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Isospectrum of non-self-adjoint almost-periodic Schrödinger operators

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Abstract. For non-self-adjoint almost-periodic Schrödinger operators, a criterion is given to guarantee that they have both the same spectrum and the same Lyapunov exponents with the discrete free Laplacian. As a byproduct, we show that the Moser–Pöschel argument for opening gaps may not be valid for non-self-adjoint operators.

1. Introduction

Benefiting from methods of dynamical systems and harmonic analysis, enormous breakthroughs have been made in recent years in the study of *self-adjoint* almost-periodic Schrödinger operators on $\ell^2(\mathbb{Z})$ (resp. $L^2(\mathbb{R})$)

$$H_V = \Delta + V(\cdot), \tag{1.1}$$

where $V(\cdot)$ are almost-periodic on $\mathbb{Z}$ (resp. $\mathbb{R}$). These breakthroughs include, for example, the Cantor spectrum [5, 9, 30], the interval spectrum [21, 31], the Anderson localization [6, 8, 17, 39, 40], the reducibility of Schrödinger cocycles [7, 24, 35], and Avila's global theory [2]. However, little progress has been made for *non-self-adjoint* almost-periodic operators (non-Hermitian quasicrystals in physical literature), and even the fundamental spectral theorem has not been established so far (may not be possible). In comparison, non-Hermitian Hamiltonians received wide attention from physicists in recent years because (i) the recent experimental advances in controlling dissipation have brought about unprecedented flexibility in engineering non-Hermitian Hamiltonians in open classical and quantum systems [1, 32]; (ii) non-Hermitian Hamiltonians exhibit rich phenomena without Hermitian counterparts, e.g., $\mathcal{PT}$ (parity-time) symmetry breaking [11], topological phase transition [49], non-Hermitian skin effects [36]. Of course, these observations and predictions in physical literature deserve rigorous mathematical proofs.

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There are also other motivations for the mathematical study of the non-self-adjoint almost-periodic operators. Firstly, a striking result worth highlighting is Avila's global theory of the one-frequency quasi-periodic Schrödinger operators [2]. However, if one wants to establish the quantitative global theory [27], the core is to study

$$(H\psi)_n = \psi_{n+1} + \psi_{n-1} + v(x + n\alpha + \mathrm{i}\epsilon)\psi_n,$$

which is a family of non-self-adjoint operators. Secondly, the spectrum of non-self-adjoint Schrödinger operators has deep connection with problems of the elliptic operators, such as ground states, steady states and averaging theory [45, 47].

1.1. Isospectrum

In this paper, we study the spectrum of the following non-self-adjoint almost-periodic Schrödinger operator on $\ell^2(\mathbb{Z})$:

$$(H_{\lambda v,\alpha,x}\psi)_n = \psi_{n+1} + \psi_{n-1} + \lambda v(x + n\alpha)\psi_n, \tag{1.2}$$

where $\lambda \in \mathbb{R}$ is the coupling constant, $x \in \mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$ is called the *phase* with $d \in \mathbb{N}^+ \cup \{\infty\}$, $v\colon \mathbb{T}^d \to \mathbb{C}$ is the potential, $\alpha \in \mathbb{T}^d$ is the frequency satisfying that $(1, \alpha)$ is independent among $\mathbb{Q}$. In this case, the spectrum of $H_{\lambda v,\alpha,x}$ is independent of x [27, 41], and thus we denote it by $\Sigma_{\lambda v,\alpha}$.

$\mathcal{PT}$ symmetric operators constitute an important class of non-self-adjoint operators coming from quantum mechanics [44]. Recall that (1.1) is $\mathcal{PT}$ symmetric, if $\bar{V}(n) = V(-n)$ [10]. In the almost-periodic setting where the potential $v\colon \mathbb{T}^d \to \mathbb{C}$, one can extend the definition to $\bar{v}(x) = v(-x)$, since the spectrum $\Sigma_{v,\alpha}$ is independent of x.

It was first observed by Bender and Boettcher [10] that a large class of $\mathcal{PT}$ symmetric operators have real spectrum. This observation has a profound significance in that it not only suggests a possibility of $\mathcal{PT}$ symmetric modification of the conventional quantum mechanics that considers observables as self-adjoint operators [44], but also goes far beyond quantum mechanics and has spread to many branches of physics [54]. Thus, a basic mathematical question is to ask which class of $\mathcal{PT}$ symmetric operators have real spectrum.

As a warm-up example, let us first look at the heuristic example

$$(H_{\lambda \exp,\alpha}\psi)_n = \psi_{n+1} + \psi_{n-1} + \lambda \mathrm{e}^{\mathrm{i}(x+n\alpha)}\psi_n, \tag{1.3}$$

proposed by Sarnak [55], whose spectrum has already been completely known.

Theorem 1.1 ([13, 15, 55]). *For any $\alpha/2\pi \in \mathbb{R}\backslash\mathbb{Q}$, we have the following.*

(1) *If $|\lambda| \leqslant 1$, then the spectrum of* (1.3) *is a real interval:*

$$\Sigma_{\lambda \exp,\alpha} = [-2, 2].$$

(2) *If $|\lambda| > 1$, denote $\xi = \ln|\lambda|$, then the spectrum of* (1.3) *is an ellipse given by*

$$\Sigma_{\lambda \exp,\alpha} = \left\{ E \in \mathbb{C} : \left(\frac{\operatorname{Re} E}{\cosh \xi}\right)^2 + \left(\frac{\operatorname{Im} E}{\sinh \xi}\right)^2 = 4 \right\}.$$

Theorem 1.1 was proved by Sarnak [55] in the case $|\lambda| \neq 1$ and α is Diophantine. It was generalized to all $\lambda \in \mathbb{R}$ and $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ by Boca [13] and by Borisov and Fedotov [15] independently. In this paper, we will give a simple proof of Theorem 1.1 by Avila's global theory of one-frequency analytic $\mathrm{SL}(2,\mathbb{C})$ cocycles [2].

The phenomenon of being isospectral to the free Laplacian H_0, described by Theorem 1.1 (1), is of particular interest. It has roots in the study of the non-self-adjoint differential operator with periodic potential, see [57, 58]. The famous Borg's uniqueness theorem [14] for the Hill operator

$$(H_v\psi)(t) = -\psi''(t) + v(t)\psi(t),$$

states that if $v \in L^2_{\mathrm{loc}}(\mathbb{R})$ is periodic and real-valued, then $\Sigma_v = [0, \infty)$ if and only if $v \equiv 0$ a.e. The modern proofs of Borg's uniqueness statement replace the condition of periodicity of the potential by the property of being reflectionless. Details can be found in [22, Theorem 4.1] and the comment following its proof. [22, Theorem 4.4] provides a more general version in the matrix-valued context. The actual trace formula on which these considerations rely on is due to [28]. For the discrete Borg's theorem, see [29, 56] for the $\mathbb{Z}^1$ version and [48] for the $\mathbb{Z}^d$ version. However, in the case of a complex-valued periodic potential v, the situation is very different, as it was proved by Gasymov [26] (see also [34]) that if the Hill operator satisfies

$$v(t) = \sum_{\mathbf{k}=1}^{\infty} \hat{v}_{\mathbf{k}} e^{\mathrm{i}\mathbf{k}t} \quad \text{with} \sum_{\mathbf{k}=1}^{\infty} |\hat{v}_{\mathbf{k}}| < \infty, \tag{1.4}$$

then $\Sigma_v = [0, \infty)$ or say H_v is isospectral to H_0. However, it is still open whether, under some smoothness requirements, any operator with periodic potential $v(x)$ isospectral to H_0 must be a "Gasymov potential", i.e., of the form given by (1.4), or the complex conjugate of a Gasymov potential [52].

In the non-periodic discrete setting, Killip and Simon [43] largely extended Borg's uniqueness theorem [14], and proved that for self-adjoint discrete Schrödinger operator (1.1) with $V: \mathbb{Z} \to \mathbb{R}$, H_V is isospectral to H_0 if and only if $V \equiv 0$. Theorem 1.1 shows that being isospectral to H_0 does not imply $v \equiv 0$ for complex quasi-periodic potential (1.2).

The main ambition of this paper is to explore the structure of complex potential (not necessarily $\mathcal{PT}$ symmetric), and to give a criterion to ensure the corresponding operators (1.2) are isospectral to H_0. Before stating the results, we first introduce some notations. For any $\mathbf{k} \in \mathbb{Z}^d$, we define $|\mathbf{k}|_\eta = \sum_{j\in\mathbb{N}} \langle j\rangle^\eta |\mathbf{k}_j|$, where $\langle j\rangle := \max\{1, j\}$ and $\eta > 0$ is a fixed constant. Let $\mathbb{Z}^d_*$ be the set of integer vectors with finite support $\mathbb{Z}^d_* = \{\mathbf{k} : 0 < |\mathbf{k}|_\eta < \infty\}$. Clearly, if $d \in \mathbb{N}^+$, then $\mathbb{Z}^d_* = \mathbb{Z}^d \backslash \{0\}$. Let $\mathbb{T}^d_h$ be the complexified torus defined by

$$\mathbb{T}^d_h := \{x \in \mathbb{C}^d : \operatorname{Re} x_j \in \mathbb{T}, |\operatorname{Im} x_j| < h\langle j\rangle^\eta\},$$

and denote by $C^\omega(\mathbb{T}^d_h, \mathbb{C})$ the space of bounded analytic complex-valued functions equipped with norm $\|v\|_h = \sum_{\mathbf{k}} |\hat{v}_{\mathbf{k}}| \mathrm{e}^{h|\mathbf{k}|_\eta}$.

Let $d \in \mathbb{N}^+ \cup \{\infty\}$. We assume that the frequency $\alpha = (\alpha_j)$ belongs to the d-dimensional cube $\mathcal{R}_0 := [1, 2]^d$, which is endowed with the probability measure $\mathcal{P}$ induced by the product measure of the d-dimensional cube $\mathcal{R}_0$. The following almost-periodic Diophantine frequencies were first defined by Bourgain [16]:

Definition 1.1 ([16]). Given $\gamma \in (0, 1)$, $\tau > 1$, we denote by $\mathrm{DC}^d_{\gamma,\tau}$ the set of Diophantine frequencies

$$\inf_{n\in\mathbb{Z}} |\langle \mathbf{k}, \alpha\rangle - 2\pi n| \geqslant \gamma \prod_{j\in\mathbb{N}} \frac{1}{1 + \langle j\rangle^\tau |\mathbf{k}_j|^\tau}, \quad \text{for all } \mathbf{k} \in \mathbb{Z}^d_*, \tag{1.5}$$

and denote $\mathrm{DC}^d = \bigcup_{\gamma>0} \mathrm{DC}^d_{\gamma,\tau}$.

As proved in [12, 16], for any $\tau > 1$, Diophantine frequencies $\mathrm{DC}^d_{\gamma,\tau}$ are typical in the set $\mathcal{R}_0$ in the sense that there exists a positive constant $C(\tau)$ such that

$$\mathcal{P}(\mathcal{R}_0 \backslash \mathrm{DC}^d_{\gamma,\tau}) \leqslant C(\tau)\gamma.$$

We also denote

$$\Gamma_r = \mathbb{Z}^d_* \cap \Big\{\mathbf{k} : \sum_j \langle j\rangle^\eta \mathbf{k}_j w_j \geqslant r|\mathbf{k}|_\eta, \text{ with } \sum_j w_j = 1, w_j > 0\Big\}, \tag{1.6}$$

where Γ_r is an integer cone whose angle is less than π strictly, as shown in Figure 1. Once we have these, now we are ready to state our main theorem.

Theorem 1.2. *Let $d \in \mathbb{N}^+ \cup \{\infty\}$, $\eta > 0$, $h > 0$, $r \in (0, 1]$, $\alpha \in \mathrm{DC}^d$. Suppose that*

$$v(x) = \sum_{\mathbf{k}\in\Gamma_r} \hat{v}_{\mathbf{k}} \mathrm{e}^{\mathrm{i}\langle \mathbf{k}, x\rangle} \in C^\omega(\mathbb{T}^d_h, \mathbb{C}).$$

Then there exists $\lambda_0 = \lambda_0(\eta, h, r, \alpha, \|v\|_h)$ such that $\Sigma_{\lambda v,\alpha} = [-2, 2]$ if $|\lambda| < \lambda_0$.

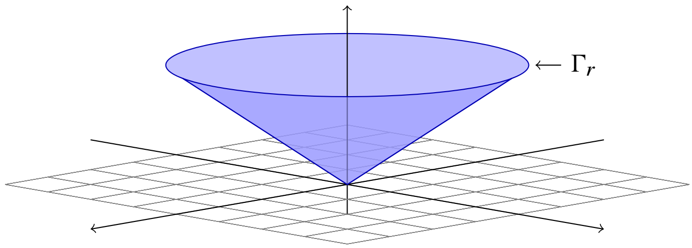

Figure 1. Integer cone.

Remark 1.1. The smallness condition of the coupling constant $|\lambda|$ is necessary due to Theorem 1.1.

Remark 1.2. If $d < \infty$, then the assumption (1.5) can be replaced by the standard Diophantine condition

$$\mathrm{DC}^d_{\gamma,\tau'} := \Big\{\alpha \in \mathbb{R}^d : \inf_{n\in\mathbb{Z}} |\langle \mathbf{k}, \alpha\rangle - 2\pi n| > \frac{\gamma}{|\mathbf{k}|^{\tau'}} \text{ for all } \mathbf{k} \in \mathbb{Z}^d\backslash\{0\}\Big\}.$$

Remark 1.3. If $\hat{v}_{\mathbf{k}} \in \mathbb{R}$ for any $\mathbf{k} \in \Gamma_r$, then the potential v is $\mathcal{PT}$ symmetric. However, the key assumption for us is the cone structure Γ_r, whether $\hat{v}_{\mathbf{k}}$ is real or not is not important.

Note that Sarnak [55] also extended his result to multi-frequency case: for any Diophantine frequency α, he constructed one $\mathcal{PT}$ symmetric v, and showed that $H_{\lambda v,\alpha,x}$ is isospectral to the discrete free Laplacian if $|\lambda|$ is small enough.[1] Our result not only generalizes Sarnak's result [55] to the almost-periodic case, but also (more importantly) finds that the cone structure Γ_r in (1.6) for the Fourier coefficients of v is a sufficient (almost optimal) condition to ensure that $H_{\lambda v,\alpha,x}$ is isospectral to the discrete free Laplacian.

To understand Theorem 1.2 more clearly, we look at the case $d = 1$, where $\Gamma_r = \mathbb{Z}^+$, consequently we have the following.

Corollary 1.1. *Suppose that $h > 0$, $\alpha \in \mathrm{DC}^1$, and that $v(x) = \sum_{\mathbf{k}>0} \hat{v}_{\mathbf{k}} e^{i\mathbf{k}x}$. If $|\lambda| < \lambda_1(h, \alpha, \|v\|_h)$ which is small enough, then $\Sigma_{\lambda v,\alpha} = [-2, 2]$.*

We remark that the phenomenon described in Theorem 1.2 is totally different from the self-adjoint case where having open gaps is a typical phenomenon [3, 24, 30, 31, 53]. The most important example is the almost Mathieu operator:

$$(H_{2\lambda\cos,\alpha,x}\psi)_n = \psi_{n+1} + \psi_{n-1} + 2\lambda\cos(x + n\alpha)\psi_n,$$

[1] While Sarnak stated the result in the continuous setting, the method can clearly be carried out in the discrete case, as point out by him at the end of [55, Section 2].

whose spectrum is a Cantor set for all $\lambda \neq 0$, $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and all $x \in \mathbb{T}$ [5]. We also remark that the cone structure assumption (1.6) is necessary due to the following counter-example where the angle of the cone is π.

Proposition 1.1. *Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $|\lambda| \in (0,1)$, $|\epsilon| < -\log|\lambda|$, and*

$$v_\epsilon(x) = 2\lambda\cos(x + \mathrm{i}\epsilon).$$

Then $\Sigma_{v_\epsilon,\alpha} = \Sigma_{2\lambda\cos,\alpha}$ is a Cantor set.

1.2. Lyapunov exponent

The Lyapunov exponents of $H_{\lambda v,\alpha,x}$ in Theorem 1.2 can be exactly calculated. Recall that the eigenvalue equations $H_{\lambda v,\alpha,x}\psi = E\psi$ are equivalent to a certain family of the discrete dynamical systems, the so called *Schrödinger cocycle* $(\alpha, S_{E,\lambda v}) \in \mathbb{T}^d \times \mathrm{SL}(2,\mathbb{C})$, i.e.,

$$\begin{pmatrix} \psi_{n+1} \\ \psi_n \end{pmatrix} = S_{E,\lambda v}(x + n\alpha)\begin{pmatrix} \psi_n \\ \psi_{n-1} \end{pmatrix}, \quad \text{where } S_{E,\lambda v}(x) = \begin{pmatrix} E - \lambda v(x) & -1 \\ 1 & 0 \end{pmatrix}.$$

Any formal solution $(\psi_n)_{n\in\mathbb{Z}}$ can be reconstructed from the transfer matrix S_n, which is defined by $S_0 = \mathrm{id}$, and for $n \geqslant 1$, by

$$S_n(x) = S_{E,\lambda v}(x + (n-1)\alpha)\cdots S_{E,\lambda v}(x), \quad S_{-n}(x) = S_n(x - n\alpha)^{-1}.$$

The Lyapunov exponent of $(\alpha, S_{E,\lambda v})$, denoted by $L(E)$, is defined by

$$L(E) = \lim_{n\to\infty}\frac{1}{n}\int_{\mathbb{T}^d}\log\|S_n(x)\|\,\mathrm{d}x.$$

In general, it is hard to give a precise expression of the Lyapunov exponent $L(E)$ except for some very special examples. It is well known that $L(E) = \max\{0, \log|\lambda|\}$ in the spectrum [2,18] for the almost Mathieu operator, however, the formula of $L(E)$ for E outside the spectrum is not known. For other analytic quasi-periodic operators, it is almost impossible to have a precise expression of $L(E)$ even though E is in the spectrum [2]. Up to now, the only quasi-periodic Schrödinger operator whose Lyapunov exponent can be calculated explicitly for all $E \in \mathbb{C}$ is the Maryland model, but this is due to the unboundedness and monotonicity of the potential [33,38].

