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Global weak solutions of the Serre–Green–Naghdi equations with surface tension

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Abstract. In this paper we consider the Serre–Green–Naghdi equations with surface tension. Smooth solutions of this system conserve an H^1-equivalent energy. We prove the existence of global weak dissipative solutions for any relatively small-energy initial data. We also prove that the Riemann invariants of the solutions satisfy a one-sided Oleinik inequality.

1. Introduction

The Euler equations are usually used to describe water waves in oceans and channels. Due to the difficulties in resolving the Euler equations both numerically and analytically, several simpler approximations have been proposed in the literature for different regimes. In the shallow-water regime, the main assumption is on the ratio of the mean water depth $\bar{h}$ to the wavelength ι: the shallowness parameter $\sigma = \bar{h}^2/\iota^2$ is considered to be small. Besides the shallowness condition, a restriction on the amplitude of the wave a can be considered, assuming that the nonlinearity (or the amplitude) parameter $\epsilon = a/\bar{h}$ is small. Consider the shallow-water regime with the small-amplitude condition [31, 39] ($\sigma \ll 1$, $\epsilon \ll 1$). Many equations have been derived to model the propagation of the waves, such as the Camassa–Holm equation [10], the Korteweg–de Vries (KdV) equation [38] and some variants of the Boussinesq equations [8, 9, 58]. Considering shallow water with possibly large-amplitude waves ($\sigma \ll 1$, $\epsilon \approx 1$), by neglecting the terms of order $\mathcal{O}(\sigma)$ in the water-wave equations, Saint-Venant obtained the nonlinear shallow water (or Saint-Venant) equations [57]. Smooth solutions of the Saint-Venant equations have a precision of order $\mathcal{O}(t\sigma)$, where t denotes the time [39]. In order to obtain a better precision, one can keep the $\mathcal{O}(\sigma)$ terms in the equations and only neglect the $\mathcal{O}(\sigma^2)$ terms. This leads to the Serre–Green–Naghdi equations. Those equations were first derived by Serre [53], rediscovered independently by Su and Gardner [56] and again by Green, Laws and Naghdi [23, 24]. The Serre–Green–Naghdi equations are the most general and most precise, but also the most complicated of the models of shallow-water equations presented above. One can always keep higher-order terms in the equation (keeping terms of order $\mathcal{O}(\sigma^2)$ for

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example); this will lead to equations with better precision, but with higher-order derivatives. These equations are not accurate due to the high-order derivative terms, which make their numerical resolution much slower.

The influence of the surface tension is generally neglected on water-wave problems. However, in certain cases, the effect of the surface tension is appreciable. Indeed, Longuet-Higgins [45] showed that the surface tension is significant in certain localised regions, and cannot be neglected near the sharp crest of the breaking wave. Other experimental studies showed the importance of the surface tension on thin layers [21, 48, 49]. Those experimentations have been done for different fluids, including water and mercury. Various mathematical studies of water-wave equations with surface tension exist in the literature; we refer to [1, 3, 4, 7, 12, 13, 29, 46, 52, 54, 60].

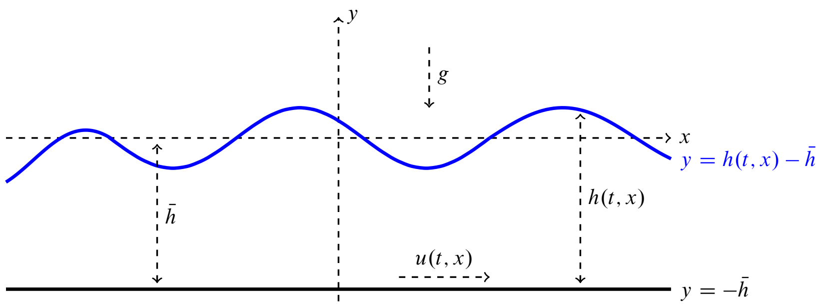

Figure 1. Fluid domain.

Consider a two-dimensional coordinate system Oxy (Figure 1) and an incompressible fluid layer. Considering the still fluid level at $y = 0$, the fluid layer is bounded between the flat bottom at $y = -\bar{h}$ and a free surface $y = h(t, x) - \bar{h}$, where h is the total water depth. Assume long waves in shallow water with possibly large amplitude. The Serre–Green–Naghdi system (without neglecting the surface tension influence) reads

$$h_t + [hu]_x = 0, \tag{1.1a}$$

$$[hu]_t + [hu^2 + \frac{1}{2}gh^2 + \mathcal{R}]_x = 0, \tag{1.1b}$$

$$\mathcal{R} \stackrel{\text{def}}{=} \frac{1}{3}h^3(-u_{tx} - uu_{xx} + u_x^2) - \gamma\left(hh_{xx} - \frac{1}{2}h_x^2\right), \tag{1.1c}$$

where u denotes the depth-averaged horizontal velocity, g is the gravitational acceleration and $\gamma > 0$ is a constant (the ratio of the surface tension coefficient to the density). The classical Serre–Green–Naghdi equations (without surface tension) are recovered taking $\gamma = 0$. The Serre–Green–Naghdi (SGN$_\gamma$) equations (1.1) have been derived in [17] as a generalisation of the classical SGN equations ($\gamma = 0$). As mentioned above, the Serre–Green–Naghdi equations are obtained in the shallow-water regime by neglecting all the $\mathcal{O}(\sigma^2)$ terms. An extension of the Serre–Green–Naghdi system with surface tension (1.1) have been derived in [37] by neglecting only the $\mathcal{O}(\sigma^3)$ terms; local well-posedness and justification of the extended system have been studied in [35–37].

Due to the appearance of time derivatives in (1.1c), it is convenient to apply the inverse of the Sturm–Liouville operator

$$\mathcal{L}_h \stackrel{\text{def}}{=} h - \frac{1}{3}\partial_x h^3 \partial_x; \tag{1.2}$$

system (1.1) then becomes

$$h_t + [hu]_x = 0, \tag{1.3a}$$

$$u_t + uu_x + gh_x = -\mathcal{L}_h^{-1}\partial_x\left\{\frac{2}{3}h^3u_x^2 - \left[\gamma h - \frac{1}{3}gh^3\right]h_{xx} + \frac{1}{2}\gamma h_x^2\right\}. \tag{1.3b}$$

When $h > 0$, the operator $\mathcal{L}_h^{-1}$ is well defined and smooths two derivatives (see Lemma 5.2 below). This is not enough to control the term containing h_{xx} on the right-hand side of (1.3b). To overcome this problem, we use the definition of $\mathcal{L}_h$ to rewrite system (1.3) in the equivalent form

$$\begin{aligned} &h_t + [hu]_x = 0,\\ &u_t + uu_x + 3\gamma h^{-2}h_x = -\mathcal{L}_h^{-1}\partial_x\left\{\frac{2}{3}h^3u_x^2 - \frac{3}{2}\gamma h_x^2 + \frac{1}{2}gh^2 - 3\gamma\ln(h)\right\}. \end{aligned} \tag{1.4}$$

Smooth solutions of the SGN$_\gamma$ equations (1.4) satisfy the energy equation (see Appendix B)

$$\mathcal{E}_t + \mathcal{D}_x = 0, \tag{1.5}$$

where

$$\begin{aligned} \mathcal{E} &\stackrel{\text{def}}{=} \frac{1}{2}hu^2 + \frac{1}{2}g(h-\bar{h})^2 + \frac{1}{6}h^3u_x^2 + \frac{1}{2}\gamma h_x^2,\\ \mathcal{D} &\stackrel{\text{def}}{=} u\mathcal{E} + u\left(\mathcal{R} + \frac{1}{2}gh^2 - \frac{1}{2}g\bar{h}^2\right) + \gamma h h_x u_x. \end{aligned} \tag{1.6}$$

Linearising the SGN$_\gamma$ equations (1.4) around the constant state $(h,u) = (\bar{h},0)$ and looking for travelling waves having the form $\exp\{(kx-\omega t)i\}$, we obtain the dispersion relation $\omega^2 = g\bar{h}k^2(1+\gamma k^2/g)/(1+\bar{h}^2k^2/3)$. Defining the Bond number $B = g\bar{h}^2/\gamma$, the SGN$_\gamma$ equations are linearly dispersive if and only if $B \neq 3$. In the dispersionless case ($B = 3$), the SGN$_\gamma$ equations admit weakly singular peakon travelling wave solutions [19, 47]. More travelling wave solutions are obtained in [41]. The Serre–Green–Naghdi equations, with or without surface tension, have been widely studied in the literature. We refer to [2, 28, 30, 34, 39, 42] for the case $\inf h_0 > 0$ and to [40] for the shoreline problem ($\operatorname{sign}(h) = \mathbb{1}_{x>x_0}$). In [2, 30, 42], a proof of the local well-posedness of the SGN equations without surface tension ($\gamma = 0$) is given. Kazerani has proved in [34] the existence of global smooth solutions of the SGN equations with viscosity for small initial data. A full justification of model (1.4) is given in [28, 39]. By "full justification" we mean local well-posedness of the system and that the solution is close to the solution of the water-wave equations with the same initial data. In a recent work [25], we have obtained a precise blow-up criterion of (1.4) (Theorem 2.3 below) and we proved that such a scenario occurs for a class of small-energy initial data (Theorem 2.6 below). Then, in general, smooth solutions cannot exist globally in time.

This paper investigates the existence of global weak solutions of (1.4) with $\gamma > 0$. To the best of the author's knowledge, the existence of global weak solutions for all the different variants of the inviscid Serre–Green–Naghdi equations has not been established before. Here, the existence of global weak solutions is established by approximating system (1.4) with another system that admits global smooth solutions. We recover weak solutions of (1.4) by taking the limit. The proof involves several steps.

We consider initial data satisfying $\int \mathcal{E}_0 \, \mathrm{d}x < \sqrt{g\gamma}\bar{h}^2$, which is propagated due to the energy conservation (1.5). Using the fact that the energy is equivalent to $\|(h - \bar{h}, u)\|_{H^1}^2$ and a Sobolev-like inequality (essentially $H^1 \hookrightarrow L^\infty$; see Proposition 2.4 below) we obtain a uniform lower bound of h. This is important for ensuring the invertibility of the operator $\mathcal{L}_h$ defined in (1.2). Smooth solutions of (1.4) blow up in finite time due to the presence of quadratic terms in the associated Riccati-type equations. In order to approximate the SGN$_\gamma$ system, we use a cut-off to obtain a linear growth that leads to global smooth solutions (due to Grönwall's inequality). However, cutting off directly as in [63, 64] violates the energy conservation (1.5). The choice of the approximated system is crucial and must conserve the properties of the SGN$_\gamma$ system. In Section 4 below, we carefully chose a suitable approximated system that is globally well posed and satisfies the energy equation (5.8). In order to pass to the limit, some uniform estimates are needed. In previous studies of smooth solutions of the SGN equations, some estimates of the operator $\mathcal{L}_h^{-1}$ have been obtained; those estimates usually depend on the L^∞ norm of h_x, which may blow up for weak solutions. In Lemma 5.2 below, we present some new estimates of $\mathcal{L}_h^{-1}$ depending only on the L^∞ norms of h and $1/h$. As in [59, 63, 64], an L^p_{loc} estimate of (h_x, u_x) with $p < 3$ is also needed. In our case and due to the complexity of the SGN$_\gamma$ equations, we have to use a change of coordinates to obtain this estimate (see Lemma 5.6 below). We then use some classical compactness arguments with Young measures [32] and a generalised compensated compactness result [22] to pass to the limit. We follow in this step the techniques developed in [59] for the Camassa–Holm equation and in [63, 64] for the variational wave equation. The structure of the SGN$_\gamma$ system being more complex, we have to handle the weak limit of some nonlinear terms that do not exist in [63, 64] (see Lemma 6.4 for example). Finally, the global weak solutions of (1.4) are obtained by taking the limit in the approximated system, and are shown to dissipate the energy and satisfy the one-sided Oleinik inequality (3.4).

The existence of global solutions to the Boussinesq equations [9, 58]

$$h_t + [hu]_x = 0, \quad u_t + uu_x + gh_x = u_{txx} \tag{1.7}$$

have been studied in [5, 51]. Schonbek [51] regularised the conservation of the mass by adding a diffusion term, i.e., $h_t + [hu]_x = \varepsilon h_{xx}$, with $\varepsilon > 0$. She proved the global well-posedness of the regularised system, and she obtained global weak solutions of (1.7) by taking $\varepsilon \to 0$. In [5], Amick proved that if the initial data, (h_0, u_0), is smooth, then the solution, (h, u), obtained by Schonbek [51] is also smooth and is the unique smooth solution of the Boussinesq equations (1.7).

The SGN$_\gamma$ equations (1.1) can be compared with the dispersionless regularised Saint-Venant (rSV) system presented in [11]. The rSV system can be obtained replacing $\mathcal{R}$ in (1.1c) by $\varepsilon\mathcal{R}_{\text{rSV}}$, with

$$\mathcal{R}_{\text{rSV}} \stackrel{\text{def}}{=} h^3(u_x^2 - u_{xt} - uu_{xx}) - gh^2\Big(hh_{xx} + \frac{1}{2}h_x^2\Big)$$

and $\varepsilon \geqslant 0$; the classical Saint-Venant system is recovered by taking $\varepsilon = 0$. Weakly singular shock profiles of the rSV equations are studied in [50]. In [44], Liu et al. proved the local well-posedness of the rSV equations and identified a class of initial data such that the corresponding solutions blow up in finite time. The rSV system has been generalised recently to obtain a regularisation of any unidimensional barotropic Euler (rE) system [26]. System (1.1) can also be compared with the modified Serre–Green–Naghdi (mSGN) equations derived in [14] to improve the dispersion relation of the classical SGN system. The mSGN system presented in [14] can be obtained replacing $\mathcal{R}$ in (1.1c) by

$$\mathcal{R}_{\text{mSGN}} \stackrel{\text{def}}{=} \frac{1}{3}\Big(1 + \frac{3}{2}\beta\Big)h^3(-u_{tx} - uu_{xx} + u_x^2) - \frac{1}{2}\beta g h^2\Big(hh_{xx} + \frac{1}{2}h_x^2\Big),$$

where β is a positive parameter. The rSV, rE and mSGN systems conserve H^1-equivalent energies and have similar properties to the SGN$_\gamma$ system (1.1). One may obtain the existence of global weak solutions of those equations following the proof given in this paper.

The study of the classical Serre–Green–Naghdi equations is more challenging. Indeed, when $\gamma = 0$, the energy (1.6) fails to control the H^1 norm of $h - \bar{h}$; then a lower bound of h cannot be obtained. This bound is crucial to obtaining the blow-up result [25] and the global existence in this paper for $\gamma > 0$. To the author's knowledge, the questions of the blow-up of smooth solutions and the existence of global solutions of the SGN equations without surface tension are still open. However, Bae and Granero-Belinchón [6] proved recently that for a class of periodic initial data satisfying $\inf h_0 = 0$, smooth solutions cannot exist globally in time. For this class of initial data, it is not known whether smooth solutions exist locally in time, but if they do, a singularity must appear in finite time.

This paper is organised as follows. In Section 2 we present the local well-posedness of (1.4) and some blow-up results. Section 3 is devoted to defining weak solutions of (1.4) and to presenting the main result, which is the existence of global dissipative weak solutions. We discuss in Section 4 the properties needed for the approximated system and we propose suitable choices. Section 5 is devoted to proving the existence of global smooth solutions of the approximated system and to obtaining some uniform estimates. We obtain strong precompactness results in Section 6. The existence of global weak solutions is proved in Section 7. In Appendix A we recall some classical lemmas that are used in this paper. Appendix B is devoted to obtaining the energy equations of the approximated system and of (1.4).

2. Review of previous results

We consider the Serre–Green–Naghdi equations with surface tension in the form

$$h_t + [hu]_x = 0, \tag{2.1a}$$

$$u_t + uu_x + 3\gamma h^{-2} h_x = -\mathscr{L}_h^{-1} \partial_x \{\mathcal{C} + F(h)\}, \tag{2.1b}$$

$$u(0,x) = u_0(x), \quad h(0,x) = h_0(x), \tag{2.1c}$$

with

$$\mathcal{C} \stackrel{\text{def}}{=} \frac{2}{3} h^3 u_x^2 - \frac{3}{2} \gamma h_x^2,$$

$$F(h) \stackrel{\text{def}}{=} \frac{1}{2} g h^2 - \frac{1}{2} g \bar{h}^2 - 3\gamma \ln(h/\bar{h}).$$

The system (2.1) is locally well posed in the Sobolev space

$$H^s(\mathbb{R}) \stackrel{\text{def}}{=} \left\{ f,\ \|f\|_{H^s(\mathbb{R})}^2 \stackrel{\text{def}}{=} \textstyle\int_{\mathbb{R}} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 \,\mathrm{d}\xi < \infty \right\},$$

where $s \geqslant 2$ is a real number.

Theorem 2.1. *Let $\gamma > 0$, $\bar{h} > 0$ and $s \geqslant 2$. Then, for any $(h_0 - \bar{h}, u_0) \in H^s(\mathbb{R})$ satisfying $\inf_{x \in \mathbb{R}} h_0(x) > 0$, there exist $T > 0$ and $(h - \bar{h}, u) \in C([0,T], H^s(\mathbb{R})) \cap C^1([0,T], H^{s-1}(\mathbb{R}))$ a unique solution of (2.1) such that*

$$\inf_{(t,x) \in [0,T] \times \mathbb{R}} h(t,x) > 0.$$

Moreover, the solution satisfies the conservation of the energy

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} \left(\frac{1}{2} h u^2 + \frac{1}{2} g (h - \bar{h})^2 + \frac{1}{6} h^3 u_x^2 + \frac{1}{2} \gamma h_x^2 \right) \mathrm{d}x = 0. \tag{2.2}$$

Remark 2.2. The solution given in Theorem 2.1 depends continuously on the initial data, i.e., if $(h_0^1 - \bar{h}, u_0^1), (h_0^2 - \bar{h}, u_0^2) \in H^s$, such that $h_0^1, h_0^2 \geqslant h_{\min} > 0$, then for all $t \leqslant T$ there exists a constant $C(\|(h^2 - \bar{h}, u^2)\|_{L^\infty([0,t],H^s)}, \|(h^1 - \bar{h}, u^1)\|_{L^\infty([0,t],H^s)}) > 0$, such that

$$\|(h^1 - h^2, u^1 - u^2)\|_{L^\infty([0,t],H^{s-1})} \leqslant C \|(h_0^1 - h_0^2, u_0^1 - u_0^2)\|_{H^s}.$$

The proof of Theorem 2.1 is classic and omitted in this paper; see [26–28, 30, 39, 44] for more details. It is clear from Theorem 2.1 that if the solution at time T remains in H^s and $\inf_x h(T,x) > 0$, then one can extend the interval of existence. This leads to the blow-up criterion

$$T_{\max} < \infty \implies \liminf_{t \to T_{\max}} \inf_{x \in \mathbb{R}} h(t,x) = 0 \quad \text{or} \quad \limsup_{t \to T_{\max}} \|(h - \bar{h}, u)\|_{H^s} = \infty,$$

where $T_{\max}$ is the maximum time existence of the solution. This criterion has been improved in [25] to the following:

Theorem 2.3 ([25]). *Let $T_{\max}$ be the maximum time existence of the solution given by Theorem 2.1. Then*

$$T_{\max} < \infty \implies \liminf_{t\to T_{\max}} \inf_{x\in\mathbb{R}} h(t,x) = 0 \quad or \quad \begin{cases} \liminf\limits_{t\to T_{\max}} \inf\limits_{x\in\mathbb{R}} u_x(t,x) = -\infty \\ \quad and \\ \limsup\limits_{t\to T_{\max}} \|h_x(t,x)\|_{L^\infty} = \infty, \end{cases}$$

which is equivalent to the second criterion

$$T_{\max} < \infty \implies \limsup_{t\to T_{\max}} \|u_x(t,x)\|_{L^\infty} = \infty \quad and \quad \begin{cases} \liminf\limits_{t\to T_{\max}} \inf\limits_{x\in\mathbb{R}} h(t,x) = 0 \\ \quad or \\ \limsup\limits_{t\to T_{\max}} \|h_x(t,x)\|_{L^\infty} = \infty. \end{cases}$$

Note that the energy conserved in (2.2) is equivalent to the H^1 norm of $(h-\bar{h}, u)$. Due to the continuous embedding $H^1 \hookrightarrow L^\infty$, we can obtain a uniform (in time) estimate of $\|(h-\bar{h}, u)\|_{L^\infty}$, and, if the initial energy is not very large compared to $\bar{h}$, we can obtain a lower bound of h. For that purpose, we present the following proposition.

Proposition 2.4. *For $\gamma > 0$, $\bar{h} > 0$, let E be a positive number such that*

$$0 < E < \sqrt{g\gamma}\,\bar{h}^2. \tag{2.3}$$

Define

$$h_{\min} \overset{\text{def}}{=} \bar{h} - (g\gamma)^{-1/4}\sqrt{E}, \qquad h_{\max} \overset{\text{def}}{=} \bar{h} + (g\gamma)^{-1/4}\sqrt{E},$$
$$u_{\max} \overset{\text{def}}{=} -u_{\min} \overset{\text{def}}{=} 3^{1/4}\sqrt{E}/h_{\min}.$$

Then, for any $(h-\bar{h}, u) \in H^1$ satisfying $\int \mathcal{E}\,\mathrm{d}x \leqslant E$, we have

$$0 < h_{\min} \leqslant h \leqslant h_{\max} < 2\bar{h}, \quad u_{\min} \leqslant u \leqslant u_{\max},$$

Remark 2.5. Taking an initial data satisfying $\int_{\mathbb{R}} \mathcal{E}_0\,\mathrm{d}x \leqslant E$, then, due to the energy conservation (2.2) and Proposition 2.4, the depth h cannot vanish. The blow-up criteria given in Theorem 2.3 become then

$$T_{\max} < \infty \implies \inf_{[0,T_{\max})\times\mathbb{R}} u_x(t,x) = -\infty \quad \text{and} \quad \limsup_{t\to T_{\max}} \|h_x(t,x)\|_{L^\infty} = \infty.$$

Proof of Proposition 2.4. The Young inequality $\frac{1}{2}a^2 + \frac{1}{2}b^2 \geqslant \pm ab$ implies that

$$\begin{aligned} E \geqslant \int_{\mathbb{R}} \mathcal{E}\,\mathrm{d}y &\geqslant \int_{\mathbb{R}} \Big(\frac{1}{2} g(h-\bar{h})^2 + \frac{1}{2}\gamma h_x^2\Big)\,\mathrm{d}x \\ &\geqslant \sqrt{g\gamma}\Big(\int_{-\infty}^{x} (h-\bar{h})h_x\,\mathrm{d}y - \int_x^{\infty} (h-\bar{h})h_x\,\mathrm{d}y\Big) \\ &\geqslant \sqrt{g\gamma}\,|h-\bar{h}|^2, \end{aligned}$$

which implies that $h_{\min} \leqslant h \leqslant h_{\max}$. Making the same estimates with u one obtains

$$\begin{aligned} E \geqslant \int_{\mathbb{R}} \mathcal{E}\, \mathrm{d}y &\geqslant \int_{\mathbb{R}} \Big(\frac{1}{2} h u^2 + \frac{1}{6} h^3 u_x^2\Big)\, \mathrm{d}y \\ &\geqslant \frac{1}{\sqrt{3}} h_{\min}^2 \Big(\int_{-\infty}^{x} u u_x \, \mathrm{d}y - \int_{x}^{\infty} u u_x \, \mathrm{d}y\Big) \\ &\geqslant \frac{1}{\sqrt{3}} h_{\min}^2 |u|^2, \end{aligned}$$

where the last inequality ends the proof of $u_{\min} \leqslant u \leqslant u_{\max}$. ∎

As in [25], we can build some initial data with small initial data such that the corresponding solutions blow up in small time.

