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# On local energy decay for solutions of the Benjamin–Ono equation

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**Abstract.** We consider the long time dynamics of large solutions to the Benjamin–Ono equation. Using virial techniques, we describe regions of space where every solution in a suitable Sobolev space must decay to zero along sequences of times. Moreover, in the case of exterior regions, we prove complete decay for any sequence of times. The remaining regions not treated here are essentially the strong dispersion and soliton regions.

## 1. Introduction

We consider the initial value problem (IVP) associated to the Benjamin–Ono (BO) equation

$$\begin{cases} \partial_t u - \mathcal{H}\partial_x^2 u + u\partial_x u = 0, \quad x, t \in \mathbb{R}, \\ u(x,0) = u_0(x) \end{cases} \tag{1.1}$$

where $u = u(x,t)$ is a real-valued function and $\mathcal{H}$ is the Hilbert transform, defined on the line as

$$\mathcal{H} f(x) = \text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(y)}{x-y}\,dy. \tag{1.2}$$

The BO equation was first deduced in the context of long internal gravity waves in a stratified fluid [4, 35]. Later, the BO equation was shown to be completely integrable (see [2] and references therein).

In particular, it possesses an infinite number of conservation laws, the first three being the following:

$$\begin{aligned} I_1(u) &= \int_{\mathbb{R}} u\,dx, \\ I_2(u) &= M(u) = \int_{\mathbb{R}} u^2\,dx, \\ I_3(u) &= E(u) = \int_{\mathbb{R}} \Big(\frac{1}{2}|D^{1/2}u|^2 + \frac{1}{6}u^3\Big)\,dx, \end{aligned} \tag{1.3}$$

where $\widehat{D^s f}(\xi) = |\xi|^s \hat{f}(\xi)$.

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The $k$-conservation law, $I_k(\cdot)$, $k \geq 2$, provides a global-in-time a priori estimate of the norm $\|D^{(k-2)/2}u(t)\|_{L^2}$ of the solution $u = u(x,t)$ of (1.1).

The IVP (1.1) has been extensively studied, especially the local well-posedness and global well-posedness measured in the Sobolev scale $H^s(\mathbb{R}) = (1-\partial_x^2)^{-s/2}L^2(\mathbb{R})$, $s \in \mathbb{R}$. In this regard, one has the following list of works: Iório [16], Abdelouhab et al. [1], Ponce [36], Koch–Tzvetkov [23], Kenig–Koenig [19], Tao [38], Burq–Planchon [7], Ionescu–Kenig [15], Molinet–Pilod [30] and Ifrim–Tataru [14], among others. In particular, in [15] the global well-posedness in $L^2(\mathbb{R})$ of IVP (1.1) was established. For further details and results concerning the IVP associated to the BO equation we refer to Saut [37].

It should be pointed out that in [31] it was proved that no well-posedness for IVP (1.1) in $H^s(\mathbb{R})$ for any $s \in \mathbb{R}$ can be established by an argument based only on the contraction principle argument.

We recall that the BO equation possesses traveling wave solutions (solitons) $u(x,t) = \phi(x-t)$ of the form

$$\phi(x) = \frac{4}{1+x^2}, \tag{1.4}$$

which is smooth and exhibits mild decay.

In this work, we are interested in the asymptotic behavior of solutions to IVP (1.1). In fact, we will deduce some decay properties for solutions of (1.1) as time evolves.

Our main results in this work are the following:

**Theorem 1.1.** *Let $u_0 \in L^2(\mathbb{R})$ and $u = u(x,t)$ be the global-in-time solution of IVP (1.1) such that*

$$u \in C(\mathbb{R} : L^2(\mathbb{R})) \cap L^\infty(\mathbb{R} : L^2(\mathbb{R})).$$

*Then*

$$\liminf_{t\to\infty} \int_{B_{t^b}(0)} u^2(x,t)\,dx = 0, \tag{1.5}$$

*where $B_{t^b}(0)$ denotes the ball centered at the origin with radius $t^b$,*

$$B_{t^b}(0) := \left\{x \in \mathbb{R} : |x| < t^b\right\} \quad \textit{with } 0 < b < \tfrac{2}{3}. \tag{1.6}$$

*Moreover, there exist a constant $C > 0$ and an increasing sequence of times $t_n \to \infty$ such that*

$$\int_{B_{t_n^b}(0)} u^2(x,t_n)\,dx \leq \frac{C}{\log^{\frac{(1-b)}{b}}(t_n)}. \tag{1.7}$$

As a consequence of the proof of this theorem we have the following corollary:

**Corollary 1.2.** *Let $u_0 \in L^2(\mathbb{R})$ and $u = u(x,t)$ be the global-in-time solution of IVP (1.1) such that*

$$u \in C(\mathbb{R} : L^2(\mathbb{R})) \cap L^\infty(\mathbb{R} : L^2(\mathbb{R})).$$

*Then*

$$\liminf_{t\to\infty} \int_{B_{t^b}(t^m)} u^2(x,t)\,dx = 0, \tag{1.8}$$

*where*

$$B_{t^b}(t^m) := \{x \in \mathbb{R} : |x - t^m| < t^b\}, \tag{1.9}$$

*with*

$$0 < b < \tfrac{2}{3} \quad \text{and} \quad 0 \le m < 1 - \tfrac{3}{2}b. \tag{1.10}$$

**Remark 1.3.** Under the additional hypothesis that

*there exist* $a \in [0, 1/2)$ *and* $c_0 > 0$ *such that for all* $T > 0$,

$$\sup_{t\in[0,T]} \int_{-\infty}^{\infty} |u(x,t)|\,dx \le c_0(1+T^2)^{a/2}, \tag{1.11}$$

a result related to those in Theorem 1.1 and Corollary 1.2 was established in [33]. The argument in the proof in [33] was based on virial identities (or weighted energy estimates), first appearing in [32] in the study of the long time behavior of solutions of the generalized Korteweg–de Vries (KdV) equation. In [24,34] this was extended, adapted and generalized to other one-dimensional dispersive nonlinear systems under an assumption similar to that in (1.11).

In [29] a key idea was introduced to remove hypothesis (1.11) and to extend the argument to higher-dimensional dispersive models. This approach was further implemented in [28] and [25] for systems.

Next, we present a result concerning the decay of solutions in the energy space:

**Theorem 1.4.** *Let* $u_0 \in H^{1/2}(\mathbb{R})$ *and* $u = u(x,t)$ *be the global-in-time solution of IVP* (1.1) *such that*

$$u \in C(\mathbb{R} : H^{1/2}(\mathbb{R})) \cap L^\infty(\mathbb{R} : H^{1/2}(\mathbb{R})).$$

*Then*

$$\liminf_{t\to\infty} \int_{B_{t^b}(0)} (u^2(x,t) + |D_x^{1/2}u(x,t)|^2)\,dx = 0, \quad 0 < b < \tfrac{2}{3}. \tag{1.12}$$

Now, we consider the asymptotic decay of the solution in a domain moving in time to the right:

**Theorem 1.5.** *There exists a constant* $C_0 > 0$ *depending only on* $\|u_0\|_{H^1}$ *such that the global solution*

$$u \in C(\mathbb{R} : H^1(\mathbb{R})) \cap L^\infty(\mathbb{R} : H^1(\mathbb{R}))$$

*of IVP* (1.1) *satisfies*

$$\lim_{t\to\infty} \|u(t)\|_{L^2(x \ge C_0 t)} = 0. \tag{1.13}$$

**Remark 1.6.** (1) This result is inspired by a similar one found in [26] for the generalized KdV equation. In fact, the proof in [26] is a generalization of the identity used in [17] to establish the so-called Kato local smoothing effect in solutions of the generalized KdV equation. The proof for the KdV is significantly simpler. In the case of the BO equation, the proof follows the virial identity obtained in Lemma 5.1, (5.1), and some commutator estimates; see the comments in Remark 5.2 below.

(2) From a scaling argument, i.e. if $u(x,t)$ is a solution of the BO equation, then for any $\lambda > 0$, $u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$ is also a solution, one sees that for any $c > 0$ one has a traveling wave solution (soliton) (see (1.4))

$$u_c(x,t) = c\phi(cx - c^2 t) = c\phi(c(x-ct)).$$

The speed of propagation of the soliton $c$ is proportional to its amplitude $c\|\phi\|_\infty$.

