J. Spectr. Theory 14 (2024), 185–205
DOI 10.4171/JST/482

Quantitative magnetic isoperimetric inequality

Rohan Ghanta, Lukas Junge, and Léo Morin

Abstract. In 1996 Erdős showed that among planar domains of fixed area, the smallest principal eigenvalue of the Dirichlet Laplacian with a constant magnetic field is uniquely achieved on the disk. We establish a quantitative version of this inequality, with an explicit remainder term depending on the field strength that measures how much the domain deviates from the disk.

1. Introduction

To solve a problem in probability and mathematical physics [11, 12], Erdős developed the magnetic isoperimetric inequality [10]. It generalizes the Faber–Krahn inequality to the magnetic Laplacian. Starting with Pólya and Szegő [20], Faber–Krahn-type results have been established by proving rearrangement inequalities. The inclusion of a magnetic field, however, makes it notoriously difficult to implement the standard symmetrization methods. Erdős met the challenge head on: he managed to prove a magnetic rearrangement inequality, which is reminiscent of the celebrated Pólya–Szegő inequality, but with an interesting caveat. Such symmetry results with a magnetic field are – alas! – very few and far between [1, 5].

Still another compelling feature is that rearrangements alone are not sufficient for arguing the magnetic isoperimetric inequality. This stands in sharp contrast to the classical Faber–Krahn setting. To complete the proof, Erdős introduced a new inequality, tailored specifically for a magnetic Schrödinger operator on a disk, and for which there exists no analog in the absence of a magnetic field.

We improve Erdős' result. He showed that if a planar domain is not a disk, then the principal eigenvalue of the Dirichlet magnetic Laplacian is strictly larger on that domain than on the disk of same area. We take the next step and establish stability: if the principal eigenvalue of the magnetic Laplacian is just slightly larger on a planar domain than on the disk of same area, then that domain is only slightly different from the disk. Faint perturbations of the smallest principal eigenvalue will not induce a

Mathematics Subject Classification 2020: 35P15.
Keywords: magnetic Laplacian, spectral geometry, Faber–Krahn, stability.

dramatic change in the underlying geometry – and this dynamic is very sensitive to the field strength. We prove our stability estimate with a remainder term that quantifies the difference between the domain and the disk.

Quantitative Faber–Krahn-type inequalities have been developed almost exclusively around the classical theory of rearrangements. Fueled in large part by the seminal work of Fusco, Maggi, and Pratelli [13], the last decade has given rise to an entire industry now devoted to the stability of a remarkable range of geometric and functional inequalities. Our paper provides the first stability result with a magnetic field. And here, the well-established rearrangement framework is no longer sufficient.

2. Statement of the problem and main result

Let $\Omega \subset \mathbb{R}^2$ be a bounded, connected open set with a smooth boundary. The principal eigenvalue of the Dirichlet magnetic Laplacian on the planar domain Ω is

$$\lambda(B,\Omega) := \inf_{f \in H_0^1(\Omega)} \frac{\int_\Omega |(-i\nabla - \alpha) f|^2 dx}{\int_\Omega |f|^2 dx},$$

where $\alpha = \frac{B}{2}(-x_2, x_1)$ is a magnetic vector potential generating a homogeneous magnetic field of strength $B \geq 0$, i.e., $\operatorname{rot}(\alpha) = B$. We denote by D_R a disk of radius R, centered at the origin, with the same area as Ω, i.e., $|\Omega| = |D_R| = \pi R^2$.

In 1996, Erdős [10] proved the magnetic isoperimetric inequality

$$\lambda(B,\Omega) \geq \lambda(B, D_R), \tag{2.1}$$

with equality if and only if Ω is a disk. In the absence of a magnetic field, i.e., $B = 0$, his result reduces to the usual Faber–Krahn inequality.

In this paper, we want to add to the right-hand side of (2.1) a remainder term that measures how much the planar domain Ω deviates from being a disk. This would make it possible to understand the shape of Ω now in terms of how close it is to achieving equality in (2.1). Cf. [7] and references therein.

We measure the difference between Ω and the disk in the usual way in terms of the interior deficiency and the Fraenkel asymmetry of the domain.

Definition. The interior deficiency (asymmetry) of a set is defined as

$$\mathcal{A}_I(\Omega) := \frac{R - \rho_-(\Omega)}{R},$$

where $\rho_-(\Omega)$ denotes the radius of the largest ball contained in Ω, and R as above is the radius of D_R.

Definition. The Fraenkel asymmetry of a set is defined as

$$\mathcal{A}_F(\Omega) := \inf_{x_0 \in \mathbb{R}^2} \frac{|\Omega \Delta (x_0 + D_R)|}{2|\Omega|},$$

where Δ denotes the symmetric difference.

Both asymmetries are bounded by one and vanish if and only if the set is a disk.

Our main result is a quantitative version of the magnetic isoperimetric inequality.

Theorem 2.1. *Let $\mathcal{A}(\Omega)$ denote either the interior asymmetry or the Fraenkel asymmetry. In the case of the interior asymmetry we also assume that Ω is simply connected. Then there is a universal constant $c > 0$, independent of Ω and B, such that*

$$\lambda(B, \Omega) \geq \lambda(B, D_R)(1 + ce^{-\frac{5}{6}BR^2} \mathcal{A}(\Omega)^{\frac{10}{3}}). \tag{2.2}$$

Moreover, if $0 \leq BR^2 \leq \frac{1}{\pi}$, then

$$\lambda(B, \Omega) \geq \lambda(B, D_R)(1 + c\mathcal{A}(\Omega)^3). \tag{2.3}$$

Remark 2.2. The quantity $\mathcal{A}(\Omega)$ is scale invariant. Furthermore, λ scales like

$$t^2 \lambda(B, t\Omega) = \lambda(t^2 B, \Omega)$$

for $t > 0$, so the factor BR^2 appearing in our constant is the natural parameter for this problem.

In the absence of a magnetic field, i.e., $B = 0$, the estimate in (2.3) reduces to Hansen and Nadirashvili's quantitative Faber–Krahn inequality with the asymmetry cubed [3, 15]. More recently, Brasco, De Philippis, and Velichkov [8] proved it with the square power: this is the sharp form, because the exponent cannot be any smaller [4,18]. Our magnetic version in (2.2) should likewise instead have the square of the asymmetry and, in principle, one could adapt Brasco, De Philippis, and Velichkov's argument to achieve this. Their state-of-the-art methods, however, are nonconstructive and will not yield an explicit constant. This would make it impossible to understand the pertinent role of the magnetic field strength B in the stability of Erdős' inequality.

Our methods, on the other hand, yield an explicit constant with a natural dependence on the field strength. Physical intuition suggests that as $B \to \infty$ the principal eigenfunctions start to localize on a length scale proportional to $1/\sqrt{B}$, away from the boundary, and therefore $\lambda(B, \cdot)$ becomes less sensitive to the shape of the domain: it can but faintly distinguish between even very dissimilar shapes, and the little sensitivity that remains comes from the fact that these eigenfunctions can still feel about near the boundary with their exponentially small tails. Now, Ω can look rather different from D_R and yet $\lambda(B, \Omega) \approx \lambda(B, D_R)$: a strong magnetic field compromises stability. We manage to capture this picture in (2.2) with our constant which vanishes, exponentially, as $B \to \infty$.

