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Almost classical solutions of Hamilton-Jacobi equations

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Abstract

We study the existence of everywhere differentiable functions which are almost everywhere solutions of quite general Hamilton-Jacobi equations on open subsets of $\mathbb{R}^d$ or on d-dimensional manifolds whenever $d \geq 2$. In particular, when M is a Riemannian manifold, we prove the existence of a differentiable function u on M which satisfies the Eikonal equation $\|\nabla u(x)\|_x = 1$ almost everywhere on M.

1. Introduction

It has been proved by Z. Buczolich [4] that if $d \geq 2$, there exists $u : \mathbb{R}^d \to \mathbb{R}$, differentiable at every point, such that $\nabla u(0) = 0$ and $\|\nabla u(x)\| \geq 1$ almost everywhere, thus giving a negative answer to the gradient problem of C. E. Weil [10]. Malý and Zelený [8] gave an elegant proof of this result using a new mathematical game. Then Deville and Matheron [6], refining the methods introduced by the above authors, proved that if Ω is a bounded open subset of $\mathbb{R}^d$ with $d \geq 2$, there exists a function $u : \overline{\Omega} \to \mathbb{R}$, continuous on $\overline{\Omega}$, differentiable at every point of Ω, such that $u(x) = 0$ for all $x \in \partial\Omega$, and such that $\|\nabla u(x)\| = 1$ almost everywhere on Ω. Notice that because of Rolle's theorem, there exists $x_0 \in \Omega$ such that $\nabla u(x_0) = 0$, so the function u cannot be $\mathcal{C}^1$-smooth. We shall call u an almost-classical solution of the Eikonal equation $\|\nabla u\| = 1$. This equation has also a unique viscosity solution, which is the function $x \mapsto dist(x, \partial\Omega)$, where $dist(x, \partial\Omega) = \inf\{\|x - y\|; y \in \partial\Omega\}$. The viscosity solution is not everywhere differentiable on Ω. Therefore, an almost classical solution of the Eikonal equation is not equal to the viscosity solution of the Eikonal equation. Nevertheless in optimal control, where

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this equation arises naturally, the viscosity solution is the "right" solution of the Eikonal equation. We refer to [2] and [5] for an account on viscosity solutions of Hamilton-Jacobi equations.

The contents of the paper are as follows. In Section 2, we recall some technical results from [6] which will be needed in this paper. In Sections 3 and 4, we study the existence of almost-classical solutions for more general Hamilton-Jacobi equations on open subsets of $\mathbb{R}^d$. Finally, Section 5 is devoted to Hamilton-Jacobi equations on manifolds, and in particular we will consider the Eikonal equation $\|\nabla u(x)\|_x = 1$ on a Riemannian manifold. See e.g. [1], [7] and [9] for further information about Hamilton-Jacobi equations on Riemannian manifolds.

Now we introduce some terminology. Let Ω be an open subset of $\mathbb{R}^d$, and let $F : \mathbb{R}\times\Omega\times\mathbb{R}^d \to \mathbb{R}$ and $u_0 : \partial\Omega \to \mathbb{R}$ be continuous. As usual, we say that a continuous function $u : \overline{\Omega} \to \mathbb{R}$ is a *classical solution* of $F\big(u(x), x, \nabla u(x)\big) = 0$ with Dirichlet condition $u\,|_{\partial\Omega}= u_0$ if for all $x \in \partial\Omega$, $u(x) = u_0(x)$, and for all $x \in \Omega$, u is differentiable at x and $F(u(x), x, \nabla u(x)) = 0$.

We say that u is a *classical subsolution* of $F\big(u(x), x, \nabla u(x)\big) = 0$ if for all $x \in \Omega$, u is differentiable at x and $F\big(u(x), x, \nabla u(x)\big) \leq 0$.

Definition 1.1. We say that a continuous function $u : \overline{\Omega} \to \mathbb{R}$ is an *almost classical solution* of $F(u(x), x, \nabla u(x)) = 0$ with Dirichlet condition $u\,|_{\partial\Omega}= u_0$ if:

- $u(x) = u_0(x)$ for all $x \in \partial\Omega$,
- u is a classical subsolution of $F(u(x), x, \nabla u(x)) = 0$,
- and u satisfies $F(u(x), x, \nabla u(x)) = 0$ for almost every $x \in \Omega$ (in the sense of Lebesgue measure on $\mathbb{R}^d$).

Notice that a classical solution is an almost classical solution, and that if u is an almost classical solution, then u is continuous on $\overline{\Omega}$ and differentiable at every point of Ω. In many natural examples, classical solutions of the Hamilton-Jacobi equation $F(u(x), x, \nabla u(x)) = 0$ exist only under very restrictive conditions on F. We prove the existence of almost classical solutions under quite general hypotheses on F. Observe that our results imply the existence of an almost classical solution u of the Eikonal equation satisfying the boundary condition $u\,|_{\partial\Omega}= 0$. In particular, we have:

Theorem 1.2. *Let Ω be an open subset of $\mathbb{R}^d$ with $d \geq 2$, and let $F : \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a continuous function. Suppose that the following conditions hold:*

(A) *There exists a continuous function $u_0 : \overline{\Omega} \to \mathbb{R}$, which is $\mathcal{C}^1$-smooth on Ω and such that $F\big(x, \nabla u_0(x)\big) \leq 0$, for every $x \in \Omega$.*

(B) *For each compact subset $K \subset \Omega$, there exists $M_K > 0$ such that*

$$\inf \big\{F(x,p) : x \in K,\, p \in \mathbb{R}^d,\, \|p\| \geq M_K\big\} > 0.$$

Then there exists an almost classical solution of $F\big(x, \nabla u(x)\big) = 0$, with Dirichlet condition $u|_{\partial\Omega} = u_0$.

The above result will actually follow from the more general Theorem 3.1 that will also provide other existence results of almost classical solutions of Hamilton-Jacobi equations. The proof of Theorem 3.1 will be given in Section 4.

In the last section, we consider Hamilton-Jacobi equations defined on a smooth manifold M of dimension $d \geq 2$, which always will be assumed to be Hausdorff and second countable. As usual, TM denotes the tangent bundle of M. A point in TM will be (x, v), where $x \in M$ and v belongs to the tangent space T_xM. In the same way, T^*M denotes the cotangent bundle of M. A point in T^*M will be (x, ξ), where $x \in M$ and $\xi \in T_x^*M$ is a linear form on the tangent space T_xM. If $u : M \to \mathbb{R}$ is differentiable at $x \in M$ we denote its differential at x by $du(x)$. Under suitable hypotheses on $F : T^*M \to \mathbb{R}$, we obtain the existence of almost classical solutions of an equation of the form $F\big(x, du(x)\big) = 0$. In particular, we obtain:

Theorem 1.3. *Let M be a smooth manifold of dimension $d \geq 2$, and let $F : T^*M \to \mathbb{R}$ be a $\mathcal{C}^1$-smooth function. Suppose that the following conditions hold:*

(A) *There exists a $\mathcal{C}^1$ function $u_0 : M \to \mathbb{R}$ such that $F\big(x, du_0(x)\big) \leq 0$, for every $x \in M$.*

(B) *For each $x \in M$, the set $B(x) = \{\xi \in T_x^*M : F(x,\xi) \leq 0\}$ is compact, the set $S(x) = \{\xi \in T_x^*M : F(x,\xi) = 0\}$ is connected, and the function $F(x,\cdot)$ has maximal rank on the set $S(x)$.*

Then there exists a differentiable function $u : M \to \mathbb{R}$ such that $F\big(x, du(x)\big) = 0$ for almost every $x \in M$.

If now we have a Riemannian manifold (M, g) and $u : M \to \mathbb{R}$ is differentiable, for every $x \in M$ we identify in the usual way the differential $du(x)$ with the gradient $\nabla u(x)$ by means of the scalar product $g_x(\cdot,\cdot)$ on the tangent space T_xM. In this case we obtain the following analogue of Theorem 1.2:

Theorem 1.4. *Let (M, g) be a Riemannian manifold of dimension $d \geq 2$, and let $F : TM \to \mathbb{R}$ be a continuous function. Suppose that the following conditions hold:*

(*A*) *There exists a* $\mathcal{C}^1$ *function* $u_0 : M \to \mathbb{R}$, *such that* $F\big(x, \nabla u_0(x)\big) \leq 0$, *for every* $x \in M$.

