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Thurston's asymmetric metric on the space of singular flat metrics with a fixed quadrangulation

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Abstract. Consider a compact surface equipped with a fixed quadrangulation. One may identify each quadrangle on the surface with a Euclidean rectangle to obtain a singular flat metric on the surface with conical singularities. We call such a singular flat metric a rectangular structure. We study a metric on the space of unit area rectangular structures which is analogous to Thurston's asymmetric metric on the Teichmüller space of a surface of finite type. We prove that the distance between two rectangular structures is equal to the logarithm of the maximum of ratios of edges of these rectangular structures. We give a sufficient condition for a path between two points of the this Teichmüller space to be geodesic and we prove that any two points of this space can be joined by a geodesic. We also prove that this metric is Finsler and give a formula for the infinitesimal weak norm on the tangent space of each point. We identify the space of unit area rectangular structures with a submanifold of a Euclidean space and we show that the subspace topology and the topology induced by the metric we introduced coincide. We show that the space of unit area rectangular structures on a surface with a fixed quadrangulation is in general not complete.

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1. Introduction

The Teichmüller metric on the space of marked Riemann surfaces was introduced in 1940 by Teichmüller in his paper [16] (cf. the English translation [19]). In this metric, the distance between two surfaces is the logarithm of the least quasiconformal dilatation of quasiconformal mappings between them. In 1985, Thurston introduced an asymmetric metric on Teichmüller space, seen as a space of marked hyperbolic surfaces, defined in a way that is analogous to the Teichmüller metric but where the stress is on the hyperbolic geometry of the surfaces instead of its conformal geometry. According to Thurston's definition, the distance between two marked hyperbolic surfaces is the logarithm of the Lipschitz constant of the best Lipschitz map between them, that is, the map realizing the least Lipschitz dilatation (see [20]). Thurston's metric, like Teichmüller's metric, has been studied from different points of view during the last decades; see, e.g., the papers [4,7,8,11], see also the recent survey [10] and the problem set [15].

It turns out that the theory of Thurston's metric is also interesting for Teichmüller spaces of surfaces equipped with flat structures and singular flat structures. One of the first published papers in this setting is [2] in which the authors gave explicit formulae for two different versions of the analogue of Thurston's asymmetric metric on the Teichmüller space of marked flat tori (identified with the upper half-space). Several properties of this asymmetric metric are proved, and in particular, its symmetrization is shown to be the hyperbolic metric, that is, the Teichmüller metric on the Teichmüller space of flat tori (a fact which does not hold for Thurston's metric on higher-dimensional Teichmüller spaces). In the recent paper [9], the authors show that this metric is weak Finsler and they prove several results on its unit ball at each point in the tangent space to Teichmüller space. The results are analogues of Thurston's results on the Finsler structure of the metric he introduced in his paper [20].

It turns out that the analogue of Thurston's metric between marked Euclidean triangles leads to a variety of questions with interesting developments. In the paper [13], the author gives a model of the hyperbolic plane based on the notion of stretch maps between triangles. In the paper [14], the authors study an analogue of Thurston's metric on the Teichmüller space of marked acute Euclidean triangles with fixed area, showing that it is a symmetric Finsler metric, giving a necessary and sufficient condition for a path in this metric space to be geodesic and determining the isometry group of this metric space. Finally, we mention the work [21] in which the author studies analogues of Thurston's metric on Teichmüller spaces of certain semi-translation surfaces.

Working out analogues of Thurston's theory in a new setting is always a rewarding activity. As it often happens in mathematics, considering elementary cases reveals the essence of a theory.

In the present paper, we study the Euclidean analogue of Thurston's metric on the moduli space of singular flat metrics on a surface having a fixed combinatorial quadrangulation. The L-shaped Euclidean polygons (see Figure 3) are examples of such metrics. It is interesting to recall in this respect that Teichmüller, after he developed his general theory of extremal quasiconformal mappings between surfaces, dedicated a paper on the special case of L-shaped surfaces, see [17]. It turned out that the theory he developed in this special case is very rich and leads to results which do not show up in the general case, see the recent English translation in [18] and the commentary in [1].

We now present more precisely the main results of the present paper.

Let S be a compact surface equipped with a quadrangulation Q, that is, a cellular decomposition into quadrangles, i.e., topological discs with 4 distinguished points on their boundary. A distinguished point of a quadrangle is called a vertex. A *rectangular structure* on (S, Q) is a singular flat metric with conical singularities obtained by identifying each quadrangle of Q with a Euclidean rectangle. Two such rectangular structures are said to be equivalent if they are related by an isotopy which preserves the vertices and which sends each rectangle isometrically to itself. We denote by $\mathcal{R}(Q)$ the set of rectangular structures on (S, Q).

Let $f : S \to S$ be a homeomorphism which fixes the vertices of Q and which is isotopic to the identity map relative to the vertices.

Given two elements μ and μ' of $\mathcal{R}(Q)$, we define

$$L(f) = \sup_{x \neq y \in S} \frac{d_{\mu'}\big(f(x), f(y)\big)}{d_{\mu}(x, y)} \quad \text{and} \quad L(\mu, \mu') = \inf_{f \simeq \mathrm{id}} \log L(f),$$

where the infimum is taken over all homeomorphisms which are isotopic to the identity and which preserve the vertices.

We also define another function K on $\mathcal{R}(Q) \times \mathcal{R}(Q)$, by setting $K(\mu, \mu')$ to be the logarithm of the maximum of ratios of lengths of edges connecting the same vertices, with respect to the rectangular structures μ and μ'. Here are the main results that we obtain:

(1) We prove that $L = K$.

(2) We prove that L is an asymmetric metric when it is restricted to the space of unit area rectangular structures on (S, Q). We denote this space by $\mathcal{R}(Q)_1$.

(3) We identify $\mathcal{R}(Q)_1$ with a subset of some Euclidean space and show that the subspace topology and the topology induced by L coincide.

(4) We give a sufficient condition for a path in $\mathcal{R}(Q)_1$ to be a geodesic. We prove that any two points on $\mathcal{R}(Q)_1$ can be a joined by a geodesic.