To prove his Faber–Krahn-type inequality in (2.1), Erdős started out in the usual way by establishing a rearrangement inequality. See Lemma 3.1. While there are certainly nontrivial magnetic aspects to the argument, Erdős essentially mimicked the standard proof [21] of the analogous Pólya–Szegő inequality using the coarea formula and the isoperimetric inequality. But in imposing the Pólya–Szegő scheme on his problem, he was forced to change the magnetic field on the disk. The vector potential on the right-hand side of (3.1) is no longer the same, and thus his magnetic rearrangement inequality cannot readily imply (2.1) in the same way that the Pólya–Szegő inequality yields Faber–Krahn.

To deal with this mis-match between the magnetic fields on Ω and D_R, Erdős developed the *comparison lemma* on the disk. See Remark 4.2. It compares the ground-state energies of the operator on the right-hand side of (3.1) corresponding to different magnetic fields. This in turn allowed him to recover the original magnetic field on D_R and finish proving (2.1). His comparison lemma is built on the variational principle and has nothing to do with rearrangements. Unlike his rearrangement inequality, it has no analog in the absence of a magnetic field.

To prove our stability estimate in Theorem 2.1, we also start out in the usual way by establishing a quantitative version of Erdős' rearrangement inequality. See Proposition 3.2. This is nothing new: in the absence of a magnetic field, i.e., $B = 0$, it just reduces to the quantitative version of the Pólya–Szegő inequality that was used in proving stability of Faber–Krahn [7]. Here we mimic Erdős' proof but instead apply the *quantitative isoperimetric inequality* on the level sets.

Theorem 2.3. *Let $U \subset \mathbb{R}^2$ be a bounded set with smooth boundary, and let $\mathcal{P}(U)$ denote the perimeter of U. Let $\mathcal{A}(U)$ denote either the interior asymmetry or the Fraenkel asymmetry. In the case of the interior asymmetry, we also assume U is simply connected. Then there is a universal constant $c > 0$ such that*

$$\mathcal{P}(U) \geq 2\sqrt{\pi}|U|^{\frac{1}{2}}(1 + c\mathcal{A}(U)^2).$$

This was first proved by Bonnesen in 1924 for simply connected planar sets using the interior asymmetry [6, 19]. In 2008, Fusco, Maggi, and Pratelli proved a more general version using the Fraenkel asymmetry [13]. Theorem 2.3 forms the backbone of the first part of the paper.

In Lemma 4.1 we establish a quantitative version of Erdős' comparison lemma. Now, this is really a new estimate, which stands completely outside of the rearrangement framework–and it only enters the scene when B is large.

In Corollary 4.3 we present two very different lower bounds on the quantity $\lambda(B, \Omega) - \lambda(B, D_R)$, both involving the asymmetry of the level sets of the principal eigenfunction corresponding to $\lambda(B, \Omega)$. The first bound, (4.7), is based on our quantitative version of the rearrangement inequality. The second bound, (4.8), is based on our quantitative version of the comparison lemma.

As usual, the main difficulty lies in going from the asymmetry of these level sets in Corollary 4.3 to the asymmetry of the whole domain. We deal with this in the second part of the paper. When B is small, we operate entirely within the rearrangement framework just as in the classical Faber–Krahn setting. Here our argument is a direct perturbation of Hansen and Nadirashvili's proof of their quantitative Faber–Krahn inequality [15]. We only use the first bound, given in (4.7), of Corollary 4.3 which is based on the quantitative version of the rearrangement inequality. This is enough to prove the estimate in (2.3) of Theorem 2.1.

But as B increases, our weak-field adaptation of Hansen and Nadirashvili's technique breaks down: with a strong magnetic field, the rearrangement framework alone is no longer sufficient for establishing stability. Here we make full use of both the quantitative version of the rearrangement inequality *and now* our quantitative version of the comparison lemma. A distinctive feature of our argument is the necessary interplay between the traditional bound in (4.7) – rooted firmly within the paradigmatic framework of rearrangement inequalities – and our *magnetic bound* in (4.8), which is unique to our problem and irreducible to any other estimate used in establishing stability of a Faber–Krahn-type inequality.

Part I
The magnetic isoperimetric inequality

Here we re-prove Erdős' magnetic isoperimetric inequality, but with a remainder term involving the asymmetry of the level sets of the principal eigenfunction corresponding to $\lambda(B, \Omega)$. This is given as Corollary 4.3. The quantitative isoperimetric inequality plays an essential role.

3. The magnetic rearrangement inequality

Standard elliptic theory tells us that the principal eigenfunction corresponding to $\lambda(B, \Omega)$ is a complex-valued analytic function. The first ingredient in Erdős' proof is a rearrangement inequality. He proved the following.

Lemma 3.1. *Let f, $\|f\|_2 = 1$ be a complex-valued analytic function on Ω that vanishes on the boundary, and let $|f|^*$ denote the symmetric decreasing rearrangement of $|f|$. Then there exists a vector potential $\tilde{\alpha}(x) = \frac{a(|x|)}{|x|}(-x_2, x_1)$, where $a(|x|)$ is a* *function satisfying* $0 \leq a(|x|) \leq \frac{B|x|}{2}$, *such that*

$$\int_{\Omega} |(-i\nabla - \alpha) f|^2 dx \geq \int_{D_R} |(-i\nabla - \tilde{\alpha})|f|^*|^2 dx + B - \int_{D_R} \operatorname{rot}(\tilde{\alpha})|f|^{*2} dx. \quad (3.1)$$

This is analogous to the celebrated Pólya–Szegő inequality but with some caveats.

(1) The magnetic field on the disk is no longer the same. Our vector potential $\alpha = \frac{B}{2}(-x_2, x_1)$ corresponds to a homogeneous field of strength B. Now, $\tilde{\alpha}$ corresponds to a radially symmetric but *inhomogeneous* field.

(2) The potential $\tilde{\alpha}$ depends on f, because Erdős constructed $a(|x|)$ from the level sets of $|f|$.

(3) In particular, if $a(|x|) = \frac{B|x|}{2}$, then the level set $\{|f| > |f|^*(x)\}$ is a disk.

Lemma 3.1 yields a lower bound on $\lambda(B, \Omega)$. Had the vector potential remained unchanged, (3.1) would have readily implied $\lambda(B, \Omega) \geq \lambda(B, D_R)$.

In this section we prove a quantitative version of his rearrangement inequality, and we write the right-hand side more conveniently in terms of polar coordinates.

Proposition 3.2. *Let* f, $\|f\|_2 = 1$ *be as in the statement of Lemma* 3.1, *and* $q(|x|) := |f|^*(x)$. *Then there exists a bounded function* $a(|x|)$, *depending on* f *and* B, *such that*[1]

$$\int_{\Omega} |(-i\nabla - \alpha) f|^2 dx$$
$$\geq B + 2\pi \int_0^R (q'(r) + a(r)q(r))^2 (1 + c\mathcal{A}^2(\{|f| > q(r)\}))^2 r\, dr,$$

and

$$0 \leq a(r) \leq \frac{Br}{2}(1 + c\mathcal{A}^2(\{|f| > q(r)\}))^{-2} \leq \frac{Br}{2}, \quad (3.2)$$

where $c > 0$ *is a universal constant independent of* B *and* Ω.

In the absence of the asymmetry term, the expression on the right-hand side indeed coincides with that of (3.1). See Proof of Lemma A.2 in Appendix A.

