|x|}) \begin{pmatrix} \beta \\ \gamma \end{pmatrix}(x),
$$

$$
\frac{\mathrm{d}}{\mathrm{d}x}\beta(x) = (\mu_+^\varepsilon(\lambda; \underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda; \underline{u}_{+\infty}) + \omega_+(x))\beta(x) + \mathcal{O}(\mathrm{e}^{-\theta|x|}) \begin{pmatrix} \boldsymbol{\alpha} \\ \gamma \end{pmatrix}(x),
$$

$$
\frac{\mathrm{d}}{\mathrm{d}x}\gamma(x) = -(\mu_+^\varepsilon(\lambda; \underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda; \underline{u}_{+\infty}) + \omega_-(x))\gamma(x) + \mathcal{O}(\mathrm{e}^{-\theta|x|}) \begin{pmatrix} \boldsymbol{\alpha} \\ \beta \end{pmatrix}(x),
$$

with $(1, 0, 0, 0)$ as the limiting value at $+\infty$, for some[9] $\theta > 0$, where $\mathcal{B}_0(x)$, $\omega_+(x)$, $\omega_-(x)$ are also of the form $\mathcal{O}(\mathrm{e}^{-\theta|x|})$.

---

[9]Throughout the proof we allow ourselves to change the precise value of θ from line to line.

It follows that when $\mu_+^\varepsilon(\lambda; \underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda; \underline{u}_{+\infty})$ is sufficiently large, by a further change of variables differing from I_4 by a block off-diagonal term

$$\begin{pmatrix} \boldsymbol{\alpha}_{\mathrm{bis}} \\ \beta_{\mathrm{bis}} \\ \gamma_{\mathrm{bis}} \end{pmatrix}(x) = \Big(I_4 + \mathcal{O}\Big(\frac{e^{-\theta|x|}}{\mu_+^\varepsilon(\lambda; \underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda; \underline{u}_{+\infty})} \Big) \Big) \begin{pmatrix} \boldsymbol{\alpha} \\ \beta \\ \gamma \end{pmatrix}(x),$$

one may transform the problem to

$$\frac{\mathrm{d}}{\mathrm{d}x} \boldsymbol{\alpha}_{\mathrm{bis}}(x) = \widetilde{\mathcal{B}}_0(x) \boldsymbol{\alpha}_{\mathrm{bis}}(x) + \mathcal{O}\Big(\frac{e^{-\theta|x|}}{\mu_+^\varepsilon(\lambda; \underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda; \underline{u}_{+\infty})} \Big) \begin{pmatrix} \beta_{\mathrm{bis}} \\ \gamma_{\mathrm{bis}} \end{pmatrix}(x),$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \beta_{\mathrm{bis}}(x) = (\mu_+^\varepsilon(\lambda; \underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda; \underline{u}_{+\infty}) + \widetilde{\omega}_+(x)) \beta_{\mathrm{bis}}(x)$$
$$+ \mathcal{O}\Big(\frac{e^{-\theta|x|}}{\mu_+^\varepsilon(\lambda; \underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda; \underline{u}_{+\infty})} \Big) \begin{pmatrix} \boldsymbol{\alpha}_{\mathrm{bis}} \\ \gamma_{\mathrm{bis}} \end{pmatrix}(x),$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \gamma_{\mathrm{bis}}(x) = -(\mu_+^\varepsilon(\lambda; \underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda; \underline{u}_{+\infty}) + \widetilde{\omega}_-(x)) \gamma_{\mathrm{bis}}(x)$$
$$+ \mathcal{O}\Big(\frac{e^{-\theta|x|}}{\mu_+^\varepsilon(\lambda; \underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda; \underline{u}_{+\infty})} \Big) \begin{pmatrix} \boldsymbol{\alpha}_{\mathrm{bis}} \\ \beta_{\mathrm{bis}} \end{pmatrix}(x),$$

with $(1, 0, 0, 0)$ as the limiting value at $+\infty$, where $\widetilde{\mathcal{B}}_0 - \mathcal{B}_0$, $\widetilde{\omega}_- - \omega_-$ and $\widetilde{\omega}_+ - \omega_+$ are all of the form

$$\mathcal{O}\Big(\frac{e^{-\theta|x|}}{\mu_+^\varepsilon(\lambda; \underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda; \underline{u}_{+\infty})} \Big).$$

Now we point out that there is a single solution to the leading-order part,

$$\frac{\mathrm{d}}{\mathrm{d}x} \boldsymbol{\alpha}_{\mathrm{main}}(x) = \widetilde{\mathcal{B}}_0(x) \boldsymbol{\alpha}_{\mathrm{main}}(x),$$

with $(1, 0)$ as the limiting value at $+\infty$. This follows from a fixed point argument on $x \geq x_0$ for x_0 large, followed by a continuation argument. We may be even more explicit. Indeed, an explicit computation yields

$$\widetilde{\mathcal{B}}_0(x) = \mathcal{B}_{\mathrm{main}}(x) + \mathcal{O}\Big(\frac{e^{-\theta|x|}}{\mu_+^\varepsilon(\lambda; \underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda; \underline{u}_{+\infty})} \Big),$$

with

$$\mathcal{B}_{\mathrm{main}}(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2} (f'(\underline{U}_\varepsilon(x)) - f'(\underline{u}_{+\infty})) \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

so that

$$\boldsymbol{\alpha}_{\mathrm{main}}(x) = e^{-\frac{1}{2} \int_x^{+\infty} (f'(\underline{U}_\varepsilon(y)) - f'(\underline{u}_{+\infty})) \, \mathrm{d}y} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathcal{O}\Big(\frac{e^{-\theta|x|}}{\mu_+^\varepsilon(\lambda; \underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda; \underline{u}_{+\infty})} \Big).$$

The proof is thus achieved by a fixed-point argument on a problem of type

$$\boldsymbol{\alpha}_{\mathrm{bis}}(x) = \boldsymbol{\alpha}_{\mathrm{main}}(x) - \int_x^{+\infty} \Phi_0(x, y) \mathcal{O}\left(\frac{\mathrm{e}^{-\theta|y|}}{\mu_+^\varepsilon(\lambda; \underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda; \underline{u}_{+\infty})}\right) \begin{pmatrix} \beta_{\mathrm{bis}} \\ \gamma_{\mathrm{bis}} \end{pmatrix}(y) \, \mathrm{d}y,$$

$$\beta_{\mathrm{bis}}(x) = -\int_x^{+\infty} \mathcal{O}\left(\frac{\mathrm{e}^{-(y-x)(\mu_+^\varepsilon(\lambda; \underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda; \underline{u}_{+\infty})) + \int_y^x \widetilde{\omega}_+} \mathrm{e}^{-\theta|y|}}{\mu_+^\varepsilon(\lambda; \underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda; \underline{u}_{+\infty})}\right) \begin{pmatrix} \boldsymbol{\alpha}_{\mathrm{bis}} \\ \gamma_{\mathrm{bis}} \end{pmatrix}(y) \, \mathrm{d}y,$$

$$\gamma_{\mathrm{bis}}(x) = \int_0^x \mathcal{O}\left(\frac{\mathrm{e}^{-(x-y)(\mu_+^\varepsilon(\lambda; \underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda; \underline{u}_{+\infty})) - \int_y^x \widetilde{\omega}_-} \mathrm{e}^{-\theta|y|}}{\mu_+^\varepsilon(\lambda; \underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda; \underline{u}_{+\infty})}\right) \begin{pmatrix} \boldsymbol{\alpha}_{\mathrm{bis}} \\ \beta_{\mathrm{bis}} \end{pmatrix}(y) \, \mathrm{d}y,$$

where Φ_0 denotes the solution operator associated with $\widetilde{\mathcal{B}}_0$. ∎

In the following we shall complete Proposition 3 which provides $P^{r,\mathrm{HF}}$ on Ω_δ with an application of Proposition 2 on

$$K_\delta := \left\{\lambda; \ d(\lambda, \mathcal{D}_0(\underline{u}_{+\infty})) \geq \delta \text{ and } |\lambda| \leq \tfrac{1}{\delta}\right\}.$$

When δ is sufficiently small, Ω_δ and K_δ overlap. Yet a priori P^r and $P^{r,\mathrm{HF}}$ differ from each other even in regions where both exist. Fortunately, the implied possible mismatch disappears at the level of Green functions.

2.2. Evans function and its asymptotics

Now, wherever it makes sense, we set

$$\mathbf{V}_\varepsilon^{r,s}(\lambda, x) := \mathrm{e}^{x\mu_-^\varepsilon(\lambda; \underline{u}_{+\infty})} P_\varepsilon^r(\lambda, x) \mathbf{R}_-^\varepsilon(\lambda; \underline{u}_{+\infty}),$$

$$\mathbf{V}_\varepsilon^{r,u}(\lambda, x) := \mathrm{e}^{x\mu_+^\varepsilon(\lambda; \underline{u}_{+\infty})} P_\varepsilon^r(\lambda, x) \mathbf{R}_+^\varepsilon(\lambda; \underline{u}_{+\infty}),$$

$$\mathbf{V}_\varepsilon^{\ell,s}(\lambda, x) := \mathrm{e}^{x\mu_-^\varepsilon(\lambda; \underline{u}_{-\infty})} P_\varepsilon^\ell(\lambda, x) \mathbf{R}_-^\varepsilon(\lambda; \underline{u}_{-\infty}),$$

$$\mathbf{V}_\varepsilon^{\ell,u}(\lambda, x) := \mathrm{e}^{x\mu_+^\varepsilon(\lambda; \underline{u}_{-\infty})} P_\varepsilon^\ell(\lambda, x) \mathbf{R}_+^\varepsilon(\lambda; \underline{u}_{-\infty}),$$

and similarly for $\mathbf{V}_\varepsilon^{r,s,\mathrm{HF}}$, $\mathbf{V}_\varepsilon^{r,u,\mathrm{HF}}$, $\mathbf{V}_\varepsilon^{\ell,s,\mathrm{HF}}$ and $\mathbf{V}_\varepsilon^{\ell,u,\mathrm{HF}}$. Note that notation is used here to recall stable and unstable spaces; we also use them in areas of the spectral plane where they do not match with stable and unstable spaces. Instead, this fits analytic continuation of generators of stable/unstable spaces.

Correspondingly we define the Evans function

$$D_\varepsilon(\lambda) := \det \left(\mathbf{V}_\varepsilon^{r,s}(\lambda, 0) \quad \mathbf{V}_\varepsilon^{\ell,u}(\lambda, 0)\right)$$

and its high-frequency counterpart $D_\varepsilon^{\mathrm{HF}}$. Note that we define the Evans function at point 0, but on one hand we do not make use of any particular property due to normalization so that the point 0 could be replaced with any other point, and on the other hand relations between Evans functions at different points are simply derived from Liouville's formula for Wronskians. For instance,

$$\det \left(\mathbf{V}_\varepsilon^{r,s}(\lambda, x) \quad \mathbf{V}_\varepsilon^{\ell,u}(\lambda, x)\right) = D_\varepsilon(\lambda) \mathrm{e}^{\int_0^x \mathrm{Tr}(\mathbf{A}_\varepsilon(\lambda, \cdot))} = D_\varepsilon(\lambda) \mathrm{e}^{\int_0^x (f'(\underline{U}_\varepsilon) - \sigma_\varepsilon)}.$$

A simple corollary to Proposition 3 follows:

Corollary 1. *Uniformly in ε (sufficiently small),*

$$\lim_{\Re(\sqrt{\lambda})\to\infty} \frac{D_\varepsilon^{\mathrm{HF}}(\lambda)}{\sqrt{\lambda}} = 2\mathrm{e}^{-\frac{1}{2}\int_0^{+\infty}(f'(\underline{U}_\varepsilon(y))-f'(\underline{u}_{+\infty}))\,\mathrm{d}y+\frac{1}{2}\int_{-\infty}^0(f'(\underline{U}_\varepsilon(y))-f'(\underline{u}_{-\infty}))\,\mathrm{d}y}.$$

To complete Corollary 1, we derive information on compacts sets of λ in the limit $\varepsilon \to 0$.

Proposition 4. *There exists $\eta_0 > 0$ such that for any $\delta > 0$ there exist positive (ε_0, c_0) such that for any $\varepsilon \in [0, \varepsilon_0]$, $D_\varepsilon(\cdot)$ is well defined on*

$$K_{\eta_0,\delta} := \left\{\lambda;\ d(\lambda, (-\infty, -\eta_0]) \geq \min(\{\delta, \tfrac{\eta_0}{2}\})\ and\ |\lambda| \leq \tfrac{1}{\delta}\right\},$$

and for any $\lambda \in K_{\eta_0,\delta}$,

$$|D_\varepsilon(\lambda)| \geq c_0 \min(\{1, |\lambda|\}).$$

Proof. We derive the result from Sturm–Liouville theory and regularity in ε. To apply Sturm–Liouville theory, we introduce the weight

$$\omega_\varepsilon(x) := \mathrm{e}^{\frac{1}{2}\int_0^x(f'(\underline{U}_\varepsilon(y))-\sigma_\varepsilon)\,\mathrm{d}y}.$$

We observe that, considered as an operator on $L^2(\mathbf{R})$ with domain $H^2(\mathbf{R})$, the operator

$$L_\varepsilon := \frac{1}{\omega_\varepsilon}\mathcal{L}_\varepsilon(\omega_\varepsilon\cdot)$$

is self-adjoint and in the region of interest it possesses no essential spectrum and its eigenvalues agree in location and algebraic multiplicity with the roots of D_ε. As a consequence the zeros of D_ε are real, and since $\underline{U}'_\varepsilon/\omega_\varepsilon$ is a nowhere-vanishing eigenvector for the eigenvalue 0, 0 is a simple root of D_ε and D_ε does not vanish on $(0, +\infty)$. From here the corresponding bound is deduced through a continuity-compactness argument in ε. ∎

3. Green functions

Now we use the introduced spectral objects to obtain representation formulas for linearized solution operators.

3.1. Duality

To begin with, to provide explicit formulas for spectral Green functions to be introduced below, we extend the conclusions of Section 2 to dual problems. Note that the duality we are referring to is not related to any particular choice of a specific Banach space but rather is distributional/algebraic.