(5) We prove that the restriction of the metric L to $\mathcal{R}(Q)_1$ is Finsler and we give a formula for the associated the infinitesimal weak norm on the tangent space of each point in $\mathcal{R}(Q)_1$.

(6) We prove that the space $\mathcal{R}(Q)_1$ is not complete when its dimension is greater than 1.

The rest of the paper is organized as follows. In Section 2 we deal with the case of rectangles. In Section 3 we introduce quadrangulations and rectangular structures on a surface. We identify the space of rectangular structures with a subset of some Euclidean space. The analogues of Thurston's asymmetric metric and the metric K are introduced in Section 4. In Section 5 we give a family of geodesics in the space of unit area rectangular structures. In Section 6, we prove that the metric L is Finsler. In Section 7, we discuss the topology of $\mathcal{R}(Q)_1$ and some metric properties of L.

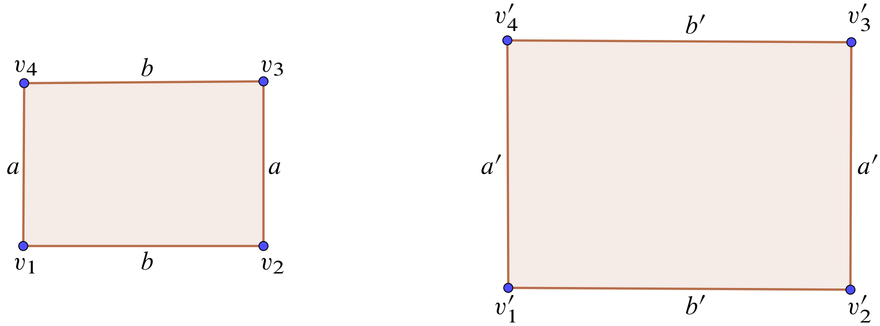

FIGURE 1
The affine map between the two rectangles is a best Lipschitz map.

2. The case of rectangles

Let R and R' be two rectangles in the Euclidean plane. Label their vertices by v_1, v_2, v_3, v_4 and v_1', v_2', v_3', v_4', respectively. Let a, b and a', b' be the lengths of the edges of R and R', respectively; see Figure 1.

A homeomorphism from R to R' is said to be label-preserving if, for every $i = 1, \ldots, 4$, we have $f(v_i) = v_i'$. For such a homeomorphism we define

$$L(f) = \sup_{x,y\in R, x\neq y} \frac{d_{\text{euc}}\big(f(x), f(y)\big)}{d_{\text{euc}}(x, y)}.$$

We then define $L(R, R') := \inf \log(L(f))$, where the infimum is taken over the set of label-preserving homeomorphisms. If for some homeomorphism g we have $L(g) = \exp(L(R, R'))$, then we call g a *best Lipschitz map*.

Proposition 1. *We have* $L(R, R') = \log(\max\{\frac{a'}{a}, \frac{b'}{b}\})$, *and the affine map between* R *and* R' *is a best Lipschitz map.*

Proof. Any label-preserving homeomorphism sends v_i to v_i', therefore it is evident that

$$\exp(L(R, R')) \geq \max\Big\{\frac{a'}{a}, \frac{b'}{b}\Big\}.$$

Now we prove that for the affine map $\mathcal{A}$, we have $L(\mathcal{A}) \leq \max\{\frac{a'}{a}, \frac{b'}{b}\}$. This will imply both statements of the proposition.

By performing some isometries of the Euclidean plane $\mathbb{R}^2$ we may suppose that v_1 and v_1' are at the origin, v_2, v_2' are on the x-axis and v_4, v_4' are on the y-axis. Consider the following affine map $\mathcal{A}\colon R \to R'$:

$$\mathcal{A}(x, y) = \Big(\frac{b'}{b}x, \frac{a'}{a}y\Big).$$

Let $q_1 = (x_1, y_1)$ and $q_2 = (x_2, y_2)$. We have

$$\begin{aligned} d_{\text{euc}}\big(\mathcal{A}(q_1), \mathcal{A}(q_2)\big) &= d_{\text{euc}}\Big(\Big(\frac{b'}{b}x_1, \frac{a'}{a}y_1\Big), \Big(\frac{b'}{b}x_2, \frac{a'}{a}y_2\Big)\Big) \\ &= \sqrt{\Big(\frac{b'}{b}\Big)^2 (x_2 - x_1)^2 + \Big(\frac{a'}{a}\Big)^2 (y_2 - y_1)^2} \\ &\leq \max\Big\{\frac{a'}{a}, \frac{b'}{b}\Big\} \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \max\Big\{\frac{a'}{a}, \frac{b'}{b}\Big\} d_{\text{euc}}(q_1, q_2). \end{aligned}$$

Hence, we obtain $L(\mathcal{A}) \leq \max\{\frac{a'}{a}, \frac{b'}{b}\}$. ■

Let us denote by R_a a rectangle having unit area. This means that the edge lengths of R_a are a and $1/a$. In this way, we may naturally identify the set of unit area rectangles with the set of positive real numbers $\mathbb{R}_+^*$, using the map sending R_a to a. In this case, we have

$$L(R_a, R_{a'}) = L(a, a') = \log\Big(\max\Big\{\frac{a'}{a}, \frac{a}{a'}\Big\}\Big) = |\log a' - \log a|.$$

Remarks 1. (1) The metric L on the space of rectangles comes from the infinitesimal metric $|da|/a$ on $\mathbb{R}_+^*$ and it is Finsler. (We shall discuss more thoroughly Finsler structure on manifolds and Finsler metrics in Section 6.)

(2) The case of best Lipschitz maps between rectangles considered here is the analogue of the case of best quasiconformal maps between rectangles considered by Grötzsch in the paper [5] (English translation in [6]).

3. Quadrangulations and rectangular structures

Let S be a compact surface, possibly with boundary. Note that we do not exclude the case where S is not orientable.