In the following, for any $E \in \mathbb{C}$, we give the exact expression of $L(E)$ for the operators defined in Theorem 1.1 and Theorem 1.2. For Sarnak's example (1.3), we have the next theorem.

Theorem 1.3. *Let* $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ *and* $v(x) = \lambda e^{ix}$. *Then for any* $\lambda \in \mathbb{R}\backslash\{0\}$, *its Lyapunov exponent satisfies*

$$L(E) = \max\{0, \log|\lambda|\}, \quad \text{for all } E \in \Sigma_{\lambda \exp,\alpha}. \tag{1.7}$$

Moreover,

$$L(E) = \max\Big\{\log\Big|\frac{E}{2} + \frac{\sqrt{E^2-4}}{2}\Big|, \log|\lambda|\Big\} \quad \text{for all } E \in \mathbb{C}. \tag{1.8}$$

Remark 1.4. Equation (1.7) was also proved by Borisov and Fedotov [15], while (1.8) is totally new. Our proofs are new and based on Avila's global theory [2].

If the potential has cone structure Γ_r, Theorem 1.2 states that the corresponding Schrödinger operator is isospectral to the discrete free Laplacian. The following theorem shows that they also share the same Lyapunov exponent with the discrete free Laplacian.

Theorem 1.4. *Under the same assumptions as in Theorem* 1.2, *we have*

$$L(E) = \log\Big|\frac{E}{2} + \frac{\sqrt{E^2-4}}{2}\Big| \quad \text{for all } E \in \mathbb{C}.$$

In particular,

$$L(E) = 0 \quad \text{for all } E \in \Sigma_{\lambda v,\alpha}.$$

1.3. Failure of Moser–Pöschel argument

Almost reducibility is an effective approach to deal with the spectral problems of almost-periodic operators, especially for the small potentials [6, 19, 20, 23, 24, 46, 51]. Now, we give the definition of almost reducibility. Recall that two cocycles $(\alpha, A), (\alpha, A') \in \mathbb{T}^d \times C^\omega(\mathbb{T}^d, \mathrm{SL}(2,\mathbb{C}))$ are analytically conjugated if there exists $B \in C^\omega(2\mathbb{T}^d, \mathrm{SL}(2,\mathbb{C}))$ such that

$$B(x+\alpha)^{-1}A(x)B(x) = A'(x).$$

We say the cocycle (α, A) is *almost reducible* if the closure of its analytic conjugates contains a constant matrix. Moreover, we say the cocycle is *reducible* if it is analytically conjugated to a constant matrix. For the self-adjoint operator, the reducibility of $(\alpha, S_{E,\lambda v})$ is closely related to the rotation number of the cocycle. Specifically, for small analytic potentials and $\alpha \in \mathrm{DC}^d$ with $d \in \mathbb{N}^+$, Eliasson's famous result [24] states that $(\alpha, S_{E,\lambda v})$ is reducible if the rotation number is Diophantine with respect to α or rationally dependent. For the non-self-adjoint operator, the rotation number is not well defined since the projection of $\mathrm{SL}(2,\mathbb{C})$ cocycle is not a circle diffeomorphism. However, we can find another object ρ that plays the same role as the rotation number. Indeed, for any $E \in [-2,2]$, if we define

$$\rho = \rho(E) := \arccos\Big(\frac{E}{2}\Big) \mod \pi,$$

then we have the following.

Theorem 1.5. *Under the same assumptions as in Theorem 1.2, we have that*

(1) *if* $\rho \in \mathrm{DC}(\alpha) = \bigcup_{\kappa>0} \mathrm{DC}_{\kappa,\tau}(\alpha)$, *where*

$$\mathrm{DC}_{\kappa,\tau}(\alpha) := \Big\{\rho \in \mathbb{R} : \|2\rho - \langle \mathbf{k},\alpha\rangle\|_{\mathbb{T}} > \kappa \prod_{j\in\mathbb{N}} \frac{1}{1+\langle j\rangle^{\tau}|\mathbf{k}_j|^{\tau}} \text{ for all } \mathbf{k}\in\mathbb{Z}^d_*\Big\},$$

then $(\alpha, S_{E,\lambda v})$ *is reducible to* $\Big(\alpha, \begin{pmatrix} e^{i\rho} & 0 \\ 0 & e^{-i\rho}\end{pmatrix}\Big)$;

(2) *if* $2\rho = \langle \mathbf{k},\alpha\rangle \mod 2\pi$ *for some* $\mathbf{k}\in\mathbb{Z}^d_*$, *then* $(\alpha, S_{E,\lambda v})$ *is reducible to* (α, A) *where* $A = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}$ *or* $A = \begin{pmatrix} 1 & \zeta \\ 0 & 1\end{pmatrix}$ *with* $\zeta \neq 0$.

In the self-adjoint case, by the well-known Moser–Pöschel argument [51], it is known that $(\alpha, S_{E,\lambda v})$ is reducible to the identity if and only if E is located at the edges of the collapsed gaps. However, in the non-self-adjoint case, one may need more caution due to the following result.

Theorem 1.6. *Let* $d \in \mathbb{N}^+ \cup \{\infty\}$, $\alpha \in \mathrm{DC}^d$, *and* $v(x) = e^{i\langle \mathbf{m},x\rangle}$. *Then there exists* $E \in (-2,2)$ *such that* $(\alpha, S_{E,\lambda v})$ *is reducible to* $\begin{pmatrix} 1 & \zeta \\ 0 & 1\end{pmatrix}$ *with* $\zeta \neq 0$, *provided that* $|\lambda|$ *is sufficiently small.*

By Theorem 1.2 and Theorem 1.6, we see that even though all gaps are collapsed, the cocycle $(\alpha, S_{E,\lambda v})$ may still not be reducible to identity. Therefore, the Moser–Pöschel argument [51] does not work for non-self-adjoint almost-periodic operators.

1.4. Methods and mechanism

Although Theorem 1.2 is an extension of Sarnak's result [55], our method is completely different. In fact, the operator (1.3) is very special, as taking Fourier transforms in trying to solve $(H_{\lambda \exp,\alpha} - E)\psi = 0$, one finds

$$\lambda\hat{\psi}(p+\alpha) = (E - 2\cos p)\hat{\psi}(p),$$

which is easily iterated. Sarnak [55] studied the behavior of $\prod_{p=0}^{n}(E - 2\cos(p+\alpha))$ by combining Birkhoff ergodic theorem and nature of $\alpha\beta$-sets studied initially by Engelking [25] and Katznelson [42]. As we can see, the method of [55] depends on the duality transformation, clearly fails if $\alpha \in \mathbb{T}^d$ with $d = \infty$, i.e., the true almost-periodic case.

We know that the spectrum of self-adjoint operators always stays in the real axis, and the non-self-adjointness would push the spectrum out. In this paper, we give a criterion for non-self-adjoint almost-periodic Schrödinger operators (1.2) to have real interval spectrum. More importantly, one can see the mechanism of the spectrum being real and staying an interval from our proof. Let us explain the main ideas. Our approach is based on the quantitative almost reducibility of the Schrödinger cocycle. In the self-adjoint case [24, 46], the potential v is real ($\hat{v}_{-\mathbf{k}} = \overline{\hat{v}_{\mathbf{k}}}$) which implies that *double resonances*[2]

$$\|\langle \mathbf{k}, \alpha \rangle \pm 2\rho\|_{\mathbb{T}} \sim 0$$

must occur, which causes the uniform hyperbolicity of the Schrödinger cocycle and makes the corresponding gap open. In the non-self-adjoint case, the potential is not real anymore, which gives us a chance to avoid the *double resonances*. For the potential v defined in Theorem 1.2, we have $\hat{v}_{\mathbf{k}} \cdot \hat{v}_{-\mathbf{k}} = 0$. During the KAM iteration steps, we will prove that at the cost of shrinking r, the cone structure Γ_r is preserved and there exists only *single resonance* ($\mathbf{k}$ or $-\mathbf{k}$) in each iteration step. Thus, the interval spectrum may survive.

More precisely, we can prove that the Schrödinger cocycle is reducible to $(\alpha, A_n e^{F_n})$ where $A_n \in \mathrm{SL}(2, \mathbb{C})$ with eigenvalues $e^{\pm i\xi_n}$ and F_n goes to zero. The structure of Γ_r guarantees that the average of the perturbation F_n is always zero, and thus $\operatorname{Im} \xi_n$ is fixed during the KAM iteration, this is the reason why the Schrödinger operator (1.2) has real spectrum. In addition, we will prove that the Schrödinger cocycle $(\alpha, S_{E,\lambda v})$ is always almost reducible to the Laplace cocycle $(\alpha, S_{E,0})$, which implies that the Schrödinger operator shares both the spectrum and the Lyapunov exponent with the discrete free Laplacian.

Organization of the paper. The rest of this paper is organized in the following way. Some basic definitions are given in Section 2. In Section 3, we study the one step of KAM iteration for $\mathrm{SL}(2, \mathbb{C})$-valued cocycle with integer cone condition. In Section 4, we obtain the reducibility of $\mathrm{SL}(2, \mathbb{C})$-valued cocycle. In Section 5, as applications, we prove Theorem 1.2, Theorem 1.4, Theorem 1.5 and Theorem 1.6. Finally, we prove Theorem 1.1, Theorem 1.3 and Proposition 1.1 in Section 6. In Appendix A, we give the proof of Lemma 3.1.

[2] Second Melnikov condition in Hamiltonian systems.

2. Preliminary

2.1. Almost-periodic cocycle, Lyapunov exponent

Let Ω be a compact metric space and (Ω, ν, T) be ergodic. A cocycle $(\alpha, A) \in \mathbb{R}\backslash\mathbb{Q} \times C^{\omega}(\Omega, \mathrm{SL}(2,\mathbb{C}))$ is a linear skew-product:

$$\begin{aligned}(T, A)\colon \Omega \times \mathbb{C}^2 &\to \Omega \times \mathbb{C}^2,\\ (x, \phi) &\mapsto (Tx, A(x)\phi).\end{aligned}$$

For $n \in \mathbb{Z}$, A_n is defined by $(T, A)^n = (T^n, A_n)$. Thus, $A_0(x) = \mathrm{id}$,

$$A_n(x) = \prod_{j=n-1}^{0} A(T^j x) = A(T^{n-1}x) \cdots A(Tx)A(x) \quad \text{for all } n \geqslant 1,$$

and $A_{-n}(x) = A_n(T^{-n}x)^{-1}$. The Lyapunov exponent is defined as

$$L(T, A) = \lim_{n\to\infty} \frac{1}{n} \int_{\Omega} \log \|A_n(x)\| \, \mathrm{d}\,\nu(x).$$

We are mainly interested in the case $\Omega = \mathbb{T}^d$, $\mathrm{d}\,\nu = \mathrm{d}\,x$ is Lebesgue measure, and $T = R_\alpha$, with $(1, \alpha)$ rationally independent. If $d \in \mathbb{N}^+$, then $(\alpha, A) =: (R_\alpha, A)$ defines a quasi-periodic cocycle. If $d = \infty$, then (α, A) defines an almost-periodic cocycle.

We say (α, A) is *uniformly hyperbolic* if there exist two continuous functions $u, s\colon \mathbb{T}^d \to \mathbb{PC}^2$, called the *unstable* and *stable directions* such that for any $n \geqslant 0$,

$$\begin{aligned}\|A_n(x)\phi\| &\leqslant C\mathrm{e}^{-cn}\|\phi\| \qquad \text{for all } \phi \in s(x),\\ \|A_{-n}(x)\phi\| &\leqslant C\mathrm{e}^{-cn}\|\phi\| \quad \text{for all } \phi \in u(x),\end{aligned}$$

for some constants $C, c > 0$. Moreover, $u(\cdot), s(\cdot)$ are invariant under the dynamics:

$$A(x) \cdot u(x) = u(x + \alpha), \quad A(x) \cdot s(x) = s(x + \alpha),$$

where $A \cdot x$ denoted the $\mathrm{SL}(2, \mathbb{C})$ action on the projective space $\mathbb{PC}^2$. If (α, A) is uniformly hyperbolic, then $L(A) > 0$. From now on, $(\alpha, A) \in \mathcal{UH}$ means (α, A) is uniformly hyperbolic.

2.2. Schrödinger operators and Schrödinger cocycles

Let Ω be a compact metric space, $T\colon \Omega \to \Omega$ a homeomorphism, and $v\colon \Omega \to \mathbb{C}$ a complex-valued continuous function. We consider the following complex-valued dynamical defined Schrödinger operators:

$$(H_x\psi)_n = \psi_{n+1} + \psi_{n-1} + v(T^n x)\psi_n, \quad n \in \mathbb{Z},$$

and denote Σ_x by the spectrum of H_x. Note that any formal solution $\psi = (\psi_n)_{n\in\mathbb{Z}}$ of $H_x\psi = E\psi$ satisfies

$$\begin{pmatrix}\psi_{n+1}\\ \psi_n\end{pmatrix} = S_{E,v}(T^n x)\begin{pmatrix}\psi_n\\ \psi_{n-1}\end{pmatrix}, \quad n \in \mathbb{Z},$$

where

$$S_{E,v}(x) := \begin{pmatrix}E - v(x) & -1\\ 1 & 0\end{pmatrix}, \quad E \in \mathbb{C}.$$

We call $(T, S_{E,v})$ *Schrödinger cocycles*. The spectrum Σ_x is closely related with the dynamical behavior of the Schrödinger cocycle $(T, S_{E,v})$. In the self-adjoint case, i.e., the potential v is real-valued, then by the well-known result of Johnson [41], $E \notin \Sigma_x$ if and only if $(T, S_{E,v}) \in \mathcal{UH}$. The following result extends Johnson's result [41] to the non-self-adjoint case.

Proposition 2.1 ([27]). *Suppose that $v\colon \Omega \to \mathbb{C}$ a complex-valued continuous function, (Ω, T) is minimal. Then there is some $\Sigma \subset \mathbb{C}$ such that $\Sigma_x = \Sigma$ for all $x \in \Omega$. Moreover, $E \notin \Sigma$ if and only if $(T, S_{E,v}) \in \mathcal{UH}$.*

2.3. Global theory of one-frequency quasi-periodic cocycles.

We give a short review of Avila's global theory of one-frequency quasi-periodic $SL(2,\mathbb{C})$ cocycles [2]. Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, suppose that $A \in C^\omega(\mathbb{T}, SL(2,\mathbb{C}))$ admits a holomorphic extension to $\{|\operatorname{Im} x| < h\}$. Then for $|\epsilon| < h$, we define $A_\epsilon \in C^\omega(\mathbb{T}, SL(2,\mathbb{C}))$ by $A_\epsilon(\cdot) = A(\cdot + i\epsilon)$, and define the acceleration of (α, A_ϵ) as follows:

$$\omega(\alpha, A_\epsilon) = \lim_{h\to 0+} \frac{L(\alpha, A_{\epsilon+h}) - L(\alpha, A_\epsilon)}{h}.$$

It follows from the convexity and continuity of the Lyapunov exponent that the acceleration is an upper semi-continuous function of parameter ϵ. The key property of the acceleration is that it is quantized.

Theorem 2.1 (Quantization of acceleration [2]). *Suppose that*

$$(\alpha, A) \in \mathbb{R}\backslash\mathbb{Q} \times C^\omega(\mathbb{T}, SL(2,\mathbb{C})),$$

then $\omega(\alpha, A_\epsilon) \in \mathbb{Z}$.

For uniformly hyperbolic cocycles, Avila [2] proved the following equivalent characterization.

Proposition 2.2 ([2]). *Let $(\alpha, A) \in \mathbb{R}\backslash\mathbb{Q} \times C^\omega(\mathbb{T}, SL(2,\mathbb{C}))$. Assume that $L(\alpha, A) > 0$. Then $(\alpha, A) \in \mathcal{UH}$ if and only if $L(\alpha, A(\cdot + i\epsilon))$ is affine with respect to ϵ around $\epsilon = 0$.*

3. Quantitative almost reducibility

As mentioned in the introduction, our approach is based on quantitative almost reducibility. The philosophy is that nice quantitative almost reducibility brings the precise estimates of the growth on the Schrödinger cocycle.

3.1. Auxiliary Banach space

We first introduce the auxiliary Banach space related to the integer cone Γ_r. Recall that the integer cone Γ_r is defined as

$$\Gamma_r = \mathbb{Z}_*^d \cap \{\mathbf{k} : [\![\mathbf{k}]\!] \geqslant r|\mathbf{k}|_\eta\},$$

where $[\![\mathbf{k}]\!] = \sum_j \langle j\rangle^\eta \mathbf{k}_j w_j$ with $\sum_j w_j = 1, w_j > 0$. For a given integer cone Γ_r, we define the space

$$\mathcal{B}_{h,r}[*] = \{F \in C^\omega(\mathbb{T}_h^d, *) : \widehat{F}_{\mathbf{k}} = 0 \text{ for all } \mathbf{k} \in \mathbb{Z}^d \backslash \Gamma_r\},$$

where $*$ could be $\mathbb{C}$ or $\mathrm{sl}(2,\mathbb{C})$, and we abbreviate $\mathcal{B}_{h,r}[\mathrm{sl}(2,\mathbb{C})]$ by $\mathcal{B}_{h,r}$ without ambiguity. Since $\mathbf{0} \notin \Gamma_r$, it holds that $\widehat{F}_{\mathbf{0}} = 0$ for any $F \in \mathcal{B}_{h,r}[*]$.

For any set $W \subset \Gamma_r$ and $N > 0$, we define the truncated set and residual set of W as

$$\mathcal{T}_N W = \{\mathbf{k} \in W : |\mathbf{k}|_\eta \leqslant N\}, \quad \mathcal{R}_N W = \{\mathbf{k} \in W : |\mathbf{k}|_\eta > N\}.$$

And we also define the truncated operator $\mathcal{T}_N$ and residual operator $\mathcal{R}_N$ by

$$(\mathcal{T}_N F)(x) = \sum_{\mathbf{k} \in \mathcal{T}_N \Gamma_r} \widehat{F}_{\mathbf{k}} e^{\mathrm{i}\langle \mathbf{k}, x\rangle}, \quad (\mathcal{R}_N F)(x) = \sum_{\mathbf{k} \in \mathcal{R}_N \Gamma_r} \widehat{F}_{\mathbf{k}} e^{\mathrm{i}\langle \mathbf{k}, x\rangle}.$$

The following are some basic properties of the space $\mathcal{B}_{h,r}[*]$.

