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Spectral properties of Schrödinger operators with locally H^{-1} potentials

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Abstract. We study half-line Schrödinger operators with locally H^{-1} potentials. In the first part, we focus on a general spectral theoretic framework for such operators, including a Last–Simon-type description of the absolutely continuous spectrum and sufficient conditions for different spectral types. In the second part, we focus on potentials which are decaying in a local H^{-1} sense; we establish a spectral transition between short-range and long-range potentials and an ℓ^2 spectral transition for sparse singular potentials. The regularization procedure used to handle distributional potentials is also well suited for controlling rapid oscillations in the potential; thus, even within the class of smooth potentials, our results apply in situations which would not classically be considered decaying or even bounded.

Contents

1. Introduction

Schrödinger operators in one dimension $H_V = -\frac{d^2}{dx^2} + V$ are often considered in the setting of locally L^2 or locally L^1 potentials; however, there are several reasons to investigate more general potentials. One is the ubiquity of non-integrable singularities such as Coulomb- or δ-type potentials in models from mathematical physics; another is the Lax pair representation of the KdV equation, where $H^{-1}(\mathbb{R})$ and $H^{-1}(\mathbb{T})$ is the optimal regularity for well-posedness [33, 36]. Non-integrable singularities are often studied by specialized methods such as those for the Kronig–Penney model, and

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inverse scattering arguments in the distributional setting are considered in ways that circumvent the underlying Schrödinger operators. One of the goals of this paper is to extend some robust techniques in spectral theory to the greater generality of locally H^{-1} potentials, defined precisely below.

Schrödinger and Sturm–Liouville operators with distributional coefficients are often treated via the regularization method introduced in the pioneering work of Savchuk and Shkalikov [56]. This approach has materialized into the main tool in the spectral theory of ordinary differential operators with measure and distributional coefficients. Indeed, it was employed, for example, by Eckhardt and Teschl [21] in the setting of measure coefficients; by Eckhardt, Gesztesy, Roger, and Teschl [18, 19] for $L^1_{\mathrm{loc}}((a,b))$ four coefficient Sturm–Liouville operators; by Eckhardt, Kostenko, Malamud, and Teschl [20] for δ' potentials supported on Cantor sets; by Hryniv and Mykytyuk [29, 30] for periodic singular potentials $H^{-1}_{\mathrm{per}}(\mathbb{R}) = H^{-1}(\mathbb{T})$; and by many other authors, see [19] for an extensive reference list. Most of the papers in this direction address foundational questions such as self-adjointness, Weyl–Titchmarsh theory, spectral decomposition, as well as some inverse spectral problems. We emphasize that the study of spectral types such as in the current paper, and of the associated dynamics for operators with singular coefficients, have received much less attention. The review of such results for deterministic Kronig–Penney-type models can be found in [39, Section 2.5] and [2, III.2.3]; some ergodic Hamiltonians modeling point interactions are discussed in [10–12, 15].

In particular, Hryniv and Mykytyuk [29, 30] introduced a class of uniformly locally H^{-1} potentials on $\mathbb{R}$ by the condition

$$\sup_n \| V\rho_n \|_{H^{-1}(\mathbb{R})} < \infty,$$

with the help of compactly supported H^1 multipliers

$$\rho_n(x) = \begin{cases} 1 - 2|x-n|^2, & |x-n| \le 1/2, \\ 2(|x-n|-1)^2, & 1/2 < |x-n| \le 1, \\ 0, & 1 < |x-n|, \end{cases}$$

and showed that real distributions in this class are precisely those with a representation

$$V = \sigma' + \tau,$$

where σ, τ are real-valued functions on $\mathbb{R}$ such that

$$\sup_x \int_x^{x+1} \sigma(t)^2 \, dt < \infty, \quad \sup_x \int_x^{x+1} |\tau(t)| \, dt < \infty. \tag{1.1}$$

Note that this class includes the potentials $V \in H^{-1}(\mathbb{R})$ and $V \in H^{-1}(\mathbb{T})$ (when viewed as periodic distributions on $\mathbb{R}$). In particular, the study of Schrödinger operators with locally H^{-1} potentials helps to bridge spectral theory with scattering arguments. This decomposition is related to the Miura transformation and the Riccati representation [32,38] for periodic V, in which every $V \in H^{-1}(\mathbb{T})$ with zero average is represented uniquely in the form $V = \sigma' + \sigma^2 - \int_{\mathbb{T}} \sigma^2(t)\,dt$. In the non-periodic case, in the construction of [29], τ takes the role of a local average, so the decomposition really requires two functions.

Several classes of singular potentials are modeled by a suitable choice of σ, τ. For example, a Coulomb-type term $|x - x_0|^{-1}$, $x_0 \in (0,\infty)$ is realized by setting $\sigma(x) = \log|x - x_0|$, $\tau(x) = 0$, and the point interaction $\delta(x - x_0)$ is realized by the characteristic function $\sigma(x) = \chi_{[x_0,\infty)}$ and $\tau(x) = 0$.

Remark 1.1. Of course, the decomposition $V = \sigma' + \tau$ is not unique; the procedure in [29] provides σ, τ such that

$$\begin{aligned} C^{-1} \sup_x (\|\sigma\chi_{[x,x+1)}\|_2 + \|\tau\chi_{[x,x+1)}\|_1) \\ \le \|V\|_{H^{-1}_{\mathrm{unif}}(\mathbb{R})} \le C \sup_x (\|\sigma\chi_{[x,x+1)}\|_2 + \|\tau\chi_{[x,x+1)}\|_1) \end{aligned}$$

with some universal constant C (the second inequality is general; the first is a consequence of the choice of σ, τ starting from V). Accordingly, the quantity

$$\|\sigma\chi_{[x,x+1)}\|_2 + \|\tau\chi_{[x,x+1)}\|_1$$

is interpreted as the local size of the potential.

By Dirichlet decoupling and Weyl matrix arguments, many spectral properties of Schrödinger operators on $\mathbb{R}$ are reduced to spectral properties of half-line Schrödinger operators. For this reason, spectral properties are often naturally considered in the half-line setting. In this paper, we consider half-line Schrödinger operators with real-valued distributional potentials $V = \sigma' + \tau$. The formal rewriting

$$-u'' + Vu = -(u' - \sigma u)' - \sigma u' + \tau u = -(u' - \sigma u)' - \sigma(u' - \sigma u) + (\tau - \sigma^2)u$$

produces Schrödinger operators as follows.

Hypothesis 1.2. *Denote* $\mathbb{R}_+ = (0,\infty)$ *and assume that* $\sigma, \tau : \mathbb{R}_+ \to \mathbb{R}$ *obey* (1.1). *Let*

$$u^{[1]} := u' - \sigma u$$

denote the quasi-derivative of $u \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}_+)$ *and introduce*

$$\begin{aligned} \mathfrak{D} &:= \{u \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}_+) : u^{[1]} \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}_+)\}, \\ \ell u &:= -(u^{[1]})' - \sigma u^{[1]} + (\tau - \sigma^2)u, \quad u \in \mathfrak{D}. \end{aligned}$$

This leads to self-adjoint operators on the Hilbert space $L^2(\mathbb{R}_+)$ with a regular endpoint at 0, limit point at ∞, given by

$$\begin{aligned} \operatorname{dom}(H^\alpha) := \{u \in L^2(\mathbb{R}_+) : u \in \mathfrak{D},\ \ell u \in L^2(\mathbb{R}_+), \\ u(0)\cos(\alpha) + u^{[1]}(0)\sin(\alpha) = 0\}, \\ H^\alpha u := \ell u, \end{aligned}$$

where α parametrizes the boundary condition at 0. We will discuss their self-adjointness and corresponding quadratic forms in Section 2. Note that this is consistent with standard ways of defining the operator if the potential is locally integrable (corresponding to $\sigma = 0$) or with δ-singularities (corresponding to jumps in u'), see [29].

Using the quasi-derivative, the eigenfunction equation can be written as a first-order system for $\binom{u^{[1]}}{u}$. This is encoded by a family of transfer matrices $T(z,x)$ which is locally absolutely continuous in x and solves the initial value problem

$$\partial_x T(z,x) = \begin{pmatrix} -\sigma(x) & \tau(x) - \sigma(x)^2 - z \\ 1 & \sigma(x) \end{pmatrix} T(z,x), \quad T(z,0) = I.$$

There is a corresponding Weyl function m_α and a canonical spectral measure μ^α. We will provide all definitions in Section 2; for the purpose of this introduction, it suffices to know that μ^α is a maximal spectral measure for H^α, and we are using it to make precise statements about the spectral type of H^α. We will use the Lebesgue decomposition

$$\mu^\alpha = \mu^\alpha_{\mathrm{ac}} + \mu^\alpha_{\mathrm{sc}} + \mu^\alpha_{\mathrm{pp}}.$$

One of the goals of this paper is to establish sufficient conditions for different spectral types, including a criterion for a.c. spectrum which extends the results of Last and Simon [42] for locally integrable V. One is a description of an essential support for the a.c. spectrum in terms of Cesarò-boundedness of the transfer matrices.

Theorem 1.3. *Assume Hypothesis 1.2. Then, for arbitrary $\alpha \in [0,\pi)$, the set*

$$\Sigma_{\mathrm{ac}} := \left\{ E \in \mathbb{R} \;\middle|\; \liminf_{l\to\infty} \frac{1}{l} \int_0^l \|T(E;x)\|^2 dx < \infty \right\} \tag{1.2}$$

is an essential support for the a.c. spectrum of H^α in the sense that μ^α_{ac} is mutually absolutely continuous with the measure $\chi_{\Sigma_{\mathrm{ac}}}(E)\,dE$. In particular,

$$\operatorname{Spec}_{\mathrm{ac}}(H^\alpha) = \overline{\Sigma_{\mathrm{ac}}}^{\mathrm{ess}}.$$

Above we denoted the essential support of a Borel set S by

$$\overline{S}^{\mathrm{ess}} := \{E \in \mathbb{R} : |S \cap (E-\varepsilon, E+\varepsilon)| > 0 \text{ for all } \varepsilon > 0\}.$$

A closely related result gives a sufficient criterion for absence of a.c. spectrum.

Theorem 1.4. *Assume Hypothesis 1.2 and fix arbitrary $\alpha \in [0, \pi)$. Let $\mathcal{F} \subset \mathbb{R}$ be a measurable set and suppose there exist sequences $\{x_j\}_{j=1}^{\infty} \subset \mathbb{R}_+$, $\{y_j\}_{j=1}^{\infty} \subset \mathbb{R}_+$ such that for Lebesgue almost every $E \in \mathcal{F}$,*

$$\lim_{j \to \infty} \|T(E; x_j, y_j)\| = \infty.$$

Then, $\mu_{\mathrm{ac}}^{\alpha}(\mathcal{F}) = 0$.

In the other direction one has the following result.

Theorem 1.5. *Assume Hypothesis 1.2 and fix $\alpha \in [0, \pi)$. Suppose that for some $p > 2$,*

$$\liminf_{x \to \infty} \int_{E_1}^{E_2} \|T(E; x)\|^p \, dE < \infty.$$

Then, H^{α} has purely absolutely continuous spectrum on (E_1, E_2).

Theorems 1.3, 1.4, and 1.5 generalize results of Last and Simon [42]. The proofs are given in Section 2, which also includes a Carmona-type formula, subordinacy, and a Simon–Stolz criterion for absence of eigenvalues.

An important ingredient are new pointwise eigenfunction estimates which are stated and derived in Section 2. These relate the pointwise behavior of a formal eigenfunction and its derivative to its local L^2 behavior. For $V \in L^2_{\mathrm{loc}}$ they follow from Sobolev embedding theorems, but, for $V \notin L^2_{\mathrm{loc}}$, the local domain becomes V-dependent and different arguments are needed; estimates of this form were previously considered for locally L^1 potentials [45, 60]. The pointwise estimates are given in Lemma 2.7; here we point out one corollary of these estimates.

Theorem 1.6. *Assume Hypothesis 1.2 and let $w\colon (0, \infty) \to (0, \infty)$ obey*

$$\sup_{\{x,y : |x-y| \le 1\}} \frac{w(y)}{w(x)} < \infty. \tag{1.3}$$

For any $E \in \mathbb{R}$, there exists a positive constant C such that for any $l > 1$ and any solution $u \in \mathfrak{D}$, $\ell u = Eu$, one has

$$\int_1^l w(x) \|\vec{u}(x)\|^2 dx \le C \int_0^{l+1} w(x) |u(x)|^2 dx, \tag{1.4}$$

where

$$\vec{u}(x) := \begin{pmatrix} u^{[1]}(x) \\ u(x) \end{pmatrix}, \quad \|\vec{u}(x)\|^2 := |u^{[1]}(x)|^2 + |u(x)|^2. \tag{1.5}$$

In particular, if

$$\int_0^\infty w(x)|u(x)|^2\,dx < \infty,$$

then

$$\int_0^\infty w(x)|u^{[1]}(x)|^2\,dx < \infty$$

and

$$\lim_{x\to\infty}\sqrt{w(x)}|u(x)| = \lim_{x\to\infty}\sqrt{w(x)}|u^{[1]}(x)| = 0. \tag{1.6}$$

In this paper, we will only use the case $w = 1$; however, polynomial weights $w(x) = (x+1)^c$ and exponential weights $w(x) = e^{cx}$ for $c \in \mathbb{R}$ are also relevant for various criteria about the spectrum, spectral type, and dynamical properties which we expect to have a generalization to the current setting.

Remark 1.1 indicates that decay at ∞ should be quantified by the local L^2-norm on σ and local L^1-norm on τ. Thus, the following result generalizes the Blumenthal–Weyl criterion for preservation of essential spectrum under decaying perturbations.

Lemma 1.7. *Assume Hypothesis 1.2 and suppose that*

$$\lim_{x\to\infty}\int_x^{x+1}(\sigma^2(t) + |\tau(t)|)\,dt = 0. \tag{1.7}$$

Then, for arbitrary $\alpha \in [0,\pi)$*,* $\operatorname{Spec}_{\mathrm{ess}}(H^\alpha) = [0,\infty)$.

We note in particular that Lemma 1.7 gives a more robust criterion even for locally L^1-potentials. Any locally uniformly L^1 potential V can be decomposed as $\sigma = 0$, $\tau = V$, but choosing a different decomposition can give better results. For instance, Lemma 1.7 implies the following result.

Corollary 1.8. *If* $V \in L^1_{\mathrm{loc}}([0,\infty))$ *is real-valued and the limit*

$$\lim_{x\to\infty}\int_0^x V(t)\,dt$$

is convergent, then the operator $-\frac{d^2}{dx^2} + V$ *is limit point at* ∞ *and its arbitrary self-adjoint realization* H_V *in* $L^2(\mathbb{R}_+)$ *satisfies* $\operatorname{Spec}_{\mathrm{ess}}(H_V) = [0,\infty)$.

This corollary applies to oscillatory potentials such as

$$V(x) := (-1)^{\lfloor 2n(x-n)\rfloor}, \quad x \in [n-1,n), n \in \mathbb{N}$$

which was considered in [23] by a more specialized argument, and to potentials

$$V(x) := x^\alpha \sin(x^\beta), \quad \alpha \geq 0, \beta > \alpha + 1 \tag{1.8}$$

which are not even locally uniformly integrable if $\alpha > 0$. Similar growing oscillatory potentials were considered in [13, 64]. To prove Lemma 1.7, we employ classical perturbation theory for quadratic forms [6, 22, 49].

The description of the essential spectrum is the starting point in the theory of Schrödinger operators with decaying potentials, which are a classical subject and have been extensively studied over the past 30 years [1, 4, 14–17, 35, 37, 40, 51, 53, 62–64]. Their spectral properties show a subtle competition between the rate of decay (with faster decay leading to absolutely continuous spectrum) and the disorder and oscillation in the potential (which promote more singular spectrum). Spectral transitions dependent on the rate of the decay have been studied by many authors, in particular: Pearson [51] in deterministic setting; Kiselev, Last, and Simon [37], central to this paper; Delyon, Simon, and Souillard [15] for discrete Schrödinger operators and Kronig–Penney models with decaying random potentials; and Kotani, Ushiroya [40] for continuous Schrödinger operators with decaying random potentials. This collection of papers gave rise to a number of subsequent investigations many of which are referenced in the review paper by Denisov and Kiselev [16].

We first prove that short-range perturbations preserve pure a.c. spectrum. In situations where different exponents are used to control local integrability and decay, the spaces of functions

$$\ell^p(L^q) = \left\{ f : \mathbb{R}_+ \to \mathbb{C} \;\middle|\; \sum_{n=0}^{\infty} \| f \chi_{[n,n+1)} \|_q^p < \infty \right\}$$

are useful, cf. [8, 54, 55]. The classical result about short-range perturbations is that $V \in L^1(\mathbb{R}_+)$ implies purely a.c. spectrum on $(0, \infty)$. The distributional analog of this criterion, informally speaking, is $V \in \ell^1(H^{-1})$; following Remark 1.1, we find the correct formulation.

Theorem 1.9. *Assume Hypothesis 1.2. If $\sigma \in \ell^1(L^2)$ and $\tau \in \ell^1(L^1) = L^1(\mathbb{R}_+)$, then H^α has purely a.c. spectrum on $(0, \infty)$ for every $\alpha \in [0, \pi)$.*

In fact, we prove a more general result than Theorem 1.9.

Theorem 1.10. *Assume Hypothesis 1.2 and*

$$\sigma \in L^1(\mathbb{R}_+), \quad (\sigma^2 - \tau) \in L^1(\mathbb{R}_+). \tag{1.9}$$

Then, for arbitrary $\alpha \in [0, \pi)$, the spectral measure on $(0, \infty)$ is of the form

$$\chi_{(0,\infty)}(E) d\mu^\alpha(E) = w_\alpha(E)\, dE$$

with w_α continuous on $(0,\infty)$ and strictly positive there. In particular, the spectrum of H^α is purely absolutely continuous on $(0,\infty)$.

To see that Theorem 1.10 implies Theorem 1.9, note that $\ell^1(L^2) \subset \ell^2(L^2) = L^2(\mathbb{R}_+)$ and $\ell^1(L^2) \subset \ell^1(L^1) = L^1(\mathbb{R}_+)$; thus, $\sigma \in \ell^1(L^2)$ and $\tau \in L^1(\mathbb{R}_+)$ implies (1.9). These results apply, for instance, to potentials (1.8) with $\beta > \alpha + 2$.

We note that neither condition in these theorems can be relaxed. For $\sigma = 0$, it is well known that decay of τ weaker than L^1 can introduce singular spectrum in $(0,\infty)$; e.g., Wigner–von Neumann-type potentials [46,50,57,58,62] can exhibit eigenvalues embedded into ac spectrum with $\tau(x) = \mathcal{O}(1/x)$ as $x \to \infty$. Similarly, we note the following.

Example 1.11. There exists $\sigma \in \mathrm{AC}_{\mathrm{loc}}([0,\infty))$ with $\sigma(x) = \mathcal{O}(1/x)$ as $x \to \infty$ such that for $\tau = 0$ and some $\alpha \in [0,\pi)$, the spectrum of H^α is not purely absolutely continuous on $(0,\infty)$.

Since such an example obeys $\sigma \in L^2(\mathbb{R}_+)$, it shows that the condition $\sigma \in \ell^1(L^2)$ cannot be relaxed in Theorem 1.9 and that the condition $\sigma^2 - \tau \in L^1(\mathbb{R}_+)$ cannot be relaxed in Theorem 1.10.

In the second part of the paper, we specialize to decaying sparse potentials and prove the following theorem.

Theorem 1.12. *Let $W_n \in H^{-1}(\mathbb{R})$ be real distributions with $\operatorname{supp} W_n \subset [-\Delta,\Delta]$. Assume that $W_n \to W$ in $H^{-1}(\mathbb{R})$, with $W \neq 0$. Let $d_n \to 0$, let $\{x_n\}_{n=1}^\infty \subset \mathbb{R}_+$ be a monotonically increasing sequence such that $x_1 > \Delta$ and $\frac{x_n}{x_{n+1}} \to 0$, and let*

$$V(x) = \sum_{n=1}^{\infty} d_n W_n(x - x_n).$$

For any $\alpha \in [0,\pi)$, $\mathrm{Spec}_{\mathrm{ess}}(H^\alpha) = [0,\infty)$ and, moreover, we have the following.