(*B*) *There exists a locally bounded function* $\rho : M \to (0, \infty)$ *such that, for every* $x \in M$, *the set* $B(x) = \{v \in T_xM : F(x, v) \leq 0\}$ *is contained in the ball of center* 0 *and radius* $\rho(x)$ *in* T_xM.

Then there exists a differentiable function $u : M \to \mathbb{R}$ *such that* $F\big(x, \nabla u(x)\big) = 0$ *for almost every* $x \in M$.

Thus if for a Riemannian manifold (M, g) we consider the function $F : TM \to \mathbb{R}$ given by

$$F(x, v) = \|v\|_x - 1 = \big(g_x(v, v)\big)^{1/2} - 1,$$

it is clear that the constant functions $u_0 \equiv 0$ and $\rho \equiv 1$ satisfy the above requirements. Therefore we obtain that there exists a differentiable function u on M which satisfies the Eikonal equation $\|\nabla u(x)\|_x = 1$ almost everywhere on M. Whenever the manifold M is compact, there exists a point $x_0 \in M$ such that $\nabla u(x_0) = 0$. Therefore, there is no classical solution of this equation, and an almost classical solution u of this equation cannot be $\mathcal{C}^1$-smooth. So almost classical solutions of Hamilton-Jacobi equations are often exotic.

2. Preliminary results

We recall three lemmas from [6] that we shall use here. The first lemma is a criterium of differentiability for the sum of a series of $\mathcal{C}^1$-smooth functions. We shall use the following notation: if X and Z are Banach spaces and $f : X \to Z$, then the oscillation of f with respect to $\delta > 0$ is defined by

$$osc(f, \delta) = \sup\big\{\|f(x_1) - f(x_2)\| : x_1, x_2 \in X, \|x_1 - x_2\| \leq \delta\big\}$$

Lemma 2.1. *Let* $(u_n)_{n\geq 1}$ *be a sequence of* $\mathcal{C}^1$ *functions between two Banach spaces* X *and* Y. *Assume that:*

(a) *the series* $\big(\sum \nabla u_n(x)\big)$ *is pointwise convergent;*

(b) *the sequence* $\big(\nabla u_n\big)$ *converges uniformly to* 0*;*

(c) $\|u_{n+1}\|_\infty = o(\|u_n\|_\infty)$*;*

(d) $\displaystyle\lim_{n\to\infty} osc\left(\sum_{k=1}^{n} \nabla u_k, \|u_{n+1}\|_\infty\right) = 0.$

Then the series $\big(\sum u_n\big)$ *is uniformly convergent, the function* $u := \sum_{n=1}^\infty u_n$ *is everywhere differentiable, and* $\nabla u(x) = \sum_{n=1}^\infty \nabla u_n(x)$ *for all* $x \in X$.

We say that a subset Q of $\mathbb{R}^d$ is a cube if $Q = \prod_{i=1}^d [a_i, b_i[$, where each $[a_i, b_i]$ is a closed and bounded interval of $\mathbb{R}$. And we say that Q is a closed cube if $Q = \prod_{i=1}^d [a_i, b_i]$. A function v defined on a cube Q is said to be *piecewise constant* if there is a finite partition $\mathcal{Q}$ of Q into cubes such that v is constant on every cube of the partition $\mathcal{Q}$. The following result gives the existence of a $\mathcal{C}^\infty$-smooth function $u : \mathbb{R}^d \to \mathbb{R}$, which vanishes in a neighbourhood of the exterior of a cube Q and such that its derivative is equal to a or $-a$ (where a is a given non zero vector in $\mathbb{R}^d$) on a subset of Q of measure almost equal to the measure of Q. The Lebesgue measure on $\mathbb{R}^d$ will be denoted λ_d.

Lemma 2.2. *Let $a \in \mathbb{R}^d$ be a non zero vector, let Q be a cube in $\mathbb{R}^d$, and let $\varepsilon > 0$. Then, there exists a bounded, $\mathcal{C}^\infty$-smooth function $u : \mathbb{R}^d \to \mathbb{R}$ satisfying the following properties:*

(a) *u vanishes in a neighbourhood of ∂Q and $\|u\|_\infty \leq \varepsilon$;*

(b) *$\lambda_d\big(\{x \in Q : \nabla u(x) = -a \text{ or } \nabla u(x) = a\}\big) \geq (1-\varepsilon)\lambda_d(Q)$;*

(c) *one can write $\nabla u = v + w$ with $\|w\|_\infty < \varepsilon$; the set $\{v(x) : x \in Q\}$ is included in the segment $[-a, a]$, and the function v is piecewise constant on Q.*

The last lemma relies on ideas due to J. Maly and M. Zeleny [8], and is also from [6]. The mapping $\mathbf{t}$ is defined using that a suitable game has a winning strategy.

Lemma 2.3. *Let B be a closed ball of $\mathbb{R}^d$. Then, there exists a map $\mathbf{t} : B \to \mathbb{R}^d$ such that if a sequence $(\sigma_n) \in B$ satisfies $\langle \mathbf{t}(\sigma_n), \sigma_{n+1} - \sigma_n \rangle \geq 0$ for all n, then (σ_n) converges.*

3. Almost classical solutions on open subsets of $\mathbb{R}^d$

For a wide class of Hamilton-Jacobi equations, we give an existence theorem of almost classical solutions defined on the closure of on an open subset of $\mathbb{R}^d$, and satisfying an homogeneous Dirichlet condition.

Theorem 3.1. *Let Ω be an open subset of $\mathbb{R}^d$ with $d \geq 2$, and $F : \mathbb{R} \times \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a continuous function. Suppose that the following conditions hold:*

(A) *$F(0, x, 0) \leq 0$, for every $x \in \Omega$; that is, the function u_0 identically equal to 0 is a classical subsolution of $F\big(u(x), x, \nabla u(x)\big) = 0$.*

(B) *For each compact subset $K \subset \Omega$, there exist $\alpha_K > 0$ and $M_K > 0$ such that for all $x \in K$, for all $u \in [0, \alpha_K]$ and for all $p \in \mathbb{R}^d$ satisfying $\|p\| \geq M_K$, we have $F(u, x, p) > 0$.*

Then there exists a function $u \geq 0$ on $\overline{\Omega}$ which is an almost classical solution of $F\big(u(x), x, \nabla u(x)\big) = 0$, with Dirichlet condition $u\,|_{\partial\Omega} = 0$. Moreover, the extension $\widetilde{u}$ of u to $\mathbb{R}^d$ satisfying $\widetilde{u}(x) = 0$ if $x \notin \overline{\Omega}$ is differentiable at every point of $\mathbb{R}^d$.

The proof of Theorem 3.1 will be postponed until Section 4. Along this section, we will obtain several consequences of this result.

Remark 3.2. It will be useful to note that condition (B) in Theorem 3.1 is equivalent to condition(B') and also to condition (B") below:

(B') *For each compact subset $K \subset \Omega$, there exists $\alpha_K > 0$ such that the set*

$$B(K; \alpha_K) = \big\{(u, x, p) \in [0, \alpha_K] \times K \times \mathbb{R}^d : \ F(x, u, p) \leq 0\big\}$$

is compact in $\mathbb{R} \times \Omega \times \mathbb{R}^d$.

(B") *For each $x_0 \in \Omega$, there exist a compact neighborhood V^{x_0} and $\alpha > 0$, such that the set*

$$B(V^{x_0}; \alpha) = \big\{(u, x, p) \in [0, \alpha] \times V^{x_0} \times \mathbb{R}^d : F(u, x, p) \leq 0\big\}$$

is compact in $\mathbb{R} \times \Omega \times \mathbb{R}^d$.

We now consider the case of general Dirichlet conditions.

