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Remarks on discrete Dirac operators and their continuum limits

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Abstract. We discuss possible definitions of discrete Dirac operators, and discuss their continuum limits. It is well known in the lattice field theory that the straightforward discretization of the Dirac operator introduces unwanted spectral subspaces, and it is known as *the fermion doubling*. In oder to overcome this difficulty, two methods were proposed. The first one is to introduce a new term, called *the Wilson term*, and the second one is *the KS-fermion model* or *the staggered fermion model*. We discuss mathematical formulations of these, and study their continuum limits.

1. Introduction

In a recent paper by Cornean, Garde, and Jensen [3], they studied the continuum limit of discretized Dirac operators in the sense of norm resolvent convergence, and they found that they do not converges to the (usual) Dirac operators. They found that if one adds another term, then these operators converge to the Dirac operators. This corresponds to *the Wilson term* in the lattice field theory. We discuss this method briefly, and then discuss another method, *the KS-fermion* (or *the staggered fermion*) model, which is mathematically ingenious and interesting in itself. Thus, this note is partly a survey of these methods, but they are rigorously reformulated in relatively general settings, and we prove some new results on their continuum limits.

The continuum limit of a quantum Hamiltonian on the square lattice in the sense of (generalized) norm resolvent convergence was studied by Nakamura and Tadano [8], and several papers have been published based on the idea of the norm resolvent convergence (see also [9] for the concept of generalized resolvent convergence). Cornean, Garde and Jensen [2] studied the convergence for more general Fourier multipliers, and Exner, Nakamura, and Tadano [4] considered continuum limit for quantum graph Hamiltonians. As mentioned above, Cornean, Garde, and Jensen [3] considered the

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continuum limit of discretized Dirac operators, and identified the main difficulty in showing the convergence to the continuous Dirac operators. See also Schmidt and Umeda [11] and Isozaki and Jensen [5] for closely related results.

It turned out that such difficulty was widely known in the lattice field (gauge) theory (see, e.g., [1, 10]), and it is generally called *the fermion doubling*. There are two standard methods to avoid the problem. The first one is adding an additional term to the Hamiltonian (or the Lagrangian), and it is called the *Wilson term.* The other method is called the *KS-fermion model* after Kogut and Susskind [6, 12]. We try to reformulate these methods, especially the KS-fermion method so that it is appropriate to study the continuum limit in the norm resolvent sense, and prove the convergence of the continuum limit.

The paper is constructed as follows. In Section 2, we prepare several basic tools. At first we explain the notations concerning the square lattices, function spaces, Fourier transforms, and several kinds of difference operators. Then in Section 2.2, we introduce an embedding operator from the function space on the lattice to the Lebesgue space on the Euclidean space, which is necessary to study the continuum limit following [8]. In Section 2.3, we recall the definition of the Dirac operators on the Euclidean spaces. In Section 3, we consider the discretization of the Dirac operator using the symmetric difference operators, and explain why it is not appropriate to consider the continuum limit. In Section 4, we introduce the Wilson term, and show the convergence of the continuum limit in the norm resolvent sense for Hamiltonians with the Wilson term under suitable conditions. Section 5 is devoted to the discussion of the KS-fermion model. We introduce the one-component KS-Hamiltonian on d-dimensional lattice (with fermion doubling problem), and then transform it to a 2^d components operator without the fermion doubling problem in Section 5.1. We briefly examine the spectral properties of this operator in Section 5.2, and prove the convergence to a continuum limit in Section 5.3. Here the number of components, 2^d, can be higher than those of the standard Dirac operator on $\mathbb{R}^d$. We discuss the model for the dimensions 1, 2, and 3 in Section 6, and show that for $d = 1$ the model is appropriate (and in fact studied in [3] already), and for $d = 2$ and 3, the continuum limit is decomposed to a direct sum of two standard Dirac operators.

2. Preliminaries

2.1. Notations

We denote the square lattice in $\mathbb{R}^d$ with the lattice spacing $h > 0$ by $h\mathbb{Z}^d = \{hn \mid n \in \mathbb{Z}^d\}$. Let $\{e_j\}_{j=1}^d$ be the standard basis of $\mathbb{R}^d$, i.e., $e_j = (\delta_{j,k})_{k=1}^d \in \mathbb{Z}^d$, $j = 1, \dots, d$, where $\delta_{j,k}$ denotes the Kronecker delta symbol. The basis (or the generators) of $h\mathbb{Z}^d$

is given by $\{he_1, \dots, he_d\}$. We recall the dual space (or the dual module) of $h\mathbb{Z}^d$ is given by $h^{-1}\mathbb{T}^d = \mathbb{R}^d/(h^{-1}\mathbb{Z}^d)$. We note the dual lattice of $h\mathbb{Z}^d$ is $h^{-1}\mathbb{Z}^d$, and hence the inner product $z \cdot \xi$ is well-defined modulo $\mathbb{Z}$ for $z \in h\mathbb{Z}^d$, $\xi \in h^{-1}\mathbb{T}^d$.