Proposition 3.1. *For any $0 < r < r' \leqslant 1$, $h > 0$, we have the following:*

(1) *$(\mathcal{B}_{h,r}[*], \|\cdot\|_h)$ is a Banach space with $\mathcal{B}_{h,r'} \subset \mathcal{B}_{h,r}$;*

(2) *$\|[F,G]\|_h \leqslant 2\|F\|_h\|G\|_h$ where $[\cdot,\cdot]$ is the Lie bracket defined by $[F,G] = FG - GF$, and thus $(\mathcal{B}_{h,r}, [\cdot,\cdot])$ is a Lie algebra.*

Proof. Proposition 3.1 (1) follows directly from the definition, so we only need to check the second one. Let $F, G \in \mathcal{B}_{h,r}$ with expansions

$$F(x) = \sum_{\mathbf{k} \in \Gamma_r} \widehat{F}_{\mathbf{k}} e^{\mathrm{i}\langle \mathbf{k}, x\rangle}, \quad G(x) = \sum_{\mathbf{n} \in \Gamma_r} \widehat{G}_{\mathbf{n}} e^{\mathrm{i}\langle \mathbf{n}, x\rangle},$$

where $\widehat{F}_{\mathbf{k}}, \widehat{G}_{\mathbf{n}} \in \mathrm{sl}(2,\mathbb{C})$ for any $\mathbf{n}, \mathbf{k} \in \Gamma_r$. By direct calculation,

$$[F,G](x) = \sum_{\mathbf{k},\mathbf{n} \in \Gamma_r} [\widehat{F}_{\mathbf{k}}, \widehat{G}_{\mathbf{n}}] e^{\mathrm{i}\langle \mathbf{k}+\mathbf{n}, x\rangle}. \tag{3.1}$$

Since $[\mathbf{k}] > r|\mathbf{k}|_\eta$, $[\mathbf{n}] > r|\mathbf{n}|_\eta$, we have $[\mathbf{k}+\mathbf{n}] > r(|\mathbf{k}|_\eta + |\mathbf{n}|_\eta) \geqslant r|\mathbf{k}+\mathbf{n}|_\eta$, which means that $\mathbf{k}+\mathbf{n} \in \Gamma_r$. On the other hand, rewrite (3.1) as

$$[F,G](x) = \sum_{\mathbf{m}\in\Gamma_r} \Big(\sum_{\mathbf{n}+\mathbf{k}=\mathbf{m}} [\widehat{F}_{\mathbf{k}}, \widehat{G}_{\mathbf{n}}] \Big) e^{i\langle \mathbf{m}, x\rangle}.$$

Then Proposition 3.1 (2) follows from $[\widehat{F}_{\mathbf{k}}, \widehat{G}_{\mathbf{n}}] \in \mathrm{sl}(2,\mathbb{C})$,

$$\begin{aligned} \|FG\|_h &= \sum_{\mathbf{n}} \Big(\sum_{\mathbf{k}} |\widehat{F}_{\mathbf{k}} \widehat{G}_{\mathbf{n}-\mathbf{k}}| \Big) e^{h|\mathbf{n}|_\eta} \\ &\leqslant \sum_{\mathbf{m}} \Big(\sum_{\mathbf{k}} |\widehat{F}_{\mathbf{k}}||\widehat{G}_{\mathbf{m}}| \Big) e^{h|\mathbf{m}+\mathbf{k}|_\eta} \\ &\leqslant \Big(\sum_{\mathbf{k}} |\widehat{F}_{\mathbf{k}}| e^{h|\mathbf{k}|_\eta} \Big) \Big(\sum_{\mathbf{m}} |\widehat{F}_{\mathbf{m}}| e^{h|\mathbf{m}|_\eta} \Big) = \|F\|_h \|G\|_h, \end{aligned}$$

and the same estimate on $\|GF\|_h$. ∎

3.2. Non-resonance cancellation lemma

We give a non-resonance cancellation lemma, which serves as the starting point of our proof. Let $A \in \mathrm{SL}(2,\mathbb{C})$. For any $Y \in \mathcal{B}_{h,r}$, we define the linear operator L_A by

$$(L_A Y)(x) := A^{-1} Y(x+\alpha) A - Y(x).$$

Suppose that $\mathcal{B}_{h,r} = \mathcal{B}_{h,r}^{\mathrm{nre}}(\sigma) \oplus \mathcal{B}_{h,r}^{\mathrm{re}}(\sigma)$, where $\mathcal{B}_{h,r}^{\mathrm{nre}}(\sigma)$ is the closed invariant subspace in $\mathcal{B}_{h,r}$ such that L_A restricted on $\mathcal{B}_{h,r}^{\mathrm{nre}}(\sigma)$ is invertible and

$$\|L_A^{-1}\| \leqslant \frac{1}{\sigma} \quad \text{on } \mathcal{B}_{h,r}^{\mathrm{nre}}(\sigma).$$

In the following lemma, we prove that all non-resonant terms in the perturbation can be eliminated.

Lemma 3.1 ([19,35]). *Let $d \in \mathbb{N}^+ \cup \{\infty\}$, $h > 0$, $r \in (0,1]$, $\alpha \in \mathbb{T}^d$, $\sigma > 0$. Suppose that $A \in \mathrm{SL}(2,\mathbb{C})$, and $F \in \mathcal{B}_{h,r}$ with*

$$\|F\|_h < \varepsilon < \min\{10^{-8}, \sigma^2\}.$$

Then there exist $Y \in \mathcal{B}_{h,r}^{\mathrm{nre}}(\sigma)$ and $F^{\mathrm{re}} \in \mathcal{B}_{h,r}^{\mathrm{re}}(\sigma)$ such that e^Y conjugates the cocycle (α, Ae^F) to $(\alpha, Ae^{F^{\mathrm{re}}})$, i.e.,

$$e^{-Y(x+\alpha)} A e^{F(x)} e^{Y(x)} = A e^{F^{\mathrm{re}}(x)},$$

with $\|Y\|_h \leqslant \varepsilon^{\frac{1}{2}}$, $\|F^{\mathrm{re}}\|_h \leqslant 2\varepsilon$ and $\|F^{\mathrm{re}} - \mathbb{P}_{\mathrm{re}} F\|_h \leqslant 2\varepsilon^{\frac{4}{3}}$.

3.3. One step of KAM iteration

In this section, we give one step of KAM iteration for $(\alpha, A\mathrm{e}^{F(x)})$ with $A \in \mathcal{M} \subset \mathrm{SL}(2,\mathbb{C})$ and $F(x) \in \mathcal{B}_{h,r}$, where

$$\mathcal{M} := \left\{\begin{pmatrix} \mathrm{e}^{\mathrm{i}\xi} & \zeta \\ 0 & \mathrm{e}^{-\mathrm{i}\xi} \end{pmatrix} : \xi, \zeta \in \mathbb{C}\right\} \cup \left\{\begin{pmatrix} \mathrm{e}^{\mathrm{i}\xi} & 0 \\ \zeta & \mathrm{e}^{-\mathrm{i}\xi} \end{pmatrix} : \xi, \zeta \in \mathbb{C}\right\}.$$

To eliminate the perturbation $F(x)$ in the cocycle, we need to deal with non-resonant case and resonant case separately. Here we say A is *non-resonant up to N*, denoted by $A \in \mathcal{NR}(N,\delta)$, if for any $\mathbf{k} \in \mathcal{T}_N\Gamma_r$,

$$|\mathrm{e}^{\mathrm{i}(\langle \mathbf{k},\alpha\rangle \pm 2\xi)} - 1| \geqslant \delta.$$

Otherwise, we say A is *resonant* and denoted by $A \in \mathcal{RS}(N,\delta)$, which means there is a $\mathbf{k}^* \in \mathcal{T}_N\Gamma_r$ such that

$$|\mathrm{e}^{\mathrm{i}(\langle \mathbf{k}^*,\alpha\rangle + 2\xi)} - 1| < \delta \quad \text{or} \quad |\mathrm{e}^{\mathrm{i}(\langle \mathbf{k}^*,\alpha\rangle - 2\xi)} - 1| < \delta.$$

In this subsection, we always fix $N = \frac{2|\log\varepsilon|}{h-h_+}$ where $h_+ \in (0,h)$. Once we have these, we introduce the following key quantitative almost reducibility result, which gives the one step of KAM iteration.

Proposition 3.2. *Let $d \in \mathbb{N}^+ \cup \{\infty\}$, $\eta > 0$, $h > 0$, $r \in (0,1]$, $\gamma \in (0,1)$, $\tau > 1$, $\alpha \in \mathrm{DC}^d_{\gamma,\tau}$. Suppose that $A \in \mathcal{M}$ with $|\operatorname{Im}\xi| \leqslant \frac{1}{2}$, $F \in \mathcal{B}_{h,r}$, then for any $h_+ \in (0,h)$, $r_+ \in (0,r)$, there exist $\varepsilon = \varepsilon(\eta, h, h_+, r, r_+, \gamma, \tau, |\zeta|)$, $c = c(\eta,\gamma,\tau)$ such that if*

$$\|F\|_h < \varepsilon < \frac{c}{(1+|\zeta|)^{10}} \min\{\mathrm{e}^{-\left(\frac{1}{h-h_+}\right)^{\frac{10}{\eta}}}, \mathrm{e}^{-\left(\frac{1}{r-r_+}\right)^{\frac{10}{\eta}}}\}, \tag{3.2}$$

then there exist $B \in C^\omega(2\mathbb{T}^d_h, \mathrm{SL}(2,\mathbb{C}))$, $A_+ \in \mathcal{M}$, and $F_+ \in \mathcal{B}_{h_+,r_+}$ such that

$$B(x+\alpha)^{-1} A\mathrm{e}^{F(x)} B(x) = A_+\mathrm{e}^{F_+(x)}.$$

Moreover, we have the following estimates.

Non-resonant case. *If $A \in \mathcal{NR}(N, \varepsilon^{\frac{1}{10}})$, then $B(\cdot) = \mathrm{e}^{Y(\cdot)}$ with*

$$\|Y\|_h \leqslant \varepsilon^{\frac{1}{2}}, \quad \|F_+\|_{h_+} \leqslant 2\varepsilon^3, \quad A_+ = A.$$

Resonant case. *If $A \in \mathcal{RS}(N, \varepsilon^{\frac{1}{10}})$, then there exists $\mathbf{k}^* \in \mathcal{T}_N\Gamma_r$ such that*

(1) *A_+ takes the form*

$$A_+ = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\xi_+} & \zeta_+ \\ 0 & \mathrm{e}^{-\mathrm{i}\xi_+} \end{pmatrix} \quad \text{or} \quad A_+ = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\xi_+} & 0 \\ \zeta_+ & \mathrm{e}^{-\mathrm{i}\xi_+} \end{pmatrix},$$

where $\zeta_+ \in \mathbb{C}$, $\xi_+ = \xi - \frac{\langle \mathbf{k}^*, \alpha\rangle}{2}$ *with estimates*

$$|\xi_+| \leqslant \varepsilon^{\frac{1}{10}}, \quad |\zeta_+| \leqslant \varepsilon^{\frac{9}{10}};$$

(2) *it holds that*

$$\|B\|_0 \leqslant e^{|\log \varepsilon|^{\frac{2}{2+\eta}}}, \quad \|F_+\|_{h_+} \leqslant \varepsilon^{100}.$$

Proof. We distinguish the proof into two cases.

Case 1. *Non-resonant case.* Let $\sigma = \varepsilon^{\frac{1}{3}}$ and decompose $\mathcal{B}_{h,r}$ as $\mathcal{B}_{h,r}^{\mathrm{nre}}(\sigma) \oplus \mathcal{B}_{h,r}^{\mathrm{re}}(\sigma)$, where

$$\mathcal{B}_{h,r}^{\mathrm{nre}}(\sigma) = \{F \in \mathcal{B}_{h,r} : F(x) = \mathcal{T}_N F(x)\},$$
$$\mathcal{B}_{h,r}^{\mathrm{re}}(\sigma) = \{F \in \mathcal{B}_{h,r} : F(x) = \mathcal{R}_N F(x)\}. \tag{3.3}$$

It is easy to see that $\mathcal{B}_{h,r}^{\mathrm{nre}}(\sigma)$ is a closed invariant subspace of $\mathcal{B}_{h,r}$. Moreover, we have the following simple observation.

Lemma 3.2. *The operator* $L_A^{-1}\colon \mathcal{B}_{h,r}^{\mathrm{nre}}(\sigma) \to \mathcal{B}_{h,r}^{\mathrm{nre}}(\sigma)$ *is bounded with* $\|L_A^{-1}\| \leqslant \frac{1}{\sigma}$.

Proof. We only consider the case $A = \begin{pmatrix} e^{i\xi} & \zeta \\ 0 & e^{-i\xi} \end{pmatrix}$, the proof for the case $A = \begin{pmatrix} e^{i\xi} & 0 \\ \zeta & e^{-i\xi} \end{pmatrix}$ is similar. For any $F \in \mathcal{B}_{h,r}^{\mathrm{nre}}(\sigma)$, we only need to solve

$$A^{-1} Y(x+\alpha) A - Y(x) = F(x).$$

Expand $Y(x) = \sum_{\mathbf{k}} \widehat{Y}_{\mathbf{k}} e^{i\langle \mathbf{k}, x\rangle}$ and $F(x) = \sum_{\mathbf{k}} \widehat{F}_{\mathbf{k}} e^{i\langle \mathbf{k}, x\rangle}$ respectively. Comparing the Fourier coefficients, one obtains that for $\mathbf{k} \in \Gamma_r$,

$$\widehat{Y}_{\mathbf{k}}^{2,1} = \frac{\widehat{F}_{\mathbf{k}}^{2,1}}{e^{i(\langle \mathbf{k}, \alpha\rangle + 2\xi)} - 1}, \tag{3.4a}$$

$$\widehat{Y}_{\mathbf{k}}^{1,1} = -\widehat{Y}_{\mathbf{k}}^{2,2} = \frac{\widehat{F}_{\mathbf{k}}^{1,1} + \zeta e^{i(\langle \mathbf{k}, \alpha\rangle + \xi)} \widehat{Y}_{\mathbf{k}}^{2,1}}{e^{i\langle \mathbf{k}, \alpha\rangle} - 1}, \tag{3.4b}$$

$$\widehat{Y}_{\mathbf{k}}^{1,2} = \frac{\widehat{F}_{\mathbf{k}}^{1,2} + \zeta^2 e^{i\langle \mathbf{k}, \alpha\rangle} \widehat{Y}_{\mathbf{k}}^{2,1} - 2\zeta e^{i(\langle \mathbf{k}, \alpha\rangle - \xi)} \widehat{Y}_{\mathbf{k}}^{1,1}}{e^{i(\langle \mathbf{k}, \alpha\rangle - 2\xi)} - 1}. \tag{3.4c}$$

Recall the following estimate for $\alpha \in \mathrm{DC}_{\gamma,\tau}^d$:

Lemma 3.3 (Small denominators [50]). *Let* $d \in \mathbb{N}^+ \cup \{\infty\}$, $\tau > 1$, $\eta > 0$, *then for any* $\mathbf{k} \in \mathbb{Z}_*^d$ *we have the following estimate:*

$$\sup_{\mathbf{k} \in \mathbb{Z}_*^d, |\mathbf{k}|_\eta \leqslant N} \prod_{j \in \mathbb{N}} (1 + \langle j\rangle^\tau |\mathbf{k}_j|^\tau) \leqslant (1+N)^{C_1 N^{\frac{1}{\eta+1}}},$$

where $C_1 = C_1(\eta, \tau)$. *Moreover,*

$$\prod_{j\in\mathbb{N}}(1+\langle j\rangle^{\tau}|\mathbf{k}_j|^{\tau}) \leqslant (1+|\mathbf{k}|_{\eta})^{C_1|\mathbf{k}|_{\eta}^{\frac{1}{\eta+1}}}.$$

Combining Lemma 3.3 with (3.2), for any $\mathbf{k}\in\mathcal{T}_N\Gamma_r$ we have

$$\|\langle\mathbf{k},\alpha\rangle\|_{\mathbb{T}} \geqslant \gamma(1+N)^{-C_1N^{\frac{1}{\eta+1}}} > \varepsilon^{\frac{1}{10}}.$$

Besides, it follows from $A\in\mathcal{NR}(N,\varepsilon^{\frac{1}{10}})$ that $|e^{i(\langle\mathbf{k},\alpha\rangle\pm2\xi)}-1| \geqslant \varepsilon^{\frac{1}{10}}$ for any $\mathbf{k}\in\mathcal{T}_N\Gamma_r$. Thus, the denominators in (3.4) are well controlled and Lemma 3.2 follows. ■

By Lemma 3.1, there exist $Y\in\mathcal{B}_{h,r}^{\mathrm{nre}}(\sigma)$ and $F^{\mathrm{re}}\in\mathcal{B}_{h,r}^{\mathrm{re}}(\sigma)$ such that

$$e^{-Y(x+\alpha)}Ae^{F(x)}e^{Y(x)} = Ae^{F^{\mathrm{re}}(x)},$$

with the following estimates:

$$\|Y\|_h \leqslant \varepsilon^{\frac{1}{2}}, \quad \|F^{\mathrm{re}}\|_h \leqslant 2\varepsilon.$$

Let $B=e^{Y}$, $A_+=A$, and $F_+(x)=F^{\mathrm{re}}(x)$. By the construction in (3.3), F_+ can be expressed as

$$F_+(x) = \sum_{\mathbf{k}\in\mathcal{R}_N\Gamma_r}\widehat{F}^{\mathrm{re}}_{\mathbf{k}}e^{i\langle\mathbf{k},x\rangle}.$$

Therefore, for any $h_+\in(0,h)$, we have the estimate

$$\begin{aligned}\|F_+\|_{h_+} = \sum_{\mathbf{k}\in\mathcal{R}_N\Gamma_r}\|\widehat{F}^{\mathrm{re}}_{\mathbf{k}}\|e^{h_+|\mathbf{k}|_\eta} &\leqslant e^{-(h-h_+)N}\sum_{\mathbf{k}\in\mathcal{R}_N\Gamma_r}\|\widehat{F}^{\mathrm{re}}_{\mathbf{k}}\|e^{h|\mathbf{k}|_\eta}\\ &\leqslant 2e^{-(h-h_+)N}\|F\|_h < 2\varepsilon^3,\end{aligned}$$

where the last inequality follows from our choice that $N=\frac{2|\log\varepsilon|}{h-h_+}$.