Also, for the associated nonlinear problem, i.e. the inviscid Burgers equation, the maximum speed of propagation of the (implicit) solution is given by the $L^\infty$-norm of the data.

Combining the Sobolev embedding theorem and the conservation laws for the BO equation, one controls the $L^\infty$-norm by the $H^1$-norm of the data. This justifies estimate (1.13).

(3) Combining (1.13) and the conservation laws for the BO equation, and under the same hypothesis as Theorem 1.5, one gets that for any $p \in (2,\infty]$ and any $C_0' > C_0$,

$$\lim_{t\to\infty} \|u(t)\|_{L^p(x \geq C_0' t)} = 0 \tag{1.14}$$

and for any $s \in (0,1)$,

$$\lim_{t\to\infty} \Big\| D_x^s \Big( u(x,t)\chi\Big(\frac{x}{2C_0'}\Big)\Big)\Big\|_{L^2} = 0, \tag{1.15}$$

with $\chi \in C^\infty(\mathbb{R})$, $0 \leq \chi(x) \leq 1$ for all $x \in \mathbb{R}$, $\chi(x) \equiv 0$ if $x \leq 1$, $\chi(x) \equiv 1$ if $x \geq 2$ and $\chi' \geq 0$.

(4) If, in addition, one assumes that the global solution $u = u(x,t)$ satisfies

$$u \in C(\mathbb{R} : H^{3/2}(\mathbb{R})) \cap L^\infty(\mathbb{R} : H^{3/2}(\mathbb{R})),$$

then

$$\lim_{t\to\infty} \|\partial_x u(t)\|_{L^2(x \geq C_0' t)} = 0, \tag{1.16}$$

and (1.15) holds for $s \in (0, 3/2)$. This result extends to global solutions $u \in C(\mathbb{R} : H^{k/2}(\mathbb{R}))$, $k \in \mathbb{N}$ with $k > 3$.

The next result studies the decay of the $L^2$-norm of the solution in the far left region:

**Theorem 1.7.** *For any constant $C_1 > 0$ and any $\eta > 0$, the global-in-time solution*

$$u \in C(\mathbb{R} : H^1(\mathbb{R})) \cap L^\infty(\mathbb{R} : H^1(\mathbb{R}))$$

*of IVP* (1.1) *satisfies*

$$\lim_{t\to\infty} \|u(t)\|_{L^2(x\leq -C_1 t \log^{1+\eta} t)} = 0. \tag{1.17}$$

**Remark 1.8.** (1) To our knowledge the result in Theorem 1.7 is totally new. From its proof below it will be clear that it applies with minor modifications to solutions of the generalized KdV equation,

$$\partial_t u + \partial_x^3 u + u^k \partial_x u = 0, \quad x, t \in \mathbb{R},\ k = 1, 2, \ldots,$$

to solutions of the generalized BO equation (see (1.19)) and solutions to other one-dimensional dispersive models.

(2) Statements (1.14), (1.15) and (1.16) in Remark 1.6 apply to the result in Theorem 1.7 with the appropriate modifications.

(3) Collecting the information in Theorems 1.1, 1.5 and 1.7 one can deduce several estimates. In particular, one has that there exists $C_0 = C_0(\|u_0\|_{H^1}) > 0$ and an increasing sequence of times $(t_n)_{n=1}^\infty$ with $t_n \uparrow \infty$ as $n \to \infty$ such that for any constants $c > 0$, $\gamma > 0$,

$$\liminf_{n\to\infty} \int_{\Omega(t_n)} |u(x, t_n)|^2\, dx = \|u_0\|_2^2, \tag{1.18}$$

with

$$\Omega(t) := \{x \in \mathbb{R} : -ct \log^{1+\gamma} t < x < -ct^{\frac{2}{3}^-} \text{ or } ct^{\frac{2}{3}^-} < x < C_0 t\}.$$

Finally, we will consider the possible extensions of the above results to solutions of the IVP associated to the $k$-generalized BO (k-gBO) equation

$$\begin{cases} \partial_t u - \mathcal{H}\partial_x^2 u + u^k \partial_x u = 0, & x, t \in \mathbb{R}, k = 2, 3, \ldots, \\ u(x, 0) = u_0(x). \end{cases} \tag{1.19}$$

In this case, the equations in (1.19) are not completely integrable and satisfy (in general) only three conservation laws : $I_1(u)$, $I_2(u)$ in (1.3) and

$$I_3(u) = \int_{\mathbb{R}} \Big(\frac{1}{2}|D^{1/2}u|^2 + \frac{u^{k+1}}{(k+1)(k+2)}\Big)\, dx.$$

A scaling argument, see Remark 1.6, says that if $u(x, t)$ is a solution of the k-gBO equation in (1.19), then for any $\lambda > 0$, $u_\lambda(x, t) = \lambda^{1/k} u(\lambda x, \lambda^2 t)$ is also a solution. This suggests that the critical Sobolev space for the well-posedness should be $H^{s_k}(\mathbb{R})$ with $s_k = 1/2 - 1/k$.

The results considered here are concerned with global solutions of (1.19). Thus, for the cases $k \geq 2$ these are only known under appropriate smallness assumptions on the data. More precisely, if $k = 2$, local well-posedness in $H^{1/2}(\mathbb{R})$ was established in [22]. This local result extends globally in time if one assumes that the $L^2$-norm of the initial data $u_0$ is small enough (the blow-up result in [27] shows that this restriction is necessary).

In [39], local well-posedness was proved in $H^s(\mathbb{R})$ for $s > 1/3$ if $k = 3$, and for $s \geq s_k = 1/2 - 1/k$ if $k \geq 4$. These local results extend to global ones under a smallness assumption of the $H^{1/2}$-norm of the initial data $u_0$ (see [10]). In all these global results one only has an a priori bound of the $H^{1/2}$-norm of the solution.

Our argument of proof of Theorem 1.5 depends on a global bound of the $L^\infty$-norm of the solution. Hence, the proof of Theorem 1.5 provided below does not extend to these small global solutions of IVP (1.19).

The approach to obtain Theorem 1.7 only requires a global bound of the $L^{k+2}$-norm of the solution, which follows from that of the $H^{1/2}$-norm. Hence, the result in Theorem 1.7 expands to all small global solutions of IVP (1.19) commented on above.

**Remark 1.9.** In the cases when $k \geq 1$ is *odd*, the arguments utilized to prove Theorem 1.1, Corollary 1.2 and Theorem 1.4 apply to get the results in (1.5), (1.8) and (1.12) with the term $u^2$ in the integrand substituted by $u^{k+1}$. However, in this case $k \geq 2$ and small data (in a weighted space), stronger asymptotic results were accomplished in [13].

The rest of this paper is ordered as follows: Section 2 contains the statements of some general estimates to be used in the proofs of the main results. Theorem 1.1 and Corollary 1.2 will be proved in Section 3. Section 4 involves the proof of Theorem 1.4 and Section 5 those of Theorems 1.5 and 1.7. Appendix A consists of the proof a commutator estimate stated in Section 2 and used in Sections 3-5.

## 2. Preliminaries

We present a series of estimates that we will employ in the proofs of our results.

**Lemma 2.1.** *For any $k, m \in \mathbb{N} \cup \{0\}$, $k + m \geq 1$, and any $p \in (1, \infty)$,*

$$\|\partial_x^k [\mathcal{H}; a] \partial_x^m f\|_p \leq c_{p,k,m} \|\partial_x^{k+m} a\|_\infty \|f\|_p. \tag{2.1}$$

The case $k + m = 1$ corresponds to the first Calderón commutator estimate [8]. The general case of (2.1) was established in [3]. For a different proof see [9].

The next estimate is an inequality of Gagliardo–Nirenberg type whose proof can be found in [5].

