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The L^p-boundedness of wave operators for fourth order Schrödinger operators on $\mathbb{R}^4$

Artbazar Galtbayar and Kenji Yajima

Abstract. We prove that the wave operators of the scattering theory for the fourth order Schrödinger operator $\Delta^2 + V(x)$ on $\mathbb{R}^4$ are bounded in $L^p(\mathbb{R}^4)$ for the set of p's of $(1,\infty)$ depending on the kind of spectral singularities of H at zero which can be described by the space of bounded solutions of $(\Delta^2 + V(x))u(x) = 0$.

1. Introduction

Let $H = \Delta^2 + V$, $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_4^2$, be the fourth order Schrödinger operator on $\mathbb{R}^4$ with real potentials $V(x)$ which satisfy the short-range condition that

$$\sup_{y\in\mathbb{R}^4} (1+|y|)^\delta \|V(x)\|_{L^q(|x-y|<1)} < \infty \quad \text{for a } q > 1 \text{ and } \delta > 1. \tag{1.1}$$

The operator H is defined via the closed and bounded-from-below quadratic form $q(u) = \int_{\mathbb{R}^4}(|\Delta u(x)|^2 + V(x)|u(x)|^2)dx$ with domain $D(q) = H^2(\mathbb{R}^4)$ and is selfadjoint in $L^2(\mathbb{R}^4)$ (cf. [18]). The spectrum of H consists of the absolutely continuous (AC for short) part $[0,\infty)$ and the bounded set of eigenvalues which are discrete in $\mathbb{R}\setminus\{0\}$ and accumulate possibly at zero; it generates a unique unitary propagator e^{itH} on $L^2(\mathbb{R}^d)$ and the wave operators $W_\pm$ defined by the strong limits in $L^2(\mathbb{R}^4)$

$$W_\pm = \lim_{t\to\pm\infty} e^{itH}e^{-itH_0}, \quad H_0 = \Delta^2$$

exist and Range $W_\pm = L^2_{\mathrm{ac}}(H)$, the AC subspace of $L^2(\mathbb{R}^4)$ for H ([20]). They are unitary operators from $L^2(\mathbb{R}^4)$ to $L^2_{\mathrm{ac}}(H)$.

The wave operators satisfy the intertwining property: for Borel functions f on $\mathbb{R}$,

$$f(H)P_{\mathrm{ac}}(H) = W_\pm f(H_0)W_\pm^*, \tag{1.2}$$

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where $P_{\rm ac}(H)$ is the projection to $L^2_{\rm ac}(H)$. It follows that, if $W_\pm$ are bounded in $L^p(\mathbb{R}^4)$,

$$\|W_\pm u\|_{L^p(\mathbb{R}^4)} \le C_p \|u\|_{L^p(\mathbb{R}^4)}, \quad u \in L^2(\mathbb{R}^4) \cap L^p(\mathbb{R}^4) \tag{1.3}$$

for $p \in I \subset [1,\infty]$ and $I^* = \{p/(p-1) : p \in I\}$, then

$$\|f(H)P_{\rm ac}(H)\|_{\mathbf{B}(L^q,L^p)} \le C\|f(H_0)\|_{\mathbf{B}(L^q,L^p)},$$

for $p \in I$ and $q \in I^*$ with the constant C independent of f and L^p-mapping properties of the AC part of $f(H)$, $f(H)P_{\rm ac}(H)$ may be deduced from those of $f(H_0)$ which is the Fourier multiplier by $f(|\xi|^4)$. Here for Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, $\mathbf{B}(\mathcal{X},\mathcal{Y})$ is the Banach space of bounded operators from $\mathcal{X}$ to $\mathcal{Y}$ and $\mathbf{B}(\mathcal{X}) = \mathbf{B}(\mathcal{X},\mathcal{X})$.

In this paper, we study whether or not $W_\pm$ satisfy (1.3) for p in a certain range of $p \in [1,\infty]$. For $1 \le p \le \infty$, and $D \subset \mathbb{R}^4$, $\|u\|_{L^p(D)}$ is the norm of $L^p(D)$, $\|u\|_p = \|u\|_{L^p(\mathbb{R}^4)}$, $\|u\| = \|u\|_2$ and (u,v) is the inner product of $L^2(\mathbb{R}^4)$; the notation (u,v) will be used whenever the integral $\int_{\mathbb{R}^4} u(x)\overline{v(x)}dx$ makes sense, e.g., for $u \in \mathcal{S}(\mathbb{R}^4)$ and $v \in \mathcal{S}'(\mathbb{R}^4)$;

$$L^p_{{\rm loc},u}(\mathbb{R}^4) = \{u : \|u\|_{L^p_{{\rm loc},u}} := \sup\{\|u(x)\|_{L^p(|x-y|\le 2)} : y \in \mathbb{R}^4\} < \infty\}.$$

We define the Fourier transform $\mathcal{F}u(\xi)$ or $\hat{u}(\xi)$ of u by

$$\hat{u}(\xi) = \mathcal{F}u(\xi) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} e^{-ix\xi}u(x)dx;$$

M_f is the multiplication operator with $f(\xi)$; $f(D) := \mathcal{F}^* M_f \mathcal{F}$ is the Fourier multiplier. We choose and fix smooth functions $\chi_\le(\lambda)$ and $\chi_\ge(\lambda)$ on $[0,\infty)$ such that

$$\chi_\le(\lambda) = \begin{cases} 1, & \lambda \le 1, \\ 0, & \lambda \ge 2, \end{cases} \qquad \chi_\le(\lambda) + \chi_\ge(\lambda) = 1$$

and let, for $a > 0$, $\chi_{\le a}(\lambda) = \chi_\le(\lambda/a)$ and $\chi_{\ge a}(\lambda) = \chi_\ge(\lambda/a)$.

We define the "high" and the "low" energy parts of $W_\pm$ respectively by

$$W_\pm \chi_{\ge a}(|D|) \quad \text{and} \quad W_\pm \chi_{\le a}(|D|).$$

For the high energy part we have the following theorems. Let $\langle x\rangle = (1+|x|^2)^{1/2}$ for $x \in \mathbb{R}^d$, $d \in \mathbb{N}$.

Theorem 1.1. *Suppose $V \in L^q_{{\rm loc},u}(\mathbb{R}^4)$ for a $q > 1$ and $\langle \log|x|\rangle^2 V \in L^1(\mathbb{R}^4)$. Let $a > 0$ and $1 < p < \infty$. Then, there exists a constant c_0 such that $W_\pm \chi_{\ge a}(|D|)$ are bounded in $L^p(\mathbb{R}^4)$ whenever V satisfies $\|V\|_{L^q_{{\rm loc},u}} + \|\langle \log|x|\rangle^2 V\|_{L^1} \le c_0$.*

Remark 1.2. In Theorem 1.1, V does not in general satisfy (1.1), however, for any $a > 0$, $|V|^{\frac{1}{2}}$ is H_0-smooth on $[a, \infty)$ in the sense of Kato (Lemma 2.2) and, if c_0 is small enough, it is also H-smooth on $[a, \infty)$ and $W_\pm \chi_{\geq a}(|D|)$ exist ([17, 24]).

The same result holds for larger V if V decays faster at infinity.

Theorem 1.3. *Suppose that $\langle x \rangle^3 V \in L^1(\mathbb{R}^4)$ and $V \in L^q(\mathbb{R}^4)$ for a $q > 1$. Suppose further that H has no positive eigenvalues. Then, for any $a > 0$, $W_\pm \chi_{\geq a}(|D|)$ are bounded in $L^p(\mathbb{R}^4)$ for $1 < p < \infty$.*

We remark that H can have positive eigenvalues for "very nice" potentials V ([8, 22]) in contrast to the case of ordinary Schrödinger operators $-\Delta + V$ which have no positive eigenvalues for the large class of short-range potentials ([15, 19]). We refer to [8, 22] and reference therein for more informations on positive eigenvalues for $(-\Delta)^m + V$, $m = 2, 3, \ldots$. We shall assume in this paper that *positive eigenvalues are absent* from H. For small V as in Theorem 1.1, H has no positive eigenvalues.

The range of p for which the low energy parts $W_\pm \chi_{\leq a}(|D|)$ are bounded in $L^p(\mathbb{R}^4)$ depends on the space $\mathcal{N}_\infty(H)$ of bounded solutions of $(\Delta^2 + V(x))u = 0$:

$$\mathcal{N}_\infty(H) := \{u : u \in L^\infty(\mathbb{R}^4) : (\Delta^2 + V(x))u = 0\}.$$

We call $\varphi \in \mathcal{N}_\infty(H)$ *zero energy resonance* of H. In Section 6 we shall prove the following lemma which is a version of the result in [14].

Lemma 1.4. *Suppose $\langle \log |x| \rangle^2 \langle x \rangle^3 V \in (L^1 \cap L^q)(\mathbb{R}^4)$ for a $q > 1$. Then, $\mathcal{N}_\infty(H)$ is finite-dimensional real vector space. For $\varphi \in \mathcal{N}_\infty(H)$, there exist $c_0 \in \mathbb{C}$, $\mathbf{a} \in \mathbb{C}^4$ and symmetric matrix A such that*

$$\varphi(x) = -c_0 + \frac{\mathbf{a} \cdot x}{|x|^2} + \frac{Ax \cdot x}{|x|^4} + O(|x|^{-3}) \quad (|x| \to \infty). \tag{1.4}$$

We call $\varphi \in \mathcal{N}_\infty(H) \setminus \{0\}$ *s-wave, p-wave, or d-wave resonance,* respectively, if $c_0 \neq 0$, $c_0 = 0$ and $\mathbf{a} \neq \mathbf{0}$ or $c_0 = 0$, $\mathbf{a} = \mathbf{0}$ and $A \neq 0$; if $c_0 = 0$, $\mathbf{a} = \mathbf{0}$, $A = 0$, then φ is zero energy eigenfunction of H.

Theorem 1.5. *Assume that H has no positive eigenvalues. Let $q > 1$.*

(1) *Suppose that $\langle x \rangle^4 V \in (L^1 \cap L^q)(\mathbb{R}^4)$. Let $\mathcal{N}_\infty(H) = \{0\}$ or $\mathcal{N}_\infty(H)$ consist only of s-wave resonances. Then, $W_\pm$ are bounded in $L^p(\mathbb{R}^4)$ for $1 < p < \infty$.*

(2) *Suppose that $\langle \log |x| \rangle^2 \langle x \rangle^8 V \in (L^1 \cap L^q)(\mathbb{R}^4)$. Let $\mathcal{N}_\infty(H)$ consist only of s- and p-wave resonances. Then, $W_\pm$ are bounded in $L^p(\mathbb{R}^4)$ for $1 < p < 4$ and are unbounded if $4 \leq p \leq \infty$.*

(3) *Suppose that $\langle \log |x| \rangle^2 \langle x \rangle^{12} V \in (L^1 \cap L^q)(\mathbb{R}^4)$. Let $\mathcal{N}_\infty(H)$ contain d-wave resonances. Then, $W_\pm$ are bounded in $L^p(\mathbb{R}^4)$ for $1 < p \leq 2$.*

(4) *Suppose that* $\langle \log|x|\rangle^2\langle x\rangle^{12}V \in (L^1\cap L^q)(\mathbb{R}^4)$. *Let* $\mathcal{N}_\infty(H)$ *consist only of* s-, p-*wave resonances and zero energy eigenfunctions. Then,* $W_\pm$ *are bounded in* $L^p(\mathbb{R}^4)$ *for* $1<p<4$.

Remark 1.6. We believe that $W_\pm$ are unbounded in $L^p(\mathbb{R}^4)$ for $p>2$ in (3) and for $p\geq 4$ in (4), however, we were not able to prove this. The end point cases $p=1$ and $p=\infty$ are out of reach of the theorems, whose proof heavily depends on the harmonic analysis machinery.

We rephrase Theorem 1.5 in terms of the singularities of the resolvent $R(\lambda^4)=(H-\lambda^4)^{-1}$ at $\lambda=0$, which is more directly connected to the proof given below. For stating this version of the theorem, we need some more notation. In what follows, we assume $u\in\mathcal{D}_*=\{u\in\mathcal{S}(\mathbb{R}^4):\hat{u}\in C_0^\infty(\mathbb{R}^4\setminus\{0\})\}$ unless otherwise stated explicitly; $\mathcal{D}_*$ is dense in $L^p(\mathbb{R}^4)$ for $1\leq p<\infty$. For $z\in\mathbb{C}\setminus[0,\infty)$ and $\mathbb{C}^+=\{z\in\Im z>0\}$,

$$R_0(z)=(H_0-z)^{-1} \quad\text{and}\quad G_0(z)=(-\Delta-z^2)^{-1}$$

respectively are resolvents of H_0 and $-\Delta$;

$$R_0^\pm(\lambda^4)=\lim_{\varepsilon\downarrow 0}R_0(\lambda^4\pm i\varepsilon) \quad\text{and}\quad G_0(\lambda)=\lim_{\varepsilon\downarrow 0}G_0(\lambda+i\varepsilon)$$

for $\lambda>0$. For $z\in\overline{\mathbb{C}}^{++}=\{z\in\mathbb{C}:\Re z\geq 0,\Im z\geq 0\}$, $z\neq 0$, we have

$$R_0(z^4)u(x)=\frac{1}{2z^2}(G_0(z)-G_0(iz))u(x), \quad u\in\mathcal{D}_*. \tag{1.5}$$

It is well known (e.g., [1]) that $G_0(z)$, $z\in\overline{\mathbb{C}}^{++}$, is the convolution with

$$\mathcal{G}_z(x):=\frac{1}{(2\pi)^4}\int_{\mathbb{R}^4}\frac{e^{ix\xi}d\xi}{|\xi|^2-z^2}=\frac{i}{4}\left(\frac{z}{2\pi|x|}\right)H_1^{(1)}(z|x|), \tag{1.6}$$

where $H_1^{(1)}(z)$ is the Hankel function and its series expansion shows

$$\mathcal{G}_z(x)=\frac{1}{4\pi^2|x|^2}+\frac{z^2}{4\pi}\sum_{n=0}^\infty\left(g(z)+\frac{c_n}{2\pi}-\frac{\log|x|}{2\pi}\right)\frac{(-z^2|x|^2/4)^n}{n!(n+1)!}. \tag{1.7}$$

Here $c_n=1/(2(n+1))+\sum_{j=1}^n j^{-1}$ and, with the principal branch,

$$g(z)=-\frac{1}{2\pi}\log\left(\frac{z}{2}\right)-\frac{\gamma}{2\pi}+\frac{i}{4},$$

γ being Euler's constant. Thus, $R_0^+(\lambda^4)$, $\lambda > 0$, is the convolution with

$$\mathcal{R}_\lambda(x) = \frac{1}{2\lambda^2}(\mathcal{G}_\lambda(x) - \mathcal{G}_{i\lambda}(x)) = \mathcal{R}(\lambda|x|),$$
$$\mathcal{R}(\lambda) := \frac{1}{4\pi}\sum_{n,\text{even}}\Big(g(\lambda) + \frac{c_n}{2\pi} - \frac{i}{8}\Big)\frac{(\lambda^2/4)^n}{n!(n+1)!} - \frac{i}{32\pi}\sum_{n,\text{odd}}\frac{(\lambda^2/4)^n}{n!(n+1)!}. \tag{1.8}$$

by virtue of (1.7) . Reordering (1.8) in the descendent order as $\lambda \to 0$, we obtain

$$\begin{aligned}\mathcal{R}_\lambda(x) &= \tilde{g}_0(\lambda) - \frac{\log|x|}{8\pi^2} - i\frac{\lambda^2|x|^2}{4^4\pi} + \frac{\lambda^4\tilde{g}_2(\lambda)|x|^4}{3\cdot 4^3} - \frac{\lambda^4|x|^4\log|x|}{6\cdot 4^4\pi^2} + \cdots \\ &=: \tilde{g}_0(\lambda) + N_0(x) + \lambda^2 G_2(x) + \lambda^4\tilde{g}_2(\lambda)G_4(x) + \lambda^4 G_{4,l}(x) + \cdots,\end{aligned} \tag{1.9}$$

$$\tilde{g}_n(\lambda) := \frac{1}{4\pi}\Big(g(\lambda) + \frac{c_n}{2\pi} - \frac{i}{8}\Big), \quad n = 0, 1, \ldots,$$

where, if n is odd, $G_{2n,l}(x) = 0$ and no factor $\tilde{g}_{2n}(\lambda)$ in front of $G_{2n}(x)$ appears.

We denote the convolution operators with $N_0(x), G_{2n}(x), G_{2n,l}(x)$ by N_0, G_{2n}, $G_{2n,l}$ respectively for $n = 1, 2, \ldots$ and with $v(x) := |V(x)|^{\frac{1}{2}}$,

$$N_0^{(v)} = M_v N_0 M_v, \quad G_{2n}^{(v)} = M_v G_{2n} M_v, \quad G_{2n,l}^{(v)} = M_v G_{2n,l} M_v.$$

Let $\operatorname{sign} a = 1$ if $a \geq 0$ and $\operatorname{sign} a = -1$ if $a < 0$;

$$U(x) = \operatorname{sign} V(x) \quad \text{and} \quad w(x) = U(x)v(x)$$

so that $V(x) = v(x)w(x)$. Define $g_0(\lambda) = \|V\|_1\tilde{g}_0(\lambda)$ and $\tilde{v} = \|v\|_2^{-1}v$;

$$P = \tilde{v}\otimes\tilde{v}, \quad Q = 1 - P, \quad T_0 = M_U + N_0^{(v)}.$$

Define the function $\mathcal{M}^+(\lambda^4)$ of $\lambda > 0$ with values in $\mathbf{B}(L^2)$ by

$$\mathcal{M}^+(\lambda^4) = M_U + M_v R_0^+(\lambda^4)M_v.$$

Here and hereafter we simply write L^2 for $L^2(\mathbb{R}^4)$. From (1.9) we have

$$\mathcal{M}^+(\lambda^4) = g_0(\lambda)P + T_0 + \lambda^2 G_2^{(v)} + \lambda^4\tilde{g}_2(\lambda)G_4^{(v)} + \lambda^4 G_{4,l}^{(v)} + \cdots. \tag{1.10}$$

It follows ([20]) from the absence of positive eigenvalues of H that under the short range condition (1.1) $\mathcal{M}^+(\lambda^4)^{-1}$ exists in $\mathbf{B}(L^2)$ for $\lambda > 0$ and is locally Hölder continuous. The operator $M_v\mathcal{M}^+(\lambda^4)^{-1}M_v$ will play the central role in the paper and we introduce the short notation

$$\mathcal{Q}_v(\lambda) = M_v\mathcal{M}^+(\lambda^4)^{-1}M_v. \tag{1.11}$$

Let $R^+(\lambda^4) = R(\lambda^4 + i0)$. Then, as is well known, for $\lambda > 0$, we have

$$R^+(\lambda^4) = R_0^+(\lambda^4) - R_0^+(\lambda^4)\mathcal{Q}_v(\lambda)R_0^+(\lambda^4). \tag{1.12}$$

The following Definition 1.7 is due to [14] where it is tacitly assumed that relevant operators are bounded in appropriate spaces (see Lemma 2.1); $\mathrm{Ker} QT_0Q|_{QL^2}$ is finite-dimensional (cf. Lemma 6.1), where for an operator A on L^2 and A-invariant subspace $\mathcal{H} \subset L^2$, $A|_{\mathcal{H}}$ is the part of A in $\mathcal{H}$. As is seen from (1.10), (1.11), and (1.12), the kind of singularities of H at zero as defined below is closely related to the singularities of $\mathcal{M}^+(\lambda^4)^{-1}$ and $R^+(\lambda^4)$ at $\lambda = 0$.

Definition 1.7. (1) We say that H is *regular* at zero if $QT_0Q|_{QL^2}$ is invertible and is *singular* at zero otherwise. If H is singular at zero, let S_1 be the projection in QL^2 to $\mathrm{Ker} QT_0Q|_{QL^2}$.

(2) Suppose that H is singular at zero.

(2-1) We say H has *singularity of the first kind* if $T_1 := S_1T_0PT_0S_1|_{S_1L^2}$ is invertible.

(2-2) If $T_1|_{S_1L^2}$ is not invertible, let S_2 be the projection in S_1L^2 to $\mathrm{Ker} T_1|_{S_1L^2}$. We say H has *singularity of the second kind* if $T_2 := S_2G_2^{(v)}S_2|_{S_2L^2}$ is invertible.

(2-3) If $T_2|_{S_2L^2}$ is not invertible, let S_3 be the projection in S_2L^2 to $\mathrm{Ker} T_2|_{S_2L^2}$. We say H has *singularity of the third kind* if $T_3 := S_3G_4^{(v)}S_3|_{S_3L^2}$ is invertible.

(2-4) If T_3 is not invertible, we say H has *singularity of the fourth kind*. Let S_4 be the projection in S_3L^2 to $\mathrm{Ker} T_3|_{S_3L^2}$ and $T_4 := S_4G_{4,l}^{(v)}M_vS_4|_{S_4L^2}$.

It is known ([14]) that T_4 is invertible. We have $Q =: S_0 \supset S_1 \supset \cdots \supset S_4$. We denote by the same letter the extension of S_j to L^2 which is defined as the zero operator on $L^2 \ominus S_jL^2$. The kind of singularities of H at zero is closely connected to the structure of $\mathcal{N}_\infty(H)$. The following lemma is a slight improvement of the result of [14] and will be proved in Section 6.

Lemma 1.8. *The following statements hold.*

(1) *Let* $\langle \log|x|\rangle^2 V \in (L^1 \cap L^q)(\mathbb{R}^4)$ *for a* $q > 1$. *Then* H *is singular at zero if and only if* $\mathcal{N}_\infty(H) \neq \{0\}$. *In this case, the map* Φ *defined by*

$$\Phi(\zeta) = N_0M_v\zeta - \|v\|^{-2}(PT_0\zeta, v), \quad \zeta \in S_1L^2 \tag{1.13}$$

is isomorphic from S_1L^2 *to* $\mathcal{N}_\infty(H)$ *and* $\Phi^{-1}(\varphi) = -w\varphi$, $\varphi \in \mathcal{N}_\infty(H)$.

(2) *Let* V *be as in* (1). *Suppose* H *has singularity of the first kind, then* rank $S_1 = 1$ *and* $\mathcal{N}_\infty(H)$ *consists only of* s*-wave resonances.*

(3) *Let* $\langle \log|x|\rangle^2 \langle x\rangle^3 V \in (L^1 \cap L^q)(\mathbb{R}^4)$. *Then,* Φ *maps* $\zeta \in S_1L^2 \ominus S_2L^2$, $S_2L^2 \ominus S_3L^2$, $S_3L^2 \ominus S_4L^2$ *and* S_4L^2 *to s-wave, p-wave, d-wave resonance and zero energy eigenfunction, respectively.*

By virtue of Lemma 1.8, Theorem 1.5 can be rephrased as follows.

Theorem 1.9. *Assume that* H *has no positive eigenvalues. Let* $q > 1$.

(1) *Suppose* $\langle x\rangle^4 V \in (L^1 \cap L^q)(\mathbb{R}^4)$. *If* H *is regular or has singularity of the first kind at zero, then* $W_\pm$ *are bounded in* $L^p(\mathbb{R}^4)$ *for* $1 < p < \infty$.

(2) *Suppose* $\langle \log|x|\rangle^2 \langle x\rangle^8 V \in (L^1 \cap L^q)(\mathbb{R}^4)$. *If* H *has singularity of the second kind at zero, then* $W_\pm$ *are bounded in* $L^p(\mathbb{R}^4)$ *for* $1 < p < 4$ *and are unbounded for* $4 \le p \le \infty$.

(3) *Suppose* $\langle \log|x|\rangle^2 \langle x\rangle^{12} V \in (L^1 \cap L^q)(\mathbb{R}^4)$. *If* H *has singularity of the third kind at zero, then* $W_\pm$ *are bounded in* $L^p(\mathbb{R}^4)$ *for* $1 < p \le 2$.

(4) *Suppose* $\langle \log|x|\rangle^2 \langle x\rangle^{12} V \in (L^1 \cap L^q)(\mathbb{R}^4)$. *If* H *has singularity of the fourth kind at zero, then* $W_\pm$ *are bounded in* $L^p(\mathbb{R}^4)$ *for* $1 < p \le 2$ *if* $T_3 \ne 0$ *and for* $1 < p < 4$ *if otherwise.*

Because of the intertwining property (1.2), the problem of L^p boundedness of wave operators has attracted interest of many authors and, for ordinary Schrödinger operators $H = -\Delta + V$ on $\mathbb{R}^d$, various results have been obtained which depend on the dimension d and on the singularities of H at zero. For some more information, we refer to the introduction of [35], [34] and the references therein, and [2, 3, 5, 9, 10, 29–33] among others.

For $H = \Delta^2 + V(x)$, the investigation started only recently and the following results have been obtained under suitable conditions on the decay at infinity and the smoothness of $V(x)$ in addition to the absence of positive eigenvalues of H. When $d = 1$, $W_\pm$ are bounded in $L^p(\mathbb{R}^1)$ for $1 < p < \infty$ but not for $p = 1$ and $p = \infty$; they are bounded from the Hardy space H^1 to L^1 and from L^1 to L^1_{weak} ([22]); if $d = 3$ and $\mathcal{N}_\infty := \{u \in L^\infty(\mathbb{R}^3) : (\Delta^2 + V)u = 0\} = 0$, then $W_\pm$ are bounded in $L^p(\mathbb{R}^3)$ for $1 < p < \infty$ ([11]); if $d \ge 5$ and $\mathcal{N}_\infty := \cap_{\varepsilon>0}\{u \in \langle x\rangle^{-\frac{d}{2}+2+\varepsilon} L^2(\mathbb{R}^d) : (\Delta^2 + V)u = 0\} = 0$, then they are bounded in $L^p(\mathbb{R}^d)$ for all $1 \le p \le \infty$ ([6, 7]). However, no results on L^p-boundedness of $W_\pm$ are known when $d = 2, 4$. We should mention, however, detailed study on dispersive estimates has been carried out by Li, Soffer, and Yao [21] for $d = 2$ and Green and Toprak [13, 14] for $d = 4$, and we borrow some results from [13, 14].

The rest of the paper is devoted to the proof of the theorems. We explain here the basic idea of the proof, introducing some more notation and displaying the plan of the paper. Various constants whose specific values are not important will be denoted by the same letter C and it may differ from line to line. We prove the theorems only for

W_- because the complex conjugation changes W_- to W_+. We often identify integral operators with their kernels and say integral operator $K(x,y)$ for the operator defined by $K(x,y)$; we say $\mu(\lambda)$, $\lambda>0$ is *good multiplier* (GMU for short) if $\mu(|D|)$ is bounded in $L^p(\mathbb{R}^4)$ for all $1<p<\infty$; if $|\mu^{(j)}(\lambda)|\le C\lambda^{-j}$ for $0\le j\le 3$, then $\mu(\lambda)$ is a GMU ([26], p.96).

In Section 2 we prove that operators in Definition 1.7 are bounded (Lemma 2.1) and give some estimates on the remainders of the series (1.9) (Lemma 2.2). We then prove (Lemma 2.3) that the spectral projection $\Pi(\lambda)$ for H_0 at λ^4 defined by

$$\Pi(\lambda)u(x) := \frac{2}{\pi i}\lim_{\varepsilon\downarrow 0}(R_0(\lambda^4-i\varepsilon)-R_0(\lambda^4+i\varepsilon))u(x) \tag{1.14}$$

satisfies that, with τ_a being the translation by $a\in\mathbb{R}^4$: $\tau_a u(x)=u(x-a)$,

$$\Pi(\lambda)u(x)=\frac{1}{(2\pi)^2}\int_{\mathbb{S}^3} e^{i\lambda x\omega}\hat{u}(\lambda\omega)d\omega=(\Pi(\lambda)\tau_{-x}u)(0), \tag{1.15}$$

which attributes the x-dependence of $\Pi(\lambda)u(x)$ to that of $\tau_{-x}u$ and simplifies some estimates in later sections (see e.g., (1.22)), and that $\Pi(\lambda)$ transforms the multiplications to the Fourier multipliers,

$$f(\lambda)\Pi(\lambda)u(x)=\Pi(\lambda)f(|D|)u(x), \tag{1.16}$$

which is particularly useful when $f(\lambda)$ is GMU (see e.g., Lemma 1.12). Note that, for $u\in\mathcal{D}_*$, $\Pi(\lambda)u(x)=0$ for λ outside a compact interval of $(0,\infty)$ and $|\Pi(\lambda)u(x)|\le C\langle x\rangle^{-\frac{3}{2}}$ uniformly with respect to $\lambda\in(0,\infty)$.

We then introduce the *stationary representation formula* of W_-,

$$W_-u=u-\int_0^\infty R_0^+(\lambda^4)\mathcal{Q}_v(\lambda)\Pi(\lambda)u\lambda^3 d\lambda \tag{1.17}$$

(cf. [24]) which is valid under assumptions of the theorems (except Theorem 1.1 where the restriction to the high energy part is necessary) and is the starting point of the proof of the theorems. As we shall exclusively deal with W_-, *we shall often omit the superscript* $+$ *from* $R_0^+(\lambda^4)$ *and* $\mathcal{M}^+(\lambda^4)$. At the end of Section 2, we prove that the Fourier multiplier defined via $\mathcal{R}(\lambda)$ satisfies

$$\|\mathcal{R}(|y||D|)\chi_{\ge a}(|D|)\|_{\mathbf{B}(L^p)}\le C(1+|\log|y||),\quad 1<p<\infty. \tag{1.18}$$

We remark that $\Omega(T)u$ and $\widetilde{\Omega}(T(\lambda))$ in the following definition are the operators defined by the integral in the stationary formula (1.17) with T and $T(\lambda)$ in replace of $\mathcal{Q}_v(\lambda)$ respectively.

Definition 1.10. (1) We say an operator is *good operator* (GOP for short) if it is bounded in $L^p(\mathbb{R}^4)$ for all $1 < p < \infty$.

(2) An operator T or operator-valued function $T(\lambda)$ of $\lambda > 0$ is *good producer* (GPR for short) if the following operators are GOP respectively:

$$\Omega(T)u := \int_0^\infty R_0(\lambda^4)\, T\, \Pi(\lambda) u \lambda^3 d\lambda, \tag{1.19}$$

$$\widetilde{\Omega}(T(\lambda))u := \int_0^\infty R_0(\lambda^4)\, T(\lambda)\, \Pi(\lambda) u \lambda^3 d\lambda. \tag{1.20}$$

In Section 3 we introduce the operator K and prove that it is a GOP (Lemma 3.4):

$$Ku(x) = \int_0^\infty \mathcal{R}_\lambda(x)(\Pi(\lambda)u)(0)\lambda^3 d\lambda. \tag{1.21}$$

The operator K is of fundamental importance: When $T = T(x, y)$, $\Omega(T)$ is the superposition of translations of K with weight $T(x, y)$,

$$\Omega(T)u(x) = \iint_{\mathbb{R}^8} T(y, z)(\tau_y K \tau_{-z} u) dz dy, \tag{1.22}$$

and $\Omega(T)$ becomes GOP if $T \in \mathcal{L}^1 := L^1(\mathbb{R}^4 \times \mathbb{R}^4)$:

$$\|\Omega(T)u\|_p \le C\|T\|_{\mathcal{L}^1}\|u\|_p, \tag{1.23}$$

(cf. Lemma 3.5). This also implies

$$\|\widetilde{\Omega}(T(\lambda))u\|_p \le C_p \int_0^\infty \lambda^2 \|T^{(3)}(\lambda)\|_{\mathcal{L}^1}\|u\|_p d\lambda, \tag{1.24}$$

where $f^{(j)}(\lambda) = (d^j f/d\lambda^j)(\lambda)$ for $j = 0, 1, \dots$ (see Proposition 3.6 for the precise statement). These estimates willl be repeatedly used in the following sections.

Definition 1.11. We say $T(\lambda, x, y)$ is *variable separable* ($\mathcal{VS}$ for short) if it has the form $T(\lambda, x, y) = \sum_{j=1}^N \mu_j(\lambda) T_j(x, y)$; it is *good variable separable* ($\mathcal{GVS}$ for short) if μ_j are GMU and $T_j(x, y) \in \mathcal{L}^1$ for $j = 1, \dots, N$.

The following is a direct consequence of (1.23).

Lemma 1.12. *If $T(\lambda, x, y)$ is $\mathcal{GVS}$, then $T(\lambda)$ is GPR.*

In Section 4 we shall prove Theorem 1.1 and Theorem 1.3. We have from (1.16) and (1.17) that

$$W_{-}\chi_{\geq a}(|D|)u = \chi_{\geq a}(|D|)u - \int_0^\infty R_0^+(\lambda^4)\mathcal{Q}_v(\lambda)\Pi(\lambda)u\lambda^3\chi_{\geq a}(\lambda)d\lambda. \tag{1.25}$$

Formally, expanding $\mathcal{Q}_v(\lambda)$ as $\mathcal{Q}_v(\lambda) = V - VR_0(\lambda^4)V + \cdots$ produces the well-known Born series for $W_{-}\chi_{\geq a}(|D|)$:

$$W_{-}\chi_{\geq a}(|D|)u = \chi_{\geq a}(|D|)u - W_1\chi_{\geq a}(|D|)u + \cdots, \tag{1.26}$$

$$W_n\chi_{\geq a}(|D|)u = \int_0^\infty R_0(\lambda^4)(M_V R_0(\lambda^4))^{n-1}M_V\Pi(\lambda)u\lambda^3\chi_{\geq a}(\lambda)d\lambda. \tag{1.27}$$

Then, $W_1\chi_{\geq a}(|D|)u = \Omega(M_V)\chi_{\geq a}(|D|)$ and, since M_V is the integral operator with the kernel $V(x)\delta(x-y) \in \mathcal{L}^1$, $W_1\chi_{\geq a}(|D|)$ is GOP by (1.23). For $n = 2$, we have

$$W_2\chi_{\geq a}(|D|)u = \int_{\mathbb{R}^4} \Omega(M_{V_y^{(2)}})\mathcal{R}(|y||D|)\chi_{\geq a}(|D|)\tau_y u dy \tag{1.28}$$

with $V_y^{(2)}(x) = V(x)V(x-y)$. Then, (1.23) with $T = M_{V_y^{(2)}}$ and (1.18) yield that

$$\begin{aligned}\|W_2\chi_{\geq a}(|D|)u\|_p &\leq C\int_{\mathbb{R}^8} |V(x)V(x-y)|(1+|\log|y||)\|u\|_p dxdy \\ &\leq C(\|V\|_{L^q_{\mathrm{loc},u}} + \|\langle\log|x|\rangle^2 V\|_{L^1})^2\|u\|_p.\end{aligned} \tag{1.29}$$

Iterating this procedure, we shall show that for $n = 3, 4, \ldots$

$$\|W_n\chi_{\geq a}(|D|)u\|_p \leq C^n(\|V\|_{L^q_{\mathrm{loc},u}} + \|\langle\log|x|\rangle^2 V\|_{L^1})^n\|u\|_p \tag{1.30}$$

with $C > 0$ independent of V and n. Thus, if $C(\|V\|_{L^q_{\mathrm{loc},u}} + \|\langle\log|x|\rangle^2 V\|_{L^1}) < 1$, the series (1.26) converges in $\mathbf{B}(L^p)$ for $1 < p < \infty$, which proves Theorem 1.1.

For proving Theorem 1.3, we expand $\mathcal{Q}_v(\lambda)$ with the remainder:

$$\mathcal{Q}_v(\lambda) = \sum_{n=0}^{N-1}(-1)^n M_V(R_0(\lambda^4)M_V)^n + (-1)^N D_N(\lambda), \tag{1.31}$$

$$D_N(\lambda) = M_v(M_w R_0(\lambda^4)M_v)^N(1 + M_w R_0(\lambda^4)M_v)^{-1}M_w. \tag{1.32}$$

The sum on the right of (1.31) produces $\sum_{n=0}^{N-1}(-1)^n W_n\chi_{\geq a}(|D|)$ which is GOP by (1.30). The decay of $\mathcal{R}_\lambda(x)$ as $\lambda \to \infty$ yields

$$\|\partial_\lambda^j D_N(\lambda)\|_{\mathcal{L}^1} \leq C\lambda^{-\frac{2N}{q'}}(\|\langle x\rangle^{(2j-3)_+}V\|_{L^1} + \|V\|_{L^q})^N, \quad j = 0, 1, 2, 3$$

for $1 < q < 4/3$ and $4 < q' = q/(q-1) < \infty$. If we take N such that $2N/q' > 3$, then $\chi_{\geq a}(\lambda) D_N(\lambda)$ becomes GPR for any $a > 0$ by (1.24) and Theorem 1.3 follows.

In Section 5 we begin studying the low energy part and prove Theorem 1.9 for the case that H is regular at zero. From (1.17), we have

$$W_- \chi_{\leq a}(|D|)u = \chi_{\leq a}(|D|)u - \int_0^\infty R_0(\lambda^4)\mathcal{Q}_v(\lambda)\Pi(\lambda)u\lambda^3\chi_{\leq a}(\lambda)d\lambda. \tag{1.33}$$

Definition 1.13. For a Banach space $\mathcal{X}$, an integer $k \geq 0$ and a function $f(\lambda) > 0$ defined for small $\lambda > 0$, say, for $\lambda \in (0, a)$, $a > 0$, $\mathcal{O}_{\mathcal{X}}^{(k)}(f)$ is the space of $\mathcal{X}$-valued C^k-functions of $\lambda \in (0, a)$ such that

$$\|(d/d\lambda)^j T(\lambda)\|_{\mathcal{X}} \leq C_j \lambda^{-j} |f(\lambda)|, \quad j = 0, \dots, k.$$

We shall abuse notation and write $\mathcal{O}_{\mathcal{X}}^{(k)}(f)$ also for an element of $\mathcal{O}_{\mathcal{X}}^{(k)}(f)$.