Definition 2. A *quadrangulation* Q of S is a collection of maps $e_i\colon [0,1] \to S$, $i \in I$, where I is a finite set, such that

(1) each e_i is continuous;

(2) $e_i|_{(0,1)}$ is a homeomorphism onto its image for all $i \in I$;

(3) $e_i([0,1]) \cap e_j([0,1]) \subset e_i(\{0,1\}) \cap e_j(\{0,1\})$ for all $i \neq j$;

(4) when we cut the surface S along the images of the e_is, we get quadrangles whose vertices are at the images of the endpoints of the e_is.

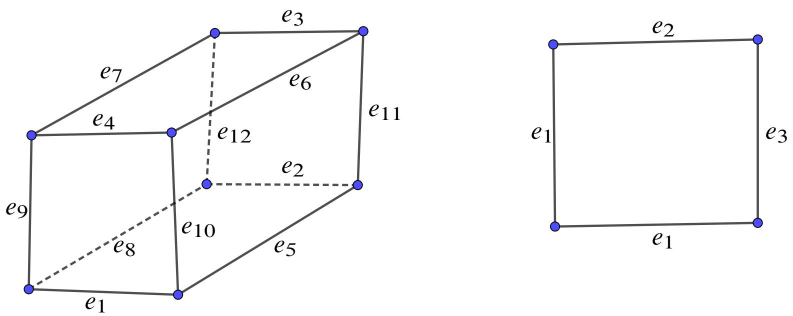

FIGURE 2
A quadrangulation of a sphere and a quadrangulation of a disc.

Given a quadrangulation as above, a quadrangle is called a *face* of it; the set of faces of the quadrangulation Q will be denoted by $F(Q)$. The e_is (or their images) form the *edges* of the quadrangulation and their union will be denoted by $E(Q)$. Each element in $e_i(\{0,1\})$ is called an endpoint of e_i. The union of the endpoints is called the *vertex set* of the quadrangulation and will be denoted by $V(Q)$.

Fix a quadrangulation Q of S. For each face of Q, we say that opposite edges are *parallel*. This generates an equivalence relation on $E(Q)$ where we say that $e_i \sim e_j$ if $i = j$ or there are edges $e_{i_1}, \dots, e_{i_k}$ such that

$$i_1 = i,\ i_k = j \text{ and } e_{i_l} \text{ and } e_{i_{l+1}} \text{ are parallel for all } 1 \le l \le k-1.$$

By abuse of language, we say that e_i and e_j are *parallel* if $e_i \sim e_j$.

Example 3. In Figure 2 we have two examples of quadrangulations of surfaces. On the left of this figure, we have a quadrangulation of the sphere. We see that $e_i \sim e_j$ if $1 \le i, j \le 4$, $5 \le i, j \le 8$ or $9 \le i, j \le 12$. On the right of Figure 2, after identifying the two edges having label e_1, we obtain a quadrangulation of a disc with three edges. In this case all edges e_1, e_2 and e_3 are parallel.

Fix a surface S with a quadrangulation Q. In this paper we study rectangular structures on quadrangulated surfaces. A rectangular structure can be obtained by filling the quadrangulation with Euclidean rectangles. Here is the precise definition.

Definition 4. A *rectangular structure* on (S, Q) is a singular flat metric with conical singularities whose set of singular points is a subset of $V(Q)$ such that if we cut the surface along the edges e_i of (S, Q) we get Euclidean rectangles.

In other words, a rectangular structure is obtained by filling in the surface with Euclidean rectangles according to the combinatorics specified by the quadrangulation.

A rectangular structure μ on (S, Q) comes equipped with several additional structures since it is a singular flat metric. First of all, there is a well-defined area measure which coincides with the 2-dimensional Lebesgue measure on each rectangle. We can also define the length $l_\mu(c)$ of a piecewise smooth curve $c\colon [a, b] \to S$ using the following rules:

(1) If c is contained in a face of Q, then $l_\mu(c)$ is its Euclidean length.

(2) If c is a concatenation of curves c_1 and c_2, then

$$l_\mu(c) = l_\mu(c_1) + l_\mu(c_2).$$

We can then define a metric d_μ for each rectangular structure on a surface where the distance between two points is given by the infimum of lengths of piecewise smooth paths joining these two points. Observe that by compactness, this infimum is attained for any two distinct points on S.

We say that two rectangular structures μ and μ' are equivalent if by cutting the surface through the edges, we get the same Euclidean rectangles. We denote the edge length of e_i with respect to μ by $l_\mu(e_i)$. It follows that two rectangular structures μ and μ' on S are equivalent if and only if $l_\mu(e_i) = l_{\mu'}(e_i)$ for all i. We denote an equivalence class of μ by $[\mu]$.

We may also define this equivalence relation in terms of d_μ. By a homeomorphism of (S, Q) we mean a homeomorphism $h\colon S \to S$ which leaves each edge of Q invariant. It follows that μ and μ' are equivalent if and only if there is a homeomorphism of h of (S, Q) such that $h^*(d_{\mu'}) = d_\mu$.

We denote the set of equivalence classes of rectangular structures by $\mathcal{R}(Q)$. For a given $A > 0$, the set $\mathcal{R}(Q)$ has a subset $\mathcal{R}(Q)_A$ which consists of equivalence classes of rectangular structures having fixed area A.

From now on we fix a set of representatives $e_{i_1}, \dots, e_{i_k}$ for the equivalence relation of parallelism on $E(Q)$ defined on the set of edges after Definition 3.