(a) *If $\sum_{n=1}^\infty |d_n|^2 < \infty$, then the spectrum of H^α is purely absolutely continuous on $(0,\infty)$, in the sense that $\chi_{(0,\infty)}\, d\mu^\alpha$ is mutually absolutely continuous with Lebesgue measure on $(0,\infty)$. In particular, $\mathrm{Spec}_{\mathrm{sc}}(H^\alpha) = \emptyset$, $\mathrm{Spec}_{\mathrm{pp}}(H^\alpha) \subset (-\infty,0]$, $\mathrm{Spec}_{\mathrm{ac}}(H^\alpha) = [0,\infty)$.*

(b) *If $\sum_{n=1}^\infty |d_n|^2 = \infty$, then the spectrum of H^α is purely singular continuous on $(0,\infty)$. In particular,*

$$\mathrm{Spec}_{\mathrm{sc}}(H^\alpha) = [0,\infty), \quad \mathrm{Spec}_{\mathrm{pp}}(H^\alpha) \subset (-\infty,0], \quad \mathrm{Spec}_{\mathrm{ac}}(H^\alpha) = \emptyset.$$

The special choice $W_n \equiv W \in L^\infty((-\Delta,\Delta))$ yields Pearson-type classical potentials; that case of Theorem 1.12 was proved by Kiselev, Last, and Simon [37]. Our extension allows more singular potentials; for instance, as an illustration of Theorem 1.12, we claim a Kiselev–Last–Simon-type spectral transition for the Kronig–Penney model. Concretely, let H be the Schrödinger operator acting on $L^2(\mathbb{R}_+)$ given by

$$H = -\frac{d^2}{dx^2} + \sum_{n=1}^{\infty} d_n \delta(x - x_n),$$

where $\{x_n\}_{n=1}^{\infty} \subset (0, \infty)$ is a sparse sequence satisfying $x_n/x_{n+1} \to 0$ as $n \to \infty$, subject to any self-adjoint condition at 0. Then for any decaying sequence $(d_n)_{n=1}^{\infty}$, $\mathrm{Spec}_{\mathrm{ess}}(H) = [0, \infty)$; moreover, the spectrum is purely a.c. on $(0, \infty)$ if $(d_n)_{n=1}^{\infty}$ is square-summable and purely s.c. on $(0, \infty)$ otherwise. Related to this example is the paper [44], cf. also [3, 39], where the spectral types of Kronig–Penney-type models are discussed. In contrast to [44], however, the above example indicates spectral transition within the class of decaying coupling constants d_n, while [44] studies different spectral types (without transition between them) for growing d_n.

Another new feature of our result is that the profile W_n may vary with n. Note that this allows examples such as the locally integrable potential

$$V = \sum_{n=1}^{\infty} d_n n \chi_{[x_n, x_n + 1/n]},$$

where χ denote characteristic functions. Since $n\chi_{[0,1/n]} \to \delta_0$ in $H^{-1}(\mathbb{R})$, by Theorem 1.12, the spectrum is purely a.c. on $(0, \infty)$ if the decaying sequence $(d_n)_{n=1}^{\infty}$ is square-summable and purely s.c. on $(0, \infty)$ otherwise.

Although stated in terms of $H^{-1}(\mathbb{R})$, the starting point in our analysis is a decomposition $W_n = S_n' + T_n$ and the proof must treat these contributions to σ and τ separately. As in the classical case [37], our proof is based on the analysis of Prüfer variables. However, in the present case this analysis is more intricate due to the appearance of new terms in the differential equations obeyed by Prüfer variables. Namely, in the setting of H^{-1} potential $V = \sigma' + \tau$, as shown in Proposition 2.13, one has

$$\theta' = k - \frac{\tau - \sigma^2}{k} \sin^2(\theta) + \sigma \sin(2\theta), \quad (\log R)' = \frac{\tau - \sigma^2}{2k} \sin(2\theta) - \sigma \cos(2\theta),$$

whereas in the classical case $V \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$, as discussed in [37],

$$\theta' = k - \frac{V}{k} \sin^2(\theta), \quad (\log R)' = \frac{V}{2k} \sin(2\theta).$$

An important ingredient in the proof of Theorem 1.12 is given by the fact that $\|T(k^2, x)\|$ is comparable to $R(x)$, see Proposition 2.13. Hence, in order to establish growth or boundedness of eigensolutions and, respectively, the absence or existence of purely absolutely continuous spectrum on $[E_1, E_2]$, it suffices to study the asymptotics for $R(x)$. In Sections 3.3 and 3.4, we describe the asymptotic behavior of $R(x)$ depending on whether or not $\{d_n\}_{n=1}^{\infty} \in \ell^2(\mathbb{N})$.

2. Spectral analysis of Schrödinger operators with distributional potentials

In this section, we consider Schrödinger operators in the setting of Hypothesis 1.2. We start with the general properties of the self-adjoint operators and quadratic forms for locally square-integrable σ and locally integrable τ, and then establish general ways to study the spectral type.

2.1. Self-adjointness and form bounds

For an interval $I \subset \mathbb{R}$ and $\sigma \in L^2_{\mathrm{loc}}(I)$, $\tau \in L^1_{\mathrm{loc}}(I)$, differential expressions of the form

$$\ell u = -(u' - \sigma u)' - \sigma(u' - \sigma u) + (\tau - \sigma^2)u$$

are within the very general setting of four-coefficient Sturm–Liouville operators with locally integrable coefficients considered by Eckhardt, Gesztesy, Nicols, and Teschl [19]; in the notation of [19], this is obtained by setting $p = 1, q = \tau - \sigma^2$, $r = 1, s = -\sigma$. Thus, the following general properties are known.

We denote for $u \in \mathrm{AC}_{\mathrm{loc}}(I)$ the quasiderivative

$$u^{[1]} = u' - \sigma u.$$

Associated with the differential expression ℓ is the local domain

$$\mathfrak{D} = \{u \in \mathrm{AC}_{\mathrm{loc}}(I) : u^{[1]} \in \mathrm{AC}_{\mathrm{loc}}(I)\}$$

and three linear, densely defined, unbounded operators $H_0, H_{\min}, H_{\max}$ acting on $L^2(I)$ defined as follows:

$$\begin{aligned}
&H_{\max} u = \ell u, \\
&u \in \mathrm{dom}(H_{\max}) := \{u \in L^2(I) : u \in \mathfrak{D},\ \ell u \in L^2(I)\}, \\
&H_0 u = \ell u, \\
&u \in \mathrm{dom}(H_0) := \{u \in \mathrm{dom}(H_{\max}) : u \text{ has compact support}\},
\end{aligned}$$

and $H_{\min} := \overline{H_0}$, the closure of H_0 in $L^2(I)$. By [19, Section 3],

$$H_{\min} = \overline{H_0} = H_0^{**} = H_{\max}^*; \quad H_{\min} \subset H_{\min}^* = H_{\max}.$$

Moreover, there is a limit point/limit circle dichotomy at each endpoint, i.e., the space of solutions of $\ell u = zu$ which are square-integrable in a neighborhood of the endpoint is two-dimensional for all $z \in \mathbb{C} \setminus \mathbb{R}$ (the limit circle case) or one-dimensional for all $z \in \mathbb{C} \setminus \mathbb{R}$ (the limit point case). In this setting, the Wronskian is defined for $u, v \in \mathfrak{D}$ by

$$W(u, v)(x) = u(x)v^{[1]}(x) - u^{[1]}(x)v(x).$$

For any $u, v \in \operatorname{dom}(H_{\max})$, the Wronskian has a limit as x approaches an endpoint. The endpoint $\sup I$ (respectively, $\inf I$) is limit point if and only if

$$\lim_{x \to \sup I} W(u, v)(x) = 0, \quad u, v \in \operatorname{dom}(H_{\max}).$$

(respectively, as $x \to \inf I$), cf. [19, Lemma 4.4].

In the setting of Hypothesis 1.2, the endpoint 0 is regular because

$$\int_0^1 \sigma^2(t)\, dt < \infty, \quad \int_0^1 |\tau(t)|\, dt < \infty \tag{2.1}$$

(informally, a regular endpoint is a point with the same local integrability properties of the coefficients as an internal point; formally, (2.1) matches the general definition of regular endpoint used in [19]). Thus, by [19, Section 3], ℓ is limit circle at 0, for every $u \in \operatorname{dom}(H_{\max})$ the limits

$$u(0) = \lim_{x \to 0} u(x), \quad u^{[1]}(0) = \lim_{x \to 0} u^{[1]}(x) \tag{2.2}$$

exist and are finite, and self-adjoint boundary conditions at 0 are of the form

$$u(0)\cos\alpha + u^{[1]}(0)\sin\alpha = 0$$

for some $\alpha \in [0, \pi)$.

Full-line Schrödinger operators with H^{-1}_{loc} potentials were studied in detail by Hryniv and Mykytyuk, for example, in [29,30]; in particular, full-line operators obeying the local uniform bounds (1.1) are limit point at $\pm\infty$. Since the limit point/limit circle dichotomy is a local property of the endpoint by [19, Section 3], in the setting of Hypothesis 1.2, ℓ is limit point at ∞. The following lemma summarizes these known facts.

Lemma 2.1. *Under the assumptions of Hypothesis 1.2, for every $u \in \operatorname{dom}(H_{\max})$, the limits (2.2) exist and are finite, all self-adjoint extensions of $H_{\min}$ are parametrized by $\alpha \in [0, \pi)$ as follows:*

$$H^\alpha u = -(u^{[1]})' - \sigma u^{[1]} + (\tau - \sigma^2)u, \quad u \in \operatorname{dom}(H^\alpha),$$
$$\operatorname{dom}(H^\alpha) = \{u \in \operatorname{dom}(H_{\max}) : u(0)\cos(\alpha) + u^{[1]}(0)\sin(\alpha) = 0\}.$$

Moreover, Hryniv and Mykytyuk [29, Section 3] described semiboundedness and quadratic forms associated with the full-line Schrödinger operators; we adapt this proof to the half-line setting, in order to describe the quadratic forms associated with H^α.

Theorem 2.2. *Under the assumptions of Hypothesis 1.2, for every $\alpha \in [0, \pi)$, the operator H^α is bounded from below and there exist $C = C(\sigma, \tau) > 1, \lambda = \lambda(\sigma, \tau) > 0$ such that for $E < \min\{\inf \operatorname{Spec}(H^\alpha), 0\}$,*

$$(H^\alpha - E)^{-1} \le C(-\Delta_X - E + \lambda)^{-1}, \tag{2.3}$$

where $-\Delta_X$ is the Dirichlet Laplacian on $\mathbb{R}_+$ if $\alpha = 0$ and Neumann Laplacian if $\alpha \in (0, \pi)$.

The quadratic form $\mathfrak{h}^\alpha$ of H^α is given by

$$\mathfrak{h}^\alpha[u, v] = \begin{cases} \int_0^\infty (\bar{u}'v' - \bar{u}'\sigma v - \bar{u}\sigma v' + \tau\bar{u}v)(x)\, dx - \cot(\alpha)\overline{u(0)}v(0), & \alpha \in (0, \pi), \\ \int_0^\infty (\bar{u}'v' - \bar{u}'\sigma v - \bar{u}\sigma v' + \tau\bar{u}v)(x)\, dx, & \alpha = 0, \end{cases} \tag{2.4}$$

for $u, v \in \operatorname{dom}(\mathfrak{h}^\alpha)$, where

$$\operatorname{dom}(\mathfrak{h}^\alpha) := \begin{cases} H^1(\mathbb{R}_+), & \alpha \in (0, \pi), \\ H_0^1(\mathbb{R}_+) := \{f \in H^1(\mathbb{R}_+) : f(0) = 0\}, & \alpha = 0. \end{cases} \tag{2.5}$$

Proof. Recall from [29, Lemma 3.1] that for arbitrary interval $I \subset \mathbb{R}_+$ of length 1, $\varepsilon \in (0, 1)$, and $\psi \in H^1(I)$,

$$\|\psi\|^2_{L^\infty(I)} \le \varepsilon\|\psi'\|^2_{L^2(I)} + 8\varepsilon^{-1}\|\psi\|^2_{L^2(I)}, \tag{2.6}$$

$$\|\psi\psi'\|_{L^2(I)} \le \varepsilon\|\psi'\|^2_{L^2(I)} + 4\varepsilon^{-3}\|\psi\|^2_{L^2(I)}. \tag{2.7}$$

In particular, as in the proof of [29, Theorem 3.4], for any $u \in H^1(\mathbb{R}_+)$,

$$\int_0^\infty |\sigma u|^2 = \sum_{n=0}^\infty \int_n^{n+1} |\sigma u|^2 \le \|\sigma\|^2_{2,\mathrm{unif}} \sum_{n=0}^\infty \|u\|^2_{L^\infty(n,n+1)} \underset{(2.6)}{\le} C\|u\|^2_{H^1(\mathbb{R}_+)} < \infty.$$

That is, $\sigma u \in L^2(\mathbb{R}_+)$. Due to this and $H^1(\mathbb{R}_+) \subset L^\infty(\mathbb{R}_+)$, the form $\mathfrak{h}^\alpha$ defined by (2.4), (2.5) is well defined.

Let us now prove that $\mathfrak{h}^\alpha$ is relatively bounded with respect to the quadratic form of the Dirichlet or Neumann free Laplacian on $\mathbb{R}_+$, depending on the value of α. Note that for arbitrary $\varepsilon > 0$, employing (2.6), (2.7) as in the proof of [29, Lemma 3.2],

$$\begin{aligned}\Big|\int_0^\infty \sigma \bar{u}' u\Big| &\le \sum_{n=0}^\infty \int_n^{n+1} |\sigma \bar{u}' u| \underset{(2.7)}{\le} \|\sigma\|_{2,\mathrm{unif}}(\varepsilon\|u'\|^2_{L^2(\mathbb{R}_+)} + 4\varepsilon^{-3}\|u\|^2_{L^2(\mathbb{R}_+)}),\\ \Big|\int_0^\infty \tau \bar{u} u\Big| &\le \sum_{n=0}^\infty \int_n^{n+1} |\tau|\, \|u\|^2_{L^\infty(n,n+1)} \underset{(2.6)}{\le} \|\tau\|_{1,\mathrm{unif}}(\varepsilon\|u'\|^2_{L^2(\mathbb{R}_+)} + 8\varepsilon^{-1}\|u\|^2_{L^2(\mathbb{R}_+)}),\end{aligned} \tag{2.8}$$

for arbitrary $u \in H^1(\mathbb{R}_+)$; moreover, by (2.6),

$$|u(0)|^2 \le \varepsilon\|u'\|^2_{L^2(\mathbb{R}_+)} + 4\varepsilon^{-3}\|u\|^2_{L^2(\mathbb{R}_+)}. \tag{2.9}$$

Let $\mathfrak{h}^X$, $X \in \{D, N\}$ denote the quadratic form corresponding to Dirichlet or Neumann free Laplacian on $\mathbb{R}_+$; i.e.,

$$\mathfrak{h}^X(u,u) = \|u'\|^2_{L^2(\mathbb{R}_+)}, \quad u \in \mathrm{dom}(\mathfrak{h}^X),$$

where

$$\mathrm{dom}(\mathfrak{h}^D) := H^1_0(\mathbb{R}_+) \quad \text{and} \quad \mathrm{dom}(\mathfrak{h}^N) := H^1(\mathbb{R}_+).$$

We will proceed with assuming $\alpha \in (0,\pi)$, the second case $\alpha = 0$ can be handled similarly. For any $a \in (0,1)$, the inequalities (2.8), (2.9) yield $b \in \mathbb{R}$ such that

$$|\mathfrak{h}^\alpha[u,u] - \mathfrak{h}^N[u,u]| \le a\|u'\|^2_{L^2(\mathbb{R}_+)} + b\|u\|^2_{L^2(\mathbb{R}_+)}, \quad u \in H^1(\mathbb{R}_+).$$

That is, the lower order terms and the boundary term in the definition of $\mathfrak{h}^\alpha$, considered as quadratic form on $H^1(\mathbb{R}_+)$, are relatively bounded with respect to Neumann form $\mathfrak{h}^N$, with relative bound less than one, see [34, Section VI.3.3] or [52, Chapter X]. Thus, by [52, Theorem X.17], $\mathfrak{h}^\alpha$ is closed bounded from below quadratic form and there is a unique self-adjoint operator T^α acting in $L^2(\mathbb{R}_+)$ which satisfies

$$\langle T^\alpha u, v\rangle_{L^2(\mathbb{R}_+)} = \mathfrak{h}^\alpha(u,v), \quad u \in \mathrm{dom}(T^\alpha),\ v \in \mathrm{dom}(\mathfrak{h}^\alpha). \tag{2.10}$$

We claim that $H^\alpha \subset T^\alpha$. Assume this claim, we note that both operators are self-adjoint and therefore must coincide. This implies that $\mathfrak{h}^\alpha$ is the quadratic form of the operator H^α which is consequently bounded from below. Let us now proof $H^\alpha \subset T^\alpha$ for $\alpha \ne 0$ (the case $\alpha = 0$ can be handled analogously). It suffices to establish (2.10) for all $u \in \mathrm{dom}(H^\alpha)$ and all $v \in H^1(\mathbb{R}_+) \cap C^\infty(\mathbb{R}_+)$ with bounded[1] support $\mathrm{supp}(v)$. Indeed, both sides of (2.10) are continuous with respect to v in

[1] In case $\alpha = 0$, take v with compact support, i.e., $v \in C_0^\infty(\mathbb{R}_+)$

$H^1(\mathbb{R}_+)$ norm and the set of $v \in H^1(\mathbb{R}_+) \cap C^\infty(\mathbb{R}_+)$ with bounded support is dense in $H^1(\mathbb{R}_+)$, see, for example, [22, Corollary 3.3 in Chapter V, Section 3]. Then using $u^{[1]}(0) = -\cot(\alpha)u(0)$, $v(x) = 0$ for sufficiently large $x > 0$, and integration by parts,

$$\begin{aligned} \mathfrak{h}^\alpha(u, v) &= \int_0^\infty \big(\overline{u^{[1]}} v' - \sigma \overline{u^{[1]}} v + (\tau - \sigma^2)\bar{u} v\big)(x)\, dx + \overline{u^{[1]}(0)} v(0) \\ &= \int_0^\infty \big(-\overline{(u^{[1]})'} v - \sigma \overline{u^{[1]}} v + (\tau - \sigma^2)\bar{u} v\big)(x)\, dx \\ &= \langle H^\alpha u, v\rangle_{L^2(\mathbb{R}_+)}. \end{aligned}$$

In order to prove (2.3) (again we focus on the case $\alpha \in (0, \pi)$), we invoke (2.8), (2.9) to obtain some $C = C(\sigma, \tau) > 1$ such that

$$\mathfrak{h}^\alpha(u, u) \le C(\|u'\|^2_{L^2(\mathbb{R}_+)} + \lambda\|u\|^2_{L^2(\mathbb{R}_+)}), \quad u \in H^1(\mathbb{R}_+).$$

Noting that the left-hand side above is the quadratic form of H^α and the right-hand side is the quadratic form of $C(-\Delta_N + \lambda)$, assertion (2.3) follows from [34, Theorem VI 2.21], where it is shown that the ordering of quadratic forms implies the ordering of resolvents. ∎

Remark 2.3. (i) The representation $V = \sigma' + \tau$ is not unique; given two pairs $(\sigma_i, \tau_i) \in L^2_{\text{loc}}(\mathbb{R}_+) \times L^1_{\text{loc}}(\mathbb{R}_+)$, $i = 1, 2$ with $\sigma_1' + \tau_1 = V = \sigma_2' + \tau_2$ one has

$$\theta := \sigma_1 - \sigma_2, \quad \theta' = \tau_2 - \tau_1,$$

so that $\theta \in W^{1,1}_{\text{loc}}(\mathbb{R}_+)$.

(ii) Fix $\theta \in W^{1,1}_{\text{loc}}(\mathbb{R}_+)$ and $(\sigma, \tau) \in L^2_{\text{loc}}(\mathbb{R}_+) \times L^1_{\text{loc}}(\mathbb{R}_+)$. We say that the pair $(\sigma + \theta, \tau - \theta')$ is *a gauge change of* (σ, τ). The domain $\operatorname{dom}(H_{\max})$ is gauge change invariant since for $u \in \mathrm{AC}_{\text{loc}}(\mathbb{R}_+)$ one has

$$(u' - \sigma u) \in \mathrm{AC}_{\text{loc}}(\mathbb{R}_+) \iff (u' - (\sigma + \theta)u) \in \mathrm{AC}_{\text{loc}}(\mathbb{R}_+)$$

and a direct calculation shows that the action of the maximal operator $H_{\max}$ is also gauge change invariant. The gauge change affects the definition of the quasi-derivative $u_j^{[1]} = u' - \sigma_j u$ so that $u_1^{[1]} = u_2^{[1]} - \theta u$. Therefore, the self-adjoint boundary conditions $u(0)\cos\alpha_j + u_j^{[1]}(0)\sin\alpha_j = 0$ are relabeled by the formula

$$\cot\alpha_2 = \cot\alpha_1 - \theta(0).$$

Remark 2.4. In the setting of Theorem 2.2, two distinct self-adjoint extensions of $H_{\min}$ are rank-one perturbations of each other. Concretely, one has

$$\dim\operatorname{rank}((H^\alpha - \mathbf{i})^{-1} - (H^\beta - \mathbf{i})^{-1}) \le 1.$$

This is due to the fact that the deficiency indices of $H_{\min}$ are $(1, 1)$ and the abstract Krein's resolvent formula [9, Theorem A.1].

We can now prove our version of the Blumenthal–Weyl criterion.

Proof of Lemma 1.7. By Remark 2.4 and [22, Theorem 2.4], it suffices to prove the statement for $\alpha = 0$. Let H_D and $\mathfrak{h}_D$ denote respectively the Dirichlet Laplacian and its quadratic form on $\mathbb{R}_+$; i.e., using the notation of Theorem 2.2 with $\alpha = 0$, $\sigma = \tau = 0$, write $H_D := H^0$, $\mathfrak{h}_D = \mathfrak{h}^0$. Our goal is to show that for σ, τ as in (1.7), the quadratic form $\mathfrak{h}^0$ is a relative compact perturbation of $\mathfrak{h}_D$ (see e.g., [49, Definition 2.12], [22, Section IV.4]). This assertion together with [49, Theorem 2.13] yields $\operatorname{Spec}_{\mathrm{ess}}(H^0) = \operatorname{Spec}_{\mathrm{ess}}(H_D)$ and, when combined with $\operatorname{Spec}_{\mathrm{ess}}(H_D) = [0, \infty)$, proves the statement.