[1] We use here the following convention for the interior asymmetry. If the open set U is not simply connected, we define $\mathcal{A}_I(U)$ to be the asymmetry of the smallest simply connected set containing U. Since Ω is simply connected, this will not change the final value of $\mathcal{A}_I(\Omega)$. This convention allows us to use Theorem 2.3 for the level sets of $|f|$.

3.1. Proof of Proposition 3.2

Erdős proved his rearrangement inequality within the standard Pólya–Szegő scheme [21] using the coarea formula and the isoperimetric inequality, which we replace with its quantitative version.

To use the coarea formula, first we need a real-valued function. By modifying the magnetic vector potential, we can work with $|f|$ instead.

Lemma 3.3. *Let f be as in the statement of Lemma 3.1, and $\Omega_0 := \Omega \setminus \{f = 0\}$. Let $\theta\colon \Omega_0 \mapsto [0, 2\pi)$ be such that $f = |f|e^{i\theta}$. Since Ω_0 has full measure, $w := \alpha - \nabla\theta$ is defined almost everywhere and* $\mathrm{rot}(w) = B$. *Then, with $w^\perp := (-w_2, w_1)$,*

$$\int_\Omega |(-i\nabla - \alpha)f|^2 dx = B + \int_\Omega |\nabla|f| + w^\perp |f||^2 dx.$$

Proof. Since $|f| \in H_0^1(\Omega)$ and w is real-valued,

$$\int_\Omega |(-i\nabla - \alpha)f|^2 dx = \int_\Omega |(-i\nabla - w)|f||^2 dx = \int_\Omega (|\nabla|f||^2 + |w^\perp|^2|f|^2) dx.$$

Note w is smooth a.e. By completing the square and integrating by parts,

$$\begin{aligned}\int_\Omega |(-i\nabla - \alpha)f|^2 dx &= \int_\Omega (|\nabla|f| + w^\perp|f||^2 - 2|f|w^\perp \cdot \nabla|f|) dx \\ &= \int_\Omega (|\nabla|f| + w^\perp|f||^2 + |f|^2 \operatorname{div}(w^\perp)) dx.\end{aligned}$$

Since $\operatorname{div}(w^\perp) = \mathrm{rot}(w) = B$, the lemma follows. ∎

Then we use the coarea formula and arrive at an expression involving an integral over the level sets of $|f|$.

Lemma 3.4. *Let $f, w^\perp$ be as in the statement of Lemma 3.3. Then,*

$$\int_\Omega |\nabla|f| + w^\perp|f||^2 dx \geq \int_0^\infty dz\, (1 - B\Phi(z)z)^2 \int_{\{|f|=z\}} |\nabla|f||, \tag{3.3}$$

with

$$\Phi(z) := \frac{|\{|f| > z\}|}{\int_{\{|f|=z\}} |\nabla|f||}. \tag{3.4}$$

If there is no magnetic field, i.e., $B = 0$, and f is a positive function, then the relation in (3.3) reduces to the usual coarea formula used in the proof of the Pólya–Szegő inequality [21].

Proof of Lemma 3.4. There exists w' orthogonal to $\nabla|f|$ and $\varphi\colon \Omega \mapsto \mathbb{R}$ such that $w^\perp = -\varphi\nabla|f| + w'$. By the Pythagorean theorem,

$$\int_\Omega |\nabla|f| + w^\perp|f||^2 dx = \int_\Omega (|(1-\varphi|f|)\nabla|f||^2 + |w'|f||^2)dx$$
$$\geq \int_\Omega |(1-\varphi|f|)\nabla|f||^2 dx.$$

Now, we are in a position to use the coarea formula:

$$\int_\Omega |(1-\varphi|f|)\nabla|f||^2 dx = \int_0^\infty dz \int_{\{|f|=z\}} (1-\varphi z)^2|\nabla|f||$$
$$\geq \int_0^\infty dz\, \frac{(\int_{\{|f|=z\}}(1-\varphi z)|\nabla|f||)^2}{\int_{\{|f|=z\}}|\nabla|f||}.$$

We use Stokes' theorem on the level sets. For almost all $z > 0$, the level set $\{|f| = z\}$ is a finite union of smooth closed curves by Sard's theorem. Thus,

$$B|\{|f| > z\}| = \int_{\{|f|>z\}} \operatorname{rot}(w) = \int_{\{|f|=z\}} w \cdot \tau,$$

where $\tau = \frac{(\nabla|f|)^\perp}{|(\nabla|f|)^\perp|}$. Since $w \cdot \tau = \varphi|\nabla|f||$, we conclude

$$\int_\Omega |\nabla|f| + w^\perp|f||^2 dx \geq \int_0^\infty dz\, \frac{(\int_{\{|f|=z\}}|\nabla|f|| - Bz|\{|f| > z\}|)^2}{\int_{\{|f|=z\}}|\nabla|f||}.$$

The lemma follows from the definition of Φ in (3.4). ∎

With the coarea-type estimate in (3.3), Erdős applied the isoperimetric inequality on the level sets of $|f|$ to prove his rearrangement inequality; and when $B = 0$, his argument reduces to the standard proof of the Pólya–Szegő inequality [21]. Below we instead apply the quantitative isoperimetric inequality on these level sets.

Proof of Proposition 3.2. From Lemma 3.3, Lemma 3.4, and Hölder's inequality,

$$\int_\Omega |(-i\nabla - \alpha)f|^2 dx \geq B + \int_0^\infty dz\, (1 - B\Phi(z)z)^2 \frac{|\{|f| = z\}|^2}{\int_{\{|f|=z\}}|\nabla|f||^{-1}}.$$

By Sard's theorem, the denominator is non-vanishing for almost all $z > 0$. And since q is the rearrangement of $|f|$,

$$q(r) = F^{-1}(\pi r^2) \quad \text{where } F(z) := |\{|f| > z\}|.$$

By the coarea formula, again for almost all $z > 0$

$$F(z) = \int_z^\infty d\xi \int_{\{|f|=\xi\}} |\nabla|f||^{-1} \quad \text{and} \quad F'(z) = -\int_{\{|f|=z\}} |\nabla|f||^{-1}.$$

Then,

$$\int_\Omega |(-i\nabla - \alpha)f|^2 dx \geq B - \int_0^\infty (1 - B\Phi(z)z)^2 |\{|f| = z\}|^2 F'(z)^{-1}\, dz.$$

Now, we do a change of variable $z = q(r)$ and apply the isoperimetric inequality, Theorem 2.3, on the level sets: $|\{|f| = q(r)\}| \geq 2\pi r(1 + c\mathcal{A}^2(\{|f| > q(r)\}))$. We write $\mathcal{A}^2$ for short. Then,

$$\int_\Omega |(-i\nabla - \alpha)f|^2 dx \geq B + \int_0^R (1 - B\Phi(q(r))q(r))^2 \frac{(2\pi r)^2 q'(r)}{F'(q(r))}(1 + c\mathcal{A}^2)^2 dr.$$

Since $q'(r) = 2\pi r F'(q(r))^{-1}$,

$$\int_\Omega |(-i\nabla - \alpha)f|^2 dx \geq B + 2\pi \int_0^R \Big[q'(r) - \frac{2\pi r B\Phi(q(r))}{F'(q(r))} q(r)\Big]^2 (1 + c\mathcal{A}^2)^2 r dr.$$

Writing $a(r) := -2\pi r B F'(q(r))^{-1}\Phi(q(r))$, we deduce our rearrangement inequality.