Corollary 3.3. *Let Ω be an open subset of $\mathbb{R}^d$ with $d \geq 2$, and $F : \mathbb{R} \times \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a continuous function. Suppose that the following conditions hold:*

(A) *There exists a continuous function $u_0 : \overline{\Omega} \to \mathbb{R}$, which is $\mathcal{C}^1$-smooth on Ω and such that $F\big(u_0(x), x, \nabla u_0(x)\big) \leq 0$, for every $x \in \Omega$.*

(B) *For each compact subset $K \subset \Omega$, there exist $M_K > 0$ and $\alpha_K > 0$ such that for all $x \in K$, for all $u \in [0, \alpha_K]$ and for all $p \in \mathbb{R}^d$ satisfying $\|p\| \geq M_K$, we have $F(u_0(x) + u, x, p) > 0$.*

Then there exists an almost classical solution u of $F\big(u(x), x, \nabla u(x)\big) = 0$, with Dirichlet condition $u\,|_{\partial\Omega} = u_0$. Moreover, if u_0 is $\mathcal{C}^1$-smooth on $\mathbb{R}^d$, the function u can be extended to a differentiable function on $\mathbb{R}^d$.

Proof . Define $G(u, x, p) = F\big(u + u_0(x), x, p + \nabla u_0(x)\big)$. Conditions ($A$) and ($B$) of Theorem 3.1 are satisfied for G. Thus, there exists an almost classical solution v of $G(v(x), x, \nabla v(x)) = 0$, with Dirichlet condition $v\,|_{\partial\Omega} = 0$, and furthermore v can be extended to a differentiable function on $\mathbb{R}^d$. The function $u : \overline{\Omega} \to \mathbb{R}$ defined by $u(x) = u_0(x) + v(x)$ is then an almost classical solution of $F\big(u(x), x, \nabla u(x)\big) = 0$, with Dirichlet condition $u\,|_{\partial\Omega} = u_0$. ∎

Notice that Theorem 1.2 is a straightforward consequence of Corollary 3.3. Another easy consequence of Corollary 3.3 is the following existence result of almost classical solutions for stationary Hamilton-Jacobi equations:

Corollary 3.4. *Let Ω be an open subset of $\mathbb{R}^d$ with $d \geq 2$, and let $F : \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a continuous function. Suppose that the following conditions hold:*

(A) *There exists a continuous function $u_0 : \overline{\Omega} \to \mathbb{R}$, which is $\mathcal{C}^1$-smooth on Ω and such that $u_0(x) + F\big(x, \nabla u_0(x)\big) \leq 0$, for every $x \in \Omega$.*

(B) *For each compact $K \subset \Omega$, there exists $M_K > 0$ such that*

$$\inf \big\{ u_0(x) + F(x,p);\ x \in K,\ p \in \mathbb{R}^d,\ \|p\| \geq M_K \big\} > 0$$

Then there exists an almost classical solution of $u(x) + F\big(x, \nabla u(x)\big) = 0$, with Dirichlet condition $u\,|_{\partial\Omega} = u_0$.

Next we give a further application of Theorem 3.1. We shall need the following notions. If A is a subset of $\mathbb{R}^d$, we denote its complement by $A^c = \mathbb{R}^d \setminus A$. Let us recall the definition of the *Hausdorff distance* between closed sets of a metric space. If X is a metric space, for each $A \subset X$ and $r > 0$ we denote $B(A,r) = \{x \in X : dist(x,A) < r\}$. We denote $\mathcal{C}(X)$ the set of all closed bounded subsets of X. If C and D are in $\mathcal{C}(X)$, the Hausdorff distance between them is

$$d_H(C,D) = \inf \big\{ r \in (0,\infty] : C \subset B(D,r) \text{ and } D \subset B(C,r) \big\}.$$

Theorem 3.5. *Let Ω be an open subset of $\mathbb{R}^d$ with $d \geq 2$. For each $x \in \Omega$ let $U(x)$ be an open bounded subset of $\mathbb{R}^d$ containing 0. Assume that the set-valued mapping $x \mapsto \partial U(x)$ from Ω into $(\mathcal{C}(\mathbb{R}^d), d_H)$ is continuous on Ω. Then there exists a differentiable function $u : \mathbb{R}^d \to \mathbb{R}$ such that:*

1. *$u\,|_{\Omega^c} \equiv 0$ and $\nabla u\,|_{\Omega^c} \equiv 0$.*

2. *$\nabla u(x) \in \overline{U(x)}$ for every $x \in \mathbb{R}^d$.*

3. *$\nabla u(x) \in \partial U(x)$ for almost every $x \in \Omega$.*

Proof. Consider the function $F : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ defined by $F(u,x,p) = -dist\big(p, \partial U(x)\big)$ if $x \in U(x)$, and $F(u,x,p) = dist\big(p, \partial U(x)\big)$ otherwise. Since the mapping $x \mapsto \partial U(x)$ from Ω into $(\mathcal{C}(\mathbb{R}^d), d_H)$ is continuous on Ω, is is easy to see that F is continuous on $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$. The function identically equal to 0 is a classical subsolution of $F(u,x,p)=0$. On the other hand, for each compact subset $K\subset\mathbb{R}^d$, there exists $R>0$ such that

$$\bigcup_{x\in K}\partial U(x)\subset B(0,R),$$

and therefore

$$\bigcup_{x\in K}\overline{U(x)}\subset B(0,R).$$

Hence for all $u\in\mathbb{R}$, for all $p\in\mathbb{R}^d$ satisfying $\|p\|\geq 2R$ and for all $x\in K$, we have $F(u,x,p)>0$. The two hypothesis of Theorem 3.1 are then satisfied. The almost classical solution of $F(u,x,p)=0$ given by Theorem 3.1 satisfies the required properties. ■

4. Proof of Theorem 3.1.

In order to prove Theorem 3.1, we first consider the case of a cube, on which almost classical solutions will be obtained as the sum of a series of $\mathcal{C}^\infty$-smooth functions, and the general case will then follow easily.

Lemma 4.1. *Assume that the hypotheses of Theorem 3.1 are satisfied, and let C be a cube such that $\overline{C}$ is contained in Ω. Then there exists a differentiable function $u_C:\mathbb{R}^d\to\mathbb{R}$ such that:*

1. $u_C\geq 0$ *and* $u_C(x)=0$ *for all x which is not in the interior of C.*
2. $F\big(u_C(x),x,\nabla u_C(x)\big)\leq 0$ *for every $x\in C$.*
3. $F\big(u_C(x),x,\nabla u_C(x)\big)=0$ *for almost every $x\in C$.*

Proof of Theorem 3.1. We first fix an increasing sequence $(K_n)_{n\geq1}$ of compact subsets of Ω such that the union of all K_n's is equal to Ω. We also assume that each K_n is the closure of a finite union of cubes. By assumption (B), for each $n\geq 1$ there exist $M_n>0$ and $\alpha_n>0$ such that, for all $x\in K_n$, for all $u\in[0,\alpha_n]$ and for all $p\in\mathbb{R}^d$ satisfying $\|p\|\geq M_n$, we have $F(u,x,p)>0$. We consider a decomposition

$$\Omega=\bigcup_{j=1}^{\infty}C_j,$$

where $(C_j)_{j\geq1}$ is a locally finite family of cubes such that:

(a) $C_j\cap C_k=\emptyset$ if $j\neq k$.

(b) for each j, there exists n such that $C_j\subset\overline{K_n\backslash K_{n-1}}$.

Refining if necessary this decomposition, we can also assume:

(c) $\operatorname{diam}(C_j) \leq \frac{1}{2^n M_n} d_H(K_n, \partial\Omega)$ whenever $C_j \subset \overline{K_n \backslash K_{n-1}}$.

By Lemma 4.1, for each $j \geq 1$ there exists a differentiable function $u_j : \mathbb{R}^d \to \mathbb{R}$ such that:

1. $u_j \geq 0$ and $u_j(x) = 0$ for all x which is not in the interior of C_j.
2. $F\big(u_j(x), x, \nabla u_j(x)\big) \leq 0$ for every $x \in C_j$.
3. $F\big(u_j(x), x, \nabla u_j(x)\big) = 0$ for almost every $x \in C_j$.