**Lemma 2.2.** *There exists $C > 0$ such that for any $f \in H^{1/2}(\mathbb{R})$,*

$$\|f\|_{L^3} \leq C \|f\|_{L^2}^{\frac{2}{3}} \|D^{1/2} f\|_{L^2}^{\frac{1}{3}}. \tag{2.2}$$

Throughout the proofs of our results we will use the following general version of the Leibniz rule for fractional derivatives:

**Lemma 2.3.** *Let* $r \in [1,\infty]$ *and* $p_1, p_2, q_1, q_2 \in (1,\infty]$ *with*

$$\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}. \tag{2.3}$$

*Given* $s > 0$ *there exists* $c = c(n,s,r,p_1,p_2,q_1,q_2) > 0$ *such that for all* $f, g \in \mathcal{S}(\mathbb{R}^n)$ *one has*

$$\|D^s(fg)\|_r \le c(\|f\|_{p_1}\|D^s g\|_{q_1} + \|g\|_{p_2}\|D^s f\|_{q_2}). \tag{2.4}$$

For the proof of Lemma 2.3 we refer to [12]. The case $r = p_1 = p_2 = q_1 = q_2 = \infty$ was established in [6]; see also [11]. For earlier versions of this result see [18, 21].

Finally, we consider a commutator estimate whose proof will be given in the appendix.

**Lemma 2.4.** *Let* $a \in C^2(\mathbb{R})$ *with* $a', a'' \in L^\infty(\mathbb{R})$. *There exists* $c > 0$ *such that for all* $f \in L^2(\mathbb{R})$,

$$\|D^{1/2}[D^{1/2}; a]f\|_{L^2} \le c\|\widehat{a'}\|_{L^1}\|f\|_{L^2} \le c\|a'\|_{L^2}^{1/2}\|a''\|_{L^2}^{1/2}\|f\|_{L^2}. \tag{2.5}$$

## 3. Proofs of Theorem 1.1 and Corollary 1.2

First we will introduce some notation and definitions.

Let $\phi$ be a smooth even and positive function such that

$$\begin{cases} \phi'(x) \le 0 \text{ for } x \ge 0, \\ \phi(x) \equiv 1 \text{ for } 0 \le x \le 1,\ \phi(x) = e^{-x} \text{ for } x \ge 2 \\ \quad \text{and } e^{-x} \le \phi(x) \le 3e^{-x} \text{ for } x \ge 0, \\ |\phi'(x)| \le c\phi(x) \text{ and } |\phi''(x)| \le c\phi(x) \text{ for some positive constant } c. \end{cases} \tag{3.1}$$

Let $\psi(x) = \int_0^x \phi(s)\,ds$. In particular, $|\psi(x)| \le 1 + 3\int_1^\infty e^{-t}\,dt < \infty$.

Next we consider a smooth cut-off function $\zeta\colon \mathbb{R} \to \mathbb{R}$ such that

$$\zeta \equiv 1 \text{ in } [0,1],\ 0 \le \zeta \le 1 \quad \text{and} \quad \zeta \equiv 0 \text{ in } (\infty,-1] \cup [2,\infty), \tag{3.2}$$

and define $\zeta_n(x) := \zeta(x-n)$.

For the parameters $\delta, \sigma \in \mathbb{R}^+$, we define

$$\phi_\delta = \delta\phi\Big(\frac{x}{\delta}\Big) \quad \text{and} \quad \psi_\sigma(x) = \sigma\psi\Big(\frac{x}{\sigma}\Big).$$

The proof of Theorem 1.1 will be deduced as a consequence of the following lemmas, which we will prove below.

First, we start by considering some useful parameters involved in our argument of proof:

$$\rho(t) = \pm t^m, \quad \mu_1(t) = \frac{t^b}{\log t} \quad \text{and} \quad \mu(t) = t^{(1-b)} \log^2 t, \tag{3.3}$$

where $m$ and $b$ are positive constants satisfying the relations

$$0 \leq m \leq 1 - \frac{b}{2} \quad \text{and} \quad 0 < b \leq \min\left\{\frac{2}{3}, \frac{2}{2+q}\right\}, \quad q > 1. \tag{3.4}$$

Since

$$\frac{\mu_1'(t)}{\mu_1(t)} = \frac{b}{t} - \frac{1}{t \log t} \quad \text{and} \quad \frac{\mu'(t)}{\mu(t)} = \frac{(1-b)}{t} + \frac{2}{t \log t}$$

it readily follows that

$$\frac{\mu_1'(t)}{\mu_1(t)} \sim \frac{\mu'(t)}{\mu(t)} = O\Big(\frac{1}{t}\Big), \quad \text{for } t \gg 1 \tag{3.5}$$

where $t \gg 1$ means the values of $t$ such that $\mu_1'(t)$ is positive. In particular, $[10, +\infty) \subset \{t \gg 1\}$.

For $u = u(x,t)$ a solution of IVP (1.1) we define the functional

$$\mathcal{I}(t) := \frac{1}{\mu(t)} \int_{\mathbb{R}} u(x,t) \psi_\sigma\Big(\frac{x}{\mu_1(t)}\Big) \phi_\delta\Big(\frac{x}{\mu_1^q(t)}\Big)\, dx, \tag{3.6}$$

for $q > 1$.

**Lemma 3.1.** *Let $u(\cdot,t) \in L^2(\mathbb{R})$, $t \gg 1$. The functional $\mathcal{I}(t)$ is well defined and bounded in time.*

*Proof.* The Cauchy–Schwarz inequality and the definitions of the functions $\mu(t)$ and $\mu_1(t)$ imply that

$$\begin{aligned} |\mathcal{I}(t)| &\leq \frac{1}{\mu(t)} \|u(t)\|_{L^2} \Big\|\psi_\sigma\Big(\frac{\cdot}{\mu_1(t)}\Big)\Big\|_{L^\infty} \Big\|\phi_\delta\Big(\frac{\cdot}{\mu_1^q(t)}\Big)\Big\|_{L^2} \\ &= \frac{\mu_1^{q/2}(t)}{\mu(t)} \|u(t)\|_{L^2} \|\psi_\sigma\|_{L^\infty} \|\phi_\delta\|_{L^2} \\ &\lesssim_{\sigma,\delta} \frac{1}{t^{(2-2b-bq)/2}} \frac{1}{\log^{(4+q)/2}(t)} \|u_0\|_{L^2}. \end{aligned} \tag{3.7}$$

Since $b$ satisfies condition (3.4) we have

$$\sup_{t \gg 1} |\mathcal{I}(t)| < \infty. \qquad \blacksquare$$

**Lemma 3.2.** *For any $t \gg 1$, it holds that*

$$\frac{1}{\mu_1(t)\mu(t)} \int_{\mathbb{R}} u^2(x,t) \psi_\sigma'\Big(\frac{x}{\mu_1(t)}\Big) \phi_\delta\Big(\frac{x}{\mu_1^q(t)}\Big)\, dx \leq 4 \frac{d}{dt}\mathcal{I}(t) + h(t), \tag{3.8}$$

*where $h(t) \in L^1(t \gg 1)$.*

*Proof.* We have that

$$\begin{aligned}\frac{d}{dt}\mathcal{I}(t) = &\ \frac{1}{\mu(t)}\int_{\mathbb{R}} \partial_t\Big(u\psi_\sigma\Big(\frac{x}{\mu_1(t)}\Big)\phi_\delta\Big(\frac{x}{\mu_1^q(t)}\Big)\Big)\,dx \\ &- \frac{\mu'(t)}{\mu^2(t)}\int_{\mathbb{R}} u\psi_\sigma\Big(\frac{x}{\mu_1(t)}\Big)\phi_\delta\Big(\frac{x}{\mu_1^q(t)}\Big)\,dx \\ =:&\ A(t)+B(t). \end{aligned} \tag{3.9}$$

The Cauchy–Schwarz inequality and the conservation of mass, $I_2$ in (1.3), yield

$$\begin{aligned}|B(t)| \ \leq&\ \Big|\frac{\mu'(t)}{\mu^2(t)}\Big|\|u(t)\|_{L^2}\Big\|\psi_\sigma\Big(\frac{\cdot}{\mu_1(t)}\Big)\Big\|_{L^\infty}\Big\|\phi_\delta\Big(\frac{\cdot}{\mu_1^q(t)}\Big)\Big\|_{L^2} \\ \lesssim_{\sigma,\delta}&\ \frac{1}{t^{(4-2b-bq)/2}}\frac{1}{\log^{(4+q)/2} t}\|u_0\|_{L^2}. \end{aligned} \tag{3.10}$$

Hence $B(t)\in L^1(\{t\gg 1\})$ whenever $b\leq \frac{2}{2+q}$. We remark that this term is bounded in $\{t\gg 1\}$.