Consider the quadratic form

$$\mathfrak{s}[u, v] := \int_0^\infty (-\overline{u'}\sigma v - \bar{u}\sigma v' + \tau\bar{u}v)(x)dx, \quad u, v \in \operatorname{dom}(\mathfrak{s}) = H_0^1(\mathbb{R}_+).$$

In order to show that $\mathfrak{h}^0$ is a relative compact perturbation of $\mathfrak{h}_D$, it suffices to verify that

(i) one has

$$|\mathfrak{s}[u, u]| \le C(\mathfrak{h}^0[u, u] + \|u\|^2_{L^2(\mathbb{R}_+)}) \quad \text{for any } u \in H_0^1(\mathbb{R}_+); \tag{2.11}$$

(ii) if $\sup_j \|u_j\|_{H^1(\mathbb{R}_+)} \le 1$, then there exists a subsequence $\{u_{j_m}\}_{m=1}^\infty$ such that for $\varepsilon > 0$ there exists $K > 1$ such that

$$|\mathfrak{s}[u_{j_m} - u_{j_n}, u_{j_m} - u_{j_n}]| < \varepsilon \quad \text{for all } m, n > K, \tag{2.12}$$

cf. [49, Theorem 2.14].

The first inequality (2.11) follows from (2.8), so it suffices to prove (2.12). First, let $\chi_{[a,b]}$ denote the characteristic function of $[a, b]$ and note that

$$\begin{aligned} |\mathfrak{s}[u, u]| \le 2\left|\int_0^\infty (\chi_{[0,t]}\overline{u'}\sigma u)(x)dx\right| + \left|\int_0^\infty (\chi_{[0,t]}\tau|u|^2)(x)dx\right| \\ + 2\left|\int_0^\infty (\chi_{[t,\infty)}\overline{u'}\sigma u)(x)dx\right| + \left|\int_0^\infty (\chi_{[t,\infty)}\tau|u|^2)(x)dx\right|, \quad u \in \operatorname{dom}(\mathfrak{s}). \end{aligned} \tag{2.13}$$

Fix arbitrary $\varepsilon > 0$, then for a sequence $\{u_j\}_{j=1}^{\infty} \subset \operatorname{dom}(\mathfrak{s})$ with $\sup_j \|u_j\|_{H^1(\mathbb{R}_+)} \le 1$ and sufficiently large, j-independent, $t = t(\varepsilon, \sigma, \tau) > 0$,

$$
\begin{aligned}
&2\Big|\int_0^\infty (\chi_{[t,\infty)}\overline{(u_j-u_k)'}\sigma(u_j-u_k))(x)dx\Big| + \Big|\int_0^\infty (\chi_{[t,\infty)}\tau|u_j-u_k|^2)(x)dx\Big| \\
&\underset{(2.8)}{\le} C(\|\chi_{[t,\infty)}\sigma\|_{2,\mathrm{unif}} + \|\chi_{[t,\infty)}\tau\|_{1,\mathrm{unif}})\|u_j-u_k\|_{H^1(\mathbb{R}_+)} \le \varepsilon/2, \qquad (2.14)
\end{aligned}
$$

for all $j \in \mathbb{N}$, where, in the last inequality, we used (1.7) and $\sup_j \|u_j\|_{H^1(\mathbb{R}_+)} \le 1$. Next, for t defined above, note that $\sup_j \|\chi_{[0,t]}u_j\|_{H^1(\mathbb{R}_+)} \le 1$ and, due to compactness of the embedding $H^1((0,t)) \hookrightarrow L^2((0,t))$, there exists a subsequence $\{u_{j_k}\}_{k=1}^{\infty}$ which is Cauchy in $L^2(\mathbb{R}_+)$. For such a subsequence and arbitrary $\varepsilon > 0$, there exists $K > 1$ such that

$$
\begin{aligned}
&2\Big|\int_0^\infty (\chi_{[0,t]}\overline{(u_{j_m}-u_{j_n})'}\sigma(u_{j_m}-u_{j_n}))(x)dx\Big| \\
&\quad + \Big|\int_0^\infty (\chi_{[0,t]}\tau|u_{j_m}-u_{j_n}|^2)(x)dx\Big| < \varepsilon/2, \quad \text{for } m,n > K, \qquad (2.15)
\end{aligned}
$$

where we used the Cauchy–Schwarz inequality and $\sup_j \|\chi_{[0,t]}u_j\|_{H^1(\mathbb{R}_+)} \le 1$. It follows from (2.13) with $u := u_{j_m} - u_{j_n}$, (2.14) and (2.15) that

$$
|\mathfrak{s}[u_{j_m}-u_{j_n}, u_{j_m}-u_{j_n}]| < \varepsilon, \quad \text{for } m,n > K,
$$

which yields (2.12) as required. ■

Proof of Corollary 1.8. Let $\sigma(x) := \int_0^x V(t)dt - \int_0^\infty V(t)dt$, $\tau = 0$. Then $\sigma(x) \to 0$, $x \to \infty$, hence (1.1) holds; hence, by Lemma 2.1 H_V is limit point at infinity. In addition one has (1.7), thus, by Lemma 1.7, $\operatorname{Spec}_{\mathrm{ess}}(H^\alpha) = [0,\infty)$ which combined with $\sigma' + \tau = V$ yield $\operatorname{Spec}_{\mathrm{ess}}(H_V) = [0,\infty)$ as asserted. ■

At this point, let us prove the assertion made in Example 1.11.

Proof of Example 1.11. The Wigner–von Neumann potential V, explicitly defined in [57, Section 3, Part B], admits a real-valued non-trivial eigenfunction $u \in L^2(\mathbb{R}_+)$ corresponding to eigenvalue 1. In particular, for the choice of boundary condition at 0 corresponding to u, the Schrödinger operator $-\frac{d^2}{dx^2} + V$ has an eigenvalue in $(0,\infty)$. We set $\sigma(x) := -\int_x^\infty V(t)dt$ and $\tau = 0$. Then $V = \sigma' + \tau$ so this is a gauge change of the Wigner–von Neumann potential; in particular, spectral type is unchanged. To prove $\sigma(x) \underset{x\to\infty}{=} \mathcal{O}(1/x)$, we recall the asymptotic formula

$$
V(t) = -\frac{8\sin(2t)}{t} + \mathcal{O}(t^{-2}), \quad t \to \infty.
$$

Hence, for some $C, c > 0$ and sufficiently large x we have

$$\left|\int_x^\infty V(t)dt\right| \le \left|\int_x^\infty \frac{8\sin(2t)}{t}dt\right| + \left|\frac{c}{x}\right|$$
$$\le \left|\frac{4\cos(2x)}{x}\right| + \left|\int_x^\infty \frac{4\cos(2t)}{t^2}dt\right| + \frac{c}{x} \le \frac{C}{x}. \qquad \blacksquare$$

Remark 2.5. (i) The invariance of the essential spectrum under small at infinity perturbations of the coefficients has been investigated by many authors in various settings, see e.g., [3, 25, 28, 65] and especially [43], which contains many relevant references. The central fact in the classical treatment of this problem via Weyl-type sequences, see [28, Section 10], is that H^α has a locally compact resolvent; i.e., $\chi_{(0,n)}(H^\alpha - \mathbf{i})^{-1}, n \ge 0$ is compact in $L^2(\mathbb{R}_+)$. This still holds in our case, as readily seen from the explicit form of Green's function. However, there is a major obstacle in using the classical approach since $\operatorname{dom}(H^\alpha)$, as a subset of $L^2(\mathbb{R}_+)$, depends on σ, τ. Notably, one does not even have the inclusion $C_0^\infty(\mathbb{R}_+) \subset \operatorname{dom}(H^\alpha)$ in general; e.g., such an inclusion does not hold when V is not locally L^2. The key feature of our proof of Lemma 1.7 is that the form domain $\mathfrak{h}^\alpha$ does not depend on σ, τ. Interestingly, the latter does depend on α, though the invariance of essential spectrum under perturbation of the boundary condition is handled by Krein's formula for the difference of resolvents of two self-adjoint extensions of the minimal operator $H_{\min}$, as discussed in Remark 2.4.

(ii) Relevant to this discussion are [26, Theorem 3.2] (see also [47,48]), where the full-line version of (1.7) is shown to be equivalent to compactness of the multiplier given by an $H^{-1}(\mathbb{R})$ potential, and [3], where Birman's perturbation theory is used to prove invariance of the essential spectrum for Kronig–Penney-type models with decaying coupling constants.

2.2. Weyl–Titchmarsh theory

For $z \in \mathbb{C}$, $g \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$, the differential equation

$$-(u^{[1]})' - \sigma u^{[1]} + (\tau - \sigma^2)u - zu = g, \quad u \in \mathfrak{D}$$

is rewritten as the first order system

$$\frac{d}{dx}\begin{bmatrix} u^{[1]}(x) \\ u(x) \end{bmatrix} = A(z,x)\begin{bmatrix} u^{[1]}(x) \\ u(x) \end{bmatrix} - \begin{bmatrix} g(x) \\ 0 \end{bmatrix}, \quad A(z,x) := \begin{bmatrix} -\sigma & (\tau - \sigma^2) - z \\ 1 & \sigma \end{bmatrix}.$$

Assuming Hypothesis 1.2, since the matrix coefficients in $A(z,x)$ lie in $L^1_{\mathrm{loc}}(\mathbb{R};\mathbb{C}^{2\times 2})$, the corresponding initial value problem has a unique locally absolutely continuous solution. In particular, for $g = 0$, $\alpha \in [0, \pi)$, we consider the initial value problem $\ell u - zu = 0$ and denote by $\phi_{\alpha,z}, \theta_{\alpha,z}$ its solutions satisfying the initial conditions

$$\begin{bmatrix} \phi_{\alpha,z}^{[1]}(0) & \theta_{\alpha,z}^{[1]}(0) \\ \phi_{\alpha,z}(0) & \theta_{\alpha,z}(0) \end{bmatrix} = R_\alpha^{-1}, \quad R_\alpha := \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}. \tag{2.16}$$

The solutions are entire with respect to z. In the special case $\alpha = 0$, we denote $\theta_z := \theta_{\alpha,z}, \phi_z := \phi_{\alpha,z}$. Note that any $u \in \mathfrak{D}$ solving $\ell u = zu$ satisfies

$$\begin{bmatrix} u^{[1]}(x) \\ u(x) \end{bmatrix} = T(z; x, 0) \begin{bmatrix} u^{[1]}(0) \\ u(0) \end{bmatrix}, \quad T(z; x, 0) := \begin{bmatrix} \phi_z^{[1]}(x) & \theta_z^{[1]}(x) \\ \phi_z(x) & \theta_z(x) \end{bmatrix}.$$

Since $W(\phi_z, \theta_z)(x)$ is constant (due to Lagrange identity [19, Lemma 2.3]),

$$\det T(z; x, 0) = 1.$$

Thus, the transfer matrix can be defined as

$$T(z; x, y) := \begin{bmatrix} \phi_z^{[1]}(x) & \theta_z^{[1]}(x) \\ \phi_z(x) & \theta_z(x) \end{bmatrix} \begin{bmatrix} \phi_z^{[1]}(y) & \theta_z^{[1]}(y) \\ \phi_z(y) & \theta_z(y) \end{bmatrix}^{-1},$$

where for any $u \in \mathfrak{D}$ solving $\ell u = zu$, and any $x, y \geq 0$,

$$T(z; x, y) \begin{bmatrix} u^{[1]}(y) \\ u(y) \end{bmatrix} = \begin{bmatrix} u^{[1]}(x) \\ u(x) \end{bmatrix}.$$

We will often denote $T(z; x) := T(z; x, 0)$.

Next, we recall the Weyl–Titchmarsh theory for ℓ. Assuming Hypothesis 1.2, since ℓ is limit point at infinity, for any $z \in \mathbb{C} \setminus \mathbb{R}$, there is a 1-dim set of solutions in $L^2(\mathbb{R}_+)$ to $\ell u = zu$, where any such non-trivial solution is called a Weyl solution at infinity and denoted by ψ_z. Fix any ψ_z; the Weyl–Titchmarsh m-function is given by

$$m_\alpha(z) = -\frac{W(\psi_z, \theta_{\alpha,z})}{W(\psi_z, \phi_{\alpha,z})} = \frac{\cos(\alpha)\psi_z^{[1]}(0) - \sin(\alpha)\psi_z(0)}{\sin(\alpha)\psi_z^{[1]}(0) + \cos(\alpha)\psi_z(0)}.$$

The boundary condition affects the Weyl function by a rotation matrix: denoting by $\simeq$ the projective relation on $\mathbb{C}^2 \setminus \{0\}$,

$$\begin{bmatrix} m_\alpha(z) \\ 1 \end{bmatrix} \simeq R_\alpha \begin{bmatrix} m_0(z) \\ 1 \end{bmatrix}. \tag{2.17}$$

The Weyl disks are defined as

$$D_x^\alpha(z) := \{\mathcal{U}_\alpha \mid u \neq 0 \in \mathfrak{D}, \ \ell u - zu = 0, \ \mathbf{i} W(\bar{u}, u)(x) < 0\},$$
$$\mathcal{U}_\alpha := \frac{\cos(\alpha)u^{[1]}(0) - \sin(\alpha)u(0)}{\sin(\alpha)u^{[1]}(0) + \cos(\alpha)u(0)}.$$

Proposition 2.6. *Assume Hypothesis 1.2 and fix $z \in \mathbb{C}_+$, $\alpha \in [0, \pi)$.*

(i) *For $x > 0$, the set $D_x^\alpha(z)$ is a disk in $\mathbb{C}_+$.*

(ii) *The disks from* (i) *are strictly nested; i.e.,*

$$\overline{D_y^\alpha(z)} \subset D_x^\alpha(z), \quad x < y.$$

(iii) *The intersection of these disks is a single element set consisting of the Weyl–Titchmarsh coefficient $m_\alpha(z)$; i.e.,*

$$\bigcap_{x \geq 0} \overline{D_x^\alpha(z)} = \{m_\alpha(z)\}.$$

Moreover, $\theta_{\alpha,z} + m_\alpha(z)\phi_{\alpha,z}$ is a Weyl–Titchmarsh solution.

(iv) *The mapping $z \mapsto m_\alpha(z)$ is a Herglotz function; i.e., analytic function*

$$\mathbb{C}_+ \to \mathbb{C}_+.$$

Proof. In this setting, the Lagrange identity for $u \in \mathfrak{D}$ with $\ell u = zu$, $z \in \mathbb{C}_+$, is

$$2 \operatorname{Im} z \int_0^x |u(t)|^2 \, dt = \mathbf{i} W(\bar{u}, u)(x) - \mathbf{i} W(\bar{u}, u)(0).$$

In particular, if $u \neq 0$, then u has only isolated zeros so the function

$$\mathbf{i} W(\bar{u}, u)(x) = -2 \operatorname{Im}(\overline{u(x)} u^{[1]}(x))$$

is real-valued and strictly increasing in x. The strict increasing property above corresponds to the fact that the operator obeys the Atkinson condition, or equivalently, the corresponding canonical system has no singular intervals. The other conclusions are general consequences of the fact that the operator is limit point at ∞ (see, e.g., [41]). ∎

To conclude Section 2.2, we recall from [19, Section 9] the spectral decomposition for the operator H^α. The Herglotz function m_α discussed in Proposition 2.6 (iv) gives rise to a Borel measure μ^α via the Stieltjes–Livsic inversion formula

$$\mu^\alpha((E_1, E_2]) := \lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_{E_1+\delta}^{E_2+\delta} \operatorname{Im} m_\alpha(\alpha + \mathbf{i}\varepsilon) d\lambda,$$

for real numbers $E_1 < E_2$. The operator H^α is unitarily equivalent to the operator of multiplication by the independent variable in the space $L^2(\mathbb{R}, \mu^\alpha)$ and the classical spectral description via boundary values of $m_\alpha(z)$ holds, see [19, Section 9]. For instance, as in the classical setting, if $\alpha - \beta \notin \pi\mathbb{Z}$, then a.c. parts of μ^α, μ^β are mutually a.c., and their singular parts are mutually singular.

We will return to a detailed analysis of the absolutely continuous part of spectral measure μ^α in Section 2.4, where we will rely on estimates for eigensolutions discussed next.

2.3. Eigensolution estimates

In this section, we derive auxiliary estimates for solutions of $\ell u = Eu$, $E \in \mathbb{R}$, $u \in \mathfrak{D}$. To describe the main assertions, let us fix $\lambda > 0$ and recall (1.5). In the estimates that follow, we give bounds with explicit dependence on the parameter E; we do not optimize these estimates, but we will use explicit estimates in some of the proofs that follow.

Lemma 2.7. *Assume Hypothesis 1.2. There exist constants $C_1, C_2, C_3, C_4, C_5 \in (0,\infty)$ which depend only on $\|\sigma\|_{2,\mathrm{unif}}$, $\|\tau\|$ such that, for every $E \in \mathbb{R}$ and every real-valued solution $u \in \mathfrak{D}$ of $\ell u = Eu$,*

(i) *on every interval $I \subset \mathbb{R}_+$ with $|I| = \lambda \le 1$,*

$$\|\vec{u}(y)\| \le C_1 e^{\lambda|E|} \|\vec{u}(x)\|, \quad \text{for all } x, y \in I;$$

(ii) *on every closed interval $I \subset \mathbb{R}_+$ with $|I| = \lambda \le 1$,*

$$\max_{x \in I} |u^{[1]}(x)| \le C_2 \frac{e^{\lambda|E|}}{\lambda} \max_{x\in I} |u(x)|;$$

(iii) *for $\delta = C_3(1+|E|)^{-1}$, at least one of the infimums*

$$\inf_{[y-\delta,y]\cap\mathbb{R}_+} |u(x)|, \qquad \inf_{[y,y+\delta]} |u(x)|$$

is larger or equal to $|u(y)|/2$;

(iv) *for every $\varepsilon \ge \frac{1}{4}$ and every $y \ge \varepsilon$,*

$$|u(y)|^2 \le C_4 (1+|E|)^2 \int_{y-\varepsilon}^{y+\varepsilon} |u(x)|^2 \, dx;$$

(v) *for every $\varepsilon \ge \frac{1}{2}$ and every $y \ge \varepsilon$,*

$$|u^{[1]}(y)|^2 \le C_5 e^{2\varepsilon|E|} (1+|E|)^2 \int_{y-\varepsilon}^{y+\varepsilon} |u(x)|^2 \, dx$$

Proof. (i) From $\vec{u}' = A\vec{u}$ for $x < y$, we obtain by Gronwall's inequality [5, Lemma 1.3]

$$\|\vec{u}(y)\| \leq e^{\int_x^y \|A(t,E)\|\,dt} \|\vec{u}(x)\|.$$

The operator norm bound

$$\|A(t,E)\| \leq 1 + 2|\sigma(t)| + |\tau(t)| + |\sigma^2(t)| + |E|$$

implies that $\|A(t,E)\|$ is uniformly locally integrable: on every interval I of length $|I| = \lambda \leq 1$,

$$\int_I \|A(t,E)\|dt \leq 1 + 2\|\sigma\|_{2,\mathrm{unif}} + \|\tau\| + \|\sigma\|_{2,\mathrm{unif}}^2 + \lambda|E|.$$

The case $y < x$ follows analogously.

(ii) We fix

$$S := \frac{1}{C_1(3 + \|\sigma\|_{2,\mathrm{unif}})} \lambda e^{-\lambda|E|} \tag{2.18}$$

and assume that for some $y_0 \in I$,

$$\|\vec{u}(y_0)\| \geq |u^{[1]}(y_0)| > \frac{1}{S}|u(x)| \quad \text{for all } x \in I.$$

Combining, we conclude that for all $x, y \in I$,

$$\|\vec{u}(y)\| \geq \frac{1}{C_1 e^{\lambda|E|}} \|\vec{u}(y_0)\| > \frac{1}{C_1 e^{\lambda|E|} S}|u(x)|.$$

Since $C_1 e^{\lambda|E|} S < 1$, this implies

$$|u^{[1]}(y)| > \Big(\frac{1}{C_1^2 e^{2\lambda|E|} S^2} - 1\Big)^{1/2} |u(x)|, \quad \text{for all } x, y \in I.$$

In particular, $u^{[1]}$ has no zeros on the interval I, so it has constant sign there. Thus,

$$\Big|\int_I u^{[1]}(t)\,dt\Big| = \int_I |u^{[1]}(t)|\,dt > \lambda\Big(\frac{1}{C_1^2 e^{2\lambda|E|} S^2} - 1\Big)^{1/2} \max_{x\in I}|u(x)|. \tag{2.19}$$

On the other hand, denoting the end points of I by $j^- < j^+$, one has

$$\begin{aligned} \Big|\int_I u^{[1]}(t)\,dt\Big| &= \Big|u(j^+) - u(j^-) - \int_I \sigma(t)u(t)\,dt\Big| \\ &\leq 2\max_{x\in I}|u(x)| + \sqrt{\lambda}\|\sigma\|_{2,\mathrm{unif}} \max_{x\in I}|u(x)|. \end{aligned} \tag{2.20}$$

Since u is not identically zero on I, combining (2.19) and (2.20), we obtain

$$\lambda\Big(\frac{1}{C_1^2 e^{2\lambda|E|} S^2} - 1\Big)^{1/2} < 2 + \|\sigma\|_{2,\mathrm{unif}}\sqrt{\lambda} \le 2 + \|\sigma\|_{2,\mathrm{unif}}$$

which implies

$$\frac{1}{C_1^2 e^{2\lambda|E|} S^2} < \frac{(2+\|\sigma\|_{2,\mathrm{unif}})^2}{\lambda^2} + 1 \le \frac{(3+\|\sigma\|_{2,\mathrm{unif}})^2}{\lambda^2}$$

and contradicts (2.18).