Case 2. *Resonant case.* In view of $\alpha\in\mathrm{DC}^d_{\gamma,\tau}$ and $A\in\mathcal{RS}(N,\varepsilon^{\frac{1}{10}})$, Lemma 3.3 and (3.2) imply

$$\|2\,\mathrm{Re}\,\xi\|_{\mathbb{T}} > \|\langle\mathbf{k}^*,\alpha\rangle\|_{\mathbb{T}} + 2|\,\mathrm{Im}\,\xi| - \varepsilon^{\frac{1}{10}} \geqslant \frac{\gamma}{2}(1+N)^{-C_1N^{\frac{1}{\eta+1}}}, \tag{3.5}$$

as a consequence,

$$(|e^{i(\langle\mathbf{k}^*,\alpha\rangle+2\xi)}-1|-\varepsilon^{\frac{1}{10}})\cdot(|e^{i(\langle\mathbf{k}^*,\alpha\rangle-2\xi)}-1|-\varepsilon^{\frac{1}{10}}) < 0, \tag{3.6}$$

which shows that the concept of resonance is well defined. In fact, if

$$|e^{i(\langle\mathbf{k}^*,\alpha\rangle-2\xi)}-1| < \varepsilon^{\frac{1}{10}},$$

then (3.6) directly follows from (3.5) that

$$|e^{i(\langle \mathbf{k}^*,\alpha\rangle+2\xi)}-1| = \|\langle \mathbf{k}^*,\alpha\rangle-2\xi\|_{\mathbb{T}} \geqslant |4\xi|-\varepsilon^{\frac{1}{10}} \gg \varepsilon^{\frac{1}{10}}.$$

Note that (3.5) also implies that in the resonant case $\|2\,\mathrm{Re}\,\xi\|_{\mathbb{T}}$ always has a lower bound, which allows us to diagonalize the constant matrix A. Just assume $A = \begin{pmatrix} e^{i\xi} & \zeta \\ 0 & e^{-i\xi} \end{pmatrix}$, then there exists $P = \begin{pmatrix} 1 & \frac{\zeta}{e^{-i\xi}-e^{i\xi}} \\ 0 & 1 \end{pmatrix}$, such that

$$P^{-1}AP = \begin{pmatrix} e^{i\xi} & 0 \\ 0 & e^{-i\xi} \end{pmatrix} = \widetilde{A}.$$

Moreover, just note

$$\begin{aligned} |e^{-i\xi}-e^{i\xi}| &= |\cos\mathrm{Re}\,\xi\cdot(e^{\mathrm{Im}\,\xi}-e^{-\mathrm{Im}\,\xi}) - i\sin\mathrm{Re}\,\xi\cdot(e^{\mathrm{Im}\,\xi}+e^{-\mathrm{Im}\,\xi})| \\ &\geqslant \frac{1}{4}\|2\mathrm{Re}\xi\|_{\mathbb{T}}, \end{aligned}$$

then we have estimate

$$\|P\| \leqslant 1+\frac{4|\zeta|}{\|2\,\mathrm{Re}\,\xi\|_{\mathbb{T}}} \leqslant 1+\frac{8|\zeta|}{\gamma}(1+N)^{C_1 N^{\frac{1}{\eta+1}}} \leqslant \frac{1}{2}e^{|\log\varepsilon|^{\frac{2}{2+\eta}}}.$$

Moreover, $P^{-1}Ae^{F(x)}P = \widetilde{A}e^{\widetilde{F}(x)}$, where $\widetilde{F} = P^{-1}FP \in \mathcal{B}_{h,r}$ satisfies

$$\|\widetilde{F}\|_h \leqslant \|F\|_h\|P\|^2 \leqslant e^{2|\log\varepsilon|^{\frac{2}{2+\eta}}}\varepsilon =: \tilde{\varepsilon}. \tag{3.7}$$

By the choice of ε in (3.2) we have $\tilde{\varepsilon} \leqslant \varepsilon^{\frac{9}{10}}$.

After the diagonalization, we are ready to solve the non-resonant terms of the perturbation. For this purpose, we need to analyze the fine structure of the small denominators. We just consider the case

$$|e^{i(\langle \mathbf{k}^*,\alpha\rangle-2\xi)}-1| < \varepsilon^{\frac{1}{10}}, \tag{3.8}$$

since the other case can be dealt with similarly. The following lemma shows that the integer cone Γ_r implies that the resonant site in $\mathcal{T}_{N'}\Gamma_r$ is unique under the proper truncation $N' \gg N$.

Lemma 3.4 (Uniqueness). *Let $N' = C_2|\log\varepsilon|^{1+\frac{\eta}{2}} - N$ and $C_3 = \frac{1}{10}C_2^{-\frac{2}{2+\eta}}$, where $C_2 = C_2(\eta,\gamma,\tau)$ is the constant such that*

$$\frac{1}{10}\Big(\frac{x}{C_2}\Big)^{\frac{1}{1+\frac{\eta}{2}}} \geqslant -\log(\frac{\gamma}{2}) + C_1 x^{\frac{1}{1+\eta}}\log(1+x) \quad \textit{for all } x>0. \tag{3.9}$$

Then for any $\mathbf{k} \in \mathcal{T}_{N'}\Gamma_r$ we have

$$|e^{i\langle \mathbf{k},\alpha\rangle}-1| \geqslant \varepsilon^{\frac{1}{10}}, \tag{3.10}$$

$$|e^{i(\langle \mathbf{k},\alpha\rangle\pm2\xi)}-1| \geqslant \varepsilon^{\frac{1}{10}}, \quad \textit{when}\, \mathbf{k} \neq \mathbf{k}^*. \tag{3.11}$$

Proof. If (3.10) does not hold, then by using $\alpha \in \mathrm{DC}^d_{\gamma,\tau}$, Lemma 3.3 and (3.9),

$$\varepsilon^{\frac{1}{10}} > \|\langle \mathbf{k}, \alpha\rangle\|_{\mathbb{T}} \geqslant \gamma(1 + |\mathbf{k}|_\eta)^{-C_1|\mathbf{k}|_\eta^{\frac{1}{\eta+1}}} \geqslant \mathrm{e}^{-C_3|\mathbf{k}|_\eta^{\frac{2}{2+\eta}}}.$$

Thus, combining the above inequality with the choice of N', we have

$$|\mathbf{k}|_\eta > C_2|\log\varepsilon|^{1+\frac{\eta}{2}} > N', \tag{3.12}$$

which shows a contradiction to $\mathbf{k} \in \mathcal{T}_{N'}\Gamma_r$.

If (3.11) does not hold, then there exists $\mathbf{k}' \neq \mathbf{k}^*$ such that $|\mathrm{e}^{\mathrm{i}(\langle \mathbf{k}',\alpha\rangle+2\xi)} - 1| < \varepsilon^{\frac{1}{10}}$ or $|\mathrm{e}^{\mathrm{i}(\langle \mathbf{k}',\alpha\rangle-2\xi)} - 1| < \varepsilon^{\frac{1}{10}}$. This implies that

$$2\varepsilon^{\frac{1}{10}} > \max\{\|\langle \mathbf{k}',\alpha\rangle + 2\xi + (\langle \mathbf{k}^*,\alpha\rangle - 2\xi)\|_{\mathbb{T}}, \|\langle \mathbf{k}',\alpha\rangle - 2\xi - (\langle \mathbf{k}^*,\alpha\rangle - 2\xi)\|_{\mathbb{T}}\}.$$

Since $\mathbf{k}' \in \Gamma_r$, it follows from the structure of the integer cone Γ_r that

$$[\![\mathbf{k}' + \mathbf{k}^*]\!] \geqslant r(|\mathbf{k}'|_\eta + |\mathbf{k}^*|_\eta) > 0,$$

which implies that $\mathbf{k}' + \mathbf{k}^* \neq \mathbf{0}$. Moreover, by $\alpha \in \mathrm{DC}^d_{\gamma,\tau}$ and Lemma 3.3,

$$2\varepsilon^{\frac{1}{10}} > \|\langle \mathbf{k}' \mp \mathbf{k}^*, \alpha\rangle\|_{\mathbb{T}} \geqslant \gamma(1 + |\mathbf{k}' \mp \mathbf{k}^*|_\eta)^{-C_1|\mathbf{k}'\mp\mathbf{k}^*|_\eta^{\frac{1}{\eta+1}}}.$$

Same as (3.12), the inequality (3.9) would imply that

$$|\mathbf{k}' \mp \mathbf{k}^*|_\eta > C_2|\log\varepsilon|^{1+\frac{\eta}{2}},$$

and consequently

$$|\mathbf{k}'|_\eta > C_2|\log\varepsilon|^{1+\frac{\eta}{2}} - N = N'.$$

This contradicts to $\mathbf{k}' \in \mathcal{T}_{N'}\Gamma_r$, and thus we finish the proof. ■

Let $\sigma = \tilde{\varepsilon}^{\frac{1}{3}}$ and rewrite the $\mathbf{k}$-th Fourier coefficient of $\widetilde{F}$ by $\widehat{F}_{\mathbf{k}} = \begin{pmatrix} a_{\mathbf{k}} & b_{\mathbf{k}} \\ c_{\mathbf{k}} & -a_{\mathbf{k}} \end{pmatrix}$ for any $\widetilde{F} \in \mathcal{B}_{h,r}$. By (3.8) and Lemma 3.4, the space decomposition with respect to $\widetilde{A}, \sigma$ takes the form as

$$\mathcal{B}^{\mathrm{nre}}_{h,r}(\sigma) = \left\{\widetilde{F}(x) = \mathcal{T}_{N'}\widetilde{F}(x) - \begin{pmatrix} 0 & b^*_{\mathbf{k}} \\ 0 & 0 \end{pmatrix}\mathrm{e}^{\mathrm{i}\langle \mathbf{k}^*,x\rangle}\right\},$$

$$\mathcal{B}^{\mathrm{re}}_{h,r}(\sigma) = \left\{\widetilde{F}(x) = \mathcal{R}_{N'}\widetilde{F}(x) + \begin{pmatrix} 0 & b^*_{\mathbf{k}} \\ 0 & 0 \end{pmatrix}\mathrm{e}^{\mathrm{i}\langle \mathbf{k}^*,x\rangle}\right\}.$$

It follows directly that $\mathcal{B}^{\mathrm{nre}}_{h,r}(\sigma)$ is a closed invariant subspace of $\mathcal{B}_{h,r}$. Moreover, we have the following.

Lemma 3.5. *The operator* $L^{-1}_{\widetilde{A}} \colon \mathcal{B}^{\mathrm{nre}}_{h,r}(\sigma) \to \mathcal{B}^{\mathrm{nre}}_{h,r}(\sigma)$ *is bounded with* $\|L^{-1}_{\widetilde{A}}\| \leqslant \frac{1}{\sigma}$.

Proof. For any $\widetilde{F} \in \mathcal{B}^{\mathrm{nre}}_{h,r}(\sigma)$, in order to solve $\widetilde{F}(x) = L_{\widetilde{A}}Y(x)$, we only need to expand $Y(x) = \sum_{\mathbf{k}} \widehat{Y}_{\mathbf{k}} e^{i\langle \mathbf{k},x\rangle}$ and $\widetilde{F}(x) = \sum_{\mathbf{k}} \widehat{F}_{\mathbf{k}} e^{i\langle \mathbf{k},x\rangle}$ respectively. Direct calculation shows

$$\widehat{Y}_{\mathbf{k}^*} = \begin{pmatrix} a_{\mathbf{k}^*}/(e^{i\langle \mathbf{k}^*,\alpha\rangle} - 1) & 0 \\ c_{\mathbf{k}^*}/(e^{i(\langle \mathbf{k}^*,\alpha\rangle+2\xi)} - 1) & -a_{\mathbf{k}^*}/(e^{i\langle \mathbf{k}^*,\alpha\rangle} - 1) \end{pmatrix},$$

$$\widehat{Y}_{\mathbf{k}} = \begin{pmatrix} a_{\mathbf{k}}/(e^{i\langle \mathbf{k},\alpha\rangle} - 1) & b_{\mathbf{k}}/(e^{i(\langle \mathbf{k},\alpha\rangle-2\xi)} - 1) \\ c_{\mathbf{k}}/(e^{i(\langle \mathbf{k},\alpha\rangle+2\xi)} - 1) & -a_{\mathbf{k}}/(e^{i\langle \mathbf{k},\alpha\rangle} - 1) \end{pmatrix} \quad \text{for all } \mathbf{k} \neq \mathbf{k}^*.$$

Then the result follows from Lemma 3.4. ∎

Once we have Lemma 3.5, we then apply Lemma 3.1 to obtain $Y \in \mathcal{B}^{\mathrm{nre}}_{h,r}(\sigma)$ and $F^{\mathrm{re}} \in \mathcal{B}^{\mathrm{re}}_{h,r}(\sigma)$ such that

$$e^{-Y(x+\alpha)}\widetilde{A}e^{\widetilde{F}(x)}e^{Y(x)} = \widetilde{A}e^{F^{\mathrm{re}}(x)},$$

with the following estimates

$$\|Y\|_h \leqslant \tilde{\varepsilon}^{\frac{1}{2}} \leqslant \varepsilon^{\frac{9}{20}}, \quad \|F^{\mathrm{re}}\|_h \leqslant 2\tilde{\varepsilon} \leqslant 2\varepsilon^{\frac{9}{10}}.$$

Next, the resonant term $\begin{pmatrix} 0 & b^*_{\mathbf{k}} \\ 0 & 0 \end{pmatrix} e^{i\langle \mathbf{k}^*,x\rangle}$ in $F^{\mathrm{re}}(x)$ can be eliminated by the rotation conjugation $Q_{\mathbf{k}^*}(x)$, where

$$Q_{\mathbf{k}}(x) := R_{\frac{\langle \mathbf{k},x\rangle}{2}} = \begin{pmatrix} e^{\frac{i}{2}\langle \mathbf{k},x\rangle} & 0 \\ 0 & e^{-\frac{i}{2}\langle \mathbf{k},x\rangle} \end{pmatrix},$$

which is defined on $2\mathbb{T}^d$. Indeed, direct calculation shows that

$$Q_{\mathbf{k}^*}(x+\alpha)^{-1}\widetilde{A}Q_{\mathbf{k}^*}(x) = \begin{pmatrix} e^{i(\xi - \frac{\langle \mathbf{k}^*,\alpha\rangle}{2})} & 0 \\ 0 & e^{-i(\xi - \frac{\langle \mathbf{k}^*,\alpha\rangle}{2})} \end{pmatrix} =: \bar{A},$$

and

$$\begin{aligned} Q_{\mathbf{k}^*}(x)^{-1}F^{\mathrm{re}}(x)Q_{\mathbf{k}^*}(x) &= \begin{pmatrix} 0 & b^*_{\mathbf{k}} \\ 0 & 0 \end{pmatrix} + Q_{-\mathbf{k}^*}(x)\mathcal{R}_{N'}F^{\mathrm{re}}(x)Q_{\mathbf{k}^*}(x) \\ &=: L + G(x) =: \bar{F}(x). \end{aligned}$$

Let $B = P \cdot e^Y \cdot Q_{\mathbf{k}^*} \in C^\omega(2\mathbb{T}^d_h, \mathrm{SL}(2,\mathbb{C}))$, then

$$B(x+\alpha)^{-1}Ae^{F(x)}B(x) = e^{\bar{A}}e^{\bar{F}(x)},$$

with estimate

$$\|B(x)\|_0 \leqslant \|P\| \cdot \|e^{Y(x)}\|_0 \leqslant e^{|\log\varepsilon|^{\frac{2}{2+\eta}}}.$$

Rewrite the cocycle as

$$\bar{A}e^{\bar{F}(x)} = \bar{A}e^{L}e^{-L}e^{\bar{F}(x)} = \begin{pmatrix} e^{i\xi_+} & \zeta_+ \\ 0 & e^{-i\xi_+} \end{pmatrix} e^{F_+(x)} =: A_+ e^{F_+(x)},$$

where $\xi_+ = \xi - \frac{\langle \mathbf{k}^*, \alpha \rangle}{2}$, $\zeta_+ = b_{\mathbf{k}}^* e^{i\xi_+}$. By the decay of Fourier coefficient $|b_{\mathbf{k}}^*| \leqslant \|F^{\mathrm{re}}\|_h e^{-h|\mathbf{k}^*|_\eta}$ and (3.7), it follows that

$$|\zeta_+| \leqslant \|F^{\mathrm{re}}\|_h e^{-h|\mathbf{k}^*|_\eta} e^{\varepsilon^{\frac{1}{10}}} \leqslant \varepsilon^{\frac{9}{10}}.$$

Furthermore, by the Baker–Campbell–Hausdorff formula, we have

$$F_+(x) = G(x) + \frac{1}{2}[-L, G(x)] + \frac{1}{12}[-L, [-L, G(x)]] + \cdots . \tag{3.13}$$

The following result is important for us, which says that the rotation $Q_{\mathbf{k}^*}(x)$ preserves the cone structure, at the cost of shrinking r a little, as shown in Figure 2. Consequently, F_+ also has the cone structure. This is the key step why this modified KAM iteration can be iterated.