To begin with, we introduce the formal adjoint

$$\mathcal{L}_\varepsilon^{\mathrm{adj}} := (f'(\underline{U}_\varepsilon) - \sigma_\varepsilon)\partial_x + \partial_x^2 + \varepsilon g'(\underline{U}_\varepsilon)$$

and note that for any sufficiently smooth v, w, and any points (x_0, x_1),

$$\mathbf{V} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} w \\ \partial_x w \end{pmatrix}(x_1) - \mathbf{V} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} w \\ \partial_x w \end{pmatrix}(x_0) = \int_{x_0}^{x_1} (w\mathcal{L}_\varepsilon v - v\mathcal{L}_\varepsilon^{\mathrm{adj}} w), \quad (3.1)$$

with $\mathbf{V} = (v, \partial_x v - (f'(\underline{U}_\varepsilon) - \sigma_\varepsilon)v)$. As a first simple consequence of (3.1) note that if $(\lambda, y, v_y^r, v_y^\ell)$ are such that $(\lambda - \mathcal{L}_\varepsilon)v_y^r = 0$ and $(\lambda - \mathcal{L}_\varepsilon)v_y^\ell = 0$, then the function

$$\varphi_y : \mathbf{R} \to \mathbf{C}, \quad x \mapsto \begin{cases} v_y^r(x) & \text{if } x > y, \\ v_y^\ell(x) & \text{if } x < y, \end{cases}$$

solves $(\lambda - \mathcal{L}_\varepsilon)\varphi_y = \delta_y$ if and only if

$$v_y^r(y) = v_y^\ell(y),$$
$$\partial_x v_y^r(y) - (f'(\underline{U}_\varepsilon)(y) - \sigma_\varepsilon)v_y^r(y) = \partial_x v_y^\ell(y) - (f'(\underline{U}_\varepsilon)(y) - \sigma_\varepsilon)v_y^\ell(y) - 1.$$

Specializing to the tensorized case where $v_y^r(x) = v^r(x)\alpha(y)$, $v_y^\ell(x) = v^\ell(x)\beta(y)$, note that the foregoing conditions are equivalent to

$$\begin{pmatrix} \mathbf{V}^r(y) & \mathbf{V}^\ell(y) \end{pmatrix} \begin{pmatrix} \alpha(y) \\ -\beta(y) \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix},$$

where $\mathbf{V}^\sharp = (v^\sharp, \partial_x v^\sharp - (f'(\underline{U}_\varepsilon) - \sigma_\varepsilon)v^\sharp)$, $\sharp \in \{r, \ell\}$. Hence, we need to find vectors satisfying some orthogonality property to identify the inverse of the matrix:

$$\begin{pmatrix} \mathbf{V}^r(y)\mathbf{V}^\ell(y) \end{pmatrix}.$$

To go further, we identify

$$(\lambda - \mathcal{L}_\varepsilon^{\mathrm{adj}})w = 0$$

and the system of ODEs

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{W}(x) = \tilde{\mathbf{A}}_\varepsilon(\lambda, x)\mathbf{W}(x)$$

for the vector $\mathbf{W} = (w, \partial_x w)$, where

$$\tilde{\mathbf{A}}_\varepsilon(\lambda, x) := \begin{pmatrix} 0 & 1 \\ \lambda - \varepsilon g'(\underline{U}_\varepsilon) & -(f'(\underline{U}_\varepsilon) - \sigma_\varepsilon) \end{pmatrix}.$$

Note that

$$\tilde{\mathbf{A}}_\varepsilon(\lambda, x) = \mathbf{A}_\varepsilon(\lambda, x) - (f'(\underline{U}_\varepsilon) - \sigma_\varepsilon)\mathrm{I}_2$$

so that all the proofs of Section 2 purely based on limiting-matrix spectral-gap arguments apply equally well to the corresponding dual problems under the exact same assumptions. Alternatively, one may derive results on dual problems by directly using the relation between solution operators

$$\widetilde{\boldsymbol{\Phi}}_\varepsilon^\lambda(x, y) = \boldsymbol{\Phi}_\varepsilon^\lambda(x, y) \mathrm{e}^{-\int_y^x (f'(\underline{U}_\varepsilon) - \sigma_\varepsilon)}.$$

Here and elsewhere throughout the text from now on we denote with a $\tilde{\ }$ all quantities arising from dual problems. Let us point out that our choices lead to

$$\tilde{\mu}_\pm^\varepsilon(\lambda; u) = \mu_\pm^\varepsilon(\lambda; u) - (f'(u) - \sigma_\varepsilon),$$

$$\widetilde{\mathbf{R}}_\pm^\varepsilon(\lambda; u) = \mathbf{R}_\pm^\varepsilon(\lambda; u), \quad \widetilde{\mathbf{L}}_\pm^\varepsilon(\lambda; u) = \mathbf{L}_\pm^\varepsilon(\lambda; u),$$

and

$$\widetilde{P}_\varepsilon^r(\lambda, x) = P_\varepsilon^r(\lambda, x) \mathrm{e}^{\int_x^{+\infty}(f'(\underline{U}_\varepsilon) - f'(\underline{u}_{+\infty}))},$$

$$\widetilde{P}_\varepsilon^\ell(\lambda, x) = P_\varepsilon^\ell(\lambda, x) \mathrm{e}^{-\int_{-\infty}^x(f'(\underline{U}_\varepsilon) - f'(\underline{u}_{-\infty}))}$$

(and likewise for high-frequency versions).

Proposition 5. *Let K be a compact subset of $\mathbf{C} \setminus (\mathcal{D}_0(\underline{u}_{+\infty}) \cup \mathcal{D}_0(\underline{u}_{+\infty}))$. There exists $\varepsilon_0 > 0$ such that there exist smooth maps*

$$\tau^r: [0, \varepsilon_0] \times K \mapsto \mathbf{C}, \quad (\varepsilon, \lambda) \mapsto \tau_\varepsilon^r(\lambda), \quad \rho^r: [0, \varepsilon_0] \times K \mapsto \mathbf{C}, \quad (\varepsilon, \lambda) \mapsto \rho_\varepsilon^r(\lambda),$$

$$\tau^\ell: [0, \varepsilon_0] \times K \mapsto \mathbf{C}, \quad (\varepsilon, \lambda) \mapsto \tau_\varepsilon^\ell(\lambda), \quad \rho^\ell: [0, \varepsilon_0] \times K \mapsto \mathbf{C}, \quad (\varepsilon, \lambda) \mapsto \rho_\varepsilon^\ell(\lambda),$$

$$\tilde{\tau}^r: [0, \varepsilon_0] \times K \mapsto \mathbf{C}, \quad (\varepsilon, \lambda) \mapsto \tilde{\tau}_\varepsilon^r(\lambda), \quad \tilde{\rho}^r: [0, \varepsilon_0] \times K \mapsto \mathbf{C}, \quad (\varepsilon, \lambda) \mapsto \tilde{\rho}_\varepsilon^r(\lambda),$$

$$\tilde{\tau}^\ell: [0, \varepsilon_0] \times K \mapsto \mathbf{C}, \quad (\varepsilon, \lambda) \mapsto \tilde{\tau}_\varepsilon^\ell(\lambda), \quad \tilde{\rho}^\ell: [0, \varepsilon_0] \times K \mapsto \mathbf{C}, \quad (\varepsilon, \lambda) \mapsto \tilde{\rho}_\varepsilon^\ell(\lambda),$$

locally uniformly analytic in λ on a neighborhood of K and such that, for any $(\varepsilon, \lambda) \in [0, \varepsilon_0] \times K$, for any $x \in \mathbf{R}$,

$$\mathbf{V}_\varepsilon^{r,s}(\lambda, x) = \rho_\varepsilon^r(\lambda) \mathbf{V}_\varepsilon^{\ell,s}(\lambda, x) + \tau_\varepsilon^r(\lambda) \mathbf{V}_\varepsilon^{\ell,u}(\lambda, x),$$

$$\mathbf{V}_\varepsilon^{\ell,u}(\lambda, x) = \rho_\varepsilon^\ell(\lambda) \mathbf{V}_\varepsilon^{r,u}(\lambda, x) + \tau_\varepsilon^\ell(\lambda) \mathbf{V}_\varepsilon^{r,s}(\lambda, x),$$

$$\widetilde{\mathbf{V}}_\varepsilon^{r,s}(\lambda, x) = \tilde{\rho}_\varepsilon^r(\lambda) \widetilde{\mathbf{V}}_\varepsilon^{\ell,s}(\lambda, x) + \tilde{\tau}_\varepsilon^r(\lambda) \widetilde{\mathbf{V}}_\varepsilon^{\ell,u}(\lambda, x),$$

$$\widetilde{\mathbf{V}}_\varepsilon^{\ell,u}(\lambda, x) = \tilde{\rho}_\varepsilon^\ell(\lambda) \widetilde{\mathbf{V}}_\varepsilon^{r,u}(\lambda, x) + \tilde{\tau}_\varepsilon^\ell(\lambda) \widetilde{\mathbf{V}}_\varepsilon^{r,s}(\lambda, x),$$

and

$$\rho_\varepsilon^r(\lambda) = D_\varepsilon(\lambda) \frac{\mathrm{e}^{-\int_{-\infty}^0 (f'(\underline{U}_\varepsilon) - f'(\underline{u}_{-\infty}))}}{\mu_+^\varepsilon(\lambda; \underline{u}_{-\infty}) - \mu_-^\varepsilon(\lambda; \underline{u}_{-\infty})},$$

$$\rho_\varepsilon^\ell(\lambda) = D_\varepsilon(\lambda) \frac{\mathrm{e}^{\int_0^{+\infty}(f'(\underline{U}_\varepsilon) - f'(\underline{u}_{+\infty}))}}{\mu_+^\varepsilon(\lambda; \underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda; \underline{u}_{+\infty})},$$

$$\tilde{\rho}_\varepsilon^r(\lambda) = D_\varepsilon(\lambda) \frac{e^{\int_0^{+\infty}(f'(\underline{U}_\varepsilon)-f'(\underline{u}_{+\infty}))}}{\mu_+^\varepsilon(\lambda;\underline{u}_{-\infty}) - \mu_-^\varepsilon(\lambda;\underline{u}_{-\infty})},$$

$$\tilde{\rho}_\varepsilon^\ell(\lambda) = D_\varepsilon(\lambda) \frac{e^{-\int_{-\infty}^0(f'(\underline{U}_\varepsilon)-f'(\underline{u}_{-\infty}))}}{\mu_+^\varepsilon(\lambda;\underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda;\underline{u}_{+\infty})}.$$

As a consequence for such an (ε,λ) and any $x \in \mathbf{R}$,

$$\mathbf{V}_\varepsilon^{\ell,u}(\lambda,x) \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tilde{\mathbf{V}}_\varepsilon^{\ell,u}(\lambda,x) = 0,$$

$$\mathbf{V}_\varepsilon^{r,s}(\lambda,x) \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tilde{\mathbf{V}}_\varepsilon^{r,s}(\lambda,x) = 0,$$

$$\mathbf{V}_\varepsilon^{r,s}(\lambda,x) \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tilde{\mathbf{V}}_\varepsilon^{\ell,u}(\lambda,x) = -D_\varepsilon(\lambda)e^{-\int_{-\infty}^0(f'(\underline{U}_\varepsilon)-f'(\underline{u}_{-\infty}))},$$

$$\mathbf{V}_\varepsilon^{\ell,u}(\lambda,x) \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tilde{\mathbf{V}}_\varepsilon^{r,s}(\lambda,x) = D_\varepsilon(\lambda)e^{\int_0^{+\infty}(f'(\underline{U}_\varepsilon)-f'(\underline{u}_{+\infty}))}.$$

The notation τ, ρ is used here to echo the transmission/reflection coefficients of the classical scattering framework.

A corresponding proposition holds for the high-frequency regime.

Proof of Proposition 5. All the properties are readily obtained by combining the fact that both pairs of functions, $(\mathbf{V}_\varepsilon^{r,s}(\lambda,\cdot), \mathbf{V}_\varepsilon^{\ell,u}(\lambda,\cdot))$ and $(\mathbf{V}_\varepsilon^{\ell,s}(\lambda,\cdot), \mathbf{V}_\varepsilon^{\ell,u}(\lambda,\cdot))$, form a basis of solutions of the spectral system of ODEs, the Liouville formula for Wronskians and duality relation (3.1). ∎

We thus have

$$\left(\mathbf{V}_\varepsilon^{r,s}(y) \quad \mathbf{V}_\varepsilon^{\ell,u}(y) \right)^{-1}$$

$$= \begin{pmatrix} \dfrac{e_2 \cdot \tilde{\mathbf{V}}_\varepsilon^{\ell,u}(\lambda,y)}{D_\varepsilon(\lambda)e^{-\int_{-\infty}^0(f'(\underline{U}_\varepsilon)-f'(\underline{u}_{-\infty}))}} & \dfrac{e_1 \cdot \tilde{\mathbf{V}}_\varepsilon^{\ell,u}(\lambda,y)}{-D_\varepsilon(\lambda)e^{-\int_{-\infty}^0(f'(\underline{U}_\varepsilon)-f'(\underline{u}_{-\infty}))}} \\ -\dfrac{e_2 \cdot \tilde{\mathbf{V}}_\varepsilon^{r,s}(\lambda,y)}{D_\varepsilon(\lambda)e^{\int_0^{+\infty}(f'(\underline{U}_\varepsilon)-f'(\underline{u}_{+\infty}))}} & \dfrac{e_1 \cdot \tilde{\mathbf{V}}_\varepsilon^{r,s}(\lambda,y)}{D_\varepsilon(\lambda)e^{\int_0^{+\infty}(f'(\underline{U}_\varepsilon)-f'(\underline{u}_{+\infty}))}} \end{pmatrix}.$$

Proposition 6. *There exist positive constants $(\varepsilon_0, C, \delta)$ such that setting with Ω_δ as in Proposition 3,*

$$\Omega_\delta := \{\lambda; \Re(\sqrt{\lambda}) \geq \tfrac{1}{\delta}\},$$

there exist on $[0,\varepsilon_0] \times \Omega_\delta$ maps $\tau^{r,\mathrm{HF}}, \rho^{r,\mathrm{HF}}, \tau^{\ell,\mathrm{HF}}, \rho^{\ell,\mathrm{HF}}, \tilde{\tau}^{r,\mathrm{HF}}, \tilde{\rho}^{r,\mathrm{HF}}, \tilde{\tau}^{\ell,\mathrm{HF}}, \tilde{\rho}^{\ell,\mathrm{HF}}$, satisfying high-frequency versions of the conclusions of Proposition 5 and moreover all these functions are uniformly bounded on $[0,\varepsilon_0] \times \Omega_\delta$.