We write $\mathcal{R}_{\lambda,2n}(x)$ for the remainder of (1.9): $\mathcal{R}_{\lambda,0}(x) = \mathcal{R}_\lambda(x)$ and

$$\mathcal{R}_{\lambda,2n}(x) = \lambda^{2n}\tilde{g}_{2n}(\lambda)G_{2n}(x) + \lambda^{2n}G_{2n,l}(x) + \cdots, \ , n = 1, 2, \dots; \tag{1.34}$$

$R_{2n}(\lambda^4)$ is the convolution with $\mathcal{R}_{\lambda,2n}(x)$ and $R_{2n}^{(v)}(\lambda^4) = M_v R_{2n}(\lambda^4) M_v$:

$$\begin{aligned} R_{2n}(\lambda^4) &= \lambda^{2n}\tilde{g}_n(\lambda)G_{2n} + \lambda^{2n}G_{2n,l} + \cdots, \\ R_{2n}^{(v)}(\lambda^4) &= M_v(\lambda^{2n}\tilde{g}_n(\lambda)G_{2n} + \lambda^{2n}G_{2n,l} + \cdots)M_v, \end{aligned} \tag{1.35}$$

where G_0 is the identity and $G_{0,l}(x) = N_0(x)$. By virtue of Lemma 2.2 and (1.10),

$$\mathcal{M}(\lambda^4) = T_0 + g_0(\lambda)P + \lambda^2 G_2^{(v)} + R_4^{(v)}(\lambda), \quad R_4^{(v)}(\lambda) \in \mathcal{O}_{\mathcal{L}^1}^{(4)}(\lambda^4 \log \lambda).$$

If H is regular at zero, then we obtain (Lemma 5.4) via Feshbach formula that, for small $\lambda > 0$, with $D_0 = Q(QT_0Q)^{-1}Q \in \mathcal{L}^1$ and L_0 of rank two,

$$(T_0 + g_0(\lambda)P)^{-1} = h(\lambda)L_0 + D_0, \quad h(\lambda) = (g_0(\lambda) + c_1)^{-1}.$$

It follows via the perturbation expansion that $\chi_{\leq a}(\lambda)\mathcal{Q}_v(\lambda)$ is the sum of $\mathcal{GVS}$ and $\mathcal{O}_{\mathcal{L}^1}^{(4)}(\lambda^4 \log \lambda)$ and, $W_-\chi_{\leq a}(|D|)$ is GOP for small $a > 0$ by Proposition 3.6.

We begin studying the case when H has singularities at zero in Section 6 where we prove Lemmas 1.4 and 1.8. If H is singular at zero, then $\mathcal{M}(\lambda^4)^{-1}$ is singular at $\lambda = 0$ and the singularities become stronger as the order of the type of singularities increases from the first to the fourth. We shall study them by repeatedly and inductively applying Lemma 7.1 due to Jensen and Nenciu.

In Section 7 we shall prove Theorem 1.9 when the singularity is of the first kind. Then, $\mathcal{Q}_v(\lambda)$ has logarithmic singularity at zero and, in terms of the basis vector ζ of S_1L^2 which is one-dimensional,

$$\mathcal{Q}_v(\lambda) \equiv (a\log\lambda + b)(v\zeta \otimes v\zeta) \quad \text{(modulo GPR)},$$

and hence, the integral of (1.33) becomes, modulo GOP,

$$\Omega_{\mathrm{low},a}u \equiv \int_0^\infty R_0(\lambda^4)(v\zeta)(x)(v\zeta, \Pi(\lambda)u)(a\log\lambda + b)\lambda^3\chi_{\le a}(\lambda)d\lambda. \tag{1.36}$$

The point here is that the singularity of $a\log\lambda + b$ can be canceled by the property

$$\int_{\mathbb{R}^4} v(x)\zeta(x)dx = 0 \tag{1.37}$$

of $\zeta \in S_1L^2$: equation (1.37) implies that $\Pi(\lambda)u(x)$ in $(v\zeta, \Pi(\lambda)u)$ of (1.36) may be replaced by $\Pi(\lambda)u(x) - \Pi(\lambda)u(0)$ and Taylor's formula

$$e^{i\lambda x\omega} - 1 = \sum_{l=1}^4 i\lambda x_l \int_0^1 \omega_l e^{i\theta\lambda x\omega}d\theta \tag{1.38}$$

implies that

$$\Pi(\lambda)u(x) - \Pi(\lambda)u(0) = i\lambda\sum_{l=1}^4 x_l\int_0^1(\Pi(\lambda)R_l u)(\theta x)d\theta, \tag{1.39}$$

where R_j, $1 \le j \le 4$ are Riesz transforms. We observe that the factor λ on the right of (1.39) produces a GMU $\mu(\lambda) := \lambda(a\log\lambda + b)$ and $\Omega_{\mathrm{low},a}u$ becomes

$$i\sum_{l=1}^4\int_0^1\left(\int_0^\infty R_0(\lambda^4)((v\zeta)\otimes(x_l v\zeta))\Pi(\lambda)(\tau_{-\theta x}R_l\mu(|D|)u)(0)\lambda^3 d\lambda\right)d\theta. \tag{1.40}$$

Then, recalling the definition (1.21) of K, we obtain

$$\Omega_{\mathrm{low},a}u(x) = -i\sum_{l=1}^4\int_0^1\left(\int_{\mathbb{R}^8}(v\zeta)(y)z_l(v\zeta)(z)\tau_y(K\tau_{-\theta z}R_l\mu(|D|)u)(x)dydz\right)d\theta,$$

and (1.23) and Minkowski's inequality imply for all $1 < p < \infty$ that

$$\|\Omega_{\mathrm{low},a}u\|_p \le \sum_{j=1}^4\|(v\zeta)(y)z_l(v\zeta)(z)\|_{\mathcal{L}^1}\|K\|_{\mathbf{B}(L^p)}\|R_l\mu(|D|)\|_{\mathbf{B}(L^p)}\|u\|_p.$$

In Section 8, we prove Theorem 1.9 when H has singularity of the second kind. Then, $\mathcal{Q}_v(\lambda)$ has much stronger singularity at zero and with the basis $\zeta_1, \dots, \zeta_m$ of S_2L^2

$$\mathcal{Q}_v(\lambda) \equiv \sum_{j,k=1}^{m} \lambda^{-2}\eta_{jk}(\zeta_j \otimes \zeta_k) \quad \text{(modulo GPR)},$$

where η_{jk} are constants. Recall that $\zeta \in S_2L^2$ also satisfies (1.37). For dealing with this λ^{-2}-singularity, we expand $e^{i\lambda x\omega}$ to the second order in (1.38) so that $\Pi(\lambda)u(x) - \Pi(\lambda)u(0)$ becomes

$$\sum_{l=1}^{4} i\lambda x_l(\Pi(\lambda)R_l u)(0) - \sum_{m,l=1}^{4} x_m x_l \lambda^2 \int_0^1 (1-\theta)(\Pi(\lambda)\tau_{-\theta x} R_m R_l u)(0)d\theta. \quad (1.41)$$

Thanks to the factor λ^2 which cancels λ^{-2}-singularity, the second term of (1.41) produces GOP for (1.33). The first term does $\sum_{j,k=1}^{m}\sum_{l=1}^{4} W_{B,jkl}u(x)$, where

$$W_{B,jkl}u(x) := i\langle x_l v, \zeta_k\rangle \int_0^\infty (R_0(\lambda^4)M_v\zeta_j)(x)(\Pi(\lambda)R_l u)(0)\lambda^2\chi_{\le a}(\lambda)d\lambda. \quad (1.42)$$

Ignoring harmless factors $i\langle x_l v, \zeta_k\rangle$ and R_l and omitting the indices of (1.42), we consider for $\omega(x) = v(x)\zeta(x)$, $\zeta \in S_2L^2$,

$$W_B u = \int_0^\infty R_0(\lambda^4)\omega(x)(\Pi(\lambda)u)(0)\lambda^2\chi_{\le a}(\lambda)d\lambda. \quad (1.43)$$

We multiply both side of (1.43) by $\chi_{\ge 4a}(|D|) + \chi_{\le 4a}(|D|)$ which is identity so that $W_B u = \chi_{\ge 4a}(|D|)W_B u + \chi_{\le 4a}(|D|)W_B u$ and move $\chi_{\ge 4a}(|D|)$ and $\chi_{\le 4a}(|D|)$ inside the integral on the right. Let $\mu(\xi) = \chi_{\ge 4a}(|\xi|)|\xi|^{-4}$. Then,

$$\chi_{\ge 4a}(|D|)R_0(\lambda^4)\omega(x) = \mu(D)\omega(x) + \lambda^4\mu(D)R_0(\lambda^4)\omega(x).$$

Thanks to the factor λ^4, the second member on the right-hand side produces GOP for $\chi_{\ge 4a}(|D|)W_B$ and the first one does the rank one operator

$$\mu(|D|)\omega(x)(u, f), \quad f(x) = \mathcal{F}(\chi_{\le a}(\xi)|\xi|^{-1})(x).$$

Here $\mu(D)\omega(x) \in L^p(\mathbb{R}^4)$ for $1 \le p \le \infty$ (cf. Lemma 8.8) and $f \in L^q(\mathbb{R}^4)$ if and only if $4/3 < q \le \infty$. Thus, $\chi_{\ge 4a}(|D|)W_B$ is bounded in $L^p(\mathbb{R}^4)$ for $1 \le p < 4$ and is unbounded for $p \ge 4$, which already proves that W_- is unbounded in $L^p(\mathbb{R}^4)$

if $p \geq 4$. We then study $\chi_{\leq 4a}(|D|)W_B u$. Since $\hat{\omega}(0)=0$, $\chi_{\leq 4a}(|D|)R_0(\lambda^4)\omega(x)$ is equal to

$$\sum_{m=1}^{4}\frac{-i}{(2\pi)^4}\int_0^1\int_{\mathbb{R}^4} z_m\omega(z)\tau_{\theta z}R_m\left(\int_{\mathbb{R}^4} e^{ix\xi}\frac{|\xi|\chi_{\leq 4a}(|\xi|)}{(|\xi|^4-\lambda^4-i0)}d\xi\right)dzd\theta.$$

It follows that

$$\chi_{\leq 4a}(|D|)W_B u(x)=\sum_{m=1}^{4}\frac{-i}{(2\pi)^4}\int_0^1\int_{\mathbb{R}^4} z_m\omega(z)\tau_{\theta z}R_m\mathcal{Y}u(x)dzd\theta,$$

$$\mathcal{Y}u(x)=\int_0^\infty\left(\int_{\mathbb{R}^4} e^{ix\xi}\frac{|\xi|\chi_{\leq 4a}(|\xi|)}{(|\xi|^4-\lambda^4-i0)}d\xi\right)(\Pi(\lambda)u)(0)\lambda^2\chi_{\leq a}(\lambda)d\lambda.$$

Substitute

$$\frac{|\xi|}{|\xi|^4-\lambda^4-i0}=\frac{\lambda}{|\xi|^4-\lambda^4-i0}+\frac{1}{(|\xi|+\lambda)(|\xi|^2+\lambda^2)}$$

in the $d\xi$-integral and recall (1.21). We obtain

$$\mathcal{Y}u(x)=\chi_{\leq 4a}(|D|)K\chi_{\leq a}(|D|)u(x)+Lu(x),$$

where L is the integral operator with the kernel

$$L(x,y)=\iint_{\mathbb{R}^8}\frac{e^{ix\xi-iy\eta}\chi_{\leq 4a}(|\xi|)\chi_{\leq a}(|\eta|)}{(|\xi|^2+|\eta|^2)(|\xi|+|\eta|)|\eta|}d\xi d\eta.$$

By virtue of Lemma 3.5, $\chi_{\leq 4a}(|D|)K\chi_{\leq a}(|D|)$ is GOP and we shall prove in the appendix that L is bounded in $L^p(\mathbb{R}^4)$ for $1<p<4$. Hence, $\mathcal{Y}$ is bounded in $L^p(\mathbb{R}^4)$ for $1<p<4$ and so is $\chi_{\leq 4a}(|D|)W_B$. Thus, W_- is bounded in $L^p(\mathbb{R}^4)$ for $1<p<4$ but unbounded for $p \geq 4$ if H has singularity of the second kind.

In Section 9, we shall study the case when H has singularities of the third or the fourth kind at zero. Then leading singularities of $\mathcal{Q}_v(\lambda)$ as $\lambda \to 0$ are of orders of $\lambda^{-4}(\log\lambda)^{-1}$ and λ^{-4} respectively. However, they act in subspaces S_3L^2 and S_4L^2 and functions ζ in S_3L^2 and S_4L^2 satisfy additional cancellation properties that $(x^\alpha v,\zeta)=0$ for $|\alpha|\leq 1$ and $|\alpha|\leq 2$ respectively, which partly cancel the singularities as previously. Thus, we can proceed by following the line of ideas of previous sections, however, the argument becomes much more complicated. We shall avoid outlining it here and proceed to the text as we do not want to make the introduction too long.

2. Preliminaries

2.1. Free resolvents

In this section we present some estimates related to $R_0^+(\lambda^4)$ or the expansion (1.9). We begin with the following lemma which in particular implies that operators that appear in Definition 1.7 are bounded in L^2. We denote by $\mathcal{H}_2$ the Hilbert space of Hilbert–Schmidt operators in L^2.

Lemma 2.1. *Let $q > 1$ and $j = 1, 2, \ldots$. We have*

$$\|N_0^{(v)}\|_{\mathcal{H}_2} \le C(\|V\|_{L^q_{\mathrm{loc},u}} + \|\langle \log|x|\rangle^2 V\|_1). \tag{2.1}$$

$$\|G_{2j}^{(v)}\|_{\mathcal{H}_2} \le C\|\langle x\rangle^{4j} V\|_1. \tag{2.2}$$

$$\|G_{2j,l}^{(v)}\|_{\mathcal{H}_2} \le C(\|\langle x\rangle^{4j} V\|_{L^q_{\mathrm{loc},u}} + \|\langle \log|x|\rangle^2 \langle x\rangle^{4j} V\|_1). \tag{2.3}$$

Proof. Let $q' = q/(q-1)$. Then, Hölder's inequality implies that

$$\int_{|x-y|\le 2} |V(x)(\log|x-y|)^2 V(y)| dx dy \tag{2.4}$$
$$\le \|V\|_1 \|V\|_{L^q_{\mathrm{loc},u}} \|\log|x|\|^2_{L^{2q'}(|x|\le 2)} \le C(\|V\|_1^2 + \|V\|^2_{L^q_{\mathrm{loc},u}}).$$

If $|x-y| \ge 2$, then $\log|x-y| \le \log\langle x\rangle + \log\langle y\rangle$ and

$$\int_{|x-y|\ge 2} |V(x)(\log|x-y|)^2 V(y)| dx dy \le C\|(\log\langle x\rangle)^2 V\|_1 \|V\|_1.$$

This proves (2.1). We omit the proof for (2.2) which is obvious and the one for (2.3) which is similar to that of (2.1). ∎

For $\mathcal{G}_z(x)$, $\Im z \ge 0$, we have the integral representation ([4] 10.9.21):

$$\mathcal{G}_z(x) = \frac{e^{iz|x|}}{2(2\pi)^{\frac{3}{2}}\Gamma(\frac{3}{2})|x|^2} \int_0^\infty e^{-t} t^{\frac{1}{2}} \Big(\frac{t}{2} - iz|x|\Big)^{\frac{1}{2}} dt, \tag{2.5}$$

where $z^{\frac{1}{2}}$ is the branch such that $z^{\frac{1}{2}} > 0$ for $z > 0$. Thus, if we let

$$\mathcal{H}(\lambda) = \frac{e^{i\lambda}}{4(2\pi)^{\frac{3}{2}}\Gamma(\frac{3}{2})\lambda^2} \int_0^\infty e^{-t} t^{\frac{1}{2}} \Big(\frac{t}{2} - i\lambda\Big)^{\frac{1}{2}} dt, \quad \Im\lambda \ge 0, \tag{2.6}$$

then, $\mathcal{R}(\lambda) = \mathcal{H}(\lambda) - \mathcal{H}(i\lambda)$ for $\lambda > 0$ and

$$\mathcal{R}_\lambda(x) = \mathcal{H}(\lambda|x|) - \mathcal{H}(i\lambda|x|). \tag{2.7}$$

Recall the definitions (1.34) and (1.35) for $\mathcal{R}_{\lambda,2n}(x)$ and $R_{2n}^{(v)}(\lambda^4)$, $n = 0, 1, \ldots$ respectively.

Lemma 2.2. *The following statements hold.*

(1) *For* $j = 0, 1, \ldots,$

$$|\partial_\lambda^j \mathcal{R}_\lambda(x)| \le C_j \begin{cases} \langle \log \lambda|x|\rangle \lambda^{-j}, & 0 < \lambda|x| \le 1, \\ |x|^j \langle \lambda|x|\rangle^{-\frac{3}{2}}, & 1 \le \lambda|x|. \end{cases} \tag{2.8}$$

(2) *If* $V \in (L^q_{\mathrm{loc},u} \cap L^r)(\mathbb{R}^4)$ *for some* $q > 1$ *and* $1 \le r \le 8/5$, *then* M_v *is* H_0*-smooth on* $[a, \infty)$ *for any* $a > 0$.

(3) *Let* $j = 0, \ldots, 2n$ *and* $a > 0$. *Then, for* $0 < \lambda < a$,

$$|\partial_\lambda^j \mathcal{R}_{\lambda,2n}(x)| \le C_j \langle \log \lambda|x|\rangle \lambda^{2n-j} |x|^{2n}, \tag{2.9}$$

where, if n is odd, $\langle \log \lambda|x|\rangle$ *should be removed from the right.*

(4) *Let* $0 < \lambda < a$. *For the operator* $R_{2n}^{(v)}(\lambda^4)$, *we have*

$$\|(d/d\lambda)^j R_{2n}^{(v)}(\lambda^4)\|_{\mathcal{H}_2} \le C\lambda^{2n-j} \langle \log \lambda\rangle \|\langle x\rangle^{4n} \langle \log |x|\rangle^2 V\|_1, \tag{2.10}$$

where, if n is odd, $\langle \log \lambda\rangle$ *and* $\langle \log |x|\rangle^2$ *should be removed from the right.*

Proof. (1) For $0 < \lambda|x| \le 1$, (2.8) follows from (1.8). For $\lambda|x| \ge 1$, (2.5) implies that

$$|\partial_\lambda^j (\lambda^{-2}\mathcal{G}_\lambda(x))| \le C_j |x|^j \langle \lambda|x|\rangle^{-\frac{3}{2}}, \quad j = 0, 1, \ldots,$$
$$|\partial_\lambda^j (\lambda^{-2}\mathcal{G}_{i\lambda}(x))| \le C e^{-\lambda|x|} |x|^j \langle \lambda|x|\rangle^{-\frac{3}{2}}, \quad j = 0, 1, \ldots.$$

Then, (2.8) follows since $\mathcal{R}_\lambda(x) = (2\lambda^2)^{-1}(\mathcal{G}_\lambda(x) - \mathcal{G}_{i\lambda}(x))$.

(2) Since $v \in L^2$ and $H^4(\mathbb{R}^4) \subset L^\infty(\mathbb{R}^4)$ by the Sobolev embedding theorem, M_v is H_0-bounded. Let $\lambda > a$. We estimate

$$\|M_v R_0^+(\lambda^4) M_v\|_{\mathcal{H}_2}^2 = \int_{\mathbb{R}^4} |V(x)\mathcal{R}_\lambda(x-y)^2 V(y)| dx dy$$

by using (2.8). The integral over $\lambda|x-y| \ge 1$ is bounded by $C\|V\|_1^2$; since

$$\langle \log \lambda|x|\rangle \le C_\varepsilon (\lambda|x|)^{-\varepsilon}$$

for any $\varepsilon > 0$ for $\lambda|x| \le 1$; the one over $\lambda|x-y| \le 1$ is bounded by

$$C_\varepsilon a^{-\varepsilon} \int_{\mathbb{R}^4} |V(x)| \Big(\sup_{x \in \mathbb{R}^4} \int_{|x-y| \le 1/a} \frac{|V(y)| dy}{|x-y|^\varepsilon} \Big) dx \le C_a \|V\|_{L^q_{\mathrm{loc},u}} \|V\|_1.$$

Thus, $\|M_v R_0^+(\lambda^4) M_v\|_{\mathbf{B}(L^2)} \le C$ for $\lambda \in [a, \infty]$ for any $a > 0$ and by complex conjugation $\|M_v R_0^-(\lambda^4) M_v\|_{\mathbf{B}(L^2)} \le C$. Thus, M_v is H_0-smooth on $[a, \infty)$ in the sense of Kato (see [24]).

(3) If $\lambda|x| \le 1$, then (2.9) is obvious. Let $\lambda|x| \ge 1$ and $\lambda < a$, then $|x| \ge 1/a$ and the right side of (2.8) is bounded by that of (2.9) if $j \le 2n$. Let $\widetilde{G}_{2m,\lambda}(x)$ denote $\lambda^{2m}|x|^{2m}$ for odd m and $\lambda^{2m}\tilde{g}_m(\lambda|x|)|x|^{2m}$ for even m. Then for $0 \le m \le n-1$ and $j \le 2n$

$$|\partial_\lambda^j \widetilde{G}_{2m,\lambda}(x)| \le C\lambda^{2m-j}\langle \log \lambda|x|\rangle |x|^{2m} \le C\lambda^{2n-j}|x|^{2n}$$

and $\mathcal{R}_{\lambda,2n}(x) = \mathcal{R}_{\lambda,0}(x) - \tilde{g}_0(\lambda|x|) - \lambda^2 G_2(x) - \cdots - \widetilde{G}_{2(n-1),\lambda}(x)$ also satisfies (2.9) for $\lambda|x| \ge 1$.

(4) By virtue of (2.9) and the estimate $|\log(\lambda|x|)| \le \langle \log \lambda\rangle\langle \log|x|\rangle$, the obvious modification of the proof of (2.1) implies (2.10). ∎

For shortening formula, we define, for $1 \le m < n$,

$$\begin{aligned} R_{2m\to 2n}(\lambda) &= \lambda^{2m}\tilde{g}_m(\lambda)G_{2m} + \cdots + \lambda^{2n}G_{2n,l}, \\ R^{(v)}_{2m\to 2n}(\lambda) &= M_v(\lambda^{2m}\tilde{g}_m(\lambda)G_{2m} + \cdots + \lambda^{2n}G_{2n,l})M_v \end{aligned} \tag{2.11}$$

where, if k is odd, $G_{2k,l} = 0$ and no $\tilde{g}_k(\lambda)$ in front of G_{2k} as previously.

2.2. Stationary representation formula

Lemma 2.3. *Let* $\Pi(\lambda)$ *be the spectral projection defined by* (1.14)*. Then,* $\Pi(\lambda)$ *satisfies* (1.15) *and* (1.16)*.*

Proof. We express the right of (1.14) via Fourier transform, use polar coordinates and change the variables. Then,

$$\Pi(\lambda)u(x) = \lim_{\varepsilon\to 0}\frac{\varepsilon}{4\pi^3}\int_0^\infty \frac{d\sigma}{(\sigma-\lambda^4)^2+\varepsilon^2}\left(\int_{\mathbb{S}^3} e^{i\sigma^{\frac14}x\omega}\hat{u}(\sigma^{\frac14}\omega)d\omega\right).$$

The first of (1.15) follows by Poisson's formula. The second of (1.15) and (1.16) are obvious from the first. ∎

Under the condition of Theorem 1.3 or Theorem 1.5, M_v is $H_0^{\frac12}$-compact and $M_v R_0^+(\lambda^4)M_v$ is $\mathcal{H}_2$-valued function of $\lambda > 0$ of class C^1 by virtue of (2.8). Moreover, the absence of positive eigenvalues from H implies $\mathcal{M}^+(\lambda^4)$, $\lambda > 0$ is invertible in $\mathbf{B}(L^2)$ ([20]). Hence, $\mathcal{M}^+(\lambda^4)^{-1}$ is also C^1 with values in $\mathbf{B}(L^2)$ and the following theorem is well known.

Theorem 2.4. *Suppose V satisfies the short range condition* (1.1) *and* $\langle \log |x|\rangle^2 V \in L^1(\mathbb{R}^4)$. *Then, for* $u \in \mathcal{D}_*$, *we have the stationary representation formula* (1.17) *for* $W_- u$.

Remark 2.5. Lemma 2.2 (2) implies that M_v is H_0-smooth on $[a,\infty)$ for any $a > 0$ under the condition of Theorem 1.1; if V is small it is also H-smooth on $[a,\infty)$ and we have the representation formula (1.25) for the high energy part $W_- \chi_{\geq a}(|D|)u$ (see [24]).

2.3. Fourier multiplier defined by the resolvent kernel

We use the following lemma in Section 4. In what follows $a \leq_{|\cdot|} b$ means $|a| \leq |b|$.

Lemma 2.6. *Let* $a > 0$ *and* $1 < p < \infty$. *Then, there exists a constant* $C_{a,p}$ *independent of* $y \in \mathbb{R}^4$ *such that*

$$\|\mathcal{R}(|y||D|)\chi_{\geq a}(|D|)\|_{\mathbf{B}(L^p)} \leq C_{a,p}(1 + |\log |y||). \tag{2.12}$$

For the proof we use the following result due to Peral ([23]):

Lemma 2.7. *Let* $\psi(\xi) \in C^\infty(\mathbb{R}^n)$ *be such that* $\psi(\xi) = 0$ *near* $\xi = 0$ *and* $\psi(\xi) = 1$ *for* $|\xi| > a$ *for an* $a > 0$. *Then, the translation invariant Fourier integral operator*

$$\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\xi + i|\xi|} \frac{\psi(\xi)}{|\xi|^b} \hat{f}(\xi) d\xi,$$

is bounded in $L^p(\mathbb{R}^n)$ *if and only if*

$$\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{b}{n-1}.$$

Proof of Lemma 2.6. Recall (2.7) that $\mathcal{R}(\lambda|x|) = \mathcal{R}_\lambda(x) = \mathcal{H}(\lambda|x|) - \mathcal{H}(i\lambda|x|)$ and $\mathcal{H}(\lambda)$ has the integral representation (2.6). Let for $a > 0$

$$\mathcal{H}_{<a}(\lambda) = \chi_{<a}(\lambda)\mathcal{H}(\lambda), \quad \mathcal{H}_{\geq a}(\lambda) = \chi_{\geq a}(\lambda)\mathcal{H}(\lambda).$$

and

$$\mathcal{R}_{<a}(\lambda) = \mathcal{H}_{<a}(\lambda) - \mathcal{H}_{<a}(i\lambda), \quad \mathcal{R}_{\geq a}(\lambda) = \mathcal{H}_{\geq a}(\lambda) - \mathcal{H}_{\geq a}(i\lambda).$$

(1) We write (2.6) for $\lambda > 0$ in the form

$$\mathcal{H}(\lambda) = \frac{e^{i\lambda}}{4(2\pi)^{\frac{3}{2}}\Gamma\left(\frac{3}{2}\right)\lambda^{\frac{3}{2}}} F(\lambda), \quad F(\lambda) = \int_0^\infty e^{-t} t^{\frac{1}{2}} \left(\frac{t}{2\lambda} - i\right)^{\frac{1}{2}} dt.$$

Since $|\partial_\lambda^j (F(\lambda)\chi_{\geq a}(\lambda))| \leq C_a \lambda^{-j}$ for $0 \leq j \leq 3$, $F(|D|)\chi_{\geq a}(|D|)$ is a GOP. Peral's theorem implies that $e^{i|D|}|D|^{-\frac{3}{2}}\chi_{\geq a/2}(|D|)$ is also GOP. Hence, so is $\mathcal{H}_{\geq a}(|D|)$ and the norm $\|\mathcal{H}_{\geq a}(|y||D|)\|_{\mathbf{B}(L^p)}$ is independent of $|y|$ by scaling. From

$$\mathcal{H}(i\lambda) = \frac{-e^{-\lambda}}{4(2\pi)^{\frac{3}{2}}\Gamma\left(\frac{3}{2}\right)\lambda^{\frac{3}{2}}} F(i\lambda), \quad F(i\lambda) = \int_0^\infty e^{-t} t^{\frac{1}{2}} \left(\frac{t}{2\lambda} + 1\right)^{\frac{1}{2}} dt$$

it is obvious that the Fourier transform of $\mathcal{H}_{\geq a}(i|\xi|)$ is in $\mathcal{S}(\mathbb{R}^4)$ and $\mathcal{H}_{\geq a}(i|D|) \in \mathbf{B}(L^p(\mathbb{R}^4))$ for all $1 \leq p \leq \infty$ with $\|\mathcal{H}_{\geq a}(i|y||D|)\|_{\mathbf{B}(L^p)}$ being independent of $|y|$. Thus, $\mathcal{R}_{\geq a}(|y||D|)$ satisfies

$$\|\mathcal{R}_{\geq a}(|y||D|)\|_{\mathbf{B}(L^p)} \leq C_p. \tag{2.13}$$

(2) Formula (1.8) implies

$$\partial_\lambda^j \left\{\chi_{\leq a}(\lambda)\left(\mathcal{R}(\lambda) + \frac{1}{8\pi^2}\log\lambda\right)\right\} \leq_{|\cdot|} C_j, \quad 0 \leq j \leq 3.$$

It follows by Mikhlin's theorem that for any $1 < p < \infty$

$$\left\|\mathcal{R}_{\leq a}(|y||D|) + \frac{1}{8\pi^2}\log(|y||D|)\chi_{\leq a}(|y||D|)\right\|_{\mathbf{B}(L^p)} \leq C_p$$

with y-independent C_p. Thus, we have only to estimate the $\mathbf{B}(L^p)$-norm of

$$\begin{aligned} &\log(|y||D|)\chi_{\leq a}(|y||D|)\chi_{>2a}(|D|) \\ &\quad = \log|y|\chi_{\leq a}(|y||D|)\chi_{>2a}(|D|) + \log|D|\chi_{\leq a}(|y||D|)\chi_{>2a}(|D|). \end{aligned}$$

The first term on the right is evidently bounded in $\mathbf{B}(L^p)$ by $C|\log|y||$. To estimate the second, let $f(\lambda, y) = (\log\lambda)\chi_{\leq a}(|y|\lambda)\chi_{>2a}(\lambda)$. We have

$$|f^{(j)}(\lambda, y)| \leq C\lambda^{-j}(1 + |\log|y||). \quad 0 \leq j \leq 3; \tag{2.14}$$

Indeed, $f(\lambda, y) \neq 0$ only if $|y| < 2$ and $a < \lambda < 2a/|y|$ and,

$$|f(\lambda, y)| \leq \max(|\log a|, |\log 2a/|y||) \leq (|\log|y|| + C_a),$$

which implies (2.14) for $j = 0$. The proof for $j = 1, 2, 3$ is similar. Thus,

$$\|f(|D|, y)\|_{\mathbf{B}(L^p)} \leq C\langle\log|y|\rangle$$

and

$$\|\mathcal{R}_{<a}(|y||D|)\chi_{>2a}(|D|)\|_{\mathbf{B}(L^p)} \leq C_{a,p}(1 + |\log|y||). \tag{2.15}$$

Estimates (2.13) and (2.15) imply (2.12). ∎

3. Integral operators

3.1. Operator K

We prove here that the operator K defined by (1.21) is GOP. Let

$$K_1u(x) = \int_0^\infty \mathcal{G}_\lambda(x)(\Pi(\lambda)u)(0)\lambda d\lambda, \tag{3.1}$$

$$K_2u(x) = \int_0^\infty \mathcal{G}_{i\lambda}(x)(\Pi(\lambda)u)(0)\lambda d\lambda. \tag{3.2}$$

By virtue of (1.5), we have

$$Ku(x) = \frac{1}{2}(K_1 - K_2)u(x), \quad u \in \mathcal{D}_*.$$

Since $(\Pi(\lambda)u)(0) \in C_0^\infty(0,\infty)$, (2.9) implies that integrals (3.1) and (3.2) converge for $x \neq 0$ and they are smooth functions of $x \in \mathbb{R}^4 \setminus \{0\}$.

Lemma 3.1. *Let* $1 < p < 2$. *Let for* $\varepsilon > 0$ *and* $u \in \mathcal{D}_*$

$$K_{1,\varepsilon}u(x) = \frac{-1}{(4\pi^2)^2(|x|^2 + i\varepsilon)} \int_{\mathbb{R}^4} \frac{u(y)}{|x|^2 - |y|^2 + i\varepsilon} dy.$$

Then, with a constant $C_p > 0$ *independent of* $\varepsilon > 0$

$$K_1u(x) = \lim_{\varepsilon\to 0} K_{1,\varepsilon}u(x), \quad \textit{pointwise for } x \neq 0,$$
$$\|K_{1,\varepsilon}u\|_p \le C_p\|u\|_p,$$
$$\lim_{\varepsilon\to 0}\|K_{1,\varepsilon}u - K_1u\|_p = 0,$$

in particular, K_1 *is bounded in* $L^p(\mathbb{R}^4)$.

Proof. Let $u, \varphi \in \mathcal{D}_*$. Then, by (1.15) and Fubini's theorem,

$$(K_1u,\varphi) = \frac{1}{(2\pi)^2}\int_0^\infty \left(\int_{\mathbb{R}^4} \mathcal{G}_\lambda(x)\overline{\varphi(x)}dx\right)\left(\int_{\mathbb{S}^3} \hat{u}(\lambda\omega)d\omega\right)\lambda d\lambda.$$

Since the limit converges uniformly for λ on compact subsets of $(0,\infty)$ and

$$\int_{\mathbb{R}^4} \mathcal{G}_\lambda(x)\overline{\varphi(x)}dx = \lim_{\varepsilon\downarrow 0}\frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} \frac{\overline{\hat{\varphi}(\xi)}}{\xi^2 - \lambda^2 - i\varepsilon}d\xi,$$

we obtain by using polar coordinates $\eta = \lambda\omega$, $\lambda > 0$, $w \in \mathbb{S}^3$ that

$$(K_1 u, \varphi) = \lim_{\varepsilon \downarrow 0} \frac{1}{(2\pi)^4} \iint_{\mathbb{R}^8} \frac{\hat{u}(\eta)\overline{\hat{\varphi}(\xi)}}{(|\xi|^2 - |\eta|^2 - i\varepsilon)|\eta|^2} d\xi d\eta. \tag{3.3}$$

On substituting

$$\frac{1}{|\xi|^2 - |\eta|^2 - i\varepsilon} = i \int_0^\infty e^{-it(|\xi|^2 - |\eta|^2 - i\varepsilon)} dt, \quad \varepsilon > 0,$$

and by using the Fubini theorem, we see that (3.3) is equal to

$$\lim_{\varepsilon \downarrow 0} \frac{i}{(2\pi)^4} \int_0^\infty e^{-\varepsilon t} \left(\int_{\mathbb{R}^4} e^{it\eta^2} \hat{u}(\eta) \frac{d\eta}{|\eta|^2} \right) \overline{\left(\int_{\mathbb{R}^4} e^{it|\xi|^2} \hat{\varphi}(\xi) d\xi \right)} dt. \tag{3.4}$$

By the Parseval, identity we have

$$\frac{1}{(2\pi)^2} \overline{\int_{\mathbb{R}^4} e^{it|\xi|^2} \hat{\varphi}(\xi) d\xi} = \frac{-1}{(4\pi t)^2} \int_{\mathbb{R}^4} e^{\frac{i|x|^2}{4t}} \overline{\varphi(x)} dx \leq_{|\cdot|} \frac{C}{\langle t \rangle^2}. \tag{3.5}$$

For the $d\eta$-integral, substitute $e^{it|\eta|^2} = 1 + i|\eta|^2 \int_0^t e^{is|\eta|^2} ds$. Applying the Parseval identity, we have

$$\begin{aligned} &\frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} e^{it\eta^2} \hat{u}(\eta) \frac{d\eta}{|\eta|^2} \\ &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} \frac{u(y)dy}{|y|^2} - i \lim_{\varepsilon \downarrow 0} \int_0^t \left(\int_{\mathbb{R}^4} \frac{e^{-\frac{i|y|^2}{4s}}}{(4\pi s)^2} u(y) dy \right) e^{-\frac{\varepsilon}{s}} ds, \end{aligned}$$

where we have inserted the harmless factor $e^{-\frac{\varepsilon}{s}}$ in the second term for the later purpose. Then, explicitly computing the s-integral implies

$$\frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} e^{it\eta^2} \hat{u}(\eta) \frac{d\eta}{|\eta|^2} = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} (1 - e^{-\frac{i|y|^2}{4t}}) \frac{u(y)}{|y|^2} dy. \tag{3.6}$$

Since (3.6) is bounded by $C\||y|^{-2}u\|_1$, the integral with respect to t of (3.4) is absolutely convergent without the factor $e^{-\varepsilon t}$ and the limit is unchanged if $e^{-\varepsilon t}$ is replaced by $e^{-\varepsilon/t}$. Equations (3.5) and (3.6) then imply that $(K_1 u, \varphi)$ is equal to the $\lim_{\varepsilon\downarrow 0}$ of

$$\frac{-i}{4(4\pi^2)^2}\int_0^\infty\Big(\int_{\mathbb{R}^4}(1-e^{-\frac{i|y|^2}{4t}})\frac{u(y)}{|y|^2}dy\Big)\Big(\int_{\mathbb{R}^4}e^{\frac{i|x|^2}{4t}}\overline{\varphi(x)}dx\Big)e^{-\frac{\varepsilon}{4t}}t^{-2}dt$$
$$=\frac{-i}{4(4\pi^2)^2}\int_{\mathbb{R}^8}\Big(\int_0^\infty(e^{\frac{i|x|^2}{4t}}-e^{\frac{i(|x|^2-y^2)}{4t}})e^{-\frac{\varepsilon}{4t}}t^{-2}dt\Big)\frac{\overline{\varphi(x)}u(y)}{|y|^2}dydx.$$

If we compute the inner integral explicitly, this becomes

$$\int_{\mathbb{R}^8}\frac{-1}{(4\pi^2)^2}\frac{\overline{\varphi(x)}u(y)dydx}{(|x|^2+i\varepsilon)(|x|^2-|y|^2+i\varepsilon)}=(K_{1,\varepsilon}u,\varphi).$$

Thus, we have shown that for any $u, \varphi \in \mathcal{D}_*$

$$(K_1 u,\varphi)=\lim_{\varepsilon\to 0}(K_{1,\varepsilon}u,\varphi). \tag{3.7}$$

It is obvious that $K_{1,\varepsilon}u(x)$ is spherically symmetric and, if we write $K_{1,\varepsilon}u(x) = K_{1,\varepsilon}u(\rho)$ if $|x| = \rho$ and

$$Mu(r)=\frac{1}{\gamma_3}\int_{\mathbb{S}^3}u(r\omega)d\omega,\quad \gamma_3=|\mathbb{S}^3|,$$

then

$$K_{1,\varepsilon}u(\rho)=\frac{-\gamma_3}{(4\pi^2)^2(\rho^2+i\varepsilon)}\int_{\mathbb{R}^4}\frac{Mu(r)r^3}{\rho^2-r^2+i\varepsilon}dr,$$

and a change of variable implies

$$K_{1,\varepsilon}u(\sqrt{\rho})=\frac{-\gamma_3}{2(4\pi^2)^2(\rho+i\varepsilon)}\int_0^\infty\frac{Mu(\sqrt{r})r}{\rho-r+i\varepsilon}dr. \tag{3.8}$$

For $u \in \mathcal{D}_*$, $Mu(r)$ is C^∞ in $(0,\infty)$. It is then well known that the right side of (3.8) converges uniformly along with derivarives on compacts of $(0,\infty)$. Since $K_1 u(x)$ is also smooth for $x \neq 0$, then (3.7) implies $K_1 u(x) = \lim_{\varepsilon\to 0} K_{1,\varepsilon}u(x)$ for all $x \neq 0$.