Then we define $u : \mathbb{R}^d \to \mathbb{R}$ by setting

$$u = \sup_{j \geq 1} u_j.$$

By property (1) above, $u = u_j$ on each C_j. Then it is easy to see that u is differentiable on Ω, identically equal to 0 on $\mathbb{R}^d \backslash \Omega$, satisfies $F\big(u(x), x, \nabla u(x)\big) \leq 0$ for every $x \in \Omega$ and $F\big(u(x), x, \nabla u(x)\big) = 0$ for almost every $x \in \Omega$. It remains to check that u is differentiable at each point of $\partial\Omega$. Fix $n \geq 1$. We know that u vanishes on the boundary of each cube C_j, so, by the mean value theorem,

$$\sup\big\{u(x) : x \in C_j\big\} \leq \sup\big\{\nabla u(x) : x \in C_j\big\} \cdot \operatorname{diam}(C_j).$$

If $C_j \subset \overline{K_n \backslash K_{n-1}}$, then $\sup\big\{\nabla u(x) : x \in C_j\big\} \leq M_n$. In that case, using (c), we obtain:

$$\sup\big\{u(x) : x \in C_j\big\} \leq \frac{1}{2^n} d_H(K_n, \partial\Omega).$$

So whenever $x \in K_n \backslash K_{n-1}$, we have that $0 \leq u(x) \leq dist(x, \partial\Omega)/2^n$. This implies that for each point $x \in \partial\Omega$, u is differentiable at x and $\nabla u(x) = 0$. ∎

Proof of Lemma 4.1. Observe that if $F(0, x, 0) = 0$ for almost every $x \in C$, we can take $u_C = 0$ and the above assertions are satisfied. From now on, we assume that

$$\lambda_d\big(\{x \in C : F(0, x, 0) < 0\}\big) > 0$$

By assumption (B), there exists $\alpha > 0$ such that

$$r := \sup\big\{\|p\| : F(u, x, p) \leq 0 \text{ for some } x \in \overline{C} \text{ and } u \in [0, \alpha]\big\}$$

is finite. We fix a map $\mathbf{t} : B(0, 1 + r) \to \mathbb{R}^d$ satisfying the conditions of Lemma 2.3.

The function u_C will be given by a series

$$u_C = \sum_{n=1}^{\infty} u_n,$$

where each u_n is a $\mathcal{C}^\infty$-smooth function on $\mathbb{R}^d$. For each n, we will write $\nabla u_n = v_n + w_n$, and we will denote

$$U_n = \sum_{k=1}^{n} u_k \qquad \text{and} \qquad \sigma_n = \sum_{k=1}^{n} v_k.$$

Construction of the functions u_n: The functions u_n will be constructed together with a sequence $(\mathcal{Q}_n)_{n\geq 0}$ of partitions of C into cubes, where each $\mathcal{Q}_{n+1}$ is a refinement of $\mathcal{Q}_n$. We also fix a sequence $(\varepsilon_k)_{k\geq 1}$ of positive numbers, with $(\varepsilon_k) \downarrow 0$ and such that $\inf\{F(0,x,0) : x \in C\} < -\varepsilon_1$ and $\varepsilon_1 < 1$, and we construct an increasing sequence of integers $(N_k)_{k\geq 0}$ with $N_0 = 0$. The following conditions will be proved by induction:

(0) There exists $x_0 \in C$ such that, for each $n \geq 1$, $u_n(x_0) = 0$ and $\nabla u_n(x_0) = 0$.

(i) For each $n \geq 1$, u_n and v_n have their support included in the interior of C, v_n is constant on each cube of $\mathcal{Q}_n$, and $\|w_n\|_\infty \leq 2^{-n}$.

(ii) For each $n \geq 1$ and $x \in C$, $F(U_n(x), x, \nabla U_n(x)) \leq 0$.

(iii) For each $n \geq 1$ and $x \in C$, we have

$$\|\sigma_n(x)\| \leq 1 + r \qquad \text{and} \qquad \langle \mathbf{t}(\sigma_n(x)) \, , \, \sigma_{n+1}(x) - \sigma_n(x) \rangle = 0,$$

(iv) $\|u_1\|_\infty \leq \alpha/2$, and, for each $n \geq 1$, we have

$$0 < \|u_{n+1}\|_\infty \leq 2^{-n}\|u_n\|_\infty \qquad \text{and} \qquad osc(\nabla U_n, \|u_{n+1}\|_\infty) \leq 1/2^n$$

(v) For each $k \geq 1$ and each $N_{k-1} < n \leq N_k$, we have $\|v_n\|_\infty \leq \varepsilon_k$.

(vi) For each $k \geq 1$,

$$\lambda_d\big\{x \in C : F(U_{N_k}(x), x, \nabla U_{N_k}(x)) \leq -\varepsilon_k\big\} \leq 2^{-k}\lambda_d(C).$$

Construction of u_1: Fix a cube $Q_0 \subset C$ with

$$d_H(Q_0, \partial C) > 0 \qquad \text{and} \qquad \sup\big\{F(0,x,0) : x \in Q_0\big\} \leq -\varepsilon_1$$

This implies, using the uniform continuity of F on compact sets, that there exists $0 < \delta_1 \leq \varepsilon_1$ such that, whenever $x \in Q_0$, $0 \leq h \leq \delta_1$ and $\|q\| \leq 2\delta_1$, then:

$$F(h, x, q) \leq 0. \tag{4.1}$$

Choose $a = a(Q_0) \in \mathbb{R}^d$ such that $\|a(Q_0)\| = \delta_1$ and $\langle \mathbf{t}(0)\,,\, a\rangle = 0$ (this is possible since $d \geq 2$). Now applying Lemma 2.2 to the cube Q_0, we obtain a C^∞ function u_1 on $\mathbb{R}^d$, and a cube partition $\mathcal{Q}_1$ of C such that Q_0 is a union of some elements of $\mathcal{Q}_1$, such that:

• u_1 vanishes on a neighborhood of ∂Q_0 and outside of Q_0.

• $0 < \|u_1\|_\infty \leq \min\{\delta_1, a/2\}$.

• $\nabla u_1 = v_1 + w_1$, where $\|w_1\|_\infty \leq \min\{\delta_1, 1/2\}$, v_1 is constant on each cube of $\mathcal{Q}_1$, and $v_1(Q_0) \subset [-a(Q_0), a(Q_0)]$.

Fix $x_0 \in \partial Q_0$: we have $u_1(x_0) = 0$ and $\nabla u_1(x_0) = 0$, so condition (0) is satisfied. Conditions (i),(iii) and (v) are clearly satisfied, and (ii) follows from (4.1). So we can start the induction.

Inductive step: Fix $k \geq 1$, assume that N_{k-1} has been defined, and for some $n \geq N_{k-1}$ the partition $\mathcal{Q}_n$ and the function u_n have been constructed.

First, there exists $0 < \delta_k \leq \varepsilon_k$ such that whenever $x \in \overline{C}$, $u \in [0, \alpha]$, $p \in B(0, 1+r)$, $0 \leq h \leq \delta_k$ and $\|q\| \leq 2\delta_k$, then

$$F(u, x, p) \leq -\varepsilon_k/2 \quad \Longrightarrow \quad F(u+h, x, p+q) \leq 0 \tag{4.2}$$

Next, choose a cube partition $\widehat{\mathcal{Q}}_n$ of C refining $\mathcal{Q}_n$ such that:

∘ If we denote $\widehat{\mathcal{R}}_n$ the family of all cubes $Q \in \widehat{\mathcal{Q}}_n$ such that $d_H(Q, \partial C) > 0$, and $K_n = \cup\{Q : Q \in \widehat{\mathcal{R}}_n\}$, we have that

$$\lambda_d(C \setminus K_n) < 2^{-(k+1)}\lambda_d(C). \tag{4.3}$$

∘ For all $Q \in \widehat{\mathcal{R}}_n$ and every $x, y \in Q$, we have

$$\big|F\big(U_n(x), x, \nabla U_n(x)\big) - F\big(U_n(y), y, \nabla U_n(y)\big)\big| < \varepsilon_k/2. \tag{4.4}$$

The second condition above can be obtained using the uniform continuity of the mapping $x \mapsto F\big(U_n(x), x, \nabla U_n(x)\big)$ on the compact set $\overline{C}$.