To estimate $A(t)$, we first differentiate in time to write

$$\begin{aligned}A(t) = &\ \frac{1}{\mu(t)}\int_{\mathbb{R}} u_t(x,t)\psi_\sigma\Big(\frac{x}{\mu_1(t)}\Big)\phi_\delta\Big(\frac{x}{\mu_1^q(t)}\Big)\,dx \\ &- \frac{\mu_1'(t)}{\mu_1(t)\mu(t)}\int_{\mathbb{R}} u(x,t)\Big(\frac{x}{\mu_1(t)}\Big)\psi_\sigma'\Big(\frac{x}{\mu_1(t)}\Big)\phi_\delta\Big(\frac{x}{\mu_1^q(t)}\Big)\,dx \\ &- \frac{q\mu_1'(t)}{\mu_1(t)\mu(t)}\int_{\mathbb{R}} u(x,t)\psi_\sigma\Big(\frac{x}{\mu_1(t)}\Big)\Big(\frac{x}{\mu_1^q(t)}\Big)\phi_\delta'\Big(\frac{x}{\mu_1^q(t)}\Big)\,dx \\ =:&\ A_1(t)+A_2(t)+A_3(t). \end{aligned} \tag{3.11}$$

Using the equation in (1.1) and integrating by parts yields

$$\begin{aligned}A_1(t) = &\ \frac{1}{\mu(t)}\int_{\mathbb{R}} \mathcal{H}u(x,t)\partial_x^2\Big(\psi_\sigma\Big(\frac{x}{\mu_1(t)}\Big)\phi_\delta\Big(\frac{x}{\mu_1^q(t)}\Big)\Big)\,dx \\ &+ \frac{1}{2\mu(t)\mu_1(t)}\int_{\mathbb{R}} u^2(x,t)\psi_\sigma'\Big(\frac{x}{\mu_1(t)}\Big)\phi_\delta\Big(\frac{x}{\mu_1^q(t)}\Big)\,dx \\ &+ \frac{1}{2\mu(t)\mu_1^q(t)}\int_{\mathbb{R}} u^2(x,t)\psi_\sigma\Big(\frac{x}{\mu_1(t)}\Big)\phi_\delta'\Big(\frac{x}{\mu_1^q(t)}\Big)\,dx \\ =:&\ A_{1,1}(t)+A_{1,2}(t)+A_{1,3}(t). \end{aligned} \tag{3.12}$$

We remark that $A_{1,2}(t)$ is the term we want to estimate in (3.12). Then we need to show that the remaining terms are in $L^1(\{t\gg 1\})$.

Differentiating with respect to $x$, using the Cauchy–Schwarz inequality, Hilbert's transform properties, the conservation of mass and the definitions of $\mu(t)$ and $\mu_1(t)$ we deduce that

$$
\begin{aligned}
|A_{1,1}(t)| &\leq \frac{1}{\mu(t)\mu_1^{3/2}(t)}\|u(t)\|_{L^2}\|\psi''_\sigma\|_{L^2}\|\phi_\delta\|_{L^\infty} \\
&\quad + \frac{1}{\mu(t)\mu_1^{(1+2q)/2}(t)}\|u(t)\|_{L^2}\|\psi'_\sigma\|_{L^2}\|\phi'_\delta\|_{L^\infty} \\
&\quad + \frac{1}{\mu(t)\mu_1^{3q/2}(t)}\|u(t)\|_{L^2}\|\psi_\sigma\|_{L^\infty}\|\phi''_\delta\|_{L^2} \\
&\lesssim_{\sigma,\delta} \frac{\|u_0\|_{L^2}}{t^{(2+b)/2}\log^{1/2} t} + \frac{\|u_0\|_{L^2}}{t^{(2-b+2bq)/2}\log^{(\frac{3}{2}-q)}(t)} \\
&\quad + \frac{\|u_0\|_{L^2}}{t^{(2-2b+3bq)/2}\log^{(4-3q)/2} t}.
\end{aligned}
\tag{3.13}
$$

Since $q > 1$ it follows that $A_{1,1}(t) \in L^1(\{t \gg 1\})$.

The term $A_{1,3}$ can be bounded by employing the conservation of mass and the definitions of $\mu(t)$ and $\mu_1(t)$:

$$
\begin{aligned}
|A_{1,3}| &\leq \frac{\|u(t)\|_{L^2}^2}{2|\mu(t)\mu_1^q(t)|}\Big\|\psi_\sigma\Big(\frac{\cdot}{\mu_1(t)}\Big)\Big\|_{L^\infty}\Big\|\phi'_\delta\Big(\frac{\cdot}{\mu_1^q(t)}\Big)\Big\|_{L^\infty} \\
&\lesssim_{\sigma,\delta} \frac{\|u_0\|_{L^2}^2}{t^{1-b+bq}\log^{(2-q)} t},
\end{aligned}
\tag{3.14}
$$

and because $q > 1$ one has that $A_{1,3}(t) \in L^1(\{t \gg 1\})$.

Next we turn our attention to the other terms of (3.11). First, by means of Young's inequality, we have for $\varepsilon > 0$,

$$
\begin{aligned}
|A_2(t)| &\leq \Big|\frac{\mu_1'(t)}{\mu_1(t)\mu(t)}\Big|\int_{\mathbb{R}}\Big|\psi'_\sigma\Big(\frac{x}{\mu_1(t)}\Big)\phi_\delta\Big(\frac{x}{\mu_1^q(t)}\Big)\Big|\Big[\frac{u^2}{4\varepsilon} + 4\varepsilon\Big|\frac{x}{\mu_1(t)}\Big|^2\Big]\,dx \\
&\leq \frac{1}{4\varepsilon}\Big|\frac{\mu_1'(t)}{\mu_1(t)\mu(t)}\Big|\int_{\mathbb{R}} u^2(x,t)\psi'_\sigma\Big(\frac{x}{\mu_1(t)}\Big)\phi_\delta\Big(\frac{x}{\mu_1^q(t)}\Big)\,dx \\
&\quad + 4\varepsilon\Big|\frac{\mu_1'(t)}{\mu_1(t)\mu(t)}\Big|\Big\|\phi_\delta\Big(\frac{\cdot}{\mu_1^q(t)}\Big)\Big\|_{L^\infty}\Big\|\Big(\frac{\cdot}{\mu_1(t)}\Big)^2\psi'_\sigma\Big(\frac{\cdot}{\mu_1(t)}\Big)\Big\|_{L^1}.
\end{aligned}
$$

Then, taking $\varepsilon = \mu_1'(t)$, which is positive in $\{t \gg 1\}$, we get

$$
\begin{aligned}
|A_2(t)| &\leq \Big|\frac{1}{4\mu_1(t)\mu(t)}\Big|\int_{\mathbb{R}} u^2\psi'_\sigma\Big(\frac{x}{\mu_1(t)}\Big)\phi_\delta\Big(\frac{x}{\mu_1^q(t)}\Big)\,dx \\
&\quad + C_{\delta,\sigma}\frac{(b\log t - 1)^2}{t^{3-3b}\log^6 t} \\
&= \frac{1}{2}A_{1,2}(t) + C_{\delta,\sigma}\frac{(b\log t - 1)^2}{t^{3-3b}\log^6 t},
\end{aligned}
\tag{3.15}
$$

where $C_{\sigma,\delta}$ is a constant depending on $\sigma$ and $\delta$.

Notice that the last term in the last inequality of (3.15) is integrable in $t \gg 1$ since $b < \frac{2}{3}$.

Finally, we consider the term $A_3(t)$. Young's inequality and the conservation of mass tell us that

$$\begin{aligned} |A_3(t)| \;&\leq\; \Big|\frac{q\mu_1'(t)}{\mu_1(t)\mu(t)}\Big| \|\psi_\sigma\|_{L^\infty} \int_{\mathbb{R}} t^{1-b} u^2(x,t)\,dx \\ &\quad + \Big|\frac{q\mu_1'(t)}{\mu_1(t)\mu(t)}\Big| \|\psi_\sigma\|_{L^\infty} \int_{\mathbb{R}} \frac{1}{t^{1-b}} \Big[\Big(\frac{x}{\mu_1^q(t)}\Big)\phi_\delta'\Big(\frac{x}{\mu_1^q(t)}\Big)\Big]^2 dx \\ &\lesssim_{\sigma,\delta} \Big|\frac{qt^{1-b}\mu_1'(t)}{\mu_1(t)\mu(t)}\Big| + \Big|\frac{q\mu_1'(t)\mu_1^q(t)}{t^{1-b}\mu_1(t)\mu(t)}\Big|. \end{aligned} \tag{3.16}$$

Hence, the conditions on (3.5) imply

$$|A_3(t)| \lesssim_{\sigma,\delta} \frac{1}{t\log^2 t} + \frac{1}{t^{3-b(2+q)}\log^{2+q} t}. \tag{3.17}$$

Since $b \leq \frac{2}{2+q}$, $A_3(t) \in L^1(\{t \gg 1\})$.

