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On the well posedness and the stability of a thermoelastic Gurtin–Pipkin–Timoshenko system without the second spectrum

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Abstract. In this work, we establish the well-posedness and the asymptotic stability of a linear thermoelastic Gurtin–Pipkin–Timoshenko system free of second spectrum. We give a detailed proof of the well-posedness, using the semigroup theory and with the help of some new operators. Then, we prove that the system is exponentially stable irrespective of the parameters. Our result generalizes an earlier result, one in the paper of Keddi, Messaoudi, and Alahyane [Journal of Thermal Stresses 46 (2023), 823–838].

1. Introduction

In the linear theory of beams, the Timoshenko model [19] is one of the most suitable and widely used models to describe the vibrations of the majority of the elastic structures of plane beams due to its consideration of the effects of shear deformation and rotatory inertia. This is contrary to the model of Euler–Bernoulli and Rayleigh [18]. One can think of the Timoshenko beam theory then as an improvement of the classical Euler–Bernoulli theory.

The Timoshenko beam model is described by the following conservative system of hyperbolic equations:

$$\begin{cases} \rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi) = 0, \end{cases} \tag{1.1}$$

where the functions φ and ψ denote, respectively, the transverse displacement and the angular rotation and the parameters $\rho_1 = \rho A$, $\rho_2 = \rho I$, $b = EI$, and $\kappa = \kappa' GA$ are positive constants such that ρ is the mass density of material, A is the cross-sectional area, I is the inertia momentum of cross-sectional area, E is Young's modulus of elasticity, G is the shear modulus and κ' is the transverse shear factor. The asymptotic behavior of the Timchenko system (1.1) has been studied by many authors, using different damping

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mechanisms, and several stability results, depending on the structural parameters of the equations, have been established. See, for example, [11, 13, 15, 16].

Later on, in some analytical studies (for example, [1, 7]), it was shown that the "classical" Timoshenko system has a second non-physical spectrum leading to a physical paradox, known as the second spectrum. In 2010, Elishakoff [10] proposed the following modified version of the Timoshenko system:

$$\begin{cases} \rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x = 0, \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \kappa(\varphi_x + \psi) = 0, \end{cases} \tag{1.2}$$

which is free of non-physical spectrum or the second spectrum and has only one spectrum (physical spectrum).

Almeida Júnior et al. [4] considered system (1.2) with the linear frictional damping $\mu\psi_t$ acting on the angle rotation equation and proved that the energy decays exponentially for any choice of positive parameters of the system. In [5], the same exponential result was proved, where a linear frictional damping acts on the transverse displacement instead. Also, similar results were obtained in [2, 3].

Concerning the Timoshenko system (1.2), in the presence of a thermal damping, Apalara et al. [6] studied the following system:

$$\begin{cases} \rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x = 0, \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \kappa(\varphi_x + \psi) + \gamma\theta_x = 0, \\ \rho_3\theta_t - \beta\theta_{xx} + \gamma\psi_{xt} = 0. \end{cases} \tag{1.3}$$

Here, θ is the temperature difference and the parameters $\rho_3, \beta > 0$ and $\gamma \neq 0$ represent the capacity, diffusivity, and coupling constants, respectively. The authors proved an exponential decay result irrespective of the parameters of the system. Recently, Keddi et al. [14] considered the following Timoshenko system free of the second spectrum coupled with a heat system of the Cattaneo type:

$$\begin{cases} \rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x = 0, \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \kappa(\varphi_x + \psi) + \delta\theta_x = 0, \\ c\theta_t + q_x + \delta\psi_{xt} = 0, \\ \tau q_t + \beta q + \theta_x = 0, \end{cases} \tag{1.4}$$

where q is the heat flux and the positive parameter τ represents the relaxation time. The authors established an existence and uniqueness result, using the semigroup theory and with the help of some new operators, and proved that the system (1.4) is exponentially stable independently of any relationship between the parameters of the system.

In the present paper, we are interested in the study of the well-posedness and the asymptotic stability of a thermoelastic Timoshenko system with one spectrum and where the heat conduction is given by the Gurtin–Pipkin thermal law and the coupling is via the rotation angle equation. More specifically, this system is given by

$$\begin{cases} \rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x = 0 & \text{in } (0,1)\times(0,\infty), \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \kappa(\varphi_x + \psi) + \delta\theta_x = 0 & \text{in } (0,1)\times(0,\infty), \\ c\theta_t - \dfrac{1}{\beta}\displaystyle\int_0^{+\infty} g(s)\theta_{xx}(t-s)ds + \delta\psi_{xt} = 0 & \text{in } (0,1)\times(0,\infty), \end{cases} \tag{1.5}$$

supplemented with the following boundary conditions of Dirichlet–Neumann–Dirichlet type:

$$\varphi(0,t) = \varphi(1,t) = \psi_x(0,t) = \psi_x(1,t) = \theta(0,t) = \theta(1,t) = 0 \quad \forall\, t \geq 0, \tag{1.6}$$

the initial conditions

$$\begin{cases} \varphi(x,0) = \varphi_0(x), & \varphi_t(x,0) = \varphi_1(x) \\ \psi(x,0) = \psi_0(x), & \theta(x,0) = \theta_0(x) \end{cases} \quad \forall x \in (0,1), \tag{1.7}$$

and the history of θ,

$$\theta(x,-s) = w_0(x,s) \quad \forall\, x \in (0,1), s > 0. \tag{1.8}$$

The memory kernel $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a convex integrable function on $\mathbb{R}^+$ of a total mass

$$\int_0^{+\infty} g(s)ds = 1.$$

From $(1.5)_2$ and the boundary conditions (1.6), we have

$$\int_0^1 \psi dx = 0;$$

consequently, the use of Poincaré's inequality for ψ is justified.

This paper is divided into four sections, in addition to the introduction. Section 2 gives some preliminaries and the semigroup setting of the problem. The well posedness is given in detail in Section 3, and the stability result is presented in Section 4. We then conclude our work in Section 5.

2. Preliminaries and semigroup setting

In this section, we present some notations and our assumption on the memory kernel g, then, we introduce the semigroup setting of our problem.