Proposition 2. *The map $\Psi\colon \mathcal{R}(Q) \to (\mathbb{R}_+^*)^k$, sending $[\mu]$ to $(l_\mu(e_{i_l}))$ is a bijection.*

Proof. Injectivity follows from the fact that the rectangular structure is determined by the lengths of the e_is and that parallel edges have the same length. Let $(a_1, \dots, a_k) \in (\mathbb{R}_+^*)^k$. To prove surjectivity we define a function $f\colon E(Q) \to \mathbb{R}_+^*$ by declaring $f(e_i) = a_l$ if $e_i \sim e_{i_l}$. Cut the surface S through the e_is. We have a finite number of quadrangles whose edges are labeled with the e_is. Identify each such quadrangle with a Euclidean rectangle so that if an edge is labeled by e_i the corresponding edge of the rectangle has length $f(e_i)$. Glue back these rectangles along their edges appropriately to obtain a rectangular structure on the surface. In this way we get a rectangular structure μ such that $\Psi(\mu) = (a_1, \dots, a_k)$. ∎

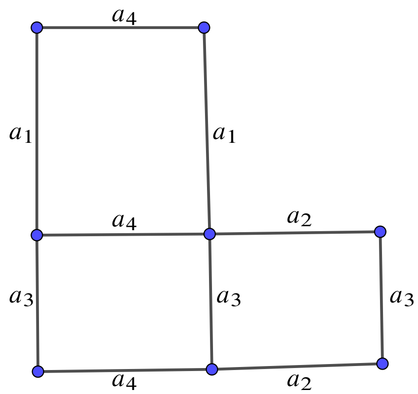

FIGURE 3
An L-shaped polygon.

Identifying $\mathcal{R}(Q)$ with $(\mathbb{R}_+^*)^k = \{(a_1 \ldots, a_k)\}$, we see that the area of a rectangular structure gives us a quadratic form with variables $a_1, \ldots, a_k$,

$$q(a_1, \ldots, a_k) = \sum_{i \leq j} c_{ij} a_i a_j,$$

so that each c_{ij} is non-negative. We also see that for each l there exists l' such that $c_{l,l'}$ is positive.

Example 5. An *L-shaped polygon* consists of three rectangles glued as in Figure 3. Note that such a polygon is determined by the edge lengths a_1, a_2, a_3, a_4. There is a corresponding area quadratic form

$$q(a_1, a_2, a_3, a_4) = a_1 a_4 + a_3 a_4 + a_2 a_3.$$

4. Thurston's asymmetric metric on the space of rectangular structures

In this section, the surface S is equipped with a fixed quadrangulation Q. We are interested in maps $S \to S$ which fix the vertices of Q. Therefore when we say that a homeomorphism $f : S \to S$ is isotopic to the identity, $f \equiv \mathrm{id}$, it should be understood that the vertices of Q are fixed during the isotopy. Keeping this in mind, we introduce two ways of providing the space of rectangular structures with a metric.

Assume that μ and μ' are two rectangular structures on (S, Q). Let $f : S \to S$ be a homeomorphism. We define

$$L(f) = \sup_{x \neq y \in S} \frac{d_{\mu'}(f(x), f(y))}{d_\mu(x, y)} \quad \text{and} \quad L(\mu, \mu') = \inf_{f \equiv \mathrm{id}} \log L(f).$$

In the last formula, the infimum is taken over all homeomorphisms isotopic to the identity. Note that the value $L(\mu, \mu')$ depends only on the equivalence classes $[\mu]$ and $[\mu']$, of μ and μ'. Thus, $L([\mu], [\mu'])$ is well defined. For $\lambda \in \mathbb{R}_+^*$, let us denote by $\lambda\mu$ the rectangular structure on (S, Q) such that $l_{\lambda\mu}(e_i) = \lambda l_\mu(e_i)$ for all $i \in I$. This gives an action of $\mathbb{R}_+^*$ on the set of equivalence classes of rectangular structures $\mathcal{R}(Q)$:

$$\lambda[\mu] = [\lambda\mu].$$

The following formula is clear:

$$\exp\bigl(L(\lambda\mu, \lambda'\mu')\bigr) = \frac{\lambda'}{\lambda} \exp\bigl(L(\mu, \mu')\bigr). \tag{1}$$

Now we define another function on $\mathcal{R}(Q) \times \mathcal{R}(Q)$.

Definition 6. Let $[\mu]$ and $[\mu']$ be two elements in $R(Q)$. We define

$$K\bigl([\mu], [\mu']\bigr) = K(\mu, \mu') := \log\left(\sup_{i \in I} \frac{l_{\mu'(e_i)}}{l_\mu(e_i)}\right).$$

Note that since $e_i \sim e_j$ implies that $l_\mu(e_i) = l_\mu(e_j)$, it follows that the supremum can be taken on the set of representatives e_{i_l} of equivalence classes of edges, that is,

$$K\bigl([\mu], [\mu']\bigr) = \log\left(\sup_{l} \frac{l_{\mu'}(e_{i_l})}{l_\mu(e_{i_l})}\right).$$

We also have the following formula:

$$\exp\bigl(K(\lambda\mu, \lambda'\mu')\bigr) = \frac{\lambda'}{\lambda} \exp\bigl(K(\mu, \mu')\bigr).$$

Since any homeomorphism $f\colon S \to S$ that we consider fixes Q, we have the following inequality:

$$K \leq L.$$

Now we fix two rectangular structures μ and μ'. Let $\{R_\alpha\}_{\alpha \in \mathcal{A}}$ be the set of rectangles of S with respect to the rectangular structure μ. Let us define

$$L(f|_{R_\alpha}) := \sup_{x \neq y \in R_\alpha} \frac{d_{\mu'}\bigl(f(x), f(y)\bigr)}{d_\mu(x, y)}.$$

Proposition 3. *We have* $L(f) = \max_\alpha\{L(f|_{R_\alpha})\}$.

