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Invariant Gibbs measures for the three-dimensional wave equation with a Hartree nonlinearity II: Dynamics

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Abstract. In this two-paper series, we prove the invariance of the Gibbs measure for a three-dimensional wave equation with a Hartree nonlinearity. The novelty lies in the singularity of the Gibbs measure with respect to the Gaussian free field.

In this paper, we focus on the dynamical aspects of our main result. The local theory is based on a paracontrolled approach, which combines ingredients from dispersive equations, harmonic analysis, and random matrix theory. The main contribution, however, lies in the global theory. We develop a new globalization argument, which addresses the singularity of the Gibbs measure and its consequences.

Keywords. Wave equations, dispersive equations, Gibbs measures, invariant measures

Contents

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Continuation of the series

This paper is the second part of a two-paper series and we refer to the first part [12] for a more detailed introduction to the series.

We study the renormalized wave equation with a Hartree nonlinearity and random initial data given by

$$\begin{cases} -\partial_{tt}^2 u - u + \Delta u = :(V * u^2)u:, \quad (t,x) \in \mathbb{R} \times \mathbb{T}^3, \\ u|_{t=0} = \phi_0, \quad \partial_t u|_{t=0} = \phi_1. \end{cases} \tag{a}$$

Here, the three-dimensional torus $\mathbb{T}^3$ is understood as $[-\pi,\pi]^3$ with periodic boundary conditions. The interaction potential $V\colon \mathbb{T}^3 \to \mathbb{R}$ satisfies $V(x) = c_\beta |x|^{-(3-\beta)}$ for all $x \in \mathbb{T}^3$ close to the origin, where $0 < \beta < 3$, satisfies $V(x) \gtrsim 1$ for all $x \in \mathbb{T}^3$, is even, and is smooth away from the origin. The nonlinearity $:(V * u^2)u:$ is a renormalization of $(V * u^2)u$ and defined in (1.16) below.

The nonlinear wave equation (a) is corresponding to the Hamiltonian H given by

$$\begin{aligned} H[u,\partial_t u](t) &= \frac{1}{2}\big(\|u(t)\|_{L_x^2}^2 + \|\nabla u(t)\|_{L_x^2}^2 + \|\partial_t u(t)\|_{L_x^2}^2\big) \\ &\quad + \frac{1}{4}\int_{\mathbb{T}^3} :(V * u^2)(t,x) u(t,x)^2 : \mathrm{d}x, \end{aligned}$$

where $L_x^2 = L_x^2(\mathbb{T}^3)$. The formal Gibbs measure $\mu^\otimes$ corresponding to the Hamiltonian has been rigorously constructed in the first paper of this series. All necessary properties of this construction will be recalled in Theorem 1.1 below.

The main result of this series is the invariance of the Gibbs measure $\mu^{\otimes}$ under the flow of the nonlinear wave equation (a). We first state a formal version of our main result and postpone a rigorous version until Theorems 1.1 and 1.3 below.

Main result (Global well-posedness and invariance, formal version). *The formal Gibbs measure $\mu^{\otimes}$ exists and, for $0 < \beta < 1/2$, is singular with respect to the Gaussian free field $g^{\otimes}$. The renormalized wave equation with Hartree nonlinearity* (a) *is globally well-posed on the support of $\mu^{\otimes}$ and the dynamics leave $\mu^{\otimes}$ invariant.*

1. Introduction

The second paper in this series deals with the dynamical aspects of our argument. As a result, it is inspired by recent advances in random dispersive equations. The interest in random dispersive equations stems from their connections to several areas of research, such as analytic number theory, harmonic analysis, random matrix theory, and stochastic partial differential equations (cf. [49]). In fact, much of the recent progress have been fueled through similar advances in singular stochastic partial differential equations, such as Hairer's *regularity structures* [40] or Gubinelli, Imkeller, and Perkowski's *paracontrolled calculus* [36].

The most classical problem in random dispersive equations is the construction of invariant measures for (periodic and defocusing) nonlinear wave and Schrödinger equations. This has been an active area of research since the 1990s, and we refer the reader to Figure 1 for an overview of some of the most important contributions.