Theorem 2.6 ([25]). *For any $T > 0$ and E satisfying (2.3), there exist*

- *$(h_0 - \bar{h}, u_0) \in C_c^\infty(\mathbb{R})$ satisfying $\int_{\mathbb{R}} \mathcal{E}_0 \, \mathrm{d}x \leqslant E$ such that the corresponding solution of (2.1) blows up at finite time $T_{\max} \leqslant T$ and*

$$\inf_{[0,T_{\max})\times\mathbb{R}} u_x(t,x) = -\infty, \qquad \sup_{[0,T_{\max})\times\mathbb{R}} h_x(t,x) = \infty, \qquad \inf_{[0,T_{\max})\times\mathbb{R}} h_x(t,x) > -\infty.$$

- *$(\tilde{h}_0 - \bar{h}, \tilde{u}_0) \in C_c^\infty(\mathbb{R})$ satisfying $\int_{\mathbb{R}} \tilde{\mathcal{E}}_0 \, \mathrm{d}x \leqslant E$ such that the corresponding solution of (2.1) blows up at finite time $\tilde{T}_{\max} \leqslant T$ and*

$$\inf_{[0,\tilde{T}_{\max})\times\mathbb{R}} \tilde{u}_x(t,x) = -\infty, \qquad \inf_{[0,\tilde{T}_{\max})\times\mathbb{R}} \tilde{h}_x(t,x) = -\infty, \qquad \sup_{[0,\tilde{T}_{\max})\times\mathbb{R}} \tilde{h}_x(t,x) < \infty.$$

3. Main results

Since smooth solutions fail to exist globally in time, even for arbitrary small-energy initial data, we shall define weak solutions of the SGN_γ system (2.1). For that purpose, we define the domain $\mathfrak{D} \subset H^1$,

$$\mathfrak{D} \stackrel{\text{def}}{=} \big\{(h - \bar{h}, u) \in H^1,\ \textstyle\int_{\mathbb{R}} \mathcal{E}\, \mathrm{d}x < \sqrt{g\gamma}\bar{h}^2\big\}.$$

Definition 3.1. We say that $(h - \bar{h}, u) \in L^\infty(\mathbb{R}^+, H^1) \cap \mathrm{Lip}(\mathbb{R}^+, L^2)$ is a weak solution of (2.1) if it satisfies the initial condition (2.1c) with (2.1a) in L^2 and for all $\varphi \in C_c^\infty((0,\infty)\times\mathbb{R})$ we have

$$\int_{\mathbb{R}^+\times\mathbb{R}} \big\{\big\{u_t + u u_x + 3\gamma (h)^{-2} h_x\big\} \mathcal{L}_h \varphi - \varphi_x \big\{\mathcal{C} + F(h)\big\}\big\}\, \mathrm{d}x\, \mathrm{d}t = 0. \tag{3.1}$$

Moreover, $(h(t,\cdot) - \bar{h}, u(t,\cdot))$ belongs to $\mathfrak{D}$ for all $t \geqslant 0$ and $(h - \bar{h}, u) \in C_r(\mathbb{R}^+, H^1)$. More precisely, for all $t_0 \geqslant 0$ we have

$$\lim_{\substack{t\to t_0\\ t>t_0}} \big\| \big(h(t,\cdot) - h(t_0,\cdot), u(t,\cdot) - u(t_0,\cdot)\big) \big\|_{H^1} = 0.$$

Now we can state the main result of this paper.

Theorem 3.2. *Let* $\bar{h}, g, \gamma > 0$ *and* $(h_0 - \bar{h}, u_0) \in \mathfrak{D}$. *Then there exists a global weak solution* $(h - \bar{h}, u) \in L^\infty([0,\infty), H^1(\mathbb{R})) \cap C([0,\infty) \times \mathbb{R})$ *of* (2.1) *in the sense of Definition* 3.1. *Moreover,*

- *for any bounded set* $\Omega = [t_1, t_2] \times [a, b] \subset (0,\infty) \times \mathbb{R}$ *and* $\alpha \in [0, 1)$ *there exists* $C_{\alpha,\Omega} > 0$ *such that*

$$\int_\Omega \left[|h_t|^{2+\alpha} + |h_x|^{2+\alpha} + |u_t|^{2+\alpha} + |u_x|^{2+\alpha} \right] \mathrm{d}x \, \mathrm{d}t \leqslant C_{\alpha,\Omega}; \tag{3.2}$$

- *the solution dissipates the energy*

$$\int_{\mathbb{R}} \mathcal{E} \, \mathrm{d}x \leqslant \int_{\mathbb{R}} \mathcal{E}_0 \, \mathrm{d}x; \tag{3.3}$$

- *there exists* $C > 0$ *such that the solution satisfies the Oleinik inequality*

$$u_x \pm \sqrt{3\gamma} h^{-3/2} h_x \leqslant C\left(1 + \frac{1}{t}\right), \quad a.e. \ (t, x) \in (0,\infty) \times \mathbb{R}. \tag{3.4}$$

Remark 3.3. The constants $C_{\alpha,\Omega}$ and C depend on $\bar{h}$, γ, g and $\int_{\mathbb{R}} \mathcal{E}_0 \, \mathrm{d}x$, but not on the initial data.

In order to obtain global solutions of (2.1), we use a suitable approximation of system (2.1) that admits global smooth solutions. Using some compactness arguments and taking the limit, we recover a global weak solution of (2.1). In the next section we present the choice of the suitable approximated system.

4. An approximated system

The blow-up of the solutions given in Theorem 2.6 is due to the Riccati-type equations. In order to prevent the singularities from appearing, we slightly modify the Riccati-type equations.

4.1. Riccati-type equations

Define the Riemann invariants[1] R and S:

$$R \stackrel{\text{def}}{=} u + 2\sqrt{3\gamma} h^{-1/2}, \quad S \stackrel{\text{def}}{=} u - 2\sqrt{3\gamma} h^{-1/2},$$
$$\lambda \stackrel{\text{def}}{=} u - \sqrt{3\gamma} h^{-1/2}, \quad \eta \stackrel{\text{def}}{=} u + \sqrt{3\gamma} h^{-1/2}.$$

[1] These quantities are constants along the characteristics if the right-hand side of (2.1) is zero.

System (2.1) can be rewritten as

$$\begin{aligned} R_t + \lambda R_x &= -\mathcal{L}_h^{-1}\partial_x\{\mathcal{C} + F(h)\}, \\ S_t + \eta S_x &= -\mathcal{L}_h^{-1}\partial_x\{\mathcal{C} + F(h)\}. \end{aligned} \tag{4.1}$$

Defining

$$\begin{aligned} P &\stackrel{\text{def}}{=} hR_x = hu_x - \sqrt{3\gamma}h^{-1/2}h_x, \\ Q &\stackrel{\text{def}}{=} hS_x = hu_x + \sqrt{3\gamma}h^{-1/2}h_x, \end{aligned}$$

we have

$$u_x = \frac{P+Q}{2h}, \quad h_x = h^{1/2}\frac{Q-P}{2\sqrt{3\gamma}}. \tag{4.2}$$

From the definition of $\mathcal{L}_h$ in (1.2), we obtain

$$\partial_x\mathcal{L}_h^{-1}\partial_x\Psi = -3h^{-3}\Psi + 3\partial_x\mathcal{L}_h^{-1}\left(h\int_{-\infty}^{x} h^{-3}\Psi\right) \tag{4.3}$$

for any smooth function Ψ satisfying $\Psi(\pm\infty) = 0$. Then

$$\mathcal{C} + \frac{1}{3}h^3\partial_x\mathcal{L}_h^{-1}\partial_x\mathcal{C} = h^3\partial_x\mathcal{L}_h^{-1}\left(h\int_{-\infty}^{x} h^{-3}\mathcal{C}\right).$$

From (1.1c) and (2.1b) we obtain

$$\begin{aligned} \mathcal{R} &= -\frac{1}{3}h^3[u_t + uu_x + 3\gamma h^{-2}h_x]_x + \mathcal{C} \\ &= \mathcal{C} + \frac{1}{3}h^3\partial_x\mathcal{L}_h^{-1}\partial_x\{\mathcal{C} + F(h)\} \qquad (4.4) \\ &= h^3\partial_x\mathcal{L}_h^{-1}\left(h\int_{-\infty}^{x} h^{-3}\mathcal{C}\right) + \frac{1}{3}h^3\partial_x\mathcal{L}_h^{-1}\partial_x F(h). \qquad (4.5) \end{aligned}$$

Let the characteristics X_x, Y_x starting from x be defined as the solutions of the ordinary differential equations

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}X_x(t) &= \eta(t, X_x(t)), \quad X_x(0) = x, \\ \frac{\mathrm{d}}{\mathrm{d}t}Y_x(t) &= \lambda(t, Y_x(t)), \quad Y_x(0) = x. \end{aligned}$$

Differentiating (4.1) with respect to x and using (4.5), we obtain the Riccati-type equations

$$\frac{\mathrm{d}^\lambda}{\mathrm{d}t}P \stackrel{\text{def}}{=} P_t + \lambda P_x = -\frac{1}{8h}P^2 + \frac{1}{8h}Q^2 - 3h^{-2}\mathcal{R}, \tag{4.6a}$$

$$\frac{\mathrm{d}^\eta}{\mathrm{d}t}Q \stackrel{\text{def}}{=} Q_t + \eta Q_x = -\frac{1}{8h}Q^2 + \frac{1}{8h}P^2 - 3h^{-2}\mathcal{R}, \tag{4.6b}$$

where $\frac{\mathrm{d}^\lambda}{\mathrm{d}t}$, $\frac{\mathrm{d}^\eta}{\mathrm{d}t}$ denote the derivatives along the characteristics with the speed λ, η respectively. We prove below that the term $\mathcal{R}$ is bounded. Also, we obtain a bound of the integral of P^2 (respectively Q^2) on the characteristics X_x (respectively Y_x). Then the singularities given in Theorem 2.6 appear due to the term P^2 in (4.6a) and/or the term Q^2 in (4.6b).

4.2. The choice of the approximated system

In order to obtain a system that admits global smooth solutions, we linearise the negative quadratic terms on the right-hand side of (4.6) in the neighbourhood of $-\infty$. For that purpose, let $\varepsilon > 0$ and we define as in [63, 64],

$$\chi_\varepsilon(\zeta) \stackrel{\text{def}}{=} \left(\zeta + \frac{1}{\varepsilon}\right)^2 \mathbb{1}_{(-\infty,-\frac{1}{\varepsilon}]}(\zeta) = \begin{cases} \left(\zeta + \frac{1}{\varepsilon}\right)^2, & \zeta \leqslant -1/\varepsilon, \\ 0, & \zeta > -1/\varepsilon. \end{cases} \tag{4.7}$$

Note that (4.6) is like a derivative of (2.1); then adding terms to (4.6) will involve some primitive terms in (2.1) which are not uniquely defined and cannot vanish at ∞ and $-\infty$. That is why system (2.1) will not be approximated simply by adding χ_ε to (4.6) as in [63, 64].

Our goal is to obtain a system of the form

$$h_t + [hu]_x = h^+,$$
$$u_t + uu_x + 3\gamma h^{-2} h_x = -\mathscr{L}_h^{-1} \partial_x\{\mathcal{C} + F(h)\} + u^+,$$

where h^+, u^+ are terms to be chosen suitably. As in Section 4.1, we obtain

$$P_t + \lambda P_x = -\frac{1}{8h} P^2 + \frac{1}{8h}\chi_\varepsilon(P) + \frac{1}{8h} Q^2 - 3h^{-2}\mathcal{R} + \underline{P},$$
$$Q_t + \eta Q_x = -\frac{1}{8h} Q^2 + \frac{1}{8h}\chi_\varepsilon(Q) + \frac{1}{8h} P^2 - 3h^{-2}\mathcal{R} + \underline{Q},$$

where

$$\underline{P} \stackrel{\text{def}}{=} h(u^+)_x + u_x h^+ - \sqrt{3\gamma} h^{-1/2}(h^+)_x + \frac{1}{2}\sqrt{3\gamma} h^{-3/2} h_x h^+ - \frac{1}{8h}\chi_\varepsilon(P),$$
$$\underline{Q} \stackrel{\text{def}}{=} h(u^+)_x + u_x h^+ + \sqrt{3\gamma} h^{-1/2}(h^+)_x - \frac{1}{2}\sqrt{3\gamma} h^{-3/2} h_x h^+ - \frac{1}{8h}\chi_\varepsilon(Q).$$

Due to definition (4.7), when ζ is near $-\infty$, the term $\chi_\varepsilon(\zeta) - \zeta^2$ behaves as a linear map. This prevents singularities from appearing in finite time. From (1.6), we have

$$\mathcal{E} = \frac{1}{2} h u^2 + \frac{1}{2} g(h - \bar{h})^2 + \frac{1}{12} h P^2 + \frac{1}{12} h Q^2.$$

Then the energy equation (1.5) becomes

$$\begin{aligned} \mathcal{E}_t + \mathcal{D}_x &= huu^+ + \frac{1}{2}u^2 h^+ + g(h - \bar{h})h^+ + \left(\frac{1}{6}h^2 u_x^2 + \frac{\gamma}{2h} h_x^2\right) h^+ \\ &\quad + \frac{1}{6} h P \underline{P} + \frac{1}{6} h Q \underline{Q} + \frac{1}{48} P \chi_\varepsilon(P) + \frac{1}{48} Q \chi_\varepsilon(Q) \\ &\leqslant huu^+ + \frac{1}{2}u^2 h^+ + g(h - \bar{h})h^+ + \left(\frac{1}{6}h^2 u_x^2 + \frac{\gamma}{2h} h_x^2\right) h^+ \\ &\quad + \frac{1}{6} h P \underline{P} + \frac{1}{6} h Q \underline{Q}. \end{aligned} \tag{4.8}$$

The goal is to find h^+ and u^+ such that

- the right-hand side of (4.8) is a derivative of some quantity (i.e., $[\cdots]_x$), which will ensure that $\int_{\mathbb{R}} \mathcal{E}\,\mathrm{d}x$ is a decreasing function of time;
- when P, Q are large, we have $\underline{P} = \mathcal{O}(P)$ and $\underline{Q} = \mathcal{O}(Q)$, which ensures (with the Grönwall inequality) that no singularity will appear in finite time.

We can write the right-hand side of (4.8) as $T_1 + T_2$ such that

$$\begin{aligned} T_1 &= g(h-\bar{h})h^+ + \gamma h_x (h^+)_x + \frac{\sqrt{3\gamma}}{48h^{1/2}} h_x (\chi_\varepsilon(P) - \chi_\varepsilon(Q)) \\ &= g(h-\bar{h})h^+ + (h-\bar{h})_x \left[\gamma (h^+)_x + \frac{\sqrt{3\gamma}}{48h^{1/2}} (\chi_\varepsilon(P) - \chi_\varepsilon(Q)) \right]. \end{aligned}$$

Then a sufficient condition to obtain $T_1 = [\cdots]_x$ is

$$g h^+ = \left[\gamma (h^+)_x + \frac{\sqrt{3\gamma}}{48h^{1/2}} (\chi_\varepsilon(P) - \chi_\varepsilon(Q)) \right]_x. \tag{4.9}$$

On the other hand we have

$$\begin{aligned} T_2 &= \frac{1}{3}h^3 u_x (u^+)_x + \frac{1}{2}h^2 u_x^2 h^+ + \frac{1}{2}u^2 h^+ + huu^+ - \frac{1}{48} h(\chi_\varepsilon(P) + \chi_\varepsilon(Q)) u_x \\ &= \left(\frac{1}{2}uh^+ + hu^+ \right) u + \left[\frac{1}{3}h^3 (u^+)_x + \frac{1}{2}h^2 u_x h^+ - \frac{1}{48} h (\chi_\varepsilon(P) + \chi_\varepsilon(Q)) \right] u_x. \end{aligned}$$

Then a sufficient condition to obtain $T_2 = [\cdots]_x$ is

$$\frac{1}{2}uh^+ + hu^+ = \left[\frac{1}{3}h^3 (u^+)_x + \frac{1}{2}h^2 u_x h^+ - \frac{1}{48} h (\chi_\varepsilon(P) + \chi_\varepsilon(Q)) \right]_x. \tag{4.10}$$

In next section we prove the global existence of smooth solutions of the approximated system, and we obtain some uniform estimates that do not depend on ε.

5. Uniform estimates

In this section we consider $\gamma > 0$, $h > 0$ and $h_0 - \bar{h}, u_0 \in H^1$ such that $\int_{\mathbb{R}} \mathcal{E}_0\,\mathrm{d}x < \sqrt{g\gamma}\bar{h}^2$. Also let j_ε be a Friedrichs mollifier; we define $h_0^\varepsilon \stackrel{\text{def}}{=} ((h_0 - \bar{h}) * j_\varepsilon) + \bar{h}$ and $u_0^\varepsilon \stackrel{\text{def}}{=} (u_0 * j_\varepsilon)$, where $(f * g)(x) \stackrel{\text{def}}{=} \int_{\mathbb{R}} f(x-x')g(x')\,\mathrm{d}x'$. Using that $\|(h_0 - h_0^\varepsilon, u_0 - u_0^\varepsilon)\|_{H^1} \to 0$ as $\varepsilon \to 0$, we can prove

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}} \mathcal{E}_0^\varepsilon\,\mathrm{d}x = \int_{\mathbb{R}} \mathcal{E}_0\,\mathrm{d}x < \sqrt{g\gamma}\bar{h}^2, \tag{5.1}$$

which implies that there exists $\varepsilon_0 > 0$ such that

$$\int_{\mathbb{R}} \mathcal{E}_0^\varepsilon\,\mathrm{d}x \leqslant E \stackrel{\text{def}}{=} \frac{1}{2} \int_{\mathbb{R}} \mathcal{E}_0\,\mathrm{d}x + \frac{1}{2} g\gamma \bar{h}^2 \quad \forall \varepsilon \leqslant \varepsilon_0.$$

Following the arguments of the previous section (see (4.9) and (4.10)), we consider the system

$$h_t^\varepsilon + [h^\varepsilon u^\varepsilon]_x = \mathcal{A}_x^\varepsilon, \tag{5.2a}$$

$$u_t^\varepsilon + u^\varepsilon u_x^\varepsilon + 3\gamma (h^\varepsilon)^{-2} h_x^\varepsilon = -\mathcal{L}_{h^\varepsilon}^{-1}\partial_x\{\mathcal{C}^\varepsilon + F(h^\varepsilon)\} + \mathcal{B}^\varepsilon, \tag{5.2b}$$

$$u^\varepsilon(0,\cdot) = u_0^\varepsilon \stackrel{\text{def}}{=} j_\varepsilon * u_0, \quad h^\varepsilon(0,\cdot) = h_0^\varepsilon \stackrel{\text{def}}{=} j_\varepsilon * (h_0 - \bar{h}) + \bar{h}, \tag{5.2c}$$

where

$$\begin{aligned}\mathcal{A}^\varepsilon &\stackrel{\text{def}}{=} (g - \gamma\partial_x^2)^{-1} g \left\{ \frac{\sqrt{3\gamma}}{48(h^\varepsilon)^{1/2}} (\chi_\varepsilon(P^\varepsilon) - \chi_\varepsilon(Q^\varepsilon)) g \right\}, \\ &= \mathfrak{G} * g \left\{ \frac{\sqrt{3\gamma}}{48(h^\varepsilon)^{1/2}} (\chi_\varepsilon(P^\varepsilon) - \chi_\varepsilon(Q^\varepsilon)) g \right\},\end{aligned} \tag{5.3}$$

$$\mathcal{B}^\varepsilon \stackrel{\text{def}}{=} \mathcal{L}_{h^\varepsilon}^{-1}\left\{ -\frac{1}{2} u^\varepsilon \mathcal{A}_x^\varepsilon + \partial_x \left\{ \frac{1}{2}(h^\varepsilon)^2 u_x^\varepsilon \mathcal{A}_x^\varepsilon - \frac{1}{48} h^\varepsilon (\chi_\varepsilon(P^\varepsilon) + \chi_\varepsilon(Q^\varepsilon)) \right\}\right\}, \tag{5.4}$$

with $\mathfrak{G}$ defined as

$$\mathfrak{G} \stackrel{\text{def}}{=} \frac{1}{2\gamma} \exp\left\{ -\frac{g}{\gamma} |\cdot| \right\}.$$

Differentiating (5.2) with respect to x we obtain

$$\begin{aligned}\frac{\mathrm{d}^\lambda}{\mathrm{d}t} P^\varepsilon &\stackrel{\text{def}}{=} P_t^\varepsilon + \lambda^\varepsilon P_x^\varepsilon \\ &= -\frac{1}{8h^\varepsilon}(P^\varepsilon)^2 + \frac{1}{8h^\varepsilon}\chi_\varepsilon(P^\varepsilon) + \frac{1}{8h^\varepsilon}(Q^\varepsilon)^2 - \frac{1}{2h^\varepsilon}\mathcal{A}_x^\varepsilon P^\varepsilon + \mathcal{M}^\varepsilon,\end{aligned} \tag{5.5a}$$

$$\begin{aligned}\frac{\mathrm{d}^\eta}{\mathrm{d}t} Q^\varepsilon &\stackrel{\text{def}}{=} Q_t^\varepsilon + \eta^\varepsilon Q_x^\varepsilon \\ &= -\frac{1}{8h^\varepsilon}(Q^\varepsilon)^2 + \frac{1}{8h^\varepsilon}\chi_\varepsilon(Q^\varepsilon) + \frac{1}{8h^\varepsilon}(P^\varepsilon)^2 - \frac{1}{2h^\varepsilon}\mathcal{A}_x^\varepsilon Q^\varepsilon + \mathcal{N}^\varepsilon,\end{aligned} \tag{5.5b}$$

with

$$\begin{aligned}&\mathcal{M}^\varepsilon \stackrel{\text{def}}{=} -3(h^\varepsilon)^{-2}\mathcal{R}^\varepsilon + \mathcal{V}_1^\varepsilon - \mathcal{V}_2^\varepsilon, \quad \mathcal{N}^\varepsilon \stackrel{\text{def}}{=} -3(h^\varepsilon)^{-2}\mathcal{R}^\varepsilon + \mathcal{V}_1^\varepsilon + \mathcal{V}_2^\varepsilon, \\ &\mathcal{V}_1^\varepsilon \stackrel{\text{def}}{=} \frac{1}{2} h^\varepsilon \partial_x \mathcal{L}_{h^\varepsilon}^{-1}\left\{ -u^\varepsilon \mathcal{A}_x^\varepsilon + h^\varepsilon \int_{-\infty}^x \left[3(h^\varepsilon)^{-1} u_x^\varepsilon \mathcal{A}_x^\varepsilon \right.\right. \\ &\qquad\qquad \left.\left. - \frac{1}{8(h^\varepsilon)^2}(\chi_\varepsilon(P^\varepsilon) + \chi_\varepsilon(Q^\varepsilon)) \right] \mathrm{d}y \right\},\end{aligned} \tag{5.6}$$

$$\mathcal{V}_2^\varepsilon \stackrel{\text{def}}{=} \frac{g}{16}(h^\varepsilon)^{-1/2} \mathfrak{G} * \left\{ (h^\varepsilon)^{-1/2}(\chi_\varepsilon(P^\varepsilon) - \chi_\varepsilon(Q^\varepsilon)) \right\} = \frac{3g}{\sqrt{3\gamma}}(h^\varepsilon)^{-1/2}\mathcal{A}^\varepsilon. \tag{5.7}$$

Smooth solutions of (5.2) satisfy the energy equation (see Appendix B)

$$\mathcal{E}_t^\varepsilon + \widetilde{\mathcal{D}}_x^\varepsilon = \frac{1}{48} P^\varepsilon \chi_\varepsilon(P^\varepsilon) + \frac{1}{48} Q^\varepsilon \chi_\varepsilon(Q^\varepsilon) \leqslant 0, \tag{5.8}$$

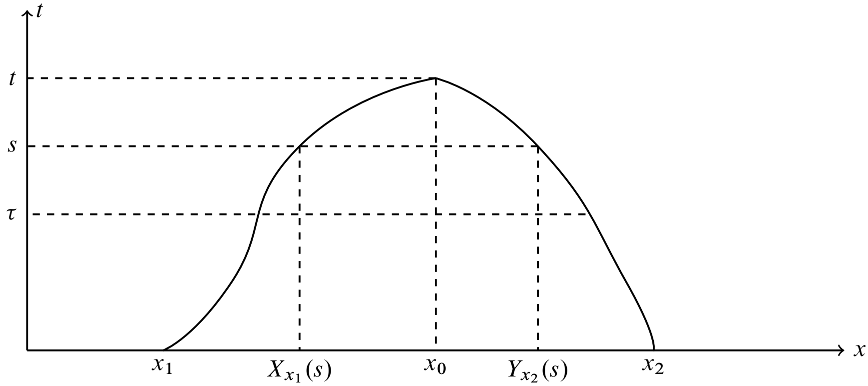

Figure 2. Characteristics.

where

$$\widetilde{\mathfrak{D}}^{\varepsilon} \stackrel{\text{def}}{=} u^{\varepsilon}\mathcal{E}^{\varepsilon} + u^{\varepsilon}\Big(\mathcal{R}^{\varepsilon} + \frac{1}{2}g(h^{\varepsilon})^2 - \frac{1}{2}g\bar{h}^2\Big) + \gamma h^{\varepsilon}h_x^{\varepsilon}u_x^{\varepsilon} - \frac{1}{3}(h^{\varepsilon})^2u^{\varepsilon}\mathcal{V}_1^{\varepsilon} - \frac{\sqrt{3\gamma}}{3}(h^{\varepsilon})^{1/2}\mathcal{V}_2^{\varepsilon}(h^{\varepsilon} - \bar{h}).$$

The first result in this section is the global well-posedness of (5.2).