We denote the standard L^2 space on the d-dimensional Euclidean space by $L^2(\mathbb{R}^d)$. We use the square summable function space on $h\mathbb{Z}^d$, namely $\ell^2(h\mathbb{Z}^d)$, and we use the norm defined by

$$\|u\|^2_{\ell^2(h\mathbb{Z}^d)} = h^d \sum_{z \in h\mathbb{Z}^d} |u(z)|^2, \quad u \in \ell^2(h\mathbb{Z}^d).$$

We denote the Fourier transform on $\mathbb{R}^d$ by

$$\mathcal{F}u(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi} u(x)dx \quad \text{for } u \in L^1(\mathbb{R}^d),\ \xi \in \mathbb{R}^d,$$

and the inverse Fourier transform by $\mathcal{F}^*$. On the lattice $h\mathbb{Z}^d$, the Fourier transform $F_h: \ell^2(h\mathbb{Z}^d) \to L^2(h^{-1}\mathbb{T}^d)$ is defined by

$$F_h u(\xi) = h^d \sum_{z \in h\mathbb{Z}^d} e^{-2\pi i z\cdot\xi} u(z), \quad \xi \in h^{-1}\mathbb{T}^d.$$

for $u \in \ell^2(h\mathbb{Z}^d)$. F_h is unitary, and the inverse is given by

$$F_h^* v(z) = \int_{h^{-1}\mathbb{T}^d} e^{2\pi i z\cdot\xi} v(\xi)d\xi, \quad z \in h\mathbb{Z}^d$$

for $v \in L^2(h^{-1}\mathbb{T}^d)$.

The partial differential operator on $\mathbb{R}^d$, or the momentum operator is denoted by

$$D_j = \frac{1}{i}\frac{\partial}{\partial x_j}, \quad j = 1, \dots, d.$$

On the lattice $h\mathbb{Z}^d$, we set the symmetric difference operators

$$D^S_{h;j} u(z) = \frac{1}{2ih}(u(z + he_j) - u(z - he_j)), \quad j = 1, \dots, d, z \in h\mathbb{Z}^d,$$

as an approximation of D_j on $h\mathbb{Z}^d$, where $u \in \ell^2(h\mathbb{Z}^d)$. We also write the forward and backward difference operators by

$$D^{\pm}_{h;j} u(z) = \pm\frac{1}{ih}(u(z \pm he_j) - u(z)), \quad z \in h\mathbb{Z}^d$$

for $u \in \ell^2(h\mathbb{Z}^d)$.

2.2. Embedding of $\ell^2(h\mathbb{Z}^d)$ into $L^2(\mathbb{R}^d)$

We need an embedding operator $J_h\colon \ell^2(h\mathbb{Z}^d) \to L^2(\mathbb{R}^d)$ when we consider the continuum limit. We employ the following operators ([8], see also [2]).

We need a function $\varphi \in \mathcal{S}(\mathbb{R}^d)$ such that $\{\varphi(\cdot - n) \mid n \in \mathbb{Z}^d\}$ is an orthonormal system in $L^2(\mathbb{R}^d)$. It is well-known that this condition is equivalent to

$$\sum_{n\in\mathbb{Z}^d} |\hat{\varphi}(\xi + n)|^2 = 1 \quad \text{for } \xi \in \mathbb{R}^d, \tag{2.1}$$

where $\hat{\varphi} = \mathcal{F}\varphi$. We then set, for $z \in h\mathbb{Z}^d$,

$$\varphi_{h;z}(x) = \varphi(h^{-1}(x - z)), \quad x \in \mathbb{R}^d,$$

and we define

$$J_h u(x) = \sum_{z\in h\mathbb{Z}^d} u(z)\varphi_{h;z}(x), \quad x \in \mathbb{R}^d.$$

It is easy to see J_h is an isometry from $\ell^2(h\mathbb{Z}^d)$ to $L^2(\mathbb{R}^d)$ provided φ satisfies (2.1), and the adjoint operator is given by

$$J_h^* v(z) = h^{-d} \int_{\mathbb{R}^d} \overline{\varphi_{h;z}(x)} v(x)dx, \quad z \in h\mathbb{Z}^d,$$

where $v \in L^2(\mathbb{R}^d)$. (We remark that our notations are slightly different from [8]. In particular, $J_h = P_h^*$ in [8]). In this paper, we make the following assumption.

Assumption A. φ satisfies the condition (2.1), and $\mathrm{supp}[\hat{\varphi}] \subset (-1,1)^d$.

We note there exists various such φ's, and we simply choose one and fix it here. See [8] for the detail.

2.3. Free Dirac operators

We recall the definition of the Dirac operators on $\mathbb{R}^d$. See, e.g., Thaller [13] for the survey on Dirac operators. For simplicity, we mainly discuss the free operators without perturbations here. Let $\alpha_1, \dots, \alpha_d$ and β be a set of $N \times N$ Hermitian matrices such that

$$\alpha_i\alpha_j + \alpha_j\alpha_i = 0, \quad \alpha_i\beta + \beta\alpha_i = 0, \quad i \neq j,$$

and $\alpha_j^2 = \beta^2 = \mathbf{1}_N$, where $N \in 2\mathbb{N}$ and $\mathbf{1}_N$ denotes the $N \times N$ identity matrix. Typical choices for $d = 1, 2, 3$ are as follows. We denote a set of Pauli matrices by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For $d = 1$, we set $N = 2$ and $\alpha_1 = \sigma_1$ and $\beta = \sigma_3$. For $d = 2$, we set $N = 2$ and $\alpha_1 = \sigma_1, \alpha_2 = \sigma_2$ and $\beta = \sigma_3$. For $d = 3$, we set $N = 4$ and

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \text{ for } j = 1,2,3; \quad \beta = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{pmatrix}.$$

We then define the (free) Dirac operator by

$$H_0 = \sum_{j=1}^{d} D_j \alpha_j + m\beta \quad \text{on } [L^2(\mathbb{R}^d)]^{\oplus N},$$

where $D_j = -i\partial/\partial x_j$, $j = 1, \ldots, d$, and $m \geq 0$.