It remains to prove the upper bound in (3.2). By Hölder's inequality

$$-F'(q(r)) = \int_{\{|f|=q(r)\}} |\nabla|f||^{-1} \geq |\{|f| = q(r)\}|^2 \Big(\int_{\{|f|=q(r)\}} |\nabla|f|| \Big)^{-1},$$

and by the isoperimetric inequality, Theorem 2.3,

$$a(r) \leq 2\pi r B \frac{|\{|f| > q(r)\}|}{|\{|f| = q(r)\}|^2} \leq \frac{Br}{2}(1 + c\mathcal{A}^2(\{|f| > q(r)\}))^{-2}.$$

This concludes the proof of Proposition 3.2. ∎

4. The comparison lemma

The second ingredient in Erdős' proof is a comparison lemma, which makes it possible to recover from the right-hand side of (3.1) the original potential α on the disk. In this section we prove a quantitative version of his comparison lemma.

For a potential $\tilde{\alpha} = \frac{a(|x|)}{|x|}(-x_2, x_1)$, with $a \in L^\infty((0,R))$, we consider the ground-state energy of the operator $(-i\nabla - \tilde{\alpha})^2 - \operatorname{rot}(\tilde{\alpha})$ restricted to radial functions on the disk, again written more conveniently in terms of polar coordinates

$$\mathrm{e}(a(r)) := \inf_{q \in H_0^{1,\mathrm{rad}}(D_R)} \frac{2\pi \int_0^R (q'(r) + a(r)q(r))^2 r\,dr}{2\pi \int_0^R q(r)^2 r\,dr}, \tag{4.1}$$

where $H_0^{1,\mathrm{rad}}(D_R) := \{q\colon [0,R] \to \mathbb{R}$ such that $x \mapsto q(|x|)$ belongs to $H_0^1(D_R)\}$.

The function $a(r) = \frac{Br}{2}$ corresponds to the original potential $\alpha = \frac{B}{2}(-x_2, x_1)$ and, since $\operatorname{rot}(\alpha) = B$,

$$B + \mathrm{e}(Br/2) = \inf_{q \in H_0^{1,\mathrm{rad}}(D_R)} \frac{\int_{D_R} |(-i\nabla - \alpha)q(|x|)|^2 dx}{\int_{D_R} q(|x|)^2 dx} \geq \lambda(B, D_R). \tag{4.2}$$

We compare the ground-state energies for different potentials on the disk.

Lemma 4.1. *Let q_a be a normalized minimizer for the energy $\mathrm{e}(a(r))$ in (4.1). Let*

$$u_a(r) := \exp\Big(-2\int_0^r a(s)\,ds\Big) \quad \text{and} \quad p_a(r) := q_a(r)u_a(r)^{-\frac{1}{2}}. \tag{4.3}$$

Then, for $a, \tilde{a} \in L^\infty((0,R))$,

$$\mathrm{e}(a(r)) \geq \mathrm{e}(\tilde{a}(r)) + \frac{2\int_0^R (\tilde{a} - a)p_a|p_a'|u_{\tilde{a}} r\,dr}{\int_0^R p_a^2 u_{\tilde{a}} r\,dr}. \tag{4.4}$$

Remark 4.2. Our bound in (4.4) implies Erdős' *comparison lemma*: if $a \leq \tilde{a}$, then $\mathrm{e}(a(r)) \geq \mathrm{e}(\tilde{a}(r))$. See [10, Lemma 3.1].

Proof. We write

$$\mathrm{e}(a(r)) = \inf_{p \in H_0^{1,\mathrm{rad}}(D_R)} \frac{\int_0^R (p')^2 u_a r\,dr}{\int_0^R p^2 u_a r\,dr} = \frac{\int_0^R (p_a')^2 u_a r\,dr}{\int_0^R p_a^2 u_a r\,dr}. \tag{4.5}$$

Since p_a is the minimizer in (4.5), it solves the Euler–Lagrange equation

$$-p_a'' u_a r - p_a' u_a' r - p_a' u_a = \mathrm{e}(a(r)) p_a u_a r. \tag{4.6}$$

Now, we consider $\mathrm{e}(\tilde{a}(r))$. It follows from the variational principle and (4.6) that

$$\begin{aligned}\mathrm{e}(\tilde{a}(r)) &\leq \frac{\int_0^R (p_a')^2 u_{\tilde{a}} r dr}{\int_0^R p_a^2 u_{\tilde{a}} r dr} \\ &= \frac{\int_0^R (-p_a'' u_a r - p_a' u_a' r - p_a' u_a)\frac{u_{\tilde{a}}}{u_a} p_a - p_a' p_a u_a r (\frac{u_{\tilde{a}}}{u_a})' dr}{\int_0^R p_a^2 u_{\tilde{a}} r dr} \\ &= \mathrm{e}(a(r)) + \frac{2\int_0^R p_a' p_a (\tilde{a} - a) u_{\tilde{a}} r dr}{\int_0^R p_a^2 u_{\tilde{a}} r dr}.\end{aligned}$$

Note that $p_a' < 0$ by Hopf's Lemma. ∎

Proposition 3.2, Lemma 4.1 and the observation in (4.2) allow us to conclude with the following corollary.

Corollary 4.3. *Let f be a principal eigenfunction corresponding to $\lambda(B, \Omega)$ and $q(|x|) := |f|^*(x)$. Let $a(r)$ be as in Proposition 3.2 above, and let q_a be a normalized minimizer for the energy $\mathrm{e}(a(r))$ in (4.1). Then there is a universal constant $c > 0$, independent of B and Ω, such that*

$$\lambda(B,\Omega) \geq \lambda(B, D_R) + c \int_0^R (q'(r) + a(r)q(r))^2 \mathcal{A}^2(\{|f| > q(r)\}) r dr, \tag{4.7}$$

and

$$\lambda(B,\Omega) \geq \lambda(B, D_R) + cB \frac{\int_0^R p_a |p_a'| e^{-\frac{Br^2}{2}} \mathcal{A}^2(\{|f| > q(r)\}) r^2 dr}{\int_0^R p_a^2 e^{-\frac{Br^2}{2}} r dr}, \tag{4.8}$$

where p_a is as given in Lemma 4.1 above.

Corollary 4.3 implies $\lambda(B,\Omega) \geq \lambda(B, D_R)$. Furthermore, if $\lambda(B,\Omega) = \lambda(B, D_R)$, then either (4.7) or (4.8) can be used to deduce that almost all of the level sets of $|f|$ are disks; and since f is an analytic function, this implies Ω is a disk.

The first bound, given in (4.7), is established with *our quantitative version of the rearrangement inequality* and with Erdős' comparison lemma. In the absence of a magnetic field, i.e., $B = 0$, this bound reduces to the usual estimate used in all the proofs of the quantitative Faber–Krahn inequality, e.g., [3, 14, 15, 17].

Our second bound, given in inequality (4.8), is established with Erdős' rearrangement inequality, *our quantitative version of the comparison lemma* and our estimate in (3.2), which follows from the quantitative isoperimetric inequality. This bound, on the other hand, has no such analog in the absence of a magnetic field.