Proof. It is clear that $L(f) \geq \max_\alpha\{L(f|_{R_\alpha})\}$. We prove the reverse inequality. Let $x, y \in S$ and let g be a length minimizing geodesic joining these two points. It suffices to prove that there exists $\alpha \in \mathcal{A}$ and $x', y' \in R_\alpha$ such that

$$\frac{d_{\mu'}\big(f(x'), f(y')\big)}{d_\mu(x', y')} \geq \frac{d_{\mu'}\big(f(x), f(y)\big)}{d_\mu(x, y)}.$$

Assume that this is not true. Choose a finite set of points $x_1 = x, x_2, \dots, x_m = y$ on the geodesic g such that for each $i = 1, \dots, m-1$, x_i and x_{i+1} lie on the same rectangle. Thus for each i, we have

$$\frac{d_{\mu'}\big(f(x_{i+1}), f(x_i)\big)}{d_\mu(x_{i+1}, x_i)} < \frac{d_{\mu'}\big(f(x), f(y)\big)}{d_\mu(x, y)},$$

$$d_{\mu'}\big(f(x_{i+1}), f(x_i)\big) < d_\mu(x_{i+1}, x_i)\frac{d_{\mu'}\big(f(x), f(y)\big)}{d_\mu(x, y)}.$$

Hence,

$$\sum_{i=1}^{m-1} d_{\mu'}\big(f(x_{i+1}), f(x_i)\big) < \frac{d_{\mu'}\big(f(x), f(y)\big)}{d_\mu(x, y)} \sum_{i=1}^{m-1} d_\mu(x_{i+1}, x_i).$$

Note that

$$d_\mu(x, y) = \sum_{i=1}^{m-1} d_\mu(x_{i+1}, x_i).$$

Therefore, we have

$$\sum_{i=1}^{m-1} d_{\mu'}\big(f(x_{i+1}), f(x_i)\big) < d_\mu\big(f(x), f(y)\big).$$

Clearly, this contradicts the triangle inequality. ∎

Now we define a homeomorphism $\phi_{\mu,\mu'}\colon S \to S$ which is isotopic to the identity. For each rectangle R_α of the rectangular structure μ, let R'_α be the corresponding rectangle in the rectangular structure μ'. We know that there is an affine map sending R_α to R'_α; see Section 2. Gluing these affine maps, we get a homeomorphism $\phi_{\mu,\mu'}$ isotopic to identity.

Now we can prove the following result.

Theorem 7. *For every two rectangular structures μ and μ' on (S, Q), we have*

$$K(\mu, \mu') = L(\mu, \mu').$$

Moreover, the map $\phi_{\mu,\mu'}$ is a best Lipschitz map.

Proof. We know that

$$\log\left(\sup_{i\in I}\frac{l_{\mu'}(e_i)}{l_\mu(e_i)}\right) = K(\mu,\mu') \le L(\mu,\mu').$$

On the other hand, by Proposition 1 and Proposition 3, we get

$$\log\big(L(\phi_{\mu,\mu'})\big) = \log\left(\sup_{i\in I}\frac{l_{\mu'}(e_i)}{l_\mu(e_i)}\right).$$

This observation implies both statements of the theorem. ■

We will show that the restriction of L to $\mathcal{R}(Q)_1 \times \mathcal{R}(Q)_1$ is a metric. First we define the notion of metric that we use. This definition is different from the usual one since we drop the symmetry axiom.

Definition 8. A *metric* on a set X is a function $\eta\colon X \times X \to \mathbb{R}$ such that

- $\eta(x,x) = 0$ for all $x \in X$;
- $\eta(x,y) > 0$ if $x \neq y$;
- $\eta(x,y) + \eta(y,z) \ge \eta(x,z)$ for all $x,y,z \in X$.

The pair (X,d) or the set X is called a *metric space*. If $\eta(x,y) = \eta(y,x)$ for all $x, y \in X$, then the metric is said to be symmetric. Otherwise, and if we want to emphasize the asymmetry, the metric is said to be asymmetric.

Lemma 1. *Let μ, μ' and μ'' be three rectangular structures on (S,Q). Then*

$$L(\mu,\mu'') \le L(\mu,\mu') + L(\mu',\mu'').$$

Proof. Let f and g be homeomorphisms $S \to S$ which are isotopic to the identity. The statement follows from the fact that

$$L(f\circ g) \le L(f)L(g).$$ ■

Lemma 2. *Let μ and μ' be two rectangular structures of equal area. If $L(\mu,\mu') \le 0$, then $L(\mu,\mu') = 0$ and $\mu \sim \mu'$.*

Proof. To prove the statement we use the identification of $\mathcal{R}(Q)$ with $(\mathbb{R}_+^*)^k$. In this identification μ and μ' correspond to two elements $(a_1,\dots,a_k)$ and $(b_1,\dots,b_k)$ such that

$$q(a_1,\dots,a_k) = q(b_1,\dots,b_k) = A > 0.$$

Note that $L = K$ implies that

$$\exp\big(L(\mu,\mu')\big) = \max\left\{\frac{b_1}{a_1},\dots,\frac{b_k}{a_k}\right\} \le 1.$$

If $\exp(L(\mu,\mu')) < 1$, then it follows that

$$q(a_1,\dots,a_k) > q(b_1,\dots,b_k),$$

which is a contradiction.

If $L(\mu,\mu') = 0$, then it follows that $a_i \le b_i$ for all $1 \le i \le k$. If there is a j such that $a_j < b_j$, then

$$q(a_1,\dots,a_k) < q(b_1,\dots,b_k),$$

which is a contradiction. Hence, $a_i = b_i$ for all i. This means that $\mu \sim \mu'$. ∎

Theorem 9. *The restriction of L to $\mathcal{R}(Q)_A \times \mathcal{R}(Q)_A$ gives a metric.*

Proof. The statement follows from Lemma 1 and Lemma 2. ∎

5. Geodesics on the spaces of rectangular structures

Let (X,d) be a metric space where d is not necessarily symmetric. Let I be an interval of $\mathbb{R}$. We say that a map $h\colon I \to X$ is geodesic if for every ordered triple $t_1 \le t_2 \le t_3$ in I, we have

$$d\big(h(x_1),h(x_3)\big) = d\big(h(x_1),h(x_2)\big) + d\big(h(x_2),h(x_3)\big).$$

Note that if the metric is not symmetric, then the map obtained from h by reversing the direction of a geodesic is not necessarily a geodesic.

In this section, we show that any two elements in $\mathcal{R}(Q)_1$ can be joined by a geodesic. Recall that we identified $\mathcal{R}(Q)$ and $(\mathbb{R}^*_+)^k$. We saw that there is a quadratic form q on $\mathbb{R}^k$ such that under this identification, $\mathcal{R}(Q)_1$ corresponds to the set

$$(\mathbb{R}^*_+)^k_1 = \{(a_1,\dots,a_k), a_i > 0, q(a_1,\dots,a_k) = 1\}.$$

Let $a' = (a'_1,\dots,a'_k)$ and $a'' = (a''_1,\dots,a''_k)$ be two elements in $(\mathbb{R}^*_+)^k_1$. We saw that

$$L(a',a'') = \log \max_i \Big\{\frac{a''_i}{a'_i}\Big\}.$$

Assume that $L(a',a'') = \log(a''_j/a'_j)$ for some $j \in \{1,2,\dots,k\}$.