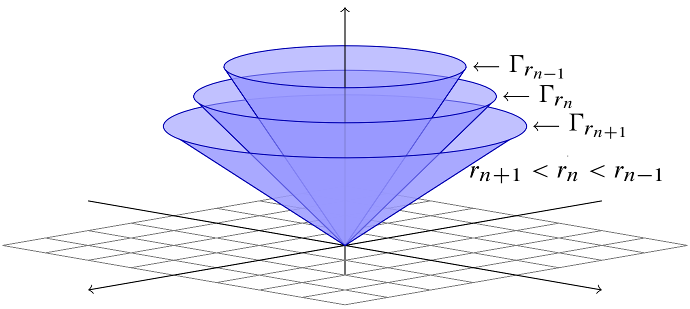

Figure 2. Integer cones in the KAM iteration.

Lemma 3.6. *For any $F^{\mathrm{re}} \in \mathcal{B}^{\mathrm{re}}_{h_+,r}(\sigma)$ and $\mathbf{k}^* \in \mathcal{T}_N \Gamma_r$, we have*

$$G(x) = Q_{-\mathbf{k}^*}(x) \cdot (\mathcal{R}_{N'} F^{\mathrm{re}}(x)) \cdot Q_{\mathbf{k}^*}(x) \in \mathcal{B}_{h_+,r_+}.$$

Consequently, we have $F_+ \in \mathcal{B}_{h_+,r_+}$.

Proof. Since $\mathcal{R}_{N'} F^{\mathrm{re}} \in \mathcal{B}^{\mathrm{re}}_{h_+,r}$, then the $\mathbf{k}$-th term in its Fourier series is

$$\widehat{F}_{\mathbf{k}} e^{i\langle \mathbf{k}, x \rangle} = \begin{pmatrix} a_{\mathbf{k}} & b_{\mathbf{k}} \\ c_{\mathbf{k}} & -a_{\mathbf{k}} \end{pmatrix} e^{i\langle \mathbf{k}, x \rangle} \quad \text{for all } \mathbf{k} \in \mathcal{R}_{N'} \Gamma_r.$$

Let $D_{\mathbf{k}}(x) = Q_{-\mathbf{k}^*}(x)\widehat{F}_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\langle \mathbf{k},x\rangle}Q^*_{\mathbf{k}}(x)$. By direct calculation we have

$$D_{\mathbf{k}}(x) = \begin{pmatrix} a_{\mathbf{k}} & 0 \\ 0 & -a_{\mathbf{k}} \end{pmatrix}\mathrm{e}^{\mathrm{i}\langle \mathbf{k},x\rangle} + \begin{pmatrix} 0 & b_{\mathbf{k}} \\ 0 & 0 \end{pmatrix}\mathrm{e}^{\mathrm{i}\langle \mathbf{k}-\mathbf{k}^*,x\rangle} + \begin{pmatrix} 0 & 0 \\ c_{\mathbf{k}} & 0 \end{pmatrix}\mathrm{e}^{\mathrm{i}\langle \mathbf{k}+\mathbf{k}^*,x\rangle}.$$

On the one hand, since $\mathbf{k}^* \in \mathcal{T}_N\Gamma_r$ and $\mathbf{k} \in \mathcal{R}_{N'}\Gamma_r$, we conclude that

$$\begin{aligned} [\![\mathbf{k} - k^*]\!] &> r|\mathbf{k}|_\eta - [\![\mathbf{k}^*]\!] \\ &\geqslant r|\mathbf{k}|_\eta - N \\ &= r_+|\mathbf{k}|_\eta + (r - r_+)|\mathbf{k}|_\eta - N \\ &\geqslant r_+|\mathbf{k}|_\eta + (r - r_+)(C_2|\log\varepsilon|^{1+\frac{\eta}{2}} - N) - N \\ &\geqslant r_+|\mathbf{k}|_\eta + C_2|\log\varepsilon|^{-\frac{\eta}{10}}|\log\varepsilon|^{1+\frac{\eta}{2}} - 2N \\ &\geqslant r_+|\mathbf{k}|_\eta + N \\ &\geqslant r_+|\mathbf{k}|_\eta + r_+|\mathbf{k}^*|_\eta \\ &\geqslant r_+|\mathbf{k} - \mathbf{k}^*|_\eta, \end{aligned}$$

where we use the fact $N \leqslant |\log\varepsilon|^{1+\frac{\eta}{8}}$ and $|\log\varepsilon| \geqslant (r - r_+)^{-\frac{10}{\eta}}$. This just means $\mathbf{k} - \mathbf{k}^* \in \Gamma_{r_+}$. On the other hand, $[\![\mathbf{k} + \mathbf{k}^*]\!] \geqslant r(|\mathbf{k}|_\eta + |\mathbf{k}^*|_\eta) > r_+|\mathbf{k} + \mathbf{k}^*|_\eta$ means $\mathbf{k} + \mathbf{k}^* \in \Gamma_{r_+}$. We conclude that $\mathbf{k}, \mathbf{k} - \mathbf{k}^*, \mathbf{k} + \mathbf{k}^* \in \Gamma_{r_+}$ and thus $D_{\mathbf{k}}(x) \in \mathcal{B}_{h_+,r_+}$.

By Proposition 3.1, we have $G(x) = \sum_{\mathbf{k}\in\mathcal{R}_{N'}\Gamma_{r_+}} D_{\mathbf{k}}(x) \in \mathcal{B}_{h_+,r_+}$. Rewrite $G = \left(\begin{smallmatrix} G_{11} & G_{12} \\ G_{21} & -G_{11} \end{smallmatrix}\right)$, then

$$[-L, G(x)] = \left[\begin{pmatrix} 0 & -b^*_{\mathbf{k}} \\ 0 & 0 \end{pmatrix}, G\right] = \begin{pmatrix} -b^*_{\mathbf{k}}G_{21} & 2b^*_{\mathbf{k}}G_{11} \\ 0 & b^*_{\mathbf{k}}G_{21} \end{pmatrix} \in \mathcal{B}_{h_+,r_+},$$

which implies that the right-hand side in (3.13) belongs to $\mathcal{B}_{h_+,r_+}$. Therefore, we finish the proof again by Proposition 3.1. ∎

By (3.13) and Lemma 3.6, we have

$$\begin{aligned} \|F_+\|_{h_+} \leqslant 2\|G\|_{h_+} &\leqslant 2\|\mathcal{R}_{N'}F^{\mathrm{re}}(x)\|_{h_+}\|Q_{\mathbf{k}^*}\|^2_{h_+} \\ &\leqslant 4\tilde{\varepsilon}\mathrm{e}^{-N'(h-h_+)}\mathrm{e}^{h_+|\mathbf{k}^*|_\eta}. \end{aligned}$$

Since $N' = C_2|\log\varepsilon|^{1+\frac{\eta}{2}} - N \gg |\log\varepsilon|^{\frac{\eta}{10}}N$ and $\mathbf{k}^* \in \mathcal{T}_N\Gamma_r$, one can get that

$$\|F_+\|_{h_+} \leqslant \varepsilon^{\frac{9}{10}}\mathrm{e}^{-100N(h-h_+)}\varepsilon^{\frac{-h_+}{h-h_+}} \leqslant \varepsilon^{100}.$$

This finishes the proof. ∎

4. Reducibility of almost-periodic cocycles

By the KAM iteration developed in the last section, we now prove the reducibility results of the almost-periodic cocycle $(\alpha, A\mathrm{e}^{F})$ with perturbation $F \in \mathcal{B}_{h,r}$. We always choose

$$A_0 = A, \quad F_0 = F, \quad \varepsilon_0 = \varepsilon, \quad h_0 = h, \quad r_0 = r.$$

For $n \geqslant 0$, we define the sequences

$$\begin{aligned} \varepsilon_{n+1} &= 2\varepsilon_n^3, \\ h_{n+1} &= h_n - \frac{h-h'}{(n+2)^2}, \\ r_{n+1} &= r_n - \frac{r-r'}{(n+2)^2}, \\ N_n &= \frac{2|\log \varepsilon_n|}{h_n - h_{n+1}}. \end{aligned} \tag{$*$}$$

Two situations need to be treated separately for $A \in \mathcal{M}$, i.e., the eigenvalues of A are $\mathrm{e}^{\pm \mathrm{i}\rho}$ with $\rho \in \mathbb{R}$ (elliptic case and parabolic case) or $\mathrm{e}^{\pm \mathrm{i}\xi}$ with $\xi \notin \mathbb{R}$ (hyperbolic case).

4.1. Elliptic case and parabolic case

Suppose that $A = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\rho} & \zeta \\ 0 & \mathrm{e}^{-\mathrm{i}\rho} \end{pmatrix}$ with $\rho \in \mathbb{R}$, the following Proposition 4.1 shows that $(\alpha, A\mathrm{e}^{F})$ is almost reducible.

Proposition 4.1. *Let $d \in \mathbb{N}^+ \cup \{\infty\}$, $\eta > 0$, $h > 0$, $h' \in (0,h)$, $r \in (0,1]$, $r' \in (0,r)$, $\gamma \in (0,1)$, $\tau > 1$, $\alpha \in \mathrm{DC}^d_{\gamma,\tau}$. Suppose that $F \in \mathcal{B}_{h,r}$. There exists*

$$\varepsilon = \varepsilon(\eta, h, h', r, r', \gamma, \tau, |\zeta|), \quad c = c(\eta, \gamma, \tau)$$

such that if

$$\|F\|_h < \varepsilon < \frac{c}{(1+|\zeta|)^{10}} \min\{\mathrm{e}^{-\left(\frac{1}{h-h'}\right)^{\frac{10}{\eta}}}, \mathrm{e}^{-\left(\frac{1}{r-r'}\right)^{\frac{10}{\eta}}}\},$$

then there exist $\Phi_n \in C^\omega(2\mathbb{T}^d_{h_n}, \mathrm{SL}(2,\mathbb{C}))$ with $\|\Phi_n\|_0 \leqslant \mathrm{e}^{C_\eta |\log \varepsilon_n|^{\frac{2}{2+\eta}}}$,

$$C_\eta := (2^{\frac{2}{2+\eta}} - 1)^{-1},$$

and $F_n \in \mathcal{B}_{h_n,r_n}$ *with* $\|F_n\|_{h_n} \leqslant \varepsilon_n$ *such that*

$$\Phi_n(x+\alpha)^{-1} A\mathrm{e}^{F(x)} \Phi_n(x) = A_n \mathrm{e}^{F_n(x)},$$

where $A_n = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\rho_n} & \zeta_n \\ 0 & \mathrm{e}^{-\mathrm{i}\rho_n} \end{pmatrix}$ *or* $A_n = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\rho_n} & 0 \\ \zeta_n & \mathrm{e}^{-\mathrm{i}\rho_n} \end{pmatrix}$ *with* $\rho_n \in \mathbb{R}$ *and* $|\zeta_n| < |\zeta|$.

Moreover, if we denote $\Theta_n = \bigcup_{\mathbf{k}\in\mathcal{T}_{N_n}\Gamma_{r_n}} \Theta_n(\mathbf{k})$, *where*

$$\Theta_n(\mathbf{k}) = \{\rho \in \mathbb{R} : |\mathrm{e}^{\mathrm{i}(\langle \mathbf{k},\alpha\rangle + 2\rho_n)} - 1| \leqslant \varepsilon_n^{\frac{1}{10}}\} \cup \{\rho \in \mathbb{R} : |\mathrm{e}^{\mathrm{i}(\langle \mathbf{k},\alpha\rangle - 2\rho_n)} - 1| \leqslant \varepsilon_n^{\frac{1}{10}}\},$$

then we have the following.

(1) *If* $\rho \notin \Theta_n$, *then* $\Phi_{n+1} = \Phi_n \cdot \mathrm{e}^{Y_n}$ *with*

$$\|Y_n\|_h \leqslant \varepsilon_n^{\frac{1}{2}}, \quad \rho_{n+1} = \rho_n, \quad \zeta_{n+1} = \zeta_n.$$

(2) *If* $\rho \in \Theta_n(\mathbf{k}_n^*)$, *then* $\Phi_{n+1} = \Phi_n \cdot B_n$ *with*

$$\|B_n\|_0 \leqslant \mathrm{e}^{|\log \varepsilon_n|^{\frac{2}{2+\eta}}}, \qquad \rho_{n+1} = \rho_n - \frac{\langle \mathbf{k}_n^*, \alpha\rangle}{2},$$
$$|\rho_{n+1}| \leqslant \varepsilon_n^{\frac{1}{10}}, \qquad |\zeta_{n+1}| \leqslant \varepsilon_n^{\frac{9}{10}}.$$

(3) *If* $\rho \in \Theta_{n_j}(\mathbf{k}_{n_j}^*) \cap \Theta_{n_{j+1}}(\mathbf{k}_{n_{j+1}}^*)$, *then*

$$|\mathbf{k}_{n_{j+1}}^*|_\eta \geqslant |\mathbf{k}_{n_j}^*|_\eta^{1+\frac{\eta}{4+\eta}}.$$

Proof. We are going to prove Proposition 4.1 inductively. Suppose that we are at n-th step, i.e., we already constructed Φ_n such that

$$\Phi_n(x+\alpha)^{-1} A\mathrm{e}^{F(x)} \Phi_n(x) = A_n \mathrm{e}^{F_n(x)},$$

with following estimates:

$$\|\Phi_n\|_0 \leqslant \mathrm{e}^{C\eta|\log \varepsilon_n|^{\frac{2}{2+\eta}}}, \quad \|F_n\|_{h_n} \leqslant \varepsilon_n, \quad \rho_n \in \mathbb{R}, \quad |\zeta_n| \leqslant |\zeta|.$$

By the selection of ($*$) and $|\zeta_n| \leqslant |\zeta|$, for any $n \geqslant 0$ we have

$$\varepsilon_n < \frac{c}{(1+|\zeta_n|)^{10}} \min\{\mathrm{e}^{-\left(\frac{1}{h_n - h_{n+1}}\right)^{\frac{10}{\eta}}}, \mathrm{e}^{-\left(\frac{1}{r_n - r_{n+1}}\right)^{\frac{10}{\eta}}}\}.$$

By Proposition 3.2, there exist $B_n \in C^\omega(2\mathbb{T}^d_{h_{n+1}}, \mathrm{SL}(2,\mathbb{C}))$, $F_{n+1} \in \mathcal{B}_{h_{n+1},r_{n+1}}$, $A_{n+1} \in \mathcal{M}$ such that

$$B_n(x+\alpha)^{-1} A_n \mathrm{e}^{F_n(x)} B_n(x) = A_{n+1} \mathrm{e}^{F_{n+1}(x)}.$$

Let $\Phi_{n+1} = \Phi_n \cdot B_n$. Then

$$\Phi_{n+1}(x+\alpha)^{-1} A e^{F(x)} \Phi_{n+1}(x) = A_{n+1} e^{F_{n+1}(x)}.$$

To obtain the estimates of Φ_{n+1}, F_{n+1}, A_{n+1}, we need to distinguish between two cases according to the nature of ρ.

Non-resonant case. If $\rho \notin \Theta_n$, which means $A_n \in \mathcal{NR}(N_n, \varepsilon_n^{\frac{1}{10}})$. Then by Proposition 3.2, we have $B_n = e^{Y_n}$ with estimates

$$\|Y_n\|_{h_n} \leqslant \varepsilon_n^{\frac{1}{2}}, \quad \|F_{n+1}\|_{h_{n+1}} \leqslant 2\varepsilon_n^3 = \varepsilon_{n+1}, \quad A_{n+1} = A_n.$$

Hence, $\rho_{n+1} = \rho_n \in \mathbb{R}$ and $|\zeta_{n+1}| = |\zeta_n| \leqslant |\zeta|$. It is obvious that

$$\|\Phi_{n+1}\|_0 = \|\Phi_n \cdot B_n\|_0 \leqslant e^{C_\eta |\log \varepsilon_{n+1}|^{\frac{2}{2+\eta}}}.$$

This proves Proposition 4.1 (1).

Resonant case. If $\rho \in \Theta_n(\mathbf{k}_n^*)$, which means $A_n \in \mathcal{RS}(N_n, \varepsilon_n^{\frac{1}{10}})$. Then by Proposition 3.2, we have following estimates:

$$\|B_n\|_0 \leqslant e^{|\log \varepsilon_n|^{\frac{2}{2+\eta}}}, \quad \|F_{n+1}\|_{h_{n+1}} \leqslant \varepsilon_n^{100} < \varepsilon_{n+1}.$$

Moreover, A_{n+1} takes the form

$$A_{n+1} = \begin{pmatrix} e^{i\rho_{n+1}} & \zeta_{n+1} \\ 0 & e^{-i\rho_{n+1}} \end{pmatrix} \quad \text{or} \quad A_{n+1} = \begin{pmatrix} e^{i\rho_{n+1}} & 0 \\ \zeta_{n+1} & e^{-i\rho_{n+1}} \end{pmatrix},$$

where $\rho_{n+1} = \rho_n - \frac{\langle \mathbf{k}, \alpha\rangle}{2} \in \mathbb{R}$ with $|\rho_{n+1}| \leqslant \varepsilon_n^{\frac{1}{10}}$ and $|\zeta_{n+1}| \leqslant \varepsilon_n^{\frac{9}{10}}$. This proves Proposition 4.1 (2). It is easy to see that

$$\|\Phi_{n+1}\|_0 = \|\Phi_n \cdot B_n\|_0 \leqslant e^{C_\eta |\log \varepsilon_n|^{\frac{2}{2+\eta}}} e^{|\log \varepsilon_n|^{\frac{2}{2+\eta}}} \leqslant e^{C_\eta |\log \varepsilon_{n+1}|^{\frac{2}{2+\eta}}}.$$

When $\rho \in \Theta_{n_j}(\mathbf{k}_{n_j}^*) \cap \Theta_{n_{j+1}}(\mathbf{k}_{n_{j+1}}^*)$, on the one hand, it follows from

$$\|2\rho_{n_{j+1}} - \langle \mathbf{k}_{n_{j+1}}^*, \alpha\rangle\|_{\mathbb{T}} \leqslant \varepsilon_{n_{j+1}}^{\frac{1}{10}}$$

and Lemma 3.3 that

$$2|\rho_{n_{j+1}}| \geqslant \gamma(1 + |\mathbf{k}_{n_{j+1}}^*|_\eta)^{-C_1 |\mathbf{k}_{n_{j+1}}^*|_\eta^{\frac{1}{\eta+1}}} - \varepsilon_{n_{j+1}}^{\frac{1}{10}} \geqslant e^{-C_3 |\mathbf{k}_{n_{j+1}}|_\eta^{\frac{2}{2+\eta}}}.$$

On the other hand, there is no resonance between n_j-th step and n_{j+1}-th step, and according to Proposition 3.2, we have

$$\rho_{1+n_j} = \rho_{n_{j+1}} \quad \text{and} \quad |\rho_{1+n_j}| \leqslant \varepsilon_{n_j}^{\frac{1}{10}}.$$

To sum up, we obtain that

$$\frac{1}{2}\exp(-C_3|\mathbf{k}^*_{n_{j+1}}|_\eta^{\frac{2}{\eta+2}}) \leqslant \varepsilon_{n_j}^{\frac{1}{10}} \leqslant \exp\Big(-\frac{1}{10}|\mathbf{k}^*_{n_j}|_\eta^{\frac{8}{8+\eta}}\Big),$$

where the second inequality uses $|\mathbf{k}^*_{n_j}|_\eta \leqslant N_{n_j} \leqslant |\log\varepsilon_{n_j}|^{1+\frac{\eta}{8}}$, which shows that

$$|\mathbf{k}^*_{n_{j+1}}|_\eta \geqslant |\mathbf{k}^*_{n_j}|_\eta^{1+\frac{\eta}{4+\eta}}.$$

Hence, we finish the whole proof. ■

4.1.1. Reducibility of almost-periodic cocycle. The following Corollary 4.1 shows that $(\alpha, A\mathrm{e}^F)$ is reducible provided that ρ belongs to at most finitely many sets Θ_n. Let $\overline{\Theta} = \limsup_{n\to\infty}\Theta_n$.