(iii) Impose $C_3 \le 1$ to ensure $\delta \le 1$. Assume that the claim is false: then $u(y) \neq 0$ and by continuity there exist $x_1 \in [y-\delta, y] \cap [0,\infty)$ and $x_2 \in [y, y+\delta]$ such that $|u(x_1)| = |u(x_2)| = |u(y)|/2$. In particular, $x_1 < y < x_2$. Pick $s \in [x_1, x_2]$ so that

$$|u(s)| = \max_{x \in [x_1,x_2]} |u(x)|.$$

By considering $\pm u$, without loss of generality we can assume $u(s) > 0$.

Moreover, let us assume $u^{[1]}(s) \ge 0$ and work on the interval $[s, x_2]$; the other case is analogous by working on $[x_1, s]$.

The first step is an upper bound for the quasiderivative. For $x \in [s, x_2]$, denote

$$h(x) = e^{\int_s^x \sigma(t)\,dt} u^{[1]}(x).$$

Then the equation for $(u^{[1]})'$ implies

$$h'(x) = e^{\int_s^x \sigma(t)\,dt}(\tau(x) - \sigma(x)^2 - E)u(x).$$

Since $x - s \le x_2 - s < 2\delta \le 1$, we use

$$\Big|\int_s^x \sigma(t)\,dt\Big| \le |x-s|^{1/2}\|\sigma\|_{2,\mathrm{unif}},$$

$$\int_s^x |\tau(t) - \sigma(t)^2 - E|dt \le \|\sigma\|_{2,\mathrm{unif}}^2 + \|\tau\| + |E|$$

to conclude that for $x \in [s, x_2]$, for some constant C,

$$\begin{aligned}|h(x) - h(s)| &\le \int_s^x e^{C|t-s|^{1/2}} |\tau(t) - \sigma(t)^2 - E||u(t)|dt \\ &\le e^{C|x-s|^{1/2}} u(s) \int_s^x |\tau(t) - \sigma(t)^2 - E|dt \\ &\le (C + |E|)e^{C|x-s|^{1/2}} u(s).\end{aligned}$$

Since $h(s) = u^{[1]}(s) \geq 0$, we turn this into a one-sided bound

$$-h(x) \leq -h(s) + (C + |E|)e^{C|x-s|^{1/2}} u(s) \leq (C + |E|)e^{C|x-s|^{1/2}} u(s)$$

and from this we finally obtain

$$-u^{[1]}(x) \leq (C + |E|)e^{2C|x-s|^{1/2}} u(s), \quad \text{for all } x \in [s, x_2]. \tag{2.21}$$

Then we expand for $x \in [s, x_2]$,

$$u(x) = u(s) + \int_s^x u'(t)\,dt = u(s) + \int_s^x u^{[1]}(t)\,dt + \int_s^x \sigma(t)u(t)\,dt$$

and by using (2.21) we get

$$u(x) \geq u(s) - |x-s|e^{2C|x-s|^{1/2}}(C + |E|)u(s) - |x-s|\|\sigma\|_{2,\mathrm{unif}} u(s).$$

Plugging in $x = x_2$, recalling that $u(x_2) \leq u(s)/2$ and $|x_2 - s| < 2\delta$, and dividing by $u(s)$ we obtain

$$\frac{1}{2} > 1 - 2\delta e^{4C\delta^{1/2}}(C + |E|) - \delta\|\sigma\|_{2,\mathrm{unif}}$$

Equivalently,

$$2\delta e^{4C\delta^{1/2}}(C + |E|) + \delta\|\sigma\|_{2,\mathrm{unif}} > \frac{1}{2}$$

which gives a contradiction if δ is small enough.

(iv) It follows from (iii) that

$$\int_{y-\varepsilon}^{y+\varepsilon} |u(x)|^2\,dx \geq \frac{|u(y)|^2}{4}\min\{\varepsilon, \delta\}.$$

(v) Without loss of generality, assume $\varepsilon \leq 1$. Starting with (ii) and then (iv), with $a = \varepsilon/2$,

$$\begin{aligned}
|u^{[1]}(y)|^2 &\leq C_2^2 \frac{e^{4a|E|}}{(2a)^2} \max_{x\in[y-a,y+a]} |u(x)|^2 \\
&\leq C_2^2 C_4 \frac{e^{4a|E|}}{(2a)^2}(1 + |E|)^2 \max_{x\in[y-a,y+a]} \int_{x-a}^{x+a} |u(t)|^2\,dt
\end{aligned}$$

which implies

$$|u^{[1]}(y)|^2 \leq C_5 e^{2\varepsilon|E|}(1 + |E|)^2 \int_{x-\varepsilon}^{x+\varepsilon} |u(t)|^2\,dt. \qquad \blacksquare$$

Proof of Theorem 1.6. By considering $\operatorname{Re} u, \operatorname{Im} u$, it suffices to consider real-valued eigensolutions. Denote by C the supremum in (1.3). By Lemma 2.7, there exists M such that

$$w(x)\|\vec{u}(x)\|^2 \le w(x)M \int_{x-1}^{x+1} u(y)^2\,dy \le CM \int_{x-1}^{x+1} w(y)u(y)^2\,dy. \tag{2.22}$$

Integrating and using Tonelli's theorem gives

$$\int_1^l w(x)\|\vec{u}(x)\|^2 dx \le CM \int_1^l \int_{x-1}^{x+1} w(y)u(y)^2 dy dx \le 2CM \int_0^{l+1} w(y)u(y)^2 dy. \tag{2.23}$$

From now on, assume $\int_0^\infty w(x)|u(x)|^2\,dx < \infty$. Letting $l \to \infty$ in (2.23) shows

$$\int_1^\infty w(x)\|\vec{u}(x)\|^2 dx < \infty.$$

By (1.3), w is bounded on $(0,1)$, so $\int_0^1 w(x)u^{[1]}(x)^2 dx < \infty$. Using decaying tails of an integrable function, (2.22) implies the pointwise decay (1.6). ∎

As a first application, we prove a Simon–Stolz-type criterion for absence of pure point spectrum, cf. [59].

Lemma 2.8. *Assume Hypothesis* 1.2. *If for some $E \in \mathbb{R}$,*

$$\int_0^\infty \frac{dx}{\|T(E;x)\|^2} = \infty, \tag{2.24}$$

then H^α has no nontrivial solutions in $L^2((0,\infty))$; in particular, H^α doesn't have an eigenvalue at E for any $\alpha \in [0,\pi)$.

Proof. Fix nontrivial $u \in \mathfrak{D}$, $\ell u = Eu$. By Theorem 1.6 with $w = 1$, it suffices to show that

$$\sqrt{(u^{[1]})^2 + u^2} \notin L^2(\mathbb{R}_+). \tag{2.25}$$

Since $T(E;x) \in \mathrm{SL}(2,\mathbb{R})$ implies $\|T(E;x)\| = \|T(E;x)^{-1}\|$,

$$\|\vec{u}(x)\| \ge C\frac{\|\vec{u}(0)\|}{\|T(E;x)\|}, \qquad \vec{u}(x) = (u^{[1]}(x), u(x))^\top,$$

which together with (2.24) yields (2.25) as required. ∎

2.4. The absolutely continuous spectrum via Last–Simon approach

The main goal of this section is to develop the Last–Simon approach, cf. [42], to absolutely continuous spectrum via growth of transfer matrices. To do this, we first discuss the relation between the subordinacy theory and the growth of transfer matrices. We say that $u \in \mathfrak{D}$ is a *subordinate solution* of $\ell u - zu = 0$ if for some solution $\ell v - zv = 0, v \in \mathfrak{D} \setminus \{0\}$,

$$\lim_{x\to\infty} \frac{\|u\|_x}{\|v\|_x} = 0, \quad \|f\|_x^2 := \int_0^x |f(y)|^2\,dy. \tag{2.26}$$

Note that if (2.26) holds for some eigensolution v, it holds for every eigensolution linearly independent with u. Moreover, taking $v = \bar{u}$, we see that if a subordinate solution exists, it must be linearly dependent with its complex conjugate, so it must be a multiple of $\phi_{\alpha,z}$ for some α.

For μ-a.e. $\lambda \in \mathbb{R}$, the normal boundary value $\lim_{\varepsilon\downarrow 0} m(\lambda + i\varepsilon)$ exists in $\overline{\mathbb{C}_+}$. Subordinacy theory relates this value to the existence of subordinate solutions [27,31]; this was recently understood to be a special case of bulk universality in a general Hamiltonian system setting [24]. To explain this, incorporate the boundary condition into the transfer matrix by defining

$$T_\alpha(z;x) = R_\alpha \begin{pmatrix} \phi_{\alpha,z}^{[1]}(0) & \theta_{\alpha,z}^{[1]}(0) \\ \phi_{\alpha,z}(0) & \theta_{\alpha,z}(0) \end{pmatrix}.$$

This transfer matrix $T_\alpha(z;x)$ obeys the initial value problem

$$\partial_x T_\alpha(z;x) = R_\alpha A(z,x) R_\alpha^{-1} T_\alpha(z;x), \quad T_\alpha(z;0) = I.$$

This is a special case of a so-called *Hamiltonian system,* and can be written as

$$j\,\partial_x T_\alpha(z;x) = R_\alpha \begin{pmatrix} 1 & \sigma(x) \\ \sigma(x) & \sigma(x)^2 - \tau(x) + z \end{pmatrix} R_\alpha^{-1} T_\alpha(z;x), \quad j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The transfer matrices generate a matrix kernel

$$\begin{aligned} \mathcal{K}_l(z,w) &= \int_0^l T_\alpha(w;x)^* R_\alpha \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} R_\alpha^{-1} T_\alpha(z;x)\,dx \\ &= \int_0^l \begin{pmatrix} \phi_{\alpha,z}(x)\overline{\phi_{\alpha,w}(x)} & \theta_{\alpha,z}(x)\overline{\phi_{\alpha,w}(x)} \\ \phi_{\alpha,z}(x)\overline{\theta_{\alpha,w}(x)} & \theta_{\alpha,z}(x)\overline{\theta_{\alpha,w}(x)} \end{pmatrix} dx. \end{aligned}$$

By the Cauchy–Schwarz inequality, the solution $\phi_{\alpha,z}$ is subordinate if and only if

$$\lim_{l\to\infty} \frac{1}{\operatorname{Tr} \mathcal{K}_l(E,E)} \mathcal{K}_l(E,E) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Scaling limits of $\mathcal{K}_l$ are related to the normal limits of m-function: by [24, Theorem 1.8],

$$\lim_{\varepsilon\downarrow 0} m_\alpha(E+i\varepsilon) = \infty \iff \lim_{l\to\infty} \frac{1}{\operatorname{Tr} \mathcal{K}_l(E,E)} \mathcal{K}_l(E,E) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Using (2.17) to restate in terms of m_0, we conclude as follows.

Lemma 2.9. *Assume Hypothesis 1.2. For any $E \in \mathbb{R}$,*

$$\lim_{\varepsilon\downarrow 0} m_0(E+i\varepsilon) = -\cot\alpha \iff \phi_{\alpha,E} \text{ is subordinate.}$$

We also denote

$$N(\ell) := \{E \in \mathbb{R} : \text{no solution of } \ell u - Eu = 0 \text{ is subordinate}\}.$$

Taking the union over α in Lemma 2.9 and taking negations, for every E for which the normal limit exists, $E \in N(\ell)$ if and only if

$$\lim_{\varepsilon\downarrow 0} m_0(E+i\varepsilon) \in \mathbb{C}_+.$$

Recall that we denote by μ_{ac}^α the absolutely continuous part of the spectral measure μ^α.

Lemma 2.10. *Assume Hypothesis 1.2. For arbitrary $\alpha \in [0,\pi)$, $N(\ell)$ is an essential support for the absolutely continuous spectrum of H^α in the sense that μ_{ac}^α is mutually absolutely continuous with $\chi_{N(\ell)}(E)\,dE$. In particular,*

$$\operatorname{Spec}_{\text{ac}}(H^\alpha) = \overline{N(\ell)}^{\text{ess}}.$$

Proof. Recall from [19, Corollary 9.4] that an essential support for μ_{ac}^α is the set

$$M_{\text{ac}} := \{E \in \mathbb{R} \mid 0 < \limsup_{\varepsilon\downarrow 0} \operatorname{Im} m_\alpha(\lambda + \mathbf{i}\varepsilon) < \infty\}.$$

Since m_α has a normal boundary value in $\overline{\mathbb{C}_+}$ for Lebesgue-a.e. E (see e.g., [61, Theorem 3.27, Corollary 3.29]), the set

$$\{E \in \mathbb{R} \mid \lim_{\varepsilon\downarrow 0} m_\alpha(E + \mathbf{i}\varepsilon) \in \mathbb{C}_+\}$$

is also an essential support for the a.c. spectrum. This set is independent of α by (2.17). By the observation proceeding the lemma, the set $N(\ell)$ is another essential support for the a.c. spectrum of H^α. ∎

Proof of Theorem 1.3. Since the spectral type of the a.c. part is independent of α (Lemma 2.10), it suffices to prove the claim for $\alpha = 0$. Assuming this value, we drop symbol α from subsequent notation.

Due to preservation of the Wronskian, we have $\|\vec{\phi}_{\alpha,E}(x)\|\|\vec{\theta}_{\alpha,E}(x)\| \geq 1$; thus,

$$(l-1)^2 \leq \|\vec{\phi}_{\alpha,E}\|^2_{L^2((1,l))}\|\vec{\theta}_{\alpha,E}\|^2_{L^2((1,l))}, \quad l > 1. \tag{2.27}$$

Then, one has

$$\begin{aligned}\frac{\|\phi_{\alpha,E}\|^2_{L^2((0,l+1))}}{\|\theta_{\alpha,E}\|^2_{L^2((0,l+1))}} &\underset{(1.4)}{\leq} C\frac{\|\vec{\phi}_{\alpha,E}\|^2_{L^2((0,l+1))}}{\|\vec{\theta}_{\alpha,E}\|^2_{L^2((1,l))}} \underset{(2.27)}{\leq} C\frac{\|\vec{\phi}_{\alpha,E}\|^4_{L^2((0,l+1))}}{(l-1)^2} \\ &\leq C\Big(\frac{\int_0^{l+1}\|T(E;x)\|^2dx}{l-1}\Big)^2,\end{aligned}$$

where in the last step, we used $\|\vec{\phi}_{\alpha,E}(x)\| \leq \|T(E,x)\|$. If the solution $\theta_{\alpha,E}$ is subordinate, taking the limit $l \to \infty$ shows

$$\lim_{l\to\infty}\frac{1}{l}\int_0^l \|T(E;x)\|^2dx = \infty.$$

In other words, for the set Σ_{ac} defined by (1.2), we conclude $\Sigma_{\mathrm{ac}} \subset N(\ell)$.

Therefore, to complete the proof, it is enough to show that

$$\liminf_{l\to\infty}\frac{1}{l}\int_0^l \|T(E;x)\|^2dx < \infty, \qquad \text{for } \mu_{\mathrm{ac}}\text{-a.e. } E.$$

To that end, let us fix $\gamma > 1$ and introduce the measure

$$d\rho := \frac{\min\{\mu^0, \mu^{\frac{\pi}{2}}\}}{e^{2\gamma|E|}}.$$

Since $d\rho$ is equivalent to μ_{ac}, in order to prove that Σ_{ac} is an essential support for μ_{ac}, it is enough to show

$$\int_{\mathbb{R}}\Big(\liminf_{l\to\infty}\frac{1}{l}\int_0^l \|T(E;x)\|^2dx\Big)d\rho(E) < \infty. \tag{2.28}$$

To that end, we will prove the following auxiliary inequalities: there exists $\Upsilon > 0$ such that for all $x \in (2,\infty)$,

$$\int_{\mathbb{R}}\frac{\int_{x-1}^{x+1}|\phi_E(t)|^2dt}{e^{\gamma|E|}}d\mu^0(E) < \Upsilon, \quad \int_{\mathbb{R}}\frac{\int_{x-1}^{x+1}|\theta_E(t)|^2dt}{e^{\gamma|E|}}d\mu^{\frac{\pi}{2}}(E) < \Upsilon, \tag{2.29}$$

$$\int_{\mathbb{R}} \frac{\int_{x-1}^{x+1} |\phi_E^{[1]}(t)|^2 dt}{e^{2\gamma|E|}} d\mu^0(E) < \Upsilon, \quad \int_{\mathbb{R}} \frac{\int_{x-1}^{x+1} |\theta_E^{[1]}(t)|^2 dt}{e^{2\gamma|E|}} d\mu^{\frac{\pi}{2}}(E) < \Upsilon. \tag{2.30}$$

We will prove the first parts of (2.29), (2.30), the second parts can be proved analogously. Since $\operatorname{supp}\mu^0$ is bounded from below, for some $\Lambda < \min\operatorname{supp}\mu^0$,

$$\int_{\mathbb{R}} \frac{|\phi_E(x)|^2}{e^{\gamma|E|}} d\mu^0(E) \le c \int_{\mathbb{R}} \frac{|\phi_E(x)|^2}{E-\Lambda} d\mu^0(E).$$

Then, using spectral representation of Green's function [19, Lemma 9.6] and the last part of Theorem 2.2, we obtain

$$\begin{aligned}\int_{\mathbb{R}} \frac{|\phi_E(x)|^2}{e^{\gamma|E|}} d\mu^0(E) \ &\le\ C \int_{\mathbb{R}} \frac{|\phi_E(x)|^2}{E-\Lambda} d\mu^0(E) \\ &= G(\Lambda; x, x) \underset{(2.3)}{\le} C G^{\text{free}}(\lambda(\sigma,\tau) - \Lambda; x, x) \le \alpha(1 - e^{-\beta|x|}),\end{aligned} \tag{2.31}$$

where $\lambda(\sigma,\tau)$ is as in (2.3), G and G^{free} denote respectively the Green's functions for H^0 and the free Dirichlet Laplacian on $\mathbb{R}_+$, i.e., for $\sigma = \tau = 0$, and the constants α, β depend only on Λ, σ, τ. Integrating (2.31) yields the first inequality in (2.29).

Next, we switch to the first inequality in (2.30). By Lemma 2.7,

$$|\phi_E^{[1]}(t)|^2 \le C(E) \int_{t-1/2}^{t+1/2} |\phi_E(y)|^2 dy,$$

with $C(E) = \mathcal{O}(e^{\gamma|E|})$, $E \to \infty$. Then, one has

$$\int_{\mathbb{R}} \frac{\int_{x-1}^{x+1} |\phi_E^{[1]}(t)|^2 dt}{e^{2\gamma|E|}} d\mu^0(E) \le \int_{\mathbb{R}} C(E) \frac{\int_{x-3/2}^{x+3/2} |\phi_E(t)|^2 dt}{e^{2\gamma|E|}} d\mu^0(E) < \Upsilon,$$

where in the last step we used (2.31). Next, (2.29) and (2.30) together yield a constant $C > 0$ such that for all $x \in (2, \infty)$,

$$\int_{\mathbb{R}} \int_{x-1}^{x+1} \|T(E;t)\|^2 dt\, d\rho(E) < C. \tag{2.32}$$

Splitting the interval $(0, l)$ into disjoint intervals of length 2, averaging over l, and applying Fatou's lemma gives

$$\int_{\mathbb{R}} \liminf_{l\to\infty} \frac{1}{l} \int_0^l \|T(E;t)\|^2 dt\, d\rho(E) \le C$$

which implies (2.28). ■

Proof of Theorem 1.4. Using estimate (2.32) together with

$$\|T(E;t,s)\| \le \|T(E;s)\|\|T(E;t)\|$$

and the Cauchy–Schwarz inequality in $L^2(\mathbb{R}, d\rho)$ gives

$$\sup_{x,y\in(2,\infty)} \int_{\mathbb{R}} \int_{x-1}^{x+1} \int_{y-1}^{y+1} \|T(E;t,s)\|^2 dt\, ds\, d\rho(E) < \infty,$$

which implies by Fatou's lemma that for ρ-a.e. E,

$$\liminf_{j\to\infty} \int_{x_j-1}^{x_j+1} \int_{y_j-1}^{y_j+1} \|T(E;t,s)\|^2 dt\, ds < \infty.$$

By Lemma 2.7, for any E there exists $C > 0$ such that

$$C^{-1} \int_{x_j-1}^{x_j+1} \int_{y_j-1}^{y_j+1} \|T(E;x,y)\|^2 dx dy \le \|T(E;x_j,y_j)\|^2$$
$$\le C \int_{x_j-1}^{x_j+1} \int_{y_j-1}^{y_j+1} \|T(E;x,y)\|^2 dx dy,$$

so for ρ-a.e. E,

$$\liminf_{j\to\infty} \|T(E;x_j,y_j)\| < \infty. \qquad \blacksquare$$

Theorem 1.4 will be our principal tool for showing the absence of absolutely continuous spectrum for a class of slowly decaying potentials, see Theorem 1.12 (b).