Concerning the memory kernel g, we assume that for $s \geq 0$

$$g(s) = \int_s^{+\infty} \mu(r)dr,$$

where

(h_1) $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-increasing absolutely continuous function, possibly unbounded near zero.

(h_2) μ is differentiable almost everywhere and there exists $\xi > 0$ such that, for almost every $s > 0$,

$$\mu'(s) + \xi\mu(s) \leq 0. \tag{2.1}$$

Remark 2.1. Note that g is a bounded and non-increasing function vanishing at infinity. Moreover, μ is integrable on $\mathbb{R}^+$ with

$$\int_0^{+\infty} \mu(s)ds = g(0).$$

In addition, the fact that g has unit total mass implies that

$$\int_0^{+\infty} s\mu(s)ds = 1.$$

Now, we consider η, the integrated past history of θ, defined by

$$\eta(x,t,s) = \int_0^s \theta(x,t-r)dr,$$

for $(x,t,s) \in (0,1) \times (0,\infty) \times (0,\infty)$. Clearly, η satisfies the equation

$$\eta_t(t,s) + \eta_s(t,s) = \theta(t).$$

Combining this latter equation with (1.5), we obtain

$$\begin{cases} \rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x = 0, \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \kappa(\varphi_x + \psi) + \delta\theta_x = 0, \\ c\theta_t - \dfrac{1}{\beta}\displaystyle\int_0^{+\infty} \mu(s)\eta_{xx}(s)ds + \delta\psi_{xt} = 0, \\ \eta_t + \eta_s = \theta, \end{cases} \tag{2.2}$$

with boundary conditions

$$\begin{cases} \varphi(0,t) = \psi_x(0,t) = \theta(0,t) = \eta(0,t,s) = 0 \\ \varphi(1,t) = \psi_x(1,t) = \theta(1,t) = \eta(1,t,s) = 0 \end{cases} \quad \forall\, t \geq 0 \tag{2.3}$$

and initial conditions

$$\begin{cases} \varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x) \\ \psi(x,0) = \psi_0(x),\ \theta(x,0) = \theta_0(x) \\ \eta(x,0,s) = \eta_0(x,s) = \displaystyle\int_0^s w_0(x,r)dr \end{cases} \quad \forall\, x \in (0,1). \tag{2.4}$$

To rewrite the problem (2.2)–(2.4) in an abstract form, we introduce the following operators:

$$
\begin{aligned}
R &= (\rho_1 I - \rho_2 \partial_{xx}),\\
S &= cI - \delta^2 \rho_1 (B^{-1} \circ \partial_{xx}),\\
T &= \frac{\kappa}{b\rho_1 + \kappa\rho_2}\big[(\rho_2 I + b\rho_1^2 B^{-1}) \circ \partial_{xx}\big],
\end{aligned}
$$

where B is the positive self-adjoint operator defined, on $H^2(0,1) \cap H_0^1(0,1)$, by

$$
B = \rho_1 \kappa I - (\rho_2 \kappa + b\rho_1)\partial_{xx}.
$$

Then, from the system (2.2), we can get the following auxiliary system:

$$
\begin{cases}
R\varphi_{tt} + bT\varphi_{xx} + \delta T\theta = 0,\\
S\theta_t - \dfrac{1}{\beta}\displaystyle\int_0^{+\infty} \mu(s)\eta_{xx}(s)ds - \delta T\varphi_t = 0,\\
\eta_t + \eta_s - \theta = 0.
\end{cases}
\tag{2.5}
$$

By introducing the state vector $\Phi = (\varphi, \phi, \theta, \eta)^T$, where $\phi = \varphi_t$, the system (2.5) becomes

$$
\begin{cases}
\Phi'(t) = \mathcal{A}\Phi(t) \ \forall\, t \geq 0,\\
\Phi(0) = \Phi_0 = (\varphi_0, \varphi_1, \theta_0, \eta_0)^T,
\end{cases}
\tag{2.6}
$$

where

$$
\mathcal{A}\Phi = \begin{pmatrix}
\phi\\
-bR^{-1}T\varphi_{xx} - \delta R^{-1}T\theta\\
S^{-1}\Big(\dfrac{1}{\beta}\displaystyle\int_0^{+\infty}\mu(s)\eta_{xx}(s)ds + \delta T\phi\Big)\\
-\eta_s + \theta
\end{pmatrix}.
$$

Now, we consider the following space:

$$
H_*^3(0,1) = \{\varphi \in H^3(0,1) \cap H_0^1(0,1) : \varphi_{xx} \in H_0^1(0,1)\}
$$

and the weighted Hilbert space

$$
\mathcal{M} = L_\mu^2(\mathbb{R}^+, H_0^1(0,1)) = \Big\{\eta : \mathbb{R}^+ \to H_0^1(0,1) \Big\backslash \int_0^{+\infty} \mu(s)\|\eta_x(s)\|_2^2 ds < +\infty\Big\},
$$

with the inner product

$$
\langle \eta, \zeta\rangle_{\mathcal{M}} = \int_0^{+\infty} \mu(s)\langle \eta_x, \zeta_x\rangle ds
$$

and the associated norm

$$
\|\eta\|_{\mathcal{M}}^2 = \int_0^{+\infty} \mu(s)\|\eta_x(s)\|_2^2 ds.
$$

Also, we define the energy space by

$$\mathcal{H} = (H^2(0,1) \cap H_0^1(0,1)) \times H_0^1(0,1) \times L^2(0,1) \times \mathcal{M},$$

equipped with the inner product

$$\langle \Phi, \Phi^* \rangle_{\mathcal{H}} = b\langle T\varphi, \varphi_{xx}^* \rangle + \langle R\phi, \phi^* \rangle + \langle S\theta, \theta^* \rangle + \frac{1}{\beta}\langle \eta, \eta^* \rangle_{\mathcal{M}},$$

for $\Phi = (\varphi, \phi, \theta, \eta)^T, \Phi^* = (\varphi^*, \phi^*, \theta^*, \eta^*)^T \in \mathcal{H}$ and the domain of $\mathcal{A}$ is given by

$$D(\mathcal{A}) = \left\{ \begin{array}{c} \Phi = (\varphi, \phi, \theta, \eta)^T \in \mathcal{H} : \varphi \in H_*^3(0,1); \phi \in H^2(0,1) \cap H_0^1(0,1); \\ \theta \in H_0^1(0,1); \eta \in \mathcal{N}, \int_0^{+\infty} \mu(s)\eta_{xx}(s)ds \in L^2(0,1) \end{array} \right\},$$

where

$$\mathcal{N} = \{\eta \in \mathcal{M} : \eta_s \in \mathcal{M}, \eta(0) = 0\}.$$