The first results in this direction were obtained in one spatial dimension by Friedlander [35], Zhidkov [75] and Bourgain [4]. Friedlander [35] and Zhidkov [75] proved the invariance of the Gibbs measure for the one-dimensional nonlinear wave equation. Inspired by earlier work of Lebowitz, Rose, and Speer [46], Bourgain [4] proved the invariance of the Gibbs measure for the one-dimensional nonlinear Schrödinger equations

$$i\partial_t u + \partial_x^2 u = |u|^{p-1}u, \quad (t,x) \in \mathbb{R} \times \mathbb{T}.$$

Dimension & nonlinearity	Wave	Schrödinger
$d = 1$, $\lvert u\rvert^{p-1}u$	[35, 75]	[4]
$d = 2$, $\lvert u\rvert^{2}u$	[58]	[5]
$d = 2$, $\lvert u\rvert^{p-1}u$		[28]
$d = 3$, $(\lvert x\rvert^{-(3-\beta)} * \lvert u\rvert^2) \cdot u$	$\beta > 1$: [55] $\beta > 0$: this paper	$\beta > 2$: [6] $2 \geq \beta > 1/2$: feasible $1/2 \geq \beta > 0$: open
$d = 3$, $\lvert u\rvert^{2}u$	open	open

Fig. 1. Invariant Gibbs measures for defocusing nonlinear wave and Schrödinger equations.

In this seminal paper, Bourgain introduced his famous *globalization argument*, which will be described in detail below. Even though Friedlander [35], Zhidkov [75] and Bourgain [4] consider random initial data (drawn from the Gibbs measure), the local theory is entirely deterministic. The reason is that the Gibbs measure is supported at spatial regularity $1/2-$, which is above the (deterministic) critical regularities $s_{\mathrm{det}} = \frac{1}{2} - \frac{1}{p}$ (cf. [15]) and $s_{\mathrm{det}} = \frac{1}{2} - \frac{2}{p-1}$ for the one-dimensional wave and Schrödinger equations (in H^s), respectively.

The first result in two spatial dimensions was obtained by Bourgain [5]. He proved the invariance of the Gibbs measure for the renormalized cubic nonlinear Schrödinger equation

$$i\partial_t u + \Delta u = {:}|u|^2 u{:}, \quad (t,x) \in \mathbb{R} \times \mathbb{T}^2. \tag{1.1}$$

In (1.1), the renormalized (or Wick-ordered) nonlinearity is given by $|u|^2 u - 2\|u\|_{L^2}^2 u$. In this specific case, the renormalized equation (1.1) is related to the cubic nonlinear Schrödinger equation through a gauge transformation. In contrast to the one-dimensional setting, the Gibbs measure is supported at spatial regularity $0-$, which is just below the (deterministic) critical regularity $s_c = 0$. To overcome this obstruction, the local theory in [5] exhibits probabilistic cancellations in several multi-linear estimates. Very recently, Fan, Ou, Staffilani, and Wang [34] extended Bourgain's result from the square torus $\mathbb{T}^2$ to irrational tori.

The situation for two-dimensional nonlinear wave equations is easier than for two-dimensional nonlinear Schrödinger equations. While the Gibbs measure is still supported at spatial regularity $0-$, this is partially compensated by the smoothing effect of the Duhamel integral. In [58], Oh and Thomann prove the invariance of the Gibbs measure for

$$-\partial_t^2 u - u + \Delta u = {:}u^p{:}, \quad (t,x) \in \mathbb{R} \times \mathbb{T}^2, \tag{1.2}$$

where $p \geq 3$ is an odd integer. The renormalized nonlinearity ${:}u^p{:}$ in (1.2) is the Wick-ordering of u^p; see e.g. [58, (1.9)]. In contrast to the nonlinear Schrödinger equation (1.1), it cannot be obtained from the original equation via a gauge transformation. However, the renormalization is likely necessary to obtain nontrivial dynamics for random low-regularity data (see e.g. [54,57]). We emphasize that their argument for the cubic ($p = 3$) and higher-order ($p \geq 5$) nonlinearity is essentially identical. Due to its clear and detailed exposition, we highly recommend [58] as a starting point for any beginning researcher in random dispersive equations.

In a recent work [28], Deng, Nahmod, and Yue proved the invariance of the Gibbs measure for the nonlinear Schrödinger equations

$$i\partial_t u + \Delta u = {:}|u|^{p-1} u{:}, \quad (t,x) \in \mathbb{R} \times \mathbb{T}^2, \tag{1.3}$$

where $p \geq 5$ is an odd integer. In contrast to the situation for the two-dimensional nonlinear wave equations, this result is much harder than its counterpart for the cubic nonlinear Schrödinger equation (1.1). The main difficulty is that all high$\times$low$\times\cdots\times$low interactions between the random initial data with itself or smoother remainders only have spatial regularity $1/2-$, which is strictly below the (deterministic) critical regularity $s_{\mathrm{det}} = 1 - \frac{2}{p-1}$.