Moreover, the maximal Hilbert transform (cf. [28, Theorem 1.4 and Lemma 1.5 of Chapter 6, pp. 218–219]) implies that, if we set $f(r) = Mu(\sqrt{r})r$, then

$$F(\sqrt{\rho}):=\sup_{\varepsilon>0}|K_{1,\varepsilon}u(\sqrt{\rho})|\le\frac{C}{\rho}(\mathcal{M}_f(\rho)+\mathcal{M}_{\tilde f}(\rho)),$$

where $\mathcal{M}_f(\rho)$ is the Hardy–Littlewood maximal function of f and $\tilde f$ is the Hilbert transform of f. Define $F(x) = F(|x|)$ for $x \in \mathbb{R}^4$. Since ρ^{1-p} is 1-dimensional $(A)_p$

weight for $1 < p < 2$ ([27, p. 218]), we obtain by the weighted inequality for the maximal functions that for $1 < p < 2$

$$
\begin{aligned}
\int_{\mathbb{R}^4} |F(x)|^p dx &= (\gamma_3/2) \int_0^\infty |F(\sqrt{\rho})|^p \rho d\rho \\
&\leq C \int_0^\infty (|\mathcal{M}_f(\rho)|^p + |\mathcal{M}_{\tilde{f}}(\rho)|^p)\rho^{1-p} d\rho \\
&\leq C \int_0^\infty (|f(r)|^p + |\tilde{f}(r)|^p) r^{1-p} dr. \qquad (3.9)
\end{aligned}
$$

If we apply the weighted inequality for the Hilbert transform, then

$$
\begin{aligned}
(3.9) &\leq C_1 \int_0^\infty |f(r)|^p r^{1-p} dr \\
&= C_1 \int_0^\infty |Mu(\sqrt{r})r|^p r^{1-p} dr \\
&= 2C_1 \int_0^\infty |Mu(r)|^p r^3 dr \leq C\|u\|_p^p.
\end{aligned}
$$

Thus,

$$\|\sup_{\varepsilon>0} |K_{1,\varepsilon}u(x)|\|_p \leq C\|u\|_p$$

and the dominated convergence theorem implies

$$\|K_{1,\varepsilon}u(x) - K_1u(x)\|_p \to 0 \quad \text{as } \varepsilon \to 0$$

for $1 < p < 2$. ∎

Remark 3.2. The operator K_1 is unbounded in $L^p(\mathbb{R}^4)$ for $2 < p < \infty$. To see this we note

$$
\begin{aligned}
\frac{1}{(|x|^2 + i\varepsilon)(|x|^2 - |y|^2 + i\varepsilon)} &= \frac{1}{(|y|^2 + i\varepsilon)(|x|^2 - |y|^2 + i\varepsilon)} \\
&\quad - \frac{|x|^2 - |y|^2}{(|x|^2 + i\varepsilon)(|y|^2 + i\varepsilon)(|x|^2 - |y|^2 + i\varepsilon)} \qquad (3.10)
\end{aligned}
$$

and recall that the integral operator produced by the first term on the right of (3.10) is uniformly bounded in $L^p(\mathbb{R}^4)$ for $\varepsilon > 0$ if $2 < p < \infty$ (cf. [34, Lemma 3.4]). Hence, if K_1 were bounded in $L^p(\mathbb{R}^4)$ for a $p \in (2, \infty)$, then it must be that for $u, w \in C_0^\infty(\mathbb{R}^4)$

$$\lim_{\varepsilon \downarrow 0} \left| \iint_{\mathbb{R}^4 \times \mathbb{R}^4} \frac{(|x|^2 - |y|^2) u(y) w(x) dx dy}{(|x|^2 + i\varepsilon)(|y|^2 + i\varepsilon)(|x|^2 - |y|^2 + i\varepsilon)} \right| \le C \|u\|_p \|w\|_q$$

for a constant $C > 0$, $q = p/(p-1)$. However, this is impossible because the left side is equal to

$$\left| \iint_{\mathbb{R}^4 \times \mathbb{R}^4} \frac{u(y) w(x) dx dy}{|x|^2 |y|^2} \right|$$

which cannot be bounded by $C\|u\|_p \|w\|_q$ for any $1 \le p \le \infty$.

Lemma 3.3. *The operator K_2 has the expression*

$$K_2 u(x) = \frac{1}{(4\pi^2)^2 |x|^2} \int_{\mathbb{R}^4} \frac{u(y)}{|x|^2 + |y|^2} dy, \quad u \in \mathcal{D}_* \tag{3.11}$$

and is bounded in $L^p(\mathbb{R}^4)$ for $1 < p < 2$ and unbounded for $2 < p < \infty$.

Proof. Denote the right side of (3.11) by $\widetilde{K}_2 u(x)$ and

$$K_{2,\varepsilon} u(x) = \frac{1}{(4\pi^2)^2 (|x|^2 + i\varepsilon)} \int_{\mathbb{R}^4} \frac{u(y)}{|x|^2 + |y|^2 + i\varepsilon} dy$$

for $\varepsilon > 0$. It is evident that for $x \ne 0$

$$\lim_{\varepsilon \to 0} K_{2,\varepsilon} u(x) = \widetilde{K}_2 u(x), \quad \sup_{\varepsilon > 0} |K_{2,\varepsilon} u(x)| \le \widetilde{K}_2 |u|(x).$$

Moreover, $\widetilde{K}_2$ is bounded in $L^p(\mathbb{R}^4)$ for $1 < p < 2$, hence,

$$\|K_{2,\varepsilon} u - \widetilde{K}_2 u\|_p \to 0 \quad \text{as } \varepsilon \to 0$$

by the dominated convergence theorem. Indeed, $\widetilde{K}_2 u(x)$ is rotationally symmetric and, if we write $\widetilde{K}_2 u(x) = \widetilde{K}_2 u(\rho)$, $\rho = |x|$ and $Mu(\rho^{\frac14}) = f(\rho)$, then

$$|(\widetilde{K}_2 u)(\rho^{\frac14})| \le \frac{1}{4(4\pi^2)^2 \rho^{\frac12}} \int_0^\infty \frac{|f(r)|}{\rho^{\frac12} + r^{\frac12}} dr = \frac{1}{4(4\pi^2)^2} \int_0^\infty \frac{|f(r\rho)|}{1 + r^{\frac12}} dr$$

and Minkowski's inequality implies for $1 < p < 2$ that

$$\|(\widetilde{K}_2 u)(\rho^{\frac14})\|_{L^p((0,\infty), d\rho)} \le C \int_0^\infty \frac{\|f\|_p}{r^{\frac1p}(1 + r^{\frac12})} dr \le C \|f\|_{L^p((0,\infty))}.$$

Since $\|f\|_{L^p((0,\infty)} \le C\|u\|_p$ by Hölder's inequality,

$$\|(\widetilde{K}_2 u)(x)\|_p = \Big(\frac{\gamma_3}{4}\int_0^\infty |(\widetilde{K}_2 u)(\rho^{\frac14})|^p d\rho\Big)^{1/p} \le C\|u\|_p^p.$$

We now show $K_2 u(x) = \widetilde{K}_2 u(x)$. Since

$$G_0(i\lambda)\overline{\varphi}(x) = \lim_{\varepsilon\downarrow 0}\frac{1}{(2\pi)^2}\int_{\mathbb{R}^4}\frac{e^{ix\xi}\overline{\hat{\varphi}(\xi)}d\xi}{|\xi|^2+\lambda^2-i\varepsilon}$$

converges uniformly with respect λ in compact subsets of $\mathbb{R}$, we obtain

$$\begin{aligned}(K_2 u,\varphi) &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4}\Big(\int_0^\infty \mathcal{G}_{i\lambda}(x)\Big(\int_{\mathbb{S}^3}\hat{u}(\lambda\omega)d\omega\Big)\lambda d\lambda\Big)\overline{\varphi(x)}dx\\ &= \lim_{\varepsilon\downarrow 0}\frac{1}{(2\pi)^4}\int_{\mathbb{R}^8}\frac{\hat{u}(\eta)\overline{\hat{\varphi}(\xi)}}{(|\xi|^2+|\eta|^2-i\varepsilon)|\eta|^2}d\xi d\eta. \qquad (3.12)\end{aligned}$$

The repetition of the proof of Lemma 3.1 with $|\eta|^2$ replacing $-|\eta|^2$ implies that the integral on the right of (3.12) is equal to $(K_{2,\varepsilon}u,\varphi)$. Thus,

$$(K_2 u,\varphi) = \lim_{\varepsilon\downarrow 0}(K_{2,\varepsilon}u,\varphi) = (\widetilde{K}_2 u,\varphi)$$

and $K_2 u(x) = \widetilde{K}_2 u(x)$. The proof of that K_2 is unbounded in $L^p(\mathbb{R}^4)$ for $p > 2$ is similar to that for K_1 and is omitted here. This completes the proof of the lemma. ∎

Lemma 3.4. *The operator K is bounded in $L^p(\mathbb{R}^4)$ for all $1 < p < \infty$.*

Proof. By Lemmas 3.1 and 3.3, K is bounded in $L^p(\mathbb{R}^4)$ for $1 < p < 2$. We prove the same for $2 < p < \infty$. Then, the lemma will follow by the interpolation. Define $K_\varepsilon u = 2(K_{1,\varepsilon}u - K_{2,\varepsilon}u)$ for $\varepsilon > 0$. Then, $Ku(x) = \lim_{\varepsilon\downarrow 0} K_\varepsilon u(x)$ for $x \ne 0$ and a simple computation implies

$$K_\varepsilon u(x) = \frac{1}{2(4\pi^2)^2}(F_{-,\varepsilon}u(x) - F_{+,\varepsilon}u(x)),$$

where $F_{\pm,\varepsilon}u(x)$ are rotationally invariant functions given by

$$F_{\pm,\varepsilon}u(x) = \int_{\mathbb{R}^4}\frac{u(y)dy}{(|x|^2 \pm |y|^2 + i\varepsilon)|y|^2}.$$

Notice that the dangerous terms $(1/2\pi^4)|x|^{-2}|y|^{-2}$ have canceled each other. We denote $F_{\pm,\varepsilon}u(x) = f_{\pm,\varepsilon}(\rho)$, $\rho = |x|$. Then,

$$\tilde{f}_+(\rho) = \sup_{\varepsilon>0} |f_{+,\varepsilon}(\rho)| \le \frac{\gamma_3}{8\pi^4}\int_0^\infty \frac{|Mu(r)|r^3 dr}{r^2(\rho^2+r^2)} = \frac{\gamma_3}{8\pi^4}\int_0^\infty \frac{|Mu(r\rho)|r dr}{1+r^2}$$

and Minkowski's inequality implies for any $2 < p < \infty$ that

$$\|\tilde{f}_+(|x|)\|_{L^p(\mathbb{R}^4)} \le C\int_0^\infty \frac{\|Mu(r\rho)\|_{L^p((0,\infty),\rho^3 d\rho)} r dr}{1+r^2} = C\|Mu(\rho)\|_{L^p((0,\infty),\rho^3 d\rho)}\int_0^\infty \frac{r^{1-4/p}dr}{1+r^2} \le C\|u\|_p.$$

It follows that $F_{+\varepsilon}(x)$ converges as $\varepsilon \to 0$ in $L^p(\mathbb{R}^4)$ for $2 < p < \infty$ to

$$F_+u(x) = \int_{\mathbb{R}^4} \frac{u(y)dy}{(|x|^2+|y|^2)|y|^2}.$$

It is shown in [34, Lemma 3.4], via the same argument as in the proof of Lemma 3.1, that, for $2 < p < \infty$, $F_{-,\varepsilon}$ is uniformly bounded in $\mathbf{B}(L^p)$ for $\varepsilon > 0$ and $F_{-,\varepsilon}u(x)$ converges as $\varepsilon \to 0$ for $x \ne 0$ and simultaneously in $L^p(\mathbb{R}^4)$. Hence, K is bounded in $L^p(\mathbb{R}^4)$ for $2 < p < \infty$ as well and the lemma follows. ■

3.2. Good operators

Recall that $\Omega(T)$ and $\widetilde{\Omega}(T(\lambda))$ are defined by (1.19) and (1.20).

Lemma 3.5. *We have* $\|\Omega(T)u\|_p \le C_p\|T\|_{\mathcal{L}^1}\|u\|_p$ *for* $1 < p < \infty$.

Proof. By using the integral kernel $\mathcal{R}_\lambda(x-y) = (\tau_y\mathcal{R}_\lambda)(x)$ of $R_0(\lambda)$, we write

$$\Omega(T)u(x) = \int_0^\infty \left(\iint_{\mathbb{R}^8} T(y,z)\tau_y\mathcal{R}_\lambda(x)(\Pi(\lambda)\tau_{-z}u)(0)dydz\right)\lambda^3 d\lambda.$$

If we may change the order of integrations, $\Omega(T)u(x)$ becomes

$$\begin{aligned}\iint_{\mathbb{R}^8} & T(y,z)\tau_y\Big(\int_0^\infty \mathcal{R}_\lambda(x)(\Pi(\lambda)\tau_{-z}u)(0)\lambda^3 d\lambda\Big)dydz \\ &= \iint_{\mathbb{R}^8} T(y,z)(\tau_y K\tau_{-z}u)(x)\end{aligned}$$

and Lemma 3.4 implies

$$\|W(T)u\|_p \le C_p\|T\|_{\mathcal{L}^1}\|u\|_p.$$

To see that the change of order of integrations is possible for almost all $x \in \mathbb{R}^4$, it suffices to show that $\mathcal{R}_\lambda(x-y)T(y,z)(\Pi(\lambda)u)(z)\lambda^3$ is (absolutely) integrable with respect to $(x,y,z,\lambda) \in B_R(0)\times\mathbb{R}^4\times\mathbb{R}^4\times(0,\infty)$ for any $R>0$, where $B_R(0)=\{x : |x|<R\}$. However, this is obvious since $\Pi(\lambda)u(z)=0$ for λ outside a compact interval $[\alpha,\beta]\Subset(0,\infty)$, $|\Pi(\lambda)u(z)|\le C\langle z\rangle^{-3/2}$ uniformly for $\lambda\in[\alpha,\beta]$ and $\int_{B(0,R)}|\mathcal{R}_\lambda(x-y)|dx$ is uniformly bounded for $y\in\mathbb{R}^4$ and $\lambda\in[\alpha,\beta]$. This completes the proof. ■

The following is the variant of [34, Proposition 3.9]. We take advantage of this chance to point out that [34, Proposition 3.9] has an error and it must be replaced by the following proposition and that some obvious modifications are necessary in the part of [34] which used that proposition. Let $a_+ = \max(a,0)$.

Proposition 3.6. *Let $T(\lambda,x,y)$ be an $\mathcal{L}^1$-valued C^2-function of $\lambda\in(0,\infty)$ such that*

$$\lim_{\lambda\to\infty}\lambda^j\|T^{(j)}(\lambda)\|_{\mathcal{L}^1}=0,\quad j=0,1,2. \tag{3.13}$$

Suppose further that $T^{(2)}(\lambda)$ is AC on compact intervals of $(0,\infty)$ and

$$\int_0^\infty \lambda^2\|T^{(3)}(\lambda)\|_{\mathcal{L}^1}d\lambda<\infty.$$

Then, for the integral operator $T(\lambda)$ with the kernel $T(\lambda,x,y)$,

$$\widetilde{\Omega}(T(\lambda))u(x)=\int_0^\infty R_0(\lambda^4)\,T(\lambda)\,\Pi(\lambda)u(x)\lambda^3d\lambda,\ u\in\mathcal{D}_* \tag{3.14}$$

satisfies the estimate (1.24) for any $1<p<\infty$.

Remark 3.7. If $a>0$, condition (3.13) is automatic for $\chi_{\le a}(\lambda)T(\lambda)$ and (1.24) is satisfied by $\widetilde{\Omega}(T(\lambda))\chi_{\le a}(|D|)$ without the condition.

Proof. Since $u \in \mathcal{D}_*$, $\Pi(\lambda)u(z) = (\Pi(\lambda)\tau_z u)(0) = 0$ outside $[\alpha, \beta] \Subset (0, \infty)$ and $|\Pi(\lambda)u(z)| \leq C\langle z\rangle^{-\frac{3}{2}}$ uniformly. It follows by virtue of (2.8) that the integral (3.14) converges absolutely and defines a continuous function of $x \in \mathbb{R}^4$. By Taylor's formula,

$$T(\lambda) = -\frac{1}{2}\int_0^\infty ((\rho-\lambda)_+)^2 T^{(3)}(\rho)d\rho, \tag{3.15}$$

where the integral is the Bochner integral in $\mathcal{L}^1$. Let

$$B(\lambda) = ((1-\lambda)_+)^2 = ((1-\lambda^2)_+)^2(1+\lambda)^{-2}.$$

Then, the Fourier transform of $B(|\xi|)$ is integrable on $\mathbb{R}^4$ (cf. [27, p. 389]). Hence, $B(|D|)$ is bounded in $L^p(\mathbb{R}^4)$ for all $1 \leq p \leq \infty$ and $\|B(|D|/\rho)\|_{\mathbf{B}(L^p)}$ is independent of $0 < \rho < \infty$. On substituting (3.15) and changing the order of the integrations, (3.14) becomes

$$\frac{-1}{2}\int_0^\infty \left(\int_0^\infty ((\rho-\lambda)_+)^2 R_0(\lambda^4)T^{(3)}(\rho)\Pi(\lambda)u\lambda^3 d\lambda\right)d\rho. \tag{3.16}$$

and, by virtue of (1.16) and (1.19), the inner integral of (3.16) is equal to

$$\rho^2\int_0^\infty R_0(\lambda^4)T^{(3)}(\rho)\Pi(\lambda)B(|D|/\rho)u\lambda^3 d\lambda = \rho^2\Omega(T^{(3)}(\rho))B(|D|/\rho)u.$$

Thus, Minkowski's inequality and Lemma 3.5 imply

$$\|(3.16)\|_p \leq C\int_0^\infty \rho^2\|T^{(3)}(\rho)\|_{\mathcal{L}^1}\|u\|_p d\rho \leq C\|u\|_p.$$

This proves the proposition. ■

4. High energy estimate

We prove here Theorems 1.1 and 1.3.

4.1. Proof of Theorem 1.1. Small potentials

By virtue of what is explained in the introduction, we have only to prove (1.30) for $n = 1, 2, \ldots$ for $W_n\chi_{\geq a}(|D|)u$ defined by (1.27).

(1) We already proved that $\|W_1\chi_{\geq a}(|D|)u\|_p \leq C\|V\|_1\|u\|_p$ in the introduction.

(2) Let $n=2$ and $V_y^{(2)}(x)=V(x)V(x-y)$. Then, as was shown by (1.29), (1.30) for $n=2$ follows from (1.28), which we prove here. We have by changing variables that

$$
\begin{aligned}
M_V R_0(\lambda^4) M_V u(x) &= \int_{\mathbb{R}^4} V(x)\mathcal{R}(\lambda|y|)V(x-y)u(x-y)dy \\
&= \int_{\mathbb{R}^4} V_y^{(2)}(x)\mathcal{R}(\lambda|y|)(\tau_y u)(x)dy \qquad (4.1)
\end{aligned}
$$

which we substitute in (1.27) for $n=2$. Then, since $\tau_y\Pi(\lambda)=\Pi(\lambda)\tau_y$ and

$$
\mathcal{R}(\lambda|y|)\Pi(\lambda)=\Pi(\lambda)\mathcal{R}(|y||D|)
$$

by virtue of (1.16), $W_2\chi_{\geq a}(|D|)u(x)$ becomes

$$
\int_0^\infty \Big(\int_{\mathbb{R}^8} \mathcal{R}(\lambda|x-y|)V_z^{(2)}(y)(\Pi(\lambda)\mathcal{R}(|z||D|)\tau_z u)(y)dzdy\Big)\lambda^3\chi_{\geq a}(\lambda)d\lambda. \qquad (4.2)
$$

If we change the order of integrations and apply (1.19), we may rewrite (4.2) in the desired form:

$$
\begin{aligned}
&\int_{\mathbb{R}^4}\Big(\int_0^\infty \big(R_0(\lambda^4)M_{V_z^{(2)}}\Pi(\lambda)\mathcal{R}(|z||D|)\chi_{\geq a}(|D|)\tau_z u\big)(x)\lambda^3 d\lambda\Big)dz \\
&\quad = \int_{\mathbb{R}^4}(\Omega(M_{V_z^{(2)}})\mathcal{R}(|z||D|)\chi_{\geq a}(|D|)\tau_z u)(x)dz.
\end{aligned}
$$

(3) Let $n\geq 3$ and $V_{y_1,\dots,y_{n-1}}^{(n)}(x)=V(x)V(x-y_1)\cdots V(x-y_1-\cdots-y_{n-1})$. Repeating the argument used for (4.1) implies

$$
\begin{aligned}
&M_w(M_v R_0(\lambda^4)M_w)^{n-1}M_v u(x) = (M_V R_0(\lambda^4))^{n-1}M_V u(x) \\
&= \int_{\mathbb{R}^{4(n-1)}} V_{y_1,\dots,y_{n-1}}^{(n)}(x)\Big(\prod_{j=1}^{n-1}\mathcal{R}(\lambda|y_j|)\Big)\tau_{y_1+\cdots+y_{n-1}}u(x)dy_1\dots dy_{n-1}.
\end{aligned}
$$

It follows that $W_n\chi_{\geq a}(|D|)u(x)$ is equal to

$$
\begin{aligned}
&\int_0^\infty\int_{\mathbb{R}^4}\int_{\mathbb{R}^{4(n-1)}} \mathcal{R}(\lambda|x-y|)V_{y_1,\dots,y_{n-1}}^{(n)}(y)\Big(\prod_{j=1}^{n-1}\mathcal{R}(\lambda|y_j|)\Big) \\
&\quad\times \Pi(\lambda)\tau_{y_1+\cdots+y_{n-1}}u(y)\lambda^3\chi_{\geq a}(\lambda)dy_1\cdots dy_{n-1}dyd\lambda. \qquad (4.3)
\end{aligned}
$$

As in the proof of Lemma 3.5, we may integrate (4.3) by $d\lambda$ first. We then apply (1.16) to $(\prod_{j=1}^{n-1} \mathcal{R}(\lambda|y_j|))\Pi(\lambda)$ and (1.19) to the resulting equation. This implies that the right of (4.3) is equal to

$$\int_{\mathbb{R}^{4(n-1)}} \Omega(M_{V^{(n)}_{y_1,\dots,y_{n-1}}})\chi_{\geq a}(|D|)\prod_{j=1}^{n-1}\mathcal{R}(|y_j||D|)\tau_{y_1+\cdots+y_{n-1}}u\,dy_1\dots dy_{n-1}.$$

Note that $\chi_{\geq a}(|D|) = \chi_{\geq a}(|D|)\chi_{\geq a/2}(|D|)^{n-2}$. Then, Minkowski's inequality and Lemmas 3.5 and 2.6 imply that $\|W_n\chi_{\geq a}(|D|)u\|_p$ is bounded by

$$\begin{aligned} &C^n_{a,p}\int_{\mathbb{R}^{4(n-1)}}\|V^{(n)}_{y_1,\dots,y_{n-1}}\|_{L^1(\mathbb{R}^4)}\prod_{j=1}^{n-1}\langle\log|y_j|\rangle\|u\|_p dy_1\dots dy_{n-1} \\ &\quad = C^n_{a,p}\int_{\mathbb{R}^{4n}}|V(x_0)|\prod_{j=1}^{n-1}|V(x_j)|\langle|\log|x_{j-1}-x_j|\rangle\|u\|_p dx_0\dots dx_{n-1} \end{aligned}$$

where we have changed variables so that $y_j = x_{j-1}-x_j$, $j=1,\dots,n-1$. We estimate the integral inductively by using Schwarz' and Hölder's inequalities n-times by

$$\begin{aligned} &\|V\|_1^{\frac12}\left(\int_{\mathbb{R}^4} V(x_0)\langle\log|x_0-x_1|\rangle^2 V(x_1)dx_0dx_1\right)^{\frac12} \\ &\quad\times\cdots\times\left(\int_{\mathbb{R}^4} V(x_{n-2})\langle\log|x_{n-2}-x_{n-1}|\rangle^2 V(x_{n-1})dx_{n-2}dx_{n-1}\right)^{\frac12}\|V\|_1^{\frac12} \\ &\leq C^n(\|V\|_{L^q_{\mathrm{loc},u}} + \|\langle\log|x|\rangle^2 V\|_{L^1})^n. \end{aligned}$$

This proves (1.30) for $n \geq 3$ and completes the proof.

4.2. Proof of Theorem 1.3

For the proof we use the following lemma.

Lemma 4.1. *Let* $1 < q < 4/3$, $q' = q/(q-1)$ *and* $j = 0, 1, \dots$. *Let us uppose that* $\langle x\rangle^{(2j-3)_+}V \in L^1(\mathbb{R}^4)$ *and* $V \in L^q(\mathbb{R}^4)$. *Then, for any* $a > 0$, $M_v R_0(\lambda^4)M_w$ *is* $\mathcal{H}_2$*-valued* C^j *function of* $\lambda > a$ *and, for* $n = 1, 2, \dots$

$$\|\partial^j_\lambda(M_vR_0(\lambda^4)M_w)^n\|_{\mathcal{H}_2} \leq \frac{C_{n,j}}{\lambda^{\frac{2n}{q'}}}((\|\langle x\rangle^{(2j-3)_+}V\|_1 + \|V\|_q)^n. \quad (4.4)$$

Proof. Let $N_{1,j}$ and $N_{2,j}$, $j = 0, 1, \ldots$ be convolution operators with

$$N_{1,j}(x) = \lambda^{-j}\langle \log \lambda|x|\rangle \chi_{\le 1}(\lambda|x|), \quad N_{2,j}(x) = \frac{\chi_{\ge 1}(\lambda|x|)|x|^j}{\lambda^{\frac{3}{2}}|x|^{\frac{3}{2}}}.$$

It follows from (2.8) that $\partial_\lambda^j \mathcal{R}(\lambda^4)(x) \le_{|\cdot|} C(N_{1,j}(x) + N_{2,j}(x))$, $j = 0, 1, \ldots$. By repeating estimate (2.4), we have for any $1 \le q \le \infty$ that

$$\begin{aligned}\|M_v N_{1,j} M_w\|_{\mathcal{H}_2}^2 &\le C\lambda^{-2j} \int_{\lambda|x-y|\le 1} |V(x)|\langle \log \lambda|x-y|\rangle^2 |V(y)| dx dy \\ &\le C\lambda^{-2j}\|V\|_1 \|V\|_{L^q_{\mathrm{loc},u}} \|\langle \log \lambda|x|\rangle\|^2_{L^{2q'}(\lambda|x|<1)}\end{aligned}$$

and

$$\|M_v N_{1,j} M_w\|_{\mathcal{H}_2} \le C\lambda^{-j-\frac{2}{q'}}\|v\|_2 \|w\|_{L^{2q}_{\mathrm{loc},u}}.$$

By Hölder's inequality, for $1 \le q < 4$,

$$\begin{aligned}\|vN_{2,0}w\|_{\mathcal{H}_2}^2 &\le C\int_{\lambda|x-y|\ge 1} \frac{|V(x)||V(y)|}{(\lambda|x-y|)^3} dx dy \\ &\le C\|V\|_1\|V\|_q \Big(\int_{\lambda|x|\ge 1} \frac{dx}{(\lambda|x|)^{3q'}}\Big)^{\frac{1}{q'}} \le C\lambda^{-\frac{4}{q'}}\|V\|_1\|V\|_q\end{aligned}$$

Likewise, for $1 \le q < 4/3$,

$$\|vN_{2,1}w\|_{\mathcal{H}_2}^2 \le C\frac{1}{\lambda^2}\int_{\lambda|x-y|\ge 1} \frac{|V(x)||V(y)|}{\lambda|x-y|} dx dy \le C\|V\|_1\|V\|_q \lambda^{-2-\frac{4}{q'}}$$

For $j \ge 2$, we evidently have $\|vN_{2,j}w\|_{\mathcal{H}_2} \le C\lambda^{-\frac{3}{2}}\|\langle x\rangle^{2j-3}V\|_1$. Combining these estimates together, we obtain for $1 \le q < \frac{4}{3}$ and $\lambda > a$ that

$$\|\partial_\lambda^j M_v R_0(\lambda^4) M_w\|_{\mathcal{H}_2} \le C\lambda^{-\min(j+\frac{2}{q'},\frac{3}{2})}(\|\langle x\rangle^{(2j-3)_+}V\|_1 + \|V\|_q)$$

This implies (4.4) for $n = 1$. For $n \ge 2$, we compute $\partial_\lambda^j (M_v R_0(\lambda^4) M_w)^n$ via Leibniz's formula and estimate each factor via (4.4) for $n = 1$. The lemma follows. ∎

Proof of Theorem 1.3. We may assume $1 < q < 4/3$. Let N be such that $2N/q' > 3$. We substitute (1.31) with (1.32) in the stationary formula (1.25) for the high energy part. Then, $W_- \chi_{\ge a}(|D|)$ becomes

$$\sum_{n=0}^{N-1} (-1)^n W_n \chi_{\ge a}(|D|)u + (-1)^N \int_0^\infty R_0^+(\lambda^4) D_N(\lambda) \Pi(\lambda) u \lambda^3 \chi_{\ge a}(\lambda) d\lambda,$$

where we set $W_0 u = u$. By virtue of (1.30), $\sum_{n=0}^{N-1}(-1)^n W_n \chi_{\geq a}(|D|)u$ is GOP; Lemma 4.1 implies that $D_N(\lambda, x, y)$ is $\mathcal{L}^1$-valued function of $\lambda \in (a, \infty)$ of class C^3 and

$$\|\partial_\lambda^j D_N(\lambda)\|_{\mathcal{L}^1} \leq C\lambda^{-\frac{2N}{q'}}(\|\langle x\rangle^3 V\|_{L^1} + \|V\|_{L^2})^N, \quad 0 \leq j \leq 3.$$

Hence, the operator $D_N(\lambda)$ is GPR for (1.25) by Proposition 3.6 and Theorem 1.3 follows. ∎

5. Low energy estimates 1. The case H is regular at zero

In what follows, we shall study $W_-\chi_{\leq a}(|D|)$ or equivalently

$$\Omega_{\text{low},a} u := \int_0^\infty R_0(\lambda^4)\mathcal{Q}_v(\lambda)\Pi(\lambda)u\lambda^3\chi_{\leq a}(\lambda)d\lambda \tag{5.1}$$

for a sufficiently small $a > 0$. As previously, we define

$$\Omega_{\text{low},a}(T)u = \int_0^\infty R_0(\lambda^4)T\Pi(\lambda)u\lambda^3\chi_{\leq a}(\lambda)d\lambda,$$

$$\tilde{\Omega}_{\text{low},a}(T(\lambda))u = \int_0^\infty R_0(\lambda^4)T(\lambda)\Pi(\lambda)u\lambda^3\chi_{\leq a}(\lambda)d\lambda.$$

Since we shall in what follows exclusively deal with small $\lambda > 0$, we shall often omit the phrase "for small $\lambda > 0$" and, abusing notation, say that T or $T(\lambda)$ is GPR if $\Omega_{\text{low},a}(T)$ or $\tilde{\Omega}_{\text{low},a}(T(\lambda))$ is GOP for a sufficiently small a. We irrespectively write $\mathcal{R}_{\text{em}}(\lambda)$ for the operator valued function which satisfies the conditions of Proposition 3.6 for small $\lambda > 0$.

We shall often use the following lemma for studying $\mathcal{M}(\lambda^4)^{-1}$ as $\lambda \to 0$. Let A be the operator matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

on the direct sum of Banach spaces $\mathcal{Y} = \mathcal{Y}_1 \oplus \mathcal{Y}_2$.

Lemma 5.1 (Feshbach formula). *Suppose a_{11}, a_{22} are closed and a_{12}, a_{21} are bounded operators. Suppose that the bounded inverse a_{22}^{-1} exists. Then A^{-1} exists if and only if $d = (a_{11} - a_{12}a_{22}^{-1}a_{21})^{-1}$ exists. In this case, we have*

$$A^{-1} = \begin{pmatrix} d & -da_{12}a_{22}^{-1} \\ -a_{22}^{-1}a_{21}d & a_{22}^{-1}a_{21}da_{12}a_{22}^{-1} + a_{22}^{-1} \end{pmatrix}.$$

In this section, we shall prove Theorem 1.9 when H is regular at zero. Thus, weassume that $\langle \log |x|\rangle^2 \langle x\rangle^4 V \in (L^1 \cap L^q)(\mathbb{R}^4)$ for a $q > 1$ and that the inverse $(QT_0Q)^{-1}$ exists in QL^2. Let

$$D_0 = Q(QT_0Q)^{-1}Q \quad \text{and} \quad L_0 = \begin{pmatrix} P & -PT_0QD_0 \\ -D_0QT_0P & D_0QT_0PT_0QD_0 \end{pmatrix}$$

in the decomposition $L^2 = PL^2 \oplus QL^2$. Notice that rank $L_0 = 2$.

Lemma 5.2. *For small $\lambda > 0$, $T_0 + g_0(\lambda)P$ is invertible and*

$$(T_0 + g_0(\lambda)P)^{-1} = D_0 + h(\lambda)L_0, \quad h(\lambda) = (g_0(\lambda) + c_1)^{-1} \tag{5.2}$$

where $c_1 = ((v, T_0v) - (QT_0v, D_0QT_0v))\|V\|_1^{-1}$ is a real constant.

Proof. In the decomposition $L^2 = PL^2 \oplus QL^2$,

$$T_0 + g_0(\lambda)P = \begin{pmatrix} g_0(\lambda) + PT_0P & PT_0Q \\ QT_0P & QT_0Q \end{pmatrix} =: \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Here $a_{22}{=}QT_0Q$ is invertible in QL^2 and

$$a_{11} - a_{12}a_{22}^{-1}a_{21} = g_0(\lambda)P + PT_0P - PT_0D_0T_0P = (g_0(\lambda) + c_1)P$$

is also invertible in $PL^2(\mathbb{R}^2)$ for small $\lambda > 0$ and $d = (g_0(\lambda) + c_1)^{-1}P$. Then, Lemma 5.1 implies that $(T_0 + g_0(\lambda)P)^{-1}$ exists and (5.2) holds. ∎

Let $\widetilde{\mathcal{D}}_0(\lambda) = (T_0 + g_0(\lambda)P)^{-1} = D_0 + h(\lambda)L_0$.

Lemma 5.3. *For small $\lambda > 0$, $\mathcal{M}(\lambda^4)$ is invertible in L^2 and*

$$\mathcal{M}(\lambda^4)^{-1} = \widetilde{\mathcal{D}}_0(\lambda) + \mathcal{O}^{(4)}_{\mathcal{H}_2}(\lambda^2 \log \lambda). \tag{5.3}$$

Proof. We have $R_2^{(v)}(\lambda)\widetilde{\mathcal{D}}_0(\lambda) \in \mathcal{O}^{(4)}_{\mathcal{H}_2}(\lambda^2 \log \lambda)$ by Lemma 2.2 and

$$\mathcal{M}(\lambda^4) = (1 + R_2^{(v)}(\lambda)\widetilde{\mathcal{D}}_0(\lambda))(g_0(\lambda)P + T_0)$$

by Lemma 5.2. It follows that $\mathcal{M}(\lambda^4)$ is invertible for small $\lambda > 0$ and

$$\mathcal{M}(\lambda^4)^{-1} = \widetilde{\mathcal{D}}_0(\lambda)(1 + R_2^{(v)}(\lambda)\widetilde{\mathcal{D}}_0(\lambda))^{-1} = \widetilde{\mathcal{D}}_0(\lambda) + \mathcal{O}^{(4)}_{\mathcal{H}_2}(\lambda^2 \log \lambda).$$

This is (5.3). ∎

Following Schlag [25], we say operator T is absolutely bounded (ABB for short) if $|T(x, y)|$ defines a bounded operator in L^2.