Finally, we put

$$H = (H^2(0,1) \cap H_0^1(0,1)) \times H_0^1(0,1) \times H^1(0,1) \times L^2(0,1) \times \mathcal{M}$$

and

$$D = \left\{ \begin{array}{l} \Phi = (\varphi, \phi, \psi, \theta, \eta)^T \in H : \varphi \in H_*^3(0,1); \phi \in H^2(0,1) \cap H_0^1(0,1); \\ \psi \in H_*^2(0,1); \theta \in H_0^1(0,1); \eta \in \mathcal{N}, \int_0^{+\infty} \mu(s)\eta_{xx}(s)ds \in L^2(0,1) \end{array} \right\},$$

where

$$H_*^2(0,1) = \{\psi \in H^2(0,1) : \psi_x(0) = \psi_x(1) = 0\}.$$

Before proceeding, we need the following lemma.

Lemma 2.2 ([12, page 348]). *For $\eta \in \mathcal{N}$ and $\theta \in L^2(0,1)$, we have*

$$-\int_0^{+\infty} \mu(s) \frac{d}{ds} \|\eta_x(s)\|_2^2 ds = \int_0^{+\infty} \mu'(s) \|\eta_x(s)\|_2^2 ds \tag{2.7}$$

and

$$-\int_0^{+\infty} \mu(s) \langle \theta, \eta_s(s) \rangle ds = \int_0^{+\infty} \mu'(s) \langle \theta, \eta(s) \rangle ds. \tag{2.8}$$

3. The well-posedness of the problem

Before stating and proving our main well-posedness theorem, we establish the following auxiliary result.

Theorem 3.1. *Let $\Phi_0 \in \mathcal{H}$. Then, there exists a unique solution $\Phi \in C(\mathbb{R}^+, \mathcal{H})$, of problem* (2.5)*. Moreover, if $\Phi_0 \in D(\mathcal{A})$, then $\Phi \in C(\mathbb{R}^+, D(\mathcal{A})) \cap C^1(\mathbb{R}^+, \mathcal{H})$.*

Proof. The proof will be based on the semigroup theory. First, we prove that $\mathcal{A}$ is dissipative. So, for any $\Phi \in D(\mathcal{A})$, we have

$$\begin{aligned}\langle \mathcal{A}\Phi, \Phi\rangle_{\mathcal{H}} &= -\langle bT\varphi_{xx} + \delta T\theta, \phi\rangle + b\langle T\phi, \varphi_{xx}\rangle \\ &\quad + \left\langle \left(\frac{1}{\beta}\int_0^{+\infty} \mu(s)\eta_{xx}(s)ds + \delta T\phi\right), \theta\right\rangle + \frac{1}{\beta}\langle(-\eta_s + \theta), \eta\rangle_{\mathcal{M}}.\end{aligned}$$

By integrating by parts and taking into account the properties of the operators R, S and T, we get

$$\langle \mathcal{A}\Phi, \Phi\rangle_{\mathcal{H}} = -\frac{1}{2\beta}\int_0^{+\infty} \mu(s)\frac{d}{ds}\|\eta_x(s)\|_2^2 ds.$$

Therefore, thanks to (2.7), we have

$$\langle \mathcal{A}\Phi, \Phi\rangle_{\mathcal{H}} = \frac{1}{2\beta}\int_0^{+\infty} \mu'(s)\|\eta_x(s)\|_2^2 ds \le 0,$$

and hence, $\mathcal{A}$ is dissipative. Secondly, we prove that the operator $I - \mathcal{A}$ is surjective. Let $G = (\varphi^*, \phi^*, \theta^*, \eta^*)^T \in \mathcal{H}$ and look for $\Phi \in D(\mathcal{A})$ satisfying

$$(I - \mathcal{A})\Phi = G. \tag{3.1}$$

This can be written as

$$\begin{cases} \varphi - \phi = \varphi^*, \\ R\phi + bT\varphi_{xx} + \delta T\theta = R\phi^*, \\ S\theta - \dfrac{1}{\beta}\displaystyle\int_0^{+\infty} \mu(s)\eta_{xx}(s)ds - \delta T\phi = S\theta^*, \\ \eta + \eta_s - \theta = \eta^*. \end{cases} \tag{3.2}$$

The solution of the differential equation $(3.2)_4$ is

$$\eta(s) = (1 - e^{-s})\theta + \int_0^s e^{r-s}\eta^*(r)dr. \tag{3.3}$$

From (3.2) and (3.3), we find that φ and θ satisfy

$$\begin{cases} R\varphi + bT\varphi_{xx} + \delta T\theta = R(\phi^* + \varphi^*) \text{ in } H^{-1}(0,1), \\ S\theta - \dfrac{1}{\beta}c_\mu\theta_{xx} - \delta T\varphi = S\theta^* - \delta T\varphi^* \\ \qquad\qquad + \dfrac{1}{\beta}\displaystyle\int_0^{+\infty} \mu(s)\Big(\int_0^s e^{r-s}\eta^*_{xx}(r)dr\Big)ds \text{ in } H^{-1}(0,1), \end{cases} \tag{3.4}$$

where

$$c_\mu = \int_0^{+\infty} \mu(s)(1-e^{-s})ds \leq g(0).$$

Therefore, the variational formulation for the system (3.4) is to find $(\varphi, \theta) \in V = (H^2(0,1) \cap H_0^1(0,1)) \times H_0^1(0,1)$ such that

$$a((\varphi,\theta),(\varphi_1,\theta_1)) = F(\varphi_1,\theta_1) \quad \forall(\varphi_1,\theta_1) \in V, \tag{3.5}$$