It is easy to see by the anti-commuting properties,

$$H_0^2 = \sum_{j=1}^{d} D_j^2 \alpha_j^2 + m^2\beta^2 = (-\triangle + m^2)\mathbf{1}_N,$$

and hence H_0 is elliptic. It is also straightforward to show H_0 is self-adjoint with $\mathcal{D}(H_0) = [H^1(\mathbb{R}^d)]^{\oplus N}$. We note

$$\mathcal{F} H_0 \mathcal{F}^* u(\xi) = \hat{H}_0(\xi) u(\xi), \quad \text{where } \hat{H}_0(\xi) = \sum_{j=1}^{d} 2\pi\xi_j \alpha_j + m\beta.$$

Since $\hat{H}_0(\xi)^2 = |2\pi\xi|^2 + m^2$, it can be easily shown that the eigenvalues of $\hat{H}_0(\xi)$ are $\pm\sqrt{|2\pi\xi|^2 + m^2}$ with multiplicities $N/2$ each.

3. Straightforward discretization of Dirac operators and the fermion doubling

We now discretize the Dirac operators on $h\mathbb{Z}^d$. Using $D_{h;j}^S$, we may define the discretized Dirac operator by

$$H_{0;h}^S = \sum_{j=1}^{d} D_{h,j}^S \alpha_j + m\beta \quad \text{on } [\ell^2(h\mathbb{Z}^d)]^{\oplus N},$$

which is a bounded symmetric operator. The symbol of $H_{0;h}^S$, $\hat{H}_{0;h}^S(\xi)$, is defined by

$$\hat{H}_{0;h}^S(\xi) v(\xi) = F_h H_{0;h}^S F_h^* v(\xi) = \Big(\sum_{j=1}^{d} h^{-1} \sin(2\pi h\xi_j) \alpha_j + m\beta \Big) v(\xi)$$

for $v \in [L^2(h^{-1}\mathbb{T}^d)]^{\oplus N}$. The eigenvalues of $\hat{H}^S_{0;h}(\xi)$ are given by

$$E_{\pm,h}(\xi) = \pm\Big(h^{-2}\sum_{j=1}^{d}\sin^2(2\pi h\xi_j) + m^2\Big)^{1/2}, \quad \xi \in h^{-1}\mathbb{T}^d.$$

We note the eigenvalues of H_0 are given by $E_\pm(\xi) = \pm\sqrt{|2\pi\xi|^2 + m^2}$ and $|E_\pm(\xi)|$ have only one critical point (local minimal point) at $\xi = 0$ in $\mathbb{R}^d$ with the minimal value m if $m > 0$. If $m = 0$, $E_\pm(\xi) = 0$ only at $\xi = 0$. On the other hand, $|E_{\pm,h}(\xi)|$ has 2^d local minimal points in $h^{-1}\mathbb{T}^d$

$$\{0, (2h)^{-1}\}^d = \{\xi \in h^{-1}\mathbb{T}^d \mid \xi_j = 0 \text{ or } (2h)^{-1}, j = 1,\dots,d\}$$

(with the minimal value m) if $m > 0$, and the 2^d zero points $\{0, (2h)^{-1}\}^d$ if $m = 0$. Hence, when $h \to 0$, the resolvent of $H^S_{0;h}$ converges to the direct sum of 2^d copies of the resolvent of H_0 with suitable identification. In particular, $H^S_{0;h}$ cannot converges to the resolvent of H_0 in the norm resolvent sense (see [3, Theorems 4.7 and 5.7]). In physics terminology, this implies $H^S_{0;h}$ describes 2^d different fermion particles, and thus this phenomenon is called *the fermion doubling* ([1, 10]). For this reason, $H^S_{0;h}$ is not considered a reasonable discretization of the Dirac operator.

4. The Wilson term

One standard procedure to avoid the fermion doubling is adding a term of the form

$$S_W = \rho(-\triangle_h)\beta,$$

to the Hamiltonian, where $\triangle_h$ is the standard difference Laplacian defined by

$$-\triangle_h u(z) = h^{-2}\sum_{j=1}^{d}(2u(z) - u(z+he_j) - u(z-he_j)), \quad z \in h\mathbb{Z}^d,$$

and $\rho > 0$ is a small coupling constant. S_W is called the *Wilson term* (see [1] and [10, Section 4.3]). We set

$$\tilde{H}_{0;h} = H^S_{0;h} + S_W.$$

If $\rho \to 0$ and $\rho h^{-2} \to \infty$ as $h \to 0$, one can show that $\tilde{H}_{0;h}$ converges to H_0 in the norm resolvent sense as $h \to 0$ (Theorem 4.1. See also [3, Sections 5.1], where the coupling constant is chosen as $\rho = h$).