Proof. Most of the proof is contained in the proof of Proposition 5. The remaining part is directly derived from the observation that Proposition 3 provides asymptotics for the operator $(P_\varepsilon^\ell(\lambda,0))^{-1} P_\varepsilon^r(\lambda,0)$, thus also for the coefficients under consideration. ∎

This leads to the following definition (wherever it makes sense):

$$G_\varepsilon(\lambda; x, y) := \begin{cases} \dfrac{\mathrm{e}^{\int_{-\infty}^0 (f'(\underline{U}_\varepsilon) - f'(\underline{u}_{-\infty}))}}{D_\varepsilon(\lambda)} \, \mathbf{e}_1 \cdot \mathbf{V}_\varepsilon^{r,s}(\lambda, x) \, \mathbf{e}_1 \cdot \widetilde{\mathbf{V}}_\varepsilon^{\ell,u}(\lambda, y) & \text{if } x > y, \\[4mm] \dfrac{\mathrm{e}^{-\int_0^{+\infty} (f'(\underline{U}_\varepsilon) - f'(\underline{u}_{+\infty}))}}{D_\varepsilon(\lambda)} \, \mathbf{e}_1 \cdot \mathbf{V}_\varepsilon^{\ell,u}(\lambda, x) \, \mathbf{e}_1 \cdot \widetilde{\mathbf{V}}_\varepsilon^{r,s}(\lambda, y) & \text{if } x < y, \end{cases}$$

where $\mathbf{e}_1 := (1, 0)$. Note that $(\lambda - \mathcal{L}_\varepsilon) G_\varepsilon(\lambda; \cdot, y) = \delta_y$ and, for $\Re(\lambda)$ sufficiently large, $G_\varepsilon(\lambda; x, y)$ is exponentially decaying as $\|x - y\| \to \infty$. To bound $G_\varepsilon(\lambda; x, y)$, we shall refine the alternative $x < y$ *versus* $x > y$. For instance, when $x > y$, more convenient equivalent representations of $G_\varepsilon(\lambda; x, y)$ are

$$\frac{\mathrm{e}^{\int_{-\infty}^0 (f'(\underline{U}_\varepsilon) - f'(\underline{u}_{-\infty}))}}{D_\varepsilon(\lambda)} \, \mathbf{e}_1 \cdot \mathbf{V}_\varepsilon^{r,s}(\lambda, x) \, \mathbf{e}_1 \cdot \widetilde{\mathbf{V}}_\varepsilon^{\ell,u}(\lambda, y) \quad \text{when } x > 0 > y,$$

$$\mathrm{e}^{\int_{-\infty}^0 (f'(\underline{U}_\varepsilon) - f'(\underline{u}_{-\infty}))} \, \mathbf{e}_1 \cdot \mathbf{V}_\varepsilon^{r,s}(\lambda, x) \, \mathbf{e}_1 \cdot \Big(\frac{\tilde{\rho}_\varepsilon^\ell(\lambda)}{D_\varepsilon(\lambda)} \widetilde{\mathbf{V}}_\varepsilon^{r,u}(\lambda, y) + \frac{\tilde{\tau}_\varepsilon^\ell(\lambda)}{D_\varepsilon(\lambda)} \widetilde{\mathbf{V}}_\varepsilon^{r,s}(\lambda, y) \Big)$$

$$\text{when } x > y > 0,$$

$$\mathrm{e}^{\int_{-\infty}^0 (f'(\underline{U}_\varepsilon) - f'(\underline{u}_{-\infty}))} \, \mathbf{e}_1 \cdot \Big(\frac{\rho_\varepsilon^r(\lambda)}{D_\varepsilon(\lambda)} \mathbf{V}_\varepsilon^{\ell,s}(\lambda, x) + \frac{\tau_\varepsilon^r(\lambda)}{D_\varepsilon(\lambda)} \mathbf{V}_\varepsilon^{\ell,u}(\lambda, x) \Big) \, \mathbf{e}_1 \cdot \widetilde{\mathbf{V}}_\varepsilon^{\ell,u}(\lambda, y)$$

$$\text{when } 0 > x > y.$$

Remark 3. The representation of spectral Green functions, thus of resolvent operators, with Evans functions is sufficient to prove classical results about the identification of spectrum – including algebraic multiplicity – at the right-hand side of the essential spectrum with zeros of Evans functions.

We use similar formulas in the high-frequency regime. Yet the Green functions of the high-frequency regime and the compact-frequency regime agree where they co-exist (by uniqueness of the spectral problem (in a suitably weighted space) in some overlapping regions and uniqueness of analytic continuation elsewhere) so that we do not need to introduce a specific piece of notation for the high-frequency regime.

We also point out that it follows from Proposition 3 that in the zone of interest,

$$|\partial_x G_\varepsilon(\lambda; x, y)| \leq C \max(\{1, \sqrt{|\lambda|}\}) |G_\varepsilon(\lambda; x, y)|$$

for some uniform constant C.

3.2. Time evolution

It follows from standard semigroup theory that the representation

$$S_\varepsilon(t) = \frac{1}{2\mathrm{i}\pi} \int_{\mathbf{R}} \mathrm{e}^{\Lambda(\xi) t} \Lambda'(\xi) (\Lambda(\xi) \mathrm{I} - \mathcal{L}_\varepsilon)^{-1} \, \mathrm{d}\xi \tag{3.2}$$

holds in $\mathcal{L}(\mathrm{BUC}^0(\mathbf{R}))$ when $\Lambda \colon \mathbf{R} \to \mathbf{C}$ is a continuous, piecewise $\mathcal{C}^1$ simple curve such that

(1) Λ is valued in the right-hand connected component of[10]

$$\{\lambda; \text{ for } u \in \{\underline{u}_{-\infty}, \underline{u}_{+\infty}\}, \ \Re(\mu_+^\varepsilon(\lambda, u)) > 0 > \Re(\mu_-^\varepsilon(\lambda, u))\};$$

(2) there hold

$$\lim_{\xi \to \pm\infty} \Im(\Lambda(\xi)) = \pm\infty, \quad \int_{\mathbf{R}} \mathrm{e}^{\Re(\Lambda(\xi))t} \frac{|\Lambda'(\xi)|}{1 + |\Lambda(\xi)|} \, \mathrm{d}\xi < +\infty$$

and there exist positive (R, c) such that for $|\xi| \geq R$,

$$\Re(\Lambda(\xi)) \geq -c |\Im(\Lambda(\xi))|;$$

(3) there is no root of D_ε on the right[11] of $\Lambda(\mathbf{R})$.

Failure of the third condition could be restored by adding positively oriented small circles to the contour Λ. This is the first condition that we want to relax by going to Green functions.

For curves as above, applying the above formula to functions in $W^{\infty,\infty}(\mathbf{R})$ and testing it against functions in $\mathcal{C}_c^\infty(\mathbf{R})$ leads to a similar representation for Green functions,

$$G_t^\varepsilon(x, y) = \frac{1}{2\mathrm{i}\pi} \int_{\mathbf{R}} \mathrm{e}^{\Lambda(\xi)t} \Lambda'(\xi) G_\varepsilon(\Lambda(\xi); x, y) \, \mathrm{d}\xi. \tag{3.3}$$

The point is that at fixed (t, x, y), the constraints on Λ ensuring the representation formula are significantly less stringent and one may use this freedom to optimize bounds. In particular, depending on the specific regime for the triplet (t, x, y) or the kind of data one has in mind, one may trade spatial localization for time decay and vice versa by adjusting contours to the right so as to gain spatial decay or to the left in order to improve time decay.

When doing so, we essentially follow the strategy of [60]. The critical decay is essentially encoded in limiting-endstate spectral spatial decay and Evans function root location. Therefore, roughly speaking, leaving aside questions related to the presence of a root of the Evans function at zero, contours are chosen here to approximately[12] optimize bounds on

$$\int_{\mathbf{R}} \mathrm{e}^{\Re(\Lambda(\xi))t + \Re(\mu_\sharp^\varepsilon(\Lambda(\xi), \underline{u}_{\mathrm{sgn}(x)\infty}))x + \Re(\tilde{\mu}_\flat^\varepsilon(\Lambda(\xi), \underline{u}_{\mathrm{sgn}(y)\infty}))y} \frac{|\Lambda'(\xi)|}{|D_\varepsilon(\Lambda(\xi))|} \, \mathrm{d}\xi$$

---

[10]This set contains $\{\lambda; \Re(\lambda) \geq \omega\}$ when ω is sufficiently large.

[11]The second condition implies that this makes sense.

[12]In some cases a genuine optimization – as in direct applications of the Riemann saddle-point method – would be impractical.

with $(\sharp, \flat) \in \{+, -\}^2$. More precisely, at fixed (t, x, y), one picks Λ_0 real in $[-\frac{1}{2}\eta_0, +\infty]$ (with η_0 as in Proposition 4), approximately minimizing

$$\Re(\lambda)t + \Re(\mu_\sharp^\varepsilon(\lambda, \underline{u}_{\mathrm{sgn}(x)\infty}))x + \Re(\tilde\mu_\flat^\varepsilon(\lambda, \underline{u}_{\mathrm{sgn}(y)\infty}))y$$

among such real λ in $[-\frac{1}{2}\eta_0, +\infty]$ and then depending on cases, one defines Λ through one of the equations

$$\Re(\mu_\sharp^\varepsilon(\Lambda(\xi), \underline{u}_{\mathrm{sgn}(x)\infty}))x + \Re(\tilde\mu_\flat^\varepsilon(\Lambda(\xi), \underline{u}_{\mathrm{sgn}(y)\infty}))y$$

$$= \Re(\mu_\sharp^\varepsilon(\Lambda_0, \underline{u}_{\mathrm{sgn}(x)\infty}))x + \Re(\tilde\mu_\flat^\varepsilon(\Lambda_0, \underline{u}_{\mathrm{sgn}(y)\infty}))y + i\xi\zeta_{\mathrm{sgn}(\xi)}(\sharp x + \flat y),$$

$$\Re(\tilde\mu_\flat^\varepsilon(\Lambda(\xi), \underline{u}_{\mathrm{sgn}(y)\infty}))y = \Re(\tilde\mu_\flat^\varepsilon(\Lambda_0, \underline{u}_{\mathrm{sgn}(y)\infty}))y + i\xi\zeta_{\mathrm{sgn}(\xi)} \times (\flat y),$$

$$\Re(\mu_\sharp^\varepsilon(\Lambda(\xi), \underline{u}_{\mathrm{sgn}(x)\infty}))x = \Re(\mu_\sharp^\varepsilon(\Lambda_0, \underline{u}_{\mathrm{sgn}(x)\infty}))x + i\xi\zeta_{\mathrm{sgn}(\xi)} \times (\sharp x),$$

with $\zeta_\pm$ conveniently chosen to ensure a condition analogous to the second condition of the semigroup representation and including

$$\lim_{|\xi|\to\infty} \Re(\sqrt{\Lambda(\xi)}) = +\infty, \qquad \lim_{\xi\to\pm\infty} \Im(\Lambda(\xi)) = \pm\infty.$$

This should be thought of as an approximate/simplified version of the saddle-point method in the sense that $\Lambda_0 = \Lambda(0)$ is an approximate maximizer of the exponential decay rate among real numbers, but a minimizer along the curve $\Lambda(\cdot)$.

Computational details – carried out in the next section – are cumbersome, but the process is rather systematic.

4. Linear stability

We now make the most of our spectral preparation to derive linear stability estimates.

To motivate the analysis, let us anticipate that our achievement is the splitting of $(S_\varepsilon(t))_{t\geq 0}$ as

$$S_\varepsilon(t)(w)(x) = \underline{U}_\varepsilon'(x)s_\varepsilon^{\mathrm{p}}(t)(w) + \tilde S_\varepsilon(t)(w)(x).$$

for some $(s_\varepsilon^{\mathrm{p}}(t))_{t\geq 0}$, $(\tilde S_\varepsilon(t))_{t\geq 0}$ with $\tilde S_\varepsilon(0) = \mathrm{Id}$, so that the following proposition holds.

Proposition 7. *There exists $\varepsilon_0 > 0$ such that*

(1) *there exists $C > 0$ such that for any $t \geq 0$, any $0 \leq \varepsilon \leq \varepsilon_0$ and any $w \in \mathrm{BUC}^0(\mathbf{R})$,*

$$\|\tilde S_\varepsilon(t)(w)\|_{L^\infty(\mathbf{R})} + \min(\{1, \sqrt{t}\})\|\partial_x \tilde S_\varepsilon(t)(w)\|_{L^\infty(\mathbf{R})} + |\partial_t s_\varepsilon^{\mathrm{p}}(t)(w)|$$

$$\leq C\mathrm{e}^{-\min(\{|g'(\underline{u}_{-\infty})|, |g'(\underline{u}_{+\infty})|\})\varepsilon t}\|w\|_{L^\infty(\mathbf{R})},$$

and when moreover $w \in \mathrm{BUC}^1(\mathbf{R})$,

$$\|\tilde S_\varepsilon(t)(w)\|_{W^{1,\infty}(\mathbf{R})} \leq C\mathrm{e}^{-\min(\{|g'(\underline{u}_{-\infty})|, |g'(\underline{u}_{+\infty})|\})\varepsilon t}\|w\|_{W^{1,\infty}(\mathbf{R})};$$

(2) *for any $\theta > 0$ there exist positive $(C_\theta, \omega_\theta)$ such that for any $t \geq 0$, any $0 \leq \varepsilon \leq \varepsilon_0$ and any $w \in \mathrm{BUC}^0(\mathbf{R})$,*

$$\|\tilde{S}_\varepsilon(t)(w)\|_{L^\infty(\mathbf{R})} + \min(\{1, \sqrt{t}\})\|\partial_x \tilde{S}_\varepsilon(t)(w)\|_{L^\infty(\mathbf{R})}$$
$$+ |\partial_t s_\varepsilon^\mathrm{p}(t)(w)|$$
$$\leq C_\theta e^{-\omega_\theta t}\|e^{\theta|\cdot|}w\|_{L^\infty(\mathbf{R})}.$$

Estimates on operators are derived through pointwise bounds on Green kernels from the trivial fact that if $\mathbf{T}$ is defined through

$$\mathbf{T}(w)(x) = \int_{\mathbf{R}} \mathbf{K}(x, y)w(y)\,\mathrm{d}y$$

then

$$\|\mathbf{T}(w)\|_{L^\infty(\mathbf{R})} \leq \|\mathbf{K}\|_{L^\infty_x(L^1_y)}\|w\|_{L^\infty(\mathbf{R})}.$$

4.1. Auxiliary lemmas

To begin with, to gain a practical grasp on the way the placement of spectral curves impacts decay rates, we provide two lemmas that will be of ubiquitous use when establishing pointwise bounds on Green functions.

Both lemmas are motivated by the fact that when $\beta \geq 0$ and $t > 0$, the minimization of

$$\Lambda_0 t + \left(\frac{\alpha}{2} - \sqrt{\frac{\alpha^2}{4} - b + \Lambda_0}\right)\beta$$

over $\Lambda_0 \in (-\frac{\alpha^2}{4} + b, +\infty)$ is equivalent to

$$2\sqrt{\frac{\alpha^2}{4} - b + \Lambda_0} = \frac{\beta}{t}. \tag{4.1}$$

The first lemma directly elucidates the consequences of this choice of Λ_0 in the approximate saddle-point method sketched above.