Now, each cube Q of $\widehat{\mathcal{Q}}_n$ is contained in a cube Q' of $\mathcal{Q}_n$ and by (i), σ_n is constant on Q'. We denote by $\sigma_n(Q)$ the constant value of $\sigma_n|_{Q'}$. Choose $a = a(Q) \in \mathbb{R}^d$ such that $\|a(Q)\| = \delta_k$ and $\langle \mathbf{t}(\sigma_n(Q))\,,\, a\rangle = 0$. Now applying Lemma 2.2, for each cube $Q \in \widehat{\mathcal{R}}_n$ we obtain a C^∞ function u_Q on $\mathbb{R}^d$, and a cube partition $\mathcal{Q}_{n+1}$ of Q_0 which is a refinement of $\widehat{\mathcal{Q}}_n$ (and therefore of $\mathcal{Q}_n$), such that:

(a) u_Q vanishes on a neighborhood of ∂Q.

(b) $0 < \|u_Q\|_\infty \leq \min\left\{2^{-n}\|u_n\|_\infty, \delta_k\right\}$ and $osc(\nabla U_n, \|u_Q\|_\infty) < 1/2^n$.

(c) $\lambda_d\big\{x \in Q : \nabla u_Q(x) = \pm a(Q)\big\} \geq (1-2^{-k})\lambda_d(Q)$.

(d) $\nabla u_Q = v_Q + w_Q$, where $\|w_Q\|_\infty \leq \min\{\delta_k, 1/2^{n+2}\}$, v_Q is constant on each cube of $\mathcal{Q}_{n+1}$, and $v_Q(Q) \subset [-a(Q), a(Q)]$. In particular, we have $\|v_Q\|_\infty \leq \|a(Q)\| = \delta_k \leq \varepsilon_k$ and $\|\nabla u_Q\|_\infty \leq 2\delta_k$.

Next we define the function u_{n+1} on $\mathbb{R}^d$. We first choose for each $Q \in \widehat{\mathcal{R}}_n$ a point x_Q in the closure of Q such that

$$F\big(U_n(x_Q), x_Q, \nabla U_n(x_Q)\big) = \inf\big\{F\big(U_n(x), x, \nabla U_n(x)\big) : x \in Q\big\}$$

We define u_{n+1} on each cube of $\widehat{\mathcal{R}}_n$ in the following way:

1. If $F\big(U_n(x_Q), x_Q, \nabla U_n(x_Q)\big) > -\varepsilon_k$, we set $u_{n+1} = 0$ and $v_{n+1} = w_{n+1} = 0$ on Q.
2. If $F\big(U_n(x_Q), x_Q, \nabla U_n(x_Q)\big) \leq -\varepsilon_k$, we set $u_{n+1} = u_Q$ on Q. In this case, $v_{n+1} = v_Q, w_{n+1} = w_Q$, and we have
$$\lambda_d\big\{x \in Q : \|\nabla u_{n+1}(x)\| = \delta_k\big\} \geq (1-2^{-k}) \cdot \lambda_d(Q).$$
3. Finally, on $(K_n)^c$, we set $u_{n+1} = 0$, and $v_{n+1} = w_{n+1} = 0$.

In this way we obtain that u_{n+1} is a C^∞ function on $\mathbb{R}^d$, which vanish on a neighborhood of ∂Q for every $Q \in \widehat{\mathcal{R}}_n$.

Next we are going to check conditions (0) to (vi) for $n+1$. Since $\widehat{\mathcal{Q}}_n$ is a refinement of $\mathcal{Q}_1$ and $x_0 \in \partial Q_0 \in \mathcal{Q}_1$, there exists $Q \in \widehat{\mathcal{Q}}_n$ such that $x_0 \in \partial Q$, so $u_{n+1}(x_0) = 0$ and $\nabla u_{n+1}(x_0) = 0$. This proves condition (0). Condition (i) is clearly satisfied. In order to prove condition (ii), fix $x \in \Omega$. By induction hypothesis, $F\big(U_n(x), x, \nabla U_n(x)\big) \leq 0$. Let us prove that $F\big(U_{n+1}(x), x, \nabla U_{n+1}(x)\big) \leq 0$:

- If $x \in (K_n)^c$, then $u_{n+1} = 0$ on a neighbourhood of x and $\nabla U_{n+1}(x) = \nabla U_n(x)$, so $F\big(U_{n+1}(x), x, \nabla U_{n+1}(x)\big) = F\big(U_n(x), x, \nabla U_n(x)\big) \leq 0$.

- If $x \in Q \in \widehat{\mathcal{R}}_n$ with $F\big(U_n(x_Q), x_Q, \nabla U_n(x_Q)\big) > -\varepsilon_k$, then $u_{n+1} = 0$ on a neighborhood of Q and

$$F\big(U_{n+1}(x_Q), x_Q, \nabla U_{n+1}(x_Q)\big) = F\big(U_n(x_Q), x_Q, \nabla U_n(x_Q)\big) \leq 0.$$

- Finally, if $x \in Q \in \widehat{\mathcal{R}}_n$ with $F\big(U_n(x_Q), x_Q, \nabla U_n(x_Q)\big) \leq -\varepsilon_k$, by 4.4, we have for all $x \in Q$, $F\big(U_n(x), x, \nabla U_n(x)\big) \leq -\varepsilon_k/2$. Since

$$|U_n(x)| \leq \sum_{k=1}^{\infty} |u_k(x)| \leq \alpha,$$

from the definition of r and the fact that $F\big(U_n(x), x, \nabla U_n(x)\big) \le 0$, it follows that $\|\nabla U_n(x)\| \le r$. Since we have also $\|u_{n+1}\|_\infty \le \delta_k$ and $\|\nabla u_{n+1}\|_\infty \le 2\delta_k$, we deduce from 4.2 that

$$F\big(U_{n+1}(x), x, \nabla U_{n+1}(x)\big) \le 0$$

In order to prove (iii), fix $x \in C$. First, $F\big(U_{n+1}(x), x, \nabla U_{n+1}(x)\big) \le 0$, so $\|\nabla U_{n+1}(x)\| \le r$, and

$$\|\nabla U_{n+1}(x) - \sigma_{n+1}(x)\| \le \sum_{k=1}^{n+1} \|w_k(x)\| \le 1.$$

Therefore $\|\sigma_n(x)\| \le 1+r$. Then, if $v_{n+1}(x) = 0$, we have $\sigma_{n+1}(x) - \sigma_n(x) = 0$, so $\langle \mathbf{t}(\sigma_n(x))\,,\, \sigma_{n+1}(x) - \sigma_n(x)\rangle = 0$. On the other hand, if $v_{n+1}(x) \ne 0$, then$x \in Q$ for some $Q \in \widehat{\mathcal{R}}_n$, and $v_{n+1} = v_Q$. In this case $\sigma_n(x) = \sigma_n(Q)$, and $v_{n+1}(x) \in [-a(Q), a(Q)]$. Thus $v_{n+1}(x) = \sigma_{n+1}(x) - \sigma_n(x)$ is proportional to $a(Q)$, and therefore orthogonal to $\mathbf{t}(\sigma_n(Q))$, and the condition $\langle \mathbf{t}(\sigma_n(x))\,,\, \sigma_{n+1}(x) - \sigma_n(x)\rangle = 0$ is again satisfied.