where a is the bilinear form on V, given by

$$\begin{aligned} a((\varphi,\theta),(\varphi_1,\theta_1)) &= \langle R\varphi, \varphi_1\rangle + b\langle T\varphi, \varphi_{1xx}\rangle + \delta\langle T\theta, \varphi_1\rangle \\ &\quad + \langle S\theta, \theta_1\rangle + \frac{1}{\beta} c_\mu \langle \theta_x, \theta_{x1}\rangle - \delta\langle T\varphi, \theta_1\rangle, \end{aligned}$$

and F is the linear functional on V, defined by

$$\begin{aligned} F(\varphi_1,\theta_1) = &\left\langle S\theta^* - \delta T\varphi^* + \frac{1}{\beta}\int_0^{+\infty} \mu(s)\left(\int_0^s e^{r-s}\eta_{xx}^*(r)dr\right)ds, \theta_1 \right\rangle \\ &+ \langle R(\phi^* + \varphi^*), \varphi_1\rangle. \end{aligned}$$

It is easy to check the boundedness of a and F. Furthermore, from the definition of R, S and T, we obtain

$$\begin{aligned} a((\varphi,\theta),(\varphi,\theta)) = \rho_1\langle\varphi,\varphi\rangle &+ \rho_2\langle\varphi_x,\varphi_x\rangle \\ &+ \frac{\kappa b\rho_2}{b\rho_1+\kappa\rho_2}\langle\varphi_{xx},\varphi_{xx}\rangle + \frac{\kappa b^2\rho_1^2}{b\rho_1+\kappa\rho_2}\langle B^{-1}\varphi_{xx},\varphi_{xx}\rangle \\ &+ c\langle\theta,\theta\rangle + \delta^2\rho_1\langle B^{-1}\theta_x,\theta_x\rangle + \frac{1}{\beta}c_\mu\langle\theta_x,\theta_x\rangle. \end{aligned}$$

Then, thanks to the properties of the operator B, we deduce that

$$\begin{aligned} a((\varphi,\theta),(\varphi,\theta)) &\geq \rho_2\langle\varphi_x,\varphi_x\rangle + \frac{\kappa b\rho_2}{b\rho_1+\kappa\rho_2}\langle\varphi_{xx},\varphi_{xx}\rangle + \frac{1}{\beta}c_\mu\langle\theta_x,\theta_x\rangle \\ &\geq M(\|\varphi_x\|_2^2 + \|\varphi_{xx}\|_2^2 + \|\theta_x\|_2^2) = M\|(\varphi,\theta)\|_V^2. \end{aligned}$$

Thus, a is coercive. Consequently, by Lax–Milgram lemma [17], problem (3.5) has a unique solution $(\varphi,\theta) \in V$.

By substituting φ into $(3.2)_1$, we get $\phi \in H^2(0,1) \cap H_0^1(0,1)$ and by using (3.3), we have

$$\begin{aligned} \int_0^{+\infty} \mu(s)\|\eta_x(s)\|_2^2 ds &\leq 2c_\mu\|\theta_x\|_2^2 + 2\int_0^{+\infty}\mu(s)\int_0^s e^{r-s}\|\eta_x^*(r)\|_2^2 drds \\ &= 2c_\mu\|\theta_x\|_2^2 + 2\int_0^{+\infty}\int_r^{+\infty}\mu(s)e^{r-s}ds\|\eta_x^*(r)\|_2^2 dr, \end{aligned}$$

and since $\mu(s) \leq \mu(r)$ for any $s \geq r$, we obtain

$$\int_0^{+\infty} \mu(s)\|\eta_x(s)\|_2^2 ds \leq 2c_\mu \|\theta_x\|_2^2 + 2\int_0^{+\infty}\left(\int_r^{+\infty} e^{r-s}ds\right)\mu(r)\|\eta_x^*(r)\|_2^2 dr$$
$$= 2c_\mu\|\theta_x\|_2^2 + 2\int_0^{+\infty}\mu(r)\|\eta_x^*(r)\|_2^2 dr < \infty.$$

This implies that $\eta \in \mathcal{M}$. Moreover, $\eta(0) = 0$ and

$$\eta_s(s) = e^{-s}\theta + \eta^*(s) - \int_0^s e^{r-s}\eta^*(r)dr = \theta + \eta^*(s) - \eta(s) \in \mathcal{M}.$$

So, $\eta \in \mathcal{N}$. Furthermore, if $\theta_1 \equiv 0$ in (3.5), then we get

$$b\langle T\varphi, \varphi_{1xx}\rangle = \langle R(\phi^* + \varphi^*) - \delta T\theta - R\varphi, \varphi_1\rangle \tag{3.6}$$

for all $\varphi_1 \in C_0^1(0,1) \subset H^2(0,1) \cap H_0^1(0,1)$, which implies that

$$bT\varphi_{xx} = R(\phi^* + \varphi^*) - \delta T\theta - R\varphi \in H^{-1}(0,1).$$

From the definition of T and the elliptic regularity theory, we conclude that

$$\varphi_{xx} \in H_0^1(0,1) \quad \text{and} \quad R\phi + bT\varphi_{xx} + \delta T\theta = R\phi^*.$$

Similarly, if $\varphi_1 \equiv 0$ in (3.5), we obtain

$$\left\langle S\theta - \frac{1}{\beta}c_\mu\theta_{xx} - \delta T\varphi - S\theta^* + \delta T\varphi^* - \frac{1}{\beta}\int_0^{+\infty}\mu(s)\left(\int_0^s e^{r-s}\eta_{xx}^*(r)dr\right)ds, \theta_1\right\rangle = 0 \tag{3.7}$$

for all $\theta_1 \in C_0^1(0,1) \subset H_0^1(0,1)$. Thus, with the help of $(3.2)_1$ and (3.3), we arrive at

$$\frac{1}{\beta}\int_0^{+\infty}\mu(s)\eta_{xx}(s)ds = S\theta - \delta T\phi - S\theta^* \in L^2(0,1).$$

Therefore, we have proved the existence of a unique solution $(\varphi, \phi, \theta, \eta) \in D(\mathcal{A})$ such that (3.2) is satisfied. Consequently, the operator $\mathcal{A}$ is maximal. Then, the result of Theorem 3.1 follows directly from the Lumer–Phillips theorem (see [8, 17]). ■

Our main well-posedness result is the following theorem:

Theorem 3.2. *Let $(\varphi_0, \varphi_1, \psi_0, \theta_0, \eta_0) \in D$, where the following compatibility condition:*

$$B(\varphi_{0x} + \psi_0) = -\rho_1(b\varphi_{0xxx} + \delta\theta_{0x}) \tag{3.8}$$

is satisfied. Then, there exists a unique solution $(\varphi, \varphi_t, \psi, \theta, \eta) \in C(\mathbb{R}^+, D) \cap C^1(\mathbb{R}^+, H)$ of problem (2.2)–(2.4).