Lemma 5.4. (1) *The operator D_0 is ABB.*
(2) *If T is ABB and $v, w \in L^2(\mathbb{R}^4)$, then $v(x)T(x, y)w(y) \in \mathcal{L}^1$.*

Proof. (1) The argument of the proof of Lemma 8 of [25] implies that D_0 is ABB.
(2) is evident by the Schwarz inequality. ∎

Proof of Theorem 1.9 when H is regular at zero. Multiply (5.3) by M_v from both sides. Then, $M_v \widetilde{\mathcal{D}}_0(\lambda) M_v$ is $\mathcal{GVS}$ since L_0 is of rank 2, $M_v D_0 M_v \in \mathcal{L}^1$ by Lemma 5.4 and $h(\lambda)$ is GMU; it is evident that $M_v \mathcal{O}^{(4)}_{\mathcal{H}_2}(\lambda^2 \log \lambda) M_v = \mathcal{R}_{\text{em}} \in \mathcal{O}^{(4)}_{\mathcal{L}^1}(\lambda^2 \log \lambda)$. Thus, $\mathcal{Q}_v(\lambda)$ is GPR and $W_- \chi_{\geq a}(|D|)$ is GOP. ∎

6. Low energy estimate 2. Resonances

In this section we prove Lemma 1.4 and Lemma 1.8. We assume only $\langle \log |x| \rangle^2 V \in (L^1 \cap L^q)(\mathbb{R}^4)$, $q > 1$ unless otherwise stated. We begin with the following lemma. Recall that S_1 is the projection in QL^2 to $\operatorname{Ker} QT_0 Q|_{QL^2}$. We shall often write $\mathcal{N}_\infty$ for $\mathcal{N}_\infty(H)$.

Lemma 6.1. *The projection S_1 is of finite rank. The operator $QT_0Q + S_1$ is invertible in QL^2.*

In what follows we denotes $D_0 = Q(QT_0Q + S_1)^{-1}Q$ in spite of Lemma 5.2 where $D_0 = Q(QT_0Q)^{-1}Q$ as the latter becomes the former when $S_1 = 0$ and as it will not appear any further.

Proof. The operator $QT_0Q = QUQ + QN_0^{(v)}Q$ is selfadjoint in the Hilbert space QL^2. Since $\mathbf{1} = U^2$, we have by comparing

$$\mathbf{1} = \begin{pmatrix} Q & 0 \\ 0 & P \end{pmatrix}, \quad U^2 = \begin{pmatrix} QUQ & QUP \\ PUQ & PUP \end{pmatrix}^2$$

that $(QUQ)^2 = Q - QUPUQ$. Since rank $QUPUQ = 1$,

$$\sigma_{\text{ess}}((QUQ)^2) = \sigma_{\text{ess}}(Q) = \{1\}$$

on QL^2 by Weyl's theorem and $\sigma_{\text{ess}}(QUQ) \subset \{1, -1\}$. The operator $N_0^{(v)}$ is compact in L^2 by Lemma 2.1 and hence so is $QN_0^{(v)}Q$ in QL^2. Thus, $\sigma_{\text{ess}}(QT_0Q)|_{QL^2} \subset \{1, -1\}$ by Weyl's theorem once more and 0 is an isolated eigenvalue of $QT_0Q|_{QL^2}$ of finite multiplicity. The rest of the lemma follows by the Riesz–Schauder theorem [36]. ∎

Proof of Lemma 1.8 (1). Let $\zeta \in S_1 L^2 \setminus \{0\}$. Then, $Q\zeta = \zeta$ and $QT_0Q\zeta = 0$. It follows that $T_0\zeta = c_0 v$ for a constant c_0, hence

$$(U + N_0^{(v)})\zeta = c_0 v, \quad c_0 = \|v\|_2^{-2}(T_0\zeta, v). \tag{6.1}$$

Thus, if we define $\varphi = \Phi(\zeta)$ by (1.13), then (6.1) implies $\varphi = -c_0 + N_0 M_v \zeta$, hence $v\varphi = -c_0 v + N_0^{(v)}\zeta = -U\zeta$ and $\zeta = -w\varphi$; applying Δ^2 to $\varphi = -c_0 + N_0 M_v \zeta$ implies $\Delta^2\varphi = v\zeta = -V\varphi$ or $(\Delta^2 + V)\varphi = 0$.

We next show that $\varphi \in L^\infty(\mathbb{R}^4)$, which will imply Φ maps $S_1 L^2$ to $\mathcal{N}_\infty$ with the inverse $\zeta = -w\varphi$ on its image, in particular, $\mathcal{N}_\infty \neq \{0\}$. The starting point is that for a $\zeta \in S_1 L^2$

$$\varphi(x) = -c_0 + N_0 M_v \zeta = -c_0 - \frac{1}{8\pi^2}\int_{\mathbb{R}^4} \log|x-y| v(y)\zeta(y)dy. \tag{6.2}$$

Recall that we are assuming $\langle \log|x|\rangle^2 V \in (L^1 \cap L^q)(\mathbb{R}^4)$ for a $q > 1$. Let $p = 2q/(q-1)$.

(i) Let first $|x| \le 10$. By Hölder's inequality,

$$\int_{|y|\le 20} |\log|x-y| v(y)\zeta(y)|dy \le \|\log|y|\|_{L^p(|y|\le 30)} \|v\|_{L^{2q}(|y|\le 20)} \|\zeta\|_2;$$

if $|y| > 20$, we have $0 < \log|x-y| \le \log(2|y|) \le 2\log|y|$ and

$$\int_{|y|>20} \log|x-y||v(y)\zeta(y)|dy \le 2\|(\log|y|)v\|_{L^2(|y|>20)}\|\zeta\|_2.$$

Thus, $|\varphi(x)| \le |c_0| + |N_0(v\zeta)(x)| \le C$ for $|x| \le 10$.

(ii) Let next $|x| > 10$. Since $P\zeta = 0$ or $\int_{\mathbb{R}^4} v(y)\zeta(y)dy = 0$, we have

$$N_0(v\zeta)(x) = -\frac{1}{8\pi^2}\int_{\mathbb{R}^4} (\log|x-y| - \log|x|)v(y)\zeta(y)dy. \tag{6.3}$$

Let $\Delta_1 = \{y : |y| > 2|x|\}$, $\Delta_2 = \{y : |y| < |x|/2\}$ and $\Delta_3 = \{y : |x|/2 \le |y| \le 2|x|\}$. If $y \in \Delta_1$, then $|x| < |x-y| < |x||y|$, $0 < \log|x-y| - \log|x| < \log|y|$ and

$$\int_{|y|>2|x|} (\log|x-y| - \log|x|)|v(y)\zeta(y)|dy \le \|\langle \log|y|\rangle v\|_2 \|\zeta\|_2$$

If $y \in \Delta_2$, then $|\log|x-y| - \log|x|| \le \log 2 < 1$ and

$$\int_{|y|<|x|/2} |(\log|x-y| - \log|x|)v(y)\zeta(y)|dy \le \int_{\mathbb{R}^4} |(v\zeta)(y)|dy \le \|v\|_2\|\zeta\|_2.$$

If $y \in \Delta_3$, then, $0 < \log|x| \le 2\log|y|$ and

$$\int_{y\in\Delta_3} |(\log|x|)v(y)\zeta(y)|dy \le 2\|\log|y|v\|_2\|\zeta\|_2;$$

$$\int_{|x-y|\le 2, y\in\Delta_3} |(\log|x-y|)v(y)\zeta(y)|dy \le \|\log|y|\|_{L^p(|y|\le 2)}\|v\|_{2q}\|\zeta\|_2;$$

if $|x-y|>2$, then $\log|x-y| \le \log(|x||y|) \le 3\log|y|$ and

$$\int_{|x-y|>2, y\in\Delta_3} |(\log|x-y|)v(y)\zeta(y)|dy \le 3\|\log|y|v\|_2\|\zeta\|_2.$$

Thus, $|N_0 v\zeta(x)| \le C$ also for $|x| \ge 10$ and $\varphi \in L^\infty(\mathbb{R}^4)$.

Finally, we prove Image $\Phi = \mathcal{N}_\infty$, which completes the proof. Let $\varphi \in \mathcal{N}_\infty \setminus \{0\}$ and define $\zeta = -w\varphi$. We have $\Delta^2(\varphi + N_0V\varphi) = (\Delta^2 + V)\varphi = 0$, hence $|\xi|^4\mathcal{F}(\varphi + N_0V\varphi)(\xi) = 0$. It follows that $\mathcal{F}(\varphi + N_0V\varphi) \in \mathcal{S}'(\mathbb{R}^4)$ vanishes outside $\{0\}$ and $\mathcal{F}(\varphi + N_0V\varphi)(\xi) = \sum_{\text{finite}} c_\alpha D^\alpha\delta(\xi)$ for constants c_α, or $(\varphi + N_0V\varphi)(x)$ is a polynomial. But, $\varphi \in L^\infty$ and $\langle\log|x|\rangle^2 V \in (L^1 \cap L^q)(\mathbb{R}^4)$ imply that

$$(N_0V\varphi)(x) = -\frac{1}{8\pi^2}\Bigg(\int_{|x-y|<2} + \int_{|x-y|\ge 2}\Bigg)\log|x-y|V(y)\varphi(y)dy$$

is bounded by $C(1+\log\langle x\rangle)$. Hence, it must be that $\varphi + N_0V\varphi = c$ for a constant c and $N_0V\varphi(x) \in L^\infty$. It follows that $\int V\varphi dx = -\int v\zeta dx = 0$ because otherwise $|N_0V\varphi(x)| \ge C|\log|x||$ for large $|x|$ for a $C > 0$. Hence, $P\zeta = 0$ or $\zeta = Q\zeta$ and

$$cv = (v + vN_0V)\varphi = -(U + N_0^{(v)})\zeta = -T_0Q\zeta.$$

Thus, $QT_0Q\zeta = 0$ or $\zeta \in S_1L^2$, $c = -\|v\|^{-2}(PT_0\zeta, v)$ and $\varphi = c + N_0v\zeta = \Phi(\zeta)$. Moreover, $\zeta \ne 0$ because $\zeta = 0$ would imply $0 \ne \varphi = c$, hence, $w = 0$ and $V = 0$, which is a contradiction. ■

Proof of Lemma 1.4. We assume here that $\langle\log|x|\rangle^2\langle x\rangle^3 V \in (L^1 \cap L^q)(\mathbb{R}^4)$ for a $q > 1$. Let $q' = q/(q-1)$. Let $\varphi \in \mathcal{N}_\infty(H)$. We have $\varphi = \Phi(\zeta)$ for $\zeta = -w\varphi \in S_1L^2$ and (6.2) and (6.3) imply

$$\varphi(x) = -c_0 + \frac{1}{8\pi^2}\int_{\mathbb{R}^4}(\log|x-y| - \log|x|)V(y)\varphi(y)dy, \qquad (6.4)$$

where c_0 is given by (6.1). We assume $|x| \ge 10^{10}$ in the sequel. Let

$$\Delta_1 = \{y : |y| > |x|/4\}$$

and

$$\Delta_2 = \{y : |y| \le |x|/4\}$$

and split the integral on the right of (6.4) as

$$\Big(\int_{\Delta_1} + \int_{\Delta_2}\Big)(\log|x-y| - \log|x|)V(y)\varphi(y)dy = I_1(x) + I_2(x).$$

(1) For $y \in \Delta_1$, we have $\log|x| \le \log 4|y|$ and $\log|x-y| \le \log 5|y|$ if $|x-y| \ge 1$. Hence, $|I_1(x)|$ is bounded by

$$2\int_{\Delta_1} \log(5|y|)|V(y)\varphi(y)|dy + \int_{|x-y|\le 1, y\in\Delta_1} \log|x-y||V(y)\varphi(y)|dy$$
$$\le C(\|\langle \log|y|\rangle V(y)\|_{L^1(\Delta_1)} + \|\log|y|\|_{L^p(|x|\le 1)}\|V\|_{L^q(\Delta_1)}) \le C\langle x\rangle^{-3}$$

Thus, $I_1(x)$ may be put into the remainder $O(|x|^{-3})$ of (1.4).

(2) For $y \in \Delta_2$, $|x - \theta y| \ge 3|x|/4 > 10^9$ for $0 \le \theta \le 1$. Let

$$f(\theta) = \log|x-\theta y| - \log|x|.$$

Then, Taylor's formula implies

$$f(1) = f(0) + f'(0) + \frac{1}{2}f''(0) + \int_0^1 \frac{(1-\theta)^2}{2} f'''(\theta)d\theta, \qquad (6.5)$$

$$f'(0) = -\sum_{j=1}^4 \frac{x_j y_j}{|x|^2}, \quad f''(0) = \frac{|y|^2}{|x|^2} - \sum_{j,k=1}^4 \frac{2x_j x_k y_j y_k}{|x|^4},$$
$$f'''(\theta) = -\frac{6(y\cdot(\theta y - x))|y|^2}{|x-\theta y|^4} + \frac{8(y\cdot(\theta y - x))^3}{|x-\theta y|^6}.$$

We substitute (6.5) for $\log|x-y| - \log|x|$ in $I_2(x)$. Since

$$R_3(x,y) := \int_0^1 \frac{(1-\theta)^2}{2} f'''(\theta)d\theta \le_{|\cdot|} (7/3)(|y|/|x|)^3,$$

the contribution of $R_3(x,y)$ to $I_2(x)$ is bounded in modulus by

$$C\langle x\rangle^{-3}\int_{\Delta_2} |y|^3|V(y)\varphi(y)|dy \le C\langle x\rangle^{-3}\|\langle y\rangle^3 V\|_1.$$

Since $|f'(0)| \le (|y|/|x|)$, $|f''(0)| \le C(|y|/|x|)^2$ and $|x|/4 \le |y|$ in $\mathbb{R}^4 \setminus \Delta_2$, we also have

$$\int_{\mathbb{R}^4\setminus\Delta_2} \Big(f'(0) + \frac{1}{2}f''(0)\Big)V(y)\varphi(y)dy \le_{|\cdot|} C\langle x\rangle^{-3}\|\langle y\rangle^3 V\|_1.$$

(3) Combining the estimates in (1) and (2), we obtain

$$\varphi(x) = -c_0 + \int_{\mathbb{R}^3} \Big(f'(0) + \frac{1}{2}f''(0)\Big)V(y)\varphi(y)dy + \mathcal{O}(\langle x\rangle^{-3})$$

which implies expansion (1.4):

$$\varphi(x) = -c_0 + \sum_{j=1}^{4} \frac{a_j x_j}{|x|^2} + \sum_{j,k=1}^{4} \frac{(2a_{jk} - b\delta_{jk})x_j x_k}{|x|^4} + O(|x|^{-3}), \tag{6.6}$$

where δ_{jk} is the Kronecker delta. For later convenience, we express the coefficients in terms of ζ by restoring $V(y)\varphi(y) = -v(y)\zeta(y)$:

$$a_j = \frac{1}{8\pi^2}\int_{\mathbb{R}^4} y_j v(y)\zeta(y)dy, \quad b = \frac{1}{8\pi^2}\int_{\mathbb{R}^4} |y|^2 v(y)\zeta(y)dy, \tag{6.7}$$

$$a_{jk} = \frac{1}{8\pi^2}\int_{\mathbb{R}^4} y_j y_k v(y)\zeta(y)dy. \tag{6.8}$$

This completes the proof. ■

Lemma 6.2. *Assume that* $\langle x\rangle^3\langle\log|x|\rangle^2 V \in (L^1 \cap L^q)(\mathbb{R}^4)$ *for a* $q > 1$.

(1) *Let* $\zeta \in S_1L^2$. *Then,*

$$\zeta \in S_2L^2 \iff T_0\zeta = 0.$$

(2) *Let* $\zeta \in S_2L^2$. *Then,*

$$\zeta \in S_3L^2 \iff (x^\alpha v, \zeta) = 0 \quad \text{for } |\alpha| \le 1.$$

(3) *Let* $\zeta \in S_3L^2$. *Then,*

$$\zeta \in S_4L^2 \iff (x^\alpha v, \zeta) = 0 \quad \text{for } |\alpha| \le 2.$$

Proof. We have $\langle x\rangle^{3/2}\langle\log|x|\rangle\zeta \in L^2$ by Lemma 1.8 (1).

(1) If $\zeta \in S_1L^2$ and $T_0\zeta = 0$, then $T_1\zeta = S_1T_0PT_0S_1\zeta = 0$ and $\zeta \in S_2L^2$. Conversely, if $\zeta \in S_1L^2$ and $T_1\zeta = 0$, then $(PT_0\zeta, PT_0\zeta) = 0$ and $PT_0\zeta = 0$; $QT_0Q\zeta = QT_0\zeta = 0$ evidently. Hence, $T_0\zeta = 0$.

(2) Let $\zeta \in S_2 L^2$. If $\zeta \in S_3 L^2$, then $(v, \zeta) = 0$ and

$$0 = (T_2\zeta, \zeta) = \frac{2i}{4^4\pi} \sum_{j=1}^{4} \left| \int_{\mathbb{R}^4} x_j v(x)\zeta(x)dx \right|^2 .$$

It follows that $(x_j v, \zeta) = 0$ for $1 \le j \le 4$. Hence, $(x^\alpha v, \zeta) = 0$ for $|\alpha| \le 1$. Conversely, if $(x^\alpha v, \zeta) = 0$ for $|\alpha| \le 1$, then $T_2\zeta(x)$ is equal to $-i(4^4\pi)^{-1}$ times

$$S_2 M_v \int_{\mathbb{R}^4} |x-y|^2 v(y)\zeta(y)dy = (S_2 v)(x) \int_{\mathbb{R}^4} y^2 v(y)\zeta(y)dy.$$

But $S_2 v = 0$ and, hence, $T_2\zeta(x) = 0$. Thus, $\zeta \in S_3 L^2$.

(3) Let $\zeta \in S_4 L^2 \subset S_3 L^2$. Then, $(x^\alpha v, \zeta) = 0$ for $|\alpha| \le 1$ by (2) and

$$0 = (T_3\zeta, \zeta) = \frac{2}{3 \cdot 4^3} \int_{\mathbb{R}^4} (|x|^2|y|^2 + 2(x \cdot y)^2) v(x)v(y)\zeta(x)\overline{\zeta(y)}dxdy$$

$$= \frac{2}{3 \cdot 4^3} \left| \int_{\mathbb{R}^4} |x|^2 v(x)\zeta(x)dx \right|^2 + \frac{1}{3 \cdot 4^2} \sum_{j,k=1}^{4} \left| \int_{\mathbb{R}^4} x_j x_k v(x)\zeta(x)dx \right|^2 .$$

It follows that $(x^\alpha v, \zeta) = 0$ also for $|\alpha| = 2$. Conversely, one has that if $\zeta \in S_2 L^2$ satisfies $(x^\alpha v, \zeta) = 0$ for $|\alpha| \le 2$, then (2) implies $\zeta \in S_3 L^2$ and

$$T_3\zeta(x) = S_3\Big(v(x) \int_{\mathbb{R}^4} \Big(|y|^4 - 4 \sum_{j=1}^{4} x_j y_j \cdot |y|^2 \Big) v(y)\zeta(y)dy \Big).$$

Since $S_3 v = 0$ and $S_3(x_j v) = 0$ for $j = 1, \ldots, 4$ by (2), $T_3\zeta = 0$. Hence, $\zeta \in S_4 L^2$. This completes the proof. ∎

Proof of Lemma 1.8 (2) *and* (3). For $\zeta \in S_1 L^2$, let $\varphi = \Phi(\zeta) \in \mathcal{N}_\infty$ and c_0, $\mathbf{a}$ and A be coefficients of the expansion (1.4) of $\varphi(x)$.

(2) Since P is one-dimensional, rank $T_1 \le 1$. Hence, if $T_1|_{S_1 L^2}$ is invertible, then rank $S_1 = 1$ and $\operatorname{Ker} T_1 = \operatorname{Ker} P T_0 S_1 = \{0\}$, which implies $c_0 = \|v\|_2^{-2}(T_0\zeta, v) \neq 0$ for $\zeta \in S_1 L^2 \setminus \{0\}$, hence H has only s-wave resonances.

(3) Here we assume $\langle x\rangle^3 \langle \log|x| \rangle^2 V \in (L^1 \cap L^q)(\mathbb{R}^4)$.

(i) Let $\zeta \in (S_1 L^2 \ominus S_2 L^2) \setminus \{0\}$. Then $c_0 = 0$ would imply $P T_0 \zeta = 0$ and $\zeta \in S_2 L^2$ which is a contradiction. Hence, $c_0 \neq 0$ and φ is s-wave resonance.

(ii) Let $\zeta \in S_2L^2$. Then, $T_0\zeta = 0$, hence $c_0 = 0$, by Lemma 6.2 and

$$i(T_2\zeta,\zeta) = \frac{1}{4^4\pi}\int_{\mathbb{R}^4\times\mathbb{R}^4} |x-y|^2(v\zeta)(x)\overline{(v\zeta)(y)}dxdy$$
$$= \frac{-1}{2^7\pi}\sum_{j=1}^4\left|\int_{\mathbb{R}^4} x_j v(x)\zeta(x)dx\right|^2 \le 0. \tag{6.9}$$

It follows that the selfadjoint operator iT_2 on S_2L^2 is non-positive and $(T_2\zeta,\zeta) = 0$ implies $T_2\zeta = 0$ and $\zeta \in S_3L^2$. Hence, for non-trivial $\zeta \in S_2L^2 \ominus S_3L^2$, $i(T_2\zeta,\zeta) < 0$, which implies $\mathbf{a} \neq 0$ in (1.4) by (6.7) and φ is p-wave resonance.

(iii) Suppose next $\zeta \in S_3L^2 \ominus S_4L^2$. Then $c_0 = 0$ as previously and $\zeta \in S_3L^2$ implies $\mathbf{a} = 0$ by Lemma 6.2 and (6.7). For $\zeta \in S_3L^2$, we have

$$(T_3\zeta,\zeta) = \frac{1}{48}\sum_{j,k=1}^4\left|\int_{\mathbb{R}^4} x_jx_k(v\zeta)(x)dx\right|^2 + \frac{1}{96}\left|\int_{\mathbb{R}^4} x^2(v\zeta)(x)dx\right|^2$$

as previously, and the selfadjoint operator T_3 on S_3L^2 is non-negative. It follows $(T_3\zeta,\zeta) > 0$ for non-trivial $\zeta \in S_3L^2 \ominus S_4L^2$. Suppose $A = 0$. Then, in the expression (6.6), $a_{jk} = 0$ for $j \neq k$ and $2a_{jj} - b = 0$ for $1 \le j \le 4$. But $\sum_{j=1}^4 a_{jj} = b$ by (6.8) and $0 = \sum_{j=1}^4(2a_{jj} - b) = -2b$. Hence, $a_{jk} = 0$ for all $1 \le j,k \le 4$ which contradicts to $(T_3\zeta,\zeta) > 0$. Thus, $A \neq 0$ for non-trivial $\zeta \in S_3L^2 \ominus S_4L^2$ and φ is d-wave resonance.

(iv) Finally, let $\zeta \in S_4L^2 \setminus \{0\}$. Then, we already have shown that $c_0 = 0$ and $\mathbf{a} = 0$. Moreover, $(T_3\zeta,\zeta) = 0$ and (6.8) implies $A = 0$. Thus, φ is zero energy eigenfunction of H. ∎

7. Singularity of the first kind

In this section we prove $W_-\chi_{\le a}(|D|)$ is GOP for sufficiently small $a > 0$ when H has singularity of the first kind at zero, assuming $\langle x\rangle^4V \in (L^1 \cap L^q)(\mathbb{R}^4)$ for a $q > 1$. In what follows, we shall repeatedly and inductively use the following lemma due to Jensen and Nenciu [16].

Lemma 7.1 ([16]). *Let A be a closed operator and S a projection in a Hilbert space $\mathcal{H}$. Suppose $A + S$ has bounded inverse. Then, A has bounded inverse if and only if*

$$B = S - S(A+S)^{-1}S$$

has bounded inverse in $S\mathcal{H}$ and, in this case,

$$A^{-1} = (A+S)^{-1} + (A+S)^{-1}SB^{-1}S(A+S)^{-1}.$$

7.1. Threshold analysis 1

We begin with lemmas which hold whenever H is singular at zero. Let $\{\zeta_1, \dots, \zeta_n\}$ be the orthonormal basis of $S_1 L^2$ so that

$$S_1 u = (\zeta_1 \otimes \zeta_1 + \cdots + \zeta_n \otimes \zeta_n)u, \quad u \in QL^2. \tag{7.1}$$

We denote by the same letter S_1 the extension of S_1 to L^2 defined by the right of (7.1) for all $u \in L^2$. The inverse $D_0 = (QT_0Q + S_1)^{-1}$ exists in QL^2 by virtue of Lemma 6.1; in the decomposition $L^2 = PL^2 \oplus QL^2$, let

$$L_0 = \begin{pmatrix} P & -PT_0QD_0 \\ -D_0QT_0P & D_0QT_0PT_0QD_0 \end{pmatrix} \tag{7.2}$$

by using the same notation as in Lemma 5.2.

Repeating the proofs of Lemmas 5.3 and 5.4 with $QT_0Q + S_1$ replacing QT_0Q we obtain the following lemma whose proof is omitted.

Lemma 7.2. *For small $\lambda > 0$, $T_0 + g_0(\lambda)P + S_1$ is invertible and*

$$(T_0 + g_0(\lambda)P + S_1)^{-1} = D_0 + h_1(\lambda)L_0, \quad h_1(\lambda) = (g_0(\lambda) + c_1)^{-1}$$

with c_1 being a constant. The operator $D_0 = (QT_0Q + S_1)^{-1}$ is ABB.

Using the notation of Lemma 5.3 once again, we let

$$\widetilde{\mathfrak{D}}_0(\lambda) := (T_0 + g_0(\lambda)P + S_1)^{-1} = D_0 + h_1(\lambda)L_0. \tag{7.3}$$
$$T_{4,l}(\lambda) := \tilde{g}_2(\lambda)G_4^{(v)} + G_{4,l}^{(v)}.$$

Lemma 7.3. *Suppose that H is singular at zero. Then, $\mathcal{M}(\lambda^4) + S_1$ is invertible in L^2 for small $\lambda > 0$.*

(1) *If $\langle x\rangle^4 V \in (L^1 \cap L^q)(\mathbb{R}^4)$ for a $q > 1$, then*

$$(\mathcal{M}(\lambda^4) + S_1)^{-1} = \widetilde{\mathfrak{D}}_0(\lambda) + Y_1(\lambda), \quad Y_1(\lambda) = \mathcal{O}^{(4)}_{\mathcal{H}_2}(\lambda^2), \tag{7.4}$$

and $M_v(\mathcal{M}(\lambda^4) + S_1)^{-1}M_v = \mathcal{G}\mathcal{V}\mathcal{S} + \mathcal{R}_{\rm em}(\lambda)$.

(2) *If V satisfies $\langle \log|x|\rangle^2\langle x\rangle^8 V \in (L^1 \cap L^q)(\mathbb{R}^4)$ for a $q > 1$, then*

$$Y_1(\lambda) = -\lambda^2 \widetilde{\mathfrak{D}}_0(\lambda)G_2^{(v)}\widetilde{\mathfrak{D}}_0(\lambda) + Y_2(\lambda), \quad Y_2(\lambda) = \mathcal{O}^{(4)}_{\mathcal{H}_2}(\lambda^4 \log\lambda). \tag{7.5}$$

(3) *If V satisfies $\langle \log|x|\rangle^2\langle x\rangle^{12} V \in (L^1 \cap L^q)(\mathbb{R}^4)$ for a $q > 1$, then*

$$Y_2(\lambda) = -\lambda^4 \widetilde{\mathfrak{D}}_0(\lambda)\{T_{4,l}(\lambda)\widetilde{\mathfrak{D}}_0(\lambda) - (G_2^{(v)}\widetilde{\mathfrak{D}}_0(\lambda))^2\} + \mathcal{O}^{(4)}_{\mathcal{H}_2}(\lambda^6 \log\lambda). \tag{7.6}$$

Proof. (1) We have $\mathcal{M}(\lambda^4) + S_1 = g_0(\lambda)P + T_0 + S_1 + R_2^{(v)}(\lambda)$ and Lemma 7.2 implies $T_0 + g_0(\lambda)P + S_1$ is invertible and we have (7.3). Then, $(\mathcal{M}(\lambda^4) + S_1)^{-1}$ exists and

$$(\mathcal{M}(\lambda^4) + S_1)^{-1} = \widetilde{\mathcal{D}}_0(\lambda)(1 + R_2^{(v)}(\lambda)\widetilde{\mathcal{D}}_0(\lambda))^{-1}. \tag{7.7}$$

By expanding the right side and by applying (2.10) we obtain (7.4). Since D_0 is ABB, $M_v(\mathcal{M}(\lambda^4) + S_1)^{-1}M_v = \mathcal{GVS} + \mathcal{R}_{\mathrm{em}}(\lambda)$.

(2) If $\langle x\rangle^8\langle \log|x|\rangle^2 V \in (L^1 \cap L^q)(\mathbb{R}^4)$, then (2.10) implies $R_2^{(v)}(\lambda) = \lambda^2 G_2(x) + R_4^{(v)}(\lambda)$ with $R_4^{(v)}(\lambda) \in \mathcal{O}^{(4)}_{\mathcal{H}_2}(\lambda^4 \log\lambda)$. We then expand (7.7) by using $(1+X)^{-1} = 1 - X + X^2(1+X)^{-1}$ with $X = R_2^{(v)}(\lambda)\widetilde{\mathcal{D}}_0(\lambda)$ and estimate the remainder by using (2.10) for $n = 1$ and $n = 2$. We obtain (7.5).

(3) If $\langle x\rangle^{12}\langle \log|x|\rangle^2 V \in (L^1 \cap L^q)(\mathbb{R}^4)$, then $R_4^{(v)}(\lambda) = \lambda^4 T_{4,l}(\lambda) + R_6^{(v)}(\lambda)$ with $R_6^{(v)}(\lambda) \in \mathcal{O}^{(4)}_{\mathcal{H}_2}(\lambda^6 \log\lambda)$. We then argue as in (2) to obtain (7.6). We omit the details. ∎

We apply Lemma 7.1 to the pair $(\mathcal{M}(\lambda^4), S_1)$. Let

$$B_1(\lambda) = S_1 - S_1(\mathcal{M}(\lambda^4) + S_1)^{-1}S_1.$$

Since $S_1 D_0 = D_0 S_1 = S_1$ and $S_1 L_0 S_1 = T_1$ by (7.2), we have $S_1\widetilde{\mathcal{D}}_0(\lambda)S_1 = h_1(\lambda)T_1 + S_1$ and (7.4) implies

$$B_1(\lambda) = -h_1(\lambda)T_1 - S_1 Y_1(\lambda) S_1. \tag{7.8}$$

If $B_1(\lambda)$ is invertible in S_1L^2, then and Lemma 7.1 implies

$$\mathcal{M}(\lambda^4)^{-1} = (\mathcal{M}(\lambda^4) + S_1)^{-1} + M^{(1)}_{\mathrm{ess}}(\lambda), \tag{7.9}$$

$$M^{(1)}_{\mathrm{ess}}(\lambda) = (\mathcal{M}(\lambda^4) + S_1)^{-1}S_1 B_1(\lambda)^{-1}S_1(\mathcal{M}(\lambda^4) + S_1)^{-1}. \tag{7.10}$$

In what follows $A(\lambda) \equiv B(\lambda)$ will mean $A(\lambda) - B(\lambda) = \mathcal{GVS} + R_{\mathrm{em}}(\lambda)$.

7.2. Singularities of the first kind

Suppose now that H has singularity of the first kind at zero. Then, $T_1 = S_1 T_0 P T_0 S_1$ is invertible and rank $S_1 = 1$. We let ζ be the normalised basis vector of S_1L^2.

Lemma 7.4. *Let H have singularity of the first kind at zero. Then,*

$$\mathcal{Q}_v(\lambda) \equiv (a\log\lambda + b)(v\zeta)\otimes(v\zeta), \tag{7.11}$$

where $a \in \mathbb{R}\setminus\{0\}$, $b \in \mathbb{C}$.

Proof. We have $T_1 = d_0^{-1}(\zeta \otimes \zeta)$ with $d_0 = c_0^{-2}\|V\|_1^{-1} > 0$, where $c_0 = \|v\|_2^{-2}(T_0\zeta, v)$ and by Lemma 7.3 $B_1(\lambda) = d(\lambda)(\zeta \otimes \zeta)$ with

$$d(\lambda) = -d_0^{-1}h_1(\lambda)(1 + \mathcal{O}_{\mathbb{C}}^{(4)}(\lambda^2 \log \lambda)).$$

Thus, $B_1(\lambda)$ is invertible and

$$B_1(\lambda)^{-1} = d(\lambda)^{-1}(\zeta \otimes \zeta). \tag{7.12}$$

Combining (7.4) and (7.12), we have

$$M_{\mathrm{ess}}^{(1)}(\lambda) = d(\lambda)^{-1}(\widetilde{\mathcal{D}}_0(\lambda) + Y_1(\lambda))(\zeta \otimes \zeta)(\widetilde{\mathcal{D}}_0(\lambda) + Y_1(\lambda)). \tag{7.13}$$

Expand the right of (7.13) and use (7.3), $d(\lambda)^{-1} = -d_0 h_1(\lambda)^{-1} + \mathrm{GMU}$, and $D_0\zeta = D_0 S_1 \zeta = \zeta$. We obtain

$$\begin{aligned} M_v M_{\mathrm{ess}}^{(1)}(\lambda) M_v &= d(\lambda)^{-1} M_v \widetilde{\mathcal{D}}_0(\lambda)(\zeta \otimes \zeta)\widetilde{\mathcal{D}}_0(\lambda) M_v + \mathcal{R}_{\mathrm{em}}(\lambda) \\ &= -d_0 h_1(\lambda)^{-1}(v\zeta) \otimes (v\zeta) + \mathcal{G}\mathcal{V}\mathcal{S} + \mathcal{R}_{\mathrm{em}}(\lambda). \end{aligned}$$

Since $M_v(\mathcal{M}(\lambda^4) + S_1)^{-1} M_v = \mathcal{G}\mathcal{V}\mathcal{S} + \mathcal{R}_{\mathrm{em}}(\lambda)$ by (7.4), (7.9) implies (7.11). ∎

Proof of Theorem 1.9 when *H* has singularity of first kind. By virtue of Lemma 7.4, $W_-\chi_{\le a}(|D|)u(x)$ is equal modulo GOP to (1.36):

$$-\int_0^\infty (a \log \lambda + b) R_0(\lambda^4)(v\zeta)(x)(v\zeta, \Pi(\lambda)u)\lambda^3 \chi_{\le a}(\lambda) d\lambda.$$

We have $v\zeta \in \langle x\rangle^{-2} L^1(\mathbb{R}^4)$ and $\int_{\mathbb{R}^4} v(x)\zeta(x)dx = 0$ for $\zeta \in S_1 L^2$. Thus, the following lemma implies Theorem 1.9 (1). The lemma is more than necessary for this purpose and we state it in this fashion for the later purpose.

Lemma 7.5. *Assume that $f, \langle x\rangle g \in L^1(\mathbb{R}^4)$ and $\int_{\mathbb{R}^4} g(x)dx = 0$. Then, operators $\widetilde{\Omega}_k$, $k = 0, 1, 2, \ldots$ defined as follows are GOP:*

$$\widetilde{\Omega}_k u(x) = \int_0^\infty (R_0^+(\lambda^4) f)(x)(g, \Pi(\lambda)u)\lambda^3 (\log \lambda)^k \chi_{\le a}(\lambda) d\lambda.$$

Proof. Let $\mu_k(\lambda) = \lambda(\log \lambda)^k \chi_{\le a}(\lambda)$ for $k = 0, 1, \ldots$; μ_k are GMU. We have

$$(g, \Pi(\lambda)u) = \int_{\mathbb{R}^4} g(z)(\Pi(\lambda)u(z) - \Pi(\lambda)u(0))dz$$

and $\Pi(\lambda)u(z) - \Pi(\lambda)u(0)$ may be expressed as in (1.39). Then, as in (1.40), $\tilde{\Omega}_k u(x)$ becomes the $\sum_{j=1}^4 \int_0^1 d\theta$ of

$$\int_0^\infty \left(\int_{\mathbb{R}^4} (R_0^+(\lambda^4) f)(x) i z_j g(z)(\Pi(\lambda) R_j u)(\theta z) dz \right) \lambda^3 \mu_k(\lambda) d\lambda. \tag{7.14}$$

Since translations commute with Fourier multipliers, (1.15) and (1.16) imply

$$\mu_k(\lambda)(\Pi(\lambda) R_j u)(\theta z) = \Pi(\lambda)(\tau_{-\theta z} R_j \mu_k(|D|)u)(0).$$

Thus, if we define $T_j(y,z) = i z_j f(y) g(z)$, then $T_j(y,z) \in \mathcal{L}^1$ and (1.22) implies

$$\begin{aligned}(7.14) &= \int_0^\infty R_0^+(\lambda^4) T_j \Pi(\lambda)(\tau_{-\theta z} R_j \mu(|D|)u)(0) \lambda^3 d\lambda \\ &= \int_{\mathbb{R}^8} T_j(y,z) \tau_y K \tau_{-\theta z} R_j \mu(|D|) u dy dz.\end{aligned}$$

It follows by virtue of Lemma 3.5 that

$$\|(7.14)\|_p \le C \|T_j\|_{\mathcal{L}^1} \|R_j \mu(|D)u\|_p \le C\|u\|_p.$$

This proves that $\tilde{\Omega}_k$, $k = 0, 1, \ldots$ are GOP. ■

8. Singularity of the second kind

We prove here Theorem 1.9 (2). Thus, we assume $\langle \log|x| \rangle^2 \langle x \rangle^8 V \in (L^1 \cap L^q)(\mathbb{R}^4)$, T_1 is singular in $S_1 L^2$ and $T_2 = S_2 G_2^{(v)} S_2$ is invertible in $S_2 L^2$. Let

$$(T_1 + S_2)^{-1} = D_1. \tag{8.1}$$

We clearly have

$$D_1 S_2 = S_2 D_1 = S_2.$$

We abuse notation below and write $\mathcal{O}^{(\ell)}_{S_j L^2}(f(\lambda))$ for $\mathcal{O}^{(\ell)}_{\mathbf{B}(S_j L^2)}(f(\lambda))$.