Lemma 1. *Let $t > 0$, $\alpha \in \mathbf{R}$, $\beta \geq 0$, $\beta_0 \geq 0$, $b < 0$ and $(\zeta_-, \zeta_+) \in \mathbf{C}^2$ such that*

$$\Re(\zeta_\pm) > |\Im(\zeta_\pm)|, \quad \mp\Im(\zeta_\pm) > 0.$$

Then the curve $\Lambda: \mathbf{R} \to \mathbf{C}$ defined through[13]

$$2\sqrt{\frac{\alpha^2}{4} - b + \Lambda(\xi)} = \frac{\beta_0}{t} + i\xi\zeta_{\mathrm{sgn}(\xi)}$$

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[13]Sign conditions on $\Im(\zeta_\pm)$ ensure that this is a licit definition.

satisfies for any $\xi \in \mathbf{R}$, when either $\beta = \beta_0$ or ($\beta \geq \beta_0$ and $\alpha \leq 0$),

$$\Re\left(\Lambda(\xi)t + \left(\frac{\alpha}{2} - \sqrt{\frac{\alpha^2}{4} - b + \Lambda(\xi)}\right)\beta\right)$$

$$\leq -\left(\frac{\alpha^2}{4} + |b|\right)t + \frac{\alpha}{2}\beta_0 - \frac{\beta_0^2}{4t} - \frac{\xi^2}{4}\Re(\zeta_{\mathrm{sgn}(\xi)}^2)t$$

$$= -|b|t - \frac{(\beta_0 - \alpha t)^2}{4t} - \frac{\xi^2}{4}\Re(\zeta_{\mathrm{sgn}(\xi)}^2)t,$$

and for any $\xi \in \mathbf{R}^*$,

$$|\Lambda'(\xi)| \leq |\zeta_{\mathrm{sgn}(\xi)}|\left(1 + \frac{\Re(\zeta_{\mathrm{sgn}(\xi)})}{|\Im(\zeta_{\mathrm{sgn}(\xi)})|}\right)\Re\left(\sqrt{\frac{\alpha^2}{4} - b + \Lambda(\xi)}\right).$$

We omit the proof of Lemma 1 as straightforward and elementary.

The second lemma is designed to deal with cases when the natural choice (4.1) is not available because of extra constraints arising from Evans functions' possible annulation in $(-\frac{\alpha^2}{4} + b, 0)$. Explicitly, it focuses on the case when $\beta/t \leq \omega_0$ when ω_0 is typically picked as either $\omega_r^{\eta_0}$ or $\omega_r^{\eta_0}$ with

$$\omega_r^{\eta_0} := 2\sqrt{\frac{(f'(\underline{u}_{+\infty}) - \sigma_\varepsilon)^2}{4} - \frac{\eta_0}{2}},$$

$$\omega_\ell^{\eta_0} := 2\sqrt{\frac{(f'(\underline{u}_{-\infty}) - \sigma_\varepsilon)^2}{4} - \frac{\eta_0}{2}}, \tag{4.2}$$

where η_0 is as in Proposition 4. Since $\beta/t \leq \omega_0$ should be thought of as a bounded-domain restriction, it is useful to let the second lemma also encode the possible trade-off between spatial localization and time decay.

Lemma 2. *Let $t > 0$, $\alpha \in \mathbf{R}$, $\beta \geq 0$, $b < 0$, $(\zeta_-, \zeta_+) \in \mathbf{C}^2$ such that*

$$\Re(\zeta_\pm) > |\Im(\zeta_\pm)|, \quad \mp\Im(\zeta_\pm) > 0,$$

and $\omega_0 \geq 0$ such that

$$\beta \leq \omega_0 t.$$

Then the curve $\Lambda: \mathbf{R} \to \mathbf{C}$, defined through

$$2\sqrt{\frac{\alpha^2}{4} - b + \Lambda(\xi)} = \omega_0 + i\xi\zeta_{\mathrm{sgn}(\xi)},$$

satisfies for any $\xi \in \mathbf{R}^*$,

$$|\Lambda'(\xi)| \leq |\zeta_{\mathrm{sgn}(\xi)}|\left(1 + \frac{\Re(\zeta_{\mathrm{sgn}(\xi)})}{|\Im(\zeta_{\mathrm{sgn}(\xi)})|}\right)\Re\left(\sqrt{\frac{\alpha^2}{4} - b + \Lambda(\xi)}\right),$$

and for any $\xi \in \mathbf{R}$ and $\eta > 0$,

$$\Re\left(\Lambda(\xi)t + \left(\frac{\alpha}{2} - \sqrt{\frac{\alpha^2}{4} - b + \Lambda(\xi)}\right)\beta\right)$$

$$\leq -\left(\frac{\alpha^2 - (1+\eta)\omega_0^2}{4} + |b|\right)t - \frac{\omega_0 - \alpha}{2}\beta$$

$$- \frac{\xi^2}{4}\left(\Re(\zeta_{\mathrm{sgn}(\xi)}^2) - \frac{1}{\eta}|\Im(\zeta_{\mathrm{sgn}(\xi)})|^2\right)t,$$

and, when moreover $\omega_0 < |\alpha|$,

$$\Re\left(\Lambda(\xi)t + \left(\frac{\alpha}{2} - \sqrt{\frac{\alpha^2}{4} - b + \Lambda(\xi)}\right)\beta\right)$$

$$\leq -|b|t - \frac{(|\alpha|t - \beta)^2}{4t}\left(1 - (1+\eta)\frac{\omega_0^2}{\alpha^2}\right)$$

$$- \frac{\xi^2}{4}\left(\Re(\zeta_{\mathrm{sgn}(\xi)}^2) - \frac{1}{\eta}|\Im(\zeta_{\mathrm{sgn}(\xi)})|^2\right)t.$$

Note that to guarantee for some $\eta > 0$ both

$$\Re(\zeta_{\mathrm{sgn}(\xi)}^2) > \frac{1}{\eta}|\Im(\zeta_{\mathrm{sgn}(\xi)})|^2 \quad \text{and} \quad 1 > (1+\eta)\frac{\omega_0^2}{\alpha^2},$$

one needs to enforce

$$\Re(\zeta_{\mathrm{sgn}(\xi)}) > \frac{|\alpha|}{\sqrt{\alpha^2 - \omega_0^2}}|\Im(\zeta_{\mathrm{sgn}(\xi)})|. \tag{4.3}$$

Likewise when $\omega_0 > |\alpha|$, one may extract large-time decay for $|\xi| \geq \xi_0 > 0$ provided that $\Re(\zeta_{\mathrm{sgn}(\xi)})$ is sufficiently large.

Proof of Lemma 2. The starting point is that for any $\eta > 0$,

$$\Re\left(\Lambda(\xi)t + \left(\frac{\alpha}{2} - \sqrt{\frac{\alpha^2}{4} - b + \Lambda(\xi)}\right)\beta\right)$$

$$\leq -\left(\frac{\alpha^2 - \omega_0^2 - \eta(\omega_0 - \frac{\beta}{t})^2}{4} + |b|\right)t - \frac{\omega_0 - \alpha}{2}\beta$$

$$- \frac{\xi^2}{4}\left(\Re(\zeta_{\mathrm{sgn}(\xi)}^2) - \frac{1}{\eta}|\Im(\zeta_{\mathrm{sgn}(\xi)})|^2\right)t$$

$$= -|b|t - \frac{(\alpha t - \beta)^2}{4t} - \frac{\xi^2}{4}\left(\Re(\zeta_{\mathrm{sgn}(\xi)}^2) - \frac{1}{\eta}|\Im(\zeta_{\mathrm{sgn}(\xi)})|^2\right)t$$

$$+ \frac{(\omega_0 t - \beta)^2}{4t}(1+\eta).$$

The first bound on the real part is then obtained by using the first formulation of the foregoing bound jointly with

$$\left(\omega_0 - \frac{\beta}{t}\right)^2 \leq \omega_0^2,$$

whereas the second bound, specialized to the case $|\alpha| > \omega_0$, stems from the second formulation and

$$(|\alpha|t - \beta)^2 \leq (\alpha t - \beta)^2, \quad (\omega_0 t - \beta)^2 \leq \frac{\omega_0^2}{\alpha^2}(|\alpha|t - \beta)^2. \qquad \blacksquare$$

4.2. First separations

We would like to split $G_t^\varepsilon(x, y)$ into pieces corresponding to different behaviors. Yet we must take into account that our description of $G_\varepsilon(\lambda; x, y)$ is different in high-frequency and compact regimes. To do so, we pick some curves and break them into pieces.

Explicitly, motivated by (4.3) with ω_0 either $\omega_r^{\eta_0}$ or $\omega_r^{\eta_0}$ – defined in (4.2) with η_0 as in Proposition 4 – we first choose $\zeta_\pm^{\mathrm{HF}}$ such that

$$\Re(\zeta_\pm^{\mathrm{HF}}) \geq 2\frac{\max(\{|f'(\underline{u}_{+\infty}) - \sigma_\varepsilon|, f'(\underline{u}_{-\infty}) - \sigma_\varepsilon\})}{\sqrt{2\eta_0}}|\Im(\zeta_\pm^{\mathrm{HF}})|, \quad \mp\Im(\zeta_\pm^{\mathrm{HF}}) > 0.$$

Then we define curves Λ_ε^r, Λ_ε^ℓ through

$$2\sqrt{\frac{(f'(\underline{u}_{+\infty}) - \sigma_\varepsilon)^2}{4} - \varepsilon g'(\underline{u}_{+\infty})} + \Lambda_\varepsilon^r(\xi) = \omega_r^{\mathrm{HF}} + i\xi\zeta_{\mathrm{sgn}(\xi)}^{\mathrm{HF}},$$

$$2\sqrt{\frac{(f'(\underline{u}_{-\infty}) - \sigma_\varepsilon)^2}{4} - \varepsilon g'(\underline{u}_{-\infty})} + \Lambda_\varepsilon^\ell(\xi) = \omega_\ell^{\mathrm{HF}} + i\xi\zeta_{\mathrm{sgn}(\xi)}^{\mathrm{HF}},$$

where ω_r^{HF} and $\omega_\ell^{\mathrm{HF}}$ are fixed such that

$$\omega_r^{\mathrm{HF}} > |f'(\underline{u}_{+\infty}) - \sigma_0|, \quad \omega_\ell^{\mathrm{HF}} > |f'(\underline{u}_{-\infty}) - \sigma_0|.$$

Note that this is sufficient to guarantee that both curves satisfy requirements ensuring (3.2) thus also (3.3).

We shall do a particular treatment of the parts of the curves corresponding to $|\xi| \leq \xi^{\mathrm{HF}}$ where we choose ξ^{HF} as

$$\xi^{\mathrm{HF}} := \frac{2\max(\{\omega_r^{\mathrm{HF}} - \omega_r^{\eta_0}, \omega_\ell^{\mathrm{HF}} - \omega_\ell^{\eta_0}\})}{\min(\{|\Im(\zeta_+^{\mathrm{HF}})|, |\Im(\zeta_-^{\mathrm{HF}})|\})}.$$

Once again the motivation for the definition of ξ^{HF} stems from Lemma 2. Indeed, the definition ensures that for $\omega \in \mathbf{R}$, a curve Λ, defined through

$$2\sqrt{\frac{(f'(\underline{u}_{+\infty}) - \sigma_\varepsilon)^2}{4} - \varepsilon g'(\underline{u}_{+\infty})} + \Lambda(\xi) = \omega + i\xi\zeta_{r,\mathrm{sgn}(\xi)}^\omega,$$

respectively through

$$2\sqrt{\frac{(f'(\underline{u}_{-\infty}) - \sigma_\varepsilon)^2}{4} - \varepsilon g'(\underline{u}_{-\infty})} + \Lambda(\xi) = \omega + i\xi\zeta_{\ell,\mathrm{sgn}(\xi)}^\omega,$$

with

$$\zeta_{\sharp,\pm}^{\omega} := \Re(\zeta_{\pm}^{\mathrm{HF}}) + \mathrm{i}\left(\Im(\zeta_{\pm}^{\mathrm{HF}}) \mp \frac{\omega_{\sharp}^{\mathrm{HF}} - \omega}{\xi^{\mathrm{HF}}}\right), \quad \sharp \in \{r, \ell\},$$

satisfies

$$\Lambda(\pm\xi^{\mathrm{HF}}) = \Lambda_{\varepsilon}^r(\pm\xi^{\mathrm{HF}}), \quad \text{respectively } \Lambda(\pm\xi^{\mathrm{HF}}) = \Lambda_{\varepsilon}^{\ell}(\pm\xi^{\mathrm{HF}}),$$

whereas, for $\sharp \in \{r, \ell\}$, $\omega \in [\omega_{\sharp}^{\eta_0}, \omega_{\sharp}^{\mathrm{HF}}]$,

$$\Re(\zeta_{\sharp,\pm}^{\omega}) = \Re(\zeta_{\pm}^{\mathrm{HF}}), \quad \mp\Im(\zeta_{\sharp,\pm}^{\omega}) > 0, \quad |\Im(\zeta_{\sharp,\pm}^{\omega})| \leq \frac{3}{2}|\Im(\zeta_{\pm}^{\mathrm{HF}})|.$$

In the following, for $\sharp \in \{r, \ell\}$, we use the notation $\Lambda_{\varepsilon}^{\sharp,\mathrm{LF}} := (\Lambda_{\varepsilon}^{\sharp})_{|[-\xi^{\mathrm{HF}}, \xi^{\mathrm{HF}}]}$ and $\Lambda_{\varepsilon}^{\sharp,\mathrm{HF}} := (\Lambda_{\varepsilon}^{\sharp})_{|\mathbf{R}\setminus[-\xi^{\mathrm{HF}}, \xi^{\mathrm{HF}}]}$.