Theorem 5.1. *Let $\bar{h} > 0$, $(h_0 - \bar{h}, u_0) \in \mathfrak{D}$ and $\varepsilon \in (0, \varepsilon_0]$. Then there exists a global smooth solution $(h^{\varepsilon} - \bar{h}, u^{\varepsilon}) \in C(\mathbb{R}^+, H^3(\mathbb{R})) \cap C^1(\mathbb{R}^+, H^2(\mathbb{R}))$ of (5.2) and for all $t > 0$ we have*

$$\int_{\mathbb{R}} \mathcal{E}^{\varepsilon}\,\mathrm{d}x - \int_0^t \int_{\mathbb{R}} \frac{1}{48}(P^{\varepsilon}\chi_{\varepsilon}(P^{\varepsilon}) + Q^{\varepsilon}\chi_{\varepsilon}(Q^{\varepsilon}))\,\mathrm{d}x\,\mathrm{d}t = \int_{\mathbb{R}} \mathcal{E}_0^{\varepsilon}\,\mathrm{d}x. \tag{5.9}$$

Moreover, there exist $A, B > 0$ depending only on $\bar{h}$, γ, g and E such that for any $t > 0$, $x_2 \in \mathbb{R}$, and for $x_1 \in (-\infty, x_2)$ the solution of $X_{x_1}(t) = Y_{x_2}(t)$ (see Figure 2), we have

$$\int_{\tau}^{t} [P^{\varepsilon}(s, X_{x_1}(s))]^2\,\mathrm{d}s + \int_{\tau}^{t} [Q^{\varepsilon}(s, Y_{x_2}(s))]^2\,\mathrm{d}s \leqslant A(t - \tau) + B \quad \forall \tau \in [0, t]. \tag{5.10}$$

In order to prove Theorem 5.1, we need to prove the invertibility of the operator $\mathcal{L}_h$ and to obtain some estimates of its inverse.

Lemma 5.2. *Let $0 < h \in H^1(\mathbb{R}) + \bar{h}$ with $h^{-1} \in L^{\infty}$. Then the operator $\mathcal{L}_h$ is an isomorphism from H^2 to L^2. Moreover, if $\psi \in C_{\lim} \stackrel{\text{def}}{=} \{f \in C(\mathbb{R}),\ f(\infty),\ f(-\infty) \in \mathbb{R}\}$, then $\mathcal{L}_h^{-1}\psi$ is well defined and there exists a constant $C = C(\bar{h}, \|h^{-1}\|_{L^{\infty}}, \|h\|_{L^{\infty}}) > 0$ such that*

$$\|\mathcal{L}_h^{-1}\psi\|_{W^{1,\infty}} \leqslant C\|\psi\|_{L^{\infty}}, \tag{5.11}$$

$$|\partial_x^2\mathcal{L}_h^{-1}\psi|(x) \leqslant C(1 + |h_x(x)|)\|\psi\|_{L^{\infty}} \quad \forall x \in \mathbb{R}, \tag{5.12}$$

$$\|\mathcal{L}_h^{-1}\psi\|_{H^1} \leqslant C\|\psi\|_{L^1}, \tag{5.13}$$

$$\|\mathcal{L}_h^{-1}\partial_x\psi\|_{L^\infty} \leqslant C\|\psi\|_{L^1}, \tag{5.14}$$

$$\|\partial_x\mathcal{L}_h^{-1}\partial_x\psi\|_{L^\infty} \leqslant C[\|\psi\|_{L^1} + \|\psi\|_{L^\infty}], \tag{5.15}$$

$$\|\mathcal{L}_h^{-1}\partial_x\psi\|_{L^2} \leqslant C\|\psi\|_{L^1}(\|h_x\|_{L^2} + 1), \tag{5.16}$$

$$\|\mathcal{L}_h^{-1}\partial_x\psi\|_{H^1} + \|\mathcal{L}_h^{-1}\psi\|_{H^1} \leqslant C\|\psi\|_{L^2}, \tag{5.17}$$

$$\|\mathcal{L}_h^{-1}\psi\|_{W^{1,\infty}} \leqslant \|\mathcal{L}_h^{-1}\psi\|_{H^2} \leqslant C[1 + \|h_x\|_{L^2}^2]\|\psi\|_{L^2}, \tag{5.18}$$

$$|\partial_x^2\mathcal{L}_h^{-1}\psi|(x) \leqslant C[(1 + \|h_x\|_{L^2})(1 + |h_x|(x))\|\psi\|_{L^2} + |\psi|(x)]. \tag{5.19}$$

Also, if $h - \bar{h} \in H^2(\mathbb{R})$ we have

$$\begin{aligned}\|\mathcal{L}_h^{-1}\partial_x\varphi\|_{H^3} &\leqslant C[(1 + \|h_x\|_{L^\infty}^2)\|\varphi\|_{H^2} + \|h - \bar{h}\|_{H^2}\|\mathcal{L}_h^{-1}\partial_x\varphi\|_{W^{1,\infty}}],\\ \|\mathcal{L}_h^{-1}\psi\|_{H^3} &\leqslant C[(1 + \|h_x\|_{L^\infty}^2)\|\psi\|_{H^2} + \|h - \bar{h}\|_{H^2}\|\mathcal{L}_h^{-1}\psi\|_{W^{1,\infty}}].\end{aligned} \tag{5.20}$$

Moreover, there exists a constant $\widetilde{C} = \widetilde{C}(\gamma, g)$ such that

$$\|(g - \gamma\partial_x^2)^{-1}\psi\|_{H^3} \leqslant \widetilde{C}\|\psi\|_{H^1}, \quad \|\partial_x(g - \gamma\partial_x^2)^{-1}\psi\|_{H^3} \leqslant \widetilde{C}\|\psi\|_{H^2}. \tag{5.21}$$

The proofs of (5.11), (5.15) and (5.20) are inspired by [44].

Proof of Lemma 5.2. Step 0. Let $(\cdot,\cdot)$ be the scalar product in L^2. Define the bilinear map $a\colon H^1 \times H^1 \to \mathbb{R}$ as

$$a(u, v) \stackrel{\text{def}}{=} (hu, v) + \frac{1}{3}(h^3u_x, v_x).$$

It is easy to check that a is continuous and coercive. Then the Lax–Milgram theorem ensures the existence of a continuous bijective linear operator $J\colon H^1 \to H^{-1}$ satisfying

$$a(u, v) = \langle Ju, v\rangle_{H^{-1}\times H^1} \quad \forall u, v \in H^1.$$

If $Ju \in L^2$, an integration by parts shows that $(h^3u_x)_x = hu - Ju \in L^2$ and $J = \mathcal{L}_h$. This implies that $u \in H^2$, which finishes the proof that $\mathcal{L}_h$ is an isomorphism from H^2 to L^2.

Now defining $C_0 \stackrel{\text{def}}{=} \{f \in C, f(\pm\infty) = 0\}$, using that $L^2 \cap C_0$ is dense in C_0 one can define $\mathcal{L}_h^{-1}$ on C_0. If φ is in C_{lim}, we use the change of functions (see [44, Lemma 4.4])

$$\varphi_0(x) \stackrel{\text{def}}{=} \varphi(x) - \mathcal{L}_h\frac{1}{\bar{h}}\left(\varphi(-\infty) + (\varphi(\infty) - \varphi(-\infty))\frac{\mathrm{e}^x}{1 + \mathrm{e}^x}\right) \in C_0,$$

the operator $\mathcal{L}_h^{-1}$ can be defined as

$$\mathcal{L}_h^{-1}\varphi \stackrel{\text{def}}{=} \mathcal{L}_h^{-1}\varphi_0 + \frac{1}{\bar{h}}\left(\varphi(-\infty) + (\varphi(\infty) - \varphi(-\infty))\frac{\mathrm{e}^x}{1 + \mathrm{e}^x}\right).$$

Step 1. Let

$$\psi = \mathcal{L}_h u = hu - \frac{1}{3}(h^3 u_x)_x. \tag{5.22}$$

Using the change of variables

$$z = \int \frac{\mathrm{d}x}{h^3}, \tag{5.23}$$

we obtain

$$\psi = hu - \frac{1}{3h^3} u_{zz}. \tag{5.24}$$

The maximum principle ensures that $\|u\|_{L^\infty} \leqslant C\|\psi\|_{L^\infty}$, which implies with (5.24) that

$$\|u_{zz}\|_{L^\infty} \leqslant C\|\psi\|_{L^\infty}. \tag{5.25}$$

Using the Landau–Kolmogorov inequality we obtain $\|u_z\|_{L^\infty} \leqslant C\|\psi\|_{L^\infty}$. Using again the change of variables (5.23) we get $\|u_x\|_{L^\infty} \leqslant C\|\psi\|_{L^\infty}$, which completes the proof of (5.11). Estimate (5.12) follows directly from the change of variables (5.23), (5.25) and (5.11).

Multiplying (5.22) by u and using integration by parts one obtains

$$\|u\|_{H^1}^2 \leqslant C\|\psi\|_{L^1}\|u\|_{L^\infty}.$$

Inequality (5.13) follows directly using the embedding $H^1 \hookrightarrow L^\infty$. Using (5.11) and

$$\mathcal{L}_h^{-1}\partial_x \psi = -3\int_{-\infty}^{x}(h^{-3}\psi) + 3\mathcal{L}_h^{-1}\left(h\int_{-\infty}^{x} h^{-3}\psi\right),$$

we obtain (5.14) and (5.15). Using the definition of $\mathcal{L}_h$ we obtain

$$\begin{aligned}
\mathcal{L}_h^{-1}\partial_x\psi &= \mathcal{L}_h^{-1}\partial_x h^3 \mathcal{L}_h \mathcal{L}_h^{-1} h^{-3}\psi \\
&= \mathcal{L}_h^{-1}\partial_x\left[h^4 \mathcal{L}_h^{-1}h^{-3}\psi - \frac{1}{3}h^3\partial_x h^3 \partial_x \mathcal{L}_h^{-1}h^{-3}\psi\right] \\
&= \mathcal{L}_h^{-1}\left[4h^3 h_x \mathcal{L}_h^{-1}h^{-3}\psi + h^4\partial_x\mathcal{L}_h^{-1}h^{-3}\psi - \frac{1}{3}\partial_x h^3\partial_x h^3\partial_x\mathcal{L}_h^{-1}h^{-3}\psi\right] \\
&= \mathcal{L}_h^{-1}[4h^3 h_x \mathcal{L}_h^{-1}h^{-3}\psi] + h^3\partial_x\mathcal{L}_h^{-1}h^{-3}\psi.
\end{aligned} \tag{5.26}$$

Inequality (5.16) follows from (5.13) and the Cauchy–Schwarz inequality.

Let $\mathcal{L}_h u = \psi + \varphi_x$. Then

$$\begin{aligned}
\|u\|_{H^1}^2 &= (u,u) + (u_x,u_x) \\
&\leqslant C[(hu,u) + \frac{1}{3}(h^3u_x,u_x)] \\
&= C(\mathcal{L}_h u, u) = C[(\psi,u) - (\varphi,u_x)] \\
&\leqslant C\|u\|_{H^1}(\|\psi\|_{L^2} + \|\varphi\|_{L^2}),
\end{aligned}$$

which implies that

$$\|u\|_{H^1} \leqslant C(\|\psi\|_{L^2} + \|\varphi\|_{L^2}). \tag{5.27}$$

Taking $\psi = 0$ (respectively $\varphi = 0$) we obtain (5.17). Replacing $h^{-3}\psi$ by ψ in (5.26), we multiply by h^{-3} and we differentiate with respect to x to obtain

$$\begin{aligned}\partial_x^2 \mathcal{L}_h^{-1}\psi &= -3h^{-2}h_x[\mathcal{L}_h^{-1}\partial_x h^3\psi - \mathcal{L}_h^{-1}[4h^3 h_x \mathcal{L}_h^{-1}\psi]] \\ &\quad + h^{-3}\partial_x\mathcal{L}_h^{-1}\partial_x h^3\psi - h^{-3}\partial_x\mathcal{L}_h^{-1}[4h^3 h_x\mathcal{L}_h^{-1}\psi]. \end{aligned}\tag{5.28}$$

Using (5.17) and the embedding $H^1 \hookrightarrow L^\infty$ we obtain

$$\begin{aligned}\|\partial_x^2\mathcal{L}_h^{-1}\psi\|_{L^2} &\leqslant C\|h_x\|_{L^2}[\|\mathcal{L}_h^{-1}\partial_x h^3\psi\|_{H^1} + \|\mathcal{L}_h^{-1}[4h^3h_x\mathcal{L}_h^{-1}\psi]\|_{H^1}] \\ &\quad + C\|\partial_x\mathcal{L}_h^{-1}\partial_x h^3\psi\|_{L^2} + C\|\partial_x\mathcal{L}_h^{-1}[4h^3h_x\mathcal{L}_h^{-1}\psi]\|_{L^2} \\ &\leqslant C\|h_x\|_{L^2}[\|\psi\|_{L^2} + \|h_x\|_{L^2}\|\mathcal{L}_h^{-1}\psi\|_{H^1}] \\ &\quad + C\|\psi\|_{L^2} + C\|h_x\|_{L^2}\|\mathcal{L}_h^{-1}\psi\|_{H^1} \\ &\leqslant C[1 + \|h_x\|_{L^2}^2]\|\psi\|_{L^2}.\end{aligned}$$

This with (5.17) implies (5.18).

Now differentiating (5.26) with respect to x, using the definition of $\mathcal{L}_h$ and replacing $h^{-3}\psi$ by ψ, we obtain

$$\partial_x\mathcal{L}_h^{-1}\partial_x h^3\psi = \partial_x\mathcal{L}_h^{-1}[4h^3h_x\mathcal{L}_h^{-1}\psi] + 3h\mathcal{L}_h^{-1}\psi - 3\psi.$$

Then (5.28) becomes

$$\partial_x^2\mathcal{L}_h^{-1}\psi = -3h^{-2}h_x[\mathcal{L}_h^{-1}\partial_x h^3\psi - \mathcal{L}_h^{-1}[4h^3h_x\mathcal{L}_h^{-1}\psi]] + 3h^{-2}\mathcal{L}_h^{-1}\psi - 3h^{-3}\psi.$$

Then, using (5.17), we obtain (5.19).

Step 2. Using $\mathcal{L}_h u = \psi + \varphi_x$ and the Young inequality $ab \leqslant \frac{1}{2\alpha}a^2 + \frac{\alpha}{2}b^2$ with $\alpha > 0$, we obtain

$$\begin{aligned}\|u_x\|_{H^1}^2 &= (u_x, u_x) + (u_{xx}, u_{xx}) \\ &\leqslant C[(hu_x, u_x) + \frac{1}{3}(h^3u_{xx}, u_{xx})] \\ &= C\Big[-(hu, u_{xx}) - (h_xu, u_x) + \frac{1}{3}((h^3u_x)_x - (h^3)_xu_x, u_{xx})\Big] \\ &= C[-(\mathcal{L}_hu, u_{xx}) - (h_xu, u_x) - (h^2h_xu_x, u_{xx})] \\ &\leqslant C[\alpha\|u_{xx}\|_{L^2}^2 + \frac{1}{\alpha}\|\mathcal{L}_hu\|_{L^2}^2 + C_\alpha(1 + \|h_x\|_{L^\infty}^2)\|u\|_{H^1}^2].\end{aligned}$$

Taking $\alpha > 0$ small enough we obtain

$$\|u_x\|_{H^1}^2 \leqslant C[\|\mathcal{L}_hu\|_{L^2}^2 + (1 + \|h_x\|_{L^\infty}^2)\|u\|_{H^1}^2],$$

then

$$\|u_x\|_{H^1} \leqslant C[\|\mathcal{L}_h u\|_{L^2} + (1 + \|h_x\|_{L^\infty})\|u\|_{H^1}].$$

Taking $\psi = 0$ (respectively $\varphi = 0$) and using (5.27), we obtain

$$\|\mathcal{L}_h^{-1}\partial_x\varphi\|_{H^2} \leqslant C(1 + \|h_x\|_{L^\infty})\|\varphi\|_{H^1}, \quad \|\mathcal{L}_h^{-1}\psi\|_{H^2} \leqslant C(1 + \|h_x\|_{L^\infty})\|\psi\|_{L^2}. \quad (5.29)$$

Let Λ be defined as $\widehat{\Lambda f} = (1 + \xi^2)^{1/2}\hat{f}$. Since $\mathcal{L}_h u = \psi + \varphi_x$, we have

$$\mathcal{L}_h \Lambda^2 u = [h, \Lambda^2]u + \Lambda^2\psi + \partial_x\Big\{-\frac{1}{3}[h^3, \Lambda^2]u_x + \Lambda^2\varphi\Big\}.$$

Defining $\tilde{u} = \Lambda^2 u$, $\tilde{\psi} = [h, \Lambda^2]u + \Lambda^2\psi$ and $\tilde{\varphi} = -\frac{1}{3}[h^3, \Lambda^2]u_x + \Lambda^2\varphi$ and using (5.27), (A.3) we obtain

$$\begin{aligned}\|\Lambda^2 u\|_{H^1} &\leqslant C[\|[h, \Lambda^2]u\|_{L^2} + \|[h^3, \Lambda^2]u_x\|_{L^2} + \|\psi\|_{H^2} + \|\varphi\|_{H^2}] \\ &\leqslant C[\|h_x\|_{L^\infty}\|u\|_{H^2} + \|h - \bar{h}\|_{H^2}\|u\|_{W^{1,\infty}} + \|\psi\|_{H^2} + \|\varphi\|_{H^2}].\end{aligned}$$

Taking $\psi = 0$ (respectively $\varphi = 0$) and using (5.27) with (5.29), we obtain (5.20).

Step 3. It remains only to prove the inequalities (5.21). Since the operator $(g - \gamma\partial_x^2)^{-1}$ is nothing but a convolution with the function $\mathfrak{G}$, the result follows directly using the Young inequality. ∎

Lemma 5.3. *Let $(h - \bar{h}, u) \in H^1(\mathbb{R})$ such that $\int_{\mathbb{R}} \mathcal{E}\,\mathrm{d}x \leqslant E < \sqrt{g\gamma}\bar{h}^2$. Then there exists a constant $C = C(\gamma, \bar{h}, E) > 0$ independent of ε and h such that*

$$\|\mathcal{L}_h^{-1}\partial_x\mathcal{C}\|_{L^\infty(\mathbb{R})} + \|\mathcal{R}\|_{L^\infty(\mathbb{R})} \leqslant C, \quad (5.30)$$

$$\int_{\mathbb{R}} (\chi_\varepsilon(P) + \chi_\varepsilon(Q))\,\mathrm{d}x \leqslant C, \quad (5.31)$$

$$\|\mathcal{L}_h^{-1}\partial_x\{h(\chi_\varepsilon(P) + \chi_\varepsilon(Q))\}\|_{L^\infty(\mathbb{R})} \leqslant C, \quad (5.32)$$

$$\|(\mathcal{L}_h^{-1}\partial_x\{h^2u_x\mathcal{A}_x^\varepsilon\}, \mathcal{L}_h^{-1}\{u\mathcal{A}_x^\varepsilon\})\|_{L^\infty(\mathbb{R})} \leqslant C, \quad (5.33)$$

$$\|\mathcal{A}_x^\varepsilon\|_{L^2} + \|(\mathcal{A}^\varepsilon, \mathcal{A}_x^\varepsilon, \mathcal{B}^\varepsilon, \mathcal{V}_1^\varepsilon, \mathcal{V}_2^\varepsilon)\|_{L^\infty(\mathbb{R})} \leqslant C, \quad (5.34)$$

where $\mathcal{A}^\varepsilon$, $\mathcal{B}^\varepsilon$, $\mathcal{V}_1^\varepsilon$ and $\mathcal{V}_2^\varepsilon$ are defined as in (5.3), (5.4), (5.6) and (5.7) by replacing $(h^\varepsilon, u^\varepsilon)$ with (h, u).

Proof. From $\int_{\mathbb{R}} \mathcal{E}\,\mathrm{d}x \leqslant E$ we have $\|(\mathcal{C}, P^2, Q^2)\|_{L^1} \leqslant C$. Then the proof of (5.30) follows from (4.5), (5.11), (5.14) and (5.18). Since $\chi_\varepsilon(\lambda) \leqslant \lambda^2$ we obtain (5.31). Then (5.31) with (5.14) implies (5.32). In remains to prove (5.34). For that purpose, we use the Young inequality, (5.3) and (5.7) to obtain

$$\|\mathcal{A}\|_{L^\infty} + \|\mathcal{A}_x\|_{L^2} + \|\mathcal{A}_x\|_{L^\infty} + \|\mathcal{V}_2\|_{L^\infty} \leqslant C. \quad (5.35)$$

The estimates (5.11), (5.14), (5.31), (5.6), (5.35), (5.4) and the Cauchy–Schwarz inequality imply (5.34). ∎

Proof of Theorem 5.1. Following [26, 27, 30, 39, 44], we can easily prove the local existence of solutions of (5.2). Integrating the energy equation (5.8) on $[0,t]\times\mathbb{R}$, we obtain (5.9).