The Wilson term destroys the fermion doubling for the following simple reason. The symbol of $\widetilde{H}_{0;h}$ is given by

$$\widehat{H}_{0;h}(\xi) = \sum_{j=1}^{d} h^{-1}\sin(2\pi h\xi_j)\alpha_j + \Big(m + \rho\sum_{j=1}^{d} 2h^{-2}(1-\cos(2\pi h\xi_j))\Big)\beta,$$

and its eigenvalues are given by

$$\widetilde{E}_{0;\pm}(\xi) = \pm\Big(\sum_{j=1}^{d} h^{-2}\sin^2(2\pi h\xi_j) + \Big(m + \rho\sum_{j=1}^{d} 2h^{-2}(1-\cos(2\pi h\xi_j))\Big)^2\Big)^{1/2}.$$

These eigenvalues $|\widetilde{E}_{0;\pm}|$ still have 2^d local minimal points in the case $m > 0$, but $|\widetilde{E}_{0;\pm}| \geq m + \rho h^{-2}$ at these local minima, except for $\xi = 0$, and they diverge to $+\infty$ as $h \to 0$. In the case $m = 0$, these eigenvalues are at least of order $O(\rho h^{-2})$ away from any neighborhood of $\xi = 0$, and hence the absolute values of eigenvalues diverges to $+\infty$ as $h \to 0$. On the other hand, if $\rho \to 0$, the Wilson term is negligible in a neighborhood of $\xi = 0$ as $h \to 0$. Specifically, we have

Theorem 4.1. *Suppose $\rho \to 0$ and $\rho h^{-2} \to \infty$ as $h \to 0$. Then for $z \in \mathbb{C} \setminus \mathbb{R}$,*

$$\|(H_0 - z)^{-1} - J_h(\widetilde{H}_{0;h} - z)^{-1}J_h^*\|_{\mathcal{B}(L^2(\mathbb{R}^d))} \to 0, \quad \textit{as } h \to 0.$$

Since J_h is an isometry, this also implies

$$\|J_h^*(H_0 - z)^{-1}J_h - (\widetilde{H}_{0;h} - z)^{-1}\|_{\mathcal{B}(\ell^2(h\mathbb{Z}^d))} \to 0, \quad \textit{as } h \to 0.$$

Proof. The proof is essentially the same as the argument in [8, Section 2], and [3, Sections 4.1 and 5.2]. We only sketch the argument. We follow the notations of [8], and we write $Q_h = F_h J_h^* \mathcal{F}^*$ and $\widehat{H}_0 = \mathcal{F} H_0 \mathcal{F}^* = \widehat{H}_0(\xi)\cdot$. Then we have

$$\|(1 - J_h J_h^*)(H_0 - z)^{-1}\| = \|(1 - Q_h^* Q_h)(\widehat{H}_0 - z)^{-1}\| \leq Ch \tag{4.1}$$

by the same proof as in [8, Lemma 2.2]. Now, we note

$$|\widehat{H}_{0;h}(\xi) - \widehat{H}_0(\xi)| \leq Ch^2|\xi|^3 + C|\rho|\,|\xi|^2$$

for $\xi \in \mathbb{R}^d$, where $|\cdot|$ in the left-hand side denotes the operator norm in $\mathbb{C}^N$. We also note

$$|(\widehat{H}_0(\xi) - z)^{-1}| \leq |\xi|^{-1}$$

and

$$|(\widehat{H}_{0;h}(\xi) - z)^{-1}| \leq \max_{\pm}|\widetilde{E}_{0;\pm}(\xi) - z|^{-1} \leq C(|\xi|^{-1} + |\rho|^{-1}h^2)$$

on the support of $\hat{\varphi}(h\xi)$. Combining these, we have

$$\begin{aligned}|(\hat{H}_0(\xi)-z)^{-1}-(\hat{H}_{0;h}(\xi)-z)^{-1}| &\le Ch^2|\xi|+C|\rho|+C|\rho|^{-1}h^4|\xi|^2\\ &\le Ch+C|\rho|+C|\rho|^{-1}h^2\end{aligned}$$

on the support of $\hat{\varphi}(h\xi)$. This implies

$$\|(\widetilde{H}_{0;h}-z)^{-1}J_h^*-J_h^*(H_0-z)^{-1}\|\le C(h+|\rho|+|\rho h^{-2}|^{-1})$$

as well as [8, Lemma 2.3]. Combining this with (4.1), we arrive at the conclusion. ■