To ensure that Lemma 2 provides exponential time decay for the part of the evolution arising from $\Lambda_{\varepsilon}^{\sharp,\mathrm{HF}}$, we reinforce the constraint on $\Re(\zeta_{\pm}^{\mathrm{HF}})$ by adding

$$\Re(\zeta_{\pm}^{\mathrm{HF}}) \geq \sqrt{\Im(\zeta_{\pm}^{\mathrm{HF}})^2 + 2\frac{\max(\{(\omega_r^{\mathrm{HF}})^2 - (f'(\underline{u}_{+\infty}) - \sigma_{\varepsilon})^2, (\omega_{\ell}^{\mathrm{HF}})^2 - (f'(\underline{u}_{-\infty}) - \sigma_{\varepsilon})^2\})}{(\xi^{\mathrm{HF}})^2}}.$$

After these preliminaries, to account for different behaviors, when $t > 0$ we break G_t^{ε} as

$$G_t^{\varepsilon} = G_t^{\varepsilon,\mathrm{pt}} + G_t^{\varepsilon,\mathrm{ess}},$$

with $G_t^{\varepsilon,\mathrm{pt}}$ and $G_t^{\varepsilon,\mathrm{ess}}$ defined as follows. First,

$$G_t^{\varepsilon,\mathrm{pt}}(x, y) = 0 \qquad\qquad \text{if } xy > 0 \text{ and } (y \geq \omega_r^{\mathrm{HF}}t \text{ or } y \leq -\omega_{\ell}^{\mathrm{HF}}t),$$

$$G_t^{\varepsilon,\mathrm{pt}}(x, y) = \frac{1}{2\mathrm{i}\pi} \int_{\Lambda} \mathrm{e}^{\lambda t} G_{\varepsilon}(\lambda; x, y) \, \mathrm{d}\lambda \quad \text{if } xy < 0,$$

and when $xy > 0$ and $-\omega_{\ell}^{\mathrm{HF}}t < y < \omega_r^{\mathrm{HF}}t$,

$$G_t^{\varepsilon,\mathrm{pt}}(x, y)$$

$$= \begin{cases} \dfrac{1}{2\mathrm{i}\pi} \displaystyle\int_{\Lambda_{\varepsilon}^{r,\mathrm{LF}}} \mathrm{e}^{\lambda t} \mathrm{e}^{\int_{-\infty}^0 (f'(\underline{U}_{\varepsilon}) - f'(\underline{u}_{-\infty}))} \mathbf{e}_1 \cdot \mathbf{V}_{\varepsilon}^{r,s}(\lambda, x) \dfrac{\tilde{\tau}_{\varepsilon}^{\ell}(\lambda)}{D_{\varepsilon}(\lambda)} \mathbf{e}_1 \cdot \tilde{\mathbf{V}}_{\varepsilon}^{r,s}(\lambda, y) \, \mathrm{d}\lambda \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } x > y > 0, \\[1em] \dfrac{1}{2\mathrm{i}\pi} \displaystyle\int_{\Lambda_{\varepsilon}^{\ell,\mathrm{LF}}} \mathrm{e}^{\lambda t} \mathrm{e}^{\int_{-\infty}^0 (f'(\underline{U}_{\varepsilon}) - f'(\underline{u}_{-\infty}))} \dfrac{\tau_{\varepsilon}^r(\lambda)}{D_{\varepsilon}(\lambda)} \mathbf{e}_1 \cdot \mathbf{V}_{\varepsilon}^{\ell,u}(\lambda, x) \mathbf{e}_1 \cdot \tilde{\mathbf{V}}_{\varepsilon}^{\ell,u}(\lambda, y) \, \mathrm{d}\lambda \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } 0 > x > y, \\[1em] \dfrac{1}{2\mathrm{i}\pi} \displaystyle\int_{\Lambda_{\varepsilon}^{\ell,\mathrm{LF}}} \mathrm{e}^{\lambda t} \mathrm{e}^{-\int_0^{+\infty} (f'(\underline{U}_{\varepsilon}) - f'(\underline{u}_{+\infty}))} \mathbf{e}_1 \cdot \mathbf{V}_{\varepsilon}^{\ell,u}(\lambda, x) \dfrac{\tilde{\tau}_{\varepsilon}^r(\lambda)}{D_{\varepsilon}(\lambda)} \mathbf{e}_1 \cdot \tilde{\mathbf{V}}_{\varepsilon}^{\ell,u}(\lambda, y) \, \mathrm{d}\lambda \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } 0 > y > x, \\[1em] \dfrac{1}{2\mathrm{i}\pi} \displaystyle\int_{\Lambda_{\varepsilon}^{r,\mathrm{LF}}} \mathrm{e}^{\lambda t} \mathrm{e}^{-\int_0^{+\infty} (f'(\underline{U}_{\varepsilon}) - f'(\underline{u}_{+\infty}))} \dfrac{\tau_{\varepsilon}^{\ell}(\lambda)}{D_{\varepsilon}(\lambda)} \mathbf{e}_1 \cdot \mathbf{V}_{\varepsilon}^{r,s}(\lambda, x) \mathbf{e}_1 \cdot \tilde{\mathbf{V}}_{\varepsilon}^{r,s}(\lambda, y) \, \mathrm{d}\lambda \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } y > x > 0, \end{cases}$$

where, here and in the definition of $G_t^{\varepsilon,\text{ess}}$, Λ is either $\Lambda = \Lambda_\varepsilon^r$ or $\Lambda = \Lambda_\varepsilon^\ell$, and we use compact notation for integrals over curves instead of explicitly parametrized versions. Second,

$$G_t^{\varepsilon,\text{ess}}(x,y) = \begin{cases} 0 & \text{if } xy < 0, \\ \dfrac{1}{2i\pi} \displaystyle\int_\Lambda e^{\lambda t} G_\varepsilon(\lambda; x, y)\, d\lambda & \text{if } xy > 0 \text{ and } (y \geq \omega_r^{\text{HF}} t \text{ or } y \leq -\omega_\ell^{\text{HF}} t), \end{cases}$$

and when $xy > 0$ and $-\omega_\ell^{\text{HF}} t < y < \omega_r^{\text{HF}} t$, we have that $G_t^{\varepsilon,\text{ess}}(x,y)$ equals

$$\begin{cases} \dfrac{1}{2i\pi} \displaystyle\int_{\Lambda_\varepsilon^{r,\text{LF}}} e^{\lambda t} e^{\int_{-\infty}^0 (f'(\underline{U}_\varepsilon) - f'(\underline{u}_{-\infty}))}\, \mathbf{e}_1 \cdot \mathbf{V}_\varepsilon^{r,s}(\lambda, x) \dfrac{\tilde{\rho}_\varepsilon^\ell(\lambda)}{D_\varepsilon(\lambda)}\, \mathbf{e}_1 \cdot \tilde{\mathbf{V}}_\varepsilon^{r,u}(\lambda, y)\, d\lambda \\ \qquad + \dfrac{1}{2i\pi} \displaystyle\int_{\Lambda_\varepsilon^{r,\text{HF}}} e^{\lambda t} G_\varepsilon(\lambda; x, y)\, d\lambda & \text{if } x > y > 0, \\[2ex] \dfrac{1}{2i\pi} \displaystyle\int_{\Lambda_\varepsilon^{\ell,\text{LF}}} e^{\lambda t} e^{\int_{-\infty}^0 (f'(\underline{U}_\varepsilon) - f'(\underline{u}_{-\infty}))} \dfrac{\rho_\varepsilon^r(\lambda)}{D_\varepsilon(\lambda)}\, \mathbf{e}_1 \cdot \mathbf{V}_\varepsilon^{\ell,s}(\lambda, x)\, \mathbf{e}_1 \cdot \tilde{\mathbf{V}}_\varepsilon^{\ell,u}(\lambda, y)\, d\lambda \\ \qquad + \dfrac{1}{2i\pi} \displaystyle\int_{\Lambda_\varepsilon^{\ell,\text{HF}}} e^{\lambda t} G_\varepsilon(\lambda; x, y)\, d\lambda & \text{if } 0 > x > y, \\[2ex] \dfrac{1}{2i\pi} \displaystyle\int_{\Lambda_\varepsilon^{\ell,\text{LF}}} e^{\lambda t} e^{-\int_0^{+\infty} (f'(\underline{U}_\varepsilon) - f'(\underline{u}_{+\infty}))}\, \mathbf{e}_1 \cdot \mathbf{V}_\varepsilon^{\ell,u}(\lambda, x) \dfrac{\tilde{\rho}_\varepsilon^r(\lambda)}{D_\varepsilon(\lambda)}\, \mathbf{e}_1 \cdot \tilde{\mathbf{V}}_\varepsilon^{\ell,s}(\lambda, y)\, d\lambda \\ \qquad + \dfrac{1}{2i\pi} \displaystyle\int_{\Lambda_\varepsilon^{\ell,\text{HF}}} e^{\lambda t} G_\varepsilon(\lambda; x, y)\, d\lambda & \text{if } 0 > y > x, \\[2ex] \dfrac{1}{2i\pi} \displaystyle\int_{\Lambda_\varepsilon^{r,\text{LF}}} e^{\lambda t} e^{-\int_0^{+\infty} (f'(\underline{U}_\varepsilon) - f'(\underline{u}_{+\infty}))} \dfrac{\rho_\varepsilon^\ell(\lambda)}{D_\varepsilon(\lambda)}\, \mathbf{e}_1 \cdot \mathbf{V}_\varepsilon^{r,u}(\lambda, x)\, \mathbf{e}_1 \cdot \tilde{\mathbf{V}}_\varepsilon^{r,s}(\lambda, y)\, d\lambda \\ \qquad + \dfrac{1}{2i\pi} \displaystyle\int_{\Lambda_\varepsilon^{r,\text{HF}}} e^{\lambda t} G_\varepsilon(\lambda; x, y)\, d\lambda & \text{if } y > x > 0. \end{cases}$$

Note that the above splitting implies

$$\partial_x G_t^\varepsilon = \partial_x G_t^{\varepsilon,\text{pt}} + \partial_x G_t^{\varepsilon,\text{ess}},$$

where here and elsewhere throughout the text, ∂_x acting on either $G_t^{\varepsilon,\text{pt}}$ or $G_t^{\varepsilon,\text{ess}}$ is understood as a pointwise derivative wherever these functions are continuous.

The rationale behind the splitting is that the large-time decay of $G_t^{\varepsilon,\text{ess}}$ is essentially limited by spatial decay, hence may be thought of as purely explained by essential spectrum considerations, whereas the large-time asymptotics of $G_t^{\varepsilon,\text{pt}}$ is driven by the presence near the spectral curves of a root of D_ε at $\lambda = 0$, hence is due to the interaction of essential and point spectra.

Some extra complications in the splitting are due to the fact that we need to prepare the identification of the most singular part as a phase modulation, which comes into a tensorized form. This explains why we define zones in terms of the size of $|y|$, instead of the otherwise more natural $|x - y|$.

4.3. First pointwise bounds

We begin our use of Lemmas 1 and 2 with short-time bounds.

Lemma 3. *There exist positive* $(\varepsilon_0, C, \omega, \theta)$ *such that for any* $t > 0$*, any* $0 \leq \varepsilon \leq \varepsilon_0$ *and any* $(x, y) \in \mathbf{R}^2$,

$$|G_t^{\varepsilon,\mathrm{pt}}(x, y)| + \min(\{1, \sqrt{t}\})|\partial_x G_t^{\varepsilon,\mathrm{pt}}(x, y)| \leq C\,\mathrm{e}^{\omega t}\mathrm{e}^{-\theta|x|}\frac{1}{\sqrt{t}}\mathrm{e}^{-\theta\frac{y^2}{t}}.$$

The foregoing lemma does not contain estimates on $G_t^{\varepsilon,\mathrm{ess}}$ because those would be redundant with the corresponding large-time estimates. The point of Lemma 3 is to show that for short-time estimates, the singularity at $\lambda = 0$ may be avoided, whereas this singularity is not present in $G_t^{\varepsilon,\mathrm{ess}}$.

Proof of Lemma 3. To bound $G_t^{\varepsilon,\mathrm{pt}}(x, y)$ when $xy < 0$, we separate $x > 0 > y$ and $x < 0 < y$. The analyses being completely similar, we only discuss here the former case. To treat it, we move curves as in Lemmas 1 and 2 with $\beta_0 = \beta = |y|$, $\alpha = f'(\underline{u}_{-\infty}) - \sigma_\varepsilon$, $b = \varepsilon g'(\underline{u}_{-\infty})$ and note that

$$\Re(\mu_-^\varepsilon(\lambda, \underline{u}_{+\infty})) \leq \frac{1}{2}(f'(\underline{u}_{+\infty}) - \sigma_\varepsilon) < 0.$$

More explicitly, we use Lemma 1 to bound the regime $|y| \geq \omega_\ell^{\mathrm{HF}}t$, which leads to the claimed heat-like bound since

$$\mathrm{e}^{-\frac{(|y|-\alpha t)^2}{4t}} \leq \mathrm{e}^{-\left(1-\frac{|\alpha|}{\omega_\ell^{\mathrm{HF}}}\right)\frac{y^2}{4t}}, \quad |y| \geq \omega_\ell^{\mathrm{HF}}t.$$

In the remaining zone where $|y| \leq \omega_\ell^{\mathrm{HF}}t$ we instead use Lemma 2 to derive a bound that may be converted into a heat-like bound through

$$\mathrm{e}^{-\frac{\omega_\ell^{\mathrm{HF}}-\alpha}{2}|y|} \leq \mathrm{e}^{-\left(1-\frac{|\alpha|}{\omega_\ell^{\mathrm{HF}}}\right)\frac{y^2}{2t}}, \quad |y| \leq \omega_\ell^{\mathrm{HF}}t.$$

The estimates on $G_t^{\varepsilon,\mathrm{pt}}(x, y)$ when $xy > 0$ are obtained in exactly the same way. ∎

We proceed with bounds on $G_t^{\varepsilon,\mathrm{ess}}$.

Lemma 4. *There exist positive* $(\varepsilon_0, C, \omega, \theta)$ *such that for any* $t > 0$*, any* $0 \leq \varepsilon \leq \varepsilon_0$ *and any* $(x, y) \in \mathbf{R}^2$,

$$\begin{aligned}
|G_t^{\varepsilon,\mathrm{ess}}&(x, y)| + \min(\{1, \sqrt{t}\})|\partial_x G_t^{\varepsilon,\mathrm{ess}}(x, y)| \\
&\leq C\mathbf{1}_{|x-y|\leq|y|}\mathrm{e}^{-\min(\{|g'(\underline{u}_{-\infty})|,|g'(\underline{u}_{+\infty})|\})\varepsilon t} \\
&\quad \times \frac{1}{\sqrt{t}}\left(\mathrm{e}^{-\theta\frac{|x-y-(f'(\underline{u}_{+\infty})-\sigma_\varepsilon)t|^2}{t}} + \mathrm{e}^{-\theta\frac{|x-y-(f'(\underline{u}_{-\infty})-\sigma_\varepsilon)t|^2}{t}}\right) \\
&\quad + C\mathrm{e}^{-\omega t}\frac{1}{\sqrt{t}}\mathrm{e}^{-\theta\frac{y^2}{t}}.
\end{aligned}$$

This also implies that there exist positive $(\varepsilon_0, C, \theta)$ such that for any $\theta' > 0$ there exists $\omega' > 0$ such that for any $t > 0$, any $0 \leq \varepsilon \leq \varepsilon_0$ and any $(x, y) \in \mathbf{R}^2$,

$$
e^{-\theta'|y|}\left(|G_t^{\varepsilon,\mathrm{ess}}(x, y)| + \min(\{1, \sqrt{t}\})|\partial_x G_t^{\varepsilon,\mathrm{ess}}(x, y)|\right)
$$
$$
\leq C e^{-\omega' t} \frac{1}{\sqrt{t}}\left(e^{-\theta \frac{|x-y-(f'(\underline{u}_{+\infty})-\sigma_\varepsilon)t|^2}{t}} + e^{-\theta \frac{|x-y-(f'(\underline{u}_{-\infty})-\sigma_\varepsilon)t|^2}{t}} + e^{-\theta \frac{y^2}{t}}\right).
$$

In the foregoing statement and throughout the text we use $\mathbf{1}_A$ to denote a characteristic function for the condition A.