Part II
The quantitative version

Here we prove Theorem 2.1 from Corollary 4.3 by extracting the asymmetry of the whole domain from the asymmetry of the level sets in (4.7) and (4.8). Let

$$|\{q(|x|) > s\}| = |\Omega|\Big(1 - \frac{1}{2}\mathcal{A}(\Omega)\Big). \tag{4.9}$$

Following Hansen and Nadirashvili [15], we split the proof into two cases, depending on whether s is small or large. Lemma B.1 in Appendix B will be useful.

5. The first case: $s \lesssim e^{-BR^2}\mathcal{A}(\Omega)$

We assume

$$s \le \frac{1}{8}|\Omega|^{-\frac{1}{2}}e^{-\frac{BR^2}{4}}\mathcal{A}(\Omega). \tag{5.1}$$

We use the representation in (4.5), which allows us to adapt the usual strategy for dealing with the Dirichlet Laplacian; and when $B = 0$, the argument reduces to Hansen and Nadirashvili's proof of their quantitative Faber–Krahn inequality [15].

We write $E(B,\Omega) := \lambda(B,\Omega) - B$. Let

$$p := q u_a^{-\frac{1}{2}}$$

with q, a as in Corollary 4.3 and u_a as in (4.3), and let

$$\tilde{p}(r) := p(r) - s e^{\int_0^{q^{-1}(s)} a(\tau)d\tau}.$$

Since $\tilde{p}' = p'$, it follows from the rearrangement inequality that

$$E(B,\Omega) \ge 2\pi \int_0^R (q' + aq)^2 r dr = 2\pi \int_0^R (\tilde{p}')^2 u_a r dr \ge 2\pi \int_0^{q^{-1}(s)} (\tilde{p}')^2 u_a r dr.$$

Since $\tilde{p}$ vanishes at $q^{-1}(s)$, it is admissible in the variational problem in (4.5) but on the disk $\{q > s\}$, and

$$\frac{E(B,\Omega)}{2\pi \int_0^{q^{-1}(s)} \tilde{p}^2 u_a r dr} \ge \inf_{p \in H_0^{1,\mathrm{rad}}(\{q>s\})} \frac{\int_0^{q^{-1}(s)} (p')^2 u_a r dr}{\int_0^{q^{-1}(s)} p^2 u_a r dr} \ge E(B, \{q > s\}),$$

where the last inequality follows from the comparison lemma and the observation in (4.2). Using the scaling property in Remark 2.2 we further estimate

$$\frac{E(B,\Omega)}{2\pi\int_0^{q^{-1}(s)}\tilde{p}^2u_a r\,dr} \geq \frac{|\Omega|}{|\{q>s\}|}E\Big(B\frac{|\{q>s\}|}{|\Omega|},D_R\Big) \geq \frac{|\Omega|}{|\{q>s\}|}E(B,D_R), \tag{5.2}$$

where the last inequality follows from Lemma A.2 in Appendix A and again the comparison lemma. Finally, we estimate the denominator

$$\begin{aligned}2\pi\int_0^{q^{-1}(s)}\tilde{p}^2u_a r\,dr &= 1-2\pi\int_{q^{-1}(s)}^{R}q^2 r\,dr\\ &\quad+2\pi\int_0^{q^{-1}(s)}\big((se^{\int_0^{q^{-1}(s)}a(\tau)d\tau})^2-2pse^{\int_0^{q^{-1}(s)}a(\tau)d\tau}\big)u_a r\,dr\\ &\geq 1-s^2|\{q<s\}|+s^2|\{q>s\}|-2se^{\frac{BR^2}{4}}|\Omega|^{\frac12}\\ &\geq 1-2se^{\frac{BR^2}{4}}|\Omega|^{\frac12}.\end{aligned}$$

At the penultimate inequality, we used that $e^{\int_0^{q^{-1}(s)}a(\tau)d\tau}\leq e^{\frac{BR^2}{4}}$ and that

$$2\pi\int_0^{q^{-1}(s)}pu_a r\,dr \leq 2\pi\int_0^R q r\,dr \leq 2\pi|\Omega|^{\frac12}\int_0^R q^2 r\,dr = |\Omega|^{\frac12}.$$

Combining the above estimate with (5.2), we have

$$E(B,\Omega)\geq E(B,D_R)\frac{|\Omega|(1-2se^{\frac{BR^2}{4}}|\Omega|^{\frac12})}{|\{q>s\}|},$$

and the choice of s in (4.9) and our assumption in (5.1) give us

$$E(B,\Omega)\geq E(B,D_R)\frac{1-\frac14\mathcal{A}(\Omega)}{1-\frac12\mathcal{A}(\Omega)}\geq E(B,D_R)\Big(1+\frac14\mathcal{A}(\Omega)\Big).$$

Then, using Lemma A.3 in Appendix A, we find

$$\lambda(B,\Omega)\geq\lambda(B,D_R)(1+c\min(1,(BR^2)^{-1}e^{-\frac34 BR^2})\mathcal{A}(\Omega)),$$

which yields the desired estimates in (2.2) and (2.3). This concludes the proof of Theorem 2.1 in the first case.

6. The second case: $s \gtrsim e^{-BR^2}\mathcal{A}(\Omega)$

We assume

$$s \geq \frac{1}{8}|\Omega|^{-\frac{1}{2}} e^{-\frac{BR^2}{4}} \mathcal{A}(\Omega). \tag{6.1}$$

Now, we have to treat weak and strong magnetic fields separately. When B is small, we only use the first bound, given in (4.7), of Corollary 4.3. As B increases, it becomes necessary to also make use of our second bound in (4.8).

6.1. Weak magnetic fields

We consider $0 \leq BR^2 \leq \frac{1}{\pi}$ and prove the stability estimate in (2.3); and when $B = 0$, the argument reduces to Hansen and Nadirashvili's proof of their quantitative Faber–Krahn inequality [15].

We work on the annulus $\{q(|x|) \leq s\}$, whose area is proportional to the asymmetry of the domain. From the first bound, given in (4.7), of Corollary 4.3, the choice of s in (4.9), and Lemma B.1 we have

$$\begin{aligned}
&\lambda(B,\Omega) - \lambda(B, D_R) \\
&\geq c \int_{q^{-1}(s)}^{R} (q'(r) + a(r)q(r))^2 \mathcal{A}^2(\{|f| > q(r)\}) r\,dr \\
&\geq cR^2\mathcal{A}^2(\Omega) \int_{q^{-1}(s)}^{R} (q'(r) + a(r)q(r))^2 r^{-1}\,dr \\
&\geq cR^2\mathcal{A}^2(\Omega)\Bigg(\sqrt{\int_{q^{-1}(s)}^{R} q'(r)^2\, r^{-1}\,dr} - \sqrt{\int_{q^{-1}(s)}^{R} \Big(\frac{B}{2}q\Big)^2 r\,dr}\Bigg)^2 \\
&\geq cR^2\mathcal{A}^2(\Omega)\Bigg(\frac{s}{\sqrt{|\{q(|x|) \leq s\}|}} - \frac{B}{2} s\sqrt{|\{q(|x|) \leq s\}|}\Bigg)^2 \\
&\geq cR^{-2}\mathcal{A}^3(\Omega)(2 - B|\Omega|)^2 \\
&\geq cR^{-2}\mathcal{A}^3(\Omega),
\end{aligned}$$

since $B \leq \frac{1}{|\Omega|} = \frac{1}{\pi R^2}$. At the penultimate inequality we also used the assumption in (6.1). Using Lemma A.3, we conclude $\lambda(B,\Omega) \geq \lambda(B, D_R)(1 + c\mathcal{A}(\Omega)^3)$.