Lemma 3. *Let $\alpha(t) = (a_1(t),\dots,a_k(t))$ be a family of rectangular structures (of not necessarily unit area) such that $\alpha(0) = a'$ and $\alpha(1) = a''$. Furthermore, assume that each a_i is C^1 and*

$$\frac{\dot a_i(s)}{a_i(s)} \le \frac{\dot a_j(s)}{a_j(s)} \tag{2}$$

for all $i = 1,\dots,k$ and for all $s \in [0,1]$.

Let $\lambda(t)$ be the unique positive real number such that $\lambda(t)\alpha(t) \in (\mathbb{R}_+^*)_1^k$. Then

$$\alpha'(t) = \lambda(t)\alpha(t) = \big(\lambda(t)a_1(t), \dots, \lambda(t)a_k(t)\big)$$

is a geodesic in $(\mathbb{R}_+^*)_1^k$ and it joins a' and a''.

Proof. It is clear that the image of α' lies in $(\mathbb{R}_+^*)_1^k$ and that α' joins a' and a''. We will show that α' is a geodesic. Before this, we need to make some calculations.

First of all, we have

$$\begin{aligned} L(\alpha(t), \alpha(t')) &= \max_i \log\left(\frac{a_i(t')}{a_i(t)}\right) = \max_i \int_t^{t'} \frac{\dot{a}_i(s)}{a_i(s)}\, ds \\ &= \int_t^{t'} \frac{\dot{a}_j(s)}{a_j(s)}\, ds = \log a_j(t') - \log a_j(t). \end{aligned}$$

By combining the above equation with equation (1), we get

$$L\big(\lambda(t)\alpha(t), \lambda(t')\alpha(t')\big) = \log\lambda(t') - \log\lambda(t) + L\big(\alpha(t'), \alpha(t)\big) \tag{3}$$

$$= \log\lambda(t') - \log\lambda(t) + \log a_j(t') - \log a_j(t). \tag{4}$$

Now we prove that α' is a geodesic, that is, we show that if $t_1 \le t_2 \le t_3$ in $[0, 1]$, then

$$L\big(\alpha'(t_1), \alpha'(t_3)\big) = L\big(\alpha'(t_1), \alpha'(t_2)\big) + L\big(\alpha'(t_2), \alpha'(t_3)\big).$$

Equations (3) and (4) imply that

$$\begin{aligned} L\big(\alpha'(t_1), \alpha'(t_3)\big) &= \log\lambda(t_3) - \log\big(\lambda(t_1)\big) + \log a_j(t_3) - \log a_j(t_1) \\ &= \log\lambda(t_3) - \log\lambda(t_2) + \log\lambda(t_2) - \log\lambda(t_1) \\ &\qquad + \log a_j(t_3) - \log a_j(t_2) + \log a_j(t_2) - \log a_j(t_1) \\ &= L\big(\alpha'(t_1), \alpha'(t_2)\big) + L\big(\alpha'(t_2), \alpha'(t_3)\big). \end{aligned}$$

■

Example 10. We note that $\alpha(t) = ((a_1')^{1-t}(a_1'')^t, \dots, (a_k')^{1-t}(a_k'')^t)$ satisfies inequality (2) (this follows from the fact that the logarithmic derivative of each component of $\alpha(t)$ is constant).

The following corollary is immediate.

Corollary 1. *For any two points in $\mathcal{R}(Q)_1$, there exists a geodesic joining them.*

6. Finsler structure on spaces of rectangular structures

In this section we show that the metric L on $\mathcal{R}(Q)_1$ is Finsler. Before doing this, we need to show that $\mathcal{R}(Q)_1$ is a differentiable manifold. Recall that we identified $\mathcal{R}(Q)$ with $(\mathbb{R}_+^*)^k$ and that there is a quadratic form q on $\mathbb{R}^k$ such that there is a one-to-one correspondence between $\mathcal{R}(Q)_1$ and the following set:

$$(\mathbb{R}_+^*)_1^k = \{(a_1, \dots, a_k), a_i > 0, q(a_1, \dots, a_k) = 1\}.$$

Thus, we will show that this set is a differentiable manifold.

Proposition 4. *$(\mathbb{R}_+^*)_1^k$ is an embedded differential submanifold of $(\mathbb{R}_+^*)^k$.*

Proof. We have

$$q(a_1, \dots, a_k) = \sum_{i \le j} c_{ij} a_i a_j.$$

Note that each c_{ij} is non-negative and for each i there is a j such that c_{ij} is positive. Consider q as a function $(\mathbb{R}_+^*)^k \to \mathbb{R}$. It follows that

$$dq = \sum_{i<j} c_{ij}(a_i da_j + a_j da_i) + \sum_i 2c_{ii} a_i da_i$$

does not vanish anywhere. Thus, $(\mathbb{R}_+^*)_1^k$ is an embedded submanifold of $(\mathbb{R}_+^*)^k$. ■

Now we introduce our setting of Finsler structures. We follow [12], in particular §6, where we define the notion of weak Finsler structure. We start by recalling the definition of a *weak norm*.

Definition 11. Let V be a real vector space. A *weak norm* on V is a function $V \to [0, \infty)$, $v \to \|v\|$ such that for every $v \in V$ the following properties hold for every v and w in V:

(1) $\|v\| = 0$ if $v = 0$;

(2) $\|tv\| = t\|v\|$ for every $t > 0$;

(3) $\|tv + (1-t)w\| \le t\|v\| + (1-t)\|w\|$ for every $t \in [0, 1]$.

Now let M be a differentiable manifold and let TM be the tangent bundle of M.