Proof of Lemma 4. To deduce the second bound from the first we observe that for any α

$$
\frac{\theta}{2t}|x - y - \alpha t|^2 + \theta'|x - y| \geq
\begin{cases}
\dfrac{1}{2}\theta'|\alpha| t & \text{if } |x - y - \alpha t| \leq \dfrac{1}{2}|\alpha| t, \\
\dfrac{\theta}{4}|\alpha|^2 t & \text{if } |x - y - \alpha t| \geq \dfrac{1}{2}|\alpha| t.
\end{cases}
$$

To prove the first bound we should distinguish between regimes defined by $0 < y < x$, $y < x < 0$, $0 > y > x$ and $y > x > 0$. Regimes $0 < y < x$ and $0 > y > x$ on one hand and $y < x < 0$ and $y > x > 0$ on the other hand may be treated similarly and we give details only for the cases $y < x < 0$ and $0 < y < x$.

Note that when $y < x < 0$, we have $|x - y| \leq |y|$. When $y < x < 0$ and $|y| \geq \omega_\ell^{\mathrm{HF}} t$, we decompose the Green function into two parts according to

$$
\mathbf{V}_\varepsilon^{\ell,s}(\lambda, x) = \rho_\varepsilon^r(\lambda)\mathbf{V}_\varepsilon^{\ell,s}(\lambda, x) + \tau_\varepsilon^r(\lambda)\mathbf{V}_\varepsilon^{\ell,u}(\lambda, x).
$$

The τ_ε^r-contribution is bounded by choosing the curve according to Lemma 1 with

$$
\beta_0 = \beta = |y|, \quad \alpha = f'(\underline{u}_{-\infty}) - \sigma_\varepsilon, \quad \zeta_\pm = \zeta_{\ell,\pm}^{|y|/t} \quad \text{and} \quad b = \varepsilon g'(\underline{u}_{-\infty}).
$$

As for the ρ_ε^r-contribution, we choose the curve according to Lemma 1 with

$$
\beta_0 = \beta = |x - y|, \quad \alpha = \sigma_\varepsilon - f'(\underline{u}_{-\infty}), \quad \zeta_\pm = \zeta_{\ell,\pm}^{|x-y|/t} \quad \text{and} \quad b = \varepsilon g'(\underline{u}_{-\infty})
$$

when $|x - y| \geq t\omega_\ell^{\eta_0}$, and according to Lemma 2 with

$$
\omega_0 = \omega_\ell^{\eta_0}, \quad \zeta_\pm = \zeta_{\ell,\pm}^{\omega_\ell^{\eta_0}}, \quad \beta = |x - y| \quad \text{and} \quad \alpha = f'(\underline{u}_{-\infty}) - \sigma_\varepsilon
$$

otherwise. To analyze the regime when $y < x < 0$ and $|y| < \omega_\ell^{\mathrm{HF}} t$, we never move the curve $\Lambda_\varepsilon^{\ell,\mathrm{HF}}$ (but bound its contribution according to Lemma 2), whereas we move $\Lambda_\varepsilon^{\ell,\mathrm{LF}}$ as in Lemma 1 with $\zeta_\pm = \zeta_{\ell,\pm}^{|x-y|/t}$ when $|x - y| \geq \omega_\ell^{\eta_0} t$ or as in Lemma 2 with $\zeta_\pm = \zeta_{\ell,\pm}^{\omega_\ell^{\eta_0}}$ when $|x - y| \leq \omega_\ell^{\eta_0} t$.

To bound the contribution of the regime $0 < y < x$, we may proceed as when $y < x < 0$ provided that $|x - y| \leq |y|$ or $-\omega_\ell^{\mathrm{HF}} t \leq y \leq \omega_r^{\mathrm{HF}} t$. The remaining case is dealt with by applying Lemma 1 with $\beta_0 = |y|$ and $\beta = |x - y|$ using the fact that $f'(\underline{u}_{+\infty}) - \sigma_\varepsilon < 0$. ∎

4.4. Linear phase separation

The large-time estimates for $G_t^{\varepsilon,\mathrm{pt}}$ require a phase separation. To carry it out we first recall that there exist $(a_\varepsilon^r, a_\varepsilon^\ell) \in \mathbf{R}^2$, each uniformly bounded from below and above, such that

$$\mathbf{e}_1 \cdot \mathbf{V}_\varepsilon^{r,s}(0,\cdot) = a_\varepsilon^r \underline{U}_\varepsilon',$$

$$\mathbf{e}_1 \cdot \mathbf{V}_\varepsilon^{\ell,u}(0,\cdot) = a_\varepsilon^\ell \underline{U}_\varepsilon'.$$

Then we split $G_t^{\varepsilon,\mathrm{pt}}$ as

$$G_t^{\varepsilon,\mathrm{pt}}(x, y) = \underline{U}_\varepsilon'(x) G_t^{\varepsilon,\mathrm{p}}(y) + \widetilde{G}_t^{\varepsilon,\mathrm{pt}}(x, y)$$

with

$$G_t^{\varepsilon,\mathrm{p}}(y) = 0 \quad \text{if } (y \geq \omega_r^{\mathrm{HF}} t \text{ or } y \leq -\omega_\ell^{\mathrm{HF}} t),$$

whereas when $-\omega_\ell^{\mathrm{HF}} t < y < \omega_r^{\mathrm{HF}} t$,

$$G_t^{\varepsilon,\mathrm{p}}(y) = \begin{cases} \dfrac{1}{2i\pi} \displaystyle\int_{\Lambda_\varepsilon^{r,\mathrm{LF}}} e^{\lambda t} e^{-\int_0^{+\infty}(f'(\underline{U}_\varepsilon) - f'(\underline{u}_{+\infty}))} a_\varepsilon^\ell \, \mathbf{e}_1 \cdot \widetilde{\mathbf{V}}_\varepsilon^{r,s}(\lambda, y) \dfrac{d\lambda}{D_\varepsilon(\lambda)} & \text{if } y > 0, \\[4ex] \dfrac{1}{2i\pi} \displaystyle\int_{\Lambda_\varepsilon^{\ell,\mathrm{LF}}} e^{\lambda t} e^{\int_{-\infty}^0 (f'(\underline{U}_\varepsilon) - f'(\underline{u}_{-\infty}))} a_\varepsilon^r \, \mathbf{e}_1 \cdot \widetilde{\mathbf{V}}_\varepsilon^{\ell,u}(\lambda, y) \dfrac{d\lambda}{D_\varepsilon(\lambda)} & \text{if } y < 0. \end{cases}$$

As a result, when $(y \geq \omega_r^{\mathrm{HF}} t \text{ or } y \leq -\omega_\ell^{\mathrm{HF}} t)$,

$$\widetilde{G}_t^{\varepsilon,\mathrm{pt}}(x, y) = \begin{cases} 0 & \text{if } xy > 0, \\[2ex] \dfrac{1}{2i\pi} \displaystyle\int_{\Lambda} e^{\lambda t} G_\varepsilon(\lambda; x, y) \, d\lambda & \text{if } xy < 0, \end{cases}$$

whereas when $xy < 0$ and $-\omega_\ell^{\mathrm{HF}} t < y < \omega_r^{\mathrm{HF}} t$,

$\widetilde{G}_t^{\varepsilon,\mathrm{pt}}(x, y)$

$$= \begin{cases} \dfrac{1}{2i\pi} \displaystyle\int_{\Lambda_\varepsilon^{r,\mathrm{LF}}} e^{\lambda t + \int_{-\infty}^0 (f'(\underline{U}_\varepsilon) - f'(\underline{u}_{-\infty}))} \, \mathbf{e}_1 \cdot (\mathbf{V}_\varepsilon^{r,s}(\lambda, x) - \mathbf{V}_\varepsilon^{r,s}(0, x)) \dfrac{\mathbf{e}_1 \cdot \widetilde{\mathbf{V}}_\varepsilon^{\ell,u}(\lambda, y) \, d\lambda}{D_\varepsilon(\lambda)} \\[2ex] \qquad + \dfrac{1}{2i\pi} \displaystyle\int_{\Lambda_\varepsilon^{r,\mathrm{HF}}} e^{\lambda t} G_\varepsilon(\lambda; x, y) \, d\lambda & \text{if } x > 0 > y, \\[4ex] \dfrac{1}{2i\pi} \displaystyle\int_{\Lambda_\varepsilon^{\ell,\mathrm{LF}}} e^{\lambda t - \int_0^{+\infty}(f'(\underline{U}_\varepsilon) - f'(\underline{u}_{+\infty}))} \, \mathbf{e}_1 \cdot (\mathbf{V}_\varepsilon^{\ell,u}(\lambda, x) - \mathbf{V}_\varepsilon^{\ell,u}(0, x)) \dfrac{\mathbf{e}_1 \cdot \widetilde{\mathbf{V}}_\varepsilon^{r,s}(\lambda, y) \, d\lambda}{D_\varepsilon(\lambda)} \\[2ex] \qquad + \dfrac{1}{2i\pi} \displaystyle\int_{\Lambda_\varepsilon^{\ell,\mathrm{HF}}} e^{\lambda t} G_\varepsilon(\lambda; x, y) \, d\lambda & \text{if } y > 0 > x, \end{cases}$$

and when $xy > 0$ and $-\omega_\ell^{\mathrm{HF}} t < y < \omega_r^{\mathrm{HF}} t$,

$$
\widetilde{G}_t^{\varepsilon,\mathrm{pt}}(x,y)
$$

$$
= \begin{cases}
\begin{aligned}
&\frac{1}{2\mathrm{i}\pi} \int_{\Lambda_\varepsilon^{r,\mathrm{LF}}} \mathrm{e}^{\lambda t \int_{-\infty}^0 (f'(\underline{U}_\varepsilon) - f'(\underline{u}_{-\infty}))} \big(\tilde{\tau}_\varepsilon^\ell(\lambda)\, \mathbf{e}_1 \cdot \mathbf{V}_\varepsilon^{r,s}(\lambda,x) - \tilde{\tau}_\varepsilon^\ell(0)\, \mathbf{e}_1 \cdot \mathbf{V}_\varepsilon^{r,s}(0,x) \big) \\
&\qquad\qquad \times \frac{\mathbf{e}_1 \cdot \widetilde{\mathbf{V}}_\varepsilon^{r,s}(\lambda,y)\, \mathrm{d}\lambda}{D_\varepsilon(\lambda)} & \text{if } x > y > 0,
\end{aligned} \\[2em]
\begin{aligned}
&\frac{1}{2\mathrm{i}\pi} \int_{\Lambda_\varepsilon^{\ell,\mathrm{LF}}} \mathrm{e}^{\lambda t + \int_{-\infty}^0 (f'(\underline{U}_\varepsilon) - f'(\underline{u}_{-\infty}))} \big(\tau_\varepsilon^r(\lambda)\, \mathbf{e}_1 \cdot \mathbf{V}_\varepsilon^{\ell,u}(\lambda,x) - \tau_\varepsilon^r(0)\, \mathbf{e}_1 \cdot \mathbf{V}_\varepsilon^{\ell,u}(0,x) \big) \\
&\qquad\qquad \times \frac{\mathbf{e}_1 \cdot \widetilde{\mathbf{V}}_\varepsilon^{\ell,u}(\lambda,y)\, \mathrm{d}\lambda}{D_\varepsilon(\lambda)} & \text{if } 0 > x > y,
\end{aligned} \\[2em]
\begin{aligned}
&\frac{1}{2\mathrm{i}\pi} \int_{\Lambda_\varepsilon^{\ell,\mathrm{LF}}} \mathrm{e}^{\lambda t - \int_0^{+\infty} (f'(\underline{U}_\varepsilon) - f'(\underline{u}_{+\infty}))} \big(\tilde{\tau}_\varepsilon^r(\lambda)\, \mathbf{e}_1 \cdot \mathbf{V}_\varepsilon^{\ell,u}(\lambda,x) - \tilde{\tau}_\varepsilon^r(0)\, \mathbf{e}_1 \cdot \mathbf{V}_\varepsilon^{\ell,u}(0,x) \big) \\
&\qquad\qquad \times \frac{\mathbf{e}_1 \cdot \widetilde{\mathbf{V}}_\varepsilon^{\ell,u}(\lambda,y)\, \mathrm{d}\lambda}{D_\varepsilon(\lambda)} & \text{if } 0 > y > x,
\end{aligned} \\[2em]
\begin{aligned}
&\frac{1}{2\mathrm{i}\pi} \int_{\Lambda_\varepsilon^{r,\mathrm{LF}}} \mathrm{e}^{\lambda t - \int_0^{+\infty} (f'(\underline{U}_\varepsilon) - f'(\underline{u}_{+\infty}))} \big(\tau_\varepsilon^\ell(\lambda)\, \mathbf{e}_1 \cdot \mathbf{V}_\varepsilon^{r,s}(\lambda,x) - \tau_\varepsilon^\ell(0)\, \mathbf{e}_1 \cdot \mathbf{V}_\varepsilon^{r,s}(0,x) \big) \\
&\qquad\qquad \times \frac{\mathbf{e}_1 \cdot \widetilde{\mathbf{V}}_\varepsilon^{r,s}(\lambda,y)\, \mathrm{d}\lambda}{D_\varepsilon(\lambda)} & \text{if } y > x > 0,
\end{aligned}
\end{cases}
$$

where, here again, Λ is either $\Lambda = \Lambda_\varepsilon^r$ or $\Lambda = \Lambda_\varepsilon^\ell$.

Proceeding as above, we obtain the following lemmas.