Corollary 4.1. *If $\rho \notin \overline{\Theta}$, then there exists $\Psi' \in C^\omega(2\mathbb{T}^d, \mathrm{SL}(2,\mathbb{C}))$ such that*

$$\Psi'(x+\alpha)^{-1}A\mathrm{e}^{F(x)}\Psi'(x) = A'.$$

Indeed, let $\tilde{n}$ such that $\rho \notin \Theta_n$ for any $n \geqslant \tilde{n}$. Then A' takes the precise form:

(1) *if $\rho_{\tilde{n}} \neq 0$, then $A' = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\rho_{\tilde{n}}} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\rho_{\tilde{n}}} \end{pmatrix}$;*

(2) *if $\rho_{\tilde{n}} = 0$, then $A' = \begin{pmatrix} 1 & \zeta_{\tilde{n}} \\ 0 & 1 \end{pmatrix}$.*

Proof. By Proposition 4.1, there exist $\Phi_{\tilde{n}}$, $F_{\tilde{n}}$, $A_{\tilde{n}}$ such that

$$\Phi_{\tilde{n}}(x+\alpha)^{-1}A\mathrm{e}^{F(x)}\Phi_{\tilde{n}}(x) = A_{\tilde{n}}\mathrm{e}^{F_{\tilde{n}}(x)}.$$

Since no resonance occurs for any $n \geqslant \tilde{n}$ by the definition of ρ, we use Proposition 4.1(1) iteratively to obtain Y_n and F_n for $n \geqslant \tilde{n}$ such that

$$\mathrm{e}^{-Y_n(x+\alpha)}A_{\tilde{n}}\mathrm{e}^{F_n(x)}\mathrm{e}^{Y_n(x)} = A_{\tilde{n}}\mathrm{e}^{F_{n+1}(x)},$$

with $\|Y_n\|_{h_n} \leqslant \varepsilon_n^{\frac{1}{2}}$ and $\|F_n\|_{h_n} \leqslant \varepsilon_n$.

If $\rho_{\tilde{n}} \neq 0$, then there exists $P \in \mathcal{M}$ such that

$$P^{-1}A_{\tilde{n}}P = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\rho_{\tilde{n}}} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\rho_{\tilde{n}}} \end{pmatrix} =: A'.$$

We let $\Psi' = \Phi_{\tilde{n}} \cdot \prod_{n=\tilde{n}}^\infty \mathrm{e}^{Y_n} \cdot P$.

If $\rho_{\tilde{n}} = 0$, then we let $\Psi' = \Phi_{\tilde{n}} \cdot \prod_{n=\tilde{n}}^\infty \mathrm{e}^{Y_n}$ for the case $A_{\tilde{n}} = \begin{pmatrix} 1 & \zeta_{\tilde{n}} \\ 0 & 1 \end{pmatrix}$. Otherwise we choose $H = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$ so that

$$H^{-1}\begin{pmatrix} 1 & 0 \\ \zeta_{\tilde{n}} & 1 \end{pmatrix}H = \begin{pmatrix} 1 & \zeta_{\tilde{n}} \\ 0 & 1 \end{pmatrix} =: A',$$

which finishes the proof by letting $\Psi' = \Phi_{\tilde{n}} \cdot \prod_{n=\tilde{n}}^\infty \mathrm{e}^{Y_n} \cdot H$. ■

4.1.2. Growth of the cocycles. Corollary 4.1 shows that the cocycle is reducible if $\rho \notin \overline{\Theta}$. In the following, we will show the cocycle has sublinear growth if $\rho \in \overline{\Theta}$.

Corollary 4.2. *If $\rho \in \overline{\Theta}$, then*

$$\|\mathcal{A}_j\|_0 \leqslant o(1+j),$$

where $(j\alpha, \mathcal{A}_j(x)) := (\alpha, Ae^{F(x)})^j$.

Proof. To control the growth of the cocycles, we need the following.

Lemma 4.1 ([4, 59]). *We have that*

$$M_l(\mathrm{id}+y_l)\cdots M_0(\mathrm{id}+y_0) = M^{(l)}(\mathrm{id}+y^{(l)}),$$

where $M^{(l)} = M_l \cdots M_0$ *and*

$$\|y^{(l)}\| \leqslant e^{\sum_{k=0}^{l}\|M^{(k)}\|^2\|y_k\|} - 1.$$

By Proposition 4.1, $(\alpha, Ae^{F(x)})$ is almost reducible. Thus, we have

$$\mathcal{A}_j(x) = \Phi_n(x+j\alpha)\Big(\prod_{s=j-1}^{0} A_n e^{F_n(x+s\alpha)}\Big)\Phi_n(x)^{-1}.$$

Then by Lemma 4.1 and $\|A_n^j\| \leqslant 1 + j|\zeta_n|$, it follows that

$$\begin{aligned}\|\mathcal{A}_j\|_0 &\leqslant \|\Phi_n\|_0^2 \cdot \|A_n\| \cdot \|A_n^{j-1}\| \cdot e^{\|F_n\|_0\|A_n\|\sum_{l=1}^{j}(1+|\zeta_n|(j-l))}\\ &\leqslant (1+2|\zeta|)\cdot(1+j|\zeta_n|)\cdot\|\Phi_n\|_0^2\cdot e^{10\varepsilon_n(j+j^2|\zeta_n|)}.\end{aligned}$$

For any $j \in \mathbb{N}$, one can construct an interval $\mathbb{I}_n$ such that

$$j \in \mathbb{I}_n := (\varepsilon_n^{-\frac{1}{8}}, \varepsilon_n^{-\frac{1}{2}}).$$

Since $\mathbb{I}_n \cap \mathbb{I}_{n+1} \neq \varnothing$, we conclude that $\{\mathbb{I}_n\}_{n\in\mathbb{N}}$ cover all the j tending to ∞, and

$$\|\mathcal{A}_j\|_0 \leqslant 2j(1+2|\zeta|)\cdot\|\Phi_n\|_0^2|\zeta_n|.$$

Note that if $\rho \in \Theta_n$, by Proposition 4.1(2) we have

$$\|\Phi_{n+1}\|_0^2 \cdot |\zeta_{n+1}| \leqslant \varepsilon_{n+1}^{\frac{1}{4}},$$

then the result follows from the assumption. ∎

4.2. Hyperbolic case

Recall that

$$\mathcal{M} := \left\{ \begin{pmatrix} e^{i\xi} & \zeta \\ 0 & e^{-i\xi} \end{pmatrix} : \xi, \zeta \in \mathbb{C} \right\} \cup \left\{ \begin{pmatrix} e^{i\xi} & 0 \\ \zeta & e^{-i\xi} \end{pmatrix} : \xi, \zeta \in \mathbb{C} \right\}.$$

To obtain the reducibility result for hyperbolic $A \in \mathcal{M}$, first we need the following simple observation.

Lemma 4.2. *Let $d \in \mathbb{N}^+ \cup \{\infty\}$, $\eta > 0$, $h > 0$, $r \in (0, 1]$, $\gamma > 0$, $\tau > 1$, $\alpha \in \mathrm{DC}^d_{\gamma,\tau}$. Suppose that $A \in \mathcal{M}$ with $\operatorname{Im} \xi \neq 0$ and $\zeta = 0$, $F \in \mathcal{B}_{h,r}$ with*

$$\|F\|_h < \varepsilon < \min\{10^{-8}, |\operatorname{Im} \xi|^3\}, \tag{4.1}$$

then (α, Ae^F) is reducible to (α, A).

Proof. Let $\sigma = \varepsilon^{\frac{1}{3}}$ and

$$\Lambda_1 = \{\mathbf{k} \in \Gamma_r : |e^{i\langle \mathbf{k},\alpha\rangle} - 1| \geqslant \sigma\},$$
$$\Lambda_2 = \{\mathbf{k} \in \Gamma_r : |e^{i(\langle \mathbf{k},\alpha\rangle \pm 2\xi)} - 1| \geqslant \sigma\}.$$

Then we define the decomposition $\mathcal{B}_{h,r} = \mathcal{B}^{\mathrm{nre}}_{h,r}(\sigma) \oplus \mathcal{B}^{\mathrm{re}}_{h,r}(\sigma)$ with respect to A, σ, where $\mathcal{B}^{\mathrm{nre}}_{h,r}(\sigma)$ is defined to be the space of all $F \in \mathcal{B}_{h,r}$ of the form

$$F(x) = \sum_{\mathbf{k}\in\Lambda_1} \begin{pmatrix} a_{\mathbf{k}} & 0 \\ 0 & -a_{\mathbf{k}} \end{pmatrix} e^{i\langle \mathbf{k},x\rangle} + \sum_{\mathbf{k}\in\Lambda_2} \begin{pmatrix} 0 & b_{\mathbf{k}} \\ c_{\mathbf{k}} & 0 \end{pmatrix} e^{i\langle \mathbf{k},x\rangle}, \tag{4.2}$$

and $\mathcal{B}^{\mathrm{re}}_{h,r}(\sigma)$ is defined to be the space of all $F \in \mathcal{B}_{h,r}$ of the form

$$F(x) = \sum_{\mathbf{k}\in\Gamma_r\setminus\Lambda_1} \begin{pmatrix} a_{\mathbf{k}} & 0 \\ 0 & -a_{\mathbf{k}} \end{pmatrix} e^{i\langle \mathbf{k},x\rangle} + \sum_{\mathbf{k}\in\Gamma_r\setminus\Lambda_2} \begin{pmatrix} 0 & b_{\mathbf{k}} \\ c_{\mathbf{k}} & 0 \end{pmatrix} e^{i\langle \mathbf{k},x\rangle}. \tag{4.3}$$

For any $Y \in \mathcal{B}^{\mathrm{nre}}_{h,r}(\sigma)$, we have

$$\begin{aligned}(L_A Y)(x) = &\sum_{\mathbf{k}\in\Lambda_1} \begin{pmatrix} a_{\mathbf{k}}(e^{i\langle \mathbf{k},\alpha\rangle} - 1) & 0 \\ 0 & -a_{\mathbf{k}}(e^{i\langle \mathbf{k},\alpha\rangle} - 1) \end{pmatrix} e^{i\langle \mathbf{k},x\rangle} \\ &+ \sum_{\mathbf{k}\in\Lambda_2} \begin{pmatrix} 0 & b_{\mathbf{k}}(e^{i\langle \mathbf{k},\alpha\rangle - 2\xi} - 1) \\ c_{\mathbf{k}}(e^{i\langle \mathbf{k},\alpha\rangle + 2\xi} - 1) & 0 \end{pmatrix} e^{i\langle \mathbf{k},x\rangle}.\end{aligned}$$

Thus, L_A is invertible on $\mathcal{B}^{\mathrm{nre}}_{h,r}(\sigma)$ and $\|L_A^{-1}\| \leqslant \frac{1}{\sigma}$, which means the decomposition for (4.2) and (4.3) is well defined.

Just note by assumption (4.1), we have

$$|e^{i(\langle \mathbf{k},\alpha\rangle \pm 2\xi)} - 1| \geqslant 2|\operatorname{Im}\xi| \geqslant \sigma \quad \text{for all } \mathbf{k} \in \mathbb{Z}_*^d,$$

which implies $\Gamma_r \backslash \Lambda_2 = \varnothing$. Thus, by Lemma 3.1, there exist $Y \in \mathcal{B}_{h,r}^{\mathrm{nre}}(\sigma)$ and $F^{\mathrm{re}} \in \mathcal{B}_{h,r}^{\mathrm{re}}(\sigma)$ such that

$$e^{-Y(x+\alpha)} A e^{F(x)} e^{Y(x)} = A e^{F'^{\mathrm{re}}(x)} =: \begin{pmatrix} e^{i\xi} e^{f(x)} & 0 \\ 0 & e^{-i\xi} e^{-f(x)} \end{pmatrix}.$$

Since $\alpha \in \mathrm{DC}_{\gamma,\tau}^d$, and $\hat{f}_{\mathbf{0}} = 0$ by $f \in \mathcal{B}_{h,r}[\mathbb{C}]$, then

$$\varphi(x+\alpha) - \varphi(x) = f(x), \quad f \in \mathcal{B}_{h,r}[\mathbb{C}],$$

always has a solution $\varphi \in C^\omega(\mathbb{T}_{h'}^d, \mathbb{C})$ with $h' \in (0,h)$. Let $\Psi = e^Y \cdot \begin{pmatrix} e^{\varphi(x)} & 0 \\ 0 & e^{-\varphi(x)} \end{pmatrix} \in C^\omega(\mathbb{T}_{h'}^d, \mathrm{SL}(2,\mathbb{C}))$. It follows that

$$\Psi(x+\alpha)^{-1} A e^{F(x)} \Psi(x) = A.$$

The proof is finished. ■

As a consequence, we have the following.

Proposition 4.2. *Let $d \in \mathbb{N}^+ \cup \{\infty\}$, $\eta > 0$, $h > 0$, $h' \in (0,h)$, $r \in (0,1]$, $r' \in (0,r)$, $\gamma > 0$, $\tau > 1$, $\alpha \in \mathrm{DC}_{\gamma,\tau}^d$. Suppose that $A = \begin{pmatrix} e^{i\xi} & \zeta \\ 0 & e^{-i\xi} \end{pmatrix}$ with $\operatorname{Im}\xi \neq 0$ and $F \in \mathcal{B}_{h,r}$. There exist $\varepsilon = \varepsilon(\eta, h, h', r, r', \gamma, \tau, |\zeta|)$ and $c = c(\eta,\gamma,\tau)$ such that if*

$$\|F\|_h < \varepsilon < \frac{c}{(1+|\zeta|)^{10}} \min\{e^{-\left(\frac{1}{h-h'}\right)^{\frac{10}{\eta}}}, e^{-\left(\frac{1}{r-r'}\right)^{\frac{10}{\eta}}}\},$$

then (α, Ae^F) is reducible to (α, A'), where $A' = \begin{pmatrix} e^{i\xi'} & 0 \\ 0 & e^{-i\xi'} \end{pmatrix}$ with $\operatorname{Im}\xi' = \operatorname{Im}\xi$.

Proof. We distinguish the proof into two cases.

Case 1. *Strong hyperbolic case.* If $|\operatorname{Im}\xi| \geqslant \frac{1}{2}$, then there exists $P \in \mathcal{M}$ with $\|P - \mathrm{id}\| \leqslant 2|\zeta|$ such that

$$P^{-1} A e^{F(x)} P = \begin{pmatrix} e^{i\xi} & 0 \\ 0 & e^{-i\xi} \end{pmatrix} e^{P^{-1}F(x)P} =: A' e^{F'(x)},$$

where $F' \in \mathcal{B}_{h,r}$ with $\|F'\|_h \leqslant \varepsilon^{\frac{9}{10}}$. By Lemma 4.2, $(\alpha, A'e^{F'})$ is reducible to (α, A') with $\xi' = \xi$.