Step 1. Defining

$$U^\varepsilon \stackrel{\text{def}}{=} (h^\varepsilon - \bar{h}, u^\varepsilon)^\top, \qquad A(U^\varepsilon) \stackrel{\text{def}}{=} \begin{pmatrix} 3\gamma(h^\varepsilon)^{-3} & 0 \\ 0 & h^\varepsilon \end{pmatrix},$$

$$B(U^\varepsilon) \stackrel{\text{def}}{=} \begin{pmatrix} u^\varepsilon & h^\varepsilon \\ 3\gamma(h^\varepsilon)^{-3} & u^\varepsilon \end{pmatrix}, \quad \mathcal{F}^\varepsilon(U^\varepsilon) \stackrel{\text{def}}{=} \begin{pmatrix} \mathcal{A}_x^\varepsilon \\ -\mathcal{L}_{h^\varepsilon}^{-1}\partial_x\{\mathcal{C}^\varepsilon + F(h^\varepsilon)\} + \mathcal{B}^\varepsilon \end{pmatrix},$$

system (5.2) becomes

$$U_t^\varepsilon + B(U^\varepsilon)U_x^\varepsilon = \mathcal{F}^\varepsilon(U^\varepsilon). \tag{5.36}$$

Let $(\cdot,\cdot)$ be the scalar product in L^2 and $E(U^\varepsilon) \stackrel{\text{def}}{=} (\Lambda^3 U^\varepsilon, A^\varepsilon \Lambda^3 U^\varepsilon)$. Since $A^\varepsilon B^\varepsilon$ is a symmetric matrix, straightforward calculations with (5.36) imply that

$$\begin{aligned} E(U^\varepsilon)_t &= -2\big([\Lambda^3, B^\varepsilon]U_x^\varepsilon, A^\varepsilon\Lambda^3U^\varepsilon\big) - 2\big(B^\varepsilon\Lambda^3U_x^\varepsilon, A^\varepsilon\Lambda^3U^\varepsilon\big) \\ &\quad + 2\big(\Lambda^3\mathcal{F}^\varepsilon, A^\varepsilon\Lambda^3U^\varepsilon\big) + \big(\Lambda^3U^\varepsilon, A_t^\varepsilon\Lambda^3U^\varepsilon\big) \\ &= -2\big([\Lambda^3, B^\varepsilon]U_x^\varepsilon, A^\varepsilon\Lambda^3U^\varepsilon\big) + \big(\Lambda^3U^\varepsilon, (A^\varepsilon B^\varepsilon)_x\Lambda^3U^\varepsilon\big) \\ &\quad + 2\big(\Lambda^3\mathcal{F}^\varepsilon, A^\varepsilon\Lambda^3U^\varepsilon\big) + \big(\Lambda^3U^\varepsilon, A_t^\varepsilon\Lambda^3U^\varepsilon\big). \end{aligned}$$

From the definition of χ_ε we have

$$|\chi_\varepsilon(\xi)| \leqslant \xi^2, \quad |\chi_\varepsilon'(\xi)| \leqslant 2|\xi|, \quad |\chi_\varepsilon''(\xi)| \leqslant 2.$$

Using the Gagliardo–Nirenberg interpolation inequality $\|f_x\|_{L^4}^2 \leqslant C\|f\|_{L^\infty}\|f_{xx}\|_{L^2}$ with (A.2), we then obtain

$$\begin{aligned} \|\chi_\varepsilon(P^\varepsilon)\|_{H^2} &\leqslant C[\|\chi_\varepsilon(P^\varepsilon)\|_{L^2} + \|\chi_\varepsilon'(P^\varepsilon)P_x^\varepsilon\|_{L^2} \\ &\quad + \|\chi_\varepsilon'(P^\varepsilon)P_{xx}^\varepsilon\|_{L^2} + \|\chi_\varepsilon''(P^\varepsilon)(P_x^\varepsilon)^2\|_{L^2}] \\ &\leqslant C\|P^\varepsilon\|_{L^\infty}\|P^\varepsilon\|_{H^2} \leqslant C\|U_x^\varepsilon\|_{L^\infty}\|U^\varepsilon\|_{H^3}. \end{aligned} \tag{5.37}$$

The same inequality can be obtained for Q^ε:

$$\|\chi_\varepsilon(Q^\varepsilon)\|_{H^2} \leqslant C\|U_x^\varepsilon\|_{L^\infty}\|U^\varepsilon\|_{H^3}. \tag{5.38}$$

Using (5.21) and (A.2), we obtain

$$\begin{aligned} \|(\mathcal{A}^\varepsilon, \mathcal{A}_x^\varepsilon)\|_{H^3} &\leqslant C[\|P^\varepsilon\|_{L^\infty}^2\|h^\varepsilon - \bar{h}\|_{H^2} + \|(\chi_\varepsilon(P^\varepsilon), \chi_\varepsilon(Q^\varepsilon))\|_{H^2}] \\ &\leqslant C(1 + \|U_x^\varepsilon\|_{L^\infty}^2)\|U^\varepsilon\|_{H^3}. \end{aligned} \tag{5.39}$$

Using (5.9), (5.14), (5.15), (5.17) and Lemma 5.3 we obtain

$$\|-\mathcal{L}_{h^\varepsilon}^{-1}\partial_x\{\mathcal{C}^\varepsilon + F(h^\varepsilon)\} + \mathcal{B}^\varepsilon\|_{W^{1,\infty}} \leqslant C(1 + \|U_x^\varepsilon\|_{L^\infty}^2). \tag{5.40}$$

Now using (5.20), (A.2), (A.4), (5.37), (5.38), (5.39) and (5.40) we obtain

$$\|\mathcal{B}^\varepsilon - \mathcal{L}_{h^\varepsilon}^{-1}\partial_x\{\mathcal{C}^\varepsilon + F(h^\varepsilon)\}\|_{H^3} \leqslant \mathcal{P}(\|U_x^\varepsilon\|_{L^\infty})\|U^\varepsilon\|_{H^3},$$

where $\mathcal{P}$ is a polynomial function. The last inequality with (5.39) implies that

$$\|\mathcal{F}^\varepsilon\|_{H^3} \leqslant \mathcal{P}(\|U_x^\varepsilon\|_{L^\infty})\|U^\varepsilon\|_{H^3}. \tag{5.41}$$

Defining $\overline{B} \stackrel{\text{def}}{=} B(\bar{h}, 0)$, and using (A.3) one obtains

$$\begin{aligned}|([\Lambda^3, B^\varepsilon]U_x^\varepsilon, A^\varepsilon\Lambda^3U^\varepsilon)| &\leqslant C\|A^\varepsilon\|_{L^\infty}\|U^\varepsilon\|_{H^3}\\ &\quad\times(\|B_x^\varepsilon\|_{L^\infty}\|U_x^\varepsilon\|_{H^2} + \|B^\varepsilon - \overline{B}\|_{H^3}\|U_x^\varepsilon\|_{L^\infty})\\ &\leqslant C\|U_x^\varepsilon\|_{L^\infty}\|U^\varepsilon\|_{H^3}^2.\end{aligned} \tag{5.42}$$

Using (5.2a) and (5.34) one obtains

$$|(\Lambda^3U^\varepsilon, (A^\varepsilon B^\varepsilon)_x\Lambda^3U^\varepsilon)| + |(\Lambda^3U^\varepsilon, A_t^\varepsilon\Lambda^3U^\varepsilon)| \leqslant C(\|U_x^\varepsilon\|_{L^\infty} + 1)\|U^\varepsilon\|_{H^3}^2. \tag{5.43}$$

Summing (5.41), (5.42) and (5.43) we obtain

$$E(U^\varepsilon)_t \leqslant \mathcal{P}(\|U_x^\varepsilon\|_{L^\infty})\|U^\varepsilon\|_{H^3} \leqslant \mathcal{P}(\|U_x^\varepsilon\|_{L^\infty})E(U^\varepsilon),$$

which implies, with the Grönwall inequality, that

$$\|U^\varepsilon\|_{H^3} \leqslant CE(U^\varepsilon) \leqslant CE(U_0^\varepsilon)\mathrm{e}^{\int_0^t \mathcal{P}(\|U_x^\varepsilon\|_{L^\infty})\,\mathrm{d}s} \leqslant C\|U_0^\varepsilon\|_{H^3}\mathrm{e}^{\int_0^t \mathcal{P}(\|U_x^\varepsilon\|_{L^\infty})\,\mathrm{d}s}.$$

This implies that if $T_{\max}^\varepsilon$ is the maximal existence time, then

$$T_{\max}^\varepsilon < \infty \implies \limsup_{t\to T_{\max}^\varepsilon}\|U_x^\varepsilon(t,\cdot)\|_{L^\infty} = \infty. \tag{5.44}$$

Step 2. Define

$$\begin{aligned}\mathcal{H}_1^\varepsilon &\stackrel{\text{def}}{=} \frac{1}{2}\sqrt{3\gamma}\big((h^\varepsilon)^{1/2}(u^\varepsilon)^2 + g(h^\varepsilon)^{-1/2}(h^\varepsilon - \bar{h})^2\big) - u^\varepsilon\Big(\mathcal{R}^\varepsilon + \frac{1}{2}g((h^\varepsilon)^2 - \bar{h}^2)\Big)\\ &\quad + u^\varepsilon\frac{1}{3}(h^\varepsilon)^2\mathcal{V}_1^\varepsilon + \frac{\sqrt{3\gamma}}{3}(h^\varepsilon)^{1/2}(h - \bar{h})\mathcal{V}_2^\varepsilon,\\ \mathcal{H}_2^\varepsilon &\stackrel{\text{def}}{=} \frac{1}{2}\sqrt{3\gamma}\big((h^\varepsilon)^{1/2}(u^\varepsilon)^2 + g(h^\varepsilon)^{-1/2}(h^\varepsilon - \bar{h})^2\big) + u^\varepsilon\Big(\mathcal{R}^\varepsilon + \frac{1}{2}g((h^\varepsilon)^2 - \bar{h}^2)\Big)\\ &\quad - u^\varepsilon\frac{1}{3}(h^\varepsilon)^2\mathcal{V}_1^\varepsilon - \frac{\sqrt{3\gamma}}{3}(h^\varepsilon)^{1/2}(h - \bar{h})\mathcal{V}_2^\varepsilon.\end{aligned}$$

We note that

$$\begin{aligned}\eta^\varepsilon\mathcal{E}^\varepsilon - \widetilde{\mathcal{D}}^\varepsilon &= \frac{\sqrt{3\gamma}}{6}(h^\varepsilon)^{1/2}(P^\varepsilon)^2 + \mathcal{H}_1^\varepsilon,\\ \widetilde{\mathcal{D}}^\varepsilon - \lambda^\varepsilon\mathcal{E}^\varepsilon &= \frac{\sqrt{3\gamma}}{6}(h^\varepsilon)^{1/2}(Q^\varepsilon)^2 + \mathcal{H}_2^\varepsilon.\end{aligned}$$

From Lemma 5.3 we deduce that $\mathcal{H}_1^\varepsilon$ and $\mathcal{H}_2^\varepsilon$ are bounded. Then, integrating (5.8) on the set (see Figure 2)

$$\big\{(s,x),\ s\in[\tau,t],\ X_{x_1}(s)\leqslant x\leqslant Y_{x_2}(s)\big\},$$

and using the divergence theorem with (5.9) one obtains (5.10) for all $t\in[0,T^\varepsilon_{\max})$.

Define $t_1\stackrel{\rm def}{=}\inf\{t\geqslant 0,\ P^\varepsilon(t,Y_{x_2}(t))\geqslant 1\}$ and let $t_2\leqslant T^\varepsilon_{\max}$ be the largest time such that $P^\varepsilon(t,Y_{x_2}(t))\geqslant 1$ on $[t_1,t_2]$. Dividing (5.5a) by P^ε and integrating on the characteristics between t_1 and $t\in[t_1,t_2]$, we obtain with (5.10) and Lemma 5.3 that

$$P^\varepsilon(t,Y_{x_2}(t))\leqslant P^\varepsilon(t_1,Y_{x_2}(t_1))\mathrm{e}^{C(1+t)t}\quad \forall t\in[t_1,t_2].$$

Using that $P^\varepsilon(t_1,Y_{x_2}(t_1))=\max\{1,P_0^\varepsilon(x_2)\}$ and doing the same for Q^ε, we obtain

$$P^\varepsilon(t,Y_{x_2}(t))\leqslant \max\{1,P_0^\varepsilon(x_2)\}\mathrm{e}^{C(1+t)t}\quad \forall (t,x_2)\in[0,T^\varepsilon_{\max})\times\mathbb{R}, \tag{5.45}$$

$$Q^\varepsilon(t,X_{x_1}(t))\leqslant \max\{1,Q_0^\varepsilon(x_1)\}\mathrm{e}^{C(1+t)t}\quad \forall (t,x_1)\in[0,T^\varepsilon_{\max})\times\mathbb{R}. \tag{5.46}$$

On the other hand, we define $\tilde t_1\stackrel{\rm def}{=}\inf\{t\geqslant 0,\ P^\varepsilon(t,Y_{x_2}(t))\leqslant -1/\varepsilon\}$ and let $\tilde t_2\leqslant T^\varepsilon_{\max}$ be the largest time such that $P^\varepsilon(t,Y_{x_2}(t))\leqslant -1/\varepsilon$ on $[\tilde t_1,\tilde t_2]$. Using (5.5a) and Lemma 5.3 one obtains

$$\frac{\mathrm{d}^\lambda}{\mathrm{d}t}P^\varepsilon\stackrel{\rm def}{=}P^\varepsilon_t+\lambda^\varepsilon P^\varepsilon_x\geqslant C\Big(\frac{1}{\varepsilon}+1\Big)P^\varepsilon-C\quad \forall t\in[\tilde t_1,\tilde t_2].$$

Using that $P^\varepsilon(\tilde t_1,Y_{x_2}(\tilde t_1))=\min\{P_0^\varepsilon(x_2),-1/\varepsilon\}$, we obtain for all $(t,x_2)\in[0,T^\varepsilon_{\max})\times\mathbb{R}$,

$$\begin{aligned}P^\varepsilon(t,Y_{x_2}(t))\geqslant \min\Big\{&-1/\varepsilon,\min\{P_0^\varepsilon(x_2)-1/\varepsilon\}\mathrm{e}^{C(1+1/\varepsilon)t}\\ &+\frac{\varepsilon}{\varepsilon+1}\big(1-\mathrm{e}^{C(1+1/\varepsilon)t}\big)\Big\}.\end{aligned} \tag{5.47}$$

Doing the same for Q^ε, we obtain for all $(t,x_1)\in[0,T^\varepsilon_{\max})\times\mathbb{R}$,

$$\begin{aligned}Q^\varepsilon(t,X_{x_1}(t))\geqslant \min\Big\{&-1/\varepsilon,\min\{Q_0^\varepsilon(x_1)-1/\varepsilon\}\mathrm{e}^{C(1+1/\varepsilon)t}\\ &+\frac{\varepsilon}{\varepsilon+1}\big(1-\mathrm{e}^{C(1+1/\varepsilon)t}\big)\Big\}.\end{aligned} \tag{5.48}$$

Finally, using (5.44), (5.45), (5.46), (5.47) and (5.48) we deduce that $T^\varepsilon_{\max}=\infty$. ∎

The remainder of this section is devoted to obtaining some uniform (on ε) estimates of the solution of (5.2) given by Theorem 5.1. These estimates are crucial to obtain the precompactness results in next section.

Lemma 5.4. *Let $(h_0-\bar h,u_0)\in\mathfrak{D}$ and let $(h^\varepsilon-\bar h,u^\varepsilon)$ be the solution given by Theorem 5.1. Then there exists a constant $C=C(\gamma,\bar h,E)>0$ independent of $\varepsilon\leqslant\varepsilon_0$ and $(h_0-\bar h,u_0)$ such that*

$$\|\mathcal{L}_{h^\varepsilon}^{-1}\partial_x\mathcal{C}^\varepsilon\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})}+\|(\mathcal{B}^\varepsilon,\mathcal{V}_1^\varepsilon,\mathcal{V}_2^\varepsilon,\mathcal{R}^\varepsilon)\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})}+\|\mathcal{A}^\varepsilon\|_{L^\infty(\mathbb{R}^+,H^1(\mathbb{R}))}\leqslant C, \tag{5.49}$$

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}}(\chi_\varepsilon(P^\varepsilon)+\chi_\varepsilon(Q^\varepsilon))\,\mathrm{d}x\,\mathrm{d}t \leqslant \varepsilon C, \tag{5.50}$$

$$\int_{\mathbb{R}^+}\|\mathcal{L}_{h^\varepsilon}^{-1}\partial_x\{h^\varepsilon(\chi_\varepsilon(P^\varepsilon)+\chi_\varepsilon(Q^\varepsilon))\}\|_{L^\infty(\mathbb{R})}\,\mathrm{d}t \leqslant \varepsilon C, \tag{5.51}$$

$$\int_{\mathbb{R}^+}\|(\mathcal{L}_{h^\varepsilon}^{-1}\partial_x\{(h^\varepsilon)^2u_x^\varepsilon\mathcal{A}_x^\varepsilon\},\mathcal{L}_{h^\varepsilon}^{-1}\{u^\varepsilon\mathcal{A}_x^\varepsilon\})\|_{L^\infty(\mathbb{R})}\,\mathrm{d}t \leqslant \varepsilon C \tag{5.52}$$

$$\int_{\mathbb{R}^+}\|(\mathcal{A}^\varepsilon,\mathcal{A}_x^\varepsilon,\mathcal{B}^\varepsilon,\mathcal{V}_1^\varepsilon,\mathcal{V}_2^\varepsilon)\|_{L^\infty(\mathbb{R})}\,\mathrm{d}t \leqslant \varepsilon C. \tag{5.53}$$

Proof. Inequality (5.49) follows from (5.9), (5.30) and (5.34). Note that

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}}(\chi_\varepsilon(P^\varepsilon)+\chi_\varepsilon(Q^\varepsilon))\,\mathrm{d}x\,\mathrm{d}t \leqslant -\varepsilon\int_{\{P^\varepsilon\leqslant -1/\varepsilon\}}P^\varepsilon\chi_\varepsilon(P^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t - \varepsilon\int_{\{Q^\varepsilon\leqslant -1/\varepsilon\}}Q^\varepsilon\chi_\varepsilon(Q^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t.$$

The last inequality with (5.9) implies (5.50). Finally, we use (5.50) and Lemma 5.2 as in the proofs of (5.32), (5.33) and (5.34). We integrate on $\mathbb{R}^+$ with respect to t to obtain (5.51), (5.52) and (5.53). ∎

Lemma 5.5 (Oleinik inequality). *There exists $C > 0$ that depends only on γ, $\bar{h}$, g and E such that for all $(t,x) \in (0,\infty)\times\mathbb{R}$ and $\varepsilon \leqslant \varepsilon_0$ we have*

$$P^\varepsilon(t,x) \leqslant C(1+t^{-1}), \quad Q^\varepsilon(t,x) \leqslant C(1+t^{-1}).$$

Proof. Let $D > 0$ be a constant such that $2D^{-1} \leqslant 16h^\varepsilon \leqslant D$, and let $A, B > 0$ be the constants given in Theorem 5.1. Using Lemma 5.4, we obtain a constant $M > 0$ depending only on γ, $\bar{h}$ and E such that

$$M \geqslant \sup_{t,x}\left\{\frac{1}{h^\varepsilon}(\mathcal{A}_x^\varepsilon)^2+\mathcal{M}^\varepsilon\right\}+DA.$$

Define

$$\mathcal{F}(s) \overset{\text{def}}{=} \frac{D}{s}+\sqrt{2MD}, \quad \mathcal{G}(s) \overset{\text{def}}{=} \mathcal{F}(s)+BD.$$

The goal is to prove that for all t and x we have $P^\varepsilon(t,X_x(t)) \leqslant \mathcal{G}(t)$ and $Q^\varepsilon(t,Y_x(t)) \leqslant \mathcal{G}(t)$. Since the proof is the same, we only prove the inequality for P^ε.

Using the inequality $-\mathcal{A}_x^\varepsilon P^\varepsilon \leqslant 2(\mathcal{A}^\varepsilon)_x^2+(P^\varepsilon)^2/8$ and using (5.5a), we obtain

$$\begin{aligned}\frac{\mathrm{d}^\lambda}{\mathrm{d}t}P^\varepsilon &\leqslant -\frac{1}{16h^\varepsilon}(P^\varepsilon)^2+\frac{1}{8h^\varepsilon}\chi_\varepsilon(P^\varepsilon)+\frac{1}{8h^\varepsilon}(Q^\varepsilon)^2+\frac{1}{h^\varepsilon}(\mathcal{A}_x^\varepsilon)^2+\mathcal{M}^\varepsilon\\ &\leqslant -\frac{1}{D}(P^\varepsilon)^2+D(Q^\varepsilon)^2+\frac{1}{8h^\varepsilon}\chi_\varepsilon(P^\varepsilon)+M-AD.\end{aligned} \tag{5.54}$$

Let $x \in \mathbb{R}$ be fixed. We suppose that there exist $t_1 > 0$ such that $P^\varepsilon(t_1,X_x(t_1)) = \mathcal{F}(t_1)$ and $P^\varepsilon(t,X_x(t)) \geqslant \mathcal{F}(t)$ for all $t \in [t_1,t_2]$. Since $P^\varepsilon \geqslant 0$ we have $\chi_\varepsilon(P^\varepsilon) = 0$. Integrating

(5.54) on the characteristics between t_1 and $t \in [t_1, t_2]$ we obtain

$$\begin{aligned} P^\varepsilon(t, X_x(t)) &\leqslant P^\varepsilon(t_1, X_x(t_1)) \\ &\quad - \int_{t_1}^{t} \frac{\mathcal{F}(s)^2}{D}\,\mathrm{d}s + AD(t-t_1) + BD + (M - AD)(t - t_1) \\ &= \mathcal{F}(t_1) + \frac{D}{t} - \frac{D}{t_1} - M(t-t_1) - 2\sqrt{2MD}\ln(t/t_1) + BD \\ &\leqslant \mathcal{G}(t). \end{aligned} \tag{5.55}$$

Since the solution $(h^\varepsilon - \bar{h}, u^\varepsilon) \in L^\infty(\mathbb{R}^+, H^3)$, initially we have $P^\varepsilon(0^+, X_x(0^+)) < \mathcal{F}(0^+) = \infty$. Inequality (5.55) shows that if P^ε crosses $\mathcal{F}$ at some $t_1 > 0$, P^ε remains always smaller than $\mathcal{G}$ for $t \geqslant t_1$. This completes the proof of $P^\varepsilon(t, X_x(t)) \leqslant \mathcal{G}(t)$ for all $t > 0$. The proof for Q^ε can be done similarly. ■

Lemma 5.6 ($L^{2+\alpha}$ estimates). *For any bounded set* $\Omega = [t_1, t_2] \times [a, b] \subset (0, \infty) \times \mathbb{R}$ *and* $\alpha \in (0, 1)$ *there exists* $C_{\alpha,\Omega} > 0$ *such that for all* $\varepsilon \leqslant \varepsilon_0$ *we have*

$$\int_\Omega [|h_t^\varepsilon|^{2+\alpha} + |h_x^\varepsilon|^{2+\alpha} + |u_t^\varepsilon|^{2+\alpha} + |u_x^\varepsilon|^{2+\alpha}]\,\mathrm{d}x\,\mathrm{d}t \leqslant C_{\alpha,\Omega}, \tag{5.56}$$

$$g\left\| \mathcal{L}_{h^\varepsilon}^{-1}\left(h^\varepsilon \int_{-\infty}^{x} (h^\varepsilon)^{-3}\mathcal{C}^\varepsilon\,\mathrm{d}y + \frac{1}{3}F(h^\varepsilon)_x \right) g \right\|_{L^\infty([t_1,t_2], W^{2,2+\alpha}([a,b]))} \leqslant C_{\alpha,\Omega}. \tag{5.57}$$

Remark 5.7. The constant $C_{\alpha,\Omega}$ depends also on $\bar{h}$, γ and E but not on ε and the initial data.

Proof of Lemma 5.6. *Step* 1. In order to prove (5.56) we use the change of variables

$$\tau \stackrel{\text{def}}{=} t, \quad z \stackrel{\text{def}}{=} \frac{1}{2}\left(\int_{-\infty}^{x} - \int_{x}^{\infty} \right)(h^\varepsilon(t, y) - \bar{h})\,\mathrm{d}y + \bar{h}x,$$

we obtain with (5.2a) that

$$\partial_x = h^\varepsilon \partial_z, \quad \partial_t = \partial_\tau + (\mathcal{A}^\varepsilon - h^\varepsilon u^\varepsilon)\partial_z, \quad \partial_t + u^\varepsilon \partial_x = \partial_\tau + \mathcal{A}^\varepsilon \partial_z.$$

The map

$$\Phi\colon \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+ \times \mathbb{R}, \quad (t, x) \mapsto \Phi(t, x) = (\tau, z)$$

is bijective. Then (5.5) becomes

$$\begin{aligned} P_\tau^\varepsilon + (\mathcal{A}^\varepsilon - \sqrt{3\gamma}(h^\varepsilon)^{1/2})P_z^\varepsilon &= -\frac{1}{8h^\varepsilon}(P^\varepsilon)^2 + \frac{1}{8h^\varepsilon}\chi_\varepsilon(P^\varepsilon) + \frac{1}{8h^\varepsilon}(Q^\varepsilon)^2 \\ &\quad - \frac{1}{2h^\varepsilon}\mathcal{A}_x^\varepsilon P^\varepsilon + \mathcal{M}^\varepsilon, \end{aligned} \tag{5.58a}$$

$$\begin{aligned} Q_\tau^\varepsilon + (\mathcal{A}^\varepsilon + \sqrt{3\gamma}(h^\varepsilon)^{1/2})Q_z^\varepsilon &= -\frac{1}{8h^\varepsilon}(Q^\varepsilon)^2 + \frac{1}{8h^\varepsilon}\chi_\varepsilon(Q^\varepsilon) + \frac{1}{8h^\varepsilon}(P^\varepsilon)^2 \\ &\quad - \frac{1}{2h^\varepsilon}\mathcal{A}_x^\varepsilon Q^\varepsilon + \mathcal{N}^\varepsilon. \end{aligned} \tag{5.58b}$$

Without loss of generality, we suppose that $\alpha = 2k/(2k+1)$ with $k \in \mathbb{N}$. Multiplying (5.58a) by $(P^\varepsilon)^\alpha$, using $\frac{2}{\alpha+1} = 1 + \frac{1-\alpha}{\alpha+1}$, $(P^\varepsilon)^\alpha \chi_\varepsilon(P^\varepsilon) \geqslant 0$ and (5.49) one obtains

$$\begin{aligned}\frac{1}{8h^\varepsilon}\Big\{\frac{1-\alpha}{\alpha+1}(P^\varepsilon)^{\alpha+1}(P^\varepsilon - Q^\varepsilon) - (P^\varepsilon)^\alpha Q^\varepsilon (P^\varepsilon - Q^\varepsilon)\Big\}\\ \leqslant \Big(\frac{(P^\varepsilon)^{\alpha+1}}{\alpha+1}\Big)_\tau + \Big(\frac{\mathcal{A}^\varepsilon - \sqrt{3\gamma}(h^\varepsilon)^{1/2}}{\alpha+1}(P^\varepsilon)^{\alpha+1}\Big)_z\\ + C(|P^\varepsilon|^{\alpha+1} + (P^\varepsilon)^\alpha).\end{aligned}$$

Doing the same with (5.5b), we obtain

$$\begin{aligned}\frac{1}{8h^\varepsilon}\Big\{\frac{1-\alpha}{\alpha+1}(Q^\varepsilon)^{\alpha+1}(Q^\varepsilon - P^\varepsilon) - (Q^\varepsilon)^\alpha P^\varepsilon (Q^\varepsilon - P^\varepsilon)\Big\}\\ \leqslant \Big(\frac{(Q^\varepsilon)^{\alpha+1}}{\alpha+1}\Big)_\tau + \Big(\frac{\mathcal{A}^\varepsilon + \sqrt{3\gamma}(h^\varepsilon)^{1/2}}{\alpha+1}(Q^\varepsilon)^{\alpha+1}\Big)_z\\ + C(|Q^\varepsilon|^{\alpha+1} + (Q^\varepsilon)^\alpha).\end{aligned}$$