Lemma 5. *There exist positive $(\varepsilon_0, C, \omega, \theta)$ such that for any $t > 0$, any $0 \le \varepsilon \le \varepsilon_0$ and any $(x,y) \in \mathbf{R}^2$,*

$$
|\widetilde{G}_t^{\varepsilon,\mathrm{pt}}(x,y)| + \min(\{1, \sqrt{t}\}) |\partial_x \widetilde{G}_t^{\varepsilon,\mathrm{pt}}(x,y)|
$$
$$
\le C \mathrm{e}^{-\min(\{|g'(\underline{u}_{-\infty})|, |g'(\underline{u}_{+\infty})|\}) \varepsilon t} \mathrm{e}^{-\theta|x|} \frac{1}{\sqrt{t}} \Big(\mathrm{e}^{-\theta \frac{|y - (f'(\underline{u}_{+\infty}) - \sigma_\varepsilon) t|^2}{t}} + \mathrm{e}^{-\theta \frac{|y + (f'(\underline{u}_{-\infty}) - \sigma_\varepsilon) t|^2}{t}} \Big)
$$
$$
+ C \mathrm{e}^{-\omega t} \mathrm{e}^{-\theta|x|} \mathrm{e}^{-\theta|x-y|}.
$$

This also implies that there exist positive $(\varepsilon_0, C, \theta)$ such that for any $\theta' > 0$ there exists $\omega' > 0$ such that for any $t > 0$, any $0 \le \varepsilon \le \varepsilon_0$ and any $(x,y) \in \mathbf{R}^2$,

$$
\mathrm{e}^{-\theta'|y|} \big(|\widetilde{G}_t^{\varepsilon,\mathrm{pt}}(x,y)| + \min(\{1, \sqrt{t}\}) |\partial_x \widetilde{G}_t^{\varepsilon,\mathrm{pt}}(x,y)| \big)
$$
$$
\le C \mathrm{e}^{-\omega' t} \Big(\frac{1}{\sqrt{t}} \big(\mathrm{e}^{-\theta \frac{|y - (f'(\underline{u}_{+\infty}) - \sigma_\varepsilon) t|^2}{t}} + \mathrm{e}^{-\theta \frac{|y + (f'(\underline{u}_{-\infty}) - \sigma_\varepsilon) t|^2}{t}} \big) + \mathrm{e}^{-\theta|x|} \mathrm{e}^{-\theta|x-y|} \mathrm{e}^{-\theta'|y|} \Big).
$$

Lemma 6. *There exist positive $(\varepsilon_0, C, \omega, \theta)$ such that for any $t > 0$, any $0 \le \varepsilon \le \varepsilon_0$ and any $y \in \mathbf{R}$,*

$$
|G_t^{\varepsilon,\mathrm{p}}(y)| \le C \mathrm{e}^{\omega t} \frac{1}{\sqrt{t}} \mathrm{e}^{-\theta \frac{y^2}{t}}.
$$

Moreover, there exist positive $(\varepsilon_0, C, \omega, \theta)$ such that for any $t > 0$, any $0 \leq \varepsilon \leq \varepsilon_0$ and any $y \in \mathbf{R}$,

$$|\partial_t G_t^{\varepsilon,\mathrm{p}}(y)|$$
$$\leq C\mathrm{e}^{-\min(\{|g'(\underline{u}_{-\infty})|,|g'(\underline{u}_{+\infty})|\})\varepsilon t} \frac{1}{\sqrt{t}}\left(\mathrm{e}^{-\theta\frac{|y-(f'(\underline{u}_{+\infty})-\sigma_\varepsilon)t|^2}{t}} + \mathrm{e}^{-\theta\frac{|y+(f'(\underline{u}_{-\infty})-\sigma_\varepsilon)t|^2}{t}}\right).$$

This also implies that there exist positive $(\varepsilon_0, C, \theta)$ such that for any $\theta' > 0$ there exists $\omega' > 0$ such that for any $t > 0$, any $0 \leq \varepsilon \leq \varepsilon_0$ and any $y \in \mathbf{R}$,

$$\mathrm{e}^{-\theta'|y|}|\partial_t G_t^{\varepsilon,\mathrm{p}}(y)| \leq C\mathrm{e}^{-\omega't} \frac{1}{\sqrt{t}}\left(\mathrm{e}^{-\theta\frac{|y-(f'(\underline{u}_{+\infty})-\sigma_\varepsilon)t|^2}{t}} + \mathrm{e}^{-\theta\frac{|y+(f'(\underline{u}_{-\infty})-\sigma_\varepsilon)t|^2}{t}}\right).$$

To conclude and prove Proposition 7, we pick a smooth cut-off function χ on $[0, +\infty)$ such that $\chi \equiv 1$ on $[2, +\infty)$ and $\chi \equiv 0$ on $[0, 1]$ and define $s_\varepsilon^{\mathrm{p}}, \widetilde{S}_\varepsilon$ by

$$s_\varepsilon^{\mathrm{p}}(t)(w) := \int_{\mathbf{R}} \chi(t) G_t^{\varepsilon,\mathrm{p}}(y) w(y) \, \mathrm{d}y,$$

$$\widetilde{S}_\varepsilon(t)(w)(x) := \int_{\mathbf{R}} (\chi(t)(G_t^{\varepsilon,\mathrm{ess}}(x, y) + \widetilde{G}_t^{\varepsilon,\mathrm{pt}}(x, y)) + (1 - \chi(t))G_t^\varepsilon(x, y)) w(y) \, \mathrm{d}y.$$

The definitions are extended to $t = 0$ by $s_\varepsilon^{\mathrm{p}}(0)(w) = 0$ and $\widetilde{S}_\varepsilon(0)(w) = w$.

As explained near its statement, Proposition 7 follows then from $L_x^\infty L_y^1$ bounds on Green kernels, which themselves are derived from pointwise bounds proved above.

5. Nonlinear stability

In the present section we conclude the proof of Theorem 2.

To do so, we seek u solving (2.1) in the form

$$u(t, x) = \underline{U}_\varepsilon(x + \psi(t)) + v(t, x + \psi(t)) \tag{5.1}$$

with (v, ψ') exponentially decaying in time. In these terms the equation becomes

$$\partial_t v + (f'(\underline{U}_\varepsilon + v) - \sigma_\varepsilon + \psi')\partial_x v - \partial_x^2 v$$
$$= \varepsilon(g(\underline{U}_\varepsilon + v) - g(\underline{U}_\varepsilon)) - (f'(\underline{U}_\varepsilon + v) - f'(\underline{U}_\varepsilon) + \psi')\underline{U}_\varepsilon'. \tag{5.2}$$

Equation (5.2) may be solved through

$$v(t, \cdot) = \widetilde{S}_\varepsilon(t)(v_0) + \int_0^t \widetilde{S}_\varepsilon(t - s)\mathcal{N}_\varepsilon[v(s, \cdot), \psi'(s)] \, \mathrm{d}s, \tag{5.3}$$

$$\psi'(t) = \partial_t s_\varepsilon^{\mathrm{p}}(t)(v_0) + \int_0^t \partial_t s_\varepsilon^{\mathrm{p}}(t - s)\mathcal{N}_\varepsilon[v(s, \cdot), \psi'(s)] \, \mathrm{d}s, \tag{5.4}$$

with

$$\mathcal{N}_\varepsilon[w, \varphi] := -(f'(\underline{U}_\varepsilon + w) - f'(\underline{U}_\varepsilon) + \varphi)\partial_x w + \varepsilon(g(\underline{U}_\varepsilon + w) - g(\underline{U}_\varepsilon) - g'(\underline{U}_\varepsilon)w)$$
$$- (f'(\underline{U}_\varepsilon + w) - f'(\underline{U}_\varepsilon) - f''(\underline{U}_\varepsilon)w)\underline{U}_\varepsilon'.$$

In the present section, for notational concision's sake, we denote

$$\omega_\infty = \min(|g'(\underline{u}_{-\infty})|, |g'(\underline{u}_{+\infty})|).$$

To begin with, we observe that the estimates of the foregoing section are almost sufficient to run a continuity argument on (5.3)–(5.4). Indeed they provide the following proposition.

Proposition 8. *There exist $\theta_0 > 0$ and $\varepsilon_0 > 0$ such that for any $0 < \theta \leq \theta_0$ and $\delta > 0$, there exists $C > 0$ such that for any $0 < \varepsilon \leq \varepsilon_0$ and $T > 0$, if (v, ψ') solves (5.3)–(5.4) on $[0, T]$, with*

$$\|v(t, \cdot)\|_{L^\infty(\mathbf{R})} \leq \delta, \quad t \in [0, T],$$

then, for any $t \in [0, T]$,

$$|\psi'(t)| + \|v(t, \cdot)\|_{W^{1,\infty}(\mathbf{R})} \leq C \|v(0, \cdot)\|_{W^{1,\infty}(\mathbf{R})} e^{-\varepsilon\omega_\infty t}$$
$$\times \exp\Big(C \sup_{0 \leq s \leq T} e^{\varepsilon\omega_\infty s}(|\psi'(s)| + \|v(s, \cdot)\|_{L^\infty(\mathbf{R})}$$
$$+ \|(\varepsilon + e^{-\theta|\cdot|})^{-1}\partial_x v(s, \cdot)\|_{L^\infty(\mathbf{R})})\Big).$$

The estimate fails to close by the fact that $\|\partial_x w\|_{L^\infty(\mathbf{R})}$ provides a weaker ε-uniform control on w than $\|(\varepsilon + e^{-\theta|\cdot|})^{-1}\partial_x w\|_{L^\infty(\mathbf{R})}$. Note however that for any $x_* > 0$,

$$\|(\varepsilon + e^{-\theta|\cdot|})^{-1}\partial_x w\|_{L^\infty([-x_*, x_*])} \leq e^{\theta x_*}\|\partial_x w\|_{L^\infty(\mathbf{R})},$$

so that we only need to improve the estimates on $\partial_x v(t, \cdot)$ on the complement of some compact neighborhood of 0.

5.1. Maximum principle and propagation of regularity

To close our nonlinear estimates without using either localization or parabolic smoothing – which would cause loss in powers of ε – we shall use a maximum principle argument.

To begin with, we state and prove a convenient classical abstract maximum principle. We provide a proof mostly to highlight that it may be thought of as an energy estimate on a suitable nonlinear function.

Lemma 7. *Let $T > 0$, $x_* \in \mathbf{R}$, $\eta > 0$, $a \in L^1([0, T]; W^{1,\infty}([x_*, +\infty)))$ bounded from above away from zero and $h \in \mathcal{C}^0([0, T] \times [x_*, +\infty) \times \mathbf{R})$. If*

$$w \in \mathcal{C}^2((0, T) \times [x_*, +\infty)) \cap \mathcal{C}^0([0, T] \times [x_*, +\infty))$$

is a bounded function such that

$$\partial_t w + a(\cdot, \cdot)\partial_x w \leq \partial_x^2 w + h(\cdot, \cdot, w) \quad on \ [0, T] \times [x_*, +\infty)$$

and M is a positive constant such that

$$
\begin{aligned}
M &\geq w(\cdot, x_*) & on \ [0, T], \\
M &\geq w(0, \cdot) & on \ [x_*, +\infty), \\
-\eta \cdot &\geq \mathbf{1}_{\cdot > M} h(t, x, \cdot),
\end{aligned}
$$

then

$$w \leq M \quad on \ [0, T] \times [x_*, +\infty).$$

Proof. When moreover

$$M > \limsup_{x \to \infty} w(\cdot, x) \quad on \ [0, T],$$

the claim is proved by a Grönwall argument on

$$t \mapsto \int_{x_*}^{+\infty} (w(t, x) - M)_+ \, dx.$$

The general case is recovered by applying this special case to $(t, x) \mapsto e^{-\theta(x-x_*)} w(t, x)$ with $\theta > 0$ sufficiently small and taking the limit $\theta \to 0$. ∎

We now use the foregoing lemma to derive a weighted bound on $\partial_x v$ outside a sufficiently large compact neighborhood of 0. We shall insert such a bound in a continuity argument so that we only need to prove that as long as $\partial_x v$ does not become too large it remains small. This is the content of the following proposition.

Proposition 9. *There exists $\theta_0 > 0$ such that for any $0 < \theta \leq \theta_0$, there exist $x_* > 0$, $\varepsilon_0 > 0$, $\delta > 0$ and $C > 0$ such that for any $0 < \varepsilon \leq \varepsilon_0$ and $T > 0$, if (v, ψ') solves (5.2) on $[0, T] \times \mathbf{R}$, with*

$$|\psi'(t)| + \|v(t, \cdot)\|_{L^\infty(\mathbf{R})} \leq \delta e^{-\varepsilon\omega_\infty t}, \quad t \in [0, T],$$

$$\frac{|\partial_x v(t, x)|}{\varepsilon + e^{-\theta|x|}} \leq \delta e^{-\varepsilon\omega_\infty t}, \quad (t, x) \in [0, T] \times \mathbf{R},$$

then for any $(t, x) \in [0, T] \times (\mathbf{R} \setminus [-x_*, x_*])$,

$$
\begin{aligned}
\frac{|\partial_x v(t, x)|}{\varepsilon + e^{-\theta|x|}} \leq C e^{-\varepsilon\omega_\infty t} \times \Big(&\sup_{0 \leq s \leq T} e^{\varepsilon\omega_\infty s} (|\psi'(s)| + \|v(s, \cdot)\|_{L^\infty(\mathbf{R})} \\
&+ \|\partial_x v(s, \cdot)\|_{L^\infty([-x_*, x_*])}) \\
&+ \|(\varepsilon + e^{-\theta|\cdot|})^{-1} \partial_x v(0, \cdot)\|_{L^\infty(\mathbf{R})} \Big).
\end{aligned}
$$

Proof. We may argue separately to deal with bounds on $x \geq x_*$ on one hand and on $x \leq -x_*$ on the other hand, and provide details only for the former. From now on we focus on $x \geq x_*$.

We would like to apply Lemma 7 to both $A_\varepsilon \partial_x v$ and $-A_\varepsilon \partial_x v$ for a suitable weight A_ε equivalent to $(t, x) \mapsto e^{\varepsilon \omega_\infty t}(\varepsilon + e^{-\theta|x|})^{-1}$. Our choice is

$$A_\varepsilon(t, x) := \frac{e^{\omega_\infty \varepsilon t} e^{\int_t^{+\infty} \varepsilon e^{-\varepsilon \omega_\infty s} \, ds}}{\varepsilon + e^{-\theta|x|}}.$$

Note that one has

$$\partial_t (A_\varepsilon \partial_x v) + \left((f'(\underline{U}_\varepsilon + v) - \sigma_\varepsilon + \psi') + 2\frac{\theta e^{-\theta|x|}}{\varepsilon + e^{-\theta|x|}} \right) \partial_x (A_\varepsilon \partial_x v) - \partial_x^2 (A_\varepsilon \partial_x v)$$

$$= (f'(\underline{U}_\varepsilon) - f'(\underline{U}_\varepsilon + v) - \psi') A_\varepsilon \underline{U}_\varepsilon'' + (f''(\underline{U}_\varepsilon) - f''(\underline{U}_\varepsilon + v)) A_\varepsilon \underline{U}_\varepsilon'^2$$

$$- A_\varepsilon \partial_x v \left(\varepsilon e^{-\varepsilon \omega_\infty t} - (f'(\underline{U}_\varepsilon + v) - \sigma_\varepsilon + \psi') \frac{\theta e^{-\theta|x|}}{\varepsilon + e^{-\theta|x|}} \right.$$

$$- \varepsilon(\omega_\infty + g'(\underline{U}_\varepsilon + v)) - \frac{\theta^2 e^{-\theta x}(2\varepsilon + e^{-\theta x})}{(\varepsilon + e^{-\theta x})^2}$$

$$\left. + f''(\underline{U}_\varepsilon + v)(2\underline{U}_\varepsilon' + \partial_x v) \right).$$

Fixing first $\theta > 0$ sufficiently small, then x_* sufficiently large and δ and ε sufficiently small, one enforces that the term in front of $\partial_x (A_\varepsilon \partial_x v)$ is bounded from above away from zero and the term in front of $A_\varepsilon \partial_x v$ is bounded from below by a multiple of $\varepsilon e^{-\varepsilon \omega_\infty t} + \theta e^{-\theta|x|}$. This is sufficient to apply Lemma 7 and derive the claimed upper bound on $x \geq x_*$. ∎

5.2. Proof of Theorem 2

Our very first task when proving Theorem 2 is to convert classical local well-posedness-yielding maximal solutions u to (2.1) into convenient local existence results for (v, ψ').