5. The KS-fermion model

5.1. The construction of the KS-Hamiltonian

Here we describe an interpretation of an idea by Susskind [12] (see also Kogut and Susskind [6] and Rothe [10, Section 4.4]), which is called the KS-fermion (or the staggered fermion) model. We write

$$s_j(n)=\sum_{k=1}^{j} n_k,\quad \text{for } n\in\mathbb{Z}^d,\ j=1,\dots,d,$$

and we also set $s_0(n)=0$. We define operators $X_{h;j}$ and Y_h on $\ell^2(h\mathbb{Z}^d)$ by

$$\begin{aligned}X_{h;j}u(z)&=(-1)^{s_{j-1}(z/h)}D^S_{h;j}u(z), && z\in h\mathbb{Z}^d,\\ Y_hu(z)&=(-1)^{s_d(z/h)}u(z), && z\in h\mathbb{Z}^d,\end{aligned}$$

where $u\in\ell^2(h\mathbb{Z}^d)$ and $j=1,\dots,d$. By direct computations, we can easily show

$$X_{h;j}X_{h;k}+X_{h;k}X_{h;j}=0\ \text{ if } j\ne k;\quad X_{h;j}Y_h+Y_hX_{h;j}=0\ \text{ for } j=1,\dots,d,$$

and $X_{h;j}^2=(D^S_{h;j})^2$, $Y_h^2=1$. These properties suggest that

$$\widetilde{H}_{\mathrm{KS};h}=\sum_{j=1}^{d}X_j+mY$$

may be considered as a discrete Dirac operator on $\ell^2(h\mathbb{Z}^d)$. In particular, we have

$$\begin{aligned}(\widetilde{H}_{\mathrm{KS};h})^2u(z)&=\sum_{j=1}^{d}(D^S_{h;j})^2u(z)+m^2u(z)\\ &=\sum_{j=1}^{d}(2h)^{-2}\big(2u(z)-u(z+2he_j)-u(z-2he_j)\big)+m^2u(z)\\ &=(-\triangle_{2h}+m^2)u(z)\end{aligned}$$

for $u \in \ell^2(h\mathbb{Z}^d)$. Whereas $\widetilde{H}_{\mathrm{KS};h}$ is a scalar operator, i.e., an operator acting on the one-component function space, it still has the fermion doubling problem. In order to solve this problem, we transform the operator $\widetilde{H}_{\mathrm{KS};h}$ to an operator $H_{\mathrm{KS};h}$ on $[\ell^2((2h)\mathbb{Z}^d)]^{\oplus 2^d}$. By doubling the lattice spacing, we reduce the period of the dual space by half, i.e., $((2h)\mathbb{Z}^d)' = (2h)^{-1}\mathbb{T}^d$, and remove the problematic periodic critical points. In order to double the lattice spacing, we increase the number of components to 2^d, in the following way. We define the set of indices Λ by

$$\Lambda = \{0,1\}^d \subset \mathbb{Z}^d, \quad |\Lambda| = 2^d,$$

and we write $a = (a_1, \dots, a_d) \in \Lambda$, where $a_j \in \{0,1\}$, $j = 1, \dots, d$. We consider $2^d \times 2^d$-matrices of the form $L = (L_{a,b})_{a,b\in\Lambda}$. We denote

$$[\ell^2((2h)\mathbb{Z}^d)]^{\oplus\Lambda} = \big\{(u_a(z))_{a\in\Lambda} \;\big|\; u_a \in \ell^2((2h)\mathbb{Z}^d), a \in \Lambda\big\}.$$

We define a unitary operator $U_h\colon \ell^2(h\mathbb{Z}^d) \to [\ell^2((2h)\mathbb{Z}^d)]^{\oplus\Lambda}$ as follows:

$$(U_h u)_a(z) = 2^{-d/2} u(z + ha), \quad z \in (2h)\mathbb{Z}^d, a \in \Lambda,$$

for $u \in \ell^2(h\mathbb{Z}^d)$. The adjoint operator is given by

$$(U_h^* w)(z + ha) = 2^{d/2} w_a(z), \quad z \in (2h)\mathbb{Z}^d, a \in \Lambda,$$

where $w = (w_a)_{a\in\Lambda} \in \ell^2((2h)\mathbb{Z}^d)]^{\oplus\Lambda}$. Now, we define the KS-Hamiltonian by

$$H_{\mathrm{KS};h} = U_h \widetilde{H}_{\mathrm{KS};h} U_h^*.$$

By direct computations, we learn the (a,b)-component of the matrix operator

$$U_h X_{h;j} U_h^*$$

is given by

$$(U_h X_{h;j} U_h^*)_{a,b} u_b(z) = \begin{cases} (-1)^{s_{j-1}(a)} D^+_{2h;j} u_b(z) & \text{if } b = a - e_j, \\ (-1)^{s_{j-1}(a)} D^-_{2h;j} u_b(z) & \text{if } b = a + e_j, \\ 0 & \text{otherwise,} \end{cases}$$

for $u_b \in \ell^2((2h)\mathbb{Z}^d)$, $a, b \in \Lambda$, $j = 1, \dots, d$. Thus, we have

$$(H_{\mathrm{KS};h})_{a,b} u_b(z) = \begin{cases} (-1)^{s_{j-1}(a)} D^+_{2h;j} u_b(z) & \text{if } b = a - e_j, \\ (-1)^{s_{j-1}(a)} D^-_{2h;j} u_b(z) & \text{if } b = a + e_j, \\ m(-1)^{s_d(a)} u_b(z) & \text{if } a = b, \\ 0 & \text{otherwise,} \end{cases}$$

for $u_b \in \ell^2((2h)\mathbb{Z}^d)$, $a, b \in \Lambda$.