Adding both equations yields

$$\begin{aligned}\frac{1}{8(h^\varepsilon)}\Big\{\frac{1-\alpha}{\alpha+1}((P^\varepsilon)^{\alpha+1} - (Q^\varepsilon)^{\alpha+1})(P^\varepsilon - Q^\varepsilon)\\ + (P^\varepsilon)^\alpha (Q^\varepsilon)^\alpha ((P^\varepsilon)^{1-\alpha} - (Q^\varepsilon)^{1-\alpha})(P^\varepsilon - Q^\varepsilon)\Big\}\\ \leqslant \Big(\frac{(P^\varepsilon)^{\alpha+1} + (Q^\varepsilon)^{\alpha+1}}{\alpha+1}\Big)_\tau\\ + \Big(\frac{\sqrt{3\gamma}(h^\varepsilon)^{1/2}((Q^\varepsilon)^{\alpha+1} - (P^\varepsilon)^{\alpha+1}) + \mathcal{A}^\varepsilon((Q^\varepsilon)^{\alpha+1} + (P^\varepsilon)^{\alpha+1})}{\alpha+1}\Big)_z\\ + C(|Q^\varepsilon|^{\alpha+1} + (Q^\varepsilon)^\alpha + |P^\varepsilon|^{\alpha+1} + (P^\varepsilon)^\alpha). \qquad (5.59)\end{aligned}$$

Let $\varphi \in C_c^\infty((t_1/2, t_2+1) \times (a-1, b+1))$ be a non-negative function such that $\varphi(t,x) = 1$ on Ω. Multiplying (5.59) by $\varphi(\Phi^{-1}(\tau,z))$ and using integration by parts with (5.9) we obtain

$$\begin{aligned}\frac{1-\alpha}{\alpha+1}\int_{\mathbb{R}^+\times\mathbb{R}} \varphi(P^\varepsilon - Q^\varepsilon)^2((P^\varepsilon)^\alpha + (Q^\varepsilon)^\alpha)\,\mathrm{d}x\,\mathrm{d}t\\ \leqslant \int_{\mathbb{R}^+\times\mathbb{R}}\Big\{\frac{1-\alpha}{\alpha+1}((P^\varepsilon)^{\alpha+1} - (Q^\varepsilon)^{\alpha+1})(P^\varepsilon - Q^\varepsilon)\Big\}\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t\\ + \int_{\mathbb{R}^+\times\mathbb{R}}\{(P^\varepsilon)^\alpha (Q^\varepsilon)^\alpha ((P^\varepsilon)^{1-\alpha} - (Q^\varepsilon)^{1-\alpha})(P^\varepsilon - Q^\varepsilon)\}\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t\\ = \int_{\mathbb{R}^+\times\mathbb{R}}\Big\{\frac{1-\alpha}{\alpha+1}((P^\varepsilon)^{\alpha+1} - (Q^\varepsilon)^{\alpha+1})(P^\varepsilon - Q^\varepsilon)\Big\}\varphi(\Phi^{-1}(\tau,z))\,\frac{\mathrm{d}z\,\mathrm{d}\tau}{h}\\ + \int_{\mathbb{R}^+\times\mathbb{R}}\{(P^\varepsilon)^\alpha (Q^\varepsilon)^\alpha ((P^\varepsilon)^{1-\alpha} - (Q^\varepsilon)^{1-\alpha})(P^\varepsilon - Q^\varepsilon)\}\varphi(\Phi^{-1}(\tau,z))\,\frac{\mathrm{d}z\,\mathrm{d}\tau}{h}\end{aligned}$$

$$
\begin{aligned}
&\leqslant 8\int_{\mathbb{R}^+\times\mathbb{R}}\Big[C(|Q^\varepsilon|^{\alpha+1}+(Q^\varepsilon)^\alpha+|P^\varepsilon|^{\alpha+1}+(P^\varepsilon)^\alpha)\varphi\,\mathrm{d}z\\
&\qquad\qquad -\Big(\frac{(P^\varepsilon)^{\alpha+1}+(Q^\varepsilon)^{\alpha+1}}{\alpha+1}\Big)\varphi_\tau\Big]\,\mathrm{d}z\,\mathrm{d}\tau\\
&\quad -8\int_{\mathbb{R}^+\times\mathbb{R}}\Big(\frac{\sqrt{3\gamma}(h^\varepsilon)^{1/2}((Q^\varepsilon)^{\alpha+1}-(P^\varepsilon)^{\alpha+1})+\mathcal{A}^\varepsilon((Q^\varepsilon)^{\alpha+1}+(P^\varepsilon)^{\alpha+1})}{\alpha+1}\Big)\varphi_z\,\mathrm{d}z\,\mathrm{d}\tau\\
&= 8\int_{\mathbb{R}^+\times\mathbb{R}} C(|Q^\varepsilon|^{\alpha+1}+(Q^\varepsilon)^\alpha+|P^\varepsilon|^{\alpha+1}+(P^\varepsilon)^\alpha)\varphi h^\varepsilon\,\mathrm{d}x\,\mathrm{d}t\\
&\quad -8\int_{\mathbb{R}^+\times\mathbb{R}}\Big(\frac{(P^\varepsilon)^{\alpha+1}+(Q^\varepsilon)^{\alpha+1}}{\alpha+1}\Big)h^\varepsilon(\varphi_t+u^\varepsilon\varphi_x-(h^\varepsilon)^{-1}\mathcal{A}^\varepsilon\varphi_x)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad -8\int_{\mathbb{R}^+\times\mathbb{R}}\Big(\frac{\sqrt{3\gamma}(h^\varepsilon)^{1/2}((Q^\varepsilon)^{\alpha+1}-(P^\varepsilon)^{\alpha+1})+\mathcal{A}^\varepsilon((Q^\varepsilon)^{\alpha+1}+(P^\varepsilon)^{\alpha+1})}{\alpha+1}\Big)\varphi_x\,\mathrm{d}x\,\mathrm{d}t\\
&\leqslant C_{\alpha,\Omega}.
\end{aligned}
$$

The last inequality follows from (5.9) and from the fact that φ is compactly supported. Then we have

$$
\int_\Omega (P^\varepsilon-Q^\varepsilon)^2((P^\varepsilon)^\alpha+(Q^\varepsilon)^\alpha)\,\mathrm{d}x\,\mathrm{d}t\leqslant C_{\alpha,\Omega}. \tag{5.60}
$$

Step 2. Multiplying (5.59) by $(h^\varepsilon)^{-1/2}$ we obtain

$$
\begin{aligned}
&\frac{1}{8(h^\varepsilon)^{3/2}}\Big\{\frac{1-\alpha}{\alpha+1}((P^\varepsilon)^{\alpha+1}-(Q^\varepsilon)^{\alpha+1})(P^\varepsilon-Q^\varepsilon)\\
&\qquad +(P^\varepsilon)^\alpha(Q^\varepsilon)^\alpha((P^\varepsilon)^{1-\alpha}-(Q^\varepsilon)^{1-\alpha})(P^\varepsilon-Q^\varepsilon)\Big\}\\
&\quad\leqslant\Big(\frac{(P^\varepsilon)^{\alpha+1}+(Q^\varepsilon)^{\alpha+1}}{(\alpha+1)(h^\varepsilon)^{1/2}}\Big)_\tau+\Big(\frac{\sqrt{3\gamma}((Q^\varepsilon)^{\alpha+1}-(P^\varepsilon)^{\alpha+1})}{\alpha+1}\Big)_z\\
&\qquad+\Big(\frac{\mathcal{A}^\varepsilon((Q^\varepsilon)^{\alpha+1}+(P^\varepsilon)^{\alpha+1})}{(\alpha+1)(h^\varepsilon)^{1/2}}\Big)_z\\
&\qquad+(h^\varepsilon)^{-1/2}C(|Q^\varepsilon|^{\alpha+1}+(Q^\varepsilon)^\alpha+|P^\varepsilon|^{\alpha+1}+(P^\varepsilon)^\alpha)\\
&\qquad+\frac{1}{2}\mathcal{A}^\varepsilon_x\frac{(P^\varepsilon)^{\alpha+1}+(Q^\varepsilon)^{\alpha+1}}{(\alpha+1)(h^\varepsilon)^{3/2}}-\frac{1}{8(h^\varepsilon)^{3/2}}\frac{4}{\alpha+1}\{(P^\varepsilon)^{\alpha+1}Q^\varepsilon+(Q^\varepsilon)^{\alpha+1}P^\varepsilon\}.
\end{aligned}
$$

Using (5.49) one obtains

$$
\begin{aligned}
&\frac{1}{8(h^\varepsilon)^{3/2}}\Big\{\frac{1-\alpha}{\alpha+1}((P^\varepsilon)^{\alpha+1}+(Q^\varepsilon)^{\alpha+1})(P^\varepsilon+Q^\varepsilon)\\
&\qquad +(P^\varepsilon)^\alpha(Q^\varepsilon)^\alpha((P^\varepsilon)^{1-\alpha}+(Q^\varepsilon)^{1-\alpha})(P^\varepsilon+Q^\varepsilon)\Big\}\\
&\quad\leqslant\Big(\frac{(P^\varepsilon)^{\alpha+1}+(Q^\varepsilon)^{\alpha+1}}{(\alpha+1)(h^\varepsilon)^{1/2}}\Big)_\tau+\Big(\frac{\sqrt{3\gamma}((Q^\varepsilon)^{\alpha+1}-(P^\varepsilon)^{\alpha+1})}{\alpha+1}\Big)_z\\
&\qquad+\Big(\frac{\mathcal{A}^\varepsilon((Q^\varepsilon)^{\alpha+1}+(P^\varepsilon)^{\alpha+1})}{(\alpha+1)(h^\varepsilon)^{1/2}}\Big)_z+C(|Q^\varepsilon|^{\alpha+1}+(Q^\varepsilon)^\alpha+|P^\varepsilon|^{\alpha+1}+(P^\varepsilon)^\alpha).
\end{aligned}
$$

As in the first step we obtain

$$\int_\Omega (P^\varepsilon + Q^\varepsilon)^2((P^\varepsilon)^\alpha + (Q^\varepsilon)^\alpha)\,\mathrm{d}x\,\mathrm{d}t \leqslant C_{\alpha,\Omega}. \tag{5.61}$$

Summing (5.60) and (5.61) one obtains

$$\int_\Omega ((P^\varepsilon)^{\alpha+2} + (Q^\varepsilon)^{\alpha+2})\,\mathrm{d}x\,\mathrm{d}t \leqslant C_{\alpha,\Omega} \quad \Longrightarrow \quad \int_\Omega [|u_x^\varepsilon|^{2+\alpha} + |h_x^\varepsilon|^{2+\alpha}]\,\mathrm{d}x\,\mathrm{d}t \leqslant C_{\alpha,\Omega}.$$

Step 3. Inequality (5.56) follows directly from (5.2) and Lemma 5.4. Finally, using (5.11), (5.12), (5.18), (5.19) and (5.56) we obtain (5.57). ■

6. Precompactness

The goal of this section is to obtain a compactness of the solution. Due to the nonlinear terms in the equations, strong precompactness is needed to pass to the limit $\varepsilon \to 0$. Strong precompactness of $(h^\varepsilon)_\varepsilon$ and $(u^\varepsilon)_\varepsilon$ is easy to obtain. However, strong precompactness of $(P^\varepsilon)_\varepsilon$ and $(Q^\varepsilon)_\varepsilon$ is more challenging. Several lemmas in this section are inspired by [15, 59, 61–64]. Throughout the section, Lemma A.2 is used many times without mentioning it.

We start with strong precompactness of $(h^\varepsilon)_\varepsilon$ and $(u^\varepsilon)_\varepsilon$.

Lemma 6.1. *There exist $(h-\bar{h}, u) \in L^\infty([0,\infty), H^1(\mathbb{R}))$ and a subsequence of $(h^\varepsilon, u^\varepsilon)_\varepsilon$ such that we have the following convergences:*

$$\begin{aligned} (h^\varepsilon - \bar{h}, u^\varepsilon) \quad &\to \quad (h-\bar{h}, u) \quad \text{in } L^\infty_{\mathrm{loc}}([0,\infty)\times\mathbb{R}),\\ (h^\varepsilon - \bar{h}, u^\varepsilon) \quad &\rightharpoonup \quad (h-\bar{h}, u) \quad \text{in } H^1([0,T]\times\mathbb{R}),\ \forall T>0. \end{aligned}$$

Proof. From the energy equation (5.9) we have that $(h^\varepsilon - \bar{h}, u^\varepsilon)$ is uniformly bounded in $L^\infty([0,\infty), H^1(\mathbb{R}))$. Then, using (5.2), and (5.49), we obtain

$$\|(h_t^\varepsilon, u_t^\varepsilon)\|_{L^2([0,T]\times\mathbb{R})} \leqslant C_T. \tag{6.1}$$

The weak convergence in $H^1([0,T]\times\mathbb{R})$ follows directly. Using the inequality

$$\|\theta(t,\cdot)-\theta(s,\cdot)\|^2_{L^2(\mathbb{R})} = \int_\mathbb{R}\left(\int_s^t \theta_t(\tau,x)\,\mathrm{d}\tau\right)^2\mathrm{d}x \leqslant |t-s|\,\|\theta_t\|^2_{L^2([0,T]\times\mathbb{R})},$$

with (6.1) we obtain

$$\lim_{t\to s}\|u^\varepsilon(t,\cdot)-u^\varepsilon(s,\cdot)\|_{L^2(\mathbb{R})} + \lim_{t\to s}\|h^\varepsilon(t,\cdot)-h^\varepsilon(s,\cdot)\|_{L^2(\mathbb{R})} = 0$$

uniformly on ε. Then, using [55, Theorem 5], we can deduce that up to a subsequence, $(h^\varepsilon, u^\varepsilon)$ converges uniformly to (h,u) on any compact set of $[0,\infty)\times\mathbb{R}$ when $\varepsilon \to 0$. ■

Now we establish the weak precompactness of $(P^\varepsilon)_\varepsilon$ and $(Q^\varepsilon)_\varepsilon$.

Lemma 6.2. *There exist a subsequence of $\{P^\varepsilon, Q^\varepsilon\}_\varepsilon$, also denoted $\{P^\varepsilon, Q^\varepsilon\}_\varepsilon$, and families of probability Young measures $\nu^1_{t,x}$, $\nu^2_{t,x}$ on $\mathbb{R}$ and $\mu_{t,x}$ on $\mathbb{R}^2$, such that for all functions $f, \phi \in C_c^\infty(\mathbb{R})$, $g \in C(\mathbb{R}^2)$ with $g(\xi,\zeta) = \mathcal{O}(|\xi|^2 + |\zeta|^2)$ at infinity, and for all $\varphi \in C_c^\infty((0,\infty)\times\mathbb{R})$ we have*

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}} \phi(x) f(P^\varepsilon(t,x))\,\mathrm{d}x = \int_{\mathbb{R}} \phi(x)\int_{\mathbb{R}} f(\xi)\,\mathrm{d}\nu^1_{t,x}(\xi)\,\mathrm{d}x, \tag{6.2}$$

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}} \phi(x) f(Q^\varepsilon(t,x))\,\mathrm{d}x = \int_{\mathbb{R}} \phi(x)\int_{\mathbb{R}} f(\zeta)\,\mathrm{d}\nu^2_{t,x}(\zeta)\,\mathrm{d}x, \tag{6.3}$$

uniformly on any compact set $[0,T]\subset[0,\infty)$, and

$$\begin{aligned}\lim_{\varepsilon\to 0}&\int_{\mathbb{R}^+\times\mathbb{R}} \varphi(t,x) g(P^\varepsilon, Q^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{\mathbb{R}^+\times\mathbb{R}} \varphi(t,x)\int_{\mathbb{R}^2} g(\xi,\zeta)\,\mathrm{d}\mu_{t,x}(\xi,\zeta)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{6.4}$$

Moreover, the map

$$(t,x)\mapsto \int_{\mathbb{R}} \xi^2\,\mathrm{d}\nu^1_{t,x}(\xi) + \int_{\mathbb{R}} \zeta^2\,\mathrm{d}\nu^2_{t,x}(\zeta) \tag{6.5}$$

belongs to $L^\infty(\mathbb{R}^+, L^1(\mathbb{R}))$, and

$$\mu_{t,x}(\xi,\zeta) = \nu^1_{t,x}(\xi)\otimes\nu^2_{t,x}(\zeta). \tag{6.6}$$

We define

$$\overline{g(P,Q)} \stackrel{\text{def}}{=} \int_{\mathbb{R}^2} g(\xi,\zeta)\,\mathrm{d}\mu_{t,x}(\xi,\zeta) \tag{6.7}$$

which is, from (6.4), the weak limit of $g(P^\varepsilon, Q^\varepsilon)$.

Proof of Lemma 6.2. *Step* 1. The pointwise convergence of (6.2) is a direct corollary of Lemma A.1 with $\mathcal{O} = \mathbb{R}$ and $p = 2$ and the energy equation (5.9). The key point in proving the uniform convergence is to show that the map

$$t\in[0,T]\mapsto \int_{\mathbb{R}} \phi(x) f(P^\varepsilon(t,x))\,\mathrm{d}x\,\mathrm{d}t \tag{6.8}$$

is equicontinuous. Multiplying (5.5a) by $f'(P^\varepsilon)$ one obtains

$$\begin{aligned}f(P^\varepsilon)_t + [\lambda^\varepsilon f(P^\varepsilon)]_x = {}& \frac{1}{4h^\varepsilon}(P^\varepsilon + 3Q^\varepsilon) f(P^\varepsilon) \\ &+ \Big[-\frac{1}{8h^\varepsilon}(P^\varepsilon)^2 + \frac{1}{8h^\varepsilon}\chi_\varepsilon(P^\varepsilon) + \frac{1}{8h^\varepsilon}(Q^\varepsilon)^2 \\ &- \frac{1}{2h^\varepsilon}\mathcal{A}^\varepsilon_x P^\varepsilon + \mathcal{M}^\varepsilon\Big] f'(P^\varepsilon).\end{aligned} \tag{6.9}$$

Multiplying by $\phi(x)$ and integrating over $[t_1, t_2] \times \mathbb{R}$ we have

$$\begin{aligned}
&\int_{\mathbb{R}} \phi(x)[f(P^\varepsilon(t_2,x)) - f(P^\varepsilon(t_1,x))]\,\mathrm{d}x \\
&\quad = \int_{t_1}^{t_2}\int_{\mathbb{R}} \Big[\phi'(x)\lambda^\varepsilon f(P^\varepsilon) + \frac{1}{4h^\varepsilon}\phi(x)(P^\varepsilon + 3Q^\varepsilon) f(P^\varepsilon)\Big]\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \int_{t_1}^{t_2}\int_{\mathbb{R}} \phi(x)\Big[-\frac{1}{8h^\varepsilon}(P^\varepsilon)^2 + \frac{1}{8h^\varepsilon}\chi_\varepsilon(P^\varepsilon) + \frac{1}{8h^\varepsilon}(Q^\varepsilon)^2 \\
&\qquad\qquad - \frac{1}{2h^\varepsilon}\mathcal{A}_x^\varepsilon P^\varepsilon + \mathcal{M}^\varepsilon\Big] f'(P^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Using that $f \in C_c^\infty$, the energy equation (5.9), Proposition 2.4 and Lemma 5.4, we find that the map (6.8) is equicontinuous. This finishes the proof of the uniform convergence of (6.2). The same proof can be used for (6.3). Using (A.1) we deduce that the map (6.5) belongs to $L^\infty(\mathbb{R}^+, L^1(\mathbb{R}))$.

Step 2. Now we suppose that g satisfies $g(\xi,\zeta) = o(|\xi|^2 + |\zeta|^2)$ at infinity. Then, using Lemma A.1 again, with $\mathcal{O} = (0,\infty)\times\mathbb{R}$ and $p = 2$, we obtain (6.4). If $g(\xi,\zeta) = \mathcal{O}(|\xi|^2 + |\zeta|^2)$, let ψ be a smooth cut-off function with $\psi(\xi) = 1$ for $|\xi| \leqslant 1$ and $\psi(\xi) = 0$ for $|\xi| \geqslant 2$. Then

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\mathbb{R}^+\times\mathbb{R}} \varphi(t,x) g_k(P^\varepsilon, Q^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_{\mathbb{R}^+\times\mathbb{R}} \varphi(t,x)\int_{\mathbb{R}^2} g_\kappa(\xi,\zeta)\,\mathrm{d}\mu_{t,x}(\xi,\zeta)\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned} \tag{6.10}$$

where $g_\kappa(\xi,\zeta) \overset{\text{def}}{=} g(\xi,\zeta)\psi(\frac{\xi}{\kappa})\psi(\frac{\zeta}{\kappa})$ with $\kappa > 0$. Using the Hölder inequality, Lemma 5.6 with $\Omega = \mathrm{supp}(\varphi)$ we obtain

$$\begin{aligned}
&\left|\int_{\mathbb{R}^+\times\mathbb{R}} \varphi(t,x)(g(P^\varepsilon,Q^\varepsilon) - g_\kappa(P^\varepsilon,Q^\varepsilon))\,\mathrm{d}x\,\mathrm{d}t\right| \\
&\quad\leqslant \int_{\mathrm{supp}(\varphi)\cap\{|P^\varepsilon|\geqslant\kappa \text{ or } |Q^\varepsilon|\geqslant\kappa\}} |\varphi(t,x)|\,|g(P^\varepsilon,Q^\varepsilon)|\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\leqslant C\left(\int_{\mathrm{supp}(\varphi)} |g(P^\varepsilon,Q^\varepsilon)|^{p/2}\,\mathrm{d}x\,\mathrm{d}t\right)^{2/p}\left(\int_{\mathrm{supp}(\varphi)\cap\{|P^\varepsilon|\geqslant\kappa \text{ or } |Q^\varepsilon|\geqslant\kappa\}} \mathrm{d}x\,\mathrm{d}t\right)^{(p-2)/p} \\
&\quad\leqslant C\Big[\big|\{(t,x)\in\mathrm{supp}(\varphi),\ |P^\varepsilon|\geqslant\kappa\}\big| + \big|\{(t,x)\in\mathrm{supp}(\varphi),\ |Q^\varepsilon|\geqslant\kappa\}\big|\Big]^{(p-2)/p} \\
&\quad\leqslant C\kappa^{2-p},
\end{aligned}$$

where $2 < p < 3$. The last inequality with (6.10) implies that we can interchange the limits $\kappa\to\infty$ and $\varepsilon\to 0$. Using that $|g_\kappa| \leqslant |g|$ and the dominated convergence theorem we obtain (6.4).

Step 3. It only remains to prove (6.6), for which we let $f \in C_c^\infty(\mathbb{R})$ and we rewrite (6.9) in the form

$$f(P^\varepsilon)_t + [\lambda f(P^\varepsilon)]_x = [(\lambda - \lambda^\varepsilon) f(P^\varepsilon)]_x + \frac{1}{4h^\varepsilon}(P^\varepsilon + 3Q^\varepsilon) f(P^\varepsilon) + \Big[-\frac{1}{8h^\varepsilon}(P^\varepsilon)^2 + \frac{1}{8h^\varepsilon}\chi_\varepsilon(P^\varepsilon) + \frac{1}{8h^\varepsilon}(Q^\varepsilon)^2 - \frac{1}{2h^\varepsilon}\mathcal{A}_x^\varepsilon P^\varepsilon + \mathcal{M}^\varepsilon\Big] f'(P^\varepsilon). \tag{6.11}$$

Lemma 6.1 implies that $(\lambda - \lambda^\varepsilon) f(P^\varepsilon) \to 0$ in $L^2_{\text{loc}}((0,\infty)\times\mathbb{R})$ when $\varepsilon \to 0$. This implies that $[(\lambda - \lambda^\varepsilon) f(P^\varepsilon)]_x$ is relatively compact in $H^{-1}_{\text{loc}}((0,\infty)\times\mathbb{R})$. Since $f \in C_c^\infty(\mathbb{R})$, using (5.49) and the energy equation (5.9) we obtain that the remaining terms of the right-hand side of (6.11) are uniformly bounded in $L^1_{\text{loc}}((0,\infty)\times\mathbb{R})$. Then, due to Lemma A.3, they are relatively compact in $H^{-1}_{\text{loc}}((0,\infty)\times\mathbb{R})$. Doing the same we can prove that for all $f, g \in C_c^\infty(\mathbb{R})$ the sequences

$$\{[f(P^\varepsilon)]_t + [\lambda f(P^\varepsilon)]_x\}_\varepsilon, \quad \{[g(Q^\varepsilon)]_t + [\eta g(Q^\varepsilon)]_x\}_\varepsilon$$

are relatively compact in $H^{-1}_{\text{loc}}((0,\infty)\times\mathbb{R})$. Then, using Lemma A.6 (a generalised compensated compactness result), we obtain

$$f(P^\varepsilon)g(Q^\varepsilon) \rightharpoonup \overline{f(P)}\,\overline{g(Q)} \quad \text{when } \varepsilon \to 0,$$

where $(\overline{f(P)}, \overline{g(Q)})$ is the weak limit of $(f(P^\varepsilon), g(Q^\varepsilon))$ defined in (6.7). Then, for any $\varphi \in C_c^\infty((0,\infty)\times\mathbb{R})$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^+\times\mathbb{R}}\int_{\mathbb{R}^2} \varphi(t,x) f(\xi) g(\zeta)\, \mathrm{d}\mu_{t,x}(\xi,\zeta)\, \mathrm{d}x\, \mathrm{d}t \\
&\quad = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^+\times\mathbb{R}} \varphi(t,x) f(P^\varepsilon) g(Q^\varepsilon)\, \mathrm{d}x\, \mathrm{d}t = \int_{\mathbb{R}^+\times\mathbb{R}} \varphi(t,x)\overline{f(P)}\,\overline{g(Q)}\, \mathrm{d}x\, \mathrm{d}t \\
&\quad = \int_{\mathbb{R}^+\times\mathbb{R}}\int_{\mathbb{R}^2} \varphi(t,x) f(\xi) g(\zeta)\, \mathrm{d}\nu^1_{t,x}(\xi)\otimes\nu^2_{t,x}(\zeta)\, \mathrm{d}x\, \mathrm{d}t,
\end{aligned}$$

which implies (6.6). The proof of Lemma 6.2 is completed. ∎

Using (4.2), Lemma 6.2, (5.2a), (5.53) and Lemma 6.1 we can obtain the identities

$$u_x = \frac{\overline{P} + \overline{Q}}{2h}, \quad h_x = h^{\frac{1}{2}}\frac{\overline{Q} - \overline{P}}{2\sqrt{3\gamma}}, \tag{6.12}$$

$$h_t + (hu)_x = 0. \tag{6.13}$$

Now we present some technical lemmas that are needed to obtain the strong precompactness of $(P^\varepsilon)_\varepsilon$ and $(Q^\varepsilon)_\varepsilon$.