Definition 12. A Finsler structure on M is a function $F \colon TM \to [0, \infty)$ such that

(1) F is continuous;

(2) for each $x \in M$, $F|_{T_x M}$ is a weak norm.

Let F be a Finsler structure on a manifold M. For each C^1 curve $c\colon [a,b] \to M$, we define its length by

$$l(c) = l_F(c) = \int_a^b F(\dot{c}(t))\,dt.$$

Definition 13. A metric d on a differentiable manifold M is said to be *Finsler* if it is the length metric associated with a Finsler structure, that is, if there exist a Finsler structure F on M such that for every $x, y \in M$, we have

$$d(x,y) = \inf\{l_F(c)\},$$

where c ranges over all piecewise C^1 curves such that $c(0) = x$ and $c(1) = y$.

Now we start the proof of the fact that the metric L on $\mathcal{R}(Q)_1$ is Finsler. Instead of $\mathcal{R}(Q)_1$, we will consider $(\mathbb{R}_+^*)_1^k$. We will consider the following function F on the tangent space $T(\mathbb{R}_+^*)_1^k$ of $(\mathbb{R}_+^*)_1^k$:

$$F\colon (a_1,\ldots,a_k,v_1,\ldots,v_k) \mapsto \max_i\Big\{\frac{v_i}{a_i}\Big\}.$$

Here, $a = (a_1,\ldots,a_k)$ is a point in $(\mathbb{R}_+^*)_1^k$ and $(v_1,\ldots,v_k)$ are coordinates of tangent vectors at the point $a = (a_1,\ldots,a_k)$. Note that it is not evident that $F|_{T_a(\mathbb{R}_+^*)_1^k}$ is non-negative. The other properties of a weak norm are clearly satisfied. We will show that for any $a' = (a_1',\ldots,a_k')$ and $a'' = (a_1'',\ldots,a_k'')$ in $(\mathbb{R}_+^*)_1^k$, we have

$$L(a',a'') = \inf\{l_F(c)\},$$

where c ranges over all C^1 curves such that $c(0) = a'$, $c(1) = a''$ and

$$l_F(c) = \int_0^1 F\big(\dot{c}(t)\big)\,dt.$$

Then we will use this to prove that $F|_{T_a(\mathbb{R}_+^*)_1^k}$ is non-negative.

Proposition 5. *Let $a' = (a_1',\ldots,a_k')$ and $a'' = (a_1'',\ldots,a_k'')$ be in $(\mathbb{R}_+^*)_1^k$. Then*

$$L(a',a'') = \inf\{l_F(c)\},$$

where c ranges over all C^1 curves such that $c(0) = a'$, $c(1) = a''$.

Proof. Let $c(t) = (a_1(t),\ldots,a_k(t))$ be a C^1 curve in $(\mathbb{R}_+^*)_1^k$ such that $c(0) = a'$ and $c(1) = a''$. Then

$$l_F(c) = \int_0^1 \max_i\Big\{\frac{\dot{a}_i(t)}{a_i(t)}\Big\}\,dt \geq \max_i\Big\{\int_0^1 \frac{\dot{a}_i(t)}{a_i(t)}\Big\} = \max_i\Big\{\log\Big(\frac{a_i''}{a_i'}\Big)\Big\} = L(a',a'').$$

Now we prove that $L_F(c) = L(a', a'')$ for the geodesics we determined in Section 5. Assume that

$$\max_i \left\{ \frac{a''_i}{a'_i} \right\} = \frac{a''_j}{a'_j}.$$

Let $d\colon [0,1] \to (\mathbb{R}^*_+)^k$ be such that $d(t) = (a_1(t), \dots, a_k(t))$ and

$$\frac{\dot{a}_i(t)}{a_i(t)} \leq \frac{\dot{a}_j(t)}{a_j(t)}$$

for all i. Let $\lambda(t)$ be the unique real number such that $c(t) = \lambda(t)d(t) \in (\mathbb{R}^*_+)^k_1$. We have

$$\frac{\frac{d}{dt}\big(\lambda(t)a_i(t)\big)}{\lambda(t)a_i(t)} = \frac{\dot{\lambda}(t)}{\lambda(t)} + \frac{\dot{a}_i(t)}{a_i(t)} \leq \frac{\frac{d}{dt}\big(\lambda(t)a_j(t)\big)}{\lambda(t)a_j(t)}.$$

It follows that

$$\begin{aligned} l_F(c) &= \int_0^1 \max_i \left\{ \frac{\frac{d}{dt}\big(\lambda(t)a_i(t)\big)}{\lambda(t)a_i(t)} \right\} dt \\ &= \int_0^1 \frac{\frac{d}{dt}\big(\lambda(t)a_j(t)\big)}{\lambda(t)a_j(t)}\, dt = \log a''_j - \log a'_j = L(a', a''). \end{aligned}$$ ∎

Proposition 6. *$F|_{T_a(\mathbb{R}^*_+)^k_1}$ is a weak norm for each $a \in (\mathbb{R}^*_+)^k_1$.*

Proof. We only need to prove that $F|_{T_a(\mathbb{R}^*_+)^k_1}$ is non-negative. If it were negative for some vector in $T_a(\mathbb{R}^*_+)^k_1$, then there would be another point a' such that $L(a, a') < 0$, since $L(a, a')$ is given by $\inf\{l_F(c)\}$, where c joins a to a', which contradicts to Lemma 2 . ∎

The following theorem follows easily from Proposition 5 and Proposition 6.

Theorem 14. *Let F be the following function on the tangent space $T(\mathbb{R}^*_+)^k_1$ of $(\mathbb{R}^*_+)^k_1$:*

$$(a_1, \dots, a_k, v_1, \dots, v_k) \mapsto \max_i \left\{ \frac{v_i}{a_i} \right\}.$$

The metric L is induced by the Finsler structure on $(\mathbb{R}^*_+)^k_1$ given by F.

7. The topology of $\mathcal{R}(Q)_1$ and the metric properties of $L = K$

H. Busemann [3] developed extensively a theory of non-symmetric metric spaces (X, δ) which satisfy additionally the following property:

$\delta(x_n, x) \to 0$ *if and only if* $\delta(x, x_n) \to 0$.