Now we are going to see that $u_{n+1} \ne 0$. Indeed, if $Q \in \widehat{\mathcal{Q}}_n$ is such that $Q \subset Q_0$ and $x_0 \in \partial Q$, then $d_H(Q, \partial C) \ge d_H(Q_0, \partial C) > 0$, so $Q \in \widehat{\mathcal{R}}_n$. Since $U_n(x_0) = 0$ and $\nabla U_n(x_0) = 0$, we have

$$F\big(U_n(x_Q), x_Q, \nabla U_n(x_Q)\big) \le F\big(U_n(x_0), x_0, \nabla U_n(x_0)\big) \le -\varepsilon_1 \le -\varepsilon_k,$$

and therefore $u_{n+1} = u_Q \ne 0$ on Q. Condition (iv) follows now from (b). Next, condition (v) also holds, although we still have to define the integer N_k.

Finally, let us prove that (vi) is satisfied. Suppose, to the contrary, that for every $n > N_{k-1}$ we have:

$$\lambda_d\big\{x \in C : F(U_n(x), x, \nabla U_n(x)) \le -\varepsilon_k\big\} > 2^{-k} \cdot \lambda_d(C).$$

By (4.3), we obtain that

$$\lambda_d\big\{x \in K_n : F(U_n(x), x, \nabla U_n(x)) \le -\varepsilon_k\big\} > 2^{-(k+1)} \cdot \lambda_d(K_n).$$

Suppose now that

$$F\big(U_n(x_Q), x_Q, \nabla U_n(x_Q)\big) \le -\varepsilon_k.$$

As we have noticed, in this case

$$\begin{aligned}\lambda_d\big\{x \in Q : \|\nabla U_{n+1}(x) - \nabla U_n(x)\| \ge \delta_k\big\} &\ge \\ \ge \lambda_d\big\{x \in Q : \|\nabla u_{n+1}(x)\| = \delta_k\big\} &\ge (1 - 2^{-k}) \cdot \lambda_d(Q).\end{aligned}$$

Now the proportion of cubes Q in $\widehat{\mathcal{R}}_n$ satisfying this has to be at least $2^{-(k+1)}$. Therefore

$$\lambda_d\big\{x \in K_n : \|\nabla U_{n+1}(x) - \nabla U_n(x)\| \geq \delta_k\big\} \geq (1 - 2^{-k})2^{-(k+1)} > 0.$$

This will be a contradiction with Lemma 2.3, since we are going to prove that the sequence $(\nabla U_n)_{n\geq 1}$ is pointwise convergent. Indeed, for each $x \in \mathbb{R}^d$, it follows from (i) that the sequence $\big(\sum_{k=1}^{n} w_k(x)\big)_{n\geq 1}$ converges, and from (iii) and Lemma 2.3 that the sequence $\big(\sigma_n(x)\big)_{n\geq 1}$ converges. Since

$$\nabla U_n(x) = \sigma_n(x) + \sum_{k=1}^{n} w_k(x)$$

we have that the sequence $\big(\nabla U_n(x)\big)_{n\geq 1}$ is convergent. This contradiction shows that there exists an integer $N_k > N_{k+1}$ satisfying (vi). This concludes the inductive step.

The function u_C: We now define

$$u_C = \sum_{n=1}^{\infty} u_n.$$

By (vi) the series is uniformly convergent on $\mathbb{R}^d$, so that $u : \mathbb{R}^d \to \mathbb{R}$ is a continuous function, and it is clear that u_C vanishes outside C. In order to see that u_C is differentiable on $\mathbb{R}^d$, we check the conditions of Lemma 2.1. For each $n \geq 1$ let k_n be an integer with $N_{k_n - 1} < n \leq N_{k_n}$. From (i) and (v), we have that

- $\|\nabla u_n\|_\infty \leq \|v_n\|_\infty + \|w_n\|_\infty \leq \varepsilon_{k_n} + 2^{-n} \to 0,$

and from (iv), we obtain:

- $\|\nabla u_{n+1}\|_\infty = o\big(\|\nabla u_n\|_\infty\big),$

- $osc\big(\nabla U_n\,, \|\nabla u_n\|_\infty\big) \leq 2^{-n} \to 0.$

Moreover, applying as before Lemma 2.3, the sequence $(\nabla U_n)_{n\geq 1}$ is pointwise convergent, that is,

$$\sum_{n=1}^{\infty} \nabla u(x)$$

is convergent for every $x \in \mathbb{R}^d$. Applying Lemma 2.1, we obtain that u_C is everywhere differentiable and ∇u_C is the pointwise limit of

$$\sum_{k=1}^{n} u_k.$$

Now (ii) implies that $F\big(u_C(x), x, \nabla u_C(x)\big) \leq 0$ for every $x \in \Omega$. Finally, let us prove that

$$F\big(u_C(x), x, \nabla u_C(x)\big) = 0$$

almost everywhere on C. Consider $x \in C$ such that $F\big(u_C(x), x, \nabla u_C(x)\big) < 0$. Taking into account that $\nabla u_C(x) = \lim_k \nabla U_{N_k}(x)$, we can find some integer k_0 such that

$$F\big(U_{N_k}(x), x, \nabla U_{N_k}(x)\big) < -\varepsilon_{k_0} \leq -\varepsilon_k,$$

for every $k \geq k_0$. Therefore the set $\big\{x \in C : F(u_C(x), x, \nabla u_C(x)) < 0\big\}$ is contained in the set

$$\limsup_k \big\{x \in C : F(U_{N_k}(x), x, \nabla U_{N_k}(x)) < -\varepsilon_k\big\}.$$

Since

$$\lambda_d\big\{x \in C : F(U_{N_k}(x), x, \nabla U_{N_k}(x)) < -\varepsilon_k\big\} \leq 2^{-k}\lambda_d(C) \to 0,$$

by Borel-Cantelli lemma $\lambda_d\Big(\big\{x \in C : F\big(u_C(x), x, \nabla u_C(x)\big) < 0\big\}\Big) = 0$. That is, $F\big(u_C(x), x, \nabla u_C(x)\big) = 0$ for almost every $x \in C$. ■

5. Almost classical solutions on Riemannian manifolds

In order to obtain our results for smooth manifolds, we will use the concept of triangulation, as it is given by Whitney in [11] (see also [3]). In what follows we assume that every smooth manifold is Hausdorff and second countable. If M is a smooth d-dimensional manifold, a *triangulation* of M is a pair (K, π), where K is a simplicial complex and $\pi : K \to M$ is a homeomorphism, such that for each d-dimensional simplex S of K there exists a local chart (W, φ) of M, where W is a neighborhood of $\pi(S)$ and $\varphi \circ \pi$ is affine on S. According to Whitney [11], every smooth manifold admits a triangulation.

Our first result is an extension of Theorem 3.1 to the setting of smooth manifolds. The Riemannian structure is not needed here. We consider an open subset Ω of the manifold M and we denote $T^*\Omega$ the corresponding cotangent bundle. Then we consider equations of the form $F\big(u(x), x, du(x)\big) = 0$, where $F : \mathbb{R} \times T^*\Omega \to \mathbb{R}$ is a continuous function. We obtain the following:

Theorem 5.1. *Let M be a smooth manifold of dimension $d \geq 2$, consider an open subset Ω of M, and let $F : \mathbb{R} \times T^*\Omega \to \mathbb{R}$ be a continuous function. Suppose that the following conditions hold:*

(A) *There exists a $\mathcal{C}^1$ function $u_0 : M \to \mathbb{R}$ such that $F\big(u_0(x), x, du_0(x)\big) \leq 0$, for every $x \in \Omega$.*

(B) *For each $x_0 \in \Omega$, there exist a compact neighborhood V^{x_0} in Ω and $\alpha > 0$, such that the set $B(V^{x_0};\alpha)$ is compact in $\mathbb{R} \times T^*M$, where*

$$B(V^{x_0};\alpha) = \big\{(u,x,\xi) \in \mathbb{R} \times T^*M : u \in [0,\alpha];\ x \in V^{x_0};\ F\big(u + u_0(x), x, \xi\big) \leq 0\big\}.$$

Then there exists a differentiable function $u : M \to \mathbb{R}$ such that:

1. *$u \geq u_0$ on M, $u = u_0$ on Ω^c and $du = du_0$ on Ω^c.*
2. *$F\big(u(x), x, du(x)\big) \leq 0$ for every $x \in \Omega$.*
3. *$F\big(u(x), x, du(x)\big) = 0$ for almost every $x \in \Omega$.*

Proof. We will consider two cases.