8.1. Threshold analysis 2

Recall (7.9), (7.10), and (7.8). We study $B_1(\lambda)$ for small $\lambda > 0$ via Lemma 7.1. In view of (7.8), let

$$\tilde{B}_1(\lambda) := -h_1(\lambda)^{-1} B_1(\lambda) = T_1 - \lambda^2 h_1(\lambda)^{-1} \tilde{T}_1(\lambda) + \tilde{T}_4(\lambda); \tag{8.2}$$

$$\tilde{T}_1(\lambda) := S_1 \tilde{\mathcal{D}}_0(\lambda) G_2^{(v)} \tilde{\mathcal{D}}_0(\lambda) S_1 \in \mathcal{O}^{(4)}_{S_1 L^2}(1),$$

$$\tilde{T}_4(\lambda) := S_1 h_1(\lambda)^{-1} Y_2(\lambda) S_1 \in \mathcal{O}^{(4)}_{S_1 L^2}(\lambda^4 (\log\lambda)^2). \tag{8.3}$$

Notice that $\widetilde{T}_1(\lambda)$ is $\mathcal{VS}$. We remark that we do not assume T_2 is invertible in S_2L^2 in Lemmas 8.1 and 8.2.

Lemma 8.1. *We have the following identities:*

$$S_2 D_0 = S_2 = D_0 S_2. \tag{8.4}$$

$$S_2 T_0 = T_0 S_2 = 0, \quad L_0 S_2 = S_2 L_0 = 0. \tag{8.5}$$

$$S_2 \widetilde{\mathcal{D}}_0(\lambda) = \widetilde{\mathcal{D}}_0(\lambda) S_2 = S_2. \tag{8.6}$$

$$S_2 \widetilde{T}_1(\lambda) S_2 = S_2 G_2^{(v)} S_2 = T_2. \tag{8.7}$$

Proof. (1) Since $S_1 D_0 = S_1 = D_0 S_1$ and $S_2 \subset S_1$, we have (8.4).

(2) Since $0 = QT_0QS_1 = QT_0S_1$, we have $T_0S_1 = PT_0S_1$ and $\mathrm{Ker}_{S_1L^2} T_1 = \mathrm{Ker}_{S_1L^2} T_0S_1$. Hence, $T_0S_2 = T_0S_1S_2 = 0$ and $S_2T_0 = 0$ by the duality. This implies the first of (8.5). Then, by using also (8.4), we obtain

$$S_2 L_0 = S_2(P - PT_0QD_0Q - QD_0QT_0P + D_0QT_0PT_0QD_0) = 0.$$

We likewise have $L_0 S_2 = 0$ and the second of (8.5) follows.

(3) Equations (8.4) and (8.5) imply $S_2\widetilde{\mathcal{D}}_0(\lambda) = S_2(h_1(\lambda)L_0 + D_0) = S_2$ and likewise $\widetilde{\mathcal{D}}_0(\lambda)S_2 = S_2$. Equation (8.7) is obvious from (8.6). ∎

Lemma 8.2. *For small $\lambda > 0$, $\widetilde{B}_1(\lambda) + S_2$ is invertible in S_1L^2 and*

$$(\widetilde{B}_1(\lambda) + S_2)^{-1} = D_1 + D_1\lambda^2 h_1(\lambda)^{-1}\widetilde{T}_1(\lambda)D_1 + \mathcal{O}^{(4)}_{S_1L^2}(\lambda^4(\log\lambda)^2). \tag{8.8}$$

Proof. From (8.2), we have $\widetilde{B}_1(\lambda) + S_2 = (\mathbf{1}_{S_1L^2} - L_1(\lambda))(T_1 + S_2)$, where

$$L_1(\lambda) := \lambda^2 h_1(\lambda)^{-1}\widetilde{T}_1(\lambda)D_1 - \widetilde{T}_4(\lambda)D_1. \tag{8.9}$$

It follows that $\widetilde{B}_1(\lambda) + S_2$ is invertible in S_1L^2 and

$$(\widetilde{B}_1(\lambda) + S_2)^{-1} = D_1 + D_1L_1(\lambda) + D_1L_1(\lambda)^2(\mathbf{1}_{S_1L^2} - L_1(\lambda))^{-1}. \tag{8.10}$$

Substituting (8.9) and using (8.3), we obtain (8.8). ∎

Lemma 8.3. *Let $B_2(\lambda) = S_2 - S_2(\widetilde{B}_1(\lambda) + S_2)^{-1}S_2$. Then*

$$B_2(\lambda) = -\lambda^2 h_1(\lambda)^{-1}(T_2 + \mathcal{O}^{(4)}_{S_2L^2}(\lambda^2\log\lambda)); \tag{8.11}$$

$B_2(\lambda)$ is invertible in S_2L^2 for small $\lambda > 0$ and

$$B_2(\lambda)^{-1} = -\lambda^{-2}h_1(\lambda)T_2^{-1} + \mathcal{O}^{(4)}_{S_2L^2}(1). \tag{8.12}$$

Proof. Multiply (8.8) by S_2 from both sides. Since $D_1S_2 = S_2D_1 = S_2$ by (8.1), (8.7) implies $S_2\widetilde{T}_1(\lambda)S_2 = T_2$ and

$$S_2(\widetilde{B}_1(\lambda)+S_2)^{-1}S_2 = S_2 + \lambda^2 h_1(\lambda)^{-1}T_2 + \mathcal{O}^{(4)}_{S_2L^2}(\lambda^4(\log\lambda)^2),$$

from which (8.11) follows (recall $h_1(\lambda) = (g_0(\lambda)+c_1)^{-1}$). Since T_2 is invertible in S_2L^2, the rest is obvious. ■

Since $B_2(\lambda)^{-1}$ exists and $B_1(\lambda)^{-1} = -h_1(\lambda)^{-1}\widetilde{B}_1(\lambda)^{-1}$, Lemma 7.1 implies

$$B_1(\lambda)^{-1} = -h_1(\lambda)^{-1}(\widetilde{B}_1(\lambda)+S_2)^{-1} - h_1(\lambda)^{-1}J_2(\lambda), \tag{8.13}$$

$$J_2(\lambda) = (\widetilde{B}_1(\lambda)+S_2)^{-1}S_2B_2(\lambda)^{-1}S_2(\widetilde{B}_1(\lambda)+S_2)^{-1}. \tag{8.14}$$

Substitute (8.13) in (7.10). Then, (7.9) yields that

$$\mathcal{M}(\lambda^4)^{-1} = (\mathcal{M}(\lambda^4)+S_1)^{-1} + \mathcal{N}_1(\lambda) + \mathcal{N}_2(\lambda),$$

$$\mathcal{N}_1(\lambda) = -h_1(\lambda)^{-1}(\mathcal{M}(\lambda^4)+S_1)^{-1}S_1(\widetilde{B}_1(\lambda)+S_2)^{-1}S_1(\mathcal{M}(\lambda^4)+S_1)^{-1}, \tag{8.15}$$

$$\mathcal{N}_2(\lambda) = -h_1(\lambda)^{-1}(\mathcal{M}(\lambda^4)+S_1)^{-1}S_1J_2(\lambda)S_1(\mathcal{M}(\lambda^4)+S_1)^{-1}. \tag{8.16}$$

Recall that $M_v(\mathcal{M}(\lambda^4)+S_1)^{-1}M_v$ is GPR by Lemma 7.3 (1).

Lemma 8.4. *The operator $M_v\mathcal{N}_1(\lambda)M_v$ is GPR.*

Proof. Substitute (7.4) for $(\mathcal{M}(\lambda^4)+S_1)^{-1}$ and (8.8) for $(\widetilde{B}_1(\lambda)+S_2)^{-1}$ in (8.15), expand the result and multiply by M_v from both sides. Then, the terms which contain $\mathcal{O}^{(4)}_{\mathcal{H}_2}(\lambda^2)$ in (7.4) or terms of order $\mathcal{O}^{(4)}_{\mathcal{H}_2}(\lambda^2\log\lambda)$ in (8.8) are $\mathcal{R}_{\rm em}(\lambda)$s. What remains is equal to $-h_1(\lambda)^{-1}M_v\widetilde{\mathcal{D}}_0(\lambda)S_1D_1S_1\widetilde{\mathcal{D}}_0(\lambda)M_v \equiv -h_1(\lambda)^{-1}M_vS_1D_1S_1M_v$ (modulo $\mathcal{GVS}$). Thus, if $S_1D_1S_1 = \sum_{j,k=1}^n c_{jk}(\zeta_j\otimes\zeta_k)$ is the matrix representation of $S_1D_1S_1$ via the basis $\{\zeta_1,\dots,\zeta_n\}$ of S_1L^2,

$$M_v\mathcal{N}_1(\lambda)M_v \equiv -\sum_{j,k=1}^n h_1(\lambda)^{-1}c_{jk}(v\zeta_j)\otimes(v\zeta_k),$$

and the lemma follows from Lemma 7.5 since $\int_{\mathbb{R}^4} v(x)\zeta_k(x)dx = 0$. ■

The following lemma is the clue to the proof of Theorem 1.9. Let $\{\zeta_1,\dots,\zeta_n\}$ be the orthonormal basis of S_1L^2 such that $\{\zeta_1,\dots,\zeta_m\}$ is the basis of the subspace S_2L^2 which is spanned by eigenfunctions of T_2 (recall (6.9)):

$$T_2\zeta_j = ia_j^2\zeta_j, \quad a_j>0,\ j=1,\dots,m.$$

Lemma 8.5. *Suppose that H has singularity of the second kind at zero. Then,*

$$\mathcal{Q}_v(\lambda) \equiv -i\lambda^{-2} \sum_{j=1}^{m} a_j^{-2} (v\zeta_j) \otimes (v\zeta_j) + h_1(\lambda)^{-1} \sum_{j,k=1}^{n} a_{jk}(\lambda)(v\zeta_j) \otimes (v\zeta_k) \quad (8.17)$$

modulo GPR, where $a_{jk}(\lambda)$, $j,k = 1,\dots,n$ are GMU.

Proof. By virtue of Lemmas 7.3 and 8.4, it suffices to prove (8.17) for $M_v \mathcal{N}_2(\lambda) M_v$ in place of $\mathcal{Q}_v(\lambda)$. We substitute (8.8) for $(\widetilde{B}_1(\lambda) + S_2)^{-1}$ and (8.12) for $B_2(\lambda)^{-1}$ in (8.14), use $S_2 D_1 = D_1 S_2 = S_2$ and express the result via the basis $\{\zeta_j\}$ of $S_1 L^2$ chosen as above. We obtain that with GMUs $\{a_{ik}(\lambda)\}_{j,k=1,\dots,n}$

$$S_1 J_2(\lambda) S_1 = i \sum_{j=1}^{m} a_j^{-2} \lambda^{-2} h_1(\lambda) \zeta_j \otimes \zeta_j + \sum_{i,k=1}^{n} a_{ik}(\lambda) \zeta_i \otimes \zeta_k. \quad (8.18)$$

We then substitute (7.4) with (7.5) for $(\mathcal{M}(\lambda^4) + S_1)^{-1}$ and (8.18) for $S_1 J_2(\lambda) S_1$ in (8.16) and expand the result. Then, $Y_1(\lambda) = \mathcal{O}^{(4)}_{\mathcal{H}_2}(\lambda^2)$ in equation (7.5) for $(\mathcal{M}(\lambda^4) + S_1)^{-1}$ cancels the singularities in (8.18) and produces $\mathcal{R}_{\text{em}}(\lambda)$. Thus, we obtain that

$$\begin{aligned} M_v \mathcal{N}_2(\lambda) M_v &\equiv -h_1(\lambda)^{-1} M_v \widetilde{\mathcal{D}}_0(\lambda) \times (8.18) \times \widetilde{\mathcal{D}}_0(\lambda) M_v \\ &\equiv -\sum_{j=1}^{m} i a_j^{-2} \lambda^{-2} (v\zeta_j) \otimes (v\zeta_j) + \sum_{i,k=1}^{n} a_{ik}(\lambda) h_1(\lambda)^{-1} (v\zeta_i) \otimes (v\zeta_k), \end{aligned}$$

where we used $\widetilde{\mathcal{D}}_0(\lambda) = D_0 + h_1(\lambda) L_0$, $D_0 = D_0^*$, $S_1 D_0 = D_0 S_1 = S_1$, $S_2 D_0 = D_0 S_2 = S_2$ and $S_2 L_0 = L_0 S_2 = 0$. Lemma 8.5 follows. ∎

8.2. Proof of Theorem 1.9 (2)

We follow the argument outlined in the introduction which patterns after the proof of [35, Theorem 5.13]. We shall, however, need some new estimates at the end of the proof. By virtue of Lemmas 7.5 and 8.5, we need to study only

$$\Omega_{\text{red}}\, u = \sum_{j=1}^{m} i a_j^{-2} \int_0^\infty R_0^+(\lambda^4)(v\zeta_j) \otimes (v\zeta_j) \Pi(\lambda) u \lambda \chi_{\le a}(\lambda) d\lambda.$$

We first deal with the terms with $j = 1,\dots,m$ separately, omitting the index j and the constant $i a_j^{-2}$:

$$\Omega u := \int_0^\infty (R_0^+(\lambda^4)(v\zeta) \otimes (v\zeta) \Pi(\lambda) u) \lambda \chi_{\le a}(\lambda) d\lambda, \quad \zeta \in S_2 L^2. \quad (8.19)$$

Since $\int_{\mathbb{R}^4} v(x)\zeta(x)dx = 0$, we may, as before, replace $\Pi(\lambda)u(z)$ by $\Pi(\lambda)u(z) - \Pi(\lambda)u(0)$ in (8.19), which we now express in the form (1.41) and denote the operators produced by the first and the second terms of (1.41) by Ω_B and Ω_G respectively. Then,

$$\Omega u = \Omega_B u + \Omega_G u$$

and we call Ω_G and Ω_B the *good* and the *bad parts* of Ω, respectively. Recall that $\zeta(x) = -w(x)\varphi(x)$ with φ being p-wave resonance, see Lemma 1.8 (1) and (3).

Good part is GOP

Lemma 8.6. *The good part Ω_G is a GOP.*

Proof. Let $T_{m,l}(x, y) = (v\zeta)(x)y_m y_l(v\zeta)(y)$ and $u_{m,l} = R_m R_l u$ for $1 \le j, l \le 4$. Then, $T_{m,l} \in \mathcal{L}^1$ and $\Omega_G u$ becomes the superposition by $\sum_{m,l=1}^4 \int_0^1 (1-\theta)d\theta$ of

$$\int_{\mathbb{R}^4\times\mathbb{R}^4} \left(\int_0^\infty \mathcal{R}_\lambda(x-y)T_{m,l}(y,z)\Pi(\lambda)(\tau_{-\theta z}\chi_{\le a}(|D|)u_{m,l})(0)\lambda^3 d\lambda \right) dzdy$$

$$= \int_{\mathbb{R}^4\times\mathbb{R}^4} T_{m,l}(y,z)\tau_y \left(\int_0^\infty \mathcal{R}_\lambda(x)\Pi(\lambda)(\tau_{-\theta z}\chi_{\le a}(|D|)u_{m,l})(0)\lambda^3 d\lambda \right) dzdy$$

$$= \int_{\mathbb{R}^4\times\mathbb{R}^4} T_{m,l}(y,z)\tau_y K(\tau_{-\theta z}\chi_{\le a}(|D|)u_{m,l})(x)dzdy.$$

Lemma 3.5 and Minkowski's inequality then imply that Ω_G is GOP. ■

Remark 8.7. The proof shows that Lemma 8.6 holds if $\zeta \otimes \zeta$ is replaced by $a \otimes \zeta$ such that $a(x)v(x) \in L^1(\mathbb{R}^4)$ and $\zeta \in QL^2$.

High energy part of the bad part. Since $\sum_{l=1}^4 i\lambda z_l(\Pi(\lambda)R_l u)(0)$, the first term of (1.41), is $\mathcal{VS}$, $\Omega_B u(x)$ becomes the sum of products

$$\Omega_B u(x) = \sum_{l=1}^4 i\langle z_l v, \zeta\rangle \Omega_{B,l}u(x),$$

$$\Omega_{B,l}u(x) = \int_0^\infty R_0^+(\lambda^4)(v\zeta)(x)(\Pi(\lambda)R_l u)(0)\lambda^2\chi_{\le a}(\lambda)d\lambda.$$

Ignoring the harmless constant $i\langle z_l v, \zeta\rangle$ and Riesz transforms R_j, we consider

$$W_B u(x) = \int_0^\infty (R_0^+(\lambda^4)\omega)(x)(\Pi(\lambda)u)(0)\lambda^2\chi_{\le a}(\lambda)d\lambda, \tag{8.20}$$

where $\omega(x) = v(x)\zeta(x)(= -V(x)\varphi(x))$ with $\zeta \in S_2L^2 \setminus \{0\}$ (and p-wave resonance $\varphi(x)$). Difficulty here is of course that (8.20) has only λ^2 instead of λ^3. We decompose

$$W_B u = \chi_{\geq 4a}(|D|)W_B u + \chi_{\leq 4a}(|D|)W_B u$$

and move $\chi_{\geq 4a}(|D|)$ and $\chi_{\leq 4a}(|D|)$ to the inside of the integral in (8.20). We first consider $\chi_{\geq 4a}(|D|)W_B u$ which is equal to (8.20) with $\chi_{\geq 4a}(|D|)R_0^+(\lambda^4)\omega(x)$ in place of $R_0^+(\lambda^4)\omega(x)$. Let $\mu_a(\xi) = \chi_{\geq 4a}(\xi)|\xi|^{-4}$. We have $\mu_a \in L^p(\mathbb{R}^4)$ for $1 < p \leq \infty$.

Lemma 8.8. *We have $\widehat{\mu_a}(x) \in L^p(\mathbb{R}^4)$ for $1 \leq p < \infty$. For all $1 \leq p \leq \infty$, $\mu_a(|D|) \in \mathbf{B}(L^p(\mathbb{R}^4))$ and $\mu_a(|D|)\omega \in L^p(\mathbb{R}^4)$*

Proof. Since $\mu_a \in C^\infty(\mathbb{R}^4)$ and $|\partial^\alpha \mu_a(\xi)| \leq C_\alpha \langle \xi \rangle^{-4-|\alpha|}$, integration by parts shows that $\widehat{\mu_a} \in C^\infty(\mathbb{R}^4 \setminus \{0\})$ and is rapidly decreasing at infinity along with derivatives; for the small $|x|$ behavior, we observe that $\widehat{\mu_a}$ is equal modulo a smooth function to

$$\frac{1}{(2\pi)^2}\int_a^\infty \left(\int_{\mathbb{S}^3} e^{irx\omega}d\omega\right)\frac{dr}{r} = \int_a^\infty \frac{J_1(r|x|)}{r^2|x|}dr$$

and the well-known property of the Bessel function implies the right side is equal to $C \log|x| + O(|x|^2)$ as $|x| \to 0$. Thus, $\widehat{\mu_a}(x) \in L^p(\mathbb{R}^4)$ for all $1 \leq p < \infty$ and $\mu_a(|D|)$ is bounded in $L^p(\mathbb{R}^4)$ for all $1 \leq p \leq \infty$. Since

$$\langle x\rangle^4\langle \log|x|\rangle v \in (L^2 \cap L^{2q})(\mathbb{R}^4),$$
$$\langle x\rangle^5\langle \log|x|\rangle \omega \in (L^1 \cap L^{\frac{2q}{q+1}})(\mathbb{R}^4)$$

and

$$\mu_a(|D|)\omega(x) = (2\pi)^{-2}(\widehat{\mu_a} * \omega)(x) \in L^p(\mathbb{R}^4) \quad \text{for all } 1 \leq p \leq \infty. \qquad \blacksquare$$

Lemma 8.9. *Let $\zeta \in S_2L^2 \setminus \{0\}$. The operator $\chi_{\geq 4a}(|D|)W_B$ is bounded in $L^p(\mathbb{R}^4)$ for $1 < p < 4$ and, if $a > 0$ is sufficiently small, it is unbounded for $4 \leq p \leq \infty$.*

Proof. By Fourier transform, we have

$$\begin{aligned}\chi_{\geq 4a}(|D|)R_0^+(\lambda^4)\omega(x) &= \mathcal{F}^*\left(\mu_a(\xi)\hat{\omega}(\xi) + \lambda^4\frac{\mu_a(\xi)\hat{\omega}(\xi)}{|\xi|^4 - \lambda^4 - i0}\right)\\ &= \mu_a(|D|)\omega(x) + \mu_a(|D|)\lambda^4 R_0^+(\lambda^4)\omega(x). \qquad (8.21)\end{aligned}$$

Accordingly, $\chi_{\geq 4a}(|D|)W_B u(x)$ becomes the sum $W_{B,\geq}^{(1)}u(x) + W_{B,\geq}^{(2)}u(x)$:

$$W_{B,\geq}^{(1)}u(x) := \mu_a(|D|)\omega(x)\int_0^\infty \Pi(\lambda)u(0)\lambda^2\chi_{\leq a}(\lambda)d\lambda, \qquad (8.22)$$

$$W_{B,\geq}^{(2)}u(x) := \int_0^\infty \mu_a(|D|)R_0^+(\lambda^4)\omega(x)\Pi(\lambda)u(0)\lambda^6\chi_{\leq a}(\lambda)d\lambda.$$

(1) Let $\nu(\lambda) = \lambda^3\chi_{\leq a}(\lambda)$. Then, $\nu(\lambda)$ is GMU and (1.21) implies

$$\begin{aligned} W_{B,\geq}^{(2)}u(x) &= \mu_a(|D|)\int_{\mathbb{R}^4} \omega(y)\tau_y\Big(\int_0^\infty \mathcal{R}_\lambda(x)\Pi(\lambda)\nu(|D|)u(0)\lambda^3 d\lambda\Big)dy \\ &= \mu_a(|D|)\int_{\mathbb{R}^4} \omega(y)(\tau_y K\nu(|D|)u)(x)dy. \end{aligned}$$

Minkowski's inequality and Lemma 3.5 then imply that $W_{B,\geq}^{(2)}$ is GOP.

(2) Let $\ell(u)$ be the linear functional defined by

$$\ell(u) = \int_0^\infty \Pi(\lambda)u(0)\lambda^2\chi_{\leq a}(\lambda)d\lambda.$$

Then, $W_{B,\geq}^{(1)}u(x) = \mu_a(|D|)\omega(x)\ell(u)$ by (8.22) and $\mu_a(|D|)\omega(x) \in L^p(\mathbb{R}^4)$ for all $1 \leq p \leq \infty$ by Lemma 8.8. It follows that, if $\mu_a(|D|)\omega(x) \neq 0$, $W_{B,\geq}^{(1)}$ is bounded in $L^p(\mathbb{R}^4)$ if and only if the functional $\ell(u)$ is bounded on $L^p(\mathbb{R}^4)$. By using polar coordinates $\xi = \lambda\omega$ and the Parseval identity, we obtain

$$\begin{aligned} \ell(u) &= \frac{1}{(2\pi)^2}\int_0^\infty\int_{\mathbb{S}^3} \hat{u}(\lambda\omega)\chi_{\leq a}(\lambda)\lambda^2 d\omega d\lambda = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4}\hat{u}(\xi)\frac{\chi_{\leq a}(|\xi|)}{|\xi|}d\xi \\ &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} u(x)f(x)dx, \quad f(x) = \mathcal{F}\Big(\frac{\chi_{\leq a}(|\xi|)}{|\xi|}\Big), \end{aligned} \tag{8.23}$$

and $f \in L^q(\mathbb{R}^4)$ if and only if $4/3 < q \leq \infty$. Hence, $\ell(u)$ is bounded on $L^p(\mathbb{R}^4)$ for $1 \leq p < 4$ and is unbounded for $4 \leq p \leq \infty$. Thus, the proof is finished if we prove $\mu_a(|D|)\omega \neq 0$ for some $a > 0$. However, if $\mu_a(|D|)\omega = 0$ for all $a > 0$, then it must be that $\omega = v\zeta = 0$ and, as $T_0\zeta = 0$ for $\zeta \in S_2L^2$, $\Phi(\zeta) = 0$ for the Φ of Lemma 1.8 (1), hence $\zeta = 0$. This is a contradiction and the lemma is proved. ■

Proof of the negative part of Theorem 1.9 (2)

Lemma 8.10. *If H has singularity of the second kind at zero, then W_- is unbounded in $L^p(\mathbb{R}^4)$ for $4 \leq p \leq \infty$*

Proof. We prove the lemma when rank $S_2 = 1$ and $S_1 = \zeta \otimes \zeta$. A modification of the general case by using Hahn–Banach theorem can be done by following the argument in [34, part (iv) of the proof of Theorem 1.4 (2b)], which we omit here. We remark that $v\zeta \neq 0$ as was shown in the proof of Lemma 8.9.

We prove the lemma by reductio ad absurdum. Suppose W_- is bounded in $L^p(\mathbb{R}^4)$ for a $4 \le p \le \infty$. Then, so must be Ω in (8.19) for all $0 < a < \infty$ and, since Ω_G is GOP by Lemma 8.6, so must be $\Omega_{B,\ge} = \chi_{\ge 4a}(|D|)\Omega_B$. Then, since $\chi_{\ge 4a}(|D|)W_B^{(2)}$ is GOP by part (1) of the proof of Lemma 8.9, we conclude that

$$\Omega_{B,\ge}^{(1)} = \sum_{l=1}^{4} \langle v z_l, \zeta\rangle W_{B,\ge}^{(1),l} u(x) = \mu_a(|D|)(v\zeta)(x)\tilde{\ell}(u),$$

must also be bounded in $L^p(\mathbb{R}^4)$ for the p, where $W_{B,\ge}^{(1),l} u = W_{B,\ge}^{(1)} R_l u$ (see (8.22)) and

$$\tilde{\ell}(u) = \Big\langle u, \sum_{l=1}^{4} \langle v z_l, \zeta\rangle f_l(x)\Big\rangle, \quad f_l(x) = \mathcal{F}\Big(\frac{\xi_l \chi_{\le a}(|\xi|)}{|\xi|^2}\Big)(x).$$

For sufficiently small $a > 0$, we have $\mu_a(|D|)(v\zeta) \neq 0$. By virtue of Lemma 6.2 (2), $\alpha := (\langle v z_1, \zeta\rangle, \dots, \langle v z_4, \zeta\rangle) \neq 0$ and, hence, $\alpha \cdot \xi$ is non-trivial linear function of ξ. It follows that

$$\sum_{l=1}^{4} \langle v z_l, \zeta\rangle f_l(x) = \mathcal{F}(\alpha \cdot \xi |\xi|^{-2} \chi_{\le a}(|\xi|)) \notin L^q(\mathbb{R}^4)$$

for any $1 \le q \le 4/3$. Thus, $\tilde{\ell}$ is unbounded on $L^p(\mathbb{R}^4)$ for any $4 \le p \le \infty$ by the Riesz theorem. This is a contradiction. ■

Low energy part of the bad part. We recall that $W_B u$ is defined by (8.20) with $\omega = v\zeta$, ζ being in $S_2 L^2$. The following lemma completes the proof of Theorem 1.9 (2). A part of the proof will be postponed to the appendix.

Lemma 8.11. *Let $\zeta \in S_2 L^2$ and $a > 0$. Then, $\chi_{\le 4a}(|D|)W_B$ is bounded in $L^p(\mathbb{R}^4)$ for* $1 < p < 4$.

Proof. Let $\rho(\lambda) = \Pi(\lambda)u(0)\lambda^2\chi_{\le a}(\lambda)$ and, for $\varepsilon > 0$,

$$W_{B,\le}^{\varepsilon} u(x) = \int_0^\infty \chi_{\le 4a}(|D|) R_0(\lambda^4 + i\varepsilon)\omega(x)\rho(\lambda)d\lambda.$$

Then, $\rho \in C_0^\infty((0,\infty))$ and

$$\mathcal{F}(W_{B,\le}^{\varepsilon} u)(\xi) = \chi_{\le 4a}(|\xi|)\hat{\omega}(\xi) \int_0^\infty \frac{\rho(\lambda)}{\xi^4 - \lambda^4 - i\varepsilon} d\lambda. \tag{8.24}$$

It is evident that

$$W_{B,\le}u = \lim_{\varepsilon\downarrow 0} W^{\varepsilon}_{B,\le}u \quad \text{in } L^2.$$

Since $\hat\omega(0) = 0$, Taylor's formula implies

$$\hat\omega(\xi) = \sum_{m=1}^{4} \xi_m \widehat{\omega_m}(\xi),\ \widehat{\omega_m}(\xi) = \frac{1}{(2\pi)^2}\int_0^1\int_{\mathbb{R}^4} e^{-i\theta z\xi} i z_m \omega(z)dzd\theta. \tag{8.25}$$

We substitute (8.25) in (8.24) and apply the inverse Fourier transform. By changing the order of integrations, we obtain

$$W^{\varepsilon}_{B,\le}u(x) = \sum_{m=1}^{4} R_m \int_0^\infty \Big(\frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} \frac{e^{ix\xi}\chi_{\le 4a}(|\xi|)|\xi|}{|\xi|^4-\lambda^4-i\varepsilon}\widehat{\omega_m}(\xi)d\xi\Big)\rho(\lambda)d\lambda,$$

where $R_m, 1 \le m \le 4$ are Riesz transforms. On substituting

$$\frac{|\xi|}{|\xi|^4-\lambda^4-i\varepsilon} = \frac{\lambda}{|\xi|^4-\lambda^4-i\varepsilon} + \frac{|\xi|-\lambda}{|\xi|^4-\lambda^4-i\varepsilon},$$

$W^{\varepsilon}_{B,\le}u(x)$ becomes

$$W^{\varepsilon}_{B,\le}u(x) = \mathcal{Z}^{\varepsilon}_1 u(x) + \mathcal{Z}^{\varepsilon}_2 u(x),$$

where the definitions of $\mathcal{Z}^{\varepsilon}_1$ and $\mathcal{Z}^{\varepsilon}_2$ are obvious. We have

$$\mathcal{Z}^{\varepsilon}_1 u(x) = \sum_{m=1}^{4} R_m \int_0^\infty \Big(\frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} \frac{e^{ix\xi}\chi_{\le 4a}(|\xi|)}{|\xi|^4-\lambda^4-i\varepsilon}\widehat{\omega_m}(\xi)d\xi\Big)\lambda\rho(\lambda)d\lambda.$$

Substitute (8.25) for $\widehat{\omega_m}(\xi)$, change the order of integrations, and integrate by $d\xi d\lambda$ first. As $\varepsilon \to 0$, $\mathcal{Z}^{\varepsilon}_1 u(x)$ converges in $L^2(\mathbb{R}^m)$ to

$$\sum_{m=1}^{4}\int_0^1 R_m \int_{\mathbb{R}^4} i z_m \omega(z)\tau_{\theta z}\Big(\int_0^\infty \mathcal{R}_\lambda(x)\Pi(\lambda)u(0)\lambda^3\chi_{\le a}(\lambda)d\lambda\Big)dzd\theta$$
$$= \sum_{m=1}^{4}\int_0^1 \Big(R_m\int_{\mathbb{R}^4}(iz_m\omega(z))\tau_{\theta z}K\chi_{\le a}(|D|)u(x)dz\Big)d\theta =: \mathcal{Z}_1 u(x).$$

Lemma 3.5 and Minkowski's inequality imply that $\mathcal{Z}_1$ is GOP.

Computing as before, we obtain

$$\mathcal{Z}^{\varepsilon}_2 u(x) = \sum_{m=1}^{4} R_m \int_0^\infty \Big(\frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} \frac{e^{ix\xi}\chi_{\le 4a}(|\xi|)(|\xi|-\lambda)}{|\xi|^4-\lambda^4-i\varepsilon}\widehat{\omega_m}(\xi)d\xi\Big)\rho(\lambda)d\lambda.$$

For $\lambda > 0$ and $\varepsilon > 0$, we have

$$\frac{|\xi| - \lambda}{|\xi|^4 - \lambda^4 - i\varepsilon} \leq_{|\cdot|} \frac{1}{(|\xi| + \lambda)(|\xi|^2 + \lambda^2)}$$

and, as $\varepsilon \to 0$, the left side converges to the right side for all (ξ, λ), $|\xi| \neq \lambda$. Thus, as $\varepsilon \to 0$, $Z_2^\varepsilon u(x)$ converges in $L^2(\mathbb{R}^4)$ to

$$Z_2 u(x) = \sum_{m=1}^{4} R_m \int_0^\infty \Big(\frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} \frac{e^{ix\xi} \chi_{\leq 4a}(|\xi|) \widehat{\omega_m}(\xi) d\xi}{(|\xi| + \lambda)(|\xi|^2 + \lambda^2)} \Big) \rho(\lambda) d\lambda.$$

We substitute (8.25) for $\widehat{\omega_m}(\xi)$, change the order of integrations, and integrate by $d\xi d\lambda$ first. This yields that

$$Z_2 u(x) = \sum_{m=1}^{4} \int_0^1 \Big(\frac{1}{(2\pi)^4} \int_{\mathbb{R}^4} i z_m \omega(z) (\tau_{\theta z} R_m L u)(x) dz \Big) d\theta, \tag{8.26}$$

where L is the integral operator defined by

$$Lu(x) = \int_0^\infty \Big(\int_{\mathbb{R}^4} e^{ix\xi} \frac{\chi_{\leq 4a}(|\xi|)}{(|\xi|^2 + \lambda^2)(|\xi| + \lambda)} d\xi \Big) \Pi(\lambda) u(0) \lambda^2 \chi_{\leq a}(\lambda) d\lambda.$$

We substitute (1.15) for $\Pi(\lambda)u(0)$, use polar coordinates $\eta = \lambda\omega$, and change the order of integrations. The result is that L is the integral operator with kernel

$$L(x, y) = \iint_{\mathbb{R}^8} \frac{e^{ix\xi + iy\eta} \chi_{\leq 4a}(|\xi|) \chi_{\leq a}(|\eta|)}{(|\xi|^2 + |\eta|^2)(|\xi| + |\eta|)|\eta|} d\xi d\eta. \tag{8.27}$$

We shall prove the following lemma in the appendix and take it for granted for the moment.

Lemma 8.12. *The operator L is bounded in $L^p(\mathbb{R}^4)$ for $1 < p < 4$.*

We apply Minkowski's inequality and Lemma 8.12 to (8.26) and obtain

$$\|Z_2 u\|_p \leq C \|\langle x \rangle \omega\|_1 \|u\|_p \quad \text{for } 1 < p < 4,$$

This completes the proof of Lemma 8.11 since $W_{B,\leq} u = Z_1 u + Z_2 u$. ∎

9. Singularities of third and fourth kinds

We prove here Theorem 1.9 (3) and (4), assuming $\langle x \rangle^{12} \langle \log |x| \rangle^2 V \in (L^1 \cap L^q)(\mathbb{R}^4)$ for a $q > 1$. We have a sequence of projections $Q \supset S_1 \supset S_2 \supset S_3 \supset S_4$. We take the basis $\{\zeta_1, \ldots, \zeta_n\}$ of $S_1 L^2$ such that $\{\zeta_1, \ldots, \zeta_m\}$, $m \le n$, spans $S_2 L^2$ and, if $T_3 = S_3 G_4^{(v)} S_3 \neq 0$,

$$T_2 \zeta_j = -i a_1^2 \zeta_j,\ 1 \le j \le r < m; \quad T_2 \zeta_j = 0,\ r+1 \le j \le m; \tag{9.1}$$

$\{\zeta_{r+1}, \ldots, \zeta_m\}$ is the basis of $S_3 L^2$. Recall $T_4 = S_4 G_{4,l}^{(v)} S_4$; T_4 is non-singular in $S_4 L^2$.

9.1. Threshold analysis 3. First step

By virtue of Lemmas 7.3 and 8.4,

$$\mathcal{Q}_v(\lambda) \equiv M_v \mathcal{N}_2(\lambda) M_v =: \mathcal{N}_2^{(v)}(\lambda) \quad \text{modulo GPR}$$

and we study $\mathcal{N}_2^{(v)}(\lambda)$ as $\lambda \to 0$. Recall $\mathcal{N}_2(\lambda)$ is given by (8.16) with (8.14) and we need to study $B_2(\lambda)^{-1}$. We have from (8.11) that

$$B_2(\lambda) = -\lambda^2 h_1(\lambda)^{-1} \widetilde{B}_2(\lambda), \quad \widetilde{B}_2(\lambda) = T_2 + \mathcal{O}^{(4)}_{S_2 L^2}(\lambda^2 \log \lambda)$$

We apply Lemma 7.1 to the pair $(\widetilde{B}_2(\lambda), S_3)$. Since $(T_2 + S_3)^{-1}$ exists in $S_2 L^2$, so does $(\widetilde{B}_2(\lambda) + S_3)^{-1}$ for small $\lambda > 0$ and

$$(\widetilde{B}_2(\lambda) + S_3)^{-1} = D_2 + \mathcal{O}^{(4)}_{S_2 L^2}(\lambda^2 \log \lambda), \quad D_2 = (T_2 + S_3)^{-1}. \tag{9.2}$$

Lemma 7.1 implies that, if $B_3(\lambda) = S_3 - S_3(\widetilde{B}_2(\lambda) + S_3)^{-1} S_3$ is invertible in $S_3 L^2$, then

$$\widetilde{B}_2(\lambda)^{-1} = (\widetilde{B}_2(\lambda) + S_3)^{-1} + (\widetilde{B}_2(\lambda) + S_3)^{-1} S_3 B_3(\lambda)^{-1} S_3 (\widetilde{B}_2(\lambda) + S_3)^{-1}. \tag{9.3}$$

On substituting (9.3) $\times (-\lambda^{-2} h_1(\lambda))$ for $B_2(\lambda)^{-1}$ in (8.14) we obtain

$$J_2(\lambda) = J_{2,1}(\lambda) + J_{2,2}(\lambda)$$

where $J_{2,1}(\lambda)$ and $J_{2,2}(\lambda)$ are equal, respectively, to

$$\begin{aligned} J_{2,1}(\lambda) &= S_1(\widetilde{B}_1(\lambda) + S_2)^{-1} S_2 (\widetilde{B}_2(\lambda) + S_3)^{-1} S_2 (\widetilde{B}_1(\lambda) + S_2)^{-1} S_1, \\ J_{2,2}(\lambda) &= S_1(\widetilde{B}_1(\lambda) + S_2)^{-1} S_2 (\widetilde{B}_2(\lambda) + S_3)^{-1} \\ &\qquad \times S_3 B_3(\lambda)^{-1} S_3 (\widetilde{B}_2(\lambda) + S_3)^{-1} S_2 (\widetilde{B}_1(\lambda) + S_2)^{-1} S_1. \end{aligned}$$

Here, we have placed S_1 on both ends of $J_{2,1}(\lambda)$ and $J_{2,2}(\lambda)$, which is allowed since $(\widetilde{B}_1(\lambda) + S_2)^{-1}$ is an operator in $S_1 L^2$ and, accordingly, we have $\mathcal{N}_2^{(v)}(\lambda) = \mathcal{N}_{2,1}^{(v)}(\lambda) + \mathcal{N}_{2,2}^{(v)}(\lambda)$ where, for $j = 1, 2$

$$\mathcal{N}_{2,j}^{(v)}(\lambda) = \lambda^{-2} M_v (\mathcal{M}(\lambda^4) + S_1)^{-1} J_{2,j}(\lambda) (\mathcal{M}(\lambda^4) + S_1)^{-1} M_v. \tag{9.4}$$

We first prove the following lemma which is irrelevant to the existence of $B_3(\lambda)^{-1}$.