2.5. Carmona formula and pure a.c. spectrum on intervals

In this section, we discuss a Carmona-type, cf. [7], approximation result for the spectral measure of H^α and use it to derive a criterion for pure a.c. spectrum on an interval. This is our main tool for showing purely absolutely continuous spectrum for a class of slowly decaying potentials, see Theorem 1.12 (a).

Theorem 2.11. *Assume Hypothesis* 1.2. *For any* $\alpha \in [0,\pi)$, *the measures*

$$d\mu_x^\alpha(E) = \frac{1}{\pi(\phi_{\alpha,E}(x)^2 + \phi_{\alpha,E}^{[1]}(x)^2)}\, dE, \quad x > 0, \tag{2.33}$$

converge vaguely to the spectral measure μ^α of H^α as $x \to \infty$ in the sense that

$$\lim_{x\to\infty} \int_{\mathbb{R}} h(E)\, d\mu_x^\alpha(E) = \int_{\mathbb{R}} h(E)\, d\mu^\alpha(E), \quad \textit{for all } h \in C_0(\mathbb{R}). \tag{2.34}$$

Proof. Recall R_α from (2.16). For $z \in \mathbb{C}_+$ and $x > 0$, let us define $m_{x,\alpha}(z) \in \mathbb{C}$ via

$$\begin{bmatrix} m_{x,\alpha}(z) \\ 1 \end{bmatrix} \simeq R_\alpha T(z;x)^{-1} \begin{bmatrix} \mathbf{i} \\ 1 \end{bmatrix}. \tag{2.35}$$

In other words, one has that $m_{x,\alpha}(z)$ is the image of $\mathbf{i}$ under the Möbius transform $\mathcal{M}[R_\alpha T(z;x,0)^{-1}]$. By Proposition 2.6, $\mathbf{i} \in \mathbb{C}_+$ implies $m_{x,\alpha}(z) \in D_x^\alpha(z) \subset \mathbb{C}_+$, and thus the function $z \mapsto m_{x,\alpha}(z)$ is Herglotz; moreover, since the disks $D_x^\alpha(z)$ shrink to a single point, for every $z \in \mathbb{C}_+$, one has $m_{x,\alpha}(z) \to m_\alpha(z)$ as $x \to \infty$. Our next objective is to compute boundary value of $\operatorname{Im} m_{x,\alpha}(E + \mathbf{i}\varepsilon)$ as $\varepsilon \downarrow 0$. Put

$$P(z,x) := \cos(\alpha)(\mathbf{i}\theta_{\alpha,z}(x) - \theta_{\alpha,z}^{[1]}(x)) + \sin(\alpha)(\mathbf{i}\phi_{\alpha,z}(x) - \phi_{\alpha,z}^{[1]}(x)),$$
$$Q(z,x) := \sin(\alpha)(\mathbf{i}\theta_{\alpha,z}(x) - \theta_{\alpha,z}^{[1]}(x)) + \cos(\alpha)(-\mathbf{i}\phi_{\alpha,z}(x) + \phi_{\alpha,z}^{[1]}(x)),$$

and rewrite (2.35) as

$$m_{x,\alpha}(z) = \frac{P(z,x)}{Q(z,x)} = \frac{P(z,x)\overline{Q(z,x)}}{|Q(z,x)|^2}.$$

Note that both the denominator and the numerator are entire functions of z. Moreover, we claim that $|Q(z,x)|^2$ does not vanish for all $z \in \mathbb{C}_+ \cup \mathbb{R}$ and $x > 0$. Since $m_{x,\alpha} \in \mathbb{C}_+$ whenever $z \in \mathbb{C}_+$, it suffices to check the claim for $z \in \mathbb{R}$. Suppose for some $x > 0$,

$$\begin{aligned} 0 = |Q(z,x)|^2 = |&(-\sin(\alpha)\theta_{\alpha,z}^{[1]}(x) + \cos(\alpha)\phi_{\alpha,z}^{[1]}(x)) \\ &+ \mathbf{i}(\sin(\alpha)\theta_{\alpha,z}(x) - \cos(\alpha)\phi_{\alpha,z}(x))|^2. \end{aligned}$$

Since $\phi_{\alpha,z}, \theta_{\alpha,z}, \phi_{\alpha,z}^{[1]}, \theta_{\alpha,z}^{[1]} \in \mathbb{R}$ for $z \in \mathbb{R}$, $|Q(z,x)|^2 = 0$ implies $\operatorname{Re} Q(z,x) = \operatorname{Im} Q(z,x) = 0$. Writing this in matrix form gives the system

$$\begin{bmatrix} \phi_{\alpha,z}^{[1]}(x) & \theta_{\alpha,z}^{[1]}(x) \\ \phi_{\alpha,z}(x) & \theta_{\alpha,z}(x) \end{bmatrix} \begin{bmatrix} \cos\alpha \\ -\sin\alpha \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which is a contradiction since the matrix is invertible. Thus, $Q(z,x) \neq 0$ for $z \in \mathbb{R}$ and $m_{x,\alpha}(z)$ has a continuous extension to $\mathbb{R}$. To summarize,

$$\lim_{\varepsilon\downarrow 0} \operatorname{Im} m_{x,\alpha}(E + \mathbf{i}\varepsilon) = \frac{\operatorname{Im}[P(E,x)\overline{Q(E,x)}]}{|Q(E,x)|^2}, \tag{2.36}$$

where

$$\begin{aligned}\operatorname{Im}[P(E,x)\overline{Q(E,x)}] &= \cos^2\alpha(\phi^{[1]}_{\alpha,E}(x)\theta_{\alpha,E}(x) - \theta^{[1]}_{\alpha,E}(x)\phi_{\alpha,E}(x)) \\ &\quad + \sin^2\alpha(-\phi_{\alpha,E}(x)\theta^{[1]}_{\alpha,E}(x) + \phi^{[1]}_{\alpha,E}(x)\theta_{\alpha,E}(x)) \\ &= (\sin^2(\alpha) + \cos^2(\alpha))W(\theta_{\alpha,E}, \phi_{\alpha,E}) = 1, \end{aligned} \tag{2.37}$$

and

$$\begin{aligned}|Q(E,x)|^2 &= \big(\cos(\alpha)\phi^{[1]}_{\alpha,z}(x) - \sin(\alpha)\theta^{[1]}_{\alpha,z}(x)\big)^2 \\ &\quad + \big(\cos(\alpha)\phi_{\alpha,z}(x) - \sin(\alpha)\theta_{\alpha,z}(x)\big)^2 \\ &= \left\| \begin{bmatrix} \phi^{[1]}_E(x) & \theta^{[1]}_E(x) \\ \phi_E(x) & \theta_E(x) \end{bmatrix} \begin{bmatrix} \cos(\alpha) \\ -\sin(\alpha) \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} \phi^{[1]}_{\alpha,E}(x) \\ \phi_{\alpha,E}(x) \end{bmatrix} \right\|^2. \end{aligned} \tag{2.38}$$

It follows from (2.36), (2.37), and (2.38) that the measure corresponding to the Herglotz function $m_{x,\alpha}(z)$ is given by (2.33). Moreover, since $m_{x,\alpha}(z) \to m_\alpha(z)$ as $x \to \infty$, by their Herglotz representations, the corresponding measures converge in the sense as asserted. ∎

Having established (2.34), the proof of Theorem 1.5 is identical to that of [42, Theorem 3.7].

Proof of Theorem 1.5. Choose a sequence $x_n \to \infty$ such that

$$\lim_{n\to\infty} \int_{E_1}^{E_2} \|T(E;x_n)\|^p \, dE < \infty.$$

Since $\det T(E;x) = 1$ and $\|v_\alpha\| = 1$,

$$\|T(E;x_n)v_\alpha\| \geq \|T^{-1}(E;x_n)\|^{-1}\|v_\alpha\| = \|T(E;x_n)\|^{-1},$$

and thus for $q = p/2$,

$$\sup_n \int_{E_1}^{E_2} \Big(\frac{1}{\pi\|T(E;x_n)v_\alpha\|^2}\Big)^q \, dE \leq \sup_n \int_{E_1}^{E_2} (\pi^{-1}\|T(E;x_n)\|^2)^q \, dE < \infty.$$

Hence, by [42, Lemma 3.8], the weak limit $d\mu^\alpha$ of measures $d\mu^\alpha_{x_n}$ is purely absolutely continuous on (E_1, E_2). ∎

In the study of decaying potentials, a variant of Carmona's formula is useful.

Theorem 2.12. *Assume Assume Hypothesis 1.2 For any $\alpha \in [0,\pi)$, the measures*

$$d\nu_x^\alpha(E) = \frac{\chi_{(0,\infty)}(E)\sqrt{E}}{\pi(E\phi_{\alpha,E}(x)^2 + \phi_{\alpha,E}^{[1]}(x)^2)}\, dE, \quad x > 0 \tag{2.39}$$

converge vaguely on $(0,\infty)$ to μ^α as $x \to \infty$ in the sense that

$$\lim_{x\to\infty} \int_0^\infty h(E)\, d\nu_x^\alpha(E) = \int_0^\infty h(E)\, d\mu^\alpha(E), \quad \textit{for all } h \in C_c((0,\infty)).$$

Proof. We use the branch of $\sqrt{-z}$ on $\mathbb{C}_+$ such that $\operatorname{Re}\sqrt{-z} > 0$, $\operatorname{Im}\sqrt{-z} < 0$. With this choice of branch, $-\sqrt{-z}$ is a Herglotz function which continuously extends to $\overline{\mathbb{C}_+}$ with values $\chi_{(0,\infty)}(E)\sqrt{E}$ on $\mathbb{R}$. For $z \in \mathbb{C}_+$, $x > 0$, define $m_{x,\alpha}(z)$ via

$$\begin{bmatrix} m_{x,\alpha}(z) \\ 1 \end{bmatrix} \simeq R_\alpha T(z;x)^{-1} \begin{bmatrix} -\sqrt{-z} \\ 1 \end{bmatrix}.$$

Since $-\sqrt{-z}$ is Herglotz, $m_{x,\alpha}(z) \in D_{x,\alpha}(z) \subset \mathbb{C}_+$ and is Herglotz as well. Moreover, since $D_x^\alpha(z)$ shrinks to a point as $x \to \infty$, $m_{x,\alpha}(z) \to m_\alpha(z)$ as $x \to \infty$. By arguments analogous to the proof of Theorem 2.11, $\operatorname{Im} m_{x,\alpha}(z)$ has a continuous extension to $(0,\infty)$ with

$$\lim_{\varepsilon\downarrow 0} \operatorname{Im} m_{x,\alpha}(E + i\varepsilon) = \frac{\sqrt{E}}{E\phi_{\alpha,E}(x)^2 + \phi_{\alpha,E}^{[1]}(x)^2}.$$

It follows from above that the measure corresponding to $m_{x,\alpha}(z)$ has the restriction to $(0,\infty)$ given by (2.39), which concludes the proof. ■

2.6. Prüfer variables

We now introduce Prüfer variables associated with real eigensolutions of ℓ and relate their growth to that of the transfer matrices. In the locally integrable setting, Prüfer variables are a well-established tool for spectral analysis for decaying potentials; we will use them in the proof of Theorem 1.12.

For $k > 0$, consider the eigenvalue equation $\ell u = k^2 u$, $u \in \mathfrak{D}$. For a non-trivial real-valued solution u, introduce $\theta\colon \mathbb{R} \to \mathbb{R}$, $R\colon \mathbb{R} \to (0,\infty)$ via the relations

$$u(x) = R(x)\sin(\theta(x)), \quad u^{[1]}(x) = kR(x)\cos(\theta(x)). \tag{2.40}$$

Since a composition of a Lipschitz function with an absolutely continuous function is absolutely continuous, this can be done so that $R, \theta \in \mathrm{AC}_{\mathrm{loc}}([0,\infty))$. The remaining non-uniqueness in the choice of θ is usually fixed by setting $\theta(0) \in [0, 2\pi)$.

Proposition 2.13. *Assume Hypothesis 1.2. For $k > 0$, in terms of Prüfer variables, the eigenfunction equation $\ell u = k^2 u$ is equivalent to the system*

$$\theta' = k - \frac{\tau - \sigma^2}{k} \sin^2(\theta) + \sigma \sin(2\theta), \tag{2.41}$$

$$(\log R)' = \frac{\tau - \sigma^2}{2k} \sin(2\theta) - \sigma \cos(2\theta). \tag{2.42}$$

Moreover, for any $\alpha, \beta \in (0, \infty)$, $\theta_1, \theta_2 \in [0, 2\pi)$, there is a constant

$$C = C(\alpha, \beta, \theta_1, \theta_2) > 1$$

such that for all $k \in (\sqrt{\alpha}, \sqrt{\beta})$,

$$\frac{1}{C} \max(R(x, \theta_1), R(x, \theta_2)) \leq \|T(k^2; x)\| \leq C \max(R(x, \theta_1), R(x, \theta_2)). \tag{2.43}$$

Proof. First, we rewrite $\ell u - k^2 u = 0$ as

$$\begin{bmatrix} u^{[1]} \\ u \end{bmatrix}' = \begin{bmatrix} -\sigma & (\tau - \sigma^2) - k^2 \\ 1 & \sigma \end{bmatrix} \begin{bmatrix} u^{[1]} \\ u \end{bmatrix}, \quad ' := \frac{d}{dx}.$$

Then, substituting (2.40) into the above equation, we obtain

$$\begin{bmatrix} kR' \cos(\theta) - R\theta' \sin(\theta) \\ R' \sin(\theta) + R\theta' \cos(\theta) \end{bmatrix} = \begin{bmatrix} -\sigma k R \cos(\theta) + R(\tau - \sigma^2) \sin(\theta) - k^2 R \sin(\theta) \\ kR \cos(\theta) + \sigma R \sin(\theta) \end{bmatrix}. \tag{2.44}$$

To derive (2.41), we take the scalar product of both sides of (2.44) and $(-\sin(\theta), k\cos(\theta))$; and to derive (2.42), we take the scalar product of sides of (2.44) and $(\cos(\theta), k \sin(\theta))$.

Let u_1, u_2 be solutions corresponding to the initial conditions $\theta(0) = \theta_1$, $\theta(0) = \theta_2$ respectively. Then,

$$\mathcal{U}(x) = T(k^2; x)\mathcal{U}(0), \quad \mathcal{U}(x) := \begin{bmatrix} u_1^{[1]}(x) & u_2^{[1]}(x) \\ u_1(x) & u_2(x) \end{bmatrix}.$$

Using the representation (2.40), one obtains

$$C_1 \max(R(x, \theta_1), R(x, \theta_2)) \leq \|\mathcal{U}(x)\| \leq C_2 \max(R(x, \theta_1), R(x, \theta_2)), \quad x \geq 0 \tag{2.45}$$

for some constants $C_1, C_2 > 0$ depending only on α, β. Finally, since $T(k^2; x) \in \mathrm{SL}(2, \mathbb{R})$, there exists constants $C_3, C_4 > 0$ depending only on θ_1, θ_2 such that

$$C_3 \|\mathcal{U}(x)\| \leq \|T(k^2; x)\| \leq C_4 \|\mathcal{U}(x)\|, \quad x \geq 0. \tag{2.46}$$

Combining (2.45), (2.46) yields (2.43). ∎

Proof of Theorem 1.10. Consider Prüfer variables $R(x,E)$ associated to the solution $\phi_{\alpha,E}$ for $E \in (0,\infty)$. By (2.42), $\log R(x,E)$ converges uniformly as $x \to \infty$ on every compact interval $[E_1, E_2] \subset (0,\infty)$. Recall $d\nu_x^\alpha(E)$ from Theorem 2.12, then for any $h \in C_c((0,\infty))$, by uniform convergence,

$$\begin{aligned}\lim_{x\to\infty} \int h(E)\,d\nu_x^\alpha(E) &= \lim_{x\to\infty} \int \frac{h(E)\sqrt{E}}{\pi(E\phi_{\alpha,E}(x)^2 + \phi_{\alpha,E}^{[1]}(x)^2)}\,dE \\ &= \int \frac{h(E)}{\sqrt{E}\pi(\lim_{x\to\infty} R(x,E))^2}\,dE.\end{aligned}$$

Thus,

$$\chi_{(0,\infty)}(E)\,d\mu^\alpha(E) = \frac{\chi_{(0,\infty)}(E)}{\sqrt{E}\pi(\lim_{x\to\infty} R(x,E))^2}\,dE. \qquad \blacksquare$$

3. Distributional sparse potentials. Investigation of spectral types

In this section we prove Theorem 1.12.

3.1. Decomposition of sparse potentials

The first step in the proof of Theorem 1.12 is to reformulate it in terms of the Hryniv–Mykytyuk decomposition in a way that is consistent with the sparse structure of the potential. If we applied their decomposition directly to V, the dependence on integers in [29] would complicate matters; instead, note that [29, Lemma 2.2] gives a decomposition of $W_n \in H^{-1}(\mathbb{R})$ with $\operatorname{supp} W_n \subset [-\Delta, \Delta]$ as $W_n = S_n' + T_n$ with $S_n \in L^2(\mathbb{R})$, $T_n \in L^1(\mathbb{R})$ supported in the same interval (the authors use $\Delta = 1$ but this is merely a matter of rescaling). Moreover, this decomposition is continuous in $H^{-1}(\mathbb{R})$-norm. Thus, we obtain

$$W_n = S_n' + T_n, \quad W = S' + T,$$

with

$$\begin{gathered}\operatorname{supp}(S_n) \cup \operatorname{supp}(S) \cup \operatorname{supp}(T_n) \cup \operatorname{supp}(T) \subset [-\Delta,\Delta], \quad S' + T \not\equiv 0, \\ S, S_n \in L^2(\mathbb{R}), T, T_n \in L^1(\mathbb{R}),\ \|S_n - S\|_{L^2(\mathbb{R})} \to 0,\ \|T_n - T\|_{L^1(\mathbb{R})} \to 0. \end{gathered} \tag{3.1}$$

In addition, without loss of generality, we can assume that $S \not\equiv 0$ and $T \not\equiv 0$: this is because if one of S, T is identically equal to zero, we can pick arbitrary $h \in W^{1,1}(\mathbb{R})$, $\operatorname{supp}(h) \subset [-\Delta,\Delta]$, $S + h \not\equiv 0$, $T - h' \not\equiv 0$. Notice that $W_n = (S_n + h)' + T_n - h'$, $W = (S + h)' + T - h'$.

In summary, we will use the following setup throughout this section.

Hypothesis 3.1. *Let $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}_+$ be a monotonically increasing sequence such that $x_1 > \Delta$ and*

$$\lim_{n\to\infty} \frac{x_n}{x_{n+1}} = 0. \tag{3.2}$$

Let $\beta > 1$ be so that $x_n \geq C\beta^n$ for a fixed constant $C > 0$. Let T, S, T_n, S_n be as in (3.1) and suppose, in addition, $T \not\equiv 0$, $S \not\equiv 0$. Furthermore, fix a sequence $\{d_n\}_{n=1}^{\infty} \subset \mathbb{R}$ with

$$\lim_{n\to\infty} d_n = 0. \tag{3.3}$$

Let sparse coefficients τ, σ be given by

$$\tau(x) = \sum_{n=1}^{\infty} d_n T_n(x - x_n),$$
$$\sigma(x) = \sum_{n=1}^{\infty} d_n S_n(x - x_n).$$

Fix arbitrary $\alpha \in [0, \pi)$, and let H^{α} be the corresponding Schrödinger operator as defined in Theorem 2.2.

The rest of this paper is dedicated to the proof of Theorem 1.12.

3.2. Auxiliary estimates for Prüfer variables

We begin with a series of auxiliary results. The first one concerns estimates for Prüfer variables and their k-derivatives near x_n for large n.

To streamline the exposition, in the remaining part of the paper, we will use C for positive constants that vary from one inequality to the other but always remain n-independent. Also, whenever an inequality involving n is mentioned without a specified range of admissible values of n, it is assumed that the range is $n \geq n_0$ for some n_0.