6.2. Strong magnetic fields

We consider $BR^2 > \frac{1}{\pi}$ and prove our stability estimate in (2.2); instead of integrating as above on $\{q(|x|) \leq s\}$, we choose to work closer to the boundary on a smaller annulus whose area is now proportional to the *spectral deficit* of the domain

$$\mathcal{D}(B, \Omega) := \frac{\lambda(B, \Omega)}{\lambda(B, D_R)} - 1.$$

We treat two cases, depending on whether q is large or small near the boundary: $q(R(1 - \mathcal{D}(B, \Omega)^\alpha)) > R^{-1}\mathcal{D}(B, \Omega)^\beta$ and $q(R(1 - \mathcal{D}(B, \Omega)^\alpha)) \leq R^{-1}\mathcal{D}(B, \Omega)^\beta$, where $\alpha = \frac{1}{5}$ and $\beta = \frac{3}{10}$ are chosen to optimize our result. For proving our estimate in (2.2), we can assume that the spectral deficit is very small

$$\mathcal{D}(B, \Omega)^\alpha < \min\Big\{\frac{1}{2BR^2}, \frac{1}{2}\Big\}. \tag{6.2}$$

6.2.1. Case $q(R(1 - \mathcal{D}(B, \Omega)^\alpha)) > R^{-1}\mathcal{D}(B, \Omega)^\beta$.

By continuity of q,

$$q(R(1 - \mathcal{D}(B, \Omega)^{\tilde\alpha})) = R^{-1}\mathcal{D}(B, \Omega)^\beta \quad \text{for some } \tilde\alpha > \alpha. \tag{6.3}$$

If $q(R(1 - \mathcal{D}(B, \Omega)^{\tilde\alpha})) \geq s$, our assumption in (6.1) readily yields

$$cR^{-1}e^{-\frac{BR^2}{4}}\mathcal{A}(\Omega) \leq s \leq q(R(1 - \mathcal{D}(B, \Omega)^{\tilde\alpha})) = R^{-1}\mathcal{D}(B, \Omega)^\beta,$$

and therefore

$$\mathcal{D}(B, \Omega) \geq ce^{-\frac{BR^2}{4\beta}}\mathcal{A}(\Omega)^{\frac{1}{\beta}}. \tag{6.4}$$

If we have $q(R(1 - \mathcal{D}(B, \Omega)^{\tilde\alpha})) < s$, then the weak-field argument from Section 6.1 applies *mutatis mutandis*. From the first bound, given in (4.7), of Corollary 4.3, the relation in (6.3), and Lemma B.1 we have

$$\begin{aligned}
\lambda(B, D_R)\mathcal{D}(B, \Omega) &\geq c\int_{R(1-\mathcal{D}(B,\Omega)^{\tilde\alpha})}^{R} (q'(r) + a(r)q(r))^2\mathcal{A}^2(\{|f| > q(r)\})r\,dr \\
&\geq cR^2\mathcal{A}(\Omega)^2\Big(\frac{q(R(1 - \mathcal{D}(B, \Omega)^{\tilde\alpha}))}{\sqrt{2R^2\mathcal{D}(B, \Omega)^{\tilde\alpha}}} \\
&\qquad - \frac{B}{2}q(R(1 - \mathcal{D}(B, \Omega)^{\tilde\alpha}))\sqrt{2R^2\mathcal{D}(B, \Omega)^{\tilde\alpha}r}\Big)^2 \\
&= cR^{-2}\mathcal{A}(\Omega)^2\mathcal{D}(B, \Omega)^{2\beta-\tilde\alpha}(1 - BR^2\mathcal{D}(B, \Omega)^{\tilde\alpha})^2.
\end{aligned}$$

However, $\tilde{\alpha}$ depends on B and Ω. Fortunately, since $\tilde{\alpha} > \alpha$ and $\mathcal{D}(B,\Omega) < 1$, we have $\mathcal{D}(B,\Omega)^{\tilde{\alpha}} < \mathcal{D}(B,\Omega)^{\alpha}$; this allows to replace $\mathcal{D}(B,\Omega)^{\tilde{\alpha}}$ in the above with $\mathcal{D}(B,\Omega)^{\alpha}$. Furthermore, the bound in (6.2) offsets the large BR^2 in the parenthetical expression, which thereby remains positive. Using Lemma A.3,

$$\mathcal{D}(B,\Omega) \geq c\frac{\mathcal{A}(\Omega)^2}{R^2\lambda(B,D_R)}\mathcal{D}(B,\Omega)^{2\beta-\alpha} \geq c\frac{\mathcal{A}(\Omega)^2}{1+BR^2}\mathcal{D}(B,\Omega)^{2\beta-\alpha},$$

and therefore

$$\mathcal{D}(B,\Omega)^{1-2\beta+\alpha} \geq c\frac{\mathcal{A}(\Omega)^2}{1+BR^2}. \tag{6.5}$$

With our above choice of α and β, the inequalities in (6.4) and (6.5) both yield the same desired estimate in (2.2).

Thus far, we have only used the first bound, given in (4.7), of Corollary 4.3 which is based on the quantitative version of the rearrangement inequality.

6.2.2. Case $q(R(1-\mathcal{D}(B,\Omega)^{\alpha})) \leq R^{-1}\mathcal{D}(B,\Omega)^{\beta}$. If $q(R(1-\mathcal{D}(B,\Omega)^{\alpha})) \geq s$, again our assumption in (6.1) readily yields

$$cR^{-1}e^{-\frac{BR^2}{4}}\mathcal{A}(\Omega) \leq s \leq q(R(1-\mathcal{D}(B,\Omega)^{\alpha})) \leq R^{-1}\mathcal{D}(B,\Omega)^{\beta}$$

and therefore, as above,

$$\mathcal{D}(B,\Omega) \geq ce^{-\frac{BR^2}{4\beta}}\mathcal{A}(\Omega)^{\frac{1}{\beta}}. \tag{6.6}$$

But, when $q(R(1-\mathcal{D}(B,\Omega)^{\alpha})) < s$, the weak-field argument from Section 6.1 is no longer useful: it requires a *lower bound* on $q(R(1-\mathcal{D}(B,\Omega)^{\alpha}))$, as above in Section 6.2.1, to be effective. That argument, however, is based wholly on the first bound, given in (4.7), of Corollary 4.3.

Now, we instead turn to our second bound, given in (4.8), which is based on our quantitative version of the comparison lemma. Here there is hope: it is possible to bound the remainder term in (4.8) from below *independently of* q.

Lemma 6.1. *Let p_a be as in Corollary 4.3. Then there exists a universal constant $c > 0$, independent of B and Ω, such that for any $0 < \varepsilon < \frac{1}{2}$*

$$\frac{\int_{R(1-\varepsilon)}^{R} p_a|p_a'|e^{-\frac{Br^2}{2}}\mathcal{A}^2(\{|f| > q(r)\})r^2dr}{\int_0^R p_a^2 e^{-\frac{Br^2}{2}}rdr} \geq ce^{-\frac{BR^2}{2}}\mathcal{M}_\varepsilon\varepsilon^2,$$

where $\mathcal{M}_\varepsilon := \inf\{\mathcal{A}^2(\{|f| > q(r)\}) : R(1-\varepsilon) < r < R\}$.