5.2. KS-Hamiltonian in the Fourier space and its eigenvalues

At first we note

$$(H_{\mathrm{KS};h})^2 = U_h(\widetilde{H}_{\mathrm{KS};h})^2U_h^* = U_h(-\triangle_{2h} + m^2)U_h^* = (-\triangle_{2h} + m^2)\mathbf{1}_{|\Lambda|}$$

since $-\triangle_{2h}$ acts on each $(2h)\mathbb{Z}^d + ha$, $a \in \Lambda$.

For simplicity, we denote $F_h\mathbf{1}_{|\Lambda|}$ on $[\ell^2(h\mathbb{Z}^d)]^{\oplus\Lambda}$ by the same symbol F_h. We set

$$\widehat{H}_{\mathrm{KS};h} = F_{2h}H_{\mathrm{KS};h}F_{2h}^*,$$

then it is a matrix with multiplication operators as the entries. Namely, if we denote the symbols of the forward/backward difference operators by

$$d_{h;j}^{\pm}(\xi) = \pm\frac{e^{\pm 2\pi i h\xi_j} - 1}{ih}, \quad j = 1, \dots, d,$$

then we have

$$(\widehat{H}_{\mathrm{KS};h}(\xi))_{a,b} = \begin{cases} (-1)^{s_{j-1}(a)}d_{2h;j}^{+}(\xi) & \text{if } b = a - e_j, \\ (-1)^{s_{j-1}(a)}d_{2h;j}^{-}(\xi) & \text{if } b = a + e_j, \\ m(-1)^{s_d(a)} & \text{if } a = b, \\ 0 & \text{otherwise.} \end{cases}$$

By the above observation, we also have

$$(\widehat{H}_{\mathrm{KS};h}(\xi))^2 = \Big(\sum_{j=1}^{d}(2h^2)^{-1}(1 - \cos(4\pi h\xi_j)) + m^2\Big)\mathbf{1}_{|\Lambda|}.$$

In particular, this implies the eigenvalues of $\widehat{H}_{\mathrm{KS};h}(\xi)$ are given by

$$\pm\widehat{E}_{\mathrm{KS};h}(\xi) = \pm\Big(\sum_{j=1}^{d}(2h^2)^{-1}(1 - \cos(4\pi h\xi_j)) + m^2\Big)^{1/2}.$$

We recall that $\widehat{E}_{\mathrm{KS};h}(\xi)$ is defined on $(2h)^{-1}\mathbb{T}^d$, and hence it has a unique minimal point $\xi = 0$. In other words, $H_{\mathrm{KS};h}$ has no fermion doubling problem in the Fourier space, though it has many components.

5.3. Continuum limit of the KS-Hamiltonian

Now, if we take the limit $h \to 0$, at least formally, $D_{2h;j}^{\pm}(\xi) \to D_j$, and hence $H_{\mathrm{KS};h} \to H_{\mathrm{KS};0}$ with a certain differential operator with constant matrix coefficients $H_{\mathrm{KS};0}$ on

$[L^2(\mathbb{R}^d)]^{\oplus\Lambda}$. For $j = 1, \dots, d$ and $a, b \in \Lambda$, we set

$$(A_j)_{a,b} = \begin{cases} (-1)^{s_{j-1}(a)} & \text{if } b = a + e_j \text{ or } b = a - e_j, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$B_{a,b} = (-1)^{s_d(a)} \delta_{a,b}.$$

Then we have

$$H_{\mathrm{KS};0} = \sum_{j=1}^{d} A_j D_j + mB \quad \text{on } [L^2(\mathbb{R}^d)]^{\oplus\Lambda}.$$

We can actually show $H_{\mathrm{KS};h}$ converges to $H_{\mathrm{KS};0}$ in the generalized norm resolvent sense.

Theorem 5.1. *For $z \in \mathbb{C} \setminus \mathbb{R}$,*

$$\|(H_{\mathrm{KS};0} - z)^{-1} - J_{2h}(H_{\mathrm{KS};h} - z)^{-1} J_{2h}^* \|_{\mathcal{B}(L^2(\mathbb{R}^d))} \to 0, \quad \text{as } h \to 0.$$

Since J_{2h} is an isometry, this also implies

$$\|J_{2h}^*(H_{\mathrm{KS};0} - z)^{-1} J_{2h} - (H_{\mathrm{KS};h} - z)^{-1}\|_{\mathcal{B}(\ell^2(h\mathbb{Z}^d))} \to 0, \quad \text{as } h \to 0.$$