Case 2. *Weak hyperbolic case.* If $|\operatorname{Im}\xi| < \frac{1}{2}$, by Proposition 3.2 there exist $B_n \in C^\omega(2\mathbb{T}^d_{h_n}, \mathrm{SL}(2,\mathbb{C}))$, $F_n \in \mathscr{B}_{h_n,r_n}$ and $A_n \in \mathcal{M}$ such that

$$B_n(x+\alpha)^{-1} A_n \mathrm{e}^{F_n(x)} B_n(x) = A_{n+1}\mathrm{e}^{F_{n+1}(x)},$$

where $A_n = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\xi_n} & \zeta_n \\ 0 & \mathrm{e}^{-\mathrm{i}\xi_n} \end{pmatrix}$ or $A_n = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\xi_n} & 0 \\ \zeta_n & \mathrm{e}^{-\mathrm{i}\xi_n} \end{pmatrix}$ with $\operatorname{Im}\xi_n = \operatorname{Im}\xi$ and $|\zeta_n| \leqslant |\zeta|$. According to the selection of $(*)$, the iteration is ensured by

$$\varepsilon_n < \frac{c}{(1+|\zeta_n|)^{10}} \min\{\mathrm{e}^{-\left(\frac{1}{h_n-h_{n+1}}\right)^{\frac{10}{\eta}}}, \mathrm{e}^{-\left(\frac{1}{r_n-r_{n+1}}\right)^{\frac{10}{\eta}}}\}.$$

Let $\Phi_0 = \mathrm{id}$ and $\Phi_n = \Phi_{n-1}\cdot B_{n-1}$. Then for $n \geqslant 0$,

$$\Phi_n(x+\alpha)^{-1} A\mathrm{e}^{F(x)}\Phi_n(x) = A_n \mathrm{e}^{F_n(x)},$$

with $\|F_n\|_{h_n} \leqslant \varepsilon_n$. Furthermore, there exists $P_n \in \mathcal{M}$ with $\|P_n\| \leqslant \mathrm{e}^{|\log\varepsilon_n|^{\frac{2}{2+\eta}}}$ such that

$$P_n^{-1} A_n \mathrm{e}^{F_n(x)} P_n = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\xi_n} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\xi_n} \end{pmatrix} \mathrm{e}^{P_n^{-1}F_n(x)P_n} =: A_n' \mathrm{e}^{F_n'(x)},$$

with $\|F_n'\|_{h_n} \leqslant \varepsilon_n^{\frac{9}{10}}$. Since $\operatorname{Im}\xi_n = \operatorname{Im}\xi$, let us choose the smallest $\tilde{n}$ such that

$$\varepsilon_{\tilde{n}}^{\frac{9}{10}} \leqslant \min\{10^{-8}, |\operatorname{Im}\xi_{\tilde{n}}|^3\}.$$

By Lemma 4.2, $(\alpha, A_{\tilde{n}}'\mathrm{e}^{F_{\tilde{n}}'(x)})$ is reducible to $(\alpha, A_{\tilde{n}}')$. Denote $A' = A_{\tilde{n}}'$ and $\xi' = \xi_{\tilde{n}}$. This finishes the proof. ∎

5. Applications in Schrödinger operators

In this section, we give the applications for Schrödinger operators. Let us rewrite the Schrödinger cocycle $S_{E,\lambda v}(x) = \begin{pmatrix} E-\lambda v(x) & -1 \\ 1 & 0 \end{pmatrix} = A_E \mathrm{e}^{F_v(x)}$, where

$$A_E = \begin{pmatrix} E & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad F_v(x) = \begin{pmatrix} 0 & 0 \\ \lambda v(x) & 0 \end{pmatrix}.$$

Since $A_E \in \mathrm{SL}(2,\mathbb{C})$ one knows that the eigenvalues of A_E are $\frac{E}{2} \pm \frac{\sqrt{E^2-4}}{2}$.

5.1. Proof of Theorem 1.2 and Theorem 1.4.

Note that one can always conjugate A_E to the upper triangular matrix A whose upper-right term is ζ. To apply Proposition 4.1 and Proposition 4.2 for all $E \in \mathbb{C}$, and to obtain uniform smallness condition of $|\lambda|$, the key observation is that $|\zeta|$ is uniformly bounded with respect to E.

Case 1. $E \in [-2,2]$. The eigenvalues of A_E are $\mathrm{e}^{\pm \mathrm{i}\rho}$ with $\rho \in \mathbb{R}$. Let

$$U_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathrm{e}^{\mathrm{i}\rho} & -1 \\ 1 & \mathrm{e}^{-\mathrm{i}\rho} \end{pmatrix};$$

then we have

$$U_0^{-1} A_E U_0 = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\rho} & \zeta \\ 0 & \mathrm{e}^{-\mathrm{i}\rho} \end{pmatrix} =: A \in \mathcal{M},$$

where $\zeta = -1 - \mathrm{e}^{-2\mathrm{i}\rho}$ and thus $|\zeta| \leqslant 2$. Let $F = U_0^{-1} \cdot F_v \cdot U_0 \in \mathcal{B}_{h,r}$. Then $\|F\|_h = \lambda \|v\|_h$. If we choose λ_0 such that

$$\lambda_0 \|v\|_h < \frac{c}{3^{10}} \min\{\mathrm{e}^{-\left(\frac{2}{h}\right)^{\frac{10}{\eta}}}, \mathrm{e}^{-\left(\frac{2}{r}\right)^{\frac{10}{\eta}}}\}, \tag{5.1}$$

then one can apply Proposition 4.1 to show $(\alpha, S_{E,\lambda v})$ is almost reducible; consequently one can use Corollary 4.1 and Corollary 4.2 to obtain that

$$L(E) = L(\alpha, A\mathrm{e}^{F}) = 0 = \log\left|\frac{E}{2} + \frac{\sqrt{E^2-4}}{2}\right|,$$

Therefore, $(\alpha, S_{E,\lambda v}) \notin \mathcal{UH}$, and by Proposition 2.1 we have $E \in \Sigma_{\lambda v,\alpha}$.

Case 2. $E \in \mathbb{C}\backslash[-2,2]$. The eigenvalues of A_E are $\mathrm{e}^{\pm \mathrm{i}\xi}$ with $\operatorname{Im}\xi < 0$. We can choose $U_0 = \frac{1}{\sqrt{|\mathrm{e}^{2\mathrm{i}\xi}|+1}}\begin{pmatrix} \mathrm{e}^{\mathrm{i}\xi} & -1 \\ 1 & \mathrm{e}^{\mathrm{i}\xi} \end{pmatrix}$ and $\zeta = -1 - \mathrm{e}^{-2\mathrm{i}\xi}$ so that

$$U_0^{-1} A_E U_0 = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\xi} & \zeta \\ 0 & \mathrm{e}^{-\mathrm{i}\xi} \end{pmatrix} =: A \in \mathcal{M}.$$

Let $F = U_0^{-1} \cdot F_v \cdot U_0$. By $|\zeta| \leqslant 2$, one can also choose λ_0 satisfying (5.1). It follows from Proposition 4.2 that the cocycle $(\alpha, S_{E,\lambda v})$ is reducible to some hyperbolic matrix $A' \in \mathrm{SL}(2,\mathbb{C})$, with

$$L(E) = L(\alpha, A_0\mathrm{e}^{F_0}) = |\operatorname{Im}\xi| = \log\left|\frac{E}{2} + \frac{\sqrt{E^2-4}}{2}\right|.$$

Therefore, $(\alpha, S_{E,\lambda v}) \in \mathcal{UH}$, and by Proposition 2.1 we have $E \notin \Sigma_{\lambda v,\alpha}$.

5.2. Proof of Theorem 1.5

If (5.1) holds, then by Proposition 4.1 and Corollary 4.1, it is enough to show that $\rho \notin \overline{\Theta}$ if $2\rho = \langle \mathbf{k}, \alpha \rangle \mod 2\pi$ or $\rho \in \mathrm{DC}_{\kappa,\tau}(\alpha)$. We show $\rho \notin \overline{\Theta}$ by contradiction. In fact, if $\rho \in \overline{\Theta}$, then we label the resonant steps $\{n_j\} \subset \mathbb{N}$ such that

$$\|2\rho_{n_j} - \langle \mathbf{k}_{n_j}^*, \alpha \rangle\|_{\mathbb{T}} \leqslant \varepsilon_{n_j}^{\frac{1}{10}}, \quad \mathbf{k}_{n_j}^* \in \mathcal{T}_{N_{n_j}} \Gamma_{r_{n_j}}. \tag{5.2}$$

Let $\mathbf{d}_j = \sum_{s=1}^{j} \mathbf{k}_{n_s}^*$ for each $j \in \mathbb{N}$. By Proposition 4.1(2) and Lemma 3.3, for sufficiently large n_j,

$$\begin{aligned} \|2\rho_{n_j} - \langle \mathbf{k}_{n_j}^*, \alpha\rangle\|_{\mathbb{T}} &= \|2\rho - \langle \mathbf{d}_j, \alpha\rangle\|_{\mathbb{T}} \\ &\geqslant \min\{\kappa, \gamma\}(1 + |2\mathbf{d}_j|_\eta)^{-C_1 |2\mathbf{d}_j|_\eta^{\frac{1}{\eta+1}}}. \end{aligned} \tag{5.3}$$

By Proposition 4.1 (3),

$$|\mathbf{d}_j|_\eta \leqslant (1 + 2|\log \varepsilon_{n_{j-1}}|^{-\frac{\eta}{4+\eta}})|\mathbf{k}_{n_j}^*|_\eta \leqslant 2|\mathbf{k}_{n_j}^*|_\eta.$$

Combining the above inequality with (5.3), one has

$$\|2\rho_{n_j} - \langle \mathbf{k}_{n_j}^*, \alpha\rangle\|_{\mathbb{T}} \geqslant C_4(\eta, \kappa, \gamma, h, h')\mathrm{e}^{-(2|\log \varepsilon_{n_j}|)^{\frac{2}{\eta+2}}},$$

which contradicts to (5.2). Let us choose $\tilde{n}$ such that $\rho \notin \Theta_{\tilde{n}}$ for any $n \geqslant \tilde{n}$.

If $\rho \in \mathrm{DC}(\alpha)$, we have $\rho_{\tilde{n}} \neq 0$ and by Corollary 4.1 (1),

$$\Psi'(x+\alpha)^{-1} A\mathrm{e}^{F(x)}\Psi'(x) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\rho_{\tilde{n}}} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\rho_{\tilde{n}}} \end{pmatrix}.$$

Let $\{n_s\}_{s=1}^{J^*}$ be the all resonant steps with $J^* < \infty$. Denote $\mathbf{m} = \sum_{s=1}^{J^*} \mathbf{k}_{n_s}^*$ and let $Q(x) = R_{\frac{\langle \mathbf{m}, x\rangle}{2}}$. Then

$$Q(x+\alpha)^{-1} \begin{pmatrix} \mathrm{e}^{\mathrm{i}\rho_{\tilde{n}}} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\rho_{\tilde{n}}} \end{pmatrix} Q(x) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\rho} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\rho} \end{pmatrix}.$$

Let $\Psi := \Psi' \cdot Q$. This finishes the proof.

If $2\rho = \langle \mathbf{k}, \alpha\rangle \mod 2\pi$, the proof follows from Corollary 4.1 (2) if $\rho_{\tilde{n}} = 0$. If $\rho_{\tilde{n}} \neq 0$, by using Corollary 4.1 (1),

$$\Psi'(x+\alpha)^{-1} A\mathrm{e}^{F(x)}\Psi'(x) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\rho_{\tilde{n}}} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\rho_{\tilde{n}}} \end{pmatrix}.$$

Choose $\mathbf{m} \in \mathbb{Z}_*^d$ such that $2\rho_{\tilde{n}} = \langle \mathbf{m}, \alpha\rangle \mod 2\pi$ and let $Q(x) = R_{\frac{\langle \mathbf{m}, x\rangle}{2}}$. Then

$$Q(x+\alpha)^{-1} \begin{pmatrix} \mathrm{e}^{\mathrm{i}\rho_{\tilde{n}}} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\rho_{\tilde{n}}} \end{pmatrix} Q(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let $\Psi := \Psi' \cdot Q$. This finishes the proof.

5.3. Proof of Theorem 1.6

Let $E = 2\cos 2\rho$ with $-2\rho = \langle \mathbf{m}, \alpha \rangle \mod 2\pi$. Choose $|\lambda|$ sufficiently small such that

$$|\lambda| < \min\{\lambda_0, \mathrm{e}^{-2h|\mathbf{m}|_\eta}\}. \tag{5.4}$$

To prove Theorem 1.6, we only need to show resonant case only appears once in the setting of Proposition 4.1. For simplicity, we denote $\varepsilon := \|\lambda v\|_h$, and thus by (5.4),

$$\varepsilon < \mathrm{e}^{-h|\mathbf{m}|_\eta}. \tag{5.5}$$

We sketch the proof into the following five steps.

Step 1. Upper triangularization. Let $U_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathrm{e}^{\mathrm{i}\rho} & -1 \\ 1 & \mathrm{e}^{-\mathrm{i}\rho} \end{pmatrix}$. Then

$$U_0^{-1} A_E \mathrm{e}^{F_v(x)} U_0 = A\mathrm{e}^{F(x)},$$

where $A = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\rho} & \zeta \\ 0 & \mathrm{e}^{-\mathrm{i}\rho} \end{pmatrix}$ with $\zeta_0 = -1 - \mathrm{e}^{\mathrm{i}\rho}$ and $F = U_0^{-1} F_v U_0$. We have

$$\widehat{F}_{\mathbf{m}} = U_0^{-1}\begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix} U_0 = \frac{\lambda}{2}\begin{pmatrix} \mathrm{e}^{\mathrm{i}\rho} & -1 \\ \mathrm{e}^{2\mathrm{i}\rho} & -\mathrm{e}^{\mathrm{i}\rho} \end{pmatrix},$$

and thus $F^{2,1}(x) = \frac{\lambda}{2}\mathrm{e}^{2\mathrm{i}\rho}\mathrm{e}^{\mathrm{i}\langle \mathbf{m}, x\rangle}$ with $|(\widehat{F}_{\mathbf{m}})^{2,1}| = \frac{\varepsilon}{2}\mathrm{e}^{-h|\mathbf{m}|_\eta}$.

Step 2. Diagonalization. Let

$$P = \begin{pmatrix} 1 & \frac{\zeta}{\mathrm{e}^{-\mathrm{i}\rho} - \mathrm{e}^{\mathrm{i}\rho}} \\ 0 & 1 \end{pmatrix}.$$

By (5.5), we have $|\mathbf{m}|_\eta \leqslant N_0 = \frac{2|\log \varepsilon|}{h - h_1}$ and thus $\rho \in \Theta_0(\mathbf{m})$. So, by Lemma 3.3, we have $\|P\| \leqslant \mathrm{e}^{|\log \varepsilon|^{\frac{2}{2+\eta}}}$. Direct calculation shows that

$$P^{-1} A \mathrm{e}^{F(x)} P = A' \mathrm{e}^{F'(x)},$$

where $A' = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\rho} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\rho} \end{pmatrix}$ and $F'(x) = \begin{pmatrix} g_1(x) & g_2(x) \\ F^{2,1}(x) & -g_1(x) \end{pmatrix}$ with $\|g_1\|_h, \|g_2\|_h \leqslant \|F'\|_h \leqslant \varepsilon^{\frac{9}{10}} =: \varepsilon'$.

Step 3. Eliminate non-resonant terms. Let $N' = C_2|\log \varepsilon|^{1+\frac{\eta}{2}} - N$ and $\sigma = \varepsilon'^{\frac{1}{3}}$. One can check that the space decomposition with respect to A', σ is well defined:

$$\mathcal{B}_{h,r}^{\mathrm{nre}}(\sigma) = \left\{ F(x) = \mathcal{T}_{N'} F(x) - \begin{pmatrix} 0 & 0 \\ c_{\mathbf{m}} & 0 \end{pmatrix} \mathrm{e}^{\mathrm{i}\langle m, x\rangle} \right\},$$

$$\mathcal{B}_{h,r}^{\mathrm{re}}(\sigma) = \left\{ F(x) = \mathcal{R}_{N'} F(x) + \begin{pmatrix} 0 & 0 \\ c_{\mathbf{m}} & 0 \end{pmatrix} \mathrm{e}^{\mathrm{i}\langle m, x\rangle} \right\}.$$

By Lemma 3.1, there exist $Y \in \mathcal{B}^{\mathrm{nre}}_{h,r}(\sigma)$ and $F^{\mathrm{re}} \in \mathcal{B}^{\mathrm{re}}_{h,r}(\sigma)$ such that

$$e^{-Y(x+\alpha)} A' e^{F'(x)} e^{Y(x)} = A' e^{F^{\mathrm{re}}(x)}$$

with $\|F^{\mathrm{re}} - \mathbb{P}_{\mathrm{re}} F'\|_h \leqslant 2\varepsilon'^{\frac{4}{3}} \leqslant 2\varepsilon^{\frac{6}{5}}$.

Step 4. Eliminate resonant terms. Let $Q(x) := R_{\frac{\langle \mathbf{m}, x\rangle}{2}}$. By Lemma 3.6, there exists $F_1 \in \mathcal{B}_{h_1,r_1}$ such that

$$Q(x+\alpha)^{-1} A' e^{F^{\mathrm{re}}(x)} Q(x) = \begin{pmatrix} 1 & 0 \\ c_{\mathbf{m}} & 1 \end{pmatrix} e^{F_1(x)} =: A_1 e^{F_1(x)},$$

where $c_{\mathbf{m}} = (\widehat{F}^{\mathrm{re}}_{\mathbf{m}})^{2,1}$ and $\|F_1\|_{h_1} \leqslant \varepsilon^{100}$. Hence,

$$|c_{\mathbf{m}} - (\widehat{F}_{\mathbf{m}})^{2,1}| = |(\widehat{F}^{\mathrm{re}}_{\mathbf{m}})^{2,1} - (\mathbb{P}_{\mathrm{re}} \widehat{F}'_{\mathbf{m}})^{2,1}| \leqslant \|F^{\mathrm{re}} - \mathbb{P}_{\mathrm{re}} F'\|_h e^{-h|\mathbf{m}|_\eta} \leqslant 2\varepsilon^{\frac{6}{5}} e^{-h|\mathbf{m}|_\eta},$$

which shows that $|c_{\mathbf{m}}| \geqslant |(\widehat{F}_{\mathbf{m}})^{2,1}| - 2\varepsilon^{\frac{6}{5}} e^{-h|\mathbf{m}|_\eta} > 0$.

Step 5. Reducibility. We claim that $\rho \notin \Theta_n$ for any $n \geqslant 1$. In fact, by $\alpha \in \mathrm{DC}^d_{\gamma,\tau}$, one can apply inductively Proposition 4.1(1) to show that for $n \geqslant 1$ and $\mathbf{k} \in \mathcal{T}_{N_n} \Gamma_{r_n}$,

$$\rho_n = 0, \text{ and } \|2\rho_n - \langle \mathbf{k}, \alpha\rangle\|_{\mathbb{T}} = \|\langle \mathbf{k}, \alpha\rangle\|_{\mathbb{T}} \geqslant e^{|\log \varepsilon_n|^{\frac{2}{2+\eta}}}.$$

Then there exist $Y_n \in \mathcal{B}_{h_n,r_n}$ and $F_n \in \mathcal{B}_{h_n,r_n}$ such that for any $n \geqslant 1$,

$$e^{-Y_n(x+\alpha)} A_1 e^{F_n(x)} e^{Y_n(x)} = A_1 e^{F_{n+1}(x)},$$

with $\|Y_n\|_{h_n} \leqslant \varepsilon_n^{\frac{1}{2}}$ and $\|F_n\|_{h_n} \leqslant \varepsilon_n$. Finally, we choose $H = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$, then

$$H^{-1} A_1 H = \begin{pmatrix} 1 & \zeta \\ 0 & 1 \end{pmatrix} \quad \text{with } \zeta = c_{\mathbf{m}}.$$

Let $\Psi = U_0 \cdot P \cdot e^Y \cdot Q \cdot \prod_{n=1}^{\infty} e^{Y_n} \cdot H$. The proof is finished by $c_{\mathbf{m}} \neq 0$.