Lemma 3.2. *Assume Hypothesis 3.1 and fix any compact interval $[E_1, E_2] \subset (0, \infty)$. Then, there exists a constant $C > 0$ such that for all $k \in [\sqrt{E_1}, \sqrt{E_2}]$ and sufficiently large n,*

$$\left|\frac{\partial\theta}{\partial k}(x_n + \Delta)\right| \leq C x_n, \tag{3.4i}$$
$$\left|\frac{\partial^2\theta}{\partial k^2}(x_n + \Delta)\right| \leq C \min\Big\{x_n^2, 1 + \sum_{m=1}^{n} d_m x_m^2\Big\}, \tag{3.4ii}$$

and

$$|\log R(x_n+\Delta)| \le C\sum_{m=1}^{n} d_m, \tag{3.4iii}$$

$$\left|\frac{\partial \log R}{\partial k}(x_n+\Delta)\right| \le C\sum_{m=1}^{n} d_m x_m. \tag{3.4iv}$$

Proof. Proof of (3.4i). Fix any compact interval $[a,b]\subset\mathbb{R}$, $f,g\in L^1([a,b])$, and suppose $h\in \mathrm{AC}_{\mathrm{loc}}([a,b])$ satisfying $h'(x)=f(x)+g(x)h(x)$. Then, for any $x\in[a,b]$,

$$|h(x)| \le |h(a)|e^{\int_a^b |g|\,dy} + \int_a^b |f|e^{\int_a^b |g|\,dt}\,dy = (|h(a)|+\|f\|_{1,[a,b]})e^{\|g\|_{1,[a,b]}}. \tag{3.5}$$

Let $h(x):=\frac{\partial\theta}{\partial k}(x)$. Differentiating (2.41) with respect to k, we have $\frac{\partial h}{\partial x}=f+gh$ with

$$f(x,k):=1+\frac{\tau-\sigma^2}{k^2}\sin^2(\theta),\quad g(x,k):=\sigma\cos(2\theta)-\frac{\tau-\sigma^2}{k}\sin(2\theta), \tag{3.6}$$

which, for $[a,b]:=[x_n-\Delta,x_n+\Delta]$ and sufficiently large n, satisfy

$$\|f(\cdot,k)\|_{L^1(a,b)}\le 2\Delta+Cd_n,\quad \|g(\cdot,k)\|_{L^1(a,b)}\le Cd_n. \tag{3.7}$$

Our objective is to prove that there exists $C>0$ such that for sufficiently large n,

$$|h(x_n+y)|\le Cx_n,\quad y\in[-\Delta,\Delta]. \tag{3.8}$$

Note that $h'(x)=1$ for $x\in(x_{n-1}+\Delta,x_n-\Delta)$; thus,

$$|h(x_n-\Delta)|\le|h(x_{n-1}+\Delta)|+x_n-x_{n-1}-2\Delta. \tag{3.9}$$

Then, using (3.5) with f,g as in (3.6), $[a,b]=[x_n-\Delta,x_n+y]$, and employing (3.7),

$$\begin{aligned}|h(x_n+y)| &\le (|h(x_n-\Delta)|+2\Delta+Cd_n)e^{Cd_n}\\ &\underset{(3.9)}{\le} (|h(x_{n-1}+\Delta)|+x_n-x_{n-1}+Cd_n)e^{Cd_n}.\end{aligned} \tag{3.10}$$

for sufficiently large n. Since $\beta>1$, $d_n\to0$, and $x_n\to\infty$, there exists $n_1\in\mathbb{N}$ such that for all $n\ge n_1$,

$$\beta^{-1}e^{Cd_n}\le\frac{1+\beta^{-1}}{2}\quad\text{and}\quad\Big(1+\frac{Cd_n}{x_n}\Big)e^{Cd_n}\le 1+\Big(\frac{1-\beta^{-1}}{2}\Big). \tag{3.11}$$

For such n_1, let $D \geq 2$ be such that

$$|h(x_{n_1-1} + \Delta)| \leq Dx_{n_1-1}.$$

We claim that for all $n \geq n_1$,

$$|h(x_n + y)| \leq Dx_n, \quad y \in [-\Delta, \Delta]. \tag{3.12}$$

Indeed, using (3.2) together with (3.10) and (3.11), for $n \geq n_1$, $y \in [-\Delta, \Delta]$, we have

$$\begin{aligned}
|h(x_n + y)| &\leq (|h(x_{n-1} + \Delta)| + (x_n - x_{n-1}) + Cd_n)e^{Cd_n} \\
&\leq ((D-1)x_{n-1} + x_n + Cd_n)e^{d_n} \\
&\leq x_n\Big((D-1)\beta^{-1} + 1 + \frac{Cd_n}{x_n}\Big)e^{Cd_n} \\
&\leq x_n\Big(D - (D-2)\frac{1-\beta^{-1}}{2}\Big) \leq Dx_n,
\end{aligned}$$

which yields (3.4i).

Proof of (3.4ii). Let $w := \frac{\partial h}{\partial k} = \frac{\partial^2 \theta}{\partial k^2}$ and differentiate (2.42) twice with respect to k, then

$$\frac{\partial w}{\partial x} = F(x) + G(x)w(x), \tag{3.13}$$

where

$$F(x) := F_1(x) + F_2(x)h(x) + F_3(x)[h(x)]^2$$

and

$$\begin{aligned}
F_1(x) &:= -2 \cdot \frac{\tau - \sigma^2}{k^3} \sin^2(\theta(x)), \\
F_2(x) &:= 2 \cdot \frac{\tau - \sigma^2}{k^2} \sin(2\theta(x)), \\
F_3(x) &:= -2 \cdot \frac{\tau - \sigma^2}{k} \cos(2\theta(x)) - 2\sigma \sin(2\theta(x)) \\
G(x) &:= -\frac{\tau - \sigma^2}{k} \sin(2\theta(x)) + \sigma \cos(2\theta(x)).
\end{aligned}$$

Note that for $[a, b] := [x_n - \Delta, x_n + \Delta]$ and sufficiently large n,

$$\|F\|_{L^1(a,b)} \leq Cd_n x_n^2, \quad \|G\|_{L^1(a,b)} \leq Cd_n, \tag{3.14}$$

where we used (3.12) in the first inequality. Then, using (3.5) with $[a, b] = [x_n - \Delta, x_n + y]$, $f = F$, $g = G$, $w(x_n - \Delta) = w(x_{n-1} + \Delta)$, and (3.14),

$$|w(x_n + y)| \leq (|w(x_{n-1} + \Delta)| + Cd_n x_n^2)e^{Cd_n}. \tag{3.15}$$

Since $\beta > 1$ and $d_n \to 0$, for any $C > 0$, there is large enough n_2 such that for all $n \geq n_2$,

$$\beta^{-2} e^{C d_n} \leq \frac{1+\beta^{-1}}{2}, \quad C d_n e^{C d_n} \leq \frac{1-\beta^{-1}}{2}. \tag{3.16}$$

For such n_2, let $\widetilde{D} \geq 2$ be such that

$$|w(x_{n_2-1}+\Delta)| \leq \widetilde{D} x_{n_2-1}^2.$$

We claim that for all $n \geq n_2$,

$$|w(x_n+y)| \leq \widetilde{D} x_n^2, \quad y \in [-\Delta, \Delta]. \tag{3.17}$$

Proceed with induction in n: suppose (3.17) holds for $n-1$; then, employing (3.15), for all $y \in [-\Delta, \Delta]$,

$$\begin{aligned}
|w(x_n+y)| &\leq (|w(x_{n-1}+\Delta)| + C d_n x_n^2) e^{C d_n} \\
&\leq (\widetilde{D} x_{n-1}^2 + C d_n x_n^2) e^{C d_n} \leq x_n^2 (\widetilde{D} \beta^{-2} + C d_n) e^{C d_n} \\
&\underset{(3.16)}{\leq} x_n^2 \Big(\widetilde{D} \frac{1+\beta^{-1}}{2} + \frac{1-\beta^{-1}}{2} \Big) \leq x_n^2 \Big(\widetilde{D} + (1-\widetilde{D}) \frac{1-\beta^{-1}}{2} \Big) \leq \widetilde{D} x_n^2.
\end{aligned} \tag{3.18}$$

To complete the proof of (3.4ii), integrate (3.13) over $[x_n - \Delta, x_n + \Delta]$ and use (3.17); then,

$$|w(x_n+\Delta) - w(x_n-\Delta)| \leq C d_n x_n^2, \quad n \geq n_2.$$

Hence,

$$\begin{aligned}
|w(x_n+\Delta)| &\leq |w(x_{n_2-1}+\Delta)| + \sum_{m=n_2}^{n} |w(x_m+\Delta) - w(x_m-\Delta)| \\
&\leq D x_{n_2-1}^2 + C \sum_{m=n_2}^{n} d_m x_m^2 \leq C \Big(1 + \sum_{m=1}^{n} d_m x_m^2 \Big).
\end{aligned}$$

Combining this with (3.18) concludes the proof for (3.4ii).

Proof of (3.4iii) and (3.4iv). Note that

$$R(x_{n-1}+\Delta) = R(x_n-\Delta)$$

as $\frac{\partial \log R}{\partial x} \equiv 0$ on $[x_{n-1}+\Delta, x_n-\Delta]$. Integrating (2.42) over $[x_n-\Delta, x_n+\Delta]$ and noting that $d_n \to 0$,

$$|\log R(x_n+\Delta) - \log R(x_n-\Delta)| \leq C d_n,$$

which implies (3.4iii). To prove (3.4iv), differentiate (2.42) with respect to k to get

$$\frac{\partial}{\partial x}\frac{\partial \log R}{\partial k} = -\frac{\tau-\sigma^2}{2k^2}\sin(2\theta) + \frac{\tau-\sigma^2}{2k}\cos(2\theta)\frac{\partial\theta}{\partial k} + \sigma\sin(2\theta)\frac{\partial\theta}{\partial k};$$

then, integrate both sides over $[x_n-\Delta, x_n+\Delta]$ while noting (3.8) and $d_n \to 0$,

$$\left|\frac{\partial}{\partial k}\log R(x_n+\Delta) - \frac{\partial}{\partial k}\log R(x_n-\Delta)\right| \le C d_n x_n.$$

The latter, in turn, yields (3.4iv). ∎

Remark 3.3. Lemma 3.2 and its proof are similar to [37, Propositions 5.1 and 5.2], where the case of $S_n = 0$ and $T_n = T \in L^\infty(\mathbb{R})$ was considered. We extend that proof to the case $S_n \neq 0$, and $T_n \in L^1(\mathbb{R})$ by using (3.5), which is an L^1 version of the key inequality [37, eq. (5.7)], and verifying new inequalities (3.7) and (3.14).

To streamline the exposition, we introduce the following notation:

$$q_n(y,k) := \frac{d_n T(y) - d_n^2 S^2(y)}{2k}, \quad \sigma_n(y) := d_n S(y). \tag{3.19}$$

Note that, due to (3.1), for a fixed interval $[\alpha,\beta] \subset (0,\infty)$, we have

$$\int_{-\Delta}^{\Delta} |q_n(y,k)| + |\sigma_n(y)| dy \underset{n\to\infty}{=} \mathcal{O}(d_n), \tag{3.20}$$

uniformly for $k \in [\alpha,\beta]$.

In the following lemma, we provide the second order expansion of variable θ with respect to d_n as $n \to \infty$. This result will be used in Lemma 3.10 and the proof of Theorem 1.12 (a).

Lemma 3.4. *Assume Hypothesis 3.1 and fix any compact interval $[E_1, E_2] \subset (0,\infty)$. Then, the asymptotic expansion*

$$\theta(x_n+y) \underset{n\to\infty}{=} \theta_n^{(0)}(y) + d_n\theta_n^{(1)}(y) + \mathcal{O}(d_n^2) \tag{3.21}$$

holds uniformly for $y \in [-\Delta,\Delta]$, $k \in [\sqrt{E_1}, \sqrt{E_2}]$, where, recalling (3.19),

$$\theta_n^{(0)}(y) := \theta(x_{n-1}+\Delta) + k(x_n + y - x_{n-1} - \Delta), \tag{3.22}$$

$$\theta_n^{(1)}(y) := \frac{1}{d_n}\int_{-\Delta}^{y} \sigma_n(s)\sin(2\theta_n^{(0)}(s)) - 2q_n(s)\sin^2(\theta_n^{(0)}(s))\, ds. \tag{3.23}$$

Proof. By (2.41),

$$
\begin{aligned}
&|\theta(x_n + y) - \theta_n^{(0)}(y)| \\
&\leq \int_{x_n-\Delta}^{x_n+y} |\theta'(s) - k|\, ds \\
&= \int_{-\Delta}^{y} |\sigma_n(s)\sin(2\theta_n(x_n + s)) - 2q_n(s)\sin^2(\theta(x_n + s))|ds \underset{n\to\infty}{=} \mathcal{O}(d_n),
\end{aligned}
\tag{3.24}
$$

where in the last step we used (3.20). The argument for the second-order asymptotic formula is similar. Note that

$$
\begin{aligned}
&|\theta(x_n + y) - \theta_n^{(0)}(y) - d_n\theta_n^{(1)}(y)| \\
&= \left| \int_{x_n-\Delta}^{x_n+y} (\theta'(s) - k)\, ds - d_n\theta_n^{(1)}(y) \right| \\
&\leq \Bigg| \int_{-\Delta}^{y} \sigma_n(s)[\sin(2\theta_n^{(0)}(s)) - \sin(2\theta_n^{(0)}(x_n + s))] \\
&\qquad - 2q_n(s)[\sin^2(\theta_n^{(0)}(s)) - \sin^2(\theta(x_n + s))]ds \Bigg|, \\
&\underset{n\to\infty}{=} \mathcal{O}(d_n^2),
\end{aligned}
$$

where we used (3.20) and (3.24) in the last step. ∎

Remark 3.5. A version of Lemma 3.4 with $S_n = 0$ and $T_n = T \in L^\infty(-\Delta, \Delta)$ is discussed in [37, Sections 5,6]. In our case, notice that when $S_n \neq 0$, the integral on the right-hand side of (3.23) contains an additional term $\sigma_n(s)\sin(2\theta_n^{(0)}(s))$. This will become relevant in the proof of Theorem 1.12 (b).

Corollary 3.6. *Assume the setting of Lemma 3.4. Then,*

$$
\frac{\partial\theta_n^{(0)}(y)}{\partial k} > \frac{x_n}{2}, \tag{3.25}
$$

holds for sufficiently large n and all $k \in [\sqrt{E_1}, \sqrt{E_2}]$, $y \in [-\Delta, \Delta]$.

Proof. Differentiating (3.22) with respect to k and using (3.4i) and (3.4ii), we get

$$\begin{aligned}\frac{\partial\theta_n^{(0)}(y)}{\partial k} &:= \frac{\partial\theta(x_{n-1}+\Delta)}{\partial k} + x_n + y - x_{n-1} - \Delta \\ &\geq x_n + y - (C+1)x_{n-1} + y > \frac{x_n}{2},\end{aligned}$$

where we used (3.2) in the last step. ∎

To conclude this section, we show that (3.3) rules out point spectrum for H^α.

Proposition 3.7. *Assume Hypothesis 3.1. Then,* $\operatorname{Spec}_{\mathrm{pp}}(H^\alpha) \cap (0,\infty) = \emptyset$ *for all* $\alpha \in [0,\pi)$.

Proof. Consider the Prüfer variables corresponding to a non-trivial real eigensolution u at $E > 0$, normalized so that $R(0) = 1$. By (3.4iii),

$$R(x_n+\Delta)^2 \geq \exp\Big(-2C\sum_{m=1}^{n} d_m\Big).$$

This means at most exponential decay of the sequence $R(x_n+\Delta)^2$, since the sequence d_n is bounded. Due to the superexponential growth (3.2), this implies

$$(x_{n+1} - x_n - 2\Delta)R(x_n+\Delta)^2 \to \infty, \quad n \to \infty.$$

Since $R(x)$ is constant on $[x_n+\Delta, x_{n+1}-\Delta]$, this implies

$$\int_0^\infty R(x)^2 dx = \infty$$

and, by Theorem 1.6, this implies $u \notin L^2(\mathbb{R}_+)$. ∎

3.3. Purely absolutely continuous spectrum

In this section, we provide the proof of Theorem 1.12 (a).

Proof of Theorem 1.12 (a). By Lemma 1.7, one has that $\operatorname{Spec}_{\mathrm{ess}}(H^\alpha) = [0,\infty)$. Then, by Theorem 2.11, it suffices to show that for every finite interval $[E_1,E_2] \subset (0,\infty)$,

$$\liminf_{n\to\infty} \int_{E_1}^{E_2} \|T(E;x_n+\Delta)\|^4\, dE < \infty. \tag{3.26}$$

In fact, we will show that for any $\theta \in [0, 2\pi)$ and any non-negative $g \in C_0^\infty(0, \infty)$ (after possibly passing to a subsequence),

$$\sup_n B_n < \infty, \quad B_n := \int_0^\infty g(k)|R(x_n + \Delta, \theta)|^4 dk < \infty. \tag{3.27}$$

The latter together with Proposition 2.13 yields (3.26). Explicitly, we will derive a recursive inequality

$$B_n \leq (1 + \rho_n) B_{n-1}, \tag{3.28}$$

for a sequence $\{\rho_n\} \in \ell^1(\mathbb{N})$, $\rho_n > 0$, which is sufficient for (3.27). To that end, we integrate (2.42) over the interval $[x_n - \Delta, x_n + \Delta]$ and use $R(k; x_n - \Delta) = R(k; x_{n-1} + \Delta)$ to obtain

$$R(k; x_n + \Delta)^4 = R(k; x_{n-1} - \Delta)^4 \exp(Q_n), \tag{3.29}$$

where

$$\begin{aligned} Q_n = \frac{2}{k} \int_{-\Delta}^{\Delta} (d_n T_n(y) - b_n^2 S^2(y)) \sin(2\theta(x_n + y))\, dy \\ - 4 \int_{-\Delta}^{\Delta} d_n S_n(y) \cos(2\theta(x_n + y))\, dy. \end{aligned}$$

Then,

$$|Q_n - \widetilde{Q}_n| \leq C d_n^2 \tag{3.30}$$

where

$$\begin{aligned} \widetilde{Q}_n := \frac{2}{k} \int_{-\Delta}^{\Delta} (d_n T_n(y) - d_n^2 S_n^2(y)) \sin(2\theta_n^{(0)}(y))\, dy \\ - 4 \int_{-\Delta}^{\Delta} d_n S_n(y) \cos(2\theta_n^{(0)}(y))\, dy. \end{aligned} \tag{3.31}$$

and $\theta_n^{(0)}$ is as in (3.22). Indeed, (3.30) follows readily from

$$|\sin(2\theta_n(y)) - \sin(2\theta_n^{(0)}(y))| \leq C|\theta_n(y) - \theta_n^{(0)}(y)| \leq C d_n^2, \quad y \in [-\Delta, \Delta].$$

Returning back to (3.29), notice that (3.30) together with $|Q_n| \leq C d_n$ yields

$$\begin{aligned} R(k; x_n + \Delta)^4 &\leq R(k; x_{n-1} - \Delta)^4 (1 + |Q_n| + C Q_n^2) \\ &\leq R(k; x_{n-1} - \Delta)^4 (1 + |\widetilde{Q}_n| + C d_n^2). \end{aligned}$$

To obtain (3.28), multiply the above inequalities by $g(k)$ and integrate over $(0,\infty)$; then,

$$B_n \le B_{n-1}(1 + Cd_n^2) + E_n, \quad E_n := \int g(k) R(k; x_{n-1} + \Delta)^4 \widetilde{Q}_n \, dk. \tag{3.32}$$

Recalling (3.31) and exchanging the order of integration, we obtain

$$\begin{aligned} E_n = & \int_{-\Delta}^{\Delta} 2d_n T_n(y) \int \frac{g(k)}{k} R(k; x_{n-1} + \Delta)^4 \sin(2\theta_n^{(0)}(y)) \, dk \, dy \\ & - \int_{-\Delta}^{\Delta} 2d_n^2 S_n^2(y) \int \frac{g(k)}{k} R(k; x_{n-1} + \Delta)^4 \sin(2\theta_n^{(0)}(y)) \, dk \, dy \\ & - \int_{-\Delta}^{\Delta} 4d_n S_n(y) \int g(k) R(k; x_{n-1} + \Delta)^4 \cos(2\theta_n^{(0)}(y)) \, dk \, dy, \end{aligned} \tag{3.33}$$

where note that all terms above are of the form

$$\mathcal{E}_n := \int_{-\Delta}^{\Delta} \gamma_n w_n(y) \int \Psi(k) R(k; x_{n-1} + \Delta)^4 u(2\theta_n^{(0)}(y)) \, dk \, dy, \tag{3.34}$$

with

$$\begin{aligned} & \gamma_n \in \{d_n, d_n^2\}, \ \Psi \in C_0^\infty(\mathbb{R}_+), \quad u \in \{\sin(x), \cos(x)\}, \\ & w_n \in \{T_n, S_n, S_n^2\} \subset L^1(\mathbb{R}), \qquad \sup_{n\ge 1} \|w_n\|_{L^1(\mathbb{R})} < \infty. \end{aligned} \tag{3.35}$$

Claim. *For B_n and $\mathcal{E}_n$ defined in (3.27) and (3.34) respectively, and $\beta > 1$ as in Hypothesis 3.1, there is a sequence $\{s_n\}_{n\ge1} \in \ell^1(\mathbb{N})$ such that*

$$\mathcal{E}_n \le C(\beta^{-n/2} + B_{n-1} s_n). \tag{3.36}$$

Proof of the claim. Let v be either sin or cos so that one has $u = v'$, and rewrite $\mathcal{E}_n$ as

$$\mathcal{E}_n = \int_{-\Delta}^{\Delta} \gamma_n w_n(y) \int \Psi(k) R(k; x_{n-1} + \Delta)^4 \frac{1}{2\frac{\partial\theta_n^{(0)}}{\partial k}(y)} \frac{\partial v}{\partial k}(2\theta_n^{(0)}(y)) \, dk \, dy.$$

Next, we integrate by parts with respect to k to obtain three integrals, each corresponding to applying ∂_k to one of the three functions in

$$\Psi(k) \ \cdot \ R(k; x_{n-1} + \Delta)^4 \ \cdot \ \frac{1}{2\frac{\partial\theta_n^{(0)}}{\partial k}(y)}. \tag{3.37}$$

Case 1. ∂_k *lands on the first term in* (3.37). Then,

$$\Big|\int_{-\Delta}^{\Delta} \gamma_n w_n(y) \int \frac{\partial \Psi(k)}{\partial k} R(k; x_{n-1}+\Delta)^4 \frac{v(2\theta_n^{(0)}(y))}{2\frac{\partial \theta_n^{(0)}}{\partial k}(y)} \, dk\, dy\Big|$$

$$\leq \frac{C d_n}{x_n} \exp\Big(\sum_{m=1}^{n-1} d_m Bigr) \leq C\beta^{-n} d_n \exp\Big(\frac{n \log \beta}{2}\Big) \leq C\beta^{-n/2},$$

where $\beta > 1$ is such that $x_n \geq C\beta^n$ (Hypothesis 3.1) and in the first inequality, we used

$$\log(R(k; x_{n-1}+\Delta)) \leq C \sum_{m=1}^{n-1} d_m \quad \text{(by (3.4iii))},$$

$$\frac{1}{2\frac{\partial \theta_n^{(0)}}{\partial k}(y)} \leq \frac{C}{x_n} \quad \text{(by (3.25))},$$

$$|v(2\theta_n^{(0)}(y))| \leq 1,$$

and

$$\int_{-\Delta}^{\Delta} \int w_n(y) \Psi(k) dy\, dk \leq C \quad \text{(by (3.35))}; \tag{3.38}$$

in the second inequality, we used $d_n = o(1)$.