Lemma 6.3. *As* $t \to 0$ *we have*

$$\|(h-h_0, u-u_0)\|_{H^1(\mathbb{R})} \to 0, \tag{6.14}$$

$$\int_{\mathbb{R}} (\overline{P^2} - \bar{P}^2)\,\mathrm{d}x + \int_{\mathbb{R}} (\overline{Q^2} - \bar{Q}^2)\,\mathrm{d}x \to 0. \tag{6.15}$$

Proof. Define

$$\begin{aligned}
W^\varepsilon(t,x) &\stackrel{\text{def}}{=} \left(\sqrt{\frac{g}{2}}(h^\varepsilon - \bar{h}), \sqrt{\frac{h^\varepsilon}{2}}u^\varepsilon, \frac{1}{\sqrt{12h^\varepsilon}}P^\varepsilon, \frac{1}{\sqrt{12h^\varepsilon}}Q^\varepsilon\right), \quad t \geqslant 0,\\
W(t,x) &\stackrel{\text{def}}{=} \left(\sqrt{\frac{g}{2}}(h - \bar{h}), \sqrt{\frac{h}{2}}u, \frac{1}{\sqrt{12h}}\bar{P}, \frac{1}{\sqrt{12h}}\bar{Q}\right), \quad t > 0,\\
\widetilde{W}(t,x) &\stackrel{\text{def}}{=} \left(\sqrt{\frac{g}{2}}(h - \bar{h}), \sqrt{\frac{h}{2}}u, \frac{1}{\sqrt{12h}}\sqrt{\overline{P^2}}, \frac{1}{\sqrt{12h}}\sqrt{\overline{Q^2}}\right), \quad t > 0,\\
W_0(x) &\stackrel{\text{def}}{=} \left(\sqrt{\frac{g}{2}}(h_0 - \bar{h}), \sqrt{\frac{h_0}{2}}u_0, \frac{1}{\sqrt{12h_0}}P_0, \frac{1}{\sqrt{12h_0}}Q_0\right).
\end{aligned}$$

From Lemmas 6.1 and 6.2 we have, for all $t > 0$,

$$\begin{aligned}
W^\varepsilon(t,\cdot) &\rightharpoonup W(t,\cdot) && \text{when } \varepsilon \to 0, \text{ in } L^2(\mathbb{R}),\\
(P^\varepsilon(t,\cdot)^2, Q^\varepsilon(t,\cdot)^2) &\rightharpoonup (\overline{P^2}(t,\cdot), \overline{Q^2}(t,\cdot)) && \text{when } \varepsilon \to 0, \text{ in } \mathcal{D}'(\mathbb{R}).
\end{aligned}$$

This, with Jensen's inequality, (5.9) and (5.1) implies that

$$\begin{aligned}
\|W(t)\|^2_{L^2(\mathbb{R})} \leqslant \|\widetilde{W}(t)\|^2_{L^2(\mathbb{R})} &\leqslant \liminf_{\varepsilon\to 0} \|W^\varepsilon(t)\|^2_{L^2(\mathbb{R})} = \liminf_{\varepsilon \to 0} \int_{\mathbb{R}} \mathcal{E}^\varepsilon(t,x)\,\mathrm{d}x\\
&\leqslant \lim_{\varepsilon\to 0} \int_{\mathbb{R}} \mathcal{E}_0^\varepsilon\,\mathrm{d}x = \int_{\mathbb{R}} \mathcal{E}_0\,\mathrm{d}x = \|W_0\|^2_{L^2}.
\end{aligned} \tag{6.16}$$

The energy inequality (5.9) with (5.1) implies that $(u^\varepsilon, P^\varepsilon, Q^\varepsilon)$ is bounded in the space $L^\infty([0,\infty), L^2(\mathbb{R}))$. Multiplying (5.2a) by 1, (5.2b) by $(h^\varepsilon)^{1/2}$ and (5.5a), (5.5b) by $(h^\varepsilon)^{-1/2}$ we obtain

$$\begin{aligned}
h_t^\varepsilon + [h^\varepsilon u^\varepsilon]_x &= \mathcal{A}_x^\varepsilon,\\
\left[\sqrt{h^\varepsilon}u^\varepsilon\right]_t &= -\frac{[h^\varepsilon u^\varepsilon]_x - \mathcal{A}_x^\varepsilon}{2(h^\varepsilon)^{1/2}}u^\varepsilon - (h^\varepsilon)^{1/2}u^\varepsilon u_x^\varepsilon - 3\gamma(h^\varepsilon)^{-3/2}h_x^\varepsilon\\
&\quad - (h^\varepsilon)^{1/2}\mathcal{L}_{h^\varepsilon}^{-1}\partial_x \mathcal{C}^\varepsilon + (h^\varepsilon)^{1/2}\mathcal{B}^\varepsilon,\\
\left[\frac{P^\varepsilon}{\sqrt{h^\varepsilon}}\right]_t + \left[\frac{\lambda^\varepsilon P^\varepsilon}{\sqrt{h^\varepsilon}}\right]_x &= \frac{1}{8(h^\varepsilon)^{3/2}}\left[(P^\varepsilon)^2 + \chi_\varepsilon(P^\varepsilon) + (Q^\varepsilon)^2 + 10P^\varepsilon Q^\varepsilon - 8\mathcal{A}_x^\varepsilon P^\varepsilon\right]\\
&\quad + \frac{\mathcal{M}^\varepsilon}{\sqrt{h^\varepsilon}},\\
\left[\frac{Q^\varepsilon}{\sqrt{h^\varepsilon}}\right]_t + \left[\frac{\eta^\varepsilon Q^\varepsilon}{\sqrt{h^\varepsilon}}\right]_x &= \frac{1}{8(h^\varepsilon)^{3/2}}\left[(Q^\varepsilon)^2 + \chi_\varepsilon(Q^\varepsilon) + (P^\varepsilon)^2 + 10P^\varepsilon Q^\varepsilon - 8\mathcal{A}_x^\varepsilon Q^\varepsilon\right]\\
&\quad + \frac{\mathcal{N}^\varepsilon}{\sqrt{h^\varepsilon}}.
\end{aligned}$$

Then, for all $T > 0$ and for all $\varphi \in H^1(\mathbb{R})$, the map

$$t \mapsto \int_{\mathbb{R}} \varphi(x) W^{\varepsilon} \, \mathrm{d}x$$

is uniformly (on $t \in [0, T]$ and $\varepsilon \leqslant \varepsilon_0$) continuous. Then Lemma A.5 implies that

$$W(t, \cdot) \rightharpoonup W_0 \quad \text{when } t \to 0, \text{ in } L^2(\mathbb{R}), \tag{6.17}$$

which implies that

$$\int_{\mathbb{R}} \mathcal{E}_0 \, \mathrm{d}x = \|W_0\|_{L^2}^2 \leqslant \liminf_{t \to 0} \|W\|_{L^2}^2.$$

On the other hand, (6.16) implies

$$\limsup_{t \to 0} \|W\|_{L^2}^2 = \limsup_{t \to 0} \int_{\mathbb{R}} \mathcal{E} \, \mathrm{d}x \leqslant \int_{\mathbb{R}} \mathcal{E}_0 \, \mathrm{d}x = \|W_0\|_{L^2}^2.$$

Then

$$\lim_{t \to 0} \|W\|_{L^2}^2 = \|W_0\|_{L^2}^2 = \int_{\mathbb{R}} \mathcal{E}_0 \, \mathrm{d}x, \tag{6.18}$$

which implies with (6.17) that

$$W(t, \cdot) \to W_0 \quad \text{when } t \to 0, \text{ in } L^2(\mathbb{R}). \tag{6.19}$$

Inequality (6.16) with (6.18) implies

$$\lim_{t \to 0} \|\widetilde{W}\|_{L^2}^2 = \lim_{t \to 0} \|W\|_{L^2}^2 = \|W_0\|_{L^2}^2. \tag{6.20}$$

Then (6.15) follows directly from (6.20). Using (6.19) and (6.12) we obtain the strong convergence

$$\left\| \left(h - h_0, \sqrt{h}u - \sqrt{h_0}u_0, h_x/h - h_0'/h_0, \sqrt{h}u_x - \sqrt{h_0}u_0'\right) \right\|_{L^2} \to 0$$

as $t \to 0$. In order to obtain (6.14), we write

$$\begin{aligned}
u - u_0 &= \frac{1}{\sqrt{h}}\left(\sqrt{h}u - \sqrt{h_0}u_0\right) + \sqrt{h_0}u_0\left(\frac{1}{\sqrt{h}} - \frac{1}{\sqrt{h_0}}\right), \\
h_x - h_0' &= h\left(\frac{h_x}{\sqrt{h}} - \frac{h_0'}{\sqrt{h_0}}\right) + \frac{h_0'}{\sqrt{h_0}}(h - h_0), \\
u_x - u_0' &= \frac{1}{\sqrt{h}}\left(\sqrt{h}u_x - \sqrt{h_0}u_0'\right) + \sqrt{h_0}u_0'\left(\frac{1}{\sqrt{h}} - \frac{1}{\sqrt{h_0}}\right).
\end{aligned}$$

On the right-hand side of the previous equations, the first term converges to 0 in L^2 as $t \to 0$. Since $h, 1/h \in L^\infty$, $u_0, h_0', u_0' \in L^2$ and $h \in C([0, \infty) \times \mathbb{R})$, the dominated convergence theorem implies that the second term goes to 0 as $t \to 0$. This ends the proof of (6.14). ∎

For any $\kappa > 0$, we define

$$S_\kappa(\xi) \stackrel{\text{def}}{=} \frac{1}{2}\xi^2 - \frac{1}{2}(\xi+\kappa)^2 \mathbb{1}_{\xi \leqslant -\kappa} - \frac{1}{2}(\xi-\kappa)^2 \mathbb{1}_{\xi \geqslant \kappa} = \begin{cases} -\kappa\left(\xi + \frac{1}{2}\kappa\right), & \xi \leqslant -\kappa, \\ \frac{1}{2}\xi^2, & |\xi| \leqslant \kappa, \\ \kappa\left(\xi - \frac{1}{2}\kappa\right), & \xi \geqslant \kappa. \end{cases} \tag{6.21}$$

$$T_\kappa(\xi) \stackrel{\text{def}}{=} S'_\kappa(\xi) = \xi - (\xi+\kappa)\mathbb{1}_{\xi \leqslant -\kappa} - (\xi-\kappa)\mathbb{1}_{\xi \geqslant \kappa} = \begin{cases} -\kappa, & \xi \leqslant -\kappa, \\ \xi, & |\xi| \leqslant \kappa, \\ \kappa, & \xi \geqslant \kappa. \end{cases} \tag{6.22}$$

Lemma 6.4. *For any $\kappa > 0$, there exist a subsequence $\{\mathcal{M}^\varepsilon, \mathcal{N}^\varepsilon, P^\varepsilon, Q^\varepsilon\}_\varepsilon$ and $\widetilde{\mathcal{M}} \in L^\infty_{\text{loc}}((0,\infty)\times\mathbb{R})$ such that, when $\varepsilon \to 0$, we have the limits in the sense of distributions on $(0,\infty)\times\mathbb{R}$*

$$\mathcal{M}^\varepsilon \rightharpoonup \widetilde{\mathcal{M}} \quad \textit{and} \quad \mathcal{M}^\varepsilon T_\kappa(P^\varepsilon) \rightharpoonup \overline{T_\kappa(P)}\widetilde{\mathcal{M}}, \tag{6.23}$$

$$\mathcal{N}^\varepsilon \rightharpoonup \widetilde{\mathcal{M}} \quad \textit{and} \quad \mathcal{N}^\varepsilon T_\kappa(Q^\varepsilon) \rightharpoonup \overline{T_\kappa(Q)}\widetilde{\mathcal{M}}. \tag{6.24}$$

Proof. *Step* 1. We define

$$\mathcal{P}^\varepsilon \stackrel{\text{def}}{=} \mathcal{L}^{-1}_{h^\varepsilon}\left(h^\varepsilon \int_{-\infty}^{x} (h^\varepsilon)^{-3}\mathcal{C}^\varepsilon \, \mathrm{d}y + \frac{1}{3}F(h^\varepsilon)_x\right).$$

From (5.49), we have that $\mathcal{P}^\varepsilon$ is bounded in $L^\infty([t_1,t_2], W^{1,\infty}([a,b]))$ for any $b > a$, $t_2 > t_1 > 0$. Thus, there exists $\widetilde{\mathcal{P}} \in L^\infty([t_1,t_2], W^{1,\infty}([a,b]))$ such that, up to a subsequence, we have

$$\mathcal{P}^\varepsilon \rightharpoonup \widetilde{\mathcal{P}}, \quad \partial_x\mathcal{P}^\varepsilon \rightharpoonup \partial_x\widetilde{\mathcal{P}}, \quad \text{as } \varepsilon \to 0,$$

in $L^p_{\text{loc}}((0,\infty)\times\mathbb{R})$ for any $p < \infty$.

Step 2. For a fixed $\varphi \in C_c^\infty((0,\infty)\times\mathbb{R})$, the inequality (5.12), Lemma 5.4 and (5.57) imply that $(1-\partial_x^2)\{\varphi\mathcal{P}^\varepsilon\}$ is uniformly bounded in $L^{2+\alpha}_{\text{loc}}((0,\infty)\times\mathbb{R})$ for all $\alpha \in [0,1)$. Then, up to a subsequence, we have

$$(1-\partial_x^2)\{\varphi\mathcal{P}^\varepsilon\} \rightharpoonup (1-\partial_x^2)\{\varphi\widetilde{\mathcal{P}}\}$$

in $L^{2+\alpha}_{\text{loc}}((0,\infty)\times\mathbb{R})$.

Step 3. Since $|T_\kappa(P^\varepsilon)| \leqslant \kappa$, the convergence $T_\kappa(P^\varepsilon) \rightharpoonup \overline{T_\kappa(P)}$ is in $L^p_{\text{loc}}((0,\infty)\times\mathbb{R})$ for any $p \in (1,\infty)$. Then, for any $\psi \in C_c^\infty((0,\infty)\times\mathbb{R})$, we have, up to a subsequence,

$$\lim_{\varepsilon\to 0}\int_{(0,\infty)\times\mathbb{R}} \psi(1-\partial_x^2)^{-1}\{\partial_x T_\kappa(P^\varepsilon)\}\,\mathrm{d}x\,\mathrm{d}t = \int_{(0,\infty)\times\mathbb{R}} \psi(1-\partial_x^2)^{-1}\{\partial_x \overline{T_\kappa(P)}\}\,\mathrm{d}x\,\mathrm{d}t.$$

This limit is stronger. Indeed, replacing f in (6.9) by T_κ we obtain

$$
\begin{aligned}
[T_\kappa(P^\varepsilon)]_t + [\lambda^\varepsilon T_\kappa(P^\varepsilon)]_x = {} & \frac{1}{4h^\varepsilon}(P^\varepsilon + 3Q^\varepsilon)T_\kappa(P^\varepsilon) \\
& + \Big[-\frac{1}{8h^\varepsilon}(P^\varepsilon)^2 + \frac{1}{8h^\varepsilon}\chi_\varepsilon(P^\varepsilon) + \frac{1}{8h^\varepsilon}(Q^\varepsilon)^2 \\
& - \frac{1}{2h^\varepsilon}\mathcal{A}_x^\varepsilon P^\varepsilon + \mathcal{M}^\varepsilon\Big]T_\kappa'(P^\varepsilon).
\end{aligned}
$$

Then the sequence $\{(1-\partial_x^2)^{-1}\{\partial_x T_\kappa(P^\varepsilon)\}\}_\varepsilon$ is uniformly bounded in $W^{1,\infty}((0,\infty)\times\mathbb{R})$. The Arzelà–Ascoli theorem implies that, up to a subsequence, we have that the convergence

$$
(1-\partial_x^2)^{-1}\{\partial_x T_\kappa(P^\varepsilon)\} \longrightarrow (1-\partial_x^2)^{-1}\{\partial_x \overline{T_\kappa(P)}\}
$$

is uniform on any compact set of $(0,\infty)\times\mathbb{R}$. Following the same proof we obtain the uniform convergence

$$
(1-\partial_x^2)^{-1}\{T_\kappa(P^\varepsilon)\} \longrightarrow (1-\partial_x^2)^{-1}\{\overline{T_\kappa(P)}\}
$$

on any compact set of $(0,\infty)\times\mathbb{R}$.

Step 4. Let $\varphi \in C_c^\infty((0,\infty)\times\mathbb{R})$. Then

$$
\begin{aligned}
\int_{(0,\infty)\times\mathbb{R}} T_\kappa(P^\varepsilon)\varphi\mathcal{P}_x^\varepsilon \,\mathrm{d}x\,\mathrm{d}t &= \int_{(0,\infty)\times\mathbb{R}} T_\kappa(P^\varepsilon)(1-\partial_x^2)^{-1}(1-\partial_x^2)[(\varphi\mathcal{P}^\varepsilon)_x - \varphi_x\mathcal{P}^\varepsilon]\,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_{(0,\infty)\times\mathbb{R}} (1-\partial_x^2)^{-1}\{\partial_x T_\kappa(P^\varepsilon)\}\cdot(1-\partial_x^2)\{\varphi\mathcal{P}^\varepsilon\}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_{(0,\infty)\times\mathbb{R}} (1-\partial_x^2)^{-1}\{T_\kappa(P^\varepsilon)\}\cdot(1-\partial_x^2)\{\varphi_x\mathcal{P}^\varepsilon\}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Taking the limit $\varepsilon\to 0$ and using Steps 2 and 3 and Lemma A.2 we obtain

$$
\begin{aligned}
\lim_{\varepsilon\to 0}\int_{(0,\infty)\times\mathbb{R}} T_\kappa(P^\varepsilon)\varphi\mathcal{P}_x^\varepsilon\,\mathrm{d}x\,\mathrm{d}t &= -\int_{(0,\infty)\times\mathbb{R}} (1-\partial_x^2)^{-1}\{\partial_x\overline{T_\kappa(P)}\}\cdot(1-\partial_x^2)\{\varphi\widetilde{\mathcal{P}}\}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_{(0,\infty)\times\mathbb{R}} (1-\partial_x^2)^{-1}\{\overline{T_\kappa(P)}\}\cdot(1-\partial_x^2)\{\varphi_x\widetilde{\mathcal{P}}\}\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{(0,\infty)\times\mathbb{R}} \overline{T_\kappa(P)}\varphi\partial_x\widetilde{\mathcal{P}}\,\mathrm{d}x\,\mathrm{d}t. \qquad (6.25)
\end{aligned}
$$

Step 5. Since $|T_\kappa(P^\varepsilon)| \leqslant \kappa$, from (5.53) we have $\mathcal{V}_1 T_\kappa(P^\varepsilon) \rightharpoonup 0$ and $\mathcal{V}_2 T_\kappa(P^\varepsilon) \rightharpoonup 0$. Then using (6.25) we obtain (6.23) with

$$
\widetilde{\mathcal{M}} \stackrel{\text{def}}{=} -3h\partial_x\widetilde{\mathcal{P}}.
$$

Following the same proof we obtain (6.24). ∎

Lemma 6.5. *For all $T > 0$, we have*

$$\lim_{\kappa\to\infty} \|\overline{T_\kappa(P)} - T_\kappa(\bar{P})\|_{L^1([0,T]\times\mathbb{R})} = \lim_{\kappa\to\infty} \|\overline{T_\kappa(Q)} - T_\kappa(\bar{Q})\|_{L^1([0,T]\times\mathbb{R})} = 0, \tag{6.26}$$

$$\lim_{\kappa\to\infty} \|\overline{T_\kappa(P)} - \bar{P}\|_{L^1([0,T]\times\mathbb{R})} = \lim_{\kappa\to\infty} \|\overline{T_\kappa(Q)} - \bar{Q}\|_{L^1([0,T]\times\mathbb{R})} = 0. \tag{6.27}$$

Moreover, for all $\kappa > 0$ we have

$$\begin{aligned} \frac{1}{2}(\overline{T_\kappa(P)} - T_\kappa(\bar{P}))^2 &\leqslant \overline{S_\kappa(P)} - S_\kappa(\bar{P}), \\ \frac{1}{2}(\overline{T_\kappa(Q)} - T_\kappa(\bar{Q}))^2 &\leqslant \overline{S_\kappa(Q)} - S_\kappa(\bar{Q}). \end{aligned} \tag{6.28}$$

Proof. Since the proofs for P and Q are the same, we only show the proof for P. From (6.22) we have

$$|T_\kappa(\xi) - \xi| \leqslant |\xi + \kappa| \mathbb{1}_{\xi \leqslant -\kappa} + |\xi - \kappa| \mathbb{1}_{\xi \geqslant \kappa} \leqslant 2|\xi| \mathbb{1}_{\kappa \leqslant |\xi|} \leqslant \frac{2}{\kappa}\xi^2.$$

Then we have

$$|\overline{T_\kappa(P)} - T_\kappa(\bar{P})| \leqslant |\overline{T_\kappa(P)} - \bar{P}| + |T_\kappa(\bar{P}) - \bar{P}| \leqslant \frac{2}{\kappa}(\overline{P^2} + \bar{P}^2).$$

Jensen's inequality implies that $\bar{P}^2 \leqslant \overline{P^2}$. Lemma 6.2 implies that $\overline{P^2} \in L^\infty(\mathbb{R}^+, L^1(\mathbb{R}))$. Then (6.26) and (6.27) follow directly.