Gathering the information in (3.9), (3.11), (3.13), (3.14), (3.15), (3.16) and (3.17) together we conclude that

$$\frac{1}{2\mu(t)\mu_1(t)} \int_{\mathbb{R}} u^2(x,t)\psi_\sigma'\Big(\frac{x}{\mu_1(t)}\Big)\phi_\delta\Big(\frac{x}{\mu_1^q(t)}\Big)\,dx \leq \frac{d}{dt}\mathcal{I}(t) + h(t), \tag{3.18}$$

where $h(t) \in L^1(\{t \gg 1\})$, as desired. ■

The next lemma will give us a key bound in our analysis.

**Lemma 3.3.** *Assume that $u_0 \in L^2(\mathbb{R})$. Let $u \in C(\mathbb{R} : L^2(\mathbb{R})) \cap L^\infty(\mathbb{R} : L^2(\mathbb{R}))$ be the solution of IVP (1.1). Then there exists a constant $0 < C < \infty$ such that*

$$\int_{\{t\gg 1\}} \frac{1}{t\log t} \int_{B_{t^b}} u^2(x,t)\,dx\,dt \leq C. \tag{3.19}$$

*Proof.* From the definition, $\mu(t)\mu_1(t) = t\log t$ and a straightforward computation involving the properties of the function $\phi$, it follows that

$$\frac{1}{\mu_1(t)\mu(t)} \int_{B_{t^b}} u^2(x,t)\,dx \leq \frac{1}{\mu_1(t)\mu(t)} \int_{\mathbb{R}} u^2\psi_\sigma'\Big(\frac{x}{\mu_1(t)}\Big)\phi_\delta\Big(\frac{x}{\mu_1^q(t)}\Big)\,dx,$$

for suitable $\sigma$ and $\delta$, whenever $q > 1$ is chosen sufficiently close to 1 and $b$ slightly smaller if necessary. Lemma 3.2 implies that

$$\int_{\{t\gg 1\}} \frac{1}{\mu_1(t)\mu(t)} \int_{B_{t^b}} u^2(x,t)\,dx\,dt \leq -\mathcal{I}(t) + \int_{\{t\gg 1\}} |h(t)|\,dt. \tag{3.20}$$

The first term on the right-hand side of inequality (3.20) is bounded because $b \leq \frac{2}{2+q} < \frac{2}{3}$, and the last one is bounded by the proof of Lemma 3.2. This completes the proof of the lemma. ■

Now we are ready to prove Theorem 1.1.

### 3.1. Proof of Theorem 1.1

Since the function $\frac{1}{t\log t} \notin L^1(B_r^c(1))$, from the previous lemma, we can ensure that there exists a sequence $(t_n) \to \infty$, such that

$$\lim_{n\to\infty} \int_{B^b_{(t_n)}} u^2(x,t_n)\,dx = 0.$$

Therefore, 0 is an accumulation point and using that $u^2 \geq 0$ we can conclude the result.

To end this section we will give a sketch of the proof of Corollary 1.2.

### 3.2. Proof of Corollary 1.2

The proof of this result follows the same argument as the proof given for Theorem 1.1. Hence, we will present the new details introduced in the proof. We consider the functional

$$\mathcal{I}_\rho(t) = \frac{1}{\mu(t)} \int u(x,t)\psi_\sigma\Big(\frac{x-\rho(t)}{\mu_1(t)}\Big)\phi_\sigma\Big(\frac{x-\rho(t)}{\mu_1^q(t)}\Big)\,dx,$$

where $\rho(t) = \pm t^m$, $m$ as in the statement of the corollary, $\mu(t)$ and $\mu_1(t)$ defined as in (3.3) and $\psi_\sigma$ and $\phi_\delta$ defined as above.

As in Lemma 3.1 we have

$$\sup_{t\gg 1} |\mathcal{I}_\rho(t)| < \infty.$$

We also obtain a similar inequality to (3.8) in Lemma 3.2, i.e.

$$\frac{1}{\mu_1(t)\mu(t)} \int u^2(x,t)\psi_\sigma'\Big(\frac{x-\rho(t)}{\mu_1(t)}\Big)\phi_\sigma\Big(\frac{x-\rho(t)}{\mu_1^q(t)}\Big)\,dx \leq 4\frac{d}{dt}\mathcal{I}_\rho(t) + h_\rho(t),$$

where $h_\rho(t) \in L^1(\{t \gg 1\})$. Besides the terms previously handled in the proof of Lemma 3.2, here we need to estimate two new terms,

$$-\frac{\rho'(t)}{\mu_1(t)\mu(t)} \int_{\mathbb{R}} u(x,t)\psi_\sigma'\Big(\frac{x-\rho(t)}{\mu_1(t)}\Big)\phi_\delta\Big(\frac{x-\rho(t)}{\mu_1^q(t)}\Big)\,dx = A(t)$$

and

$$-\frac{\rho'(t)}{\mu_1^q(t)\mu(t)} \int_{\mathbb{R}} u(x,t)\psi_\sigma\Big(\frac{x-\rho(t)}{\mu_1(t)}\Big)\phi_\delta'\Big(\frac{x-\rho(t)}{\mu_1^q(t)}\Big)\,dx = B(t).$$

The Cauchy–Schwarz inequality and the mass conservation yield

$$\begin{aligned} |A(t)+B(t)| &\leq \Big|\frac{\rho'(t)}{\mu_1^{1/2}(t)\mu(t)}\Big| \|u_0\|_{L^2}\|\psi_\sigma'\|_{L^2}\|\phi_\delta\|_{L^\infty} \\ &\quad + \Big|\frac{\rho'(t)}{\mu_1^q(t)\mu(t)}\Big| \|u_0\|_{L^2}\|\psi_\sigma\|_{L^\infty}\|\phi_\delta'\|_{L^2} \\ &\lesssim_{\sigma,\delta,m} \frac{1}{t^{(4-2m-b)/2}\log^{3/2} t} + \frac{1}{t^{(4-2b-2m+bq)/2}\log^{(4-q)/2} t}. \end{aligned}$$

We observe that the first term in the last inequality is in $L^1(\{t \gg 1\})$ since $m \leq 1 - \frac{b}{2}$. Similarly, the last term in the last inequality is also in $L^1(\{t \gg 1\})$ since $m \leq 1 - \frac{b}{2} < 1 - b\frac{1-q}{2}$.

From this point on the argument of proof to establish Theorem 1.1 can be applied to end the proof of Corollary 1.2.

## 4. Proof of Theorem 1.4 (asymptotic behavior in $H^{\frac{1}{2}}(\mathbb{R})$)

This section contains the proof of Theorem 1.4. The argument follows closely what we did in the previous section. Thus, we will give only the main new ingredients in the proof.

### 4.1. Asymptotic behavior of $\|D^{1/2}u(t)\|_{L^2}$

**Lemma 4.1.** *Let $u \in C(\mathbb{R} : H^{\frac{1}{2}}(\mathbb{R})) \cap L^\infty(\mathbb{R} : H^{\frac{1}{2}}(\mathbb{R}))$ be the solution of IVP (1.1). Then there exists a constant $C > 0$ such that*

$$\int_{\{t \gg 1\}} \frac{1}{t \log t} \int_{B_{t^b}(0)} |D^{1/2}u(x,t)|^2 \, dx \, dt \leq C. \tag{4.1}$$

*Proof.* Consider the functional

$$\mathcal{J}(t) := \frac{1}{\mu(t)} \int_{\mathbb{R}} u^2(x,t) \psi_\sigma\Big(\frac{x}{\mu_1(t)}\Big) \, dx, \tag{4.2}$$

where $\mu(t)$ and $\mu_1(t)$ were defined in (3.3).