Proof. We first note

$$|d_{2h;j}^{\pm}(\xi) - 2\pi\xi_j| \le \frac{(4\pi h \xi_j)^2}{2 \cdot 2h} \le 4\pi^2 h |\xi|^2,$$

and

$$|(\widehat{H}_{\mathrm{KS};0}(\xi) - z)^{-1}| \le \max_{\pm} |(\pm(|2\pi\xi|^2 + m^2) - z)^{-1}| \le C|\xi|^{-1},$$

uniformly in $\xi \in \mathbb{R}^d$, $h > 0$, where $\widehat{H}_{\mathrm{KS};0}(\xi) = \sum_{j=1}^d 2\pi\xi_j A_j + mB$. We also have

$$|(\widehat{H}_{\mathrm{KS};h}(\xi) - z)^{-1}| = \max_{\pm} |(\pm \widehat{E}_h(\xi) - z)^{-1}| \le C|\xi|^{-1}$$

on the support of $\hat{\varphi}(h\xi)$. Combining these, we have

$$|(\widehat{H}_{\mathrm{KS};h}(\xi) - z)^{-1} - (\widehat{H}_{\mathrm{KS};0}(\xi) - z)^{-1}| \le Ch$$

on the support of $\hat{\varphi}(h\xi)$, and then it is straightforward to show the claim as in the proof of Theorem 4.1, or [8]. See also [3, Section 3.1]. ■

We note $A_1, \dots, A_d$ and B satisfy the following properties as well as $\alpha_1, \dots, \alpha_d$ and β, i.e.,

$$A_j A_k + A_k A_j = 0 \text{ if } j \ne k, \quad A_j B + B A_j = 0,$$

and $A_j^2 = B^2 = \mathbf{1}_{|\Lambda|}$, $j = 1, \dots, d$. Thus, we may consider $H_{\mathrm{KS};0}$ as a Dirac operator, but the number of components are not necessarily the same as the standard Dirac operators. Namely, if $d = 1$, then $2^1 = 2$ and the number of components is the same as the standard one, but if $d = 2$, then $2^2 = 4 > 2$, and if $d = 3$ then $2^3 = 8 > 4$, and the number of components are twice as that of the standard Dirac operators. We expect $H_{\mathrm{KS};0}$ is decomposed to a direct sum of the standard Dirac operators, and we confirm it for $d \leq 3$ in Section 6.

6. Examples

Here we consider KS-Hamiltonians and their continuum limit for $d = 1, 2$, and 3.

6.1. 1-dimensional case

For $d = 1$, the model is transparent and easy to understand. It is also essentially the same model discussed in [2, Section 3.1] as *the 1D forward-backward difference model*.

At first, we have

$$\widetilde{H}_{\mathrm{KS};h} u(x) = \frac{1}{2ih}(u(x+h) - u(x-h)) + (-1)^{x/h} m u(x), \quad x \in h\mathbb{Z},$$

for $u \in \ell^2(h\mathbb{Z})$, and hence

$$H_{\mathrm{KS};h} = \begin{pmatrix} m & D^+_{2h,1} \\ D^-_{2h;1} & -m \end{pmatrix}.$$

Its eigenvalues are $\pm\sqrt{(2h^2)^{-1}(1-\cos(4\pi h\xi)) + m^2}$, and the continuum limit is

$$H_{\mathrm{KS};0} = \begin{pmatrix} m & D \\ D & -m \end{pmatrix} = D\sigma_1 + m\sigma_3, \quad \text{on } L^2(\mathbb{R}),$$

where $D = -i\frac{\partial}{\partial x}$, and thus we recover the standard 1D Dirac operator.

6.2. 2-dimensional case

If $d = 2$, the one component operator is given by

$$\widetilde{H}_{\mathrm{KS};h} u(x,y) = D^S_{h;1} u(x,y) + (-1)^{x/h} D^S_{h;2} u(x,y) + (-1)^{(x+y)/h} m u(x,y)$$

for $(x,y) \in h\mathbb{Z}^2$, where $u \in \ell^2(h\mathbb{Z}^2)$. We set

$$\begin{aligned} u_1(x,y) &= u(x,y), & u_2(x,y) &= u(x+h, y+h), \\ u_3(x,y) &= u(x, y+h), & u_4(x,y) &= u(x+h, y) \end{aligned}$$

for $(x, y) \in 2h\mathbb{Z}^2$ and $u \in \ell^2(h\mathbb{Z}^2)$, and then we set $U_h u = (u_j)_{j=1}^4$. Applying the formula in Section 5.1 we have

$$H_{\mathrm{KS};h} = \begin{pmatrix} m & 0 & D^-_{2h,1} & D^-_{2h;2} \\ 0 & m & -D^+_{2h;2} & D^+_{2h;1} \\ D^+_{2h;1} & -D^-_{2h;2} & -m & 0 \\ D^+_{2h;2} & D^-_{2h;1} & 0 & -m \end{pmatrix},$$

and in the continuum limit, we obtain

$$H_{\mathrm{KS};0} = \begin{pmatrix} m & 0 & D_1 & D_2 \\ 0 & m & -D_2 & D_1 \\ D_1 & -D_2 & -m & 0 \\ D_2 & D_1 & 0 & -m \end{pmatrix}.$$

This does not look like the standard 2D Dirac operator, but if we set

$$M = \begin{pmatrix} 1 & 0 & 1 & 0 \\ i & 0 & -i & 0 \\ 0 & 1 & 0 & 1 \\ 0 & i & 0 & -i \end{pmatrix},$$

then

$$\begin{aligned} M^{-1} H_{\mathrm{KS};0} M &= \begin{pmatrix} m & D_1 + iD_2 & 0 & 0 \\ D_1 - iD_2 & -m & 0 & 0 \\ 0 & 0 & m & D_1 - iD_2 \\ 0 & 0 & D_1 + iD_2 & -m \end{pmatrix} \\ &= \begin{pmatrix} \overline{D_1\sigma_1 + D_2\sigma_2 + m\sigma_3} & 0 \\ 0 & D_1\sigma_1 + D_2\sigma_2 + m\sigma_3 \end{pmatrix}. \end{aligned}$$

Thus, we arrive at a direct sum of two standard 2D Dirac operators, one of which is simply the complex conjugate.