Proof. Since $p_a' < 0$,

$$\int\limits_{R(1-\varepsilon)}^{R} p_a |p_a'| e^{-\frac{Br^2}{2}} \mathcal{A}(\{|f| > q(r)\}) r^2 dr$$

$$\geq c M_\varepsilon R^2 e^{-\frac{BR^2}{2}} \int\limits_{R(1-\varepsilon)}^{R} -p_a(r) p_a'(r) dr = c M_\varepsilon R^2 e^{-\frac{BR^2}{2}} p_a(R(1-\varepsilon))^2.$$

Furthermore,

$$p_a(R(1-\varepsilon)) = \int\limits_{R(1-\varepsilon)}^{R} -p_a'(r) dr \geq \frac{1}{R} \int\limits_{R(1-\varepsilon)}^{R} -p_a'(r) r dr \geq \frac{\varepsilon}{R} \int\limits_{0}^{R} -p_a'(r) r dr,$$

where in the last inequality we used that $r \mapsto -p_a'(r) r$ is increasing (see (4.6)). The lemma follows from the Sobolev inequality

$$\int\limits_{0}^{R} -p_a'(r) r dr \geq c \Big(\int\limits_{0}^{R} p_a^2(r) r dr \Big)^{\frac{1}{2}} \geq c \Big(\int\limits_{0}^{R} p_a^2(r)\, e^{-\frac{Br^2}{2}} r\, dr \Big)^{\frac{1}{2}}. \qquad \blacksquare$$

Before proceeding with our argument, we remark that Lemma 6.1 would not have been useful for dealing with the previous situation in Section 6.2.1.

If $q(R(1 - \mathcal{D}(B, \Omega)^\alpha)) < s$, then we use the above lemma with $\varepsilon = \mathcal{D}(B, \Omega)^\alpha$. From our second bound, given in (4.8), of Corollary 4.3, Lemma 6.1, and Lemma B.1 we have

$$\lambda(B, D_R) \mathcal{D}(B, \Omega) \geq c B e^{-\frac{BR^2}{2}} \mathcal{A}(\Omega)^2 \mathcal{D}(B, \Omega)^{2\alpha}.$$

Again, using Lemma A.3 and now that $BR^2 > \frac{1}{\pi}$,

$$\mathcal{D}(B, \Omega) \geq c \frac{e^{-\frac{BR^2}{2}}}{1 + (BR^2)^{-1}} \mathcal{A}(\Omega)^2 \mathcal{D}(B, \Omega)^{2\alpha} \geq c e^{-\frac{BR^2}{2}} \mathcal{A}(\Omega)^2 \mathcal{D}(B, \Omega)^{2\alpha},$$

and therefore

$$\mathcal{D}(B, \Omega)^{1-2\alpha} \geq c e^{-\frac{BR^2}{2}} \mathcal{A}(\Omega)^2. \tag{6.7}$$

With our above choice of α and β, the inequalities in (6.6) and (6.7) both yield the same desired estimate in (2.2). This concludes the proof of Theorem 2.1.

A. The magnetic Laplacian on the disk

It follows from Erdős' rearrangement inequality and comparison lemma, and from the observation in (4.2) that the principal eigenfunction of the magnetic Laplacian on the disk is radially symmetric.

Theorem A.1. *As above, let D_R be a disk of radius R centered at the origin. Then*

$$\lambda(B, D_R) = \inf_{q \in H_0^{1,\mathrm{rad}}(D_R)} \frac{\int_{D_R} |(-i\nabla - \alpha) q(|x|)|^2 dx}{\int_{D_R} q(|x|)^2 dx},$$

where $H_0^{1,\mathrm{rad}}(D_R) := \{q : [0, R] \to \mathbb{R}$ *such that* $x \mapsto q(|x|)$ *belongs to* $H_0^1(D_R)\}$.

Thus, we write $\lambda(B, D_R)$ more conveniently in terms of polar coordinates.

Lemma A.2. *Let $H_0^{1,\mathrm{rad}}(D_R)$ be as in Theorem A.1. Then*

$$\lambda(B, D_R) = B + \inf_{q \in H_0^{1,\mathrm{rad}}(D_R)} \frac{2\pi \int_0^R (q'(r) + \frac{Br}{2} q(r))^2 r dr}{2\pi \int_0^R q(r)^2 r dr} =: B + \mathrm{e}(Br/2).$$

Proof. First we consider a broader class of vector potentials $\tilde{\alpha}(x) := \frac{a(|x|)}{|x|}(-x_2, x_1)$ on the disk, with $a(|x|)$ bounded. These correspond to radially symmetric but possibly inhomogeneous magnetic fields that show up in the rearrangement inequality. Written in polar coordinates, $\tilde{\alpha}(r, \theta) = a(r)(-\sin\theta, \cos\theta)$ and for $f \in H_0^1(D_R)$

$$\int_{D_R} |(-i\nabla - \tilde{\alpha}) f|^2 dx = \int_0^R \int_0^{2\pi} \Big(|\partial_r f|^2 + \Big| \frac{i}{r} \partial_\theta f + a f \Big|^2 \Big) r d\theta dr.$$

Thus, for any $q \in H_0^{1,\mathrm{rad}}(D_R)$,

$$\begin{aligned}
&\int_{D_R} |(-i\nabla - \tilde{\alpha}) q(|x|)|^2 dx \\
&\quad = 2\pi \int_0^R ((q'(r)^2 + (a(r)q(r))^2) r dr \\
&\quad = 2\pi \int_0^R (q'(r) + a(r)q(r))^2 r dr - 2\pi \int_0^R (q^2)' a(r) r dr,
\end{aligned}$$

and after integrating by parts

$$\begin{aligned}
\int\limits_{D_R} &|(-i\nabla - \tilde{\alpha})q(|x|)|^2 dx \\
&= 2\pi \int\limits_0^R (q'(r) + a(r)q(r))^2 r dr + 2\pi \int\limits_0^R q^2 (a(r)r)' dr \\
&= 2\pi \int\limits_0^R (q'(r) + a(r)q(r))^2 r dr + \int\limits_{D_R} \operatorname{rot}(\tilde{\alpha}) q(|x|)^2 dx.
\end{aligned}$$

Returning to the original potential $\alpha = \frac{B}{2}(-x_2, x_1)$, the lemma follows from Theorem A.1, the above calculation and that $\operatorname{rot}(\alpha) = B$. ■

Moreover, Erdős proved the following estimates. See [10, Proposition A.1].

Lemma A.3. *There are universal constants C_1, C_2 such that*

$$B + \frac{C_1}{R^2} e^{-\frac{3}{4}BR^2} \leq \lambda(B, D_R) \leq B + C_2 B\Big(\frac{1}{BR^2} + BR^2\Big) e^{-\frac{1}{8}BR^2}.$$

Improving these estimates is an ongoing area of research [2, 9, 16], and the references therein. In the absence of a magnetic field, $\lambda(0, D_R) = j_{0,1}^2 R^{-2}$ where $j_{0,1} \approx 2.4048$ is the first zero of the Bessel function of order zero.

B. Asymmetry of large subsets

If a subset is large enough, its asymmetry is comparable to the asymmetry of the whole domain [7, 15].