Lemma 9.1. *The following statements hold.*

(1) *There exists* $\beta_{jk}(\lambda) \in \mathcal{O}^{(4)}_{\mathbb{C}}(1)$, $1 \le j,k \le n$, *such that*

$$\mathcal{N}^{(v)}_{2,1}(\lambda) \equiv -i\sum_{j=1}^{m} a_j^{-2}\lambda^{-2}(v\zeta_j)\otimes(v\zeta_j) + \sum_{j,k=1}^{n} h_1(\lambda)^{-1}\beta_{jk}(\lambda)(v\zeta_j)\otimes(v\zeta_k).$$

(2) *The operator produced by* (5.1) *with* $\mathcal{N}^{(v)}_{2,1}(\lambda)$ *in place of* $\mathcal{Q}_v(\lambda)$ *is bounded in* $L^p(\mathbb{R}^4)$ *for* $1 < p < 4$ *and unbounded for* $4 \le p \le \infty$.

Proof. Since $J_{2,1}(\lambda)$ has S_1 on both sides, (8.8) and (9.2) imply

$$J_{2,1}(\lambda) = S_2D_2S_2 + \mathcal{O}^{(4)}_{S_1L^2}(\lambda^2 h_1(\lambda)^{-1}). \tag{9.5}$$

We substitute (9.5) for $J_{2,1}(\lambda)$ and (7.4) for $(\mathcal{M}(\lambda^4) + S_1)^{-1}$ in (9.4) and expand the result. Then, $Y_1(\lambda) = \mathcal{O}^{(4)}_{\mathcal{H}_2}(\lambda^2)$ cancels the singularities and, modulo $\mathcal{GVS} + \mathcal{R}_{\rm em}(\lambda)$,

$$\mathcal{N}^{(v)}_{2,1}(\lambda) \equiv \lambda^{-2}M_v\widetilde{\mathcal{D}}_0(\lambda)J_{2,1}(\lambda)\widetilde{\mathcal{D}}_0(\lambda)M_v.$$

Then, since $\widetilde{\mathcal{D}}_0(\lambda) = D_0 + h_1(\lambda)L_0$ and $S_1D_0 = D_0S_1 = S_1$,

$$\begin{aligned}\mathcal{N}^{(v)}_{2,1}(\lambda) \equiv{}& \lambda^{-2}M_vJ_{2,1}(\lambda)M_v\\ &+ \lambda^{-2}h_1(\lambda)M_v(L_0J_{2,1}(\lambda) + J_{2,1}(\lambda)L_0)M_v\\ &+ \lambda^{-2}h_1(\lambda)^2M_vL_0J_{2,1}(\lambda)L_0.\end{aligned}$$

Here the first line on the right-hand side is of the desired form by virtue of (9.5) and the second and the third line produce $\mathcal{GVS}$ since $S_2L_0 = L_0S_2 = 0$. Thus, statement (1) follows.

(2) By virtue of (1), $\mathcal{N}^{(v)}_{2,1}$ has the same form as (8.17). Hence, it produces the operator which is bounded $L^p(\mathbb{R}^4)$ for $1 < p < 4$ and unbounded for $4 \le p$ as was shown in the proof of Theorem 1.9 (2). ■

Corollary 9.2. *If H has singularities of the third or fourth kind at zero, then, modulo the operator which is bounded in $L^p(\mathbb{R}^4)$ for $1<p<4$ and unbounded for $4\le p\le\infty$, $W_-\chi_{\le a}(|D|)$ is equal to $\mathcal{Z}$ which is defined by*

$$\mathcal{Z}u := \int_0^\infty R_0(\lambda^4)\mathcal{N}^{(v)}_{2,2}(\lambda)\Pi(\lambda)u\lambda^3\chi_{\le a}(\lambda)u\,d\lambda. \tag{9.6}$$

9.2. Key lemma

To study $\mathcal{N}^{(v)}_{2,2}(\lambda)$, we use the following lemma. We use in this section only the result that $B_3(\lambda)^{-1} = \mathcal{O}^{(4)}_{S_3L^2}(\lambda^{-2}h_1(\lambda))$ or $B_3(\lambda)^{-1} = \mathcal{O}^{(4)}_{S_3L^2}(\lambda^{-2})$ in the respective cases.

Lemma 9.3. *The following statements hold.*

(1) *If H has singularity of the third kind, then*

$$B_3(\lambda)^{-1} = \lambda^{-2}\tilde{g}_2(\lambda)^{-1}S_3T_3^{-1}S_3 + \mathcal{O}^{(4)}_{S_3L^2}(\lambda^{-2}(\log\lambda)^{-2}). \tag{9.7}$$

(2) *If the singularity is of the fourth kind, then*

$$B_3(\lambda)^{-1} = \lambda^{-2}S_4T_4^{-1}S_4 + \mathcal{O}^{(4)}_{S_3L^2}(\lambda^{-2}(\log\lambda)^{-1}). \tag{9.8}$$

(3) *If the singularity is of the fourth kind but d-wave resonances are absent from H, then modulo $\mathcal{O}^{(4)}_{S_3L^2}(\lambda^2(\log\lambda)^3)$*

$$B_3(\lambda)^{-1} \equiv \lambda^{-2}S_4T_4^{-1}S_4 + \mathcal{O}^{(4)}_{S_4L^2}((\log\lambda)^2). \tag{9.9}$$

For proving Lemma 9.3, we prepare a few lemmas.

Lemma 9.4. *The following statements hold.*

(1) *The following identities are satisfied by S_3:*

$$G_2^{(v)}S_3 = i(4^4\pi)^{-1}v\otimes S_3(x^2v), \quad S_3G_2^{(v)} = i(4^4\pi)^{-1}S_3(x^2v)\otimes v. \tag{9.10}$$

$$S_jG_2^{(v)}S_3 = S_3G_2^{(v)}S_j = 0, \ j = 0,1,2,3. \tag{9.11}$$

$$\widetilde{T}_1(\lambda)D_1S_3 = -h_1(\lambda)S_1T_0PG_2^{(v)}S_3. \tag{9.12}$$

$$S_2\widetilde{T}_1(\lambda)D_1S_3 = 0. \tag{9.13}$$

(2) *We have the following identities for S_4:*

$$G_2M_vS_4 = S_4M_vG_2 = 0, \tag{9.14}$$

$$\widetilde{T}_1(\lambda)S_4 = S_4\widetilde{T}_1(\lambda) = 0. \tag{9.15}$$

Proof. (1) Lemma 6.2 (2) evidently implies (9.10). Then, (9.11) follows since one has $S_jv = 0$, $j = 0,\dots,3$. Recall that $\widetilde{T}_1(\lambda) = S_1\widetilde{\mathcal{D}}_0(\lambda)G_2^{(v)}\widetilde{\mathcal{D}}_0(\lambda)S_1$. Then, $D_1S_2 = S_2$, $L_0S_2 = 0$ and (9.11) together imply

$$\widetilde{T}_1(\lambda)D_1S_3 = (S_1 + h_1(\lambda)S_1L_0)G_2^{(v)}S_3 = h_1(\lambda)S_1L_0G_2^{(v)}S_3.$$

Substitute (7.2) for L_0. Then, $QG_2^{(v)}S_2 = 0$ and $S_1D_0Q = S_1$ imply (9.12). Since $S_2T_0 = 0$ by (8.5), (9.13) follows from (9.12).

(2) Lemma 6.2 (3) implies (9.14). Since $S_4\widetilde{\mathcal{D}}_0(\lambda) = S_4(D_0 + h_1(\lambda)L_0) = S_4$, (9.15) follows from (9.14). ∎

The following lemma is a precise version of (9.2). The lemma is more than what necessary for the proof of Lemma 9.3; however, we need it in this form for that of Lemma 9.8.

Lemma 9.5. *Modulo* $\mathcal{O}^{(4)}_{S_2L^2}(\lambda^4 \log \lambda)$, *we have that*

$$(\widetilde{B}_2(\lambda) + S_3)^{-1} \equiv D_2 - D_2 F_3(\lambda) D_2 + F_{3,sq}(\lambda), \tag{9.16}$$

where $F_3(\lambda)$ *and* $F_{3,sq}(\lambda)$ *are given by*

$$\begin{aligned} F_3(\lambda) &= \lambda^2 S_2\{T_{4,l}(\lambda) - G_2^{(v)} \widetilde{\mathcal{D}}_0(\lambda) G_2^{(v)} + h_1(\lambda)^{-1}(\widetilde{T}_1(\lambda)D_1)^2\}S_2 \\ &\quad + \lambda^4 h_1^{-1}(\lambda)\tilde{g}_2(\lambda) S_2\{G_4^{(v)} S_1 G_2^{(v)} D_1 + G_2^{(v)} S_1 D_1 S_1 G_4^{(v)}\}S_2 \\ &\quad + \lambda^4 h_1(\lambda)^{-2} S_2(\widetilde{T}_1(\lambda)D_1)^3 S_2, \end{aligned} \tag{9.17}$$

$$F_{3,sq}(\lambda) = \lambda^4 D_2\{S_2(\tilde{g}_2(\lambda)G_4^{(v)} + h_1(\lambda)^{-1}(\widetilde{T}_1(\lambda)D_1)^2)S_2 D_2\}^2. \tag{9.18}$$

Proof. Expanding (8.10) to the third order, we have by (8.9) that

$$(\widetilde{B}_1(\lambda) + S_2)^{-1} = \sum_{j=0}^{3} D_1 L_1(\lambda)^j + \mathcal{O}^{(4)}_{S_1L^2}(\lambda^8 (\log\lambda)^4). \tag{9.19}$$

Since $S_2 D_1 = D_1 S_2 = S_2$, $B_2(\lambda) = S_2 - S_2(\widetilde{B}_1(\lambda) + S_2)^{-1} S_2$ becomes

$$B_2(\lambda) = -\sum_{j=1}^{3} S_2 L_1(\lambda)^j S_2 + \mathcal{O}^{(4)}_{S_2L^2}(\lambda^8 (\log\lambda)^4).$$

Recall (8.9), (8.3), and (7.6). We have

$$L_1(\lambda) \equiv \lambda^2 h_1(\lambda)^{-1}(A - \lambda^2 B),$$

modulo $\mathcal{O}_{S_1L^2}(\lambda^6 (\log\lambda)^2)$ where

$$A = \widetilde{T}_1(\lambda) D_1, \quad B = -S_1 \widetilde{\mathcal{D}}_0(\lambda)\{T_{4,l}(\lambda)\widetilde{\mathcal{D}}_0(\lambda) - (G_2^{(v)} \widetilde{\mathcal{D}}_0(\lambda))^2\}S_1.$$

Then, since $\widetilde{B}_2(\lambda) = -\lambda^{-2} h_1(\lambda) B_2(\lambda)$, we obtain by using $S_2 \widetilde{T}_1(\lambda) D_1 S_2 = T_2$ that

$$\begin{aligned} \widetilde{B}_2(\lambda) &\equiv T_2 + S_2\{(\lambda^2 B + \lambda^2 h_1(\lambda)^{-1} A^2) \\ &\qquad + \lambda^4 h_1(\lambda)^{-1}(AB + BA) + \lambda^4 h_1(\lambda)^{-1} A^3\}S_2 \end{aligned}$$

modulo $\mathcal{O}_{S_1L^2}(\lambda^6 \log\lambda)$. Then, identities in Lemma 8.1 and $D_0 S_1 = S_1 D_0 = S_1$ produce

$$\widetilde{B}_2(\lambda) = T_2 + \widetilde{F}_3(\lambda), \quad \widetilde{F}_3(\lambda) = F_3(\lambda) + \mathcal{O}^{(4)}_{S_2L^2}(\lambda^4 \log\lambda). \tag{9.20}$$

From (9.20) we deduce that $(\widetilde{B}_2(\lambda) + S_3)^{-1} = D_2(\mathbf{1}_{S_2L^2} + \widetilde{F}_3(\lambda) D_2)^{-1}$ and

$$\begin{aligned} (\widetilde{B}_2(\lambda) + S_3)^{-1} &\equiv D_2 - D_2 F_3(\lambda) D_2 + D_2(F_3(\lambda)D_2)^2 \\ &= D_2 - D_2 F_3(\lambda) D_2 + F_{3,sq}(\lambda) + \mathcal{O}^{(4)}_{S_2L^2}(\lambda^4 \log\lambda). \end{aligned}$$

modulo $\mathcal{O}^{(4)}_{S_2L^2}(\lambda^4 \log\lambda)$ as desired. ∎

Lemma 9.6. *Let* $\widetilde{L} = S_3 G_2^{(v)}(-L_0 + L_0 S_1 D_1 S_1 L_0) G_2^{(v)} S_3$ *and*

$$\mathcal{C}(\lambda) := T_3 + \tilde{g}_2(\lambda)^{-1} S_3 G_{4,l}^{(v)} S_3 + \tilde{g}_2(\lambda)^{-1} h_1(\lambda)\widetilde{L}.$$

Then, $B_3(\lambda) = S_3 - S_3(\widetilde{B}_2(\lambda) + S_3)^{-1} S_3$ *is equal to*

$$B_3(\lambda) = \lambda^2 \tilde{g}_2(\lambda)\mathcal{C}(\lambda) - \lambda^4 \tilde{g}_2(\lambda)^2 F(\lambda),$$
$$F(\lambda) := S_3(G_4^{(v)} S_2 D_2 S_2 G_4^{(v)}) S_3 + \mathcal{O}^{(4)}_{S_3 L^2}((\log \lambda)^{-1}).$$

Proof. On substituting (9.16), we have

$$B_3(\lambda) = S_3 F_3(\lambda) S_3 - S_3 F_{3,sq}(\lambda) S_3 + \mathcal{O}^{(4)}_{S_2 L^2}(\lambda^4 \log \lambda).$$

When sandwiched by S_3, the second line of (9.17) vanishes since (9.11) implies $Q G_2^{(v)} S_3 = S_3 G_2^{(v)} Q = 0$; the third line of (9.17) and the second term on the right of (9.18) become $\mathcal{O}^{(4)}_{S_2 L^2}(\lambda^4)$ since $S_3 \widetilde{T}_1(\lambda) D_1$ and $\widetilde{T}_1(\lambda) D_1 S_3$ are in $\mathcal{O}^{(4)}_{S_1 L^2}(h_1(\lambda))$. Hence, modulo $\mathcal{O}^{(4)}_{S_3 L^2}(\lambda^4 \log \lambda)$,

$$S_3 F_3(\lambda) S_3 \equiv \lambda^2 S_3(T_{4,l}(\lambda) + h_1(\lambda)\widetilde{L}) S_3,$$
$$S_3 F_{3,sq}(\lambda) S_3 \equiv \lambda^4 \tilde{g}_2(\lambda)^2 S_3(G_4^{(v)} S_2 D_2 S_2 G_4^{(v)}) S_3.$$

Recalling that $S_3 G_4^{(v)} S_3 = T_3$, we obtain the lemma. ■

Proof of Lemma 9.3 (1). If H has singularity of the third kind, then T_3 is invertible in $S_3 L^2$. It follows that

$$\mathcal{C}(\lambda) = (1_{S_3 L^2} + (\tilde{g}_2(\lambda)^{-1} S_3 G_{4,l}^{(v)} S_3 + \tilde{g}_2(\lambda)^{-1} h_1(\lambda)\widetilde{L}) T_3^{-1}) T_3$$

is invertible in $S_3 L^2$ for small $\lambda > 0$ and

$$\mathcal{C}(\lambda)^{-1} = T_3^{-1} - \tilde{g}_2(\lambda)^{-1} T_3^{-1} S_3 G_{4,l}^{(v)} S_3 T_3^{-1} + \mathcal{O}_{S_3 L^2}((\log \lambda)^{-2}).$$

Then, so is $B_3(\lambda) = \lambda^2 \tilde{g}_2(\lambda)(1 - \lambda^2 \tilde{g}_2(\lambda) F(\lambda)\mathcal{C}(\lambda)^{-1})\mathcal{C}(\lambda)$ and

$$B_3(\lambda)^{-1} = \lambda^{-2} \tilde{g}_2(\lambda)^{-1} \mathcal{C}(\lambda)^{-1} + \mathcal{C}(\lambda)^{-1} F(\lambda) \mathcal{C}(\lambda)^{-1} + \mathcal{O}_{S_3 L^2}(\lambda^2 \tilde{g}_2(\lambda)).$$

This implies the lemma. ■

Proof of Lemma 9.3 (2). If H has singularity of the fourth kind, then Lemma 9.6 remains to hold and $T_4 = S_4 G_{4,l}^{(v)} S_4$ is non-singular in $S_4 L^2$ ([14]). Let

$$S_4^{\perp} = S_3 \ominus S_4.$$

Lemma 9.7. *For small $\lambda > 0$, $\mathcal{C}(\lambda)^{-1}$ exists in S_3L^2 and*

$$\mathcal{C}(\lambda)^{-1} = \tilde{g}_2(\lambda)S_4T_4^{-1}S_4 + Z(\lambda), \quad Z(\lambda) = \mathcal{O}^{(4)}_{S_3L^2}(1) \tag{9.21}$$

and in the decomposition $S_3L^2 = S_4^{\perp}L^2 \oplus S_4L^2$

$$Z(\lambda) = \begin{pmatrix} d(\lambda) & -d(\lambda)S_4^{\perp}G^{(v)}_{4,l}S_4T_4^{-1} \\ -T_4^{-1}S_4G^{(v)}_{4,l}S_4^{\perp}d(\lambda) & T_4^{-1}S_4G^{(v)}_{4,l}S_4^{\perp}d(\lambda)S_4^{\perp}G^{(v)}_{4,l}S_4T_4^{-1} \end{pmatrix},$$
$$d(\lambda) = (S_4^{\perp}T_3S_4^{\perp})^{-1} + \mathcal{O}^{(4)}_{S_4^{\perp}L^2}((\log\lambda)^{-1}).$$

Proof. Since $G_2^{(v)}S_4 = S_4G_2^{(v)} = 0$ by (9.14), we have $\widetilde{L}S_4 = S_4\widetilde{L} = 0$ and, in the decomposition $S_3L^2 = S_4^{\perp} \oplus S_4L^2$,

$$\mathcal{C}(\lambda) = \begin{pmatrix} S_4^{\perp}\mathcal{C}(\lambda)S_4^{\perp} & \tilde{g}_2(\lambda)^{-1}S_4^{\perp}G^{(v)}_{4,l}S_4 \\ \tilde{g}_2(\lambda)^{-1}S_4G^{(v)}_{4,l}S_4^{\perp} & \tilde{g}_2(\lambda)^{-1}T_4 \end{pmatrix}. \tag{9.22}$$

We apply Lemma 5.1 to $\mathcal{C}(\lambda)$. Then, $a_{22} = \tilde{g}_2(\lambda)^{-1}T_4$ is invertible in S_4L^2;

$$\begin{aligned} a_{11} - a_{12}a_{22}^{-1}a_{21} &= S_4^{\perp}\mathcal{C}(\lambda)S_4^{\perp} - \tilde{g}_2(\lambda)^{-1}S_4^{\perp}G^{(v)}_{4,l}S_4T_4^{-1}S_4G^{(v)}_{4,l}S_4^{\perp} \\ &= S_4^{\perp}T_3S_4^{\perp} + \mathcal{O}^{(4)}_{S_4^{\perp}L^2}((\log\lambda)^{-1}) \end{aligned}$$

is also invertible for small $\lambda > 0$ since $S_4^{\perp}T_3S_4^{\perp}$ is invertibe in $S_4^{\perp}L^2$;

$$d(\lambda) = (a_{11} - a_{12}a_{22}^{-1}a_{21})^{-1} = (S_4^{\perp}T_3S_4^{\perp})^{-1}(1 + \mathcal{O}^{(4)}_{S_4^{\perp}}(\log\lambda)^{-1}).$$

It follows by Lemma 5.1 that $\mathcal{C}(\lambda)^{-1}$ exists for small $\lambda > 0$ and is given by (9.21). ∎

Since $\mathcal{C}(\lambda)^{-1}$ exists $B_3(\lambda) = \lambda^2\tilde{g}_2(\lambda)(1 - \lambda^2\tilde{g}_2(\lambda)F(\lambda)\mathcal{C}(\lambda)^{-1})\mathcal{C}(\lambda)$ and, since $\mathcal{C}(\lambda)^{-1} = \mathcal{O}^{(4)}_{S_3L^2}(\log\lambda)$ by (9.21), $B_3(\lambda)$ is invertible in S_3L^2 and

$$B_3(\lambda)^{-1} = \lambda^{-2}\tilde{g}_2(\lambda)^{-1}\mathcal{C}(\lambda)^{-1} + \mathcal{C}(\lambda)^{-1}F(\lambda)\mathcal{C}(\lambda)^{-1} + \mathcal{O}^{(4)}_{S_3L^2}(\lambda^2\tilde{g}_2(\lambda)^4). \tag{9.23}$$

This implies (9.8) because $F(\lambda) = \mathcal{O}^{(4)}_{S^3L^2}(1)$ and Lemma 9.3 (2) is proved. ∎

Proof of Lemma 9.3 (3). Lemma 1.8 (3) implies that d-resonances are absent from H if and only if $S_3L^2 \ominus S_4L^2 = \{0\}$ or $T_3 = 0$ on S_3L^2. Then, $S_4 = S_3$, $S_4^{\perp} = 0$ and (9.22) becomes $\mathcal{C}(\lambda) = \tilde{g}_2(\lambda)^{-1}S_4T_4S_4$. It follows that $Z = 0$ in (9.21). Then, (9.23) implies (9.9) because $\mathcal{C}(\lambda)^{-1}F(\lambda)\mathcal{C}(\lambda)^{-1} \in \mathcal{O}^{(4)}_{S^4L^2}((\log\lambda)^2)$. ∎

9.3. Simplification

In this section we want to simplify $\mathcal{N}_{2,2}^{(v)}$ of (9.4) modulo GPR. For shortening formulae we introduce

$$E_{2,l}(\lambda) = S_1(\widetilde{B}_1(\lambda) + S_2)^{-1}S_2(\widetilde{B}_2(\lambda) + S_3)^{-1}S_3, \tag{9.24}$$

$$E_{2,r}(\lambda) = S_3(\widetilde{B}_2(\lambda) + S_3)^{-1}S_2(\widetilde{B}_1(\lambda) + S_2)^{-1}S_1, \tag{9.25}$$

$$E_2(\lambda) = E_{2,l}(\lambda)B_3(\lambda)^{-1}E_{2,r}(\lambda) \tag{9.26}$$

and express $\mathcal{N}_{2,2}^{(v)}(\lambda)$ in the form

$$\mathcal{N}_{2,2}^{(v)}(\lambda) = \lambda^{-2}M_v(\mathcal{M}(\lambda^4) + S_1)^{-1}E_2(\lambda)(\mathcal{M}(\lambda^4) + S_1)^{-1}M_v. \tag{9.27}$$

Note that $E_2(\lambda)$ is sandwiched by S_1 and, hence, is $\mathcal{VS}$ but not $\mathcal{GVS}$ in general because of the strong singularities in $B_3(\lambda)^{-1}$, see Lemma 9.3.

In the following lemma, $A \approx B$ means that the factors A which appear in the right of (9.24), (9.25), and (9.26) may be replaced by B without changing $\mathcal{N}_{2,2}^{(v)}$ modulo GPR. *The proof of the lemma uses only the information on the size of $B_3(\lambda)^{-1}$ of Lemma 9.3.*

Lemma 9.8. (1) *If H has singularity of the third kind, then*

$$(\mathcal{M}(\lambda^4) + S_1)^{-1} \approx \widetilde{\mathcal{D}}_0(\lambda) - \widetilde{\mathcal{D}}_0(\lambda)\lambda^2 G_2^{(v)}\widetilde{\mathcal{D}}_0(\lambda). \tag{9.28}$$

$$(\widetilde{B}_1(\lambda) + S_2)^{-1} \approx D_1 + \lambda^2 h_1(\lambda)^{-1}\widetilde{T}_1(\lambda)D_1. \tag{9.29}$$

$$\begin{aligned}(\widetilde{B}_2(\lambda) + S_3)^{-1} \approx{}& D_2 - \lambda^2 D_2S_2(T_{4,l} - G_2^{(v)}\widetilde{\mathcal{D}}_0(\lambda)G_2^{(v)})S_2D_2 \\ & - \lambda^2 h_1(\lambda)^{-1}D_2S_2(\widetilde{T}_1(\lambda)D_1)^2S_2D_2.\end{aligned} \tag{9.30}$$

(2) *If H has singularity of the fourth kind, then*

$$\begin{aligned}(\mathcal{M}(\lambda^4) + S_1)^{-1} \approx{}& \widetilde{\mathcal{D}}_0(\lambda) - \lambda^2\widetilde{\mathcal{D}}_0(\lambda)G_2^{(v)}\widetilde{\mathcal{D}}_0(\lambda) \\ & - \lambda^4\tilde{g}_2(\lambda)D_0G_4^{(v)}D_0,\end{aligned} \tag{9.31}$$

$$\begin{aligned}(\widetilde{B}_1(\lambda) + S_2)^{-1} \approx{}& D_1 + \lambda^2 h_1(\lambda)^{-1}D_1\widetilde{T}_1(\lambda)D_1 \\ & + \lambda^4 h_1(\lambda)^{-1}\tilde{g}_2(\lambda)D_1G_4^{(v)}D_1 + \lambda^4 h_1(\lambda)^{-2}D_1(G_2^{(v)}D_1)^2,\end{aligned} \tag{9.32}$$

$$(\widetilde{B}_2(\lambda) + S_3)^{-1} \approx D_2 - D_2F_3(\lambda)D_2 + F_{3,sq}(\lambda). \tag{9.33}$$

where we wrote $D_1S_1 = S_1D_1 = D_1$ for simplicity.

Proof. We first prove (2) and explain how to obtain (1) from (2) at the end of the proof. The proof is divided into several steps. Recall notation (1.35) for $R_{2n}^{(v)}(\lambda)$ and (2.11) for $R_{2m\to 2n}^{(v)}(\lambda)$. It is important to observe that $E_2(\lambda) \in \mathcal{O}_{S_1L^2}^{(4)}(\lambda^{-2})$ is $\mathcal{VS}$.

(i) Denote the right side of (9.31) by $A_1(\lambda)$ and let

$$\widetilde{A}_1(\lambda) = \widetilde{\mathcal{D}}_0(\lambda) - \lambda^2 \widetilde{\mathcal{D}}_0(\lambda) G_2^{(v)} \widetilde{\mathcal{D}}_0(\lambda) - \lambda^4 \tilde{g}_2(\lambda) \widetilde{\mathcal{D}}_0(\lambda) G_4^{(v)} \widetilde{\mathcal{D}}_0(\lambda).$$

It is obvious that $A_1(\lambda)$ and $\widetilde{A}_1(\lambda)$ are $\mathcal{VS}$. By virtue of (1.10) and (2.10), $(\mathcal{M}(\lambda^4) + S_1)^{-1}$ may be expressed as

$$\widetilde{\mathcal{D}}_0(\lambda)(1 + R_{2\to 4}^{(v)}(\lambda)\widetilde{\mathcal{D}}_0(\lambda))^{-1}(1 + R_6^{(v)}(\lambda)(1 + R_{2\to 4}^{(v)}(\lambda)\widetilde{\mathcal{D}}_0(\lambda))^{-1})^{-1}$$

and $R_6^{(v)}(\lambda)(1 + R_{2\to 4}^{(v)}(\lambda)\widetilde{\mathcal{D}}_0(\lambda))^{-1} = \mathcal{O}_{\mathcal{H}_2}^{(4)}(\lambda^6)$. It follows that

$$\begin{aligned}(\mathcal{M}(\lambda^4) + S_1)^{-1} &= \widetilde{\mathcal{D}}_0(\lambda)(1 + R_{2\to 4}^{(v)}(\lambda)\widetilde{\mathcal{D}}_0(\lambda))^{-1} + \mathcal{O}_{\mathcal{H}_2}^{(4)}(\lambda^6) \\ &= \sum_{j=0}^{2} \widetilde{\mathcal{D}}_0(\lambda)(-R_{2\to 4}^{(v)}(\lambda)\widetilde{\mathcal{D}}_0(\lambda))^j + \mathcal{O}_{\mathcal{H}_2}^{(4)}(\lambda^6). \end{aligned} \tag{9.34}$$

On substituting (9.34) in (9.27), $\mathcal{O}_{\mathcal{H}_2}^{(4)}(\lambda^6)$ produces $\mathcal{O}_{\mathcal{L}^1}^{(4)}(\lambda^2)$ for $\mathcal{N}_{2,2}^{(v)}(\lambda)$ which is GPR by Proposition 3.6 and we may ignore it from (9.34). Then, the terms of order $\mathcal{O}_{\mathcal{H}_2}^{(4)}(\lambda^4)$ which appear in the sum on the right of (9.34) produce $\mathcal{GVS}$ for $\mathcal{N}_{2,2}^{(v)}(\lambda)$ and they may also be ignored. Thus, $(\mathcal{M}(\lambda^4) + S_1)^{-1} \approx \widetilde{A}_1(\lambda)$. Since $\mathcal{D}_0 = D_0 + h_1(\lambda) L_0$, $\lambda^4 \tilde{g}_2(\lambda)\widetilde{\mathcal{D}}_0(\lambda) G_4^{(v)} \widetilde{\mathcal{D}}_0(\lambda) = \lambda^4 \tilde{g}_2(\lambda) D_0 G_4^{(v)} D_0 + \mathcal{O}_{\mathcal{H}_2}(\lambda^4)$ and we may further replace $\widetilde{A}_1(\lambda)$ by $A_1(\lambda)$. This proves (9.31).

(ii) Let $\mathcal{N}_{2,\mathrm{red}}^{(v)} = \lambda^{-2} M_v A_1(\lambda) E_2(\lambda) A_1(\lambda) M_v$, which is $\mathcal{VS}$ and which is equal to $\mathcal{N}_{2,2}^{(v)}(\lambda)$ modulo GPR by step (i). Let $F_1(\lambda) = A_1(\lambda) S_1 - S_1 A_1(\lambda)$. Then,

$$F_1(\lambda) = h_1(\lambda)[L_0, S_1] + \mathcal{O}_{\mathcal{H}_2}^{(4)}(\lambda^2) \in \mathcal{O}_{\mathcal{H}_2}^{(4)}(h_1(\lambda)).$$

On replacing $A_1(\lambda) S_1$ on the left by $S_1 A_1(\lambda) + F_1(\lambda)$ and $S_1 A_1(\lambda)$ on the right by $A_1(\lambda) S_1 - F_1(\lambda)$, $\mathcal{N}_{2,\mathrm{red}}^{(v)}(\lambda)$ becomes

$$\begin{aligned}&\lambda^{-2} S_1 M_v A_1(\lambda) E_2(\lambda) A_1(\lambda) M_v S_1 - \lambda^{-2} M_v S_1 A_1(\lambda) E_2(\lambda) F_1(\lambda) M_v \\ &\quad + \lambda^{-2} M_v F_1(\lambda) E_2(\lambda) A_1(\lambda) S_1 M_v - \lambda^{-2} M_v F_1(\lambda) E_2(\lambda) F_1(\lambda) M_v.\end{aligned}$$

The point here is that the first term is sandwiched by $S_1 M_v$ and $M_v S_1$ and other terms carry at least one $F_1(\lambda) \in \mathcal{O}_{\mathcal{H}_2}^{(4)}(h_1(\lambda))$ and, hence, by virtue of Lemma 7.5 and by Lemma 3.5, terms of order $\mathcal{O}^{(4)}(\lambda^4 \log \lambda)$ in the formulae which will appear for $(\widetilde{B}_1(\lambda) + S_2)^{-1}$ and $(\widetilde{B}_2(\lambda) + S_3)^{-1}$ in the following step (iii) produce GPR for $\mathcal{N}_{2,2}^{(v)}(\lambda)$ and, hence, may be omitted.

(iii) We show (9.32). Since $L_1(\lambda) \in \mathcal{O}_{\mathcal{H}_2}^{(4)}(\lambda^2 \log \lambda)$, (9.19) implies

$$(\widetilde{B}_1(\lambda) + S_2)^{-1} = D_1 + D_1 L_1(\lambda) + D_1 L_1(\lambda)^2 + \mathcal{O}_{\mathcal{H}_2}^{(4)}(\lambda^6 (\log \lambda)^3). \tag{9.35}$$

Then (8.9), (8.3), and (7.6) imply that modulo $\mathcal{O}^{(4)}_{\mathcal{H}_2}(\lambda^4 \log\lambda)$

$$\begin{aligned} D_1 L_1(\lambda) &\equiv \lambda^2 h_1(\lambda)^{-1} D_1 \widetilde{T}_1(\lambda) D_1 + \lambda^4 h_1(\lambda)^{-1} \tilde{g}_2(\lambda) D_1 G_4^{(v)} D_1, \\ D_1 L_1(\lambda)^2 &\equiv \lambda^4 h_1(\lambda)^{-2} D_1 (G_2^{(v)} D_1)^2, \end{aligned}$$

where we used $D_1 S_1 = S_1 D_1 = D_1$ and $D_1 D_0 = D_0 D_1 = D_1$. Since $\mathcal{O}^{(4)}_{\mathcal{H}_2}(\lambda^4 \log\lambda)$ may be ignored from the right side of (9.35) by (ii) above, we obtain (9.32).

(iv) Since the term $\mathcal{O}^{(4)}_{S_2 L^2}(\lambda^4 \log\lambda)$ in $(\widetilde{B}_2(\lambda) + S_3)^{-1}$ may be ignored by (ii) above, (9.16) implies (9.33). This completes the proof of statement (2).

If $B_3(\lambda)^{-1} = \mathcal{O}^{(4)}_{S_3 L^2}(\lambda^{-2} \tilde{g}_2(\lambda)^{-1})$, then the proof of (2) implies that terms in the class $\mathcal{O}_{\mathcal{H}_2}(\lambda^4 \log\lambda)$ may be ignored from (9.34) and, hence, from (9.31) and those in $\mathcal{O}_{\mathcal{H}_2}(\lambda^4 (\log\lambda)^2)$ from (9.32) and (9.33). The statement (1) follows. ■

9.4. Proof of Theorem 1.9 (3). Singularity of the third kind

We have $B_3(\lambda)^{-1} = \mathcal{O}^{(4)}_{S_3 L^2}(\lambda^{-2} (\log\lambda)^{-1})$. We use Lemma 9.8 (1). Denote the right of (9.28), (9.29), and (9.30) by $\widetilde{\mathcal{D}}_0(\lambda) + a$, $D_1 + b$, and $D_2 + c$ respectively. We have $a \in \mathcal{O}^{(4)}_{L^2}(\lambda^2)$, $b \in \mathcal{O}^{(4)}_{S_1 L^2}(\lambda^2 \log\lambda)$ and $c \in \mathcal{O}^{(4)}_{S_1 L^2}(\lambda^2 \log\lambda)$; they are $\mathcal{GVS}$ and

$$\begin{aligned} \mathcal{N}^{(v)}_{2,2}(\lambda) \equiv \lambda^{-2} M_v (\widetilde{\mathcal{D}}_0(\lambda) + a_l) S_1 (D_1 + b_l) S_2 (D_2 + c_l) S_3 \\ \times B_3(\lambda)^{-1} S_3 (D_2 + c_r) S_2 (D_1 + b_r) S_1 (\widetilde{\mathcal{D}}_0(\lambda) + a_r) M_v, \end{aligned} \tag{9.36}$$

where we have added the indices l and r to a, b and c to distinguish the ones on the left and the right of $B_3(\lambda)^{-1}$. We expand the right of (9.36). The result is $\mathcal{VS}$; the terms which contain more than two of $\{a_l, a_r, b_l, b_r, c_l, c_r\}$ are in the class $\mathcal{O}^{(4)}_{\mathcal{L}^1}(\lambda^2 (\log\lambda)^2)$ and they are $\mathcal{GVS}$; those which contain two of them are also GPR because they are $\mathcal{GVS}$ if they contain a_l or a_r or, otherwise, they are of the form

$$M_v \widetilde{\mathcal{D}}_0(\lambda) \mathcal{O}^{(4)}_{S_1 L^2}(\log\lambda) \widetilde{\mathcal{D}}_0(\lambda) M_v = M_v S_1 \mathcal{O}^{(4)}_{S_1 L^2}(\log\lambda) S_1 M_v + \mathcal{GVS}.$$

Thus, modulo GPR, $\mathcal{N}^{(v)}_{2,2}(\lambda)$ is the sum of the terms which contain at most one of $\{a_l, a_r, b_l, b_r, c_l, c_r\}$. We denote the term which contains none of them by $\mathcal{O}(\emptyset)(\lambda)$ and those which contain a_l, etc. by $\mathcal{O}(a_l)(\lambda)$, etc. respectively and we individually estimate the operators produced by (5.1) with $\mathcal{O}(\emptyset)(\lambda)$, $\mathcal{O}(a_l)(\lambda), \ldots$ in place of $\mathcal{Q}_v(\lambda)$.