The Cauchy–Schwarz inequality implies that $\overline{T_\kappa(P)}^2 \leqslant \overline{T_\kappa(P)^2}$. Then, using definition (6.22), we obtain

$$\begin{aligned} (\overline{T_\kappa(P)} - T_\kappa(\bar{P}))^2 &\leqslant \overline{T_\kappa(P)^2} + T_\kappa(\bar{P})^2 - 2T_\kappa(\bar{P})\overline{T_\kappa(P)} \\ &= \overline{T_\kappa(P)^2} + T_\kappa(\bar{P})^2 - 2T_\kappa(\bar{P})\bar{P} + 2T_\kappa(\bar{P})\overline{(P+\kappa)\mathbb{1}_{P\leqslant -\kappa}} \\ &\quad + 2T_\kappa(\bar{P})\overline{(P-\kappa)\mathbb{1}_{P\geqslant \kappa}} \\ &= \overline{T_\kappa(P)^2} + 2T_\kappa(\bar{P})[\overline{(P+\kappa)\mathbb{1}_{P\leqslant -\kappa}} - (\bar{P}+\kappa)\mathbb{1}_{\bar{P}\leqslant -\kappa}] \\ &\quad - T_\kappa(\bar{P})^2 + 2T_\kappa(\bar{P})[\overline{(P-\kappa)\mathbb{1}_{P\geqslant \kappa}} - (\bar{P}-\kappa)\mathbb{1}_{\bar{P}\geqslant \kappa}] \\ &\leqslant \overline{T_\kappa(P)^2} - 2\kappa[\overline{(P+\kappa)\mathbb{1}_{P\leqslant -\kappa}} - (\bar{P}+\kappa)\mathbb{1}_{\bar{P}\leqslant -\kappa}] \\ &\quad - T_\kappa(\bar{P})^2 + 2\kappa[\overline{(P-\kappa)\mathbb{1}_{P\geqslant \kappa}} - (\bar{P}-\kappa)\mathbb{1}_{\bar{P}\geqslant \kappa}], \end{aligned} \tag{6.29}$$

where the last inequality follows from Jensen's inequality with the concavity of $\xi \mapsto (\xi+\kappa)\mathbb{1}_{\xi\leqslant -\kappa}$, the convexity of $\xi \mapsto (\xi-\kappa)\mathbb{1}_{\xi\geqslant\kappa}$ and $-\kappa \leqslant T_\kappa(\xi) \leqslant \kappa$. Since

$$S_\kappa(\xi) = \frac{1}{2}T_\kappa(\xi)^2 + \kappa(\xi-\kappa)\mathbb{1}_{\xi\geqslant\kappa} - \kappa(\xi+\kappa)\mathbb{1}_{\xi\leqslant-\kappa},$$

we have

$$\begin{aligned} \overline{S_\kappa(P)} &= \frac{1}{2}\overline{T_\kappa(P)^2} + \kappa\overline{(P-\kappa)\mathbb{1}_{P\geqslant\kappa}} - \kappa\overline{(P+\kappa)\mathbb{1}_{P\leqslant-\kappa}}, \\ S_\kappa(\bar{P}) &= \frac{1}{2}T_\kappa(\bar{P})^2 + \kappa(\bar{P}-\kappa)\mathbb{1}_{\bar{P}\geqslant\kappa} - \kappa(\bar{P}+\kappa)\mathbb{1}_{\bar{P}\leqslant-\kappa}. \end{aligned}$$

The last two identities with (6.29) imply (6.28). ∎

Now we state the main result of this section.

Lemma 6.6. *The measures ν^1, ν^2 given in Lemma 6.2 are Dirac measures, and*

$$\nu^1_{t,x}(\xi) = \delta_{\bar{P}(t,x)}(\xi), \quad \nu^2_{t,x}(\zeta) = \delta_{\bar{Q}(t,x)}(\zeta).$$

Proof. Since the proof is the same, we present here only the proof of $\nu^1_{t,x}(\xi) = \delta_{\bar{P}(t,x)}(\xi)$. Note that if $\overline{P^2} = \bar{P}^2$ then $\int_{\mathbb{R}} (\bar{P} - \xi)^2 \, d\nu^1_{t,x}(\xi) = 0$, which implies that $\operatorname{supp}(\nu^1_{t,x}) = \{\bar{P}\}$. Since $\nu^1_{t,x}$ is a probability measure, necessarily $\nu^1_{t,x} = \delta_{\bar{P}}$. It remains then only to prove that $\overline{P^2} = \bar{P}^2$. The goal is to obtain an evolutionary inequality of $\overline{P^2} - \bar{P}^2$. Then, since it is equal to zero initially, we prove that it remains zero for all time. The proof is given in several steps.

Step 1. Replacing f in (6.9) by S_κ one obtains

$$\begin{aligned} S_\kappa(P^\varepsilon)_t + [\lambda^\varepsilon S_\kappa(P^\varepsilon)]_x &= \frac{1}{4h^\varepsilon}(P^\varepsilon + 3Q^\varepsilon) S_\kappa(P^\varepsilon) \\ &\quad + \Big[-\frac{1}{8h^\varepsilon}(P^\varepsilon)^2 + \frac{1}{8h^\varepsilon}\chi_\varepsilon(P^\varepsilon) + \frac{1}{8h^\varepsilon}(Q^\varepsilon)^2 \\ &\quad - \frac{1}{2h^\varepsilon}\mathcal{A}^\varepsilon_x P^\varepsilon + \mathcal{M}^\varepsilon \Big] T_\kappa(P^\varepsilon). \end{aligned}$$

Taking $\varepsilon \to 0$, using (5.53) and Lemmas 6.2 and 6.4 we obtain

$$\begin{aligned} &\overline{S_\kappa(P)}_t + (\lambda \overline{S_\kappa(P)})_x \\ &\quad = \frac{1}{8h}\big\{ 2\overline{P S_\kappa(P)} - \overline{P^2 T_\kappa(P)} + 6\bar{Q}\,\overline{S_\kappa(P)} + \overline{Q^2}\,\overline{T_\kappa(P)} \big\} + \overline{T_\kappa(P)}\widetilde{\mathcal{M}}. \end{aligned} \tag{6.30}$$

Step 2. Replacing f in (6.9) by the identity function and taking $\varepsilon \to 0$, we obtain

$$\bar{P}_t + (\lambda \bar{P})_x = \frac{1}{8h}(\overline{P^2} + 6\bar{P}\bar{Q} + \overline{Q^2}) + \widetilde{\mathcal{M}}.$$

Let j_ε be a Friedrichs mollifier and note that $\bar{P}^\varepsilon \overset{\text{def}}{=} \bar{P} * j_\varepsilon$. Then we have

$$\bar{P}^\varepsilon_t + (\lambda \bar{P}^\varepsilon)_x = \theta_\varepsilon + \Big\{ \frac{1}{8h}(\overline{P^2} + 6\bar{P}\bar{Q} + \overline{Q^2}) \Big\} * j_\varepsilon + \widetilde{\mathcal{M}} * j_\varepsilon,$$

where $\theta_\varepsilon \overset{\text{def}}{=} (\lambda \bar{P}^\varepsilon)_x - (\lambda \bar{P})_x * j_\varepsilon$. Multiplying by $T_\kappa(\bar{P}^\varepsilon)$ and using (6.12), we obtain

$$\begin{aligned} S_\kappa(\bar{P}^\varepsilon)_t + (\lambda S_\kappa(\bar{P}^\varepsilon))_x &= \frac{1}{4h}(3\bar{Q} + \bar{P}) S_\kappa(\bar{P}^\varepsilon) - \frac{1}{4h}(3\bar{Q} + \bar{P})\bar{P}^\varepsilon T_\kappa(\bar{P}^\varepsilon) \\ &\quad + T_\kappa(\bar{P}^\varepsilon)\Big\{ \frac{1}{8h}(\overline{P^2} + 6\bar{P}\bar{Q} + \overline{Q^2}) \Big\} * j_\varepsilon \\ &\quad + T_\kappa(\bar{P}^\varepsilon)(\widetilde{\mathcal{M}} * j_\varepsilon) + T_\kappa(\bar{P}^\varepsilon)\theta_\varepsilon. \end{aligned}$$

Taking $\varepsilon \to 0$ and using Lemma A.4, one obtains

$$\begin{aligned} &S_\kappa(\bar{P})_t + (\lambda S_\kappa(\bar{P}))_x \\ &\quad = \frac{1}{8h}\big\{ 2\bar{P} S_\kappa(\bar{P}) + 6\bar{Q} S_\kappa(\bar{P}) + T_\kappa(\bar{P})(\overline{P^2} - 2\bar{P}^2 + \overline{Q^2}) \big\} + T_\kappa(\bar{P})\widetilde{\mathcal{M}}. \end{aligned} \tag{6.31}$$

Step 3. From (6.30) and (6.31) we obtain

$$
\begin{aligned}
&[\overline{S_\kappa(P)} - S_\kappa(\bar{P})]_t + [\lambda(\overline{S_\kappa(P)} - S_\kappa(\bar{P}))]_x \\
&\quad = \widetilde{\mathcal{M}}(\overline{T_\kappa(P)} - T_\kappa(\bar{P})) \\
&\qquad + \frac{1}{8h}\{2\overline{PS_\kappa(P)} - \overline{P^2T_\kappa(P)} + \bar{P}^2T_\kappa(\bar{P}) - 2\bar{P}S_\kappa(\bar{P}) + T_\kappa(\bar{P})(\bar{P}^2 - \overline{P^2})\} \\
&\qquad + \frac{1}{8h}\{6\bar{Q}(\overline{S_\kappa(P)} - S_\kappa(\bar{P})) + \overline{Q^2}(\overline{T_\kappa(P)} - T_\kappa(\bar{P}))\}.
\end{aligned} \tag{6.32}
$$

From (6.21) and (6.22) we have

$$
\begin{aligned}
\xi^2T_\kappa(\xi) - 2\xi S_\kappa(\xi) &= \xi^2T_\kappa(\xi) - 2\xi S_\kappa(\xi) + \xi^3 - \xi^3 \\
&= \xi^2[T_\kappa(\xi) - \xi] + \xi(\xi+\kappa)^2\mathbb{1}_{\xi\leqslant -\kappa} + \xi(\xi-\kappa)^2\mathbb{1}_{\xi\geqslant\kappa} \\
&= \kappa^2[T_\kappa(\xi) - \xi] - (\xi^2-\kappa^2)[(\xi+\kappa)\mathbb{1}_{\xi\leqslant-\kappa} + (\xi-\kappa)\mathbb{1}_{\xi\geqslant\kappa}] \\
&\quad + \xi(\xi+\kappa)^2\mathbb{1}_{\xi\leqslant-\kappa} + \xi(\xi-\kappa)^2\mathbb{1}_{\xi\geqslant\kappa} \\
&= \kappa^2[T_\kappa(\xi) - \xi] + \kappa(\xi+\kappa)^2\mathbb{1}_{\xi\leqslant-\kappa} - \kappa(\xi-\kappa)^2\mathbb{1}_{\xi\geqslant\kappa}.
\end{aligned}
$$

Then from (6.21) we have

$$
\begin{aligned}
&2\overline{PS_\kappa(P)} - \overline{P^2T_\kappa(P)} + \bar{P}^2T_\kappa(\bar{P}) - 2\bar{P}S_\kappa(\bar{P}) + T_\kappa(\bar{P})(\bar{P}^2 - \overline{P^2}) \\
&\quad = (T_\kappa(\bar{P})+\kappa)(\bar{P}+\kappa)^2\mathbb{1}_{\bar{P}\leqslant-\kappa} + (T_\kappa(\bar{P})-\kappa)(\bar{P}-\kappa)^2\mathbb{1}_{\bar{P}\geqslant\kappa} \\
&\qquad - (T_\kappa(\bar{P})+\kappa)\overline{(P+\kappa)^2\mathbb{1}_{P\leqslant-\kappa}} - (T_\kappa(\bar{P})-\kappa)\overline{(P-\kappa)^2\mathbb{1}_{P\geqslant\kappa}} \\
&\qquad - \kappa^2(\overline{T_\kappa(P)} - T_\kappa(\bar{P})) - 2T_\kappa(\bar{P})(\overline{S_\kappa(P)} - S_\kappa(\bar{P}))
\end{aligned} \tag{6.33}
$$

From definition (6.22) we have

$$
(T_\kappa(\bar{P})+\kappa)(\bar{P}+\kappa)^2\mathbb{1}_{\bar{P}\leqslant-\kappa} = (T_\kappa(\bar{P})-\kappa)(\bar{P}-\kappa)^2\mathbb{1}_{\bar{P}\geqslant\kappa} = 0. \tag{6.34}
$$

Since $T_\kappa(\bar{P}) \geqslant -\kappa$, we have

$$
-(T_\kappa(\bar{P})+\kappa)\overline{(P+\kappa)^2\mathbb{1}_{P\leqslant-\kappa}} \leqslant 0. \tag{6.35}
$$

Let $t_0 > 0$ and $\kappa \geqslant C(1+t_0^{-1})$. Then, from Lemma 5.5, we have for all $t \geqslant t_0$ that $P^\varepsilon \leqslant \kappa$ and $\bar{P} \leqslant \kappa$. Then, using the convexity of T_κ on $(-\infty,\kappa)$ and Jensen's inequality, we obtain

$$
-\kappa^2(\overline{T_\kappa(P)} - T_\kappa(\bar{P})) \leqslant 0 \quad \forall t \geqslant t_0, \ \ \kappa \geqslant C(1+t_0^{-1}). \tag{6.36}
$$

We take $t_0 > 0$ and $\kappa \geqslant C(1+t_0^{-1})$ again. Then, for all $\varphi \in C_c^\infty((t_0,\infty)\times\mathbb{R})$, we have

$$
\int \overline{(P-\kappa)^2\mathbb{1}_{P\geqslant\kappa}}\varphi \,\mathrm{d}x\,\mathrm{d}t = \lim_{\varepsilon\to 0}\int (P^\varepsilon-\kappa)^2\mathbb{1}_{P^\varepsilon\geqslant\kappa}\varphi\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{6.37}
$$

Then, using (6.32), (6.33), (6.34), (6.35), (6.36) and (6.37), we obtain

$$
\begin{aligned}
&[\overline{S_\kappa(P)} - S_\kappa(\bar{P})]_t + [\lambda(\overline{S_\kappa(P)} - S_\kappa(\bar{P}))]_x \\
&\quad \leqslant \widetilde{\mathcal{M}}(\overline{T_\kappa(P)} - T_\kappa(\bar{P})) \\
&\qquad + \frac{1}{8h}\{(6\bar{Q} - 2T_\kappa(\bar{P}))(\overline{S_\kappa(P)} - S_\kappa(\bar{P})) + \overline{Q^2}(\overline{T_\kappa(P)} - T_\kappa(\bar{P}))\},
\end{aligned} \tag{6.38}
$$

for any $t_0 > 0$, $\kappa \geqslant C(1+t_0^{-1})$ and $t > t_0$.

Step 4. Let $f_\kappa(t,x) \stackrel{\text{def}}{=} \overline{S_\kappa(P)} - S_\kappa(\bar{P})$ and $f_\kappa^\varepsilon \stackrel{\text{def}}{=} f_\kappa * j_\varepsilon$, where j_ε is a Friedrichs mollifier. Then, from (6.38) and Lemma A.4, we obtain

$$(f_\kappa^\varepsilon)_t + (\lambda f_\kappa^\varepsilon)_x \leqslant \frac{1}{4h}\{3\bar{Q} - T_\kappa(\bar{P})\} f_\kappa^\varepsilon + \Big(\widetilde{\mathfrak{M}} + \frac{\overline{Q^2}}{8h}\Big)(\overline{T_\kappa(P)} - T_\kappa(\bar{P})) + \theta_\varepsilon,$$

where $\theta_\varepsilon \to 0$ in $L^1_{\text{loc}}((0,\infty)\times\mathbb{R})$. Let $\beta > 0$; multiplying by $h^{3/2}(h^{3/2} f_\kappa^\varepsilon + \beta)^{-1/2}/2$ and using (6.13) one obtains

$$\begin{aligned}
&\Big[\sqrt{h^{3/2} f_\kappa^\varepsilon + \beta}\Big]_t + \Big[\lambda\sqrt{h^{3/2} f_\kappa^\varepsilon + \beta}\Big]_x \\
&\quad\leqslant \Big(\widetilde{\mathfrak{M}} + \frac{\overline{Q^2}}{8h}\Big)\frac{\overline{T_\kappa(P)} - T_\kappa(\bar{P})}{2\sqrt{h^{3/2} f_\kappa^\varepsilon + \beta}} h^{3/2} + \tilde{\theta}_\varepsilon + \frac{(\bar{P} - T_\kappa(\bar{P}))h^{1/2} f_\kappa^\varepsilon}{8\sqrt{h^{3/2} f_\kappa^\varepsilon + \beta}} \\
&\qquad + \frac{\beta\lambda_x}{\sqrt{h^{3/2} f_\kappa^\varepsilon + \beta}},
\end{aligned}$$

where $\tilde{\theta}_\varepsilon \stackrel{\text{def}}{=} \theta_\varepsilon h^{3/2}(h^{3/2} f_\kappa^\varepsilon + \beta)^{-1/2}/2 \to 0$ in $L^1_{\text{loc}}((0,\infty)\times\mathbb{R})$. Taking $\varepsilon \to 0$ we obtain

$$\begin{aligned}
&\Big[\sqrt{h^{3/2} f_\kappa + \beta}\Big]_t + \Big[\lambda\sqrt{h^{3/2} f_\kappa + \beta}\Big]_x \\
&\quad\leqslant \Big(\widetilde{\mathfrak{M}} + \frac{\overline{Q^2}}{8h}\Big)\frac{\overline{T_\kappa(P)} - T_\kappa(\bar{P})}{2\sqrt{h^{3/2} f_\kappa + \beta}} h^{3/2} + \frac{(\bar{P} - T_\kappa(\bar{P}))h^{1/2} f_\kappa}{8\sqrt{h^{3/2} f_\kappa + \beta}} \\
&\qquad + \frac{\beta\lambda_x}{\sqrt{h^{3/2} f_\kappa + \beta}}.
\end{aligned} \tag{6.39}$$

From (6.28) we have

$$\Big|\Big(\widetilde{\mathfrak{M}} + \frac{\overline{Q^2}}{8h}\Big)\frac{\overline{T_\kappa(P)} - T_\kappa(\bar{P})}{2\sqrt{h^{3/2} f_\kappa + \beta}} h^{3/2}\Big| \leqslant \frac{\sqrt{2}}{2}\Big|\widetilde{\mathfrak{M}} + \frac{\overline{Q^2}}{8h}\Big| h^{3/4}.$$

Using that $|T_\kappa(\xi)| \leqslant |\xi|$ and $S_\kappa(\xi) \leqslant \xi^2/2$ we obtain

$$g\Big|\frac{(\bar{P} - T_\kappa(\bar{P}))h^{1/2} f_\kappa}{8\sqrt{h^{3/2} f_\kappa + \beta}} g\Big| \leqslant \frac{|\bar{P}|\sqrt{f_\kappa}}{4h^{1/4}} \leqslant \frac{1}{8h^{1/4}}(\bar{P}^2 + f_\kappa) \leqslant \frac{1}{8h^{1/4}}\Big(\frac{3}{2}\bar{P}^2 + \frac{1}{2}\overline{P^2}\Big).$$

Since L^1 convergence implies pointwise convergence (up to a subsequence), using the dominated convergence theorem with (6.26) and (6.27) we obtain

$$\begin{aligned}
&\lim_{\kappa\to\infty} g\Big\|\Big(\widetilde{\mathfrak{M}} + \frac{\overline{Q^2}}{8h}\Big)\frac{\overline{T_\kappa(P)} - T_\kappa(\bar{P})}{2\sqrt{h^{3/2} f_\kappa + \beta}} h^{3/2} g\Big\|_{L^1(\Omega)} \\
&\quad + \lim_{\kappa\to\infty} g\Big\|\frac{(\bar{P} - T_\kappa(\bar{P}))h^{1/2} f_\kappa}{8\sqrt{h^{3/2} f_\kappa + \beta}} g\Big\|_{L^1(\Omega)} = 0
\end{aligned}$$

for any compact set $\Omega \subset (0,\infty)\times\mathbb{R}$. Since $|S_\kappa(\xi)| \leqslant \xi^2/2$, we have $|f_\kappa| \leqslant \overline{P}^2/2 + \overline{P^2}/2$. Taking $\kappa \to \infty$ in (6.39) and using the dominated convergence theorem again we obtain

$$\left[\sqrt{h^{3/2} f + \beta}\right]_t + \left[\lambda\sqrt{h^{3/2} f + \beta}\right]_x \leqslant \frac{\beta\lambda_x}{\sqrt{h^{3/2} f + \beta}}, \qquad f \overset{\text{def}}{=} \frac{1}{2}(\overline{P^2} - \overline{P}^2).$$

Now taking $\beta \to 0$ we obtain

$$\left[\sqrt{h^{3/2} f}\right]_t + \left[\lambda\sqrt{h^{3/2} f}\right]_x \leqslant 0 \quad \text{in } (t_0,\infty)\times\mathbb{R}. \tag{6.40}$$

Step 5. Following [64], let $g \overset{\text{def}}{=} \sqrt{h^{3/2} f} \in L^\infty((0,\infty), L^2(\mathbb{R}))$. Also let $\varphi \in C_c^\infty(\mathbb{R})$ satisfy $\varphi(x) = 1$ for $|x| \leqslant 1$ and $\varphi(x) = 0$ for $|x| \geqslant 2$. Then, for all $n \geqslant 1$, we have $g\varphi(x/n) \in L^\infty((0,\infty), L^1(\mathbb{R}))$. Then almost all $t > 0$ are Lebesgue points of $t \mapsto \int_\mathbb{R} g(t,x)\varphi(x/n)\,\mathrm{d}x$ for all $n \geqslant 1$. Let $\bar{t} > 0$ be a Lebesgue point of $t \mapsto \int_\mathbb{R} g(t,x)\varphi(x/n)\,\mathrm{d}x$ and $\delta \in (0, \bar{t}/2)$. Also let $\psi \in C_c^\infty((0,\infty))$ satisfy

$$\begin{aligned} \psi(t) &= 0 \ \text{ on } (0,\delta/2)\cup(\bar{t}+\delta,\infty), & \psi(t) &= 1 \ \text{ on } (\delta, \bar{t}-\delta),\\ 0 \leqslant \psi'(t) &\leqslant C/\delta \ \text{ on } (\delta/2,\delta), & -\psi'(t) &\geqslant C/\delta \ \text{ on } (\bar{t}-\delta, \bar{t}+\delta). \end{aligned}$$

Multiplying (6.40) by $\varphi(x/n)\psi(t)$, integrating on $(0,\infty)\times\mathbb{R}$ and using integration by parts one obtains

$$\begin{aligned} \frac{C}{\delta}\int_{\bar{t}-\delta}^{\bar{t}+\delta}\int_\mathbb{R} g(t,x)\varphi(x/n)\,\mathrm{d}x\,\mathrm{d}t &\leqslant -\int_{\bar{t}-\delta}^{\bar{t}+\delta}\int_\mathbb{R} g(t,x)\varphi(x/n)\psi'(t)\,\mathrm{d}x\,\mathrm{d}t\\ &\leqslant \frac{C}{\delta}\int_{\delta/2}^{\delta}\int_\mathbb{R} g(t,x)\varphi(x/n)\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{n}\|\lambda\|_{L^\infty}\int_{\delta/2}^{\bar{t}+\delta}\int_\mathbb{R} g(t,x)|\varphi'(x/n)|\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

From (6.15), we have

$$\lim_{t\to 0}\int_\mathbb{R} g(t,x)\varphi(x/n)\,\mathrm{d}x = 0 \quad\Longrightarrow\quad \lim_{\delta\to 0}\frac{1}{\delta}\int_{\delta/2}^{\delta}\int_\mathbb{R} g(t,x)\varphi(x/n)\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Since $\bar{t} > 0$ is a Lebesgue point of $t \mapsto \int_\mathbb{R} g(t,x)\varphi(x/n)\,\mathrm{d}x$, taking first $\delta \to 0$ and then $n \to \infty$ we obtain

$$g(\bar{t},x) = 0 \quad \text{a.e. } (\bar{t},x) \in (0,\infty)\times\mathbb{R}.$$

Hence $\overline{P^2} = \overline{P}^2$ almost everywhere, which implies that $\nu^1_{t,x}(\xi) = \delta_{\overline{P}(t,x)}(\xi)$. The proof of $\nu^2_{t,x}(\zeta) = \delta_{\overline{Q}(t,x)}(\zeta)$ can be done similarly. ■

7. The global weak solutions

In this section we use the precompactness results given in the previous section to prove that the limit (h,u) given in Lemma 6.1 is a weak solution of (2.1). All the limits in this section are up to a subsequence.