Lemma B.1. *Let $U \subseteq \Omega$ with $|U| = \pi r^2$ and $|\Omega| = \pi R^2$. If $|U| \geq |\Omega|(1 - \frac{1}{2}\mathcal{A}(\Omega))$, then $r\mathcal{A}(U) \geq \frac{1}{2}R\mathcal{A}(\Omega)$.*

Proof. First we consider the interior asymmetry. From our assumption on the area of U, we have $|U| \geq |\Omega|(1 - \frac{1}{2}\mathcal{A}_I(\Omega))^2$ and thus $r \geq R(1 - \frac{1}{2}\mathcal{A}_I(\Omega))$. We then deduce that $r - \rho_-(U) \geq r - \rho_-(\Omega) \geq \frac{1}{2}(R - \rho_-(\Omega))$, which yields the lemma.

Now, we turn to the Fraenkel asymmetry. Let D_U and D_Ω denote two concentric balls such that $|D_U| = |U|$ and $|D_\Omega| = |\Omega|$. Then, $|D_\Omega \Delta \Omega| \leq |D_U \Delta U| + 2(|\Omega| - |U|)$. Using this inequality and our assumption on the area of U, we deduce

$$\frac{|D_U \Delta U|}{2|U|} \geq \frac{|D_\Omega \Delta \Omega|}{2|U|} - \frac{|\Omega| - |U|}{|U|} \geq \frac{1}{2}\mathcal{A}_F(\Omega)\frac{|\Omega|}{|U|} \geq \frac{1}{2}\frac{R}{r}\mathcal{A}_F(\Omega).$$

Taking the infimum over all translations of D_U concludes the proof. ■

Acknowledgements. We are most grateful to Søren Fournais for encouraging our collaboration. Rohan Ghanta first suggested this problem in October 2017 to Michael Loss, whom he thanks for the initial encouragement.

Funding. The work of Lukas Junge and Léo Morin supported by the Independent Research Fund Denmark via the project grant "Mathematics of the dilute Bose gas" no. 0135-00166B.

References

[1] J. E. Avron, I. W. Herbst, and B. Simon, Schrödinger operators with magnetic fields. III. Atoms in homogeneous magnetic field. *Comm. Math. Phys.* **79** (1981), no. 4, 529–572 Zbl 0464.35086 MR 623966

[2] J.-M. Barbaroux, L. Le Treust, N. Raymond, and E. Stockmeyer, On the semiclassical spectrum of the Dirichlet-Pauli operator. *J. Eur. Math. Soc. (JEMS)* **23** (2021), no. 10, 3279–3321 Zbl 1467.35227 MR 4275474

[3] T. Bhattacharya, Some observations on the first eigenvalue of the p-Laplacian and its connections with asymmetry. *Electron. J. Differential Equations* (2001), article no. 35 Zbl 0991.35032 MR 1836803

[4] T. Bhattacharya and A. Weitsman, Estimates for Green's function in terms of asymmetry. In *Applied analysis (Baton Rouge, LA, 1996)*, pp. 31–58, Contemp. Math. 221, Amer. Math. Soc., Providence, RI, 1999 Zbl 0914.31001 MR 1647193

[5] D. Bonheure, J. Dolbeault, M. J. Esteban, A. Laptev, and M. Loss, Symmetry results in two-dimensional inequalities for Aharonov-Bohm magnetic fields. *Comm. Math. Phys.* **375** (2020), no. 3, 2071–2087 Zbl 1439.81049 MR 4091495

[6] T. Bonnesen, Über das isoperimetrische Defizit ebener Figuren. *Math. Ann.* **91** (1924), no. 3-4, 252–268 Zbl 50.0487.03 MR 1512192

[7] L. Brasco and G. De Philippis, Spectral inequalities in quantitative form. In *Shape optimization and spectral theory*, pp. 201–281, De Gruyter Open, Warsaw, 2017 Zbl 1373.49051 MR 3681151

[8] L. Brasco, G. De Philippis, and B. Velichkov, Faber-Krahn inequalities in sharp quantitative form. *Duke Math. J.* **164** (2015), no. 9, 1777–1831 Zbl 1334.49149 MR 3357184

[9] T. Ekholm, H. Kovařík, and F. Portmann, Estimates for the lowest eigenvalue of magnetic Laplacians. *J. Math. Anal. Appl.* **439** (2016), no. 1, 330–346 Zbl 1386.35295 MR 3474366

[10] L. Erdős, Rayleigh-type isoperimetric inequality with a homogeneous magnetic field. *Calc. Var. Partial Differential Equations* **4** (1996), no. 3, 283–292 Zbl 0846.35094 MR 1386738

[11] L. Erdős, Lifschitz tail in a magnetic field: the nonclassical regime. *Probab. Theory Related Fields* **112** (1998), no. 3, 321–371 Zbl 0921.60099 MR 1660914

[12] L. Erdős, Lifschitz tail in a magnetic field: coexistence of classical and quantum behavior in the borderline case. *Probab. Theory Related Fields* **121** (2001), no. 2, 219–236 Zbl 0991.60098 MR 1865486

[13] N. Fusco, F. Maggi, and A. Pratelli, The sharp quantitative isoperimetric inequality. *Ann. of Math. (2)* **168** (2008), no. 3, 941–980 Zbl 1187.52009 MR 2456887

[14] N. Fusco, F. Maggi, and A. Pratelli, Stability estimates for certain Faber-Krahn, isocapacitary and Cheeger inequalities. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **8** (2009), no. 1, 51–71 Zbl 1176.49047 MR 2512200

[15] W. Hansen and N. Nadirashvili, Isoperimetric inequalities in potential theory. In *Proceedings from the International Conference on Potential Theory (Amersfoort, 1991)*, Kluwer Academic Publishers, pp. 1–14, Kluwer Academic Publishers, Dordrecht, 1994 Zbl 0825.31003 MR 1266215

[16] B. Helffer and A. Morame, Magnetic bottles in connection with superconductivity. *J. Funct. Anal.* **185** (2001), no. 2, 604–680 Zbl 1078.81023 MR 1856278

[17] A. D. Melas, The stability of some eigenvalue estimates. *J. Differential Geom.* **36** (1992), no. 1, 19–33 Zbl 0770.35049 MR 1168980

[18] N. Nadirashvili, Conformal maps and isoperimetric inequalities for eigenvalues of the Neumann problem. In *Proceedings of the Ashkelon Workshop on Complex Function Theory (1996)*, pp. 197–201, Israel Math. Conf. Proc. 11, Bar-Ilan University, Ramat Gan, 1997 Zbl 0890.35097 MR 1476715

[19] R. Osserman, Bonnesen-style isoperimetric inequalities. *Amer. Math. Monthly* **86** (1979), no. 1, 1–29 Zbl 0404.52012 MR 519520

[20] G. Pólya and G. Szegö, *Isoperimetric inequalities in mathematical physics*. Ann. of Math. Stud. 27, Princeton University Press, Princeton, NJ, 1951 Zbl 0044.38301 MR 43486

[21] G. Talenti, Best constant in Sobolev inequality. *Ann. Mat. Pura Appl. (4)* **110** (1976), 353–372 Zbl 0353.46018 MR 463908

Received 16 June 2023; revised 24 October 2023.

Rohan Ghanta
ghanta@alumni.princeton.edu

Lukas Junge
Department of Mathematics, University of Copenhagen, Universitetsparken 5,
2100 Copenhagen Ø, Denmark; lj@math.ku.dk

Léo Morin
Department of Mathematics, University of Copenhagen, Universitetsparken 5,
2100 Copenhagen Ø, Denmark; lpdm@math.ku.dk