Differentiating (4.2) yields

$$\begin{aligned} \frac{d}{dt}\mathcal{J}(t) = & -\frac{\mu'(t)}{\mu^2(t)} \int_{\mathbb{R}} u^2(x,t)\psi_\sigma\Big(\frac{x}{\mu_1(t)}\Big) \, dx \\ & + \frac{2}{\mu(t)} \int_{\mathbb{R}} u(x,t)\partial_t u(x,t) \psi_\sigma\Big(\frac{x}{\mu_1(t)}\Big) \, dx \\ & - \frac{\mu_1'(t)}{\mu(t)\mu_1(t)} \int_{\mathbb{R}} u^2(x,t)\phi_\sigma\Big(\frac{x}{\mu_1(t)}\Big)\Big(\frac{x}{\mu_1(t)}\Big) \, dx \\ =: & \; A(t) + B(t) + C(t). \end{aligned} \tag{4.3}$$

Combining the properties $\mu(t)$ and $\mu_1(t)$, the conservation of mass, and using (3.3), it follows that

$$|A(t)| + |C(t)| \lesssim_\sigma \frac{\|u_0\|_{L^2}^2}{t^{2-b} \log^2 t}. \tag{4.4}$$

Thus, the terms $A(t)$, $C(t)$ are integrable in $\{t \gg 1\}$.

Regarding $B(t)$, we use the equation in (1.1) and integrate by parts to write

$$\begin{aligned} B(t) = & -\frac{2}{\mu(t)}\int_{\mathbb{R}} \partial_x u \mathcal{H}\partial_x u \psi_\sigma\Big(\frac{x}{\mu_1(t)}\Big)\,dx \\ & -\frac{2}{\mu(t)\mu_1(t)}\int_{\mathbb{R}} u\mathcal{H}\partial_x u\phi_\sigma\Big(\frac{x}{\mu_1(t)}\Big)\,dx \\ & +\frac{2}{3\mu(t)\mu_1(t)}\int_{\mathbb{R}} u^3\phi_\sigma\Big(\frac{x}{\mu_1(t)}\Big)\,dx \\ =: & \; B_1(t)+B_2(t)+B_3(t). \end{aligned} \tag{4.5}$$

From Hilbert's transform properties, integrating by parts and the Cauchy–Schwarz inequality we obtain

$$\begin{aligned} |B_1(t)| &= \Big|-\frac{1}{\mu(t)}\int_{\mathbb{R}} u\partial_x\Big[\mathcal{H},\psi_\sigma\Big(\frac{\cdot}{\mu_1(t)}\Big)\Big]\partial_x u\,dx\Big| \\ &\le \frac{1}{\mu(t)}\|u\|_{L^2}\Big\|\partial_x\Big[\mathcal{H},\psi_\sigma\Big(\frac{\cdot}{\mu_1(t)}\Big)\Big]\partial_x u\Big\|_{L^2}. \end{aligned} \tag{4.6}$$

Lemma 2.1 gives us

$$|B_1(t)| \le \frac{1}{\mu(t)}\|u\|_{L^2}^2\Big\|\partial_x^2\psi_\sigma\Big(\frac{\cdot}{\mu_1(t)}\Big)\Big\|_{L^\infty} \lesssim_\sigma \frac{1}{t^{1+b}}, \tag{4.7}$$

which belongs to $L^1(\{t \gg 1\})$.

To estimate $B_2(t)$, we apply Plancherel's identity to obtain

$$\begin{aligned} B_2(t) = & -\frac{2}{\mu_1(t)\mu(t)}\int_{\mathbb{R}} uD^{1/2}\Big[D^{1/2},\phi_\sigma\Big(\frac{\cdot}{\mu_1(t)}\Big)\Big]u\,dx \\ & -\frac{2}{\mu_1(t)\mu(t)}\int_{\mathbb{R}} (D^{1/2}u)^2\phi_\sigma\Big(\frac{x}{\mu_1(t)}\Big)\,dx \\ =: & \; B_{2,1}(t)+B_{2,2}. \end{aligned} \tag{4.8}$$

Notice that $B_{2,2}(t)$ is the term we want to estimate.

To bound the term $B_{2,1}(t)$ we use the Cauchy–Schwarz inequality, the conservation of mass, (2.5) and properties of the Fourier transform to deduce that

$$\begin{aligned} |B_{2,2}(t)| &\le \Big|\frac{2}{\mu_1(t)\mu(t)}\Big|\|u_0\|_{L^2}\Big\|\widehat{\Big(\partial_x\phi_\sigma\Big(\frac{\cdot}{\mu_1(t)}\Big)\Big)}\Big\|_{L^1} \\ &\lesssim_\sigma \frac{1}{t^{1+b}} \in L^1(\{t\gg 1\}). \end{aligned} \tag{4.9}$$

Finally, notice that by Lemma 2.2,

$$\begin{aligned} \int_{\mathbb{R}} |u|^3\phi_\sigma\Big(\frac{x}{\mu_1(t)}\Big)\,dx &\le \sum_{n\in\mathbb{Z}}\int_{\mathbb{R}} (|u|\zeta_n)^3\phi_\sigma\Big(\frac{x}{\mu_1(t)}\Big)\,dx \\ &\le \sum_{n\in\mathbb{Z}}\|u\zeta_n\|_{L^3}^3\Big(\sup_{x\in[n,n+1]}\phi_\sigma\Big(\frac{x}{\mu_1(t)}\Big)\Big) \\ &\lesssim \sum_{n\in\mathbb{Z}}\|u\zeta_n\|_{L^2}^2\|D^{1/2}(u\zeta_n)\|_{L^2}\Big(\sup_{x\in[n,n+1]}\phi_\sigma\Big(\frac{x}{\mu_1(t)}\Big)\Big). \end{aligned} \tag{4.10}$$

Moreover, by Lemma 2.3 and the hypotheses on $u$ and $\zeta_n$,

$$\|D^{1/2}(u\zeta_n)\|_{L^2} \lesssim \|D^{1/2}u(t)\|_{L^2}\|\zeta_n\|_{L^\infty} + \|u(t)\|_{L^2}\|D^{1/2}\zeta_n\|_{L^\infty} \\ \lesssim \|u(t)\|_{H^{1/2}(\mathbb{R})} \lesssim \|u\|_{L^\infty_t H^{1/2}}. \tag{4.11}$$

Combining these estimates we deduce that

$$\int_{\mathbb{R}} |u(x,t)|^3 \phi_\sigma\Big(\frac{x}{\mu_1(t)}\Big)\,dx \lesssim \sum_{n\in\mathbb{Z}} \|u\zeta_n\|_{L^2}^2 \Big(\sup_{x\in[n,n+1]} \phi_\sigma\Big(\frac{x}{\mu_1(t)}\Big)\Big).$$

A similar analysis to that given in [29, Lemma 4.1] (see also [20]) yields

$$\int_{\mathbb{R}} |u(x,t)|^3 \phi_\sigma\Big(\frac{x}{\mu_1(t)}\Big)\,dx \lesssim \int_{\mathbb{R}} |u(x,t)|^2 \phi_\sigma\Big(\frac{x}{\mu_1(t)}\Big)\,dx. \tag{4.12}$$

Using the properties of the function $\phi$ in (3.1) for suitable $\delta$ and $\sigma$ we can apply Lemma 3.3 to deduce that $B_3(t) \in L^1(\{t \gg 1\})$.

Collecting the information in (4.3), (4.4), (4.7), (4.9) and (4.12) we deduce that

$$\frac{1}{t\log t}\int_{B_{tb}} |D^{1/2}u(x,t)|^2\,dx\,dt \le \frac{d}{dt}\mathcal{J}(t) + g(t),$$

where $\mathcal{J}(t)$ is bounded and $g(t) \in L^1(\{t \gg 1\})$.

A similar analysis to the one implemented in the proof of Theorem 1.1 yields the desired result. ∎

The remainder of the proof of Theorem 1.4 uses a similar argument to the proof of Theorem 1.1, so we will omit it.

## 5. Proofs of Theorems 1.5 and 1.7

The proofs of Theorems 1.5 and 1.7 are based on the following virial identity.