Proof. Theorem 3.1 yields the existence of a unique solution

$$(\varphi, \varphi_t, \theta, \eta) \in C(\mathbb{R}^+, D(\mathcal{A})) \cap C^1(\mathbb{R}^+, \mathcal{H})$$

for the system

$$\begin{cases} \rho_1 \kappa \varphi_{tt} - (b\rho_1 + \rho_2 \kappa)\varphi_{ttxx} + b\kappa \varphi_{xxxx} + \delta \kappa \theta_{xx} = 0, \\ c\kappa \theta_t - \dfrac{\kappa}{\beta} \displaystyle\int_0^{+\infty} \mu(s)\eta_{xx}(s)ds + \delta(\rho_1 \varphi_{ttt} - \kappa \varphi_{xxt}) = 0, \\ \eta_t + \eta_s - \theta = 0. \end{cases} \tag{3.9}$$

From the equation $(3.9)_1$ and the definition of B, we have

$$B\varphi_{tt} + b\kappa \varphi_{xxxx} + \delta \kappa \theta_{xx} = 0,$$

which gives, by differentiation,

$$\varphi_{ttt} = -(B^{-1} \circ \partial_{xx})(b\kappa \varphi_{xxt} + \delta \kappa \theta_t) \in C(\mathbb{R}^+, L^2(0,1)).$$

Now, we take

$$\psi(x,t) = -\varphi_x(x,t) + \frac{\rho_1}{\kappa} \int_0^x \varphi_{tt}(y,t)dy,$$

hence, $\psi \in C(\mathbb{R}^+, H^2_*(0,1)) \cap C^1(\mathbb{R}^+, L^2(0,1))$ and

$$\rho_1 \varphi_{tt} - \kappa(\varphi_x + \psi)_x = 0.$$

Therefore, it follows, from (3.9) that

$$\begin{cases} \rho_1 \varphi_{tt} - \kappa(\varphi_x + \psi)_x = 0, \\ -\rho_2 \varphi_{ttxx} - b\psi_{xxx} + \kappa(\varphi_x + \psi)_x + \delta \theta_{xx} = 0, \\ c\theta_t - \dfrac{1}{\beta} \displaystyle\int_0^{+\infty} \mu(s)\eta_{xx}(s)ds + \delta \psi_{xt} = 0, \\ \eta_t + \eta_s - \theta = 0. \end{cases} \tag{3.10}$$

Thanks to the expression of the equation $(3.10)_3$, we get $\psi_{xt} \in C(\mathbb{R}^+, L^2(0,1))$, which implies that $\psi \in C^1(\mathbb{R}^+, H^1(0,1))$. This completes the proof. ■

Remark 3.3. From the equations

$$\rho_1 \varphi_{tt} - \kappa(\varphi_x + \psi)_x = 0,$$
$$B\varphi_{tt} + b\kappa \varphi_{xxxx} + \delta \kappa \theta_{xx} = 0$$

and the compatibility condition (3.8), we can define φ_{tt} at $t = 0$ as follows:

$$\rho_1 \varphi_{tt}(x,0) := \kappa(\varphi_{0x} + \psi_0)_x(x) = -\kappa \rho_1 B^{-1}(b\varphi_{0xxxx} + \delta \theta_{0xx}).$$

Consequently, $E(0)$ makes sense, where $E(t)$ is given in (4.1) below.

4. Exponential stability

In order to establish our main stability result, we begin this section by introducing some functionals and proving some auxiliary results.

Lemma 4.1. *Let* $(\varphi, \psi, \theta, \eta)$ *be the solution of problem* (2.2)–(2.4)*. Then, the energy functional, given by*

$$E(t) = \frac{1}{2}\int_0^1 (\rho_1\varphi_t^2 + \frac{\rho_2\rho_1}{\kappa}\varphi_{tt}^2 + \rho_2\varphi_{xt}^2 + b\psi_x^2 + k(\varphi_x + \psi)^2 + c\theta^2)dx \\ + \frac{1}{2\beta}\int_0^{+\infty} \mu(s)\|\eta_x(s)\|_2^2 ds, \tag{4.1}$$

satisfies

$$E'(t) = \frac{1}{2\beta}\int_0^{+\infty} \mu'(s)\|\eta_x(s)\|_2^2 ds \le 0. \tag{4.2}$$

Proof. By multiplying the first three equations in (2.2) by φ_t, ψ_t, and θ, respectively, integrating over $(0, 1)$, with respect to x, employing integration by parts, the boundary conditions (2.3) and

$$\kappa\psi_{xt} = \rho_1\varphi_{ttt} - \kappa\varphi_{xxt},$$

and adding the results together, we obtain

$$\frac{d}{dt}E(t) = \frac{1}{\beta}\int_0^1 \theta\Big(\int_0^{+\infty} \mu(s)\eta_{xx}(s)ds\Big)dx + \frac{1}{2\beta}\frac{d}{dt}\int_0^{+\infty} \mu(s)\|\eta_x(s)\|_2^2 ds. \tag{4.3}$$

But equation $(2.2)_4$ yields

$$\begin{aligned}\int_0^1 \theta\Big(\int_0^{+\infty} \mu(s)\eta_{xx}(s)ds\Big)dx &= -\int_0^{+\infty} \mu(s)\int_0^1 \theta_x\eta_x(s)dxds \\ &= -\frac{1}{2}\frac{d}{dt}\int_0^{+\infty} \mu(s)\|\eta_x(s)\|_2^2 ds \\ &\quad -\frac{1}{2}\int_0^{+\infty} \mu(s)\frac{d}{ds}\|\eta_x(s)\|_2^2 ds.\end{aligned}$$