Recall (see (9.1)) that the basis of $S_3 L^2$ is given by $\{\zeta_{r+1}, \ldots, \zeta_m\}$, $r < m$. By virtue of (9.7) we have

$$B_3(\lambda)^{-1} = \sum_{j,k=r+1}^{m} \lambda^{-2} (\log\lambda)^{-1} c_{jk}(\lambda) \zeta_j \otimes \zeta_k \tag{9.37}$$

with $c_{jk}(\lambda) \in \mathcal{O}^{(4)}_{\mathbb{C}}(1)$ for small $\lambda > 0$, $j, k = r+1, \dots, m$.

Lemma 9.9. *The operators $\mathcal{O}(\emptyset)(\lambda)$, $\mathcal{O}(a_l)(\lambda)$, $\mathcal{O}(b_l)(\lambda)$, and $\mathcal{O}(c_l)(\lambda)$ are all GPRs.*

Proof. By using Lemma 8.1, we have

$$\begin{aligned}
\mathcal{O}(\emptyset)(\lambda) &= \lambda^{-2} M_v S_3 B_3(\lambda)^{-1} S_3 M_v,\\
\mathcal{O}(a_l)(\lambda) &= -h_1(\lambda) M_v L_0 G_2^{(v)} S_3 B_3(\lambda)^{-1} S_3 M_v,\\
\mathcal{O}(b_l)(\lambda) &= M_v \widetilde{\mathfrak{D}}_0(\lambda) S_1 L_0 G_2^{(v)} S_3 B_3(\lambda)^{-1} S_3 M_v,\\
\mathcal{O}(c_l)(\lambda) &= -M_v S_2 D_2 S_2 \{T_{4,l} + G_2^{(v)}(S_1 D_1 S_1 L_0) G_2^{(v)}\\
&\qquad + h_1(\lambda) G_2^{(v)}(L_0 S_1 D_1 S_1 L_0 - L_0) G_2^{(v)}\} S_3 B_3(\lambda)^{-1} S_3 M_v.
\end{aligned}$$

We observe that all of these have $S_3 M_v$ at right ends, which we will use to cancel the singularity of $B_3(\lambda)^{-1}$. Thus, the proof is similar and we only prove that $\mathcal{O}(a_l)(\lambda)$ is GPR and comment on how to modify the argument for others at the end of the proof.

Let, for $r+1 \le j, k \le m$, $c_{jk}(\lambda)$ be as in (9.37) and

$$\mu_{jk}(\lambda) = -h_1(\lambda)(\log\lambda)^{-1} c_{jk}(\lambda),\ \rho_j(x) = (M_v L_0 G_2^{(v)} \zeta_j)(x),\ \omega_k(x) = (v\zeta_k)(x)$$

so that $\mathcal{O}(a_l)(\lambda) = \sum_{j,k=r+1}^{m} \lambda^{-2} \mu_{jk}(\lambda)(\rho_j \otimes \omega_k)$. Since $\mu_{jk}(\lambda)$ are GMU and $\rho_j,\ \omega_k \in \langle x\rangle^{-6} L^1(\mathbb{R}^4)$, we need only to show that the operator I defined by

$$Iu = \int_0^\infty R_0^+(\lambda^4)(\rho \otimes \omega)\Pi(\lambda) u \lambda d\lambda \tag{9.38}$$

for $u \in \mathcal{D}_*$ is GOP when $\rho \in \langle x\rangle^{-6} L^1(\mathbb{R}^4)$ and $\omega(x) = v(x)\zeta(x)$ with $\zeta \in S_3 L^2$. Note that the integral by $d\lambda$ is only over a compact interval of $(0, \infty)$ since $u \in \mathcal{D}_*$.

Since $\int_{\mathbb{R}^4} x^\alpha \omega(x) dx = 0$ for $|\alpha| \le 1$, we have by (1.41) that

$$(\omega, \Pi(\lambda)u) = \sum_{i,l=1}^{4} \lambda^2 \int_0^1 (1-\theta)\Big(\int_{\mathbb{R}^4} \omega_{il}(z)(\Pi(\lambda)\tau_{-\theta z} u_{il})(0) dz\Big) d\theta,$$

where $\omega_{il}(z) = z_i z_l \omega(z)$ and $u_{il} = R_i R_l u$, which we substitute in (9.38). The change of order of integrations then yields $Iu(x) = \sum_{i,l=1}^{4} \int_0^1 (1-\theta) I_{il}(\theta) u(x) d\theta$, where

$$I_{il}(\theta)u(x) = \iint_{\mathbb{R}^8} \rho(y)\omega_{il}(z)\tau_y\Big(\int_0^\infty \mathcal{R}_\lambda(x)\Pi(\lambda)\tau_{-\theta z} u_{il}(0)\lambda^3 d\lambda\Big) dy dz.$$

The integral inside the parenthesis is equal to $K\tau_{-\theta z} u_{il}$ (recall (1.21)) and (1.22) implies that for any $1 < p < \infty$

$$\|I_{il}(\theta)u\|_p \le C\|\rho\|_1 \|\omega_{il}\|_1 \|u_{il}\|_p \le C\|u\|_p.$$

Hence, I is GOP and $\mathcal{O}(a_l)(\lambda)$ is a GPR.

Since $h_1(\lambda)$ does not play any role except that it is GMU, the entire argument for $\mathcal{O}(a_l)(\lambda)$ applies for proving that $\mathcal{O}(\emptyset)(\lambda)$ and $\mathcal{O}(b_l)(\lambda)$ are GPR. The argument for $\mathcal{O}(c_l)(\lambda)$ is similar. The only point we have to note is that, instead of $h_1(\lambda)$ in $\mathcal{O}(a_l)(\lambda)$, $\mathcal{O}(c_l)(\lambda)$ contains the singularity $\tilde{g}_2(\lambda)$ which is hidden in $T_{4,l} = G_4^{(v)}\tilde{g}_2(\lambda) + G_{4,l}^{(v)}$, however, this is harmless since $\tilde{g}_2(\lambda)(\log\lambda)^{-1}c_{jk}(\lambda)$ is still GMU. ∎

We next study the operators produced by

$$\begin{aligned}\mathcal{O}(a_r)(\lambda) =& - h_1(\lambda)M_vS_3B_3(\lambda)^{-1}S_3G_2^{(v)}L_0M_v,\\ \mathcal{O}(b_r)(\lambda) =& M_vS_3B_3(\lambda)^{-1}S_3G_2^{(v)}L_0S_1D_1\widetilde{\mathcal{D}}_0(\lambda)M_v,\\ \mathcal{O}(c_r)(\lambda) =& - M_vS_3B_3(\lambda)^{-1}S_3\{T_{4,l} + G_2^{(v)}(L_0S_1D_1S_1)G_2^{(v)}\\ &+ h_1(\lambda)G_2^{(v)}(L_0S_1D_1S_1L_0 - L_0)G_2^{(v)}\}S_2D_2S_2M_v.\end{aligned} \tag{9.39}$$

The following lemma completes the proof of Theorem 1.9 (3).

Lemma 9.10. *The operators (9.6) with $\mathcal{O}(a_r)(\lambda)$, $\mathcal{O}(b_r)(\lambda)$, and $\mathcal{O}(c_r)(\lambda)$ in place of $\mathcal{N}_{2,2}^{(v)}(\lambda)$ are all bounded in $L^p(\mathbb{R}^4)$ for $1 < p < 2$.*

We use the following lemma.

Lemma 9.11. *Let $\zeta \in S_3L^2$ and $\rho \in L^1(\mathbb{R}^4)$ and let $a > 0$ be sufficiently small. Then, $\mathcal{Z}^{(r)}$ defined by*

$$\mathcal{Z}^{(r)}u = \int_0^\infty R_0^+(\lambda^4)(v\zeta \otimes \rho)\Pi(\lambda)u\lambda\chi_{\le a}(\lambda)d\lambda, \quad u \in \mathcal{D}_*.$$

is bounded in $L^p(\mathbb{R}^4)$ for $1<p<2$ and, if $\int_{\mathbb{R}^4}\rho(x)dx \neq 0$, unbounded for $2 \le p \le \infty$.

Proof. The proof patterns after that of Lemma 8.9. Let $\omega = v\zeta$ and

$$\mathcal{Z}^{(r)}u = \chi_{\le 4a}(|D|)\mathcal{Z}^{(r)}u + \chi_{>4a}(|D|)\mathcal{Z}^{(r)}u =: \mathcal{Z}_{\le 4a}^{(r)}u + \mathcal{Z}_{>4a}^{(r)}u.$$

(1) We first show that $\mathcal{Z}_{>4a}^{(r)}$ is bounded in $L^p(\mathbb{R}^4)$ for $1 < p < 2$. We have

$$\mathcal{Z}_{>4a}^{(r)}u(x) = \int_0^\infty \chi_{>4a}(|D|)R_0^+(\lambda^4)(\omega \otimes \rho)\Pi(\lambda)u\lambda\chi_{\le a}(\lambda)d\lambda.$$

Let $\mu_a(\xi) = \chi_{>4a}(\xi)|\xi|^{-4}$. Then, $\mu_a(|D|)\omega \in L^p(\mathbb{R}^4)$ for all $1 \le p \le \infty$, $\mu_a(|D|)$ is GOP (cf. Lemma 8.8) and

$$\chi_{>4a}(|D|)R_0^+(\lambda^4)\omega(x) = \mu_a(|D|)\omega(x) + \mu_a(|D|)\lambda^4R_0^+(\lambda^4)\omega(x)$$

(see (8.21)). It follows that $\mathcal{Z}^{(r)}_{>4a}u = \mathcal{Z}^{(r,1)}_{>4a}u + \mathcal{Z}^{(r,2)}_{>4a}u$, where

$$\mathcal{Z}^{(r,1)}_{>4a}u = \mu_a(|D|)\omega(x)\ell(u), \quad \ell(u) := \int_0^\infty (\rho, \Pi(\lambda)u)\lambda\chi_{\le a}(\lambda)d\lambda,$$

$$\mathcal{Z}^{(r,2)}_{>4a}u = \mu_a(|D|)\int_0^\infty R_0^+(\lambda^4)(\omega\otimes\rho)\Pi(\lambda)u\lambda^5\chi_{\le a}(\lambda)d\lambda.$$

Then, Lemma 3.5 implies $\|\mathcal{Z}^{(r,2)}_{>4a}u\|_p \le C_p\|\omega\|_1\|\rho\|_1\|u\|_p$ for $1 < p < \infty$. Changing the order of integrations, we obtain as in (8.23) that

$$\ell(u) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^8} \rho(y)u(x)J_a(x-y)dxdy, J_a(x) = \mathcal{F}\Big(\frac{\chi_{\le a}(|\xi|)}{|\xi|^2}\Big)(x),$$

Here $J_a(x)$ is smooth and $|J_a(x)| \le C\langle x\rangle^{-2}$. It follows by Young's inequality that $|\ell(u)| \le C\|\rho\|_1\|J\|_{p'}\|u\|_p$ for $1 \le p < 2$ and $p' = p/(p-1)$ and $\|\mathcal{Z}^{(r,1)}_{>4a}u\|_p \le C\|\rho\|_1\|u\|_p$. Thus, $\|\mathcal{Z}^{(r)}_{>4a}u\|_p \le C\|\rho\|_1\|u\|_p$ and $\mathcal{Z}^{(r)}_{>4a}$ is bounded in $L^p(\mathbb{R}^4)$ for $1 \le p < 2$.

(2) We note that $\omega \ne 0$ as otherwise $\zeta = 0$ by virtue of Lemma 1.8 and that $\mu_a(|D|)\omega \ne 0$ for small $a > 0$. We have that

$$\int_{\mathbb{R}^4} J_a(x-y)\rho(y)dy \notin L^{p'}(\mathbb{R}^4), \quad 1 \le p' = p/(p-1) < 2$$

unless $\hat\rho(0) = 0$ as in part (2) of the proof of Lemma 8.9. Thus, $\mathcal{Z}^{(r,1)}_{>4a}$ is unbounded in $L^p(\mathbb{R}^4)$ for $2 \le p < \infty$ and, hence, so is $\mathcal{Z}^{(r)}$.

(3) We finally show that

$$\mathcal{Z}^{(r)}_{\le 4a}u(x) = \int_0^\infty \chi_{\le 4a}(|D|)R_0^+(\lambda^4)(\omega\otimes\rho)\Pi(\lambda)u\lambda\chi_{\le a}(\lambda)d\lambda$$

satisfies $\|\mathcal{Z}^{(r)}_{\le 4a}u\|_p \le C\|\langle x\rangle^2\omega\|_1\|\rho\|_1\|u\|_p$ for $1 < p < 2$. We may assume $\omega, \rho \in C_0^\infty(\mathbb{R}^4)$. Let $\omega_{kl}(y) = y_k y_l\omega(y)$ for $k, l = 1, \dots, 4$. Since $\int_{\mathbb{R}^4} x^\alpha\omega(x)dx = 0$, $\partial^\alpha\hat\omega(0) = 0$ for $|\alpha| \le 1$ and

$$\hat\omega(\xi) = \sum_{k,l=1}^4 \frac{-\xi_k\xi_l}{(2\pi)^2}\int_0^1 (1-\theta)\Big(\int_{\mathbb{R}^4} e^{-iy\xi\theta}\omega_{kl}(y)dy\Big)d\theta.$$

Hence, $\chi_{\le 4a}(|D|)R_0^+(\lambda^4)\omega(x)$ is equal to

$$\lim_{\varepsilon\downarrow 0}\sum_{k,l=1}^{4}\int_0^1(1-\theta)\left(\frac{-1}{(2\pi)^4}\iint_{\mathbb{R}^8}\frac{e^{i(x-\theta y)\xi}\xi_k\xi_l\chi_{\le 4a}(|\xi|)}{|\xi|^4-(\lambda+i\varepsilon)^4}\omega_{kl}(y)dyd\xi\right)d\theta.$$

Let $\gamma_{kl,a}(D)=R_jR_k\chi_{\le 4a}(|D|)$. Then, the inner integral becomes

$$\begin{aligned}&\int_{\mathbb{R}^4}\omega_{kl}(y)\tau_{y\theta}\left(\frac{-1}{(2\pi)^4}\int_{\mathbb{R}^4}\frac{e^{ix\xi}\xi_k\xi_l\chi_{\le 4a}(|\xi|)}{|\xi|^4-(\lambda+i\varepsilon)^4}d\xi\right)dy\\ &\quad=\int_{\mathbb{R}^4}\omega_{kl}(y)\tau_{y\theta}\gamma_{kl,a}(D)\left(\frac{-1}{(2\pi)^4}\int_{\mathbb{R}^4}\frac{e^{ix\xi}|\xi|^2}{|\xi|^4-(\lambda+i\varepsilon)^4}d\xi\right)dy\\ &\quad=-\frac{1}{2}\int_{\mathbb{R}^4}\omega_{kl}(y)\tau_{y\theta}\gamma_{kl,a}(D)(\mathcal{G}_{i\lambda-\varepsilon}(x)+\mathcal{G}_{\lambda+i\varepsilon}(x))dy\end{aligned}$$

(recall (1.6)). Thus, by virtue of (3.1) and (3.2), $\mathcal{Z}_{\le 4a}^{(r)}u$ is equal to the superposition by

$$\sum_{k,l=1}^{4}-\frac{1}{2}\int_0^1(1-\theta)d\theta$$

of

$$\begin{aligned}&\int_{\mathbb{R}^4}\omega_{kl}(y)\tau_{\theta y}\left(\gamma_{kl,a}(D)\int_0^\infty(\mathcal{G}_{i\lambda}(x)+\mathcal{G}_{\lambda}(x))(\rho,\Pi(\lambda)u)\lambda\chi_{\le a}(\lambda)d\lambda\right)dy\\ &\quad=\int_{\mathbb{R}^4}\omega_{kl}(y)\tau_{\theta y}\left(\gamma_{kl,a}(D)\int_{\mathbb{R}^4}\rho(z)(K_1+K_2)(\tau_{-z}\chi_{\le a}(|D|)u)(x)dz\right)dy\end{aligned}$$

We then apply Lemmas 3.1 and 3.3 and Minkowski's inequality and obtain the desired estimate. ■

Proof of Lemma 9.10. The operators $\mathcal{O}(a_r)(\lambda)$, $\mathcal{O}(b_r)(\lambda)$ and $\mathcal{O}(c_r)(\lambda)$ all have M_vS_3 on the left ends. The proof is similar and we prove the lemma only for $\mathcal{O}(a_r)(\lambda)$. The modification for others is obvious. We use the notation in the proof of Lemma 9.9. Substitute $\mathcal{O}(a_r)(\lambda)$ for $\mathcal{N}_{2,2}^{(v)}(\lambda)$ in (9.6) and use (1.16). Then, by virtue (9.39) and (9.37), $\mathcal{Z}u$ of (9.6) becomes

$$\sum_{j,k=r+1}^{m}J_{jk}u,\quad J_{jk}u=\int_0^\infty R_0^+(\lambda^4)(v\zeta_j\otimes\rho_k)\Pi(\lambda)u_{a,jk}\lambda d\lambda,$$

where $\zeta_j \in S_3L^2$, $\rho_k = M_v L_0 G_2^{(v)}\zeta_k$ and $u_{a,jk} = \chi_{\le a}(|D|)\mu_{jk}(|D|)u$ with μ_{jk} being GMU. Then $\rho_k \in L^1(\mathbb{R}^4)$ for $k = r+1, \dots, m$ and Lemma 9.11 implies that $\|J_{jk}u\|_p \le C_p \|\langle x\rangle^2 v\zeta_j\|_1 \|\rho_k\|_1 \|u\|_p$ for $1 < p < 2$. By virtue of Corollary 9.2, the lemma follows. ∎

Remark 9.12. Lemma 9.11 suggests that $W_\pm$ are unbounded in $L^p(\mathbb{R}^4)$ for $2 < p \le \infty$. This holds if the sum of $\rho's$ corresponding to $\mathcal{O}(a_r)$, $\mathcal{O}(b_r)$, and $\mathcal{O}(c_r)$ has non-vanishing integral. However, we were not able to show $\int_{\mathbb{R}^4} \rho_k(x)dx \neq 0$ even for $\rho_k(x) = M_v L_0 G_2^{(v)}\zeta_k$ of $\mathcal{O}(a_r)$ above, where $G_2^{(v)}\zeta_k(x)$ is equal to a constant.

9.5. Proof of Theorem 1.9 (4)

It suffices by virtue of Corollary 9.2 to prove that operator $\mathcal{Z}$ satisfies statement (4) of the theorem.

(1) If H has singularity of the fourth kind, then $B_3(\lambda)^{-1}$ satisfies (9.8) in general and the proof in the previous subsection shows that $\mathcal{Z}$ with $\mathcal{N}_{2,2}^{(v)}(\lambda)$ being replaced by $\mathcal{O}^{(4)}_{S_3L^2}(\lambda^{-2}(\log\lambda)^{-1})$ of (9.8) is bounded in $L^p(\mathbb{R}^4)$ for $1 < p < 2$. Thus, the proof of Theorem 1.9 (4) is finished if we have proved it for the case $T_3 = 0$ *but only using the condition* $T_3 = 0$ *through* (9.9).

(2) We now prove Theorem 1.9 (4) that W_- is bounded in $L^p(\mathbb{R}^4)$ for $1 < p < 4$ if $T_3 = 0$. without using this condition explicitly but using (9.9). Recall that only the size information on $B_3(\lambda)^{-1}$ is used for proving Lemma 9.8.

We have Lemma 9.3 (3). Then, by virtue of Lemma 7.5, $\mathcal{O}^{(4)}_{S_3L^2}(\lambda^2(\log\lambda)^3)$ in $B_3(\lambda)^{-1}$ produces GPR for $\mathcal{N}_{2,2}^{(v)}(\lambda)$ and we ignore it and we may assume

$$B_3(\lambda)^{-1} = -\lambda^{-2}S_4\widetilde{B}_3(\lambda)^{-1}S_4, \quad \widetilde{B}_3(\lambda)^{-1} := T_4^{-1} + \mathcal{O}^{(4)}_{S_4L^2}(\lambda^2(\log\lambda)^2). \tag{9.40}$$

Thus, the following lemma completes the proof of Theorem 1.9 (4).

Lemma 9.13. *Let* (9.40) *be satisfied. Then,* $\mathcal{Z}$ *is bounded in* $L^p(\mathbb{R}^4)$ *for* $1 < p < 4$.

We prove Lemma 9.13 by a series of lemma. The argument is similar to but is more complicated than that of the previous subsection. Denote the right sides of (9.31), (9.32), and (9.33) by $\widetilde{\mathcal{D}}_0(\lambda) + \tilde{a}$, $D_1 + \tilde{b}$, and $D_2 + \tilde{c}$ respectively. We have

$$\tilde{a} \in \mathcal{O}^{(4)}_{\mathcal{H}}(\lambda^2), \quad \tilde{b} \in \mathcal{O}^{(4)}_{\mathcal{H}}(\lambda^2\log\lambda), \quad \tilde{c} \in \mathcal{O}^{(4)}_{\mathcal{H}}(\lambda^2\log\lambda). \tag{9.41}$$

Then, Lemma 9.8 implies that modulo GPR

$$\mathcal{N}_{2,2}^{(v)}(\lambda) \equiv \lambda^{-4}J_l(\lambda)\widetilde{B}_3(\lambda)^{-1}J_r(\lambda), \tag{9.42}$$

$$J_l(\lambda) = M_v(\widetilde{\mathcal{D}}_0(\lambda) + \tilde{a}_l)S_1(D_1 + \tilde{b}_l)S_2(D_2 + \tilde{c}_l)S_4, \tag{9.43}$$

$$J_r(\lambda) = S_4(D_2 + \tilde{c}_r)S_2(D_1 + \tilde{b}_r)S_1(\widetilde{\mathcal{D}}_0(\lambda) + \tilde{a}_r)M_v.$$

where we have added the indices l and r as previously. We expand the right of (9.42) and consider each terms separately, which are $\mathcal{V}\mathcal{S}$s.

We begin with the following remarks. By virtue of the part (ii) of the proof of Lemma 9.8, the part $\mathcal{O}^{(4)}_{S_2L^2}(\lambda^4 \log\lambda)$ of $\tilde{c}$ produces GPR for $\mathcal{N}^{(v)}_{2,2}(\lambda)$ and, hence, we may ignore it; identities (9.14) and (9.15) that $G_2M_vS_4 = S_4M_vG_2 = 0$ and $\widetilde{T}_1(\lambda)S_4 = S_4\widetilde{T}_1(\lambda) = 0$ respectively, substantially simplify the formulae:

$$
\begin{aligned}
&S_4(D_2F_3(\lambda)D_2) = S_4(\lambda^2 T_{4,l}(\lambda) + \lambda^4(h_1^{-1}\tilde{g}_2)(\lambda)G_4^{(v)}S_1G_2^{(v)})S_2D_2,\\
&S_4F_{3,sq}(\lambda) = \lambda^4\tilde{g}_2(\lambda)S_4G_4^{(v)}\widetilde{D}_2(\tilde{g}_2(\lambda)G_4^{(v)} + h_1(\lambda)^{-1}(\widetilde{T}_1(\lambda)D_1)^2)S_2D_2,\\
&(D_2F_3(\lambda)D_2)S_4 = D_2S_2(\lambda^2T_{4,l}(\lambda) + \lambda^4(h_1^{-1}\tilde{g}_2)(\lambda)G_2^{(v)}\widetilde{D}_1G_4^{(v)})S_4,\\
&F_{3,sq}(\lambda)S_4 = \lambda^4\tilde{g}_2(\lambda)D_2S_2(\tilde{g}_2(\lambda)G_4^{(v)} + h_1(\lambda)^{-1}(\widetilde{T}_1(\lambda)D_1)^2)\widetilde{D}_2G_4^{(v)}S_4
\end{aligned}
\tag{9.44}
$$

where we set $\widetilde{D}_1 = S_1D_1S_1$ and $\widetilde{D}_2 = S_2D_2S_2$.

The next lemma is evident by (9.41) and we omit the proof.

Lemma 9.14. *The terms which contain three or more of $\{\tilde{a}_l, \tilde{a}_r, \tilde{b}_l, \tilde{b}_r, \tilde{c}_l, \tilde{c}_r\}$ are $\mathcal{G}\mathcal{S}\mathcal{V}$ and, hence, are GPRs.*

We then consider the terms which contain *at most two* of $\{\tilde{a}_l, \tilde{a}_r, \tilde{b}_l, \tilde{b}_r, \tilde{c}_l, \tilde{c}_r\}$ and, in addition to the notation of Section 9.4, we introduce $\mathcal{O}(\tilde{a}_r, \tilde{b}_l)(\lambda)$, etc. to denote the terms which contain $\tilde{a}_r$ and $\tilde{b}_l$, etc. By virtue of (9.40),

$$
S_4\widetilde{B}_3(\lambda)^{-1}S_4 = \sum_{j,k=r+1}^{m} \tilde{t}_{jk}(\lambda)\zeta_j \otimes \zeta_k, \tag{9.45}
$$

where $\tilde{t}_{jk}(\lambda) = t_{jk} + \mathcal{O}^{(4)}_{\mathbb{C}}(\lambda^2(\log\lambda)^2)$ and $\{\zeta_{r+1}, \dots, \zeta_m\}$ is the basis of $S_4L^2 = S_3L^2$.

Lemma 9.15. *Let $Z_\emptyset$ be the operator defined by the right side of (9.6) with $\mathcal{O}(\emptyset)(\lambda)$ in place of $\mathcal{N}^{(v)}_{2,2}(\lambda)$. Then, $Z_\emptyset$ is bounded in $L^p(\mathbb{R}^4)$ for $1 < p < 4$.*

Proof. We have $\mathcal{O}(\emptyset)(\lambda) = \lambda^{-4}S_4\widetilde{B}_3(\lambda)^{-1}S_4$ and by (9.45)

$$
Z_\emptyset u = \sum_{j,k=r+1}^{m} \int_0^\infty R_0^+(\lambda^4)(v\zeta_j)(x)(v\zeta_k, \Pi(\lambda)u)\lambda^{-1}\tau_{jk}(\lambda)\chi_{\le a}(\lambda)d\lambda. \tag{9.46}
$$

Since $\zeta \in S_4L^2$ satisfies $\int_{\mathbb{R}^4} x^\alpha(v\zeta)(x)dx = 0$ for $|\alpha| \le 2$ (cf. Lemma 6.2), we have by expanding $e^{i\lambda z\omega}$ to the third order in (1.39) that $(v\zeta, \Pi(\lambda)u)$ is equal to

$$
\sum_{|\alpha|=3} C_\alpha \int_0^1 (1-\theta)^2\left(\lambda^3\int_{\mathbb{R}^4} z^\alpha(v\zeta)(z)\Pi(\lambda)(R^\alpha\tau_{-\theta z}u)(0)dz\right)d\theta,
$$

where C_α are unimportant constants and $R^\alpha = R_1^{\alpha_1} \cdots R_4^{\alpha_4}$ for $\alpha = (\alpha_1, \dots, \alpha_4)$. Thus, if we define

$$E_j u(x) = \int_0^\infty (R_0^+(\lambda^4)(v\zeta_j)(x)\Pi(\lambda)u(0)\lambda^2 \chi_{\le a}(\lambda) d\lambda, \tag{9.47}$$

then $\mathcal{Z}_\emptyset u(x)$ becomes the sum over $r+1 \le j, k \le m$ and $\{\alpha : |\alpha| = 3\}$ of

$$C_\alpha \int_0^1 (1-\theta)^2 \Big(\int_{\mathbb{R}^4} z^\alpha (v\zeta_k)(z) E_j (R^\alpha \tau_{-\theta z} \tilde{t}_{jk}(|D|))u(x) dz \Big) d\theta \tag{9.48}$$

However, E_j is equal to W_B in (8.20) with ζ_j in place of ζ and $\{E_j\}_{j=r',\dots,m}$ are bounded in $L^p(\mathbb{R}^4)$ for $1 < p < 4$ by Lemmas 8.9 and 8.11. Thus, Minkowski's inequality implies that for $1 < p < 4$

$$\|\mathcal{Z}_\emptyset u(x)\|_p \le C \sum_{j,k=r+1}^m \|\langle z\rangle^3 (v\zeta_k)\|_1 \|u\|_p \|E_j\|_{\mathbf{B}(L^p)}. \tag{9.49}$$

This proves the lemma. ■

Lemma 9.16. *Let $J_l'(\lambda)$ be the sum of the terms which appear when we expand the right of (9.43) and which contain at least one from $\{\tilde{a}_l, \tilde{b}_l, \tilde{c}_l\}$. Then, the operator $\mathcal{Z}_{l,\emptyset}$ produced by $\lambda^{-4} J_l'(\lambda) \widetilde{B}_3(\lambda)^{-1} S_4 M_v$, the sum of the terms which contain none of $\{\tilde{a}_r, \tilde{b}_r, \tilde{c}_r\}$ but at least one from $\{\tilde{a}_l, \tilde{b}_l, \tilde{c}_l\}$, is GOP.*

Proof. The operator $\mathcal{Z}_{l,\emptyset} u$ is equal to the right of (9.46) with $J_l'(\lambda)\zeta_j$ in place of $v\zeta_j$, hence, is $\sum_{j,k=r+1}^m \sum_{|\alpha|=3} C_\alpha \int_0^1 (1-\theta)^2 d\theta$ of (9.48) with E_j being replaced by $\widetilde{E}_j$ which is defined the right of (9.47) with $J_l'(\lambda)\zeta_j$ in place of $v\zeta_j$. Here, in view of (9.41) and that $J_l'(\lambda)$ is $\mathcal{VS}$, we have $J_l'(\lambda)\zeta_j = \sum_{\text{finite sum}} \lambda \sigma_n(\lambda) f_{jn}$ with $f_{jn} \in L^1(\mathbb{R}^4)$ and $\sigma_n \in \mathcal{O}_{\mathbb{C}}^{(4)}(\lambda \log \lambda)$. It follows that $\widetilde{E}_j$, $j = r+1, \dots, m$, become

$$\widetilde{E}_j u(x) = \sum_{\text{finite sum}} \int_0^\infty (R_0^+(\lambda^4) f_{jn})(x) (\Pi(\lambda)\sigma_j(|D|)u)(0) \lambda^3 \chi_{\le a}(\lambda) d\lambda$$

and they are GOP by virtue of Lemma 3.5. Hence, as in (9.49) we have

$$\|\mathcal{Z}_{l,\emptyset} u(x)\|_p \le C \sum_{j,k=r+1}^m \|\langle z\rangle^3 (v\zeta_k)\|_1 \|u\|_p \|\widetilde{E}_j\|_{\mathbf{B}(L^p)}$$

for all $1 < p < \infty$. This proves the lemma. ■

Next, we consider the operators produced by the terms which contain two of $\{\tilde a_r, \tilde b_r, \tilde c_r\}$ and none of $\{\tilde a_l, \tilde b_l, \tilde c_l\}$.

Lemma 9.17. *The operators $\mathcal{O}(\tilde a_r, \tilde b_r)(\lambda)$ and $\mathcal{O}(\tilde a_r, \tilde c_r)(\lambda)$ are GPR.*

Proof. (1) Since $S_4\tilde b_r = \lambda^4 h_1(\lambda)^{-1}\tilde g_2(\lambda)S_4G_4^{(v)}D_1$ by (9.14) and (9.15) and $\tilde a_r \in \mathcal{O}^{(4)}_{\mathcal{H}_2}(\lambda^2)$, $\mathcal{O}(\tilde a_r, \tilde b_r)(\lambda) = \lambda^{-4}M_vS_4\widetilde B_3(\lambda)^{-1}S_4\tilde b_r S_1\tilde a_r M_v \in \mathcal{O}^{(4)}_{\mathcal{L}^1}(\lambda^2(\log\lambda)^2)$. Hence, it is $\mathcal{GVS}$.

(2) We have $\mathcal{O}(\tilde a_r, \tilde c_r)(\lambda) = \lambda^{-4}M_vS_4\widetilde B_3(\lambda)^{-1}S_4\tilde c_r S_2\tilde a_r M_v$. By using (9.44) for $S_4\tilde c_r$, that $\tilde a_r \in \mathcal{O}^{(4)}_{\mathcal{H}_2}(\lambda^2)$ and (9.45) for $S_4\widetilde B_3(\lambda)^{-1}S_4$, we have modulo $\mathcal{GVS}$ that

$$\begin{aligned}\mathcal{O}(\tilde a_r, \tilde c_r)(\lambda) &\equiv \tilde g_2(\lambda)M_vS_4\widetilde B_3(\lambda)^{-1}S_4G_4^{(v)}S_2G_2^{(v)}D_0M_v \\ &= \tilde g_2(\lambda)\sum_{j,k=r+1}^{m}\tilde t_{jk}(\lambda)(v\zeta_j)\otimes\rho_k, \quad \rho_k := M_vD_0G_2^{(v)}S_2G_4^{(v)}\zeta_k.\end{aligned}$$

Here $\tilde t_{jk}(\lambda)$ are GMU, $\langle z\rangle\rho_k \in L^1(\mathbb{R}^4)$ for $k = 1, \dots, 4$ and $\int_{\mathbb{R}^4}\rho_k(z)dz = 0$ since $D_0 = QD_0$ and $Qv = 0$. Hence, $\mathcal{O}(\tilde a_r, \tilde c_r)(\lambda)$ is GPR by virtue of Lemma 7.5. ■

Lemma 9.18. *The operator defined by (9.6) with $\mathcal{O}(\tilde b_r, \tilde c_r)(\lambda)$ in place of $\mathcal{N}^{(v)}_{2,2}(\lambda)$ is bounded in $L^p(\mathbb{R}^4)$ for $1 < p < 4$.*

Proof. We have $\mathcal{O}(\tilde b_r, \tilde c_r)(\lambda) = \lambda^{-4}M_vS_4\widetilde B_3(\lambda)^{-1}S_4\tilde c_r S_2\tilde b_r S_1\widetilde{\mathcal{D}}_0(\lambda)M_v$ and, by virtue of (9.32) and (9.44), this is equal modulo GPR to

$$-h_1(\lambda)^{-1}M_vS_4\widetilde B_3(\lambda)^{-1}S_4(\tilde g_2(\lambda)G_4^{(v)} + G_{4,l}^{(v)})\widetilde D_2G_2^{(v)}\widetilde{\mathcal{D}}_0(\lambda)\widetilde D_1\widetilde{\mathcal{D}}_0(\lambda)M_v. \tag{9.50}$$

We may simplify (9.50) without changing it modulo GPR as follows: (i) we may first replace $\widetilde B_3(\lambda)^{-1}$ by T_4^{-1} since the remainder produces GPR; (ii) we next replace the rightmost $\widetilde{\mathcal{D}}_0(\lambda) = D_0 + h_1(\lambda)L_0$ by $h_1(\lambda)L_0$, which is possible since $D_0M_v\Pi(\lambda) = D_0M_v(\Pi(\lambda) - \Pi(0))$ can be written in the form (1.39) and produces GPR; (iii) this leaves $h_1(\lambda)L_0$, which cancels $h_1(\lambda)^{-1}$ in the front and, then, $G_{4,l}^{(v)}$ may be removed; (iv) another $\widetilde{\mathcal{D}}_0(\lambda)$ may be replaced by D_0 since $\tilde g_2(\lambda)h_1(\lambda)$ is GMU. In this way, we have modulo GPR that

$$\begin{aligned}\mathcal{O}(\tilde b_r, \tilde c_r)(\lambda) &\equiv -\tilde g_2(\lambda)M_vS_4T_4^{-1}S_4G_4^{(v)}\widetilde D_2G_2^{(v)}\widetilde D_1L_0M_v \\ &= \sum_{j,k=r+1}^{m}\tilde g_2(\lambda)t_{jk}(v\zeta_j)\otimes\rho_k, \quad \rho_k = -M_vL_0\widetilde D_1G_2^{(v)}\widetilde D_2G_4^{(v)}\zeta_k\end{aligned} \tag{9.51}$$

and the operator produces by $\mathcal{O}(\tilde{b}_r,\tilde{c}_r)(\lambda)$ is equal modulo GOP to the linear combination of

$$\int_0^\infty (R_0^+(\lambda^4)((v\zeta_j)\otimes\rho_k)\Pi(\lambda)u\lambda^3\tilde{g}_2(\lambda)\chi_{\le a}(\lambda)d\lambda$$
$$=\int_{\mathbb{R}^4}\rho_k(z)\Big(\int_0^\infty (R_0^+(\lambda^4)(v\zeta_j)(x)(\Pi(\lambda)\tau_{-z}u)(0)\lambda^2\mu(\lambda)\chi_{\le a}(\lambda)d\lambda\Big)dz,$$

where $\mu(\lambda):=\lambda\tilde{g}_2(\lambda)$ is GMU. The function inside the brackets is equal to $W_B u$ of (8.20) with ω and u being replaced by $v\zeta_j$ and $\tau_{-z}\mu(|D|)u$ respectively. Then Lemma 8.9 and Lemma 8.11 imply the lemma. ■

We next consider the terms which contain one from $\{\tilde{a}_r,\tilde{b}_r,\tilde{c}_r\}$ and another from $\{\tilde{a}_l,\tilde{b}_l,\tilde{c}_l\}$.