6. One-frequency examples

In this section, by Avila's global theory of one-frequency analytic $\mathrm{SL}(2,\mathbb{C})$ cocycles [2], we determine the spectrum of two examples of one-frequency non-self-adjoint Schrödinger operators.

6.1. Proof of Theorem 1.3

To calculate the Lyapunov exponent of (α, S) with $S(x) = \begin{pmatrix} E-\lambda e^{ix} & -1 \\ 1 & 0 \end{pmatrix}$, let us complexify the phase

$$S_\epsilon(x) := S(x + i\epsilon) = \begin{pmatrix} E - \lambda e^{i(x+i\epsilon)} & -1 \\ 1 & 0 \end{pmatrix}.$$

Denote by $L(E,\epsilon) := L(\alpha, S_\epsilon)$ the Lyapunov exponent of (α, S_ϵ) and by $\omega(E,\epsilon) := \omega(\alpha, S_\epsilon)$ the acceleration of that. For sufficiently large $\epsilon > 0$,

$$S_\epsilon(x) = \begin{pmatrix} E & -1 \\ 1 & 0 \end{pmatrix} + o(1),$$

then by the continuity of Lyapunov exponent [18, 37],

$$L(E,\epsilon) = \log\left|\frac{E}{2} + \frac{\sqrt{E^2-4}}{2}\right| + o(1).$$

According to the quantization of acceleration in Theorem 2.1, for $\epsilon > 0$ large enough,

$$\omega(E,\epsilon) = 0, \quad L(E,\epsilon) = \log\left|\frac{E}{2} + \frac{\sqrt{E^2-4}}{2}\right|. \tag{6.1}$$

A similar argument works for sufficiently small $\epsilon < 0$,

$$S_\epsilon(x) = e^{ix}e^{-\epsilon}\begin{pmatrix} -\lambda & 0 \\ 0 & 0 \end{pmatrix} + o(1),$$

and furthermore,

$$\omega(E,\epsilon) = -1, \quad L(E,\epsilon) = -\epsilon + \log|\lambda|. \tag{6.2}$$

Abbreviate the spectrum $\Sigma_{\lambda \exp,\alpha}$ by Σ. Let us calculate $L(E)$ for $E \in \Sigma$ firstly. For $|\lambda| \leqslant 1$, by the convexity of $L(E,\epsilon)$ with respect to ϵ and (6.1)–(6.2), we always have $\omega(E,0) = 0$, then by Proposition 2.1 and Proposition 2.2, we have

$$L(E) = 0.$$

For $|\lambda| > 1$, by Proposition 2.1 and Proposition 2.2, $L(E,\epsilon)$ cannot be affine, which means $\omega(E,\epsilon) \neq \omega(E,-\epsilon)$ for any small $|\epsilon|$. By the convexity and (6.1)–(6.2), we have $\omega(E,0) = 0$ and $\omega(E,\epsilon) = -1$ for any $\epsilon < 0$, which implies

$$L(E) = \log|\lambda|.$$

To show $L(E)$ for all $E \in \mathbb{C}$, we need to add the calculation for $E \notin \Sigma$. Note that, by Proposition 2.1 and Proposition 2.2, $\omega(E,\epsilon)$ is locally constant around $\epsilon = 0$. By the convexity of $L(E,\epsilon)$, it is easy to see that if $\omega(E,0) = 0$,

$$L(E) = \lim_{\epsilon\to+\infty} L(E,\epsilon) = \log\left|\frac{E}{2} + \frac{\sqrt{E^2-4}}{2}\right|. \tag{6.3}$$

For $|\lambda| \leqslant 1$, we have $\omega(E,0) = 0$; then the result follows directly from (6.3). For $|\lambda| > 1$, for better understanding the case, denote

$$\mathcal{I} = \Big\{E \in \mathbb{C} : \log|\lambda| > \log\Big|\frac{E}{2} + \frac{\sqrt{E^2-4}}{2}\Big|\Big\},$$
$$\mathcal{O} = \Big\{E \in \mathbb{C} : \log|\lambda| < \log\Big|\frac{E}{2} + \frac{\sqrt{E^2-4}}{2}\Big|\Big\}. \tag{6.4}$$

As shown in Figure 3, if $\omega(E,0) = -1$, by (6.2) we have

$$L(E) = \log|\lambda|,$$

which corresponds to $E \in \mathcal{I}$, i.e., the interior of the ellipse; if $\omega(E,0) = 0$, then the result follows from (6.3) again, which corresponds to $E \in \mathcal{O}$, i.e., the outside of the ellipse.

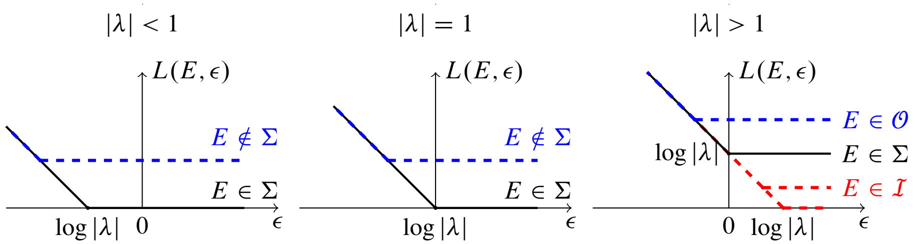

Figure 3. Lyapunov exponent $L(E,\epsilon)$ for $E \in \mathbb{C}$.

6.2. Proof of Theorem 1.1

We need to distinguish between two cases.

Case 1. $|\lambda| \leqslant 1$. Note that $\log|\frac{E}{2} + \frac{\sqrt{E^2-4}}{2}| = 0$ if and only if $E \in [-2,2]$. If $E \in \Sigma$, by Theorem 1.3 we have $L(E) = 0 = \log|\frac{E}{2} + \frac{\sqrt{E^2-4}}{2}|$ which implies that $E \in [-2,2]$. On the contrary, if $E \notin \Sigma$, then $L(E) > 0$ according to $(\alpha, S) \in \mathcal{UH}$, which implies that $\log|\frac{E}{2} + \frac{\sqrt{E^2-4}}{2}| = L(E) > 0$ by Theorem 1.3 and thus $E \notin [-2,2]$.

Case 2. $|\lambda| > 1$. Recall that $\mathcal{E}_\lambda = \{E : E = \lambda e^{i\theta} + \lambda^{-1}e^{-i\theta}, \theta \in [0,2\pi]\}$ and $E \in \mathcal{E}_\lambda$ if and only if $\log|\lambda| = \log|\frac{E}{2} + \frac{\sqrt{E^2-4}}{2}|$. If $E \in \Sigma$, by Theorem 1.3 we deduce that $E \in \mathcal{E}_\lambda$. If $E \notin \Sigma$, by Proposition 2.1 and Proposition 2.2, we have $L(E) > 0$ and $L(E,\epsilon)$ is affine with respect to ϵ around $\epsilon = 0$. Hence, we have either $E \in \mathcal{I}$ or $E \in \mathcal{O}$, see the definition in (6.4) and the explanation in Figure 3, and thus $E \notin \mathcal{E}_\lambda$.

6.3. Proof of Proposition 1.1

The proof is essentially contained in [49], we include the proof here just for completeness.

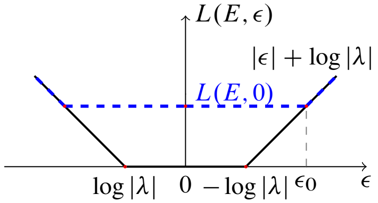

Figure 4. Lyapunov exponent $L(E,\epsilon)$ with $|\lambda| \in (0,1)$.

Denote by $L(E,\epsilon)$ the Lyapunov exponent of $H_{v_\epsilon,\alpha}$. As shown in Figure 4, Avila [2] proved that for any $E \in \mathbb{C}$, any $\epsilon \in \mathbb{R}$,

$$L(E,\epsilon) = \max\{\log|\lambda| + |\epsilon|, L(E,0)\}. \tag{6.5}$$

In particular,

$$L(E,\epsilon) = \max\{\log|\lambda| + |\epsilon|, 0\} \quad \text{for all } E \in \Sigma_{2\lambda\cos,\alpha}. \tag{6.6}$$

Suppose that $E \in \Sigma_{2\lambda\cos,\alpha}$. Since $|\lambda| < 1$ and $|\epsilon| < -\log|\lambda|$, it follows from (6.6) that $L(E,\epsilon) = 0$ and $L(E,\epsilon+t) = 0$ when $|t| \leqslant \log|\lambda| - |\epsilon|$, thus $E \in \Sigma_{v_\epsilon,\alpha}$ according to Proposition 2.1 and Proposition 2.2.

Suppose that $E \notin \Sigma_{2\lambda\cos,\alpha}$. By Proposition 2.1, we have $L(E,0) > 0$ and $(\alpha, S_{E,v_0}) \in \mathcal{UH}$. Then it follows from (6.5) that

$$L(E,\epsilon_0) = L(E,0) \quad \text{for } |\epsilon_0| \leqslant L(E,0) - \log|\lambda|.$$

Since $|\epsilon| < -\log|\lambda|$ by assumption, we have $L(E,\epsilon) = L(E,0) > 0$ and $L(E,\epsilon+t)$ is affine for $|t| < L(E,0)$, it follows from Proposition 2.1 and Proposition 2.2 again that $E \notin \Sigma_{v_\epsilon,\alpha}$. We thus finish the whole proof.

A. Proof of Lemma 3.1

We are going to use induction to show

$$e^{-Y_{j-1}(x+\alpha)} A e^{F_{j-1}^{\mathrm{nre}}(x) + F_{j-1}^{\mathrm{re}}(x)} e^{Y_{j-1}(x)} = A e^{F_j^{\mathrm{nre}}(x) + F_j^{\mathrm{re}}(x)}, \; j \geqslant 1,$$

with estimates

$$\|Y_{j-1}\|_h \leqslant \varepsilon_{j-1}^{\frac{3}{4}}, \ \|F_j^{\mathrm{re}} - F_{j-1}^{\mathrm{re}}\|_h \leqslant \varepsilon^{\frac{1}{3}}\varepsilon_{j-1}, \ \|F_j^{\mathrm{nre}}\|_h \leqslant \varepsilon_j, \tag{A.1}$$

where the sequences are defined as

$$\varepsilon_j = \varepsilon_{j-1}^{\frac{3}{2}}, \ \ \varepsilon_0 = \varepsilon, \ \ F_0 = F, \ \ F_0^{\mathrm{nre}} = \mathbb{P}_{\mathrm{nre}} F, \ \ F_0^{\mathrm{re}} = \mathbb{P}_{\mathrm{re}} F.$$

Suppose that for $j = n$, we obtain $(\alpha, A\mathrm{e}^{F_n^{\mathrm{re}}+F_n^{\mathrm{nre}}})$ and (A.1) holds. For any $Y \in \mathcal{B}_{h,r}^{\mathrm{nre}}(\sigma)$, we define

$$\begin{aligned}\widetilde{J}(Y) &:= \log \mathrm{e}^{-Y}\mathrm{e}^{F_n^{\mathrm{re}}} + Y - F_n^{\mathrm{re}},\\ \widetilde{K}(Y) &:= \log \mathrm{e}^{-Y}\mathrm{e}^{F_n^{\mathrm{re}}} + \log \mathrm{e}^{F_n^{\mathrm{re}}}\mathrm{e}^{Y} - 2F_n^{\mathrm{re}}.\end{aligned}$$

Let $J(Y)$ (resp. $K(Y)$) be the linear part of $\widetilde{J}(Y)$ (resp. $\widetilde{K}(Y)$) with respect to Y,

$$J(\cdot), K(\cdot)\colon \mathcal{B}_{h,r} \to \mathcal{B}_{h,r}.$$

Define the sequences for $j \in \mathbb{N}$ as

$$Q_{j+1} = (-1)^j J(Q_j), \ R_{j+1} = (-1)^j \mathbb{P}_{\mathrm{nre}} J(R_j), \ Q_0 = K(Y), \ R_0 = F_n^{\mathrm{nre}}.$$

Let us consider the linear operator $I_A\colon \mathcal{B}_{h,r}^{\mathrm{nre}}(\sigma) \to \mathcal{B}_{h,r}^{\mathrm{nre}}(\sigma)$ given by

$$I_A Y = L_A Y - \sum_{j=0}^{2^n-1} Q_j(Y) = A^{-1}Y(x+\alpha)A - Y(x) - \sum_{j=0}^{2^n-1} Q_j(Y(x)).$$

Since $\|F_n^{\mathrm{re}}\|_h \leqslant 2\varepsilon$, we have $\|I_A Y\| \geqslant \frac{3}{4}\varepsilon^{\frac{1}{2}}\|Y\|_h$, and thus $\|I_A^{-1}\|$ is bounded by $\frac{4}{3}\varepsilon^{-\frac{1}{2}}$. There exists Y_n such that $I_A Y_n = \mathbb{P}_{\mathrm{nre}} \sum_{j=0}^{2^n-1} R_j$, i.e.,

$$A^{-1}Y_n(x+\alpha)A - Y_n(x) - \sum_{j=0}^{2^n-1} Q_j(Y_n) = \mathbb{P}_{\mathrm{nre}} \sum_{j=0}^{2^n-1} R_j.$$

Moreover, $\|Y_n\|_h \leqslant \frac{4}{3}\varepsilon^{-\frac{1}{2}}(\varepsilon_n + 4\varepsilon\varepsilon_n) \leqslant 2\varepsilon^{-\frac{1}{2}}\varepsilon_n$. Thus,

$$\begin{aligned}\mathrm{e}^{F_{n+1}^{\mathrm{nre}}(x)+F_{n+1}^{\mathrm{re}}(x)} &= \mathrm{e}^{-A^{-1}Y_n(x+\alpha)A}\mathrm{e}^{F_n^{\mathrm{nre}}(x)+F_n^{\mathrm{re}}(x)}\mathrm{e}^{Y_n(x)}\\ &= \mathrm{e}^{-Y_n(x)-\mathbb{P}_{\mathrm{nre}}\sum_{j=0}^{2^n-1} R_j - \sum_{j=0}^{2^n-1} Q_j}\mathrm{e}^{F_n^{\mathrm{nre}}(x)+F_n^{\mathrm{re}}(x)}\mathrm{e}^{Y_n(x)},\end{aligned}$$

Let us recall the Baker–Campbell–Hausdorff formula,

$$\log(\mathrm{e}^X\mathrm{e}^W\mathrm{e}^Z) = X + W + Z + \frac{1}{2}[X,W] + \frac{1}{2}[W,Z] + \frac{1}{2}[X,Z] + O^3(X,W,Z),$$

where $O^3(X, W, Z)$ stands for the sum of terms whose Lie brackets involving three elements of X, W, Z. By the construction of R_j, Q_j and B-C-H formula,

$$F_{n+1}^{\mathrm{re}}(x) = F_n^{\mathrm{re}}(x) + \mathbb{P}_{\mathrm{re}}\Big\{-\frac{1}{2}[F_n^{\mathrm{nre}}, F_n^{\mathrm{re}}] + [F_n^{\mathrm{re}}, Y_n] + \cdots\Big\},$$
$$F_{n+1}^{\mathrm{nre}}(x) = \mathbb{P}_{\mathrm{nre}}\Big\{-\frac{1}{4}[Y_n, [F_n^{\mathrm{re}}+, Y_n]] - \frac{1}{2}[R_{2^n-1}, F_n^{\mathrm{re}}] - \frac{1}{2}[Q_{2^n-1}, F_n^{\mathrm{re}}] + \cdots\Big\}.$$

Since $\|Q_j\|_h \leqslant 2\varepsilon\|Q_{j-1}\|_h$ and $\|R_j\|_h \leqslant 2\varepsilon\|R_{j-1}\|_h$, we get that

$$\|Q_{2^n-1}\|_h \leqslant (2\varepsilon)^{2^n-1}\|Q_0\|_h \leqslant (2\varepsilon)^{2^n}\|Y_n\|_h,$$
$$\|R_{2^n-1}\|_h \leqslant (2\varepsilon)^{2^n-1}\|F_0^{\mathrm{nre}}\|_h \leqslant (2\varepsilon)^{2^n}.$$

Thus, we deduce that

$$\|F_{n+1}^{\mathrm{re}} - F_n^{\mathrm{re}}\|_h \leqslant \varepsilon^{\frac{1}{3}}\varepsilon_n, \quad \|F_{n+1}^{\mathrm{nre}}\|_h \leqslant \varepsilon_n^{\frac{3}{2}} = \varepsilon_{n+1}.$$

Now, we let $Y = \log(\prod_{n=0}^{\infty} \mathrm{e}^{Y_n})$ and $F^{\mathrm{re}} = \lim_{n\to\infty} F_n^{\mathrm{re}}$. Since $\mathcal{B}_{h,r}^{\mathrm{re}}(\sigma)$ and $\mathcal{B}_{h,r}^{\mathrm{nre}}(\sigma)$ are closed subspace in $\mathcal{B}_{h,r}$, it follows from Proposition 3.1 that $Y \in \mathcal{B}_{h,r}$ and $F^{\mathrm{re}} \in \mathcal{B}_{h,r}^{\mathrm{re}}(\sigma)$. By direct calculation, we have $\|Y\|_h \leqslant \varepsilon^{\frac{1}{2}}$ and $\|F^{\mathrm{re}} - \mathbb{P}_{\mathrm{re}}F\|_h \leqslant 2\varepsilon^{\frac{4}{3}}$.

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