Recalling (2.7), we arrive at

$$\int_0^1 \theta\Big(\int_0^{+\infty} \mu(s)\eta_{xx}(s)ds\Big)dx = -\frac{1}{2}\frac{d}{dt}\int_0^{+\infty} \mu(s)\|\eta_x(s)\|_2^2 ds \\ + \frac{1}{2}\int_0^{+\infty} \mu'(s)\|\eta_x(s)\|_2^2 ds. \tag{4.4}$$

Substituting (4.4) into (4.3), we get the desired result. ∎

Now, we consider the following two functionals:

$$
\begin{aligned}
F_1(t) &= -\frac{c}{g(0)}\int_0^{+\infty}\mu(s)\int_0^1 \theta\eta(s)dxds + \frac{\delta\rho_1}{\kappa}\int_0^1 \varphi_t\theta dx \\
&\quad - \frac{\delta\rho_1}{\kappa g(0)}\int_0^1 \varphi_{tt}\int_0^{+\infty}\mu(s)\eta(s)dsdx, \\
F_2(t) &= b\int_0^1 \psi_x\varphi_t dx - \delta\int_0^1 \theta\varphi_t dx,
\end{aligned}
$$

and the two positive constants

$$
\begin{aligned}
\mu_2 &= \frac{bc+\delta^2}{c}, \\
\mu_1 &= \frac{\delta^2\rho_1}{c\kappa}.
\end{aligned}
$$

Lemma 4.2. *Let $(\varphi, \psi, \theta, \eta)$ be the solution of problem (2.2)–(2.4). Then, the functional*

$$
\mathcal{F} = \mu_2 F_1 + \mu_1 F_2
$$

satisfies the estimate

$$
\begin{aligned}
\mathcal{F}'(t) &\le -\frac{c\mu_2}{2}\int_0^1 \theta^2 dx - \frac{\rho_2\mu_1}{2}\int_0^1 \varphi_{tt}^2 dx - \frac{\kappa^2\mu_1}{\rho_1}\int_0^1 (\varphi_x+\psi)^2 dx \\
&\quad + \varepsilon_1\int_0^1 \varphi_{xt}^2 dx - m\left(1+\frac{1}{\varepsilon_1}\right)\int_0^{+\infty}\mu'(s)\|\eta_x(s)\|_2^2 ds
\end{aligned} \tag{4.5}
$$

for any $\varepsilon_1 > 0$ and for some constant $m > 0$ independent of ε_1.

Proof. Direct differentiation, using $(2.2)_1$, $(2.2)_3$, and $(2.2)_4$, leads to

$$
\begin{aligned}
F_1'(t) &= -\frac{1}{\beta g(0)}\int_0^{+\infty}\mu(s)\int_0^1 \left(\int_0^{+\infty}\mu(s)\eta_{xx}(s)ds\right)\eta(s)dxds \\
&\quad - c\int_0^1 \theta^2 dx + \frac{c}{g(0)}\int_0^{+\infty}\mu(s)\int_0^1 \theta\eta_s(s)dxds \\
&\quad + \frac{\delta\rho_1}{c\kappa}\frac{1}{\beta}\int_0^{+\infty}\mu(s)\int_0^1 \varphi_t\eta_{xx}(s)dxds - \frac{\delta^2\rho_1}{c\kappa}\int_0^1 \varphi_t\psi_{xt}dx \\
&\quad - \frac{\delta}{g(0)}\int_0^{+\infty}\mu(s)\int_0^1 \varphi_{xxt}\eta(s)dxds \\
&\quad + \frac{\delta\rho_1}{\kappa g(0)}\int_0^{+\infty}\mu(s)\int_0^1 \varphi_{tt}\eta_s(s)dxds.
\end{aligned}
$$

Thanks to the integration by parts and (2.8), we have

$$\begin{aligned} F_1'(t) &= -c\int_0^1 \theta^2 dx + \frac{1}{\beta g(0)}\left\| \int_0^{+\infty} g'(s)\eta_x(s)ds \right\|_2^2 \\ &\quad - \frac{c}{g(0)}\int_0^{+\infty}\mu'(s)\int_0^1 \theta\eta(s)dxds - \frac{\delta^2\rho_1}{c\kappa}\int_0^1 \varphi_t\psi_{xt}dx \\ &\quad + \frac{\delta}{g(0)}\left(1-\frac{\rho_1 g(0)}{c\kappa\beta}\right)\int_0^{+\infty}\mu(s)\int_0^1 \varphi_{xt}\eta_x(s)dxds \\ &\quad - \frac{\delta\rho_1}{\kappa g(0)}\int_0^{+\infty}\mu'(s)\int_0^1 \varphi_{tt}\eta(s)dxds. \end{aligned} \tag{4.6}$$

Next, taking the derivative of F_2, using the equations of (2.2), and integrating by parts, we get

$$\begin{aligned} F_2'(t) &= -\rho_2\int_0^1 \varphi_{tt}^2 dx - \frac{\kappa^2}{\rho_1}\int_0^1 (\varphi_x+\psi)^2 dx + \left(b+\frac{\delta^2}{c}\right)\int_0^1 \psi_{xt}\varphi_t dx \\ &\quad + \frac{\delta}{c\beta}\int_0^{+\infty}\mu(s)\int_0^1 \varphi_{xt}\eta_x(s)dxds. \end{aligned} \tag{4.7}$$