Let $(h^\varepsilon - \bar{h}, u^\varepsilon)$ be the solution given in Theorem 5.1. Then, from Lemmas 6.2, 6.6, 5.6, (4.2) and (6.12) we have

$$\begin{aligned}(P^\varepsilon, Q^\varepsilon, u_x^\varepsilon, h_x^\varepsilon) \quad &\rightharpoonup \quad (\bar{P}, \bar{Q}, u_x, h_x) \quad \text{in } L^p_{\text{loc}}((0,\infty)\times\mathbb{R}), \\ \|(P^\varepsilon)^2, (Q^\varepsilon)^2, (u_x^\varepsilon)^2, (h_x^\varepsilon)^2\|_{L^1(\Omega)} \quad &\to \quad \|\bar{P}^2, \bar{Q}^2, u_x^2, h_x^2\|_{L^1(\Omega)},\end{aligned} \tag{7.1}$$

for any $p \in [2,3)$ and compact set $\Omega \subset (0,\infty)\times\mathbb{R}$. This implies that

$$(P^\varepsilon, Q^\varepsilon, u_x^\varepsilon, h_x^\varepsilon) \to (\bar{P}, \bar{Q}, u_x, h_x) \quad \text{in } L^2_{\text{loc}}((0,\infty)\times\mathbb{R}). \tag{7.2}$$

Using Lemmas 5.6 and 6.1 we obtain that, for all $p \in [2,3)$,

$$(u_t^\varepsilon, h_t^\varepsilon) \rightharpoonup (u_t, h_t) \quad \text{in } L^p_{\text{loc}}((0,\infty)\times\mathbb{R}). \tag{7.3}$$

Now, using (5.53) and taking the weak limit $\varepsilon \to 0$ in (5.2a), we obtain (2.1a). Applying $\mathcal{L}_{h^\varepsilon}$ on (5.2b) and multiplying by $\varphi \in C_c^\infty((0,\infty)\times\mathbb{R})$ we obtain

$$\begin{aligned}\int_{\mathbb{R}^+\times\mathbb{R}} &\Big\{\{u_t^\varepsilon + u^\varepsilon u_x^\varepsilon + 3\gamma(h^\varepsilon)^{-2}h_x^\varepsilon\}\Big\{h^\varepsilon\varphi - (h^\varepsilon)^2 h_x^\varepsilon\varphi_x - \frac{1}{3}(h^\varepsilon)^3\varphi_{xx}\Big\} \\ &\quad + \frac{1}{2}\varphi u^\varepsilon \mathcal{A}_x^\varepsilon\Big\}\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{\mathbb{R}^+\times\mathbb{R}} \varphi_x\Big\{\mathcal{C}^\varepsilon + F(h^\varepsilon) - \frac{1}{2}(h^\varepsilon)^2 u_x^\varepsilon \mathcal{A}_x^\varepsilon + \frac{1}{48}h^\varepsilon(\chi_\varepsilon(P^\varepsilon) + \chi_\varepsilon(Q^\varepsilon))\Big\}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

From (7.2) and Lemma 6.1 we obtain the following convergence as $\varepsilon \to 0$:

$$\Big\{h^\varepsilon\varphi - (h^\varepsilon)^2 h_x^\varepsilon\varphi_x - \frac{1}{3}(h^\varepsilon)^3\varphi_{xx}\Big\} \to \mathcal{L}_h\varphi \quad \text{in } L^2_{\text{loc}}((0,\infty)\times\mathbb{R}).$$

Using (7.2) and Lemma 6.1 again, and also (7.3), we obtain the convergence

$$\{u_t^\varepsilon + u^\varepsilon u_x^\varepsilon + 3\gamma(h^\varepsilon)^{-2}h_x^\varepsilon\} \rightharpoonup \{u_t + uu_x + 3\gamma h^{-2}h_x\} \quad \text{in } L^2_{\text{loc}}((0,\infty)\times\mathbb{R}).$$

We suppose that $\operatorname{supp}(\varphi) \subset [t_1,t_2]\times[a,b]$. Then, using the energy equation (5.9) and Lemma 5.4, we obtain

$$\begin{aligned}\left|\int_{\mathbb{R}^+\times\mathbb{R}} \varphi_x (h^\varepsilon)^2 u_x^\varepsilon \mathcal{A}_x^\varepsilon\,\mathrm{d}x\,\mathrm{d}t\right| &\leqslant C\|\varphi_x(h^\varepsilon)^2 u_x^\varepsilon\|_{L^\infty([t_1,t_2],L^2([a,b]))}\|\mathcal{A}_x^\varepsilon\|_{L^1([t_1,t_2],L^\infty([a,b]))} \\ &\leqslant \varepsilon C.\end{aligned}$$

Following the same argument we obtain

$$\left|\int_{\mathbb{R}^+\times\mathbb{R}} \varphi u^\varepsilon \mathcal{A}_x^\varepsilon\,\mathrm{d}x\,\mathrm{d}t\right| \leqslant \varepsilon C.$$

Then, taking $\varepsilon \to 0$, using Lemma A.2, (5.50) and (7.2), we obtain (3.1). Performing the proof of (6.14) for any t_0, we obtain that $(h - \bar{h}, u) \in C_r(\mathbb{R}^+, H^1(\mathbb{R}))$. Lemma 5.5 implies (3.4). Inequality (3.2) follows from Lemma 5.6, (7.1) and (7.3). Finally, the energy inequality (3.3) follows from (6.16).

A. Some classical lemmas

Here we recall simple versions of some classical lemmas that are needed in this paper. We start this section by the following lemma on Young measures.

Lemma A.1 ([32]). *Let $\mathcal{O}$ be a subset of $\mathbb{R}^n$ with a zero-measure boundary. For any bounded family $\{v^\varepsilon\}_\varepsilon \subset L^p(\mathcal{O}, \mathbb{R}^N)$ with $p > 1$ there exist a subsequence also denoted $\{v^\varepsilon\}_\varepsilon$ and a family of probability measures on $\mathbb{R}^N$, $\{\mu_y, y \in \mathcal{O}\}$, such that for all $f \in C^0(\mathbb{R}^N)$ with $f(\xi) = o(|\xi|^p)$ at infinity and for all $\phi \in C_c^\infty(\mathcal{O})$ we have*

$$\lim_{\varepsilon\to 0} \int_{\mathcal{O}} \phi(y) f(v^\varepsilon(y))\,\mathrm{d}y = \int_{\mathcal{O}} \phi(y) \int_{\mathbb{R}} f(\xi)\,\mathrm{d}\mu_y(\xi)\,\mathrm{d}y$$

with

$$\int_{\mathcal{O}} \int_{\mathbb{R}} |\xi|^p\,\mathrm{d}\mu_y(\xi)\,\mathrm{d}y \leqslant \liminf_{\varepsilon\to 0} \|u^\varepsilon\|^p_{L^p(\mathcal{O})}. \tag{A.1}$$

Other results on strong and weak precompactness are also needed, which we recall.

Lemma A.2 ([20]). *Let Ω be an open set of $\mathbb{R}^n$. Assume that $f_n \to f$ in $L^p(\Omega)$ with $p \in (1,\infty)$, g_n is bounded in L^q with $q \in (1,\infty)$ and $g_n \rightharpoonup g$ in $L^q(\Omega)$. Then, for any $\varphi \in L^r(\Omega)$ such that $1/p + 1/q + 1/r = 1$, we have*

$$\lim_{n\to\infty} \int_\Omega f_n g_n \varphi\,\mathrm{d}x = \int_\Omega f g \varphi\,\mathrm{d}x.$$

Lemma A.3 ([20]). *For any $p > 2$ we have $L^1_{\mathrm{loc}}(\mathbb{R}^2) \cap W^{-1,p}_{\mathrm{loc}}(\mathbb{R}^2) \Subset H^{-1}_{\mathrm{loc}}(\mathbb{R}^2)$. In other words, for any open, bounded, smooth set $U \subset \mathbb{R}^2$, if the sequence $(f_n)_n$ is bounded in $L^1(U) \cap W^{-1,p}(U)$, then $(f_n)_n$ is relatively compact in $H^{-1}(U)$.*

Lemma A.4 ([18, Lemma II.1]). *Let $c \in L^1_{\mathrm{loc}}(\mathbb{R}^+, H^1_{\mathrm{loc}}(\mathbb{R}))$ and $f \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+, L^2_{\mathrm{loc}}(\mathbb{R}))$. Also let j_ε be a Friedrichs mollifier. Then*

$$(c\partial_x f) * j_\varepsilon - c(\partial_x f * j_\varepsilon) \xrightarrow{\varepsilon\to 0} 0 \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}).$$

Lemma A.5 ([43, Lemma C.1]). *Let $(f_n)_n$ be a bounded sequence in $L^\infty([0,T], L^2(\mathbb{R}))$. If f_n belongs to $C([0,T], H^{-1}(\mathbb{R}))$ and, for any $\varphi \in H^1(\mathbb{R})$, the map*

$$t \mapsto \int_{\mathbb{R}} \varphi(x) f_n(t,x)\,\mathrm{d}x$$

is uniformly continuous for $t \in [0,T]$ and $n \geqslant 1$, then $(f_n)_n$ is relatively compact in the space $C([0,T], L^2_w(\mathbb{R}))$, where L^2_w is the L^2 space equipped with its weak topology.

Lemma A.6 (Compensated compactness [22]). *Let Ω be an open set of $\mathbb{R}^2$ and let $a, b \in C(\Omega, \mathbb{R})$ such that for all $(x_1, x_2) \in \Omega$ we have $a(x_1,x_2) \neq b(x_1,x_2)$. Also let (f_n), (g_n) be bounded sequences in $L^2_{\mathrm{loc}}(\Omega, \mathbb{R})$ such that $f_n \rightharpoonup f$ and $g_n \rightharpoonup g$. If the sequences*

$$\{\partial_{x_1} f_n + \partial_{x_2}(a f_n)\}_n \quad \text{and} \quad \{\partial_{x_1} g_n + \partial_{x_2}(b g_n)\}_n$$

are relatively compact in $H^{-1}_{\mathrm{loc}}(\Omega)$, then $f_n g_n \rightharpoonup fg$ in the sense of distributions.

Let Λ be defined such that $\widehat{\Lambda f} = (1+\xi^2)^{1/2}\hat{f}$ and let $[A, B] \stackrel{\text{def}}{=} AB - BA$ be the commutator of the operators A and B. We recall now some estimates of the H^s norm of the product, the commutator and the composition of functions.

Lemma A.7 ([33]). *If $r \geqslant 0$, then there exists $C > 0$ such that*

$$\|fg\|_{H^r} \leqslant C(\|f\|_{L^\infty}\|g\|_{H^r} + \|f\|_{H^r}\|g\|_{L^\infty}), \tag{A.2}$$

$$\|[\Lambda^r, f]g\|_{L^2} \leqslant C(\|f_x\|_{L^\infty}\|g\|_{H^{r-1}} + \|f\|_{H^r}\|g\|_{L^\infty}). \tag{A.3}$$

Lemma A.8 ([16]). *Let $F \in C^\infty(\mathbb{R})$ with $F(0) = 0$. Then, for any $m \in \mathbb{N}$, there exists a continuous function $\widetilde{F}$ such that, for all $f \in H^m$, we have*

$$\|F(f)\|_{H^m} \leqslant \widetilde{F}(\|f\|_{L^\infty})\|f\|_{H^m}. \tag{A.4}$$

B. The energy equation

The goal of this section is to prove that smooth solutions of (2.1) (respectively (5.2)) satisfy the energy equation (1.5) (respectively (5.8)). Taking $\varepsilon = 0$, we notice that (1.5) is a particular case of (5.8). We consider $\varepsilon \geqslant 0$ and $(h^\varepsilon, u^\varepsilon)$ smooth solutions of (5.2). Then we have

$$\frac{1}{2}g[(h^\varepsilon - \bar{h})^2]_t = -g(h^\varepsilon - \bar{h})(h^\varepsilon u^\varepsilon)_x + g(h^\varepsilon - \bar{h})\mathcal{A}^\varepsilon_x,$$

$$\frac{1}{2}\gamma[(h^\varepsilon_x)^2]_t = -\gamma h^\varepsilon_x(h^\varepsilon u^\varepsilon)_{xx} + g h^\varepsilon_x \mathcal{A}^\varepsilon - \frac{\sqrt{3\gamma}}{48(h^\varepsilon)^{1/2}}h^\varepsilon_x(\chi_\varepsilon(P^\varepsilon) - \chi_\varepsilon(Q^\varepsilon)).$$

Summing, we obtain

$$\begin{aligned}\frac{1}{2}g[(h^\varepsilon - \bar{h})^2]_t + \frac{1}{2}\gamma[(h^\varepsilon_x)^2]_t &= g[(h^\varepsilon - \bar{h})\mathcal{A}^\varepsilon]_x - g(h^\varepsilon - \bar{h})(h^\varepsilon u^\varepsilon)_x - \gamma h^\varepsilon_x(h^\varepsilon u^\varepsilon)_{xx}\\ &\quad - \frac{1}{96}(Q^\varepsilon - P^\varepsilon)(\chi_\varepsilon(P^\varepsilon) - \chi_\varepsilon(Q^\varepsilon)).\end{aligned} \tag{B.1}$$

Defining $\mathcal{X}^\varepsilon \stackrel{\text{def}}{=} \mathcal{C}^\varepsilon + F(h^\varepsilon)$, from (5.2) we have

$$\begin{aligned}\frac{1}{6}[(h^\varepsilon)^3(u^\varepsilon_x)^2]_t &= \frac{1}{2}(h^\varepsilon)^2(u^\varepsilon_x)^2\mathcal{A}^\varepsilon_x - \frac{1}{2}(h^\varepsilon)^2(u^\varepsilon_x)^2(h^\varepsilon u^\varepsilon)_x\\ &\quad - \frac{1}{3}(h^\varepsilon)^3 u^\varepsilon_x(u^\varepsilon u^\varepsilon_x)_x - \gamma(h^\varepsilon)^3 u^\varepsilon_x((h^\varepsilon)^{-2}h^\varepsilon_x)_x\\ &\quad - \frac{1}{3}u^\varepsilon_x(h^\varepsilon)^3\partial_x\mathcal{L}^{-1}_{h^\varepsilon}\partial_x\mathcal{X}^\varepsilon + \frac{1}{3}u^\varepsilon_x(h^\varepsilon)^3\partial_x\mathcal{B}^\varepsilon.\end{aligned} \tag{B.2}$$

Using (5.2) again and the definition of $\mathcal{L}_{h^\varepsilon}$, we obtain

$$\begin{aligned}\frac{1}{2}[h^\varepsilon(u^\varepsilon)^2]_t &= \frac{1}{2}(u^\varepsilon)^2\mathcal{A}^\varepsilon_x - \frac{1}{2}(u^\varepsilon)^2(h^\varepsilon u^\varepsilon)_x - h^\varepsilon(u^\varepsilon)^2 u^\varepsilon_x - 3\gamma(h^\varepsilon)^{-1}u^\varepsilon h^\varepsilon_x\\ &\quad - h^\varepsilon u^\varepsilon \mathcal{L}^{-1}_{h^\varepsilon}\mathcal{X}^\varepsilon_x + h^\varepsilon u^\varepsilon \mathcal{B}^\varepsilon\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}(u^\varepsilon)^2(h^\varepsilon u^\varepsilon)_x - h^\varepsilon (u^\varepsilon)^2 u_x^\varepsilon - 3\gamma (h^\varepsilon)^{-1} u^\varepsilon h_x^\varepsilon - u^\varepsilon \mathcal{X}_x^\varepsilon \\
&\quad - \frac{1}{3} u^\varepsilon \partial_x (h^\varepsilon)^3 \partial_x \mathcal{L}_{h^\varepsilon}^{-1} \partial_x \mathcal{X}^\varepsilon + \frac{1}{3} u^\varepsilon \partial_x (h^\varepsilon)^3 \partial_x \mathcal{B}^\varepsilon + \frac{1}{2} u^\varepsilon ((h^\varepsilon)^2 u_x^\varepsilon \mathcal{A}_x^\varepsilon)_x \\
&\quad - \frac{1}{48} u^\varepsilon [h^\varepsilon (\chi_\varepsilon(P^\varepsilon) + \chi_\varepsilon(Q^\varepsilon))]_x \\
&= -\frac{1}{2}[(u^\varepsilon)^3 h^\varepsilon]_x - 3\gamma (h^\varepsilon)^{-1} u^\varepsilon h_x^\varepsilon - [u^\varepsilon \mathcal{X}^\varepsilon]_x - \frac{1}{3}[u^\varepsilon (h^\varepsilon)^3 \partial_x \mathcal{L}_{h^\varepsilon}^{-1} \partial_x \mathcal{X}^\varepsilon]_x \\
&\quad + \frac{1}{3}[u^\varepsilon (h^\varepsilon)^3 \partial_x \mathcal{B}^\varepsilon]_x + \frac{1}{2}[u^\varepsilon (h^\varepsilon)^2 u_x^\varepsilon \mathcal{A}_x^\varepsilon]_x \\
&\quad - \frac{1}{48}[u^\varepsilon h^\varepsilon (\chi_\varepsilon(P^\varepsilon) + \chi_\varepsilon(Q^\varepsilon))]_x \\
&\quad + u_x^\varepsilon \mathcal{X}^\varepsilon + \frac{1}{3} u_x^\varepsilon (h^\varepsilon)^3 \partial_x \mathcal{L}_{h^\varepsilon}^{-1} \partial_x \mathcal{X}^\varepsilon \\
&\quad - \frac{1}{3} u_x^\varepsilon (h^\varepsilon)^3 \partial_x \mathcal{B}^\varepsilon - \frac{1}{2}(h^\varepsilon)^2 (u_x^\varepsilon)^2 \mathcal{A}_x^\varepsilon \\
&\quad + \frac{1}{96}(P^\varepsilon + Q^\varepsilon)(\chi_\varepsilon(P^\varepsilon) + \chi_\varepsilon(Q^\varepsilon)).
\end{aligned}
\tag{B.3}
$$

Summing (B.2) and (B.3) one obtains

$$
\begin{aligned}
\frac{1}{2}[h^\varepsilon (u^\varepsilon)^2]_t + \frac{1}{6}[(h^\varepsilon)^3 (u_x^\varepsilon)^2]_t &= -3\gamma (h^\varepsilon)^{-1} u^\varepsilon h_x^\varepsilon + u_x^\varepsilon \mathcal{X}^\varepsilon - \frac{1}{2}(h^\varepsilon)^2 (u_x^\varepsilon)^2 (h^\varepsilon u^\varepsilon)_x \\
&\quad - \frac{1}{3}(h^\varepsilon)^3 u_x^\varepsilon (u^\varepsilon u_x^\varepsilon)_x - \frac{1}{2}[(u^\varepsilon)^3 h^\varepsilon]_x - [u^\varepsilon \mathcal{X}^\varepsilon]_x \\
&\quad - \frac{1}{3}[u^\varepsilon (h^\varepsilon)^3 \partial_x \mathcal{L}_{h^\varepsilon}^{-1} \partial_x \mathcal{X}^\varepsilon]_x + \frac{1}{3}[u^\varepsilon (h^\varepsilon)^3 \partial_x \mathcal{B}^\varepsilon]_x \\
&\quad + \frac{1}{2}[u^\varepsilon (h^\varepsilon)^2 u_x^\varepsilon \mathcal{A}_x^\varepsilon]_x - \frac{1}{48}[u^\varepsilon h^\varepsilon (\chi_\varepsilon(P^\varepsilon) + \chi_\varepsilon(Q^\varepsilon))]_x \\
&\quad - \gamma (h^\varepsilon)^3 u_x^\varepsilon ((h^\varepsilon)^{-2} h_x^\varepsilon)_x \\
&\quad + \frac{1}{96}(P^\varepsilon + Q^\varepsilon)(\chi_\varepsilon(P^\varepsilon) + \chi_\varepsilon(Q^\varepsilon)).
\end{aligned}
\tag{B.4}
$$

Using (4.3) and (5.4) we obtain

$$
\begin{aligned}
\frac{1}{3}(h^\varepsilon)^3 \partial_x \mathcal{B}^\varepsilon &= \frac{1}{3}(h^\varepsilon)^3 \partial_x \mathcal{L}_{h^\varepsilon}^{-1} \left\{ -\frac{1}{2} u^\varepsilon \mathcal{A}_x^\varepsilon \right\} - \frac{1}{2}(h^\varepsilon)^2 u_x^\varepsilon \mathcal{A}_x^\varepsilon \\
&\quad + \frac{1}{48} h^\varepsilon (\chi_\varepsilon(P^\varepsilon) + \chi_\varepsilon(Q^\varepsilon)) \\
&\quad + (h^\varepsilon)^3 \partial_x \mathcal{L}_{h^\varepsilon}^{-1} \left\{ h^\varepsilon \int_{-\infty}^x \Big(\frac{1}{2}(h^\varepsilon)^{-1} u_x^\varepsilon \mathcal{A}_x^\varepsilon \right. \\
&\qquad \left. - \frac{1}{48 (h^\varepsilon)^2} (\chi_\varepsilon(P^\varepsilon) + \chi_\varepsilon(Q^\varepsilon)) \Big) \mathrm{d}y \right\} \\
&= \frac{1}{3}(h^\varepsilon)^2 \mathcal{V}_1^\varepsilon - \frac{1}{2}(h^\varepsilon)^2 u_x^\varepsilon \mathcal{A}_x^\varepsilon + \frac{1}{48} h^\varepsilon (\chi_\varepsilon(P^\varepsilon) + \chi_\varepsilon(Q^\varepsilon)).
\end{aligned}
$$

Now using (4.4) and (B.4) we obtain

$$\begin{aligned}\frac{1}{2}[h^\varepsilon(u^\varepsilon)^2]_t + \frac{1}{6}[(h^\varepsilon)^3(u_x^\varepsilon)^2]_t &= -3\gamma(h^\varepsilon)^{-1}u^\varepsilon h_x^\varepsilon + u_x^\varepsilon\mathcal{X}^\varepsilon - \frac{1}{2}(h^\varepsilon)^2(u_x^\varepsilon)^2(h^\varepsilon u^\varepsilon)_x \\ &\quad - \frac{1}{3}(h^\varepsilon)^3u_x^\varepsilon(u^\varepsilon u_x^\varepsilon)_x - \frac{1}{2}[(u^\varepsilon)^3h^\varepsilon]_x - [u^\varepsilon\mathcal{R}^\varepsilon]_x \\ &\quad - \Big[u^\varepsilon\Big(\frac{1}{2}g((h^\varepsilon)^2 - \bar{h}^2) - 3\gamma\ln(h^\varepsilon/\bar{h})\Big)\Big]_x \\ &\quad + \frac{1}{3}[u^\varepsilon(h^\varepsilon)^2\mathcal{V}_1^\varepsilon]_x - \gamma(h^\varepsilon)^3u_x^\varepsilon((h^\varepsilon)^{-2}h_x^\varepsilon)_x \\ &\quad + \frac{1}{96}(P^\varepsilon + Q^\varepsilon)(\chi_\varepsilon(P^\varepsilon) + \chi_\varepsilon(Q^\varepsilon)).\end{aligned} \tag{B.5}$$

Forward calculations lead to

$$\begin{aligned}&g(h^\varepsilon - \bar{h})(h^\varepsilon u^\varepsilon)_x + \gamma h_x^\varepsilon(h^\varepsilon u^\varepsilon)_{xx} + 3\gamma(h^\varepsilon)^{-1}u^\varepsilon h_x^\varepsilon - u_x^\varepsilon\mathcal{X}^\varepsilon \\ &\qquad + \frac{1}{2}(h^\varepsilon)^2(u_x^\varepsilon)^2(h^\varepsilon u^\varepsilon)_x + \frac{1}{3}(h^\varepsilon)^3u_x^\varepsilon(u^\varepsilon u_x^\varepsilon)_x + \gamma(h^\varepsilon)^3u_x^\varepsilon((h^\varepsilon)^{-2}h_x^\varepsilon)_x \\ &\quad = \frac{1}{2}g[u^\varepsilon(h^\varepsilon - \bar{h})^2]_x + \frac{1}{6}[(h^\varepsilon)^3u^\varepsilon(u_x^\varepsilon)^2]_x \\ &\qquad + 3\gamma[u^\varepsilon\ln(h^\varepsilon/\bar{h})]_x + \frac{1}{2}\gamma[u^\varepsilon(h_x^\varepsilon)^2]_x + \gamma[h^\varepsilon h_x^\varepsilon u_x^\varepsilon]_x.\end{aligned} \tag{B.6}$$

Summing (B.1), (B.5) and (B.6) we obtain

$$\begin{aligned}&\Big[\frac{1}{2}h^\varepsilon(u^\varepsilon)^2 + \frac{1}{2}g(h^\varepsilon - \bar{h})^2 + \frac{1}{6}(h^\varepsilon)^3(u_x^\varepsilon)^2 + \frac{1}{2}\gamma(h_x^\varepsilon)^2\Big]_t \\ &\quad + \Big[\frac{1}{2}(u^\varepsilon)^3h^\varepsilon - g(h^\varepsilon - \bar{h})\mathcal{A}^\varepsilon + u^\varepsilon\mathcal{R}^\varepsilon + \frac{1}{2}gu^\varepsilon((h^\varepsilon)^2 - \bar{h}^2) - \frac{1}{3}u^\varepsilon(h^\varepsilon)^2\mathcal{V}_1^\varepsilon \\ &\qquad + \frac{1}{2}gu^\varepsilon(h^\varepsilon - \bar{h})^2 + \frac{1}{6}(h^\varepsilon)^3u^\varepsilon(u_x^\varepsilon)^2 + \frac{1}{2}\gamma u^\varepsilon(h_x^\varepsilon)^2 + \gamma h^\varepsilon h_x^\varepsilon u_x^\varepsilon\Big]_x \\ &\quad = \frac{1}{48}P^\varepsilon\chi_\varepsilon(P^\varepsilon) + \frac{1}{48}Q^\varepsilon\chi_\varepsilon(Q^\varepsilon).\end{aligned}$$

This is (5.8).

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