This follows from the following simple observation. By design, $s_\varepsilon^p(t) \equiv 0$ when $0 \leq t \leq 1$. Thus if u solves (2.1) on $[0, T] \times \mathbf{R}$ then (v, ψ') satisfying (5.1)–(5.3)–(5.4) may be obtained recursively through

$$\psi(t) = \psi_0, \quad v(t, \cdot) = u(t, \cdot - \psi(t)) - \underline{U}_\varepsilon, \quad \text{when } 0 \leq t \leq \min(\{1, T\}),$$

and, for any $n \in \mathbf{N}$,

$$\psi'(t) = \partial_t s_\varepsilon^p(t)(v_0) + \int_0^{t-1} \partial_t s_\varepsilon^p(t - s) \mathcal{N}_\varepsilon[v(s, \cdot), \psi'(s)] \, ds,$$

$$\psi(t) = \psi_0 + \int_0^t \psi'(s) \, ds,$$

$$v(t, \cdot) = u(t, \cdot - \psi(t)) - \underline{U}_\varepsilon \quad \text{when } \min(\{n, T\}) \leq t \leq \min(\{n + 1, T\}).$$

Now, combining Propositions 8 and 9, one obtains that for any $\theta > 0$ sufficiently small, there exist $\varepsilon_0 > 0$, $\delta > 0$ and $C \geq 1$ such that for any $0 < \varepsilon \leq \varepsilon_0$, and (v_0, ψ_0) with

$$\|v_0\|_{L^\infty(\mathbf{R})} + \|(\varepsilon + e^{-\theta|\cdot|})^{-1}\partial_x v_0\|_{L^\infty(\mathbf{R})} \leq \delta,$$

the corresponding solution u to (2.1), in the form (5.1), satisfies that if for some $T > 0$ and any $0 \leq t \leq T$,

$$|\psi'(t)| + \|v(t, \cdot)\|_{L^\infty(\mathbf{R})} + \|(\varepsilon + e^{-\theta|\cdot|})^{-1}\partial_x v(t, \cdot)\|_{L^\infty(\mathbf{R})}$$
$$\leq 2C e^{-\varepsilon\omega_\infty t}(\|v_0\|_{L^\infty(\mathbf{R})} + \|(\varepsilon + e^{-\theta|\cdot|})^{-1}\partial_x v_0\|_{L^\infty(\mathbf{R})}),$$

then for any $0 \leq t \leq T$,

$$|\psi'(t)| + \|v(t, \cdot)\|_{L^\infty(\mathbf{R})} + \|(\varepsilon + e^{-\theta|\cdot|})^{-1}\partial_x v(t, \cdot)\|_{L^\infty(\mathbf{R})}$$
$$\leq C e^{-\varepsilon\omega_\infty t}(\|v_0\|_{L^\infty(\mathbf{R})} + \|(\varepsilon + e^{-\theta|\cdot|})^{-1}\partial_x v_0\|_{L^\infty(\mathbf{R})}).$$

From this and a continuity argument stem that u is global and that the latter estimate holds globally in time. One achieves the proof of Theorem 2 by deriving bounds on ψ by integration of those on ψ' and going back to original variables.

A. Wave profiles

In the present appendix, we prove Proposition 1. Let us first reformulate the wave profile equation in terms of
$$\underline{\widetilde{U}}_\varepsilon := \frac{U_\varepsilon - U_0}{\varepsilon}, \quad \tilde{\sigma}_\varepsilon := \frac{\sigma_\varepsilon - \sigma_0}{\varepsilon}.$$
The equation to consider is

$$\underline{\widetilde{U}}_\varepsilon'' - ((f'(\underline{U}_0) - \sigma_0)\underline{\widetilde{U}}_\varepsilon)' = -g(\underline{U}_0 + \varepsilon\underline{\widetilde{U}}_\varepsilon) - (\tilde{\sigma}_\varepsilon(\underline{U}_0 + \varepsilon\underline{\widetilde{U}}_\varepsilon))'$$
$$+ \left(\frac{f(\underline{U}_0 + \varepsilon\underline{\widetilde{U}}_\varepsilon) - f(\underline{U}_0) - \varepsilon f'(\underline{U}_0)\underline{\widetilde{U}}_\varepsilon}{\varepsilon}\right)',$$

with $\underline{\widetilde{U}}_\varepsilon(0) = 0$, $(\tilde{\sigma}_\varepsilon, e^{\theta|\cdot|}\underline{\widetilde{U}}_\varepsilon, e^{\theta|\cdot|}\underline{\widetilde{U}}_\varepsilon')$ uniformly bounded, for some sufficiently small $\theta > 0$. As announced in the introduction, the framework we first consider is suboptimal from the point of view of spatial localization, but we shall refine it in a second step. To carry out the first step we introduce spaces $W_\theta^{k,\infty}$ and their subspaces BUC_θ^k, corresponding to norms

$$\|v\|_{W_\theta^{k,\infty}(\mathbf{R})} = \sum_{j=0}^{k}\|e^{\theta|\cdot|}\partial_x^j v\|_{L^\infty(\mathbf{R})}.$$

In this first step, we just pick some $0 < \theta < \min(\{\theta_0^\ell, \theta_0^r\})$ and let all the constants depend on this particular choice.

We begin with two preliminary remarks. First, note that a simple integration yields that a necessary constraint is

$$\tilde{\sigma}_\varepsilon = -\frac{1}{\underline{u}_{+\infty} - \underline{u}_{-\infty}} \int_{\mathbf{R}} g(\underline{U}_0 + \varepsilon\tilde{\underline{U}}_\varepsilon) =: \tilde{\Sigma}_\varepsilon[\tilde{\underline{U}}_\varepsilon]$$

and that

$$\tilde{\mathcal{N}}_\varepsilon[\tilde{\underline{U}}_\varepsilon] := -g(\underline{U}_0 + \varepsilon\tilde{\underline{U}}_\varepsilon) - (\tilde{\Sigma}_\varepsilon[\tilde{\underline{U}}_\varepsilon](\underline{U}_0 + \varepsilon\tilde{\underline{U}}_\varepsilon))'$$
$$+ \left(\frac{f(\underline{U}_0 + \varepsilon\tilde{\underline{U}}_\varepsilon) - f(\underline{U}_0) - \varepsilon f'(\underline{U}_0)\tilde{\underline{U}}_\varepsilon}{\varepsilon}\right)'$$

defines a continuous map from BUC^1_θ to the closed subspace of BUC^0_θ whose range is contained in the set of functions with zero integral and that, on any ball of BUC^1_θ, has an $\mathcal{O}(\varepsilon)$-Lipschitz constant.

Second, denoting by L_0 the operator defined by

$$L_0(v) := v'' - (f'(\underline{U}_0) - \sigma_0)v)'$$

on BUC^0_θ, with domain BUC^2_θ, we observe that L_0 is Fredholm of index 0 (as a continuous operator from BUC^2_θ to BUC^0_θ), its kernel is spanned by $\underline{U}'_\varepsilon$ and the kernel of its adjoint is reduced to constant functions. The foregoing claims are easily proved by direct inspection, but may also be obtained with the arguments of Sections 2 and 3, combining the spatial dynamics point of view with a Sturm–Liouville argument. Since evaluation at 0 acts continuously on BUC^2_θ and $\underline{U}'_0(0) \neq 0$, this implies that the restriction of L_0 from the closed subspace of BUC^2_θ consisting of functions with value 0 at 0, to the closed subspace of BUC^0_θ consisting of functions with zero integral, is boundedly invertible. Indeed, the inverse of this restriction is readily seen to be given by

$$L_0^\dagger(h)(x) := -\int_0^x \int_z^{+\infty} \frac{\underline{U}'_0(x)}{\underline{U}'_0(z)} h(y)\, \mathrm{d}y\, \mathrm{d}z = \int_0^x \int_{-\infty}^z \frac{\underline{U}'_0(x)}{\underline{U}'_0(z)} h(y)\, \mathrm{d}y\, \mathrm{d}z.$$

Note that from the profile equation itself stems that if $\tilde{\underline{U}}_\varepsilon$ is a BUC^1_θ-solution, it is also a BUC^2_θ-solution, so that the problem reduces to

$$\tilde{\sigma}_\varepsilon = \tilde{\Sigma}[\tilde{\underline{U}}_\varepsilon], \quad \tilde{\underline{U}}_\varepsilon = L_0^\dagger(\tilde{\mathcal{N}}_\varepsilon[\tilde{\underline{U}}_\varepsilon]).$$

If C_0 is chosen such that $C_0 > \|L_0^\dagger\tilde{\mathcal{N}}_\varepsilon[0_\mathbf{R}]\|_{W^{1,\infty}_\theta(\mathbf{R})}$, it follows that, when ε is sufficiently small, the map $L_0^\dagger \circ \tilde{\mathcal{N}}_\varepsilon$ sends the complete space

$$\{v \in \mathrm{BUC}^1_\theta(\mathbf{R}); \; v(0) = 0 \text{ and } \|v\|_{W^{1,\infty}_\theta(\mathbf{R})} \leq C_0\}$$

into itself and is strictly contracting with an $\mathcal{O}(\varepsilon)$-Lipschitz constant. Thus, resorting to the Banach fixed-point theorem achieves the first step of the proof of Proposition 1.

Note that, in order to conclude the proof, it is sufficient to provide asymptotic descriptions of $\mathrm{e}^{\theta^\ell_\varepsilon|\cdot|}(\underline{U}_\varepsilon - \underline{u}_{-\infty})$, $\mathrm{e}^{\theta^r_\varepsilon\cdot}(\underline{U}_\varepsilon - \underline{u}_{+\infty})$, $\mathrm{e}^{\theta^\ell_\varepsilon|\cdot|}\underline{U}'_\varepsilon$ and $\mathrm{e}^{\theta^r_\varepsilon\cdot}\underline{U}'_\varepsilon$. Indeed, on one hand,

the asymptotic comparisons for $e^{\theta_\varepsilon^\ell|\cdot|}\underline{U}_\varepsilon^{(k)}$ and $e^{\theta_\varepsilon^r\cdot}\underline{U}_\varepsilon^{(k)}$, $k \geq 2$, are then deduced recursively by using the profile equation (differentiated $(k-2)$ times). On the other hand, since, for $\# \in \{\ell, r\}$, $\theta_\varepsilon^\# = \theta_0^\# + \mathcal{O}(\varepsilon)$, the asymptotic descriptions are sufficient to upgrade the existence part of the first step arbitrarily close to optimal spatial decay rates, $\alpha^\# \to \theta_0^\#$.

As a further reduction, we observe that the asymptotics for $e^{\theta_\varepsilon^\ell|\cdot|}(\underline{U}_\varepsilon - \underline{u}_{-\infty})$ and $e^{\theta_\varepsilon^r\cdot}(\underline{U}_\varepsilon - \underline{u}_{+\infty})$ may be deduced from those for $e^{\theta_\varepsilon^\ell|\cdot|}\underline{U}_\varepsilon'$ and $e^{\theta_\varepsilon^r\cdot}\underline{U}_\varepsilon'$ by integration since

$$
\begin{aligned}
&e^{-\theta_\varepsilon^\ell x}(\underline{U}_\varepsilon(x) - \underline{u}_{-\infty}) - e^{-\theta_0^\ell x}(\underline{U}_0(x) - \underline{u}_{-\infty}) \\
&\quad = \int_{-\infty}^x e^{\theta_\varepsilon^\ell(y-x)}(e^{-\theta_\varepsilon^\ell y}\underline{U}_\varepsilon'(y) - e^{-\theta_0^\ell y}\underline{U}_0'(y))\,\mathrm{d}y \\
&\qquad + \int_{-\infty}^x e^{\theta_0^\ell(y-x)}(e^{(\theta_\varepsilon^\ell - \theta_0^\ell)(y-x)} - 1)e^{-\theta_0^\ell y}\underline{U}_0'(y)\,\mathrm{d}y,
\end{aligned}
$$

and likewise near $+\infty$.

To conclude, we derive the study of $e^{\theta_\varepsilon^\ell|\cdot|}\underline{U}_\varepsilon'$ and $e^{\theta_\varepsilon^r\cdot}\underline{U}_\varepsilon'$ from the analysis of Proposition 2 (with $K = \{0\}$). Indeed,

$$
\theta_\varepsilon^\ell = \mu_+^\varepsilon(0; \underline{u}_{-\infty}), \quad \theta_\varepsilon^\ell = -\mu_-^\varepsilon(0; \underline{u}_{+\infty}),
$$

and

$$
\begin{aligned}
\underline{U}_\varepsilon'(x) &= \frac{(\underline{u}_{+\infty} - \underline{u}_{-\infty})}{2} \frac{e^{\theta_\varepsilon^\ell x}\,\mathbf{e}_1 \cdot P_\varepsilon^\ell(0, x)\mathbf{R}_+^\varepsilon(0; \underline{u}_{-\infty})}{\int_{-\infty}^0 e^{\theta_\varepsilon^\ell y}\,\mathbf{e}_1 \cdot P_\varepsilon^\ell(0, y)\mathbf{R}_+^\varepsilon(0; \underline{u}_{-\infty})\,\mathrm{d}y} \\
&= \frac{(\underline{u}_{+\infty} - \underline{u}_{-\infty})}{2} \frac{-e^{\theta_\varepsilon^r x}\,\mathbf{e}_1 \cdot P_\varepsilon^r(0, x)\mathbf{R}_-^\varepsilon(0; \underline{u}_{+\infty})}{\int_0^{+\infty} e^{-\theta_\varepsilon^r y}\,\mathbf{e}_1 \cdot P_\varepsilon^r(0, y)\mathbf{R}_-^\varepsilon(0; \underline{u}_{+\infty})\,\mathrm{d}y}.
\end{aligned}
$$

Thus the claimed expansion stems from the smoothness in ε afforded by Proposition 2.

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