From (4.6) and (4.7), we find that

$$\begin{aligned} \mathcal{F}'(t) &= -c\mu_2\int_0^1 \theta^2 dx - \rho_2\mu_1\int_0^1 \varphi_{tt}^2 dx - \frac{\kappa^2\mu_1}{\rho_1}\int_0^1 (\varphi_x+\psi)^2 dx \\ &\quad - \frac{c\mu_2}{g(0)}\int_0^{+\infty}\mu'(s)\int_0^1 \theta\eta(s)dxds - \frac{\mu_2}{\beta g(0)}\left\|\int_0^{+\infty}\mu(s)\eta_x(s)ds\right\|_2^2 \\ &\quad - \frac{\delta\rho_1\mu_2}{\kappa g(0)}\int_0^{+\infty}\mu'(s)\int_0^1 \varphi_{tt}\eta(s)dxds + \chi\int_0^{+\infty}\mu(s)\int_0^1 \varphi_{xt}\eta_x(s)dxds, \end{aligned} \tag{4.8}$$

where $\chi = \frac{\delta\mu_2}{g(0)}(1-\frac{\rho_1 g(0)}{c\kappa\beta}) + \frac{\delta}{c\beta}\mu_1$. Thus, exploiting Young's, Cauchy–Schwarz, and Poincarés inequalities, we infer that

$$\begin{aligned} &-\frac{c\mu_2}{g(0)}\int_0^{+\infty}\mu'(s)\int_0^1 \theta\eta(s)dxds \\ &\qquad \le \frac{c\mu_2}{2}\int_0^1 \theta^2 dx - \frac{c\mu_2\xi c_p}{2g(0)}\int_0^{+\infty}\mu'(s)\|\eta_x(s)\|_2^2 ds, \\ &-\frac{\delta\rho_1\mu_2}{\kappa g(0)}\int_0^{+\infty}\mu'(s)\int_0^1 \varphi_{tt}\eta(s)dxds \\ &\qquad \le \frac{\rho_2\mu_1}{2}\int_0^1 \varphi_{tt}^2 dx - \frac{(\delta\rho_1\mu_2)^2\xi c_p}{2\kappa^2 g(0)\rho_2\mu_1}\int_0^{+\infty}\mu'(s)\|\eta_x(s)\|_2^2 ds, \\ &\chi\int_0^{+\infty}\mu(s)\int_0^1 \varphi_{xt}\eta_x(s)dxds \le \varepsilon_1\int_0^1 \varphi_{xt}^2 dx + \frac{\chi^2 g(0)}{4\varepsilon_1}\int_0^{+\infty}\mu(s)\|\eta_x(s)\|_2^2 ds. \end{aligned}$$

Substituting all these latter inequalities in (4.8), we easily obtain (4.5). ∎

Lemma 4.3. *Let $(\varphi, \psi, \theta, \eta)$ be the solution of problem (2.2)–(2.4). Then, the functional*

$$\mathcal{G}(t) = -\kappa \int_0^1 \varphi_x \varphi_{xt} dx$$

satisfies the estimate

$$\mathcal{G}'(t) \leq -\kappa \int_0^1 \varphi_{xt}^2 dx + \varepsilon_2 \int_0^1 \psi_x^2 dx + m\left(1 + \frac{1}{\varepsilon_2}\right) \int_0^1 \varphi_{tt}^2 dx \tag{4.9}$$

for any $\varepsilon_2 > 0$ and for some constant $m > 0$ independent of ε_2.

Proof. Calculating the derivative of $\mathcal{G}$, exploiting the first equation in (2.2) and integrating by parts, we get

$$\mathcal{G}'(t) = -\kappa \int_0^1 \varphi_{xt}^2 dx + \rho_1 \int_0^1 \varphi_{tt}^2 dx - \kappa \int_0^1 \psi_x \varphi_{tt} dx,$$

Inequality (4.9) follows by using Young's inequality. ■

Lemma 4.4. *Let $(\varphi, \psi, \theta, \eta)$ be the solution of problem (2.2)–(2.4). Then, the functional*

$$\mathcal{H}(t) = \rho_2 \kappa \int_0^1 \varphi_{xt} \varphi_x dx$$

satisfies, for some $m > 0$, the estimate

$$\begin{aligned}\mathcal{H}'(t) \leq & -\frac{b\kappa}{2} \int_0^1 \psi_x^2 dx + \rho_2 \kappa \int_0^1 \varphi_{xt}^2 dx + m \int_0^1 \varphi_{tt}^2 dx \\ & + m \int_0^1 (\varphi_x + \psi)^2 dx + m \int_0^1 \theta^2 dx.\end{aligned} \tag{4.10}$$

Proof. The multiplication of the second equation in (2.2) by $\kappa \varphi_x$ and integration by parts give

$$\begin{aligned}\mathcal{H}'(t) = & \rho_2 \kappa \int_0^1 \varphi_{xt}^2 dx + b\kappa \int_0^1 \psi_x \varphi_{xx} dx + \kappa^2 \int_0^1 (\varphi_x + \psi)^2 dx \\ & - \kappa^2 \int_0^1 (\varphi_x + \psi)\psi dx - \delta\kappa \int_0^1 \theta \varphi_{xx} dx.\end{aligned}$$

Then, the use of the first equation in (2.2) leads to

$$\begin{aligned}\mathcal{H}'(t) = & -b\kappa \int_0^1 \psi_x^2 dx + \rho_2 \kappa \int_0^1 \varphi_{xt}^2 dx + b\rho_1 \int_0^1 \psi_x \varphi_{tt} dx + \kappa^2 \int_0^1 (\varphi_x + \psi)^2 dx \\ & - \kappa^2 \int_0^1 (\varphi_x + \psi)\psi dx - \delta\rho_1 \int_0^1 \theta \varphi_{tt} dx + \delta\kappa \int_0^1 \theta \psi_x dx.\end{aligned}$$

Finally, the application of Young's and Poincaré's inequalities yields (4.10). ■

Lemma 4.5. *Let $(\varphi, \psi, \theta, \eta)$ be the solution of problem (2.2)–(2.4). Then, the functional*

$$\mathcal{K}(t) = -\rho_1 \int_0^1 \varphi_t \varphi dx$$

satisfies, for some $m > 0$, the estimate

$$\mathcal{K}'(t) \le -\rho_1 \int_0^1 \varphi_t^2 dx + m \int_0^1 (\varphi_x + \psi)^2 dx + m \int_0^1 \psi_x^2 dx. \tag{4.11}$$

Proof. By performing a simple differentiation of $\mathcal{K}$, using the first equation in (2.2), we obtain

$$\mathcal{K}'(t) = -\rho_1 \int_0^1 \varphi_t^2 dx + \kappa \int_0^1 (\varphi_x + \psi)^2 dx - \kappa \int_0^1 (\varphi_x + \psi)\psi dx.$$

Using Young's and Poincaré's inequalities, estimate (4.11) is established. ∎

Now, we are ready to state and prove the main result.