Lemma 9.19. *If $\tilde{e}_l\in\{\tilde{a}_l,\tilde{b}_l,\tilde{c}_l\}$ and $\tilde{f}_r\in\{\tilde{a}_r,\tilde{b}_r,\tilde{c}_r\}$, then $\mathcal{O}(\tilde{e}_l,\tilde{f}_r)(\lambda)$ is GPR.*

Proof. (i) The operators $\mathcal{O}(\tilde{a}_l,\tilde{f}_r)(\lambda)$ and $\mathcal{O}(\tilde{b}_l,\tilde{f}_r)(\lambda)$ are $\mathcal{GVS}$ since

$$\tilde{a}_l S_4=-\lambda^4\tilde{g}_2(\lambda)D_0G_4^{(v)}S_4,\quad \tilde{b}_l S_4=\lambda^4 h_1(\lambda)^{-1}\tilde{g}_2(\lambda)D_1G_4^{(v)}S_4$$

by virtue of the cancellation properties (9.14) and (9.15) of S_4, $\tilde{f}_r\in\mathcal{O}^{(4)}_{\mathcal{H}_2}(\lambda^2\log\lambda)$ and M_v sandwiches them.

(ii) The operators $\mathcal{O}(\tilde{c}_l,\tilde{a}_r)(\lambda)$ and $\mathcal{O}(\tilde{c}_l,\tilde{b}_r)(\lambda)$ are also $\mathcal{GVS}$ since

$$\tilde{c}_l=\mathcal{O}_{S_2L^2}(\lambda^2\log\lambda)$$

and $S_4\tilde{a}_r, S_4\tilde{b}_r\in\mathcal{O}_{\mathcal{H}_2}(\lambda^4(\log\lambda)^2)$.

(iii) Recall $\{\zeta_1,\dots,\zeta_m\}$ is the basis of $S_2L^2(\mathbb{R}^4)$. Then, Lemma 8.1 for S_2, (9.44) for $\tilde{c}_l,\tilde{c}_r$, and (9.45) for $\widetilde{B}_3(\lambda)^{-1}$ jointly imply that $\mathcal{O}(\tilde{c}_l,\tilde{c}_r)(\lambda)$ is equal to

$$\lambda^{-4}M_vS_2\tilde{c}_lS_4\widetilde{B}_3(\lambda)^{-1}S_4\tilde{c}_rS_2M_v=\sum_{j,k=1}^m a_{jk}(\lambda)(v\zeta_j)\otimes(v\zeta_k)$$

with $a_{jk}(\lambda)\in\mathcal{O}^4_{\mathbb{C}}((\log\lambda)^2)$, $j,k=1,\dots,m$. Then, Lemma 7.5 implies that one has $\mathcal{O}(\tilde{c}_l,\tilde{c}_r)(\lambda)$ is GPR. ■

Finally, we consider the terms which contain one from $\{\tilde{a}_r,\tilde{b}_r,\tilde{c}_r\}$ but none of $\{\tilde{a}_l,\tilde{b}_l,\tilde{c}_l\}$, viz. $\mathcal{O}(\tilde{a}_r)(\lambda)$, $\mathcal{O}(\tilde{b}_r)(\lambda)$ and $\mathcal{O}(\tilde{c}_r)(\lambda)$.

Lemma 9.20. *Operator $\mathcal{O}(\tilde{a}_r)(\lambda)$ is GPR.*

Proof. We have $\mathcal{O}(\tilde{a}_r)(\lambda) = -\tilde{g}_2(\lambda)M_v S_4 \widetilde{B}_3(\lambda)^{-1} S_4 G_4^{(v)} D_0 M_v$ and (9.45) implies that, in terms of the basis $\{\zeta_{r+1}, \dots, \zeta_m\}$ of $S_4 L^2$ and with GMUs $\tilde{t}_{jk}(\lambda)$,

$$\mathcal{O}(\tilde{a}_r)(\lambda) = -\sum_{j,k=r+1}^{m} \tilde{g}_2(\lambda)\tilde{t}_{jk}(\lambda)(v\zeta_j) \otimes (M_v D_0 G_4^{(v)} \zeta_k).$$

Here $f(x) := M_v D_0 G_4^{(v)} \zeta_k(x)$ satisfies $\int_{\mathbb{R}^4} f(x)dx = 0$ as remarked in part (ii) in the proof Lemma 9.17. Thus, Lemma 7.5 implies $\mathcal{O}(\tilde{a}_r)(\lambda)$ is GPR. ∎

The next lemma concludes the proof of Theorem 1.9 (4).

Lemma 9.21. *The operators produced by (9.6) by replacing $\mathcal{N}_{2,2}^{(v)}(\lambda)$ by $\mathcal{O}(\tilde{b}_r)(\lambda)$ or $\mathcal{O}(\tilde{c}_r)(\lambda)$ are bounded in $L^p(\mathbb{R}^4)$ for $1 < p < 4$.*

Proof. (i) By virtue of (9.32) and (9.14), we have that modulo GPR

$$\mathcal{O}(\tilde{b}_r)(\lambda) \equiv h_1(\lambda)^{-1}\tilde{g}_2(\lambda)M_v S_4 T_4^{-1} S_4 G_4^{(v)} D_1 S_1 (D_0 + h_1(\lambda)L_0)M_v,$$

which is the sum of two terms. The one which contains $D_0 = D_0 Q$ is GPR as in the proof of Lemma 9.20 above; the other which contains $h_1(\lambda)L_0$ can be written in the form

$$\sum_{j,k=r+1}^{m} \tilde{g}_2(\lambda)t_{jk}(v\zeta_j) \otimes \tilde{\rho}_k, \quad \tilde{\rho}_k = M_v L_0 S_1 D_1 G_4^{(v)} \zeta_k \in L^1(\mathbb{R}^4).$$

This is of the same form as of (9.51) in the proof of Lemma 9.18 with ρ_k being replaced by $\tilde{\rho}_k$ which can play the role of the former. Hence, it produces a bounded operator in $L^p(\mathbb{R}^4)$ for $1 < p < 4$.

(ii) We have $\mathcal{O}(\tilde{c}_r)(\lambda) = \lambda^{-4} M_v S_4 \widetilde{B}_3(\lambda)^{-1} S_4 \tilde{c}_r S_2 M_v$. Since it has $S_2 M_v$ on the right end, the terms of order $\mathcal{O}_{\mathcal{H}}^{(4)}(\lambda^4(\log\lambda)^2)$ in (9.44) and the remainder term in (9.40) for $\widetilde{B}_3(\lambda)^{-1}$ produce GPR for $\mathcal{O}(\tilde{c}_r)(\lambda)$ by Lemma 7.5. It follows that modulo GPR

$$\mathcal{O}(\tilde{c}_r)(\lambda) \equiv \lambda^{-2} M_v S_4 T_4^{-1} S_4 (\tilde{g}_2(\lambda) G_4^{(v)} + G_{4,l}^{(v)}) S_2 D_2 S_2 M_v.$$

Let $\{\zeta_1, \dots, \zeta_m\}$ be the basis of $S_2 L^2$ such that $\{\zeta_{r+1} \dots, \zeta_m\}$ spans $S_4 L^2$. Then, we have with constants c_{jk} and d_{jk} that

$$\mathcal{O}(\tilde{c}_r)(\lambda) \equiv \sum_{j=r+1}^{m} \sum_{k=1}^{m} \lambda^{-2}\gamma_{jk}(\lambda)(v\zeta_j) \otimes (v\zeta_k), \quad \gamma_{jk}(\lambda) = c_{jk}\log\lambda + d_{jk}.$$

and the operator (5.1) produced by $\mathcal{O}(\tilde{c}_r)(\lambda)$ becomes modulo GOP

$$\sum_{j=r+1}^{m} \sum_{k=1}^{m} \int_0^\infty (R_0^+(\lambda^4) M_v \zeta_j)(x) \langle v\zeta_k, \Pi(\lambda)u \rangle \lambda \gamma_{jk}(\lambda) \chi_{\le a}(\lambda) d\lambda. \tag{9.52}$$

Since $\int_{\mathbb{R}^4} v(z)\zeta_k(z)dz = 0$, we may replace $\Pi(\lambda)u$ by (1.39) in (9.52). Then each summand becomes the superposition by $i\sum_{m=1}^4 \int_0^1 d\theta \int_{\mathbb{R}^4} z_m(v\zeta_k)(z)dz$ of

$$\int_0^\infty (R_0^+(\lambda^4)M_v\zeta_j)(x)\Pi(\lambda)(\tau_{-\theta z}R_l u)(0)\lambda^2\gamma_{jk}(\lambda)\chi_{\le a}(\lambda)d\lambda. \tag{9.53}$$

If we replace $\gamma_{jk}(\lambda)$ by d_{jk}, (9.53) becomes the trivial modification of (8.20) and it is bounded in $L^p(\mathbb{R}^4)$ for $1 < p < 4$ by Lemmas 8.9 and 8.11. Thus, the next lemma with Minkowski's inequality completes the proof of Lemma 9.21. ∎

Lemma 9.22. *Let $\zeta \in S_4L^2$. Then, the operator Z_{add} defined by*

$$Z_{\text{add}}u(x) = \int_0^\infty R_0^+(\lambda^4)(v\zeta)(x)\Pi(\lambda)u(0)\lambda^2(\log\lambda)\chi_{\le a}(\lambda)d\lambda,$$

is bounded in $L^p(\mathbb{R}^4)$ for $1 < p < 4$.

Proof. The proof is the modification of that of Lemmas 8.9 and 8.11. Note that Z_{add} differs from W_B of (8.20) only in that the former has stronger singularity by $\log\lambda$ than the latter and $\omega = v\zeta$ for Z_{add} enjoys better cancellation property than that for W_B.

(1) We first show that $\chi_{\ge 4a}(|D|)Z_{\text{add}} \in \mathbf{B}(L^p)$ for $1 < p < 4$. The argument of the proof of Lemma 8.9 implies that this follows if the following linear functional which replaces (8.23):

$$\tilde{\ell}(u) = \int_0^\infty \Pi(\lambda)u(0)\lambda^2(\log\lambda)\chi_{\le a}(\lambda)d\lambda = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} u(x)f(x)dx$$

is bounded on $L^p(\mathbb{R}^4)$ for $1 < p < 4$, where

$$f(x) = \mathcal{F}(\chi_{\le a}(|\xi|)|\xi|^{-1}\log|\xi|)(x).$$

However, this is obvious by Hölder's inequality since

$$f(x) = \int_{\mathbb{R}^4} \widehat{\chi_{\le a}}(x-y)|y|^3(\alpha\log|y| + \beta)dy \in L^p(\mathbb{R}^4), \quad 4/3 < p \le \infty, \tag{9.54}$$

where α and β are constants (cf. Theorem 2.4.6 of [12]).

(2) We next show that $\chi_{\le 4a}(|D|)Z_{\text{add}}$ is also bounded in $L^p(\mathbb{R}^4)$ for $1 < p < 4$:

$$\chi_{\le 4a}(|D|)Z_{\text{add}}u(x) = \int_0^\infty \big(\chi_{\le 4a}(|D|)R_0^+(\lambda^4)\omega\big)(x)\Pi(\lambda)u(0)\mu_a(\lambda)d\lambda,$$

$$\mu_a(\lambda) = \lambda^2(\log\lambda)\chi_{\le a}(\lambda).$$

We proceed as in the proof of Lemma 8.11. Since $\int_{\mathbb{R}^4} x^\alpha \omega(x)dx = 0$ for $|\alpha| \le 2$ we have

$$\hat{\omega}(\xi) = \sum_{|\alpha|=3} C_\alpha \xi^\alpha \int_0^1 (1-\theta)^2 \Big(\int_{\mathbb{R}^4} e^{-i\theta z\xi} z^\alpha \omega(z) dz \Big) d\theta,$$

where C_α are unimportant constants. It follows that $\chi_{\le 4a}(|D|)R_0^+(\lambda^4)\omega(x)$ is equal to

$$\sum_{|\alpha|=3} C_\alpha \int_0^1 (1-\theta)^2 \Big(\int_{\mathbb{R}^4} z^\alpha \omega(z) R^\alpha \tau_{\theta z} \mathcal{A}_\lambda(x) dz \Big) d\theta, \tag{9.55}$$

$$\mathcal{A}_\lambda(x) = \lim_{\varepsilon\to+0} \int_{\mathbb{R}^4} \frac{e^{ix\xi}|\xi|^3 \chi_{\le 4a}(|\xi|)}{|\xi|^4 - \lambda^4 - i\varepsilon} \frac{d\xi}{(2\pi)^4} \tag{9.56}$$

and $\chi_{\le 4a}(|D|)Z_{\text{add}}u(x)$ becomes the same superposition as in (9.55) as follows:

$$\sum_{|\alpha|=3} C_\alpha \int_0^1 \int_{\mathbb{R}^4} (1-\theta)^2 z^\alpha \omega(z) R^\alpha \tau_{\theta z} \Big(\int_0^\infty \mathcal{A}_\lambda(x) \Pi(\lambda)u(0)\mu_a(\lambda) d\lambda \Big) d\theta dz \tag{9.57}$$

We substitue

$$\frac{|\xi|^3}{|\xi|^4 - \lambda^4 - i\varepsilon} = \frac{\lambda^3}{|\xi|^4 - \lambda^4 - i\varepsilon} + \frac{|\xi|^3 - \lambda^3}{|\xi|^4 - \lambda^4 - i\varepsilon}.$$

Then, the integral on the right-hand side of (9.56), which is uniformly bounded by $C\langle x\rangle^{-\frac{3}{2}}$ for $\varepsilon > 0$ and $\lambda \in \operatorname{supp} \Pi(\lambda)u(0)$, converges compact uniformly as $\varepsilon \to 0$ to

$$\mathcal{A}_\lambda(x) = \lambda^3 \chi_{\le 4a}(|D|)\mathcal{R}_\lambda(x) + \mathcal{B}_\lambda(x),$$
$$\mathcal{B}_\lambda(x) = \frac{1}{2(2\pi)^4} \int_{\mathbb{R}^4} \Big(\frac{1}{|\xi|+\lambda} + \frac{|\xi|+\lambda}{|\xi|^2+\lambda^2} \Big) e^{ix\xi} \chi_{\le 4a(|\xi|)} d\xi.$$

Then, $\lambda^3 \chi_{\le 4a}(|D|)\mathcal{R}_\lambda(x)$ produces the superposition as in (9.57) of

$$\int_0^\infty \chi_{\le 4a}(|D|)\mathcal{R}_\lambda(x)\Pi(\lambda)u(0)\mu_a(\lambda)\lambda^3 d\lambda = \chi_{\le 4a}(|D|)K\mu_a(D)u(x),$$

which is GOP by virtue of Lemma 3.4.

Restoring $\mu_a(\lambda) = \lambda^2 \log\lambda \chi_{\le a}(\lambda)$, we see that $\mathcal{B}_\lambda(x)$ produces the same superposition as in (9.57) of

$$\mathcal{M}u(x) = \int_0^\infty \mathcal{B}_\lambda(x)\Pi(\lambda)u(0)\lambda^2 \log\lambda \chi_{\le a}(\lambda) d\lambda.$$

Then, the computation which led to (8.27) yields that

$$\mathcal{M}u(x) = L_1u(x) + L_2u(x) := \int_{\mathbb{R}^4} (L_1(x,y) + L_2(x,y))u(y),$$

$$L_1(x,y) = \int_{\mathbb{R}^8} \frac{e^{ix\xi - iy\eta}\chi_{\le 4a}(|\xi|)\chi_{\le a}(|\eta|)}{(2\pi)^2(|\xi| + |\eta|)} \frac{\log|\eta|}{|\eta|} d\xi d\eta,$$

$$L_2(x,y) = \int_{\mathbb{R}^8} \frac{e^{ix\xi - iy\eta}(|\xi| + |\eta|)\chi_{\le 4a}(|\xi|)\chi_{\le a}(|\eta|)}{(2\pi)^2(|\xi|^2 + |\eta|^2)} \frac{\log|\eta|}{|\eta|} d\xi d\eta. \tag{9.58}$$

We prove that L_1 and L_2 are bounded in $L^p(\mathbb{R}^4)$ for $1 < p < 4$ which will finish the proof of the lemma. Let $q = p/(p-1)$, $4/3 < q < \infty$.

(i) The obvious modification of the proof of Lemma 10.3 by using (9.54) instead of $|\mathcal{F}(|\eta|^{-1}\hat{\chi}_{\le a}(\eta))(y)| \le C\langle y\rangle^{-3}$ implies

$$|L_1(x,y)| \le \frac{C\log(|y|+2)}{\langle x\rangle^3\langle y\rangle(1+|x|+|y|)^2}.$$

It is obvious that $\|L_1(x,\cdot)\|_q \le C_q\langle x\rangle^{-3}$ for $q > 4/3$, which we use for small $|x| \le \max(10, C_q)$, where C_q is such that $2q(3q-4)^{-1} < \log C_q$. Let $|x| \ge \max(10, C_q)$. Then, we evidently have

$$\left(\int_{|y|\le|x|} |L_1(x,y)|^q dy\right)^{\frac{1}{q}} \le \frac{C\log|x|}{\langle x\rangle^5}\left(\int_0^{|x|} \frac{r^3 dr}{\langle r\rangle^q}\right)^{\frac{1}{q}} \le C\log|x| \begin{cases} \langle x\rangle^{-5}, & q > 4, \\ \langle x\rangle^{-2-\frac{4}{p}}, & q < 4, \end{cases}$$

and, by using integration by parts, we also have

$$\left(\int_{|x|\le|y|} |L_1(x,y)|^q dy\right)^{\frac{1}{q}} \le \frac{C}{\langle x\rangle^3}\left(\int_{|x|}^{\infty} \frac{(\log r)^q}{r^{3q-3}} dr\right)^{\frac{1}{q}} \le \frac{C\log|x|}{\langle x\rangle^{2+\frac{4}{p}}}.$$

It follows that

$$\|L_1u\|_p \le \left(\int_{\mathbb{R}^4} \|L_1(x,y)\|^p_{L^q(\mathbb{R}^4)} dx\right)^{\frac{1}{p}} \|u\|_p \le C\|u\|_p.$$

(ii) Let $\nu_{8a}(\xi) = |\xi|\chi_{\le 8a}(|\xi|)$, $\nu_{2a}(\eta) = |\eta|\chi_{\le 2a}(|\eta|)$ and

$$L_3(x,y) = \int_{\mathbb{R}^8} \frac{e^{ix\xi - iy\eta}\chi_{\le 4a}(|\xi|)\chi_{\le a}(|\eta|)}{(2\pi)^2(|\xi|^2 + |\eta|^2)} \frac{\log|\eta|}{|\eta|} d\xi d\eta.$$

Then, by replacing $|\xi| + |\eta|$ by $\nu_{8a}(\xi) + \nu_{2a}(\eta)$ in (9.58), we obtain

$$L_2 = \nu_{8a}(D)L_3 + L_3\nu_{2a}(D) \tag{9.59}$$

and $\nu_{8a}(D)$ and $\nu_{2a}(D)$ are GOP since $\hat{\nu}_b \in L^1(\mathbb{R}^4)$ for any $b > 0$. We obtain by using (9.54) once more that

$$|L_3(x,y)| \le C \int_{\mathbb{R}^4} \frac{\log(2+|y-z|)dz}{(1+|x|+|z|)^6\langle y-z\rangle^3}.$$

Then,

$$\sup_{y\in\mathbb{R}^4} \int_{\mathbb{R}^4} |L_3(x,y)|dx \le C \sup_{y\in\mathbb{R}^4} \int_{\mathbb{R}^4} \frac{\log(2+|y-z|)dz}{(1+|z|)^2\langle y-z\rangle^3} < \infty$$

and L_3 is bounded in $L^1(\mathbb{R}^4)$. For $2 < p < 4$, we have $4/3 < q < 2$ and Minkowski's and Hölder's inequalities imply

$$|L_3u(x)| \le \|u\|_p \left\| \frac{\log(2+|y|)}{\langle y\rangle^3} \right\|_q \int_{\mathbb{R}^4} \frac{dz}{(1+|x|+|z|)^6} \le C \frac{\|u\|_p}{\langle x\rangle^2}$$

and $\|L_3u\|_p \le C\|u\|_p$. Thus, L_3 is bounded in $L^p(\mathbb{R}^4)$ also for $2 < p < 4$ and, hence, for all $1 \le p < 4$ by interpolation. Thus, so is L_2 by virtue of (9.59) and the lemma is proved. ■

10. Appendix. Proof of Lemma 8.12

We admit the following lemma for the moment and complete the proof of Lemma 8.12 first.

Lemma 10.1. *There exists constant $C > 0$ such that*

$$|L(x,y)| \le \frac{C}{\langle x\rangle(1+|x|+|y|)^3}. \tag{10.1}$$

Proof of Lemma 8.12. By the change of variables $y = (1+|x|)z$ and by integrating over the spherical variables first, we have

$$|Lu(x)| \le C \int_{\mathbb{R}^4} \frac{|u((1+|x|)z)|dz}{(1+|z|)^3} = C\gamma_3 \int_0^\infty \frac{M_{|u|}((1+|x|)r)r^3dr}{(1+r)^3}$$

where γ_3 is the surface measure of $\mathbb{S}^3$. Then Hölder's inequality implies

$$\begin{aligned}\int_{\mathbb{R}^4} M_{|u|}((1+|x|)r)^p dx &= \gamma_3 \int_0^\infty M_{|u|}((1+\rho)r)^p \rho^3 d\rho \\ &\le \gamma_3 \int_1^\infty (M_{|u|}(\rho r))^p \rho^3 d\rho \\ &\le \gamma_3 r^{-4} \int_r^\infty M_{|u|}(\rho)^p \rho^3 d\rho \le C r^{-4} \|u\|_p^p.\end{aligned}$$

It follows by Minkowski's inequality that for $1 < p < 4$

$$\|Lu\|_p \le C\gamma_3 \int_0^\infty \frac{\|u\|_p r^{3-\frac{4}{p}} dr}{(1+|r|)^3} \le C' \|u\|_p.$$

This completes the proof of Lemma 8.12 and, hence, of Lemma 8.11. ■

Proof of Lemma 10.1. Denote $\chi_a(|\xi|)$, etc by $\chi_a(\xi)$, etc. It suffices to show (10.1) for the convolution of the Fourier transforms of

$$f_1(\xi,\eta) = \frac{\chi_{\le a}(\xi)\chi_{\le a}(\eta)}{|\xi|+|\eta|}, \quad f_2(\eta) = \frac{\chi_{\le a}(\eta)}{|\eta|}, \quad f_3(\xi,\eta) = \frac{\chi_{\le a}(\xi)\chi_{\le a}(\eta)}{|\xi|^2+|\eta|^2},$$

which we denote by $L(x, y)$ again. By the rotational symmetry and homogeneity we have that

$$|\hat{f}_2(x)| \le C\langle x\rangle^{-3}, \quad |\hat{f}_3(x,y)| \le C(\langle x\rangle^2 + \langle y\rangle^2)^{-3}. \tag{10.2}$$

■

Lemma 10.2. *For $a > 0$, there exists a constant $C > 0$ such that*

$$|\hat{f}_1(x,y)| \le C\langle x\rangle^{-2}\langle y\rangle^{-2}(\langle x\rangle + \langle y\rangle)^{-3}. \tag{10.3}$$

Proof. By following the argument in [26, pp. 61-62], we obtain

$$\begin{aligned}\iint_{\mathbb{R}^8} \frac{e^{ix\xi - iy\eta}}{|\xi|+|\eta|} d\xi d\eta &= \int_0^\infty \Big(\int_{\mathbb{R}^4} e^{ix\xi - t|\xi|} d\xi\Big)\Big(\int_{\mathbb{R}^4} e^{-iy\eta - t|\eta|} d\eta\Big) dt \\ &= \frac{c_4}{(|x|^2+|y|^2)^{\frac{7}{2}}} \int_0^\infty \frac{s^2 ds}{(s^4+s^2+F^2)^{\frac{5}{2}}},\end{aligned}$$

with

$$0 \le F = \frac{|x||y|}{|x|^2 + |y|^2} \le \frac{1}{2}.$$

The last integral is bounded by $\int_0^\infty s^2(s^2 + F^2)^{-\frac{5}{2}} ds = CF^{-2}$ and

$$\iint_{\mathbb{R}^8} \frac{e^{ix\xi - iy\eta}}{|\xi| + |\eta|} d\xi d\eta \le_{|\cdot|} \frac{C}{|x|^2|y|^2(|x| + |y|)^3}.$$

It follow that

$$|\hat{f}_1(x, y)| \le C \iint_{\mathbb{R}^8} \frac{|\hat{\chi}_a(x - z)\hat{\chi}_a(y - w)|}{|z|^2|w|^2(|z| + |w|)^3} dwdz \le \frac{C}{\langle x\rangle^2\langle y\rangle^2(\langle x\rangle + \langle y\rangle)^3}. \qquad ■$$

Lemma 10.3. *For $a > 0$, there exists a constant $C > 0$ such that*

$$\iint_{\mathbb{R}^8} e^{ix\xi - iy\eta} \frac{\chi_{\le a}(|\xi|)\chi_{\le a}(|\eta|)}{(|\xi| + |\eta|)|\eta|} d\xi d\eta \le_{|\cdot|} \frac{C}{\langle x\rangle^3\langle y\rangle(\langle x\rangle + \langle y\rangle)^2}. \tag{10.4}$$

Proof. By (10.2) and (10.3), it suffices to prove

$$\int_{\mathbb{R}^4} \frac{dz}{\langle z\rangle^2(1 + |x| + |z|)^3\langle y - z\rangle^3} \le \frac{C}{\langle x\rangle\langle y\rangle(\langle x\rangle + \langle y\rangle)^2}. \tag{10.5}$$

Let $\Delta_1 = \{|y - z| \le |y|/2\}$, $\Delta_2 = \{|z| \ge 2|y|\}$ and $\Delta_3 = \{|y|/2 < |y - z|,\ |z| \le 2|y|\}$ so that $\mathbb{R}^4 = \Delta_1 \cup \Delta_2 \cup \Delta_3$. Denote the integrand of (10.5) by $F(x, y, z)$. Since $|y|/2 \le |z| \le 3|y|/2$ on Δ_1, by using polar coordinates we have

$$\int_{\Delta_1} F(x, y, z)dz \le \int_0^{|y|/2} \frac{r^3 dr}{\langle y\rangle^2(1 + |x| + |y|)^3\langle r\rangle^3} \le \frac{C}{\langle y\rangle(1 + |x| + |y|)^3}.$$

Since $|y - z| \ge |z|/2$ on Δ_2,

$$\int_{\Delta_2} F(x, y, z)dz \le \int_{|z|>2|y|} \frac{dz}{\langle z\rangle^5(1 + |x| + 2|y|)^3} \le \frac{C}{\langle y\rangle(1 + |x| + |y|)^3}.$$

Since $|y|/2 < |z - y| \le 5|y|/2$ on Δ_3,

$$\int_{\Delta_3} F(x, y, z)dz \le \frac{C}{\langle y\rangle^3} \int_0^{2|y|} \frac{r dr}{(1 + |x| + r)^3} \le \frac{C}{\langle y\rangle\langle x\rangle(1 + |x| + |y|)^2}.$$

Summing up, we obtain (10.5). ■

Proof of Lemma 10.1. By (10.2) and (10.4), it suffices to show

$$\int_{\mathbb{R}^8} \frac{dwdz}{(\langle x-w\rangle + \langle y-z\rangle)^6 \langle w\rangle^3 \langle z\rangle (\langle w\rangle + \langle z\rangle)^2} \le \frac{C}{\langle x\rangle (1+|x|+|y|)^3}.$$

Denote the integrand by $F = F(x, y, w, z)$ and split $\mathbb{R}^4_w = \Delta_1 \cup \Delta_2 \cup \Delta_3$ and $\mathbb{R}^4_z = \Delta'_1 \cup \Delta'_2 \cup \Delta'_3$ where

$$\Delta_1 = \{w : |w-x| \le |x|/2\} \text{ (then } |x|/2 \le |w| \le 3|x|/2); \tag{10.6}$$

$$\Delta_2 = \{w : |x|/2 < |w-x| \le 2|x|\} \text{ (then } |w| \le 3|x|); \tag{10.7}$$

$$\Delta_3 = \{w : |w-x| \ge 2|x|\} \text{ (then } 2|w|/3 \le |w-x| \le 2|w|,\ |w| \ge |x|); \tag{10.8}$$

$$\Delta'_1 = \{z : |z-y| \le |y|/2\} \text{ (then } |y|/2 \le |z| \le 3|y|/2); \tag{10.9}$$

$$\Delta'_2 = \{z : |y|/2 < |z-y| \le 2|y|\} \text{ (then } |z| \le 3|y|); \tag{10.10}$$

$$\Delta'_3 = \{z : |z-y| \ge 2|y|\} \text{ (then } 2|z|/3 \le |z-y| \le 2|z|,\ |z| \ge |y|). \tag{10.11}$$

Here the remarks in the parentheses are obvious except possibly for the first ones for Δ_3 and Δ'_3. We prove the one for Δ_3. Since $|w| \ge |x|$, $|w-x| \le |w|+|x| \le 2|w|$; if $|w| > 3|x|$, then $|w-x| \ge |w|-|x| > 2|w|/3$; if $|w| \le 3|x|$, then $|w-x| \ge 2|x| \ge 2|w|/3$.

We shall show separately for $1 \le j, k \le 3$ that

$$L_{jk}(x, y) = \int_{\Delta_j \times \Delta'_k} F(x, y, w, z) dwdz \le \frac{C}{\langle x\rangle (1+|x|+|y|)^3}. \tag{10.12}$$

The proof is elementary and is similar for all of them. Thus, we shall be a little skippy in what follows.

(11) We have (10.6) and (10.9) for $(w, z) \in \Delta_1 \times \Delta'_1$ and

$$L_{11}(x, y) \le_{|\cdot|} \frac{1}{\langle x\rangle^3 \langle y\rangle (1+|x|+|y|)^2} \int_0^{|x|/2} \int_0^{|y|/2} \frac{r^3 \rho^3 dr d\rho}{(1+r+\rho)^6}.$$

The integral is bounded by a constant times

$$\frac{1}{4} \int_0^{|x|^4} \int_0^{|y|^4} \frac{d\sigma d\tau}{(1+\sigma+\tau)^{\frac{3}{2}}} = (1+|x|^4)^{\frac{1}{2}} + (1+|y|^4)^{\frac{1}{2}} - (1+|x|^4+|y|^4)^{\frac{1}{2}} - 1,$$

which is bounded by $C\langle x\rangle^2 \langle y\rangle^2 (\langle x\rangle^2 + \langle y\rangle^2)^{-1}$ and (10.12) for $L_{11}(x, y)$ follows.

(12) By virtue of (10.6) and (10.10) for $(w,z) \in \Delta_1 \times \Delta_2'$,

$$\begin{aligned} L_{12}(x,y) &\le \frac{C}{\langle x\rangle^3} \int_{\Delta_1\times\Delta_2'} \frac{C\,dw\,dz}{(1+|x-w|+|y|)^6\langle z\rangle(1+|x|+|z|)^2} \\ &= \frac{C}{\langle x\rangle^3}\Big(\int_{\Delta_1} \frac{dw}{(1+|x-w|+|y|)^6}\Big)\Big(\int_{\Delta_2'} \frac{dz}{\langle z\rangle(1+|x|+|z|)^2}\Big) \\ &= \frac{C}{\langle x\rangle^3} I_1(x,y) I_2(x,y) \end{aligned}$$

where definitions should be obvious. Then,

$$I_1(x,y) = \gamma_3 \int_0^{|x|/2} \frac{r^3 dr}{(1+r+|y|)^6} \le C \frac{|x|^4}{\langle y\rangle^2(\langle x\rangle^4 + \langle y\rangle^4)}. \tag{10.13}$$

$$I_2(x,y) \le C \int_0^{3|y|} \frac{\rho^2 d\rho}{(1+|x|+\rho)^2} \le \frac{C|y|^3}{(\langle x\rangle^2 + \langle y\rangle^2)}. \tag{10.14}$$

and (10.12) for $L_{12}(x,y)$ follows.

(13) Since $|x|/2 \le |w| \le 3|x|/2$ on Δ_1, we have

$$L_{13}(x,y) \le \frac{C}{\langle x\rangle^3} \int_{\Delta_3'} \Big(\int_{\Delta_1} \frac{dw}{(1+|x-w|+|y-z|)^6}\Big) \frac{dz}{\langle z\rangle(1+|x|+|z|)^2}$$

The dw-integral is equal to $I_1(x,y-z)$ of (10.13) and $2|z|/3 \le |y-z| \le 2|z|$ for $z \in \Delta_3'$. It follows that

$$L_{13}(x,y) \le C|x| \int_{|z|>|y|} \frac{dz}{\langle z\rangle^3(1+|x|+|z|)^6} \le \frac{C}{\langle x\rangle(1+|x|+|y|)^3}.$$

(21) By using (10.7) and (10.9) for $(w,z) \in \Delta_2 \times \Delta_1'$, we have

$$\begin{aligned} L_{21}(x,y) &\le \int_{\Delta_2\times\Delta_1'} \frac{C\,dw\,dz}{(1+|x|+|y-z|)^6\langle w\rangle^3\langle y\rangle(1+|w|+|y|)^2} \\ &= \frac{C}{\langle y\rangle}\Big(\int_{\Delta_2} \frac{dw}{\langle w\rangle^3(1+|w|+|y|)^2}\Big) I_1(y,x) \\ &\le \frac{C}{\langle x\rangle(1+|x|+|y|)^3}. \end{aligned}$$

(22) We have

$$\begin{aligned}
L_{22}(x,y) &\le \frac{C}{(1+|x|+|y|)^6} \int_{\Delta_2 \times \Delta_2'} \frac{dwdz}{\langle w\rangle^3 \langle z\rangle (1+|w|+|z|)^2} \\
&\le \frac{C}{(1+|x|+|y|)^6} \int_{\Delta_2 \times \Delta_2'} \frac{dwdz}{\langle w\rangle^3 \langle z\rangle^3} \\
&\le \frac{C|x||y|}{(1+|x|+|y|)^6} \le \frac{C}{\langle x\rangle (1+|x|+|y|)^3}.
\end{aligned}$$

(23) Use (10.7) and (10.11) and polar coordinates $w = r\sigma$ and $z = \rho\omega$. Then

$$\begin{aligned}
L_{23}(x,y) &\le C \int_{|z|>|y|} \Bigg(\int_{|w|\le 3|x|} \frac{dw}{\langle w\rangle^3 (\langle w\rangle + \langle z\rangle)^2} \Bigg) \frac{dz}{\langle z\rangle (1+|x|+|z|)^6} \\
&\le \int_{|y|}^{\infty} \frac{C\rho^3 d\rho}{(1+\rho)^2 (1+|x|+\rho)^6} \le \int_{|y|^2}^{\infty} \frac{C dr}{(1+|x|^2+r)^3} \\
&\le \frac{C}{\langle x\rangle (1+|x|+|y|)^3}.
\end{aligned}$$

(31) For $(w,z) \in \Delta_3 \times \Delta_1'$, we have (10.8) and (10.9). Then, by using (10.13),

$$\begin{aligned}
L_{31}(x,y) &\le C \int_{\Delta_3} \Bigg(\int_{\Delta_1} \frac{dz}{(1+|w|+|y-z|)^6} \Bigg) \frac{dw}{\langle w\rangle^3 \langle y\rangle (1+|w|+|y|)^2} \\
&= \int_{\Delta_3} \frac{I_1(y,w)dw}{\langle w\rangle^3 \langle y\rangle (1+|w|+|y|)^2} \le \frac{C|y|^4}{\langle y\rangle} \int_{|w|>|x|} \frac{dw}{\langle w\rangle^5 (1+|w|+|y|)^6} \\
&\le \frac{C|y|^3}{(1+|x|+|y|)^6} \int_{|w|>|x|} \frac{dw}{\langle w\rangle^5} \le \frac{C}{\langle x\rangle (1+|x|+|y|)^3}.
\end{aligned}$$

(32) For $(w,z) \in \Delta_3 \times \Delta_2'$, we have (10.8) and (10.10) and

$$L(x,y) \le \int_{\Delta_3} \Bigg(\int_{\Delta_2'} \frac{dz}{\langle z\rangle (1+|w|+|z|)^2} \Bigg) \frac{dw}{(1+|w|+|y|)^6 \langle w\rangle^3}.$$

Estimating the dz-integral by $I_2(w,y)$ by using (10.14), we obtain

$$\begin{aligned}
L(x,y) &\le \int_{|w|>|x|} \frac{C|y|^3 dw}{(1+|w|+|y|)^8 \langle w\rangle^3} \\
&\le \frac{C|y|^3}{(1+|x|+|y|)^7} \le \frac{C}{\langle x\rangle (1+|x|+|y|)^3}.
\end{aligned}$$

(33) For $(w, z) \in \Delta_3 \times \Delta_3'$, we have (10.8) and (10.11). Hence,

$$L(x,y) \le \int_{|z|>|y|} \left(\int_{|w|>|x|} \frac{C\,dw}{(1+|w|+|z|)^8 \langle w\rangle^3} \right) \frac{dz}{(1+|z|)}$$

$$\le \int_{|z|>|y|} \frac{C\,dz}{(1+|z|)(1+|x|+|z|)^7} \le C \int_{|y|}^{\infty} \frac{dr}{(1+|x|+r)^5}.$$

This is bounded by $C\langle x\rangle^{-1}(1+|x|+|y|)^{-3}$ and completes the proof. ■

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Artbazar Galtbayar
Center of Mathematics for Applications and Department of Applied Mathematics, National University of Mongolia, University Street 3, 10623 Ulaanbaatar, Mongolia; galtbayar@num.edu.mn

Kenji Yajima
Department of Mathematics, Gakushuin University, 1-5-1 Mejiro, Toshima-ku, Tokyo 171-8588, Japan; kenji.yajima@gakushuin.ac.jp