To overcome this difficulty, Deng, Nahmod, and Yue worked with random averaging operators, which are related to the adapted linear evolutions in [10]. Their framework was recently generalized through the *theory of random tensors* [30], which will be further discussed below.

Unfortunately, much less is known in three spatial dimensions. The reason is that the Gibbs measure is supported at spatial regularity $-1/2-$, which is far below the deterministic critical regularity $s_{\mathrm{det}} = \frac{3}{2} - \frac{2}{p-1}$. In fact, the invariance of the Gibbs measure for both the cubic nonlinear wave and Schrödinger equation are famous open problems. Previous research has instead focused on simpler models, which are obtained either through additional symmetry assumptions, a (slight) regularization of the random initial data, or a (slight) regularization of the nonlinearity.

In the radially-symmetric setting, the invariance of the Gibbs measure for the three-dimensional cubic wave and Schrödinger equation has been proven in [8, 23, 74] and [7], respectively. The radially-symmetric setting was also studied in earlier work on the two-dimensional nonlinear Schrödinger equation [26, 68, 69].

In [56], Oh, Pocovnicu, and Tzvetkov studied the cubic nonlinear wave equation with Gaussian initial data. While the Gaussian initial data in [56] does not directly correspond to a Gibbs measure, the local theory in [56] still yields partial progress towards the (local aspects of) the Gibbs-measure problem. The Gaussian initial data in [56] has regularity $s > -1/4$ and, as a result, is more than $1/4$ derivatives smoother than the Gibbs measure. Using some of the methods in this paper, Oh, Wang, and Zine [60] very recently improved the regularity condition from $s > -1/4$ to $s > -1/2$. In particular, the Gaussian data in [60] is only ϵ derivatives smoother than the support of the Gibbs measure.

In [6], Bourgain studied the defocusing and focusing three-dimensional Schrödinger equation with a Hartree nonlinearity given by

$$i\partial_t u + \Delta u = \pm{:}(V * |u|^2)u{:}, \quad (t,x) \in \mathbb{R} \times \mathbb{T}^3, \tag{1.4}$$

where the interaction potential V behaves like $+|x|^{-(3-\beta)}$. He proved the invariance of the Gibbs measure for $\beta > 2$, which corresponds to a relatively smooth interaction potential. In the focusing case, this is optimal (up to the endpoint $\beta = 2$), since the Gibbs measure is not normalizable for $\beta < 2$ (cf. [55]). From a physical perspective, the most relevant cases are the Coulomb potential $|x|^{-1}$ (corresponding to $\beta = 2$) and the Newtonian potential $|x|^{-2}$ (corresponding to $\beta = 1$). Since the cubic nonlinear Schrödinger equation formally corresponds to (1.4) with the interaction potential V given by the Dirac measure, it is also interesting (and challenging) to take β close to zero. After the first version of this manuscript appeared, Deng, Nahmod, and Yue [29] used random averaging operators (as in [28]) to cover the regime $\beta > 1 - \epsilon$ in the defocusing case, where $\epsilon > 0$ is a small unspecified constant. As discussed in [29], it is likely possible to use the more sophisticated theory of random tensors from [30] to cover the regime $\beta > 1/2$. In the regime $0 < \beta = 1/2$, the Gibbs measure becomes singular with respect to the Gaussian free field (see Theorem 1.1). As described in [29, Section 1.2.1], the extension of the theory of random tensors to singular Gibbs measures remains a challenging open problem (see also Remark 1.6).

After the completion of this series, the author learned of independent work by Oh, Okamoto, and Tolomeo [55]. The authors study (the stochastic analogue of) the focusing and defocusing three-dimensional nonlinear wave equation with a Hartree nonlinearity given by

$$-\partial_t^2 u - u + \Delta u = \pm\lambda :(V * u^2)u:, \quad (t,x) \in \mathbb{R} \times \mathbb{T}^3,$$

where $\lambda > 0$. The main focus of [55] lies on the construction and properties of the Gibbs measures, which are discussed in the first part of the series (cf. [12, Remark 1.2]). Regarding the dynamical results of [55], the authors prove the invariance of the Gibbs measure in the following cases:

(i) focusing ($-$): $\beta > 2$ or $\beta = 2$ in the weakly nonlinear regime.