In this section, we prove that this property is satisfied for $(\mathcal{R}(Q)_1, L)$. Note that we identified $\mathcal{R}(Q)_1$ with the set

$$(\mathbb{R}_+^*)_1^k = \{(a_1, \dots, a_k) : a_i > 0, q(a_1, \dots, a_k) = 1\},$$

where $q(a_1, \dots, a_k) = \sum_{i \le j} c_{ij} a_i a_j$ such that for each i there is a j such that $c_{ij} > 0$. Also, by Theorem 7, we have

$$\begin{aligned} L\big((a_1, \dots, a_k), (a_1', \dots, a_k')\big) &= K\big((a_1, \dots, a_k), (a_1', \dots, a_k')\big) \\ &= \log\Big(\max_i \Big\{\frac{a_i'}{a_i}\Big\}\Big). \end{aligned}$$

Proposition 7. *Let $a = (a_1, \dots, a_k) \in (\mathbb{R}_+^*)_1^k$ and $a_n = (a_{1,n}, \dots, a_{k,n})$ be a sequence in $(\mathbb{R}_+^*)_1^k$. Then the following properties hold:*

(1) $K(a_n, a) \to 0$ *if and only if* $a_{i,n} \to a_i$ *for each* $1 \le i \le k$;

(2) $K(a, a_n) \to 0$ *if and only if* $a_{i,n} \to a_i$ *for each* $1 \le i \le k$.

Proof. We will prove only the first part of the proposition. The proof of the other part is similar to the proof of the first part.

One implication is obvious, that is, it is clear that if for each i, $a_{i,n} \to a_i$, then

$$\max_i \Big\{\frac{a_i}{a_{i,n}}\Big\} \to 1,$$

and hence $K(a_n, a) \to 0$ as $n \to \infty$. For the other implication assume that $K(a_n, a) \to 0$. It follows that

$$\max_i \Big\{\frac{a_i}{a_{i,n}}\Big\} \to 1 \quad \text{as } n \to \infty.$$

Assume that there exists i' such that

$$\lim_{n\to\infty} \frac{a_{i'}}{a_{i',n}} \ne 1.$$

It is clear that each $a_{i,n}$ is bounded. Passing to some subsequences we may suppose that

(1) $\lim_{n\to\infty} a_{i,n}$ exists;

(2) $\lim_{n\to\infty} a_{i',n} < a_i$.

Note that it is also true that $\lim_{n\to\infty} a_{i,n} \le a_i$. Hence, we have

$$1 = q(a_1, \dots, a_k) > q\big(\lim_{n\to\infty} a_{1,n}, \dots, \lim_{n\to\infty} a_{k,n}\big) = 1,$$

which is a contradiction. ∎

The following corollary is immediate.

Corollary 2. $L(x_n, x) \to 0$ *if and only if* $L(x, x_n) \to 0$.

Note that it means that $(\mathcal{R}(Q)_1, L)$ satisfies Busemann's axioms for metric spaces.

There are at least two usual ways in which a non-symmetric metric d can be symmetrized, namely,

(1) $d_{\text{arith}}(x, y) = \frac{1}{2}(d(x, y) + d(y, x))$;

(2) $d_{\max}(x, y) = \max\{d(x, y), d(y, x)\}$.

These two metrics, d_{arith} and $d_{\max}$, are equivalent, that is, there exist constants C and C' such that for all x and y, we have

$$\frac{1}{C} d_{\text{arith}}(x, y) \leq d_{\max}(x, y) \leq C' d_{\text{arith}}(x, y).$$

Indeed, we can take the constants as $C = 1$ and $C' = 2$. It follows that these two metrics induce the same topology on the space on which they are defined. By the topology induced by an asymmetric metric, we mean the topology induced by one of its symmetrizations (following again Busemann [3]).

Remark 15. Let $a = (a_1, \ldots, a_k)$ and $a_n = (a_{1,n}, \ldots, a_{k,n})$ be in $(\mathbb{R}_+^*)_1^k$. Proposition 7 states that the following are equivalent:

(1) $a_{i,n} \to a_i$ for all $1 \leq i \leq k$;

(2) $K(a_n, a) \to 0$;

(3) $K(a, a_n) \to 0$.

It follows that $a_{i,n} \to a_i$ for each i if and only if $K_{\max}(a_n, a) \to 0$ if and only if $K_{\text{arith}}(a_n, a) \to 0$. This implies that the usual topology on $(\mathbb{R}_+^*)_1^k$ as a subspace of $\mathbb{R}^k$ and the topology induced by $K_{\max}$ (or equivalently, K_{arith}) are the same.

We will prove that $(\mathcal{R}(Q)_1, L)$ is not complete in general. Let us first define completeness in the case of asymmetric metrics.

Definition 16. A space X equipped with an asymmetric metric δ satisfying Busemann's axioms is complete if for any sequence x_n the following property holds:

> *if* $\delta(x_n, x_m) \to 0$ *as* n *and* $m \to \infty$*, then* x_n *converges to a point in* X*. This means that* $\delta(x_n, x) \to 0$*, or equivalently,* $\delta(x, x_n) \to 0$*, as* $n \to \infty$*, for some* $x \in X$.

Proposition 8. *If* $k > 2$*, then* $((\mathbb{R}_+^*)_1^k, K)$ *is not complete.*

Proof. There exists $a = (a_1, \ldots, a_k)$ such that $a_i = 0$ for some $1 \leq i \leq k$ and $a_j > 0$ when $j \neq i$. Now the point a can be chosen so that $q(a) = 1$. Let $a_n = (a_{1,n}, \ldots, a_{k,n})$

so that $a_{i,n} = 1/2^n$ and $a_{j,n} = a_j$ when $j \neq i$. Let $b_n = a_n/\sqrt{q(a_n)}$. Then,

$$K(b_n, b_m) \to 0$$

as n and $m \to \infty$, and b_n does not converge to any point in $(\mathbb{R}_+^*)_1^k$. ∎

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