First Case: Suppose first that $u_0 \equiv 0$ on M. Let (K,π) be a triangulation of M, where K is a simplicial complex and $\pi : K \to M$ is a homeomorphism, and consider the family $\{S_i\}_{i\in I}$ of all d-dimensional simplices of K. For each $i \in I$, denote $T_i = \pi(S_i)$. Then

$$M = \bigcup_{i\in I} T_i,$$

each ∂T_i has measure zero in M, and $int(T_i) \cap int(T_j) = \emptyset$ if $i \neq j$. Since M is locally compact and π is a homeomorphism, we have that the simplicial complex K is locally compact, and therefore locally finite. Thus the family $\{T_i\}_{i\in I}$ is locally finite. Since M is also σ-compact, we obtain that the index set I is countable. For each $i \in I$, denote $\Omega_i = \Omega \cap int(T_i)$. Then the set $\Omega \setminus (\cup_{i\in I}\Omega_i)$ has measure zero in M.

For each $i \in I$ there is a chart (W_i, φ_i) in M with $T_i \subset W_i$. Associated to this chart there is a natural diffeomorphism

$$\Phi_i : \mathbb{R} \times T^*W_i \to \mathbb{R} \times \varphi_i(W_i) \times \mathbb{R}^d$$

of the form $\Phi_i(u,x,\xi) = \big(u, \varphi_i(x), h_i(x,\xi)\big)$, where $h_i(x,\xi) \in \mathbb{R}^d$ satisfies that, for every $p \in \mathbb{R}^d$:

$$\langle h_i(x,\xi), p\rangle = \xi \circ d\varphi_i(x)^{-1}(p).$$

If $\varphi_i(\Omega_i) \neq \emptyset$, consider the function $G_i = F \circ \Phi_i^{-1} : \mathbb{R} \times \varphi_i(W_i) \times \mathbb{R}^d \to \mathbb{R}$. In order to apply Theorem 3.1 to the function G_i note that the following conditions hold:

(A) $G_i(0, z, 0) = F(0, \varphi_i^{-1}(z), 0) \leq 0$, for each $z \in \varphi_i(\Omega_i)$.

(B) For each compact subset H of $\varphi_i(\Omega_i)$, there exists $\alpha_H > 0$ such that the set

$$B(H; \alpha_H) = \big\{(u, z, p) \in [0, \alpha_H] \times H \times \mathbb{R}^d : G_i(u, z, p) \leq 0\big\}$$

is compact in $\mathbb{R} \times \varphi_i(W_i) \times \mathbb{R}^d$.

Therefore, taking into account Remark 3.2, we obtain that there exists a differentiable function $v_i : \varphi_i(W_i) \to \mathbb{R}$ such that:

1. $v_i \mid_{\varphi_i(\Omega_i)^c} \equiv 0$ and $\nabla v_i \mid_{\varphi_i(\Omega_i)^c} \equiv 0$;
2. $G_i\big(v_i(z), z, \nabla v_i(z)\big) \leq 0$ for every $z \in \varphi_i(\Omega_i)$.
3. $G_i\big(v_i(z), z, \nabla v_i(z)\big) = 0$ for almost every $z \in \varphi_i(\Omega_i)$.

Then the function $u_i = v_i \circ \varphi_i : W_i \to \mathbb{R}$ is differentiable on W_i, and for each $x \in W_i$ we have that

$$\begin{aligned} F\big(u_i(x), x, du_i(x)\big) &= F\big(u_i(x), x, dv_i(\varphi_i(x)) \circ d\varphi_i(x)\big) \\ &= F\big(\Phi_i^{-1}\big(v_i(\varphi_i(x)), \varphi_i(x), \nabla v_i(\varphi_i(x))\big)\big) \\ &= G_i\big(v_i(\varphi_i(x)), \varphi_i(x), \nabla v_i(\varphi_i(x))\big). \end{aligned}$$

As a consequence, we obtain that

1. $u_i \mid_{\Omega_i^c} \equiv 0$ and $\nabla u_i \mid_{\Omega_i^c} \equiv 0$.
2. $F\big(u_i(x), x, du_i(x)\big) \leq 0$ for every $x \in \Omega_i$.
3. $F\big(u_i(x), x, du_i(x)\big) = 0$ for almost every $x \in \Omega_i$.

On the other hand, if If $\varphi_i(\Omega_i) = \emptyset$, we set $u_i = 0$. Now we define $u : M \to \mathbb{R}$ by setting $u = u_i$ on each T_i. Then u is well-defined, since $\partial T_i \subset \Omega_i^c$ for each $i \in I$. Taking into account that the family $\{T_i\}_{i \in I}$ is locally finite, we see that u is differentiable on M, and it satisfies the required conditions.

General Case: In general, consider the continuous function $G : \mathbb{R} \times T^*\Omega \to \mathbb{R}$ defined by:

$$G(u, x, \eta) = F\big(u + u_0(x), x, \eta + du_0(x)\big).$$

We know that, or each $x_0 \in \Omega$, there exist a compact neighborhood V^{x_0} in Ω and $\alpha > 0$, such that

$$B_F(V^{x_0};\alpha) = \big\{(u,x,\xi) \in [0,\alpha] \times T^*\Omega : \, x \in V^{x_0};\ F(u+u_0(x),x,\xi) \leq 0\big\}$$

is compact in $\mathbb{R} \times T^*\Omega$. Since the mapping $\tau : \mathbb{R} \times T^*\Omega \to \mathbb{R} \times T^*\Omega$ given by

$$\tau(u,x,\xi) = (u,x,\xi - du_0(x))$$

is continuous, we have that the set

$$\begin{aligned} B_G(V^{x_0};\alpha) &= \big\{(u,x,\eta) \in [0,\alpha] \times T^*\Omega : \, x \in V^{x_0};\ G(u+u_0(x),x,\eta) \leq 0\big\} \\ &= \tau\big(B_F(V^{x_0};\alpha)\big) \end{aligned}$$

is compact in $\mathbb{R} \times T^*\Omega$. Thus by the first case we obtain that there exists a differentiable function $v : M \to \mathbb{R}$ such that:

1. $v\,|_{\Omega_0^c} \equiv 0$ and $dv\,|_{\Omega_0^c} \equiv 0$.
2. $F(v(x),x,dv(x)) \leq 0$ for every $x \in \Omega$.
3. $F(v(x),x,dv(x)) = 0$ for almost every $x \in \Omega$.

Now it is easy to see that the function $u = u_0 + v$ satisfies the required properties. ■

Our next Corollary, which is analogous to Corollary 3.4, is an easy consequence of Theorem 5.1.

Corollary 5.2. *Let M be a smooth manifold of dimension $d \geq 2$, consider an open subset Ω of M, and let $F : T^*\Omega \to \mathbb{R}$ be a continuous function. Suppose that the following conditions hold:*

(*A*) *There exists a $\mathcal{C}^1$ function $u_0 : M \to \mathbb{R}$ such that $u_0(x) + F\big(x, du_0(x)\big) \leq 0$, for every $x \in \Omega$.*

(*B*) *For each $x_0 \in \Omega$, there exists a compact neighborhood V^{x_0} in Ω such that the set $B(V^{x_0}) = \big\{(x,\xi) \in T^*M : \, x \in V^{x_0};\ u_0(x) + F(x,\xi) \leq 0\big\}$ is compact in T^*M.*

Then there exists a differentiable function $u : M \to \mathbb{R}$ such that:

1. *$u \geq u_0$ on M, $u = u_0$ on Ω^c and $du = du_0$ on Ω^c.*
2. *$u(x) + F\big(x, du(x)\big) \leq 0$ for every $x \in \Omega$.*
3. *$u(x) + F\big(x, du(x)\big) = 0$ for almost every $x \in \Omega$.*

Also as a consequence of Theorem 5.1 we obtain the following result, which extends Theorem 1.3:

Theorem 5.3. *Let M be a smooth manifold of dimension $d \geq 2$, consider an open subset Ω of M, and let $F : T^*\Omega \to \mathbb{R}$ be a $\mathcal{C}^1$-smooth function. Suppose that the following conditions hold:*

- (A) *There exists a $\mathcal{C}^1$-smooth function $u_0 : M \to \mathbb{R}$ such that $F\big(x, du_0(x)\big) \leq 0$, for every $x \in \Omega$.*
- (B) *For each $x \in \Omega$, the set $B(x) = \big\{\xi \in T_x^*M : F(x,\xi) \leq 0\big\}$ is compact, the set $S(x) = \big\{\xi \in T_x^*M : F(x,\xi) = 0\big\}$ is connected, and the function $F(x,\cdot)$ has maximal rank on the set $S(x)$.*

Then there exists a differentiable function $u : M \to \mathbb{R}$ such that:

1. *$u \geq u_0$ on M, $u = u_0$ on Ω^c and $du = du_0$ on Ω^c.*
2. *$F\big(x, du(x)\big) \leq 0$ for every $x \in \Omega$.*
3. *$F\big(x, du(x)\big) = 0$ for almost every $x \in \Omega$.*

Proof. We are going to see that the conditions of Theorem 5.1 are satisfied. Fix $x_0 \in \Omega$, and consider a chart (W, φ) in M with $x_0 \in W$. Associated to this chart, consider as before the natural diffeomorphism

$$\Phi : T^*W \to \varphi(W) \times \mathbb{R}^d$$

of the form $\Phi(x,\xi) = \big(\varphi(x), h(x,\xi)\big)$, where $h(x,\xi) \in \mathbb{R}^d$ satisfies that, for every $p \in \mathbb{R}^d$:

$$\langle h(x,\xi), p\rangle = \xi \circ d\varphi(x)^{-1}(p).$$

Denote $z_0 = \varphi(x_0)$. We take into account that $\Phi(S(x_0))$ is compact, and that $F \circ \Phi^{-1}$ has maximal rank on $\{z_0\} \times \Phi(S(x_0))$, and we apply the Implicit Function Theorem. Then we can find a neighborhood U^{z_0} contained in $\varphi(W)$ and a finite family $V_1, \dots V_m$ of open subsets of $\mathbb{R}^d$ with compact closure such that $\Phi(S(x_0)) \subset V_1 \cup \cdots \cup V_m$ and, for each $j = 1, \dots, m$, the set of points $(z,p) \in U^{z_0} \times V_j$ satisfying $F \circ \Phi^{-1}(z,p) = 0$ coincides, up to a permutation in the coordinates of p, with the graph of a $\mathcal{C}^1$-smooth mapping $g_j : U^{z_0} \times W_j \to \mathbb{R}$, where W_j is an open subset of $\mathbb{R}^{d-1}$.

We claim that there exists a compact neighborhood V^{x_0} such that, for every $x \in V^{x_0}$, we have that $\Phi(S(x)) \subset V_1 \cup \cdots \cup V_m$. Indeed, if this is not the case, there exist a sequence $(z_n)_n \subset U^{z_0}$ converging to z_0 and a sequence $(p_n)_n \subset (V_1 \cup \cdots \cup V_m)^c$ such that $F \circ \Phi^{-1}(z_n, p_n) = 0$ for every n. Since each

$S(x_n)$ is connected and $\Phi(S(x_n)) \cap (V_1 \cup \cdots \cup V_m) \neq \emptyset$ for every n, we can assume that, in fact, $(p_n)_n \subset \partial(V_1 \cup \cdots \cup V_m)$, which is a compact set. Then, taking a subsequence, we can assume that $(p_n)_n$ is convergent to some point $p_0 \in \partial(V_1 \cup \cdots \cup V_m)$. Now $F \circ \Phi^{-1}(z_0, p_0) = \lim_n F \circ \Phi^{-1}(z_n, p_n) = 0$, that is, $p_0 \in \Phi(S(x_0))$, and this contradicts the fact that $\Phi(S(x_0)) \subset V_1 \cup \cdots \cup V_m$. Then there exists $R > 0$ such that $\Phi(S(x)) \subset B(0, R)$ for every $x \in V^{x_0}$. Since $\Phi(S(x))$ is the boundary of $\Phi(B(x))$ we have that in fact $\Phi(B(x)) \subset B(0, R)$ for every $x \in V^{x_0}$. Thus the set

$$B(V^{x_0}) = \big\{(x, \xi) \in T^*\Omega : \, x \in V^{x_0}; \, F(x, \xi) \leq 0\big\}$$

is compact in $T^*\Omega$, and the requirements of Theorem 5.1 are satisfied. ■

In our next result we consider a Riemannian manifold (M, g). As we mentioned before, if $u : M \to \mathbb{R}$ is differentiable, for every $x \in M$ we identify in the usual way the differential $du(x)$ with the gradient $\nabla u(x)$ by means of the scalar product $g_x(\cdot, \cdot)$ on the tangent space T_xM. In this case we obtain the following extension of Theorem 1.4:

Theorem 5.4. *Let (M, g) be a Riemannian manifold of dimension $d \geq 2$, consider an open subset Ω of M, and let $F : T\Omega \to \mathbb{R}$ be a continuous function. Suppose that the following conditions hold:*

- (A) *There exists a $\mathcal{C}^1$ function $u_0 : M \to \mathbb{R}$, such that $F\big(x, \nabla u_0(x)\big) \leq 0$, for every $x \in \Omega$.*
- (B) *There exists a locally bounded function $\rho : \Omega \to (0, \infty)$ such that, for every $x \in \Omega$, the set $B(x) = \big\{v \in T_xM : F(x, v) \leq 0\big\}$ is contained in the ball of center 0 and radius $\rho(x)$ in T_xM.*

Then there exists a differentiable function $u : M \to \mathbb{R}$ such that:

1. $u \geq u_0$ *on* M, $u = u_0$ *on* Ω^c *and* $\nabla u = \nabla u_0$ *on* Ω^c.
2. $F\big(x, \nabla u(x)\big) \leq 0$ *for every* $x \in \Omega$.
3. $F\big(x, \nabla u(x)\big) = 0$ *for almost every* $x \in \Omega$.

Proof. We are going to see that the conditions of Theorem 5.1 are satisfied. Fix $x_0 \in \Omega$ and consider a chart (W, φ) in M with $x_0 \in W$. Associated to this chart, consider the natural diffeomorphism

$$\Phi : TW \to \varphi(W) \times \mathbb{R}^d$$

given by $\Phi(x, v) = (\varphi(x), d\varphi(x)(v))$. Choose a compact neighborhood V^{x_0} of x_0 contained in W and $R > 0$ such that for every $x \in V^{x_0}$ the set

$B(x) = \{v \in T_xM : F(x,v) \leq 0\}$ is contained in the closed ball of center 0 and radius R in T_xM. Now set

$$r = \inf\{\|v\|_x : x \in V^{x_0};\ \|d\varphi(x)(v)\|_{\mathbb{R}^d} = 1\}.$$

By compactness, it is clear that $r > 0$. For every $x \in V^{x_0}$ and every $v \in T_xM$, we have

$$\|d\varphi(x)(v)\|_{\mathbb{R}^d} \leq \frac{1}{r}\|v\|_x.$$

Therefore

$$B(V^{x_0}) = \{(x,v) \in TM : x \in V^{x_0};\ F(x,v) \leq 0\}$$
$$\subset \{(x,v) \in TM : x \in V^{x_0};\ \|v\|_x \leq R\} \subset \Phi^{-1}\Big(\varphi(V^{x_0}) \times \overline{B}\Big(0; \frac{R}{r}\Big)\Big).$$

It follows that $B(V^{x_0})$ is a compact subset of TM. ■

Corollary 5.5. *Let (M,g) be a Riemannian manifold of dimension ≥ 2 and let Ω be an open subset of M. Then there exists a differentiable function $u : M \to \mathbb{R}$ such that $u\,|_{\Omega^c} \equiv 0$ and $\|\nabla u(x)\|_x = 1$ for almost every $x \in \Omega$.*

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