Case 2. ∂_k *lands on the middle term in* (3.37). We employ $\partial_k R^4 = R^4 \partial_k \log R^4$ and (3.4iv) to estimate the R-term, and (3.25) to estimate the $\theta_n^{(0)}$-term as

$$\Big|\int_{-\Delta}^{\Delta} \gamma_n w_n(y) \int \Psi(k) R(k; x_{n-1}+\Delta)^4 \partial_k \log(R(k; x_{n-1}+\Delta)^4) \frac{v(2\theta_n^{(0)}(y))}{2\frac{\partial \theta_n^{(0)}}{\partial k}(y)} dk\, dy\Big|$$

$$\leq C d_n \Big(\sum_{m=1}^{n-1} d_m x_m\Big) \frac{1}{x_n} B_{n-1} \leq C \eta_n B_{n-1},$$

where in the first inequality, we used (3.38) and

$$R(k; x_{n-1}+\Delta)^4 \leq C \sum_{m=1}^{n-1} d_m x_m, \quad \text{(by (3.4iv))}; \tag{3.39}$$

in the second inequality, we used $d_m = o(1)$, (3.2), and

$$\eta_n := \frac{d_n}{x_n} \sum_{m=1}^{n-1} d_m x_m, \quad \text{with } \sum_{n=1}^{\infty} \eta_n < \infty, \tag{3.40}$$

with (3.40) proved in Remark 3.8.

Case 3. ∂_k *lands on the last term in* (3.37). In this case, we have

$$\Big|\int_{-\Delta}^{\Delta}\gamma_n w_n(y)\int \Psi(k)R(k;x_{n-1}+\Delta)^4\frac{1}{(\frac{\partial\theta_n^{(0)}}{\partial k}(y))^2}\frac{\partial^2\theta_n^{(0)}(y)}{\partial^2 k}v(2\theta_n^{(0)}(y))dk\,dy\Big|$$
$$\le Cd_n\Big(1+\sum_{m=1}^{n-1}d_m x_m^2\Big)\frac{1}{x_n^2}B_{n-1}\le C\kappa_n B_{n-1},$$

where in the first inequality, we used (3.4ii) and (3.38); in the second inequality, we set

$$\kappa_n := \frac{d_n}{x_n^2}\Big(1+\sum_{m=1}^{n-1}d_m x_m^2\Big),\quad \text{with } \sum_{n=1}^{\infty}\kappa_n<\infty, \tag{3.41}$$

with (3.41) proved in Remark 3.8.

Combining Cases 1–3, we obtain (3.36). ∎

Since all three terms on the right-hand side of (3.33) are of the type $\mathcal{E}_n$,

$$E_n\le C(\beta^{-n/2}+B_{n-1}s_n),\quad \text{with } \{s_n=\eta_n+\kappa_n\}_{n\in\mathbb{N}}\in\ell^1(\mathbb{N})$$

Combining this with (3.32), for sufficiently large n,

$$B_n\le B_{n-1}(1+Cd_n^2+Cs_n)+C\beta^{-n/2}.$$

Therefore, $\max(1,B_n)\le(1+Cd_n^2+s_n+\beta^{-n/2}))\max(1,B_{n-1})$ and thus (3.27) holds. ∎

Remark 3.8. In the setting of Theorem 1.12 (a), the numerical series introduced in (3.40) and (3.41) are convergent due to [37, Lemma 5.3]; we expand the concise proof provided therein. For a numerical sequence $d=\{d_n\}_{n\in\mathbb{N}}$, consider the convolution operator

$$(T_\gamma d)_n := \sum_{m=1}^{\infty}\gamma^{-|m-n|}d_m,\quad \text{for } \gamma>1.$$

By Young's inequality, T_γ is a bounded linear operator on $\ell^2(\mathbb{N})$. Let $\gamma>1$ be such that $\frac{x_m}{x_n}\le C\gamma^{-|m-n|}$, $m\le n$. Then, (3.40) follows from

$$\sum_{n=1}^{\infty}d_n\sum_{m=1}^{n-1}d_m\frac{x_m}{x_n}\le\sum_{n=1}^{\infty}d_n\sum_{m=1}^{\infty}d_m\gamma^{-|m-n|}=\langle d,T_\gamma d\rangle_{\ell^2}$$
$$\le\|T_\gamma\|_{\mathcal{B}(\ell^2(\mathbb{N}))}\|d\|_{\ell^2(\mathbb{N})}^2<\infty, \tag{3.42}$$

and (3.41) follows from

$$\sum_{n=1}^{\infty} d_n\Big(1+\sum_{m=1}^{n-1} d_m\Big(\frac{x_m}{x_n}\Big)^2\Big) \le \sum_{n=1}^{\infty}\Big(C\beta^{-2n}+d_n\sum_{m=1}^{n-1} d_m\gamma^{-2|m-n|}\Big)$$
$$\le C+\|T_{\gamma^2}\|_{\mathcal{B}(\ell^2(\mathbb{N}))}\|d\|_{\ell^2(\mathbb{N})}<\infty,$$

where β is as in Hypothesis 3.1 in the second inequality.

3.4. Purely singular continuous spectrum

In this section, we provide the proof of Theorem 1.12 (b). Since Proposition 3.7 rules out the presence of positive eigenvalues, to demonstrate the absence of absolutely continuous spectrum, the strategy is to verify the conditions of Theorem 1.4 via (2.43) and

$$\lim_{j\to\infty} R(x_{n_j}+\Delta,k)=\infty.$$

We begin with a set of auxiliary results concerning the Fourier transform of the potential. We will use the notation

$$\hat f(k):=\int_{-\infty}^{\infty} e^{2\mathbf{i}ky} f(y)dy,\quad \operatorname{Ar}(\hat f(z)):=\begin{cases}\arg(\hat f(z)), & \hat f(z)\neq 0,\\ 0, & \hat f(z)=0,\end{cases} \tag{3.43}$$

Lemma 3.9. *Assume Hypothesis 3.1.*

(i) *For $j=0,1,2$ one has*

$$\frac{d^j}{dz^j}\operatorname{Ar}(\widehat{T}_n(z))\to\frac{d^j}{dz^j}\operatorname{Ar}(\widehat{T}(z)),\quad n\to\infty,$$

uniformly for z in compact intervals $\mathcal{I}\subset(0,\infty)$ that contain no roots of $\widehat{T}$. In particular, for such $\mathcal{I}$ one has

$$\limsup_{n\to\infty}\ \sup_{z\in\mathcal{I}}\Big|\frac{d^j}{dz^j}\operatorname{Ar}(\widehat{T}_n(z))\Big|<\infty,\quad j=0,1,2.$$

Identical assertions hold with T replaced by S.

(ii) *Let $\Phi(z):=(2z)^{-1}\widehat{T}(z)-\mathbf{i}\widehat{S}(z)$ and suppose that a compact interval $J\subset(0,\infty)$ contains no roots of Φ. Then one has*

$$\liminf_{n\to\infty}\ \inf_{z\in J}\Big|\frac{\widehat{T}_n(z)}{2z}-\mathbf{i}\widehat{S}_n(z)\Big|^2>0. \tag{3.44}$$

Proof. (i) Denote for simplicity $f_n(z):=\widehat{T}_n(z)$, $f(z):=\widehat{T}(z)$. Clearly, f is entire function which is not identically zero and f_n converges to f uniformly on compacts. We claim that there exists $n_0\in\mathbb{N}$ such that

(1) for all $n \geq n_0$ and $z \in \mathcal{I}$, $f_n(z) \neq 0$;

(2) for $j = 0, 1, 2$, $\arg f_n^{(j)} \to \arg f^{(j)}$ uniformly on $\mathcal{I}$ and, in particular,

$$\sup_{n \geq n_0} \sup_{z \in J} |\arg f_n^{(j)}(z)| < \infty.$$

To prove these two basics facts from complex analysis, first, recall that if $f_n \to f$ uniformly on some compact K, then for any compact $K' \subset \operatorname{int} K$, $f_n' \to f'$ uniformly on K'; this holds by Cauchy's differentiation formula

$$f_n'(z) = \frac{1}{2\pi i} \oint_{|w-z|=\varepsilon} \frac{f_n(w)}{(w-z)^2}\, dw$$

applied with $\varepsilon = \operatorname{dist}(K', \mathbb{C} \setminus K)$. Next, denote $D = \{z \in \mathbb{C} \mid f(z) = 0\}$, $d = \operatorname{dist}(D, \mathcal{I}) > 0$, and $\mathcal{I}_\varepsilon := \{z \in \mathbb{C} \mid \operatorname{dist}(z, \mathcal{I}) \leq \varepsilon\}$.

On the set $\mathcal{I}_{d/2}$, f_n converge uniformly to f, so there exists n_0 such that for all $n \geq n_0$ and $z \in \mathcal{I}_{d/2}$, $f_n(z) \neq 0$. By the above argument, $f_n' \to f'$ uniformly on $\mathcal{I}_{d/3}$. Thus, $(\log f_n)' = f_n'/f_n \to f'/f = (\log f)'$ uniformly on $\mathcal{I}_{d/3}$. Thus, $(\log f_n)'' \to (\log f)''$ uniformly on $\mathcal{I}_{d/4}$. Taking imaginary parts, we conclude $\arg f_n' \to \arg f'$ and $\arg f_n'' \to \arg f''$ uniformly on $\mathcal{I}$. Choosing branches so that $\log f_n(\min \mathcal{I}) \to \log f(\min \mathcal{I})$ and taking limits of

$$\log f_n(x) = \log f_n(\min \mathcal{I}) + \int_{\min \mathcal{I}}^{x} \frac{f_n'(y)}{f_n(y)} dy$$

and taking imaginary parts shows uniform convergence of $\arg f_n$ to $\arg f$ on $\mathcal{I}$.

(ii) The proof follows directly from complex analytic facts (1), (2) stated above with $f(z) := \widehat{T}(z) - 2z\mathbf{i}\widehat{S}(z)$, $f_n(z) := \widehat{T}_n(z) - 2z\mathbf{i}\widehat{S}_n(z)$. ∎

Assuming Hypothesis 3.1, we say that a compact interval $J \subset \mathbb{R}_+ := (0, \infty)$ is (S, T)-*admissible* if J avoids zeros of $\widehat{S}(z)$, $\widehat{T}(z)$, and $(2z)^{-1}\widehat{T}(z) - \mathbf{i}\widehat{S}(z)$, that is,

$$J \cap \{z \in (0, \infty) : \widehat{T}(z) = 0 \text{ or } \widehat{S}(z) = 0 \text{ or } (2z)^{-1}\widehat{T}(z) - \mathbf{i}\widehat{S}(z) = 0\} = \emptyset.$$

In the following lemma, we derive a third order expansion for the increment of $\log R(x_n + \Delta, k)$ with respect to d_n. For $\{z_n\}_{n \geq 0} \subset \mathbb{C}$ we denote $\delta z_n := z_n - z_{n-1}$.

Lemma 3.10. *Assume Hypothesis 3.1, fix a finite interval $[E_1, E_2] \subset (0, \infty)$ such that $[\sqrt{E_1}, \sqrt{E_2}]$ is (S, T)-admissible, and define $Y_n(k) := \log R(x_n + \Delta, k)$. Then the following asymptotic expansion holds uniformly for $k \in [\sqrt{E_1}, \sqrt{E_2}]$*

$$\delta Y_n(k) \underset{n \to \infty}{=} X_n(k) + \widetilde{X}_n(k) + \mathring{X}_n(k) + \mathcal{O}(d_n^3),$$

where the oscillatory terms $X_n, \widetilde{X}_n$ and are given by

$$X_n(k) := d_n \int_{-\Delta}^{\Delta} \Big[\frac{T_n(y)}{2k}\Big] \sin(2\theta_n^{(0)}(y)) - \left[\frac{d_n T_n(y)}{4k^2} \int_{-\Delta}^{y} T_n(s) ds\right] \cos(2\theta_n^{(0)}(y))\, dy \tag{3.45}$$

$$- d_n \int_{-\Delta}^{\Delta} \left[\frac{d_n S_n(y)}{2k} \int_{-\Delta}^{y} T_n(s)\, ds\right] \sin(2\theta_n^{(0)}(y)) + S_n(y) \cos(2\theta_n^{(0)}(y))\, dy \tag{3.46}$$

$$- d_n^2 \int_{-\Delta}^{\Delta} \Big[\frac{S_n^2(y)}{2k}\Big] \sin(2\theta_n^{(0)}(y)) dy, \tag{3.47}$$

and

$$\begin{aligned} \widetilde{X}_n(k) := &\frac{d_n^2 |\widehat{T}_n(k)|^2}{8k^2} \cos\left(4\theta_n^{(0)}(0) + 4\phi_n(k)\right) \\ &- \frac{d_n^2 |\widehat{S}_n(k)|^2}{2} \cos\left(4\theta_n^{(0)}(0) + 4\psi_n(k)\right) \\ &+ \frac{d_n^2}{2k} |\widehat{S}_n(k) \widehat{T}_n(k)| \sin(4\theta_n^{(0)}(0) + 2\psi_n(k) + 2\phi_n(k)), \end{aligned} \tag{3.48}$$

where

$$\phi_n(k) := \frac{\mathrm{Ar}(\widehat{T}_n(k))}{2},$$
$$\psi_n(k) := \frac{\mathrm{Ar}(\widehat{S}_n(k))}{2},$$

cf. (3.43)*, and the non-oscillatory term $\mathring{X}_n$ is given by*

$$\mathring{X}_n(k) := \frac{d_n}{2} \left| \frac{\widehat{T}_n(k)}{2k} - \mathbf{i}\, \widehat{S}_n(k) \right|^2, \; n \geq 1. \tag{3.49}$$

Proof. Integrating both sides of (2.42) over the interval $[x_n - \Delta, x_n + \Delta]$, we get

$$\delta Y_n(k) = \int_{-\Delta}^{\Delta} q_n(s) \sin(2\theta(x_n + s)) - \sigma_n(s) \cos(2\theta(x_n + s))\, ds. \tag{3.50}$$

Combining (3.21) and Taylor expansions for sin, cos near $2\theta_n^{(0)}(s)$,

$$\sin(2\theta(x_n + s)) \underset{n\to\infty}{=} \sin(2\theta_n^{(0)}(s)) + 2d_n \theta_n^{(1)}(s) \cos(2\theta_n^{(0)}(y)) + \mathcal{O}(d_n^2), \tag{3.51a}$$

$$\cos(2\theta(x_n + s)) \underset{n\to\infty}{=} \cos(2\theta_n^{(0)}(s)) - 2d_n \theta_n^{(1)}(s) \sin(2\theta_n^{(0)}(s)) + \mathcal{O}(d_n^2), \tag{3.51b}$$

uniformly for $s \in [-\Delta, \Delta]$. Replacing the trigonometric terms in (3.50) by their second-order approximations (3.51), one infers

$$\begin{aligned}\delta Y_n(k) \underset{n\to\infty}{=} \int_{-\Delta}^{\Delta} q_n(y)\sin(2\theta_n^{(0)}(y)) - \sigma_n(y)\cos(2\theta_n^{(0)}(y))\,dy \\ + 2\int_{-\Delta}^{\Delta} d_n\theta_n^{(1)}(y)q_n(y)\cos(2\theta_n^{(0)}(y)) \\ + d_n\theta_n^{(1)}(y)\sigma_n(y)\sin(2\theta_n^{(0)}(y))dy + \mathcal{O}(d_n^3),\end{aligned} \tag{3.52}$$

where the last cubic term was obtained by combining the linear (3.20) and the quadratic (3.51) asymptotic formulas. In order to facilitate integration by parts in the subsequent argument, let us rewrite the terms in (3.52) containing $\theta_n^{(1)}$. First, use the double angle formula to replace $\sin^2$ term in (3.23),

$$\theta_n^{(1)}(y) = \frac{1}{d_n}\int_{-\Delta}^{y} [\sigma_n(s)\sin(2\theta_n^{(0)}(s)) + q_n(s)\cos(2\theta_n^{(0)}(s))] - q_n(s)\,ds. \tag{3.53}$$

Then, substitute this identity into the first term under the integral in (3.52) to get

$$\begin{aligned}&\int_{-\Delta}^{\Delta} d_n\theta_n^{(1)}(y)q_n(y)\cos(2\theta_n^{(0)}(y))dy \\ &= -\int_{-\Delta}^{\Delta}\left[\int_{-\Delta}^{y} q_n(s)ds\right]q_n(y)\cos(2\theta_n^{(0)}(y))dy \\ &\quad + \int_{-\Delta}^{\Delta}\left[\int_{-\Delta}^{y} \sigma_n(s)\sin(2\theta_n^{(0)}(s)) + q_n(s)\cos(2\theta_n^{(0)}(s))\,ds\right]q_n(y)\cos(2\theta_n^{(0)}(y))dy\end{aligned} \tag{3.54}$$

and similarly, substitute (3.53) into the second term under the same integral to get

$$\begin{aligned}&\int_{-\Delta}^{\Delta} d_n\theta_n^{(1)}(y)\sin(2\theta_n^{(0)}(y))\sigma_n(y)dy \\ &= -\int_{-\Delta}^{\Delta}\left[\int_{-\Delta}^{y} q_n(s)\,ds\right]\sigma_n(y)\sin(2\theta_n^{(0)}(y))dy \\ &\quad + \int_{-\Delta}^{\Delta}\left[\int_{-\Delta}^{y} \sigma_n(s)\sin(2\theta_n^{(0)}(s)) + q_n(s)\cos(2\theta_n^{(0)}(s))ds\right]\sigma_n(y)\sin(2\theta_n^{(0)}(y))dy.\end{aligned} \tag{3.55}$$

Returning to $\delta Y_n(k)$, we plug (3.54), (3.55) in (3.52), use (3.57) with

$$f = \sigma_n(y)\sin(2\theta_n^{(0)}(y))$$

and

$$g = q_n(s)\cos(2\theta_n^{(0)}(s));$$

then, we obtain

$$\begin{aligned}
\delta Y_n(k) \underset{n\to\infty}{=} & \int_{-\Delta}^{\Delta}\Big(q_n(y) - \int_{-\Delta}^{y} q_n(s)\,ds\,\sigma_n(y)\Big)\sin(2\theta_n^{(0)}(y))dy \\
& - \int_{-\Delta}^{\Delta}\Big(\int_{-\Delta}^{y} q_n(s)ds\,q_n(y) + \sigma_n(y)\Big)\cos(2\theta_n^{(0)}(y))\,dy \\
& \times\Big(\int_{-\Delta}^{\Delta}\sigma_n(y)\sin(2\theta_n^{(0)}(y))dy + \int_{-\Delta}^{\Delta} q_n(y)\cos(2\theta_n^{(0)}(y))dy\Big)^2 \\
& + \mathcal{O}(d_n^3).
\end{aligned} \tag{3.56}$$

Next, denote the quadratic term above by L and note that $\theta_n^{(0)}(y) = \theta_n^{(0)}(0) + ky$; then,

$$\begin{aligned}
L &= \Big(d_n \mathrm{Im} e^{2\mathbf{i}\theta_n^{(0)}(0)}\widehat{S}_n(k) + \frac{d_n}{2k}\mathrm{Re}\, e^{2\mathbf{i}\theta_n^{(0)}(0)}\widehat{T}_n(k)\Big)^2 \\
&= \Big(d_n \mathrm{Im} e^{2\mathbf{i}\theta_n^{(0)}(0)+2\mathbf{i}\psi_n(k)}|\widehat{S}_n(k)| + \frac{d_n}{2k}\mathrm{Re}\, e^{2\mathbf{i}\theta_n^{(0)}(0)+2\mathbf{i}\phi_n(k)}|\widehat{T}_n(k)|\Big)^2 \\
&= \big(d_n \sin(2\theta_n^{(0)}(0) + 2\psi_n(k))|\widehat{S}_n(k)|\big)^2 + \Big(\frac{d_n}{2k}\cos(2\theta_n^{(0)}(0) + 2\phi_n(k))|\widehat{T}_n(k)|\Big)^2 \\
&\quad + \frac{d_n^2}{k}|\widehat{S}_n(k)\widehat{T}_n(k)|\sin\big(2\theta_n^{(0)}(0) + 2\phi_n(k)\big)\cos\big(2\theta_n^{(0)}(0) + 2\phi_n(k)\big) \\
&= \frac{d_n^2|\widehat{S}_n(k)|^2}{2} - \frac{d_n^2|\widehat{S}_n(k)|^2\cos\big(4\theta_n^{(0)}(0) + 4\psi_n(k)\big)}{2} \\
&\quad + \frac{d_n^2|\widehat{T}_n(k)|^2}{8k^2} + \frac{d_n^2|\widehat{T}_n(k)|^2\cos\big(4\theta_n^{(0)}(0) + 4\phi_n(k)\big)}{8k^2} \\
&\quad + \frac{d_n^2}{k}|\widehat{S}_n(k)\widehat{T}_n(k)|\sin\big(2\theta_n^{(0)}(0) + 2\psi_n(k)\big)\cos\big(2\theta_n^{(0)}(0) + 2\phi_n(k)\big).
\end{aligned}$$

To conclude the derivation, we plug the above expression for L in (3.56), expand q_n, σ_n in terms of d_n, S_n, T_n and combine the third order terms (with respect to d_n as $n \to \infty$) with $\mathcal{O}(d_n^3)$. ∎

Remark 3.11. Suppose that $f, g \in L^1(-\Delta, \Delta)$, then

$$
\begin{aligned}
\frac{1}{2}\Big(\int_{-\Delta}^{\Delta} f(y)dy + \int_{-\Delta}^{\Delta} g(y)dy\Big)^2
&= \int_{-\Delta}^{\Delta} f(y) \int_{-\Delta}^{y} g(s)ds\, dy + \int_{-\Delta}^{\Delta} g(y) \int_{-\Delta}^{y} f(s)ds\, dy \\
&\quad + \int_{-\Delta}^{\Delta} f(y) \int_{-\Delta}^{y} f(s)ds\, dy + \int_{-\Delta}^{\Delta} g(y) \int_{-\Delta}^{y} g(s)ds\, dy. \qquad (3.57)
\end{aligned}
$$

This identity follows from

$$
\int_{-\Delta}^{\Delta} f(s)ds \int_{-\Delta}^{\Delta} g(y)dy = \int_{-\Delta}^{\Delta} f(y) \int_{-\Delta}^{y} g(s)ds\, dy + \int_{-\Delta}^{\Delta} g(y) \int_{-\Delta}^{y} f(s)ds\, dy, \qquad (3.58)
$$

which is derived by changing the order of integration in the first integral on the right-hand side of (3.58).