**Lemma 5.1.** *Let $u \in C(\mathbb{R} : H^1(\mathbb{R}))$ be the global real solution of IVP (1.1). Then for any weighted function $\varphi = \varphi(x,t)$ with*

$$\varphi \in C(\mathbb{R} : L^\infty \cap \dot{H}^4(\mathbb{R}))$$

*the following identity holds:*

$$\begin{aligned}\frac{d}{dt}\int u^2(x,t)\varphi(x,t)\,dx = & -\int u\partial_x[\mathcal{H};\varphi]\partial_x u\,dx - \int (D_x^{1/2}u)^2\partial_x\varphi\,dx \\ & -\int uD_x^{1/2}[D_x^{1/2};\partial_x\varphi]u\,dx \\ & +\frac{2}{3}\int u^3\partial_x\varphi\,dx + \int u^2\partial_t\varphi\,dx \\ =: & \ A_1 + A_2 + A_3 + A_4 + A_5.\end{aligned} \tag{5.1}$$

**Remark 5.2.** The terms $A_1$, $A_2$, $A_3$ derive from the (linear) dispersive part of the equation: $A_2$ corresponds to the local smoothing effect of Kato type, first deduced in solutions of the KdV equation [17]. As was proved in Section 2 the terms $A_1$, $A_3$ are of order 0 on $u$ and of order 2, in the homogeneous sense, on the weighted function $\varphi$.

*Proof of Lemma* 5.1. Using the equation we get

$$\begin{aligned}\frac{d}{dt}\int u^2(x,t)\varphi(x,t)\,dx &= 2\int u\partial_t u\varphi\,dx + \int u^2\partial_t\varphi\,dx\\ &= 2\int u(\mathcal{H}\partial_x^2 u - u\partial_x u)\varphi\,dx + \int u^2\partial_t\varphi\,dx\\ &= 2\int u\mathcal{H}\partial_x^2 u\varphi\,dx + \frac{2}{3}\int u^3\partial_x\varphi\,dx + \int u^2\partial_t\varphi\,dx\\ &= 2\int u\mathcal{H}\partial_x^2 u\varphi\,dx + A_4 + A_5. \end{aligned} \tag{5.2}$$

By integration by parts it follows that

$$2\int u\mathcal{H}\partial_x^2 u\varphi\,dx = -2\int \partial_x u\mathcal{H}\partial_x u\varphi\,dx - 2\int u\mathcal{H}\partial_x u\partial_x\varphi\,dx. \tag{5.3}$$

Since

$$\begin{aligned}-\int \partial_x u\mathcal{H}\partial_x u\varphi\,dx &= \int \partial_x u\mathcal{H}(\partial_x u\varphi)\,dx\\ &= \int \partial_x u\mathcal{H}\partial_x u\varphi\,dx + \int \partial_x u[\mathcal{H};\varphi]\partial_x u\,dx,\end{aligned} \tag{5.4}$$

one has that

$$\begin{aligned}-2\int \partial_x u\mathcal{H}\partial_x u\varphi\,dx &= \int \partial_x u[\mathcal{H};\varphi]\partial_x u\,dx\\ &= -\int u\partial_x[\mathcal{H};\varphi]\partial_x u\,dx = A_1.\end{aligned} \tag{5.5}$$

Also

$$\begin{aligned}-2\int u\mathcal{H}\partial_x u\partial_x\varphi\,dx &= -2\int uD_x u\partial_x\varphi\,dx\\ &= -2\int D_x^{1/2}uD_x^{1/2}(u\partial_x\varphi)\,dx\\ &= -2\int D_x^{1/2}uD_x^{1/2}u\partial_x\varphi\,dx - 2\int D_x^{1/2}u[D_x^{1/2};\partial_x\varphi]u\,dx\\ &= -2\int D_x^{1/2}uD_x^{1/2}u\partial_x\varphi\,dx - 2\int uD_x^{1/2}[D_x^{1/2};\partial_x\varphi]u\,dx\\ &= A_2 + A_3.\end{aligned} \tag{5.6}$$

Inserting (5.5) and (5.6) into (5.3), and this into (5.2), we obtain (5.1). ∎

### 5.1. Proof of Theorem 1.5

First, we fix

$$\varphi(x,t) = \chi\Big(\frac{x-c_1}{c_0 t}\Big), \tag{5.7}$$

with $\chi$ satisfying

$$\begin{cases} \chi \in C^\infty(\mathbb{R}), & 0 \le \chi \le 1 \text{ in } \mathbb{R}, \\ \chi(s) \equiv 0 \text{ if } s \le 1, & \chi(s) \equiv 1 \text{ if } s \ge 2, \\ \chi'(s) > 0 & \text{in } (1,2), \\ |\chi^{(k)}(s)| \le 2^k & \text{for any } s \in \mathbb{R}, k = 1,2,3, \end{cases} \tag{5.8}$$

and $c_0$, $c_1$ constants to be chosen later. We observe that

$$\partial_t \chi\Big(\frac{x-c_1}{c_0 t}\Big) = \chi'\Big(\frac{x-c_1}{c_0 t}\Big)\Big(\frac{x-c_1}{c_0 t}\Big)\frac{-1}{t} \le \chi'(\cdot)\frac{-1}{t} \tag{5.9}$$

and

$$\partial_x \chi\Big(\frac{x-c_1}{c_0 t}\Big) = \chi'(\cdot)\frac{1}{c_0 t}. \tag{5.10}$$

With the notation in (5.1) and using the commutator estimates in Lemmas 2.1 and 2.4 it follows that

$$\begin{aligned} |A_1| &\le \frac{c}{c_0^2 t^2}\|\chi''\|_{L^\infty}\|u(t)\|_{L^2}^2, \\ A_2 &\le 0, \\ |A_4| &\le \frac{2\|u(t)\|_\infty}{3c_0 t}\int u^2(x,t)\chi'\Big(\frac{x-c_1}{c_0 t}\Big)\,dx, \\ A_5 &\le \frac{-1}{t}\int u^2(x,t)\chi'\Big(\frac{x-c_1}{c_0 t}\Big)\,dx, \end{aligned} \tag{5.11}$$

and

$$\begin{aligned} |A_3| &\le \Big(\frac{1}{c_0 t}\Big)^{5/2}\|\chi''(\cdot)\|_{L^2}^{1/2}\|\chi'''(\cdot)\|_{L^2}^{1/2}\|u(t)\|_{L^2}^2 \\ &\le c_\chi\Big(\frac{1}{c_0 t}\Big)^2\|u(t)\|_{L^2}^2. \end{aligned} \tag{5.12}$$

Inserting the above estimates in (5.1) and combining Sobolev embedding with the conservation laws of the BO equation one finds that

$$\begin{aligned} &\frac{d}{dt}\int u^2(x,t)\chi\Big(\frac{x-c_1}{c_0 t}\Big)\,dx \\ &\quad \le \frac{c_\chi}{(c_0 t)^2}\|u_0\|_{L^2}^2 + \Big(\frac{2c_2\|u_0\|_{H^1}}{3c_0 t} - \frac{1}{t}\Big)\int u^2(x,t)\chi'\Big(\frac{x-c_1}{c_0 t}\Big)\,dx \end{aligned} \tag{5.13}$$

with $c_2$ a universal constant. Thus, we take $c_0$ such that

$$\frac{2c_2\|u_0\|_{H^1}}{3c_0} < 1,$$

and for any given $\varepsilon > 0$ we fix $t_1 > 0$ such that

$$\|u_0\|_2^2 \int_{t_1}^{\infty} \frac{c_\chi}{(c_0 t)^2}\, dt \le \varepsilon.$$

Hence, integrating (5.13) in the time interval $[t_1, t_2]$ we find that

$$\int u^2(x,t_2)\chi\Big(\frac{x-c_1}{c_0 t_2}\Big)\, dx \le \int u^2(x,t_1)\chi\Big(\frac{x-c_1}{c_0 t_1}\Big)\, dx + \varepsilon. \tag{5.14}$$

Next we fix $c_1 > 0$ such that

$$\int u^2(x,t_1)\chi\Big(\frac{x-c_1}{c_0 t_1}\Big)\, dx \le \varepsilon,$$

to get that for any $t_2 > t_1$,

$$\int_{x>c_1+2c_0t_2} u^2(x,t_2)\, dx \le \int u^2(x,t_1)\chi\Big(\frac{x-c_1}{c_0 t_1}\Big)\, dx + \varepsilon \le 2\varepsilon.$$

Finally, fixing $C_0 = 3c_0$ we have

$$\limsup_{t\to\infty} \int_{x>C_0 t} u^2(x,t)\, dx \le 2\varepsilon$$

which yields the desired result (1.13).