Theorem 4.6. *The energy functional given by (4.1) is exponentially stable. That is,*

$$E(t) \le \beta e^{-\lambda t} \quad \forall\, t \ge 0, \tag{4.12}$$

where β and λ are positive constants.

Proof. First, we define the following Lyapunov functional:

$$\mathcal{L}(t) = NE(t) + n_1 \mathcal{F} + n_2 \mathcal{G} + n_3 \mathcal{H} + \mathcal{K},$$

where n_i and N are positive constants to be properly chosen later.

Direct computations, using (4.2), (4.5), (4.9), (4.10), and (4.11), and taking $\varepsilon_i = n_i^{-1}, i = 1, 2$, imply that

$$\begin{aligned}
\mathcal{L}'(t) \le & -\rho_1 \int_0^1 \varphi_t^2 dx - \left(\frac{b\kappa}{2} n_3 - m - 1\right) \int_0^1 \psi_x^2 dx \\
& - \left(\kappa n_2 - \rho_2 \kappa n_3 - 1\right) \int_0^1 \varphi_{xt}^2 dx \\
& - \left(\frac{\kappa^2 \mu_1}{\rho_1} n_1 - m n_3 - m\right) \int_0^1 (\varphi_x + \psi)^2 dx \\
& - \left[\frac{\rho_2 \mu_1}{2} n_1 - m n_2 (1 + n_2) - m n_3\right] \int_0^1 \varphi_{tt}^2 dx \\
& - \left(\frac{c \mu_2}{2} n_1 - m n_3\right) \int_0^1 \theta^2 dx \\
& + \left[\frac{1}{2\beta} N - m n_1 (1 + n_1)\right] \int_0^{+\infty} \mu\prime(s) \|\eta_x(s)\|_2^2 ds.
\end{aligned} \tag{4.13}$$

At this point, we fix n_3 so that

$$b\kappa n_3 - 2m - 2 > 0,$$

then, we choose n_2 large enough so that

$$\kappa n_2 - \rho_2 \kappa n_3 - 1 > 0.$$

After fixing n_2, we select n_1 such that

$$\begin{aligned} \kappa^2 \mu_1 n_1 - m\rho_1 n_3 - m\rho_1 &> 0, \\ \rho_2 \mu_1 n_1 - 2mn_2(1+n_2) - 2mn_3 &> 0, \\ c\mu_2 n_1 - 2mn_3 &> 0. \end{aligned}$$

On the other hand, thanks to Young's, Poincaré's and Cauchy–Schwarz inequalities, we have

$$(N-m)E(t) \le \mathcal{L}(t) \le (N+m)E(t). \tag{4.14}$$

Now, we choose N large enough so that

$$N - 2\beta m n_1(1+n_1) > 0$$

and

$$\lambda_1 E(t) \le \mathcal{L}(t) \le \lambda_2 E(t) \tag{4.15}$$

for two positive constants λ_1 and λ_2. Therefore, there exist two positive constants λ_3 and λ_4 such that (4.13) becomes

$$\begin{aligned} \mathcal{L}'(t) \le & -\lambda_3 \int_0^1 (\varphi_t^2 + \varphi_{tt}^2 + \varphi_{xt}^2 + \psi_x^2 + (\varphi_x + \psi)^2 + \theta^2) dx \\ & + \lambda_4 \int_0^{+\infty} \mu'(s) \|\eta_x(s)\|_2^2 ds. \end{aligned} \tag{4.16}$$

From the hypotheses on μ and thanks to (4.15), we get, for some $\lambda > 0$,

$$\mathcal{L}'(t) \le -\lambda \mathcal{L}(t) \quad \forall\, t \ge 0. \tag{4.17}$$

A simple integration of (4.17) over $(0, t)$ leads to

$$\mathcal{L}(t) \le \mathcal{L}(0) e^{-\lambda t} \quad \forall\, t \ge 0. \tag{4.18}$$

A combination of (4.15) and (4.18) gives (4.12), which completes the proof. ■

Remark 4.7. It is well known that the Gurtin–Pipkin law is more general than Cattaneo's law. Therefore, this result generalizes that of [14]. In fact, if

$$g_\varepsilon(s) = \frac{1}{\varepsilon} e^{-\frac{s}{\varepsilon}},$$

with $\varepsilon = \tau\beta^{-1}$, then the heat flux, given by

$$q = -\frac{1}{\beta}\int_0^{+\infty} g_\varepsilon(s)\theta_x(t-s)ds,$$

satisfies

$$\begin{aligned}\beta q_t &= -\frac{1}{\varepsilon}\partial_t\left(e^{-\frac{t}{\varepsilon}}\int_{-\infty}^{t} e^{\frac{s}{\varepsilon}}\theta_x(s)ds\right)\\ &= -\frac{1}{\varepsilon}(\theta_x(t) - \int_0^{+\infty} g_\varepsilon(s)\theta_x(t-s)ds).\end{aligned}$$

Therefore, simple calculations show that q satisfies $(1.4)_4$. In addition, using $(1.5)_3$, we obtain $(1.4)_3$. Moreover, it is easy to check that g_ε satisfies the hypotheses (h_1) and (h_2); consequently, the exponential stability result of [14] is a particular case of this present result. In fact, as it is shown in [9], the energy functional of the Timoshenko–Gurtin–Pipkin system is exponentially stable if and only if the same holds for the energy generated by the Timoshenko–Cattaneo system.

5. Conclusion

In this article, we have investigated a thermoelastic Timoshenko–Gurtin–Pipkin system without the second spectrum. We established the existence of solution, using the semigroup theory and with the help of unusual functionals and proved that the solution decays to the rest state in an exponential rate. This result generalizes that of [14], where a Cattaneo–Timoshenko model, without the second spectrum, was discussed.

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