6.3. 3-dimensional case

For $d = 3$, the computations look somewhat complicated. We omit to write down the definition of $\widetilde{H}_{\mathrm{KS};h}$. We set

$$\begin{aligned} &u_1(x,y,z) = u(x,y,z), && u_2(x,y,z) = u(x+h,y+h,z), \\ &u_3(x,y,z) = u(x,y+h,z+h), && u_4(x,y,z) = u(x+h,y,z+h) \\ &u_5(x,y,z) = u(x,y,z+h), && u_6(x,y,z) = u(x+h,y+h,z+h), \\ &u_7(x,y,z) = u(x,y+h,z), && u_8(x,y,z) = u(x+h,y,z), \end{aligned}$$

for $(x,y,z)\in 2h\mathbb{Z}^3$, $u\in \ell^2(h\mathbb{Z}^3)$, and we set $U_h u=(u_j)_{j=1}^8$. Then we have

$$H_{\mathrm{KS};h}=\begin{pmatrix} m & 0 & 0 & 0 & D_3^- & 0 & D_2^- & D_1^- \\ 0 & m & 0 & 0 & 0 & D_3^- & D_1^+ & -D_2^+ \\ 0 & 0 & m & 0 & D_2^+ & D_1^- & -D_3^+ & 0 \\ 0 & 0 & 0 & m & D_1^+ & -D_2^- & 0 & -D_3^+ \\ D_3^+ & 0 & D_2^- & D_1^- & -m & 0 & 0 & 0 \\ 0 & D_3^+ & D_1^+ & -D_2^+ & 0 & -m & 0 & 0 \\ D_2^+ & D_1^- & -D_3^- & 0 & 0 & 0 & -m & 0 \\ D_1^+ & -D_2^- & 0 & -D_3^- & 0 & 0 & 0 & -m \end{pmatrix},$$

and the continuum limit is

$$H_{\mathrm{KS};0}=\begin{pmatrix} m & 0 & 0 & 0 & D_3 & 0 & D_2 & D_1 \\ 0 & m & 0 & 0 & 0 & D_3 & D_1 & -D_2 \\ 0 & 0 & m & 0 & D_2 & D_1 & -D_3 & 0 \\ 0 & 0 & 0 & m & D_1 & -D_2 & 0 & -D_3 \\ D_3 & 0 & D_2 & D_1 & -m & 0 & 0 & 0 \\ 0 & D_3 & D_1 & -D_2 & 0 & -m & 0 & 0 \\ D_2 & D_1 & -D_3 & 0 & 0 & 0 & -m & 0 \\ D_1 & -D_2 & 0 & -D_3 & 0 & 0 & 0 & -m \end{pmatrix}.$$

By setting

$$M=\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & -i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -i & 0 & 0 & 0 & i \end{pmatrix},$$

we have

$$\begin{aligned} &M^{-1}H_{\mathrm{KS};0}M \\ &=\begin{pmatrix} m & 0 & D_3 & -iD_1+D_2 & 0 & 0 & 0 & 0 \\ 0 & m & iD_1+D_2 & -D_3 & 0 & 0 & 0 & 0 \\ D_3 & -iD_1+D_2 & -m & 0 & 0 & 0 & 0 & 0 \\ iD_1+D_2 & -D_3 & 0 & -m & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & m & 0 & D_3 & iD_1+D_2 \\ 0 & 0 & 0 & 0 & 0 & m & -iD_1+D_2 & -D_3 \\ 0 & 0 & 0 & 0 & D_3 & iD_1+D_2 & -m & 0 \\ 0 & 0 & 0 & 0 & -iD_1+D_2 & -D_3 & 0 & -m \end{pmatrix} \\ &=\begin{pmatrix} m\mathbf{1}_2 & D_1\sigma_2+D_2\sigma_1+D_3\sigma_3 & 0 & 0 \\ D_1\sigma_2+D_2\sigma_1+D_3\sigma_3 & -m\mathbf{1}_2 & 0 & 0 \\ 0 & 0 & m\mathbf{1}_2 & D_1\overline{\sigma_2}+D_2\sigma_1+D_3\sigma_3 \\ 0 & 0 & D_1\overline{\sigma_2}+D_2\sigma_1+D_3\sigma_3 & -m\mathbf{1}_2 \end{pmatrix} \\ &=\begin{pmatrix} D_1\alpha_2+D_2\alpha_1+D_3\alpha_3+m\beta & 0 \\ 0 & \overline{D_1\alpha_2+D_2\alpha_1+D_3\alpha_3+m\beta.} \end{pmatrix}. \end{aligned}$$

This gives a direct sum of a representation of 3D Dirac operator and its complex conjugate (another representation). Of course, the final form depends on the choice of the diagonalization matrix M. These matrix computations were aided by a symbolic computation tool SymPy on Python.

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