### 5.2. Proof of Theorem 1.7

First we fix

$$\varphi(x,t) = \beta\Big(\frac{x+c_1}{\mu(t)}\Big), \tag{5.15}$$

with $\beta$ satisfying

$$\begin{cases} \beta \in C^\infty(\mathbb{R}), & 0 \le \beta \le 1 \text{ in } \mathbb{R}, \\ \beta(s) \equiv 1 \text{ if } s \le -2, & \beta(s) \equiv 0 \text{ if } s \ge -1, \\ \beta'(s) < 0 & \text{in } (-2,-1), \\ |\beta^{(k)}(s)| \le 2^k & \text{for any } s \text{ in } \mathbb{R},\ k = 1,2,3, \end{cases} \tag{5.16}$$

with

$$\mu(t) = c_2 t \log^{1+\eta} t, \quad \eta > 0, \tag{5.17}$$

$c_1$ a constant to be chosen later and $c_2 > 0$ an arbitrary constant. We observe that

$$\partial_t \beta\Big(\frac{x+c_1}{\mu(t)}\Big) = \beta'\Big(\frac{x+c_1}{\mu(t)}\Big)\Big(\frac{x+c_1}{\mu(t)}\Big)\frac{-\mu'(t)}{\mu(t)} \le 0 \tag{5.18}$$

and

$$\partial_x \beta\Big(\frac{x+c_1}{\mu(t)}\Big) = \beta'\Big(\frac{x+c_1}{\mu(t)}\Big)\frac{1}{\mu(t)}. \tag{5.19}$$

With the notation in (5.1) from commutator estimates in Lemmas 2.1 and 2.4 it follows that

$$
\begin{aligned}
|A_1| &\le \frac{c}{\mu^2(t)}\|\beta''\|_{L^\infty}\|u(t)\|_{L^2}^2 \le \frac{c_\beta}{\mu^2(t)}\|u(t)\|_{L^2}^2,\\
A_2 &\le \frac{c}{\mu(t)}\|\beta'\|_{L^\infty}\|D_x^{1/2}u(t)\|_{L^2}^2 \le \frac{c_\beta}{\mu(t)}\|D_x^{1/2}u(t)\|_{L^2}^2,\\
|A_3| &\le \frac{c}{\mu^{5/2}(t)}\|\beta''(\cdot)\|_{L^2}^{1/2}\|\beta'''(\cdot)\|_{L^2}^{1/2}\|u(t)\|_{L^2}^2 \le \frac{c_\beta}{\mu^2(t)}\|u(t)\|_{L^2}^2,\\
|A_4| &\le \frac{2\|u(t)\|_{L^3}^3\|\beta'\|_{L^\infty}}{3\mu(t)} \le \frac{c_\beta\|u(t)\|_{L^\infty}\|u(t)\|_{L^2}^2}{\mu(t)},\\
A_5 &\le 0.
\end{aligned}
\tag{5.20}
$$

Inserting (5.20) into the virial identity (5.1) and using the conservation laws of the BO equation one sees that there exists

$$K_0 = K_0(\beta; \|u_0\|_{L^2}; \|D_x^{1/2}u_0\|_{L^2}; \|\partial_x u_0\|_{L^2}) > 0$$

such that

$$\frac{d}{dt}\int u^2(x,t)\beta\Big(\frac{x+c_1}{\mu(t)}\Big)\,dx \le \frac{K_0}{t\log^{1+\eta} t}. \tag{5.21}$$

Thus, given any $\varepsilon > 0$ we take $t_0 > 1$ such that

$$\int_{t_0}^{\infty} \frac{K_0}{c_2 t\log^{1+\eta} t}\,dt \le \varepsilon \tag{5.22}$$

to get that for any $t_1 > t_0$,

$$\int u^2(x,t_1)\beta\Big(\frac{x+c_1}{c_2 t_1\log^{1+\eta} t_1}\Big)\,dx \le \int u^2(x,t_0)\beta\Big(\frac{x+c_1}{c_2 t_0\log^{1+\eta} t_0}\Big)\,dx + \varepsilon. \tag{5.23}$$

By taking $c_1$ such that

$$\int u^2(x,t_0)\beta\Big(\frac{x+c_1}{c_2 t_0\log^{1+\eta} t_0}\Big)\,dx < \varepsilon,$$

one finds that for any $t_1 > t_0$,

$$\int_{x<-c_1-2c_2 t_t\log^{1+\eta} t_1} u^2(x,t_1)\,dx \le \int u^2(x,t_1)\beta\Big(\frac{x+c_1}{c_2 t_1\log^{1+\eta} t_1}\Big)\,dx \le 2\varepsilon. \tag{5.24}$$

Hence,

$$\limsup_{t\to\infty}\int_{x<-3c_2 t\log^{1+\eta} t} u^2(x,t)\,dx \le 2\varepsilon. \tag{5.25}$$

Since $\varepsilon > 0$ and $c_2 > 0$ are arbitrary we finish the proof.

# A. Some auxiliary results

*Proof of Lemma* 2.4. One sees that

$$\begin{aligned}&\widehat{(D^{1/2}[D^{1/2};a]f)}(\xi)\\&\quad=|\xi|^{1/2}\int|\xi|^{1/2}\big(\hat{a}(\xi-\eta)\hat{f}(\eta)-\hat{a}(\xi-\eta)|\eta|^{1/2}\hat{f}(\eta)\big)\,d\eta.\end{aligned}\tag{A.1}$$

Therefore,

$$|\widehat{(D^{1/2}[D^{1/2};a]f)}(\xi)|\le\int|\xi|^{1/2}\big||\xi|^{1/2}-|\eta|^{1/2}\big||\hat{a}(\xi-\eta)\hat{f}(\eta)|\,d\eta.\tag{A.2}$$

Assuming the following claim:

$$\exists\, c>0 \text{ s.t. } \forall\xi,\eta\in\mathbb{R},\quad |\xi|^{1/2}\big||\xi|^{1/2}-|\eta|^{1/2}\big|\le c|\xi-\eta|,\tag{A.3}$$

we will conclude the proof. From (A.3) it follows that

$$\begin{aligned}E_1(\xi)&\equiv|\widehat{(D^{1/2}[D^{1/2};a]f)}(\xi)|\\&\le c\int|\xi-\eta||\hat{a}(\xi-\eta)\hat{f}(\eta)|\,d\eta=c\int|\widehat{a'}(\xi-\eta)\hat{f}(\eta)|\,d\eta.\end{aligned}\tag{A.4}$$

Thus,

$$\|E_1\|_2=\|\widehat{a'}*\hat{f}\|_{L^2}\le\|\widehat{a'}\|_{L^1}\|f\|_{L^2}.\tag{A.5}$$

Using that

$$\begin{aligned}\|\widehat{a'}\|_1&=\int_{|\xi|\le R}|\widehat{a'}|\,d\xi+\int_{|\xi|>R}\frac{|\xi|\,|\widehat{a'}|}{|\xi|}\,d\xi\\&\le cR^{1/2}\|\widehat{a'}\|_{L^2}+cR^{-1/2}\|\widehat{a''}\|_{L^2},\end{aligned}\tag{A.6}$$

then choosing $R=\|\widehat{a''}\|_{L^2}^{1/2}/\|\widehat{a'}\|_{L^2}^{1/2}$ we obtain (2.5).

It remains to prove the claim in (A.3). First, we consider the case where $\xi$ and $\eta$ have the same sign, so we assume $\xi,\eta>0$. In this setting one sees that for some $\theta\in(0,1)$,

$$|\xi^{1/2}-\eta^{1/2}|=\frac{1}{(\theta\xi+(1-\theta)\eta)^{1/2}}|\xi-\eta|.\tag{A.7}$$

Thus, if $0<\xi/10<\eta$, (A.7) yields the estimate in (A.3).

If $0<\eta\le\xi/10$, one has

$$\xi^{1/2}(\xi^{1/2}-\eta^{1/2})\le\xi\le2|\xi-\eta|.$$

In the case where $\xi$, $\eta$ have different signs, one sees that

$$|\xi-\eta|=|\xi|+|\eta|,$$

and the estimate (A.3) holds. ∎

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