Lemma 3.12. *Recall $Y_n, \mathring{X}_n$ from Lemma 3.10 and define*

$$
Q_n(k) := Y_n(k) - \sum_{m=1}^{n} \mathring{X}_m(k). \qquad (3.59)
$$

Then for arbitrary non-negative $g \in C_0^\infty(0,\infty)$ with $\operatorname{supp}(g) \subset J$ for a (S,T)-admissible interval J we have

$$
\lim_{n\to\infty} \frac{\int_0^\infty g(k)|Q_n(k)|dk}{\sum_{m=1}^n d_m^2} = 0.
$$

Proof. Setting $Q_0 = X_0 = \tilde{X}_0 = \mathring{X}_0 = 0$, we note that

$$
Q_n(k) = \sum_{m=1}^{n} \delta Q_n(k) = \sum_{m=1}^{n} \big(X_m(k) + \tilde{X}_m(k) + \mathcal{O}(d_n^3)\big). \qquad (3.60)
$$

Define

$$
B_n := \int_0^\infty g(k)\Big|\sum_{m=1}^{n} X_m(k)\Big|^2 dk, \quad \tilde{B}_n := \int_0^\infty g(k)\Big|\sum_{m=1}^{n} \tilde{X}_m(k)\Big|^2 dk;
$$

then, by Cauchy–Schwarz inequality in $L^2(\mathbb{R}_+, dk)$,

$$\int_0^\infty g(k)|Q_n(k)|dk \le \|\sqrt{g}\|_{L^2(\mathbb{R}_+)}(\sqrt{B_n} + \sqrt{\widetilde{B}_n}) + \mathcal{O}(d_n^3). \tag{3.61}$$

Following the proof of [37, Theorem 1.6], we notice that by Stolz lemma (the discrete version of L'Hospital's rule), $\sum_{m=1}^n d_m^3 / \sum_{m=1}^n d_m^2 \to 0$ as $n \to \infty$; hence, in order to show (3.60), it suffices to prove

$$\sqrt{B_n} \Big/ \sum_{m=1}^n d_m^2 \to 0, \quad \sqrt{\widetilde{B}_n} \Big/ \sum_{m=1}^n d_m^2 \to 0, \quad n \to \infty. \tag{3.62}$$

To derive the first limit, recall X_n from Lemma 3.10 and denote the integral terms in (3.45), (3.46), (3.47) by U_n, V_n, Z_n respectively; thus, $X_n = d_n U_n - d_n V_n - d_n^2 Z_n$. Put $M_{n-1}(k) := \sum_{m=1}^{n-1} X_m(k)$; then,

$$B_n \le B_{n-1} + \int g(k)|X_n(k)|^2 dk \tag{3.63}$$

$$+ 2\left|\int g(k) M_{n-1}(k) d_n U_n(k)\, dk\right| + 2\left|\int g(k) M_{n-1}(k) d_n V_n(k)\, dk\right| \tag{3.64}$$

$$+ 2\left|\int g(k) M_{n-1}(k) d_n^2 Z_n(k)\, dk\right| \tag{3.65}$$

Note that U_n, V_n, Z_n contain $\sin(2\theta_n^{(0)}(y))$, $\cos(2\theta_n^{(0)}(y))$ terms which we split in (3.64), (3.65) using the triangle inequality. The resulting terms are of the form

$$\int \int_{-\Delta}^{\Delta} \gamma_n w_n(y) \Psi(k) M_{n-1}(k) u(2\theta_n^{(0)}(y)) dy\, dk,$$

with γ_n, w_n, Ψ, u as in (3.35). As in the proof of Theorem 1.12 (a), rewrite this quantity as

$$\int_{-\Delta}^{\Delta} \gamma_n w_n(y) \int \Psi(k) M_{n-1}(k) \frac{1}{2\frac{\partial \theta_n^{(0)}}{\partial k}(y)} \frac{\partial v}{\partial k}(2\theta_n^{(0)}(y))\, dk\, dy$$

where v is either sin or cos so that $u = v'$. Next, integrate by parts with respect to k and obtain three integrals, each corresponding to applying ∂_k to one of the three functions in

$$\Psi(k) \cdot M_{n-1}(k) \cdot \frac{1}{2\frac{\partial \theta_n^{(0)}}{\partial k}(y)}. \tag{3.66}$$

Case 1. *∂_k lands on the first term in* (3.66). In this case,

$$\left| \int_{-\Delta}^{\Delta} \gamma_n w_n(y) \int \frac{\partial \Psi(k)}{\partial k} M_{n-1}(k) \frac{1}{2\frac{\partial \theta_n^{(0)}}{\partial k}(y)} v(2\theta_n^{(0)}(y))\, dk\, dy \right| \leq \frac{C d_n (n-1)}{x_n}, \tag{3.67}$$

where we used (3.38) and $M_{n-1}(k) \leq C(n-1)$.

Case 2. *∂_k lands on the second term in* (3.66). Then,

$$\left| \int_{-\Delta}^{\Delta} \gamma_n w_n(y) \int \Psi(k) \frac{\partial M_{n-1}(k)}{\partial k} \frac{1}{2\frac{\partial \theta_n^{(0)}}{\partial k}(y)} v(2\theta_n^{(0)}(y))\, dk\, dy \right| \leq \frac{C d_n}{x_n} \sum_{m=1}^{n-1} x_m \omega_m \tag{3.68}$$

where and we used (3.38) and

$$\frac{\partial M_{n-1}(k)}{\partial k} \leq \sum_{m=1}^{n-1} x_m d_m.$$

Case 3. *∂_k lands on the third term in* (3.66). We first replace Ψ by $\frac{\Psi}{g} g$ and then estimate

$$\begin{aligned} &\left| \int_{-\Delta}^{\Delta} \gamma_n w_n(y) \int g(k) M_{n-1}(k) \frac{\Psi(k)}{g(k)} \frac{1}{\left(\frac{\partial \theta_n^{(0)}}{\partial k}(y)\right)^2} \frac{\partial^2 \theta_n^{(0)}(y)}{\partial^2 k} v(2\theta_n^{(0)}(y)) dk\, dy \right| \\ &\leq \frac{C d_n}{x_n^2} \Big(1 + \sum_{m=1}^{n-1} d_m x_m^2\Big) \left| \int g(k) M_{n-1}(k) dk \right| \\ &\leq \frac{C d_n}{x_n^2} \Big(1 + \sum_{m=1}^{n-1} d_m x_m^2\Big) \Big(\int g(k) |M_{n-1}(k)|^2 dk \Big)^{1/2} =: \alpha_n \sqrt{B_{n-1}}, \end{aligned} \tag{3.69}$$

where in the second to last inequality, we used (3.38) and

$$\frac{\partial^2 \theta_n^{(0)}(y)}{\partial^2 k} \leq C \Big(1 + \sum_{m=1}^{n-1} x_m \omega_m\Big) \quad \text{(by (3.4i) and (3.4ii));}$$

in the last inequality, we used the Cauchy–Schwarz inequality in $L^2(\mathbb{R}_+, dk)$ and denoted

$$\alpha_n := \frac{C d_n}{x_n^2} \Big(1 + \sum_{m=1}^{n-1} d_m x_m^2\Big). \tag{3.70}$$

We are now ready to derive the first limit in (3.62): combine (3.64), (3.65), (3.67), (3.68), (3.69), and estimate the last term in (3.63) from above by Cd_n^2, we have

$$B_n \le B_{n-1} + 2\alpha_n \sqrt{B_{n-1}} + \beta_n, \tag{3.71}$$

where α_n is as in (3.70) and

$$\beta_n := C\Big(\frac{d_n(n-1)}{x_n} + \frac{d_n}{x_n}\sum_{m=1}^{n-1} x_m d_m + d_n^2\Big).$$

Then, (3.71) together with [37, Lemma 6.2] yields

$$\sqrt{B_n} \le \sqrt{B_0} + \sum_{m=1}^{n} \alpha_m + \Big(\sum_{m=1}^{n} \beta_m\Big)^{1/2}. \tag{3.72}$$

Consequently, the first limit in (3.62) holds as asserted due to

$$\sum_{m=1}^{n} \alpha_m \Big/ \sum_{m=1}^{n} d_m^2 \to 0, \quad \Big(\sum_{m=1}^{n} \beta_m\Big)^{1/2} \Big/ \sum_{m=1}^{n} d_m^2 \to 0, \quad n \to \infty, \tag{3.73}$$

these two limits are discussed in Remark 3.13 below.

Let us now derive the second limit in (3.62). First, we write

$$\widetilde{X}_n = d_n^2 \widetilde{U}_n + d_n^2 \widetilde{V}_n + d_n^2 \widetilde{Z}_n$$

where $\widetilde{U}_n, \widetilde{V}_n, \widetilde{Z}_n$ denote k-dependent functions in (3.48). Then, denoting

$$\widetilde{M}_{n-1}(k) := \sum_{m=1}^{n-1} \widetilde{X}_m(k),$$

we obtain

$$\begin{aligned}\widetilde{B}_n \le \widetilde{B}_{n-1} &+ \int g(k)|X_n(k)|^2 dk \\ &+ 2\Big|\int g(k)\widetilde{M}_{n-1}(k) d_n^2 \widetilde{U}_n(k)\, dk\Big| + 2\Big|\int g(k)\widetilde{M}_{n-1}(k) d_n^2 \widetilde{V}_n(k)\, dk\Big| \qquad (3.74)\\ &+ 2\Big|\int g(k)\widetilde{M}_{n-1}(k) d_n^2 \widetilde{Z}_n(k)\, dk\Big|. \qquad (3.75)\end{aligned}$$

Note that all three terms in (3.74), (3.75) are of the form

$$\int \gamma_n \Psi(k) \widetilde{M}_{n-1}(k) u(\mu_n(y))\, dk\, dy, \tag{3.76}$$

with $\Psi \in C_0^\infty(0,\infty)$, $\gamma_n := d_n^2$ and

$$\mu_n(k) \in \{4\theta_n^{(0)}(0) + 2\psi_n(k) + 2\phi_n(k),\ 4\theta_n^{(0)}(0) + 4\psi_n(k),\ 4\theta_n^{(0)}(0) + 4\phi_n(k)\}.$$

Using (3.25), (3.4ii), and Lemma 3.9 (i), we get

$$\frac{\partial \mu_n(k)}{\partial k} > C x_n, \tag{3.77}$$

$$\left|\frac{\partial^2 \mu_n(k)}{\partial k^2}\right| \le C\Big(1 + \sum_{m=1}^{n} d_m x_m^2\Big). \tag{3.78}$$

As in the first part of the proof, we proceed by rewriting (3.76) in the form

$$\int \gamma_n \Psi(k) \widetilde{M}_{n-1}(k) \frac{1}{\frac{\partial \mu_n(k)}{\partial k}} \frac{\partial v}{\partial k}(\mu_n(k))\, dk,$$

and integrating by parts with respect to k. This approach, as before, leads to three integrals, each corresponding to applying $\frac{\partial}{\partial k}$ to one of the three functions in

$$\Psi(k) \cdot \widetilde{M}_{n-1}(k) \cdot \frac{1}{\frac{\partial \mu_n(k)}{\partial k}}. \tag{3.79}$$

Case 1. *∂_k lands on the first term of* (3.79). In this case,

$$\left|\int \gamma_n \frac{\partial \Psi(k)}{\partial k} \widetilde{M}_{n-1}(k) \frac{1}{\frac{\partial \mu_n(k)}{\partial k}} v(\mu_n(k))\, dk\right| \le \frac{C d_n (n-1)}{x_n}$$

where we used $\widetilde{M}_{n-1}(k) \le C(n-1)$ and

$$\frac{1}{\frac{\partial \mu_n(k)}{\partial k}} \le \frac{C}{x_n} \text{ by (3.77)}, \quad \Psi \in C_0^\infty(0,\infty), \quad |v(2\theta_n^{(0)}(y))| \le 1. \tag{3.80}$$

Case 2. *∂_k lands on the second term of* (3.79). In this case,

$$\left|\int \gamma_n \Psi(k) \frac{\partial \widetilde{M}_{n-1}(k)}{\partial k} \frac{1}{\frac{\partial \mu_n(k)}{\partial k}} v(\mu_n(k))\, dk\right| \le \frac{C d_n}{x_n} \sum_{m=1}^{n-1} x_m d_m$$

where we used (3.80) and

$$\frac{\partial \widetilde{M}_{n-1}(k)}{\partial k} \le \sum_{m=1}^{n-1} x_m d_m.$$

Case 3. *∂_k lands on the third term of* (3.79). We first replace Ψ by $\frac{\Psi}{g} g$ and then estimate

$$\begin{aligned}
&\left|\gamma_n \int g(k) \widetilde{M}_{n-1}(k) \frac{\Psi(k)}{g(k)} \frac{1}{\big(\frac{\partial \mu_n(k)}{\partial k}\big)^2} \frac{\partial^2 \mu_n(k)}{\partial k^2} v(\mu_n(k)) dk\, dy\right| \\
&\quad \le \frac{C d_n}{x_n^2}\Big(1 + \sum_{m=1}^{n-1} d_m x_m^2\Big)\left|\int g(k) \widetilde{M}_{n-1}(k) dk\right|
\end{aligned}$$

$$\leq \frac{Cd_n}{x_n^2}\Big(1+\sum_{m=1}^{n-1} d_m x_m^2\Big)\Big(\int g(k)|\widetilde{M}_{n-1}(k)|^2 dk\Big)^{1/2} =: \alpha_n \sqrt{\widetilde{B}_{n-1}},$$

where in the second to last inequality, we used (3.78) and (3.80); in the last inequality, we used the Cauchy–Schwarz inequality in $L^2(\mathbb{R}_+, dk)$ and the notation (3.70).

Combining Cases 1–3, we get a version of (3.72) with B replaced by $\widetilde{B}$. As before, using [37, Lemma 6.2] and Proposition 3.8, we infer the second limit in (3.62). ■

Remark 3.13. Assuming the setting of Theorem 1.12 (b). To prove the first limit in (3.73), recall γ from Remark 3.8 and write

$$\begin{aligned}\sum_{n=2}^{k} d_n \sum_{m=1}^{n-1} d_m\Big(\frac{x_m}{x_n}\Big)^2 &\leq C\sum_{n=2}^{k} d_n\Big(\frac{x_{n-1}}{x_n}\Big)^2 \sum_{m=1}^{n-1} d_m\Big(\frac{x_m}{x_{n-1}}\Big)^2\\ &\leq C\sum_{n=2}^{k} d_n\Big(\frac{x_{n-1}}{x_n}\Big)^2 \sum_{m=1}^{n-1} d_m \gamma^{-2|m-n-1|}\\ &\leq C\Big(\sum_{n=2}^{k} d_n^2\Big(\frac{x_{n-1}}{x_n}\Big)^4\Big)^{1/2}\Big(\sum_{n=2}^{k} d_n^2\Big)^{1/2}\end{aligned}$$

where in the last inequality, we used boundedness of the convolution operator, as in Remark 3.8. Note that

$$\sum_{n=2}^{k} d_n^2 \to \infty, \quad k\to\infty; \quad \text{and} \quad \frac{x_{n-1}}{x_n} \to 0, \quad n\to\infty;$$

thus, first limit in (3.73) holds. To prove the second limit in (3.73), use (3.42) to get

$$\sqrt{\beta_n} \leq C\sqrt{1+\sum_{m=1}^{k} d_m^2} = o\Big(\sum_{n=2}^{k} d_n^2\Big), \quad k\to\infty.$$

Lemma 3.14. *Assume Hypothesis 3.1, fix a finite interval $[E_1, E_2] \subset (0,\infty)$ such that $[\sqrt{E_1}, \sqrt{E_2}]$ is (S,T)-admissible. Then there exists a subsequence $\{n_j\}_{j\geq 1}$ such that for Lebesgue almost every $k \in [\sqrt{E_1}, \sqrt{E_2}]$ one has*

$$\lim_{j\to\infty} R(x_{n_j}+\Delta, k) = \infty. \tag{3.81}$$

Proof. Let $g \in C_0^\infty(0,\infty)$ be a strictly positive function with

$$[\sqrt{E_1}, \sqrt{E_2}] \subset \operatorname{supp}(g) \subset J,$$

for an (S,T)-admissible J, and $\int_{\mathbb{R}_+} g(k)dk = 1$. Consider two sequences

$$\xi_n := Q_n \qquad \text{(cf. (3.59))},$$
$$\zeta_n := \sum_{m=1}^{n} \mathring{X}_m \quad \text{(cf. (3.49))},$$

of random variables in the probability space $(\Omega,\mathbb{P}) := ((0,\infty), g(k)dk)$, and denote $\alpha_n := \sum_{m=1}^{n} d_m^2$. Lemma 3.12 and (3.44) yield

$$\lim_{n\to\infty} \alpha_n^{-1}\mathbb{E}\xi_n = 0 \quad \text{and} \quad \zeta_n \geq C\alpha_n.$$

Then, by [37, Lemma 6.1 (i), (ii′)], there exists a subsequence $\{n_j\}_{j\geq 1}$ such that for almost every $k \in \mathcal{I}$,

$$\lim_{j\to\infty} (\xi_{n_j}(k) + \zeta_{n_j}(k)) = \infty;$$

that is, $\lim_{j\to\infty} Y_{n_j}(k) = \infty$ and therefore (3.81) holds as claimed. ∎

Proof of Theorem 1.12 (b). By Lemma 1.7, $\mathrm{Spec}_{\mathrm{ess}}(H^\alpha) = [0,\infty)$. Moreover, by Proposition 3.7, H^α has no positive eigenvalues.

For every (S,T)-admissible interval $[\sqrt{E_1},\sqrt{E_2}]$, by Lemma 3.14 for some subsequence $\{n_j\}_{j=1}^\infty$ we have

$$\lim_{j\to\infty} R(x_{n_j} + \Delta, k) = \infty, \quad a.e.\ k \in [\sqrt{E_1},\sqrt{E_2}].$$

Next, by Theorem 1.4, $\mathrm{Spec}_{\mathrm{ac}}(H^\alpha) \cap [E_1,E_2] = \emptyset$ and, since the union of all (S,T)-admissible intervals gives $\mathbb{R}_+$ up ot a discrete set, we conclude that $\mathrm{Spec}_{\mathrm{ac}}(H^\alpha) = \emptyset$. Therefore, the spectrum of H^α is purely singular continuous on $(0,\infty)$. ∎

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