(ii) defocusing ($+$): $\beta > 1$.

In light of the nonnormalizability of the focusing Gibbs measure for $\beta < 2$ and $\beta = 2$ in the strongly nonlinear regime (cf. [55]), the result is optimal in the focusing case. In the defocusing case, however, the restriction $\beta > 1$ excludes all Gibbs measures which are singular with respect to the Gaussian free field. In contrast, Theorem 1.3 below covers the complete range $\beta > 0$, which includes singular Gibbs measures. In fact, this is the main motivation behind our two-paper series.

In the preceding discussion, we have seen several examples of invariant Gibbs measures supported at regularities even below the deterministic critical regularity. In [28, 30], Deng, Nahmod, and Yue describe a probabilistic scaling heuristic, which takes into account the expected probabilistic cancellations. We denote the critical regularity with respect to the probabilistic scaling by s_{prob} and the spatial regularity of the support of the Gibbs measure by s_{G}. Based on the probabilistic scaling heuristic, we then expect probabilistic local well-posedness as long as $s_{\text{G}} > s_{\text{prob}}$. We record the relevant quantities for nonlinear wave and Schrödinger equations in Figure 2. For comparison, we also include the deterministic critical regularity s_{det}. The probabilistic scaling heuristic, however, does

Dimension & nonlinearity	Wave			Schrödinger		
	s_{G}	s_{prob}	s_{det}	s_{G}	s_{prob}	s_{det}
$d=1$, $\lvert u\rvert^{p-1}u$	$\frac{1}{2}-$	$-\frac{1}{2p}$	$\frac{1}{2}-\frac{1}{p}$	$\frac{1}{2}-$	$-\frac{1}{p-1}$	$\frac{1}{2}-\frac{2}{p-1}$
$d=2$, $\lvert u\rvert^{p-1}u$	$0-$	$-\frac{3}{2p}$	$1-\frac{2}{p-1}$	$0-$	$-\frac{1}{p-1}$	$1-\frac{2}{p-1}$
$d=3$, $(V*\lvert u\rvert^2)\cdot u$	$-\frac{1}{2}-$	$-\min(\frac{2+\beta}{3},\frac{3}{2})$	$\max(\frac{1-2\beta}{2},0)$	$-\frac{1}{2}-$	$-\min(\frac{1+\beta}{2},1)$	$\max(\frac{1-2\beta}{2},0)$
$d=3$, $\lvert u\rvert^2 u$	$-\frac{1}{2}-$	$-\frac{2}{3}$	$\frac{1}{2}$	$-\frac{1}{2}-$	$-\frac{1}{2}$	$\frac{1}{2}$

Fig. 2. Relevant spatial regularities for the invariance of the Gibbs measure: s_{G} (support of the Gibbs measure), s_{prob} (probabilistic scaling), s_{det} (deterministic scaling). The value of s_{prob} for power-type nonlinearities can be found in [28]. The probabilistic critical regularity s_{prob} for the wave equation with a Hartree nonlinearity is a result of high×high×high→low and (high×high→low)×high →high interactions. For the Schrödinger equation with a Hartree nonlinearity, s_{prob} is a result of (high×high→high)×high→high and (high×high→low)×high→high interactions.

not address any obstructions related to the global theory, renormalizations, or measure-theoretic aspects. As a result, it does not capture some of the difficulties for dispersive equations with singular Gibbs measures, such as the cubic nonlinear wave equation in three dimensions.

Our discussion so far has been restricted to invariant Gibbs measures for nonlinear wave and Schrödinger equations. While this is the most classical problem in random dispersive equations, there exist many more active directions of research. Since a full overview of the field is well beyond the scope of the introduction, we only mention a few directions and refer to the given references for more details.

(1) Invariance of white noise [44, 52, 62].

(2) Invariant measures (at high regularity) for completely integrable equations [31, 71, 72].

(3) Quasi-invariant Gaussian measures for non-integrable equations [39, 59, 70].

(4) Non-invariance methods related to scattering, solitons, and blow-up [9, 11, 32, 43, 61].

(5) Wave turbulence [13, 17, 18, 27].

(6) Stochastic dispersive equations [21, 22, 24, 37, 38].

After this overview of the relevant literature, we now turn to a more detailed description of the most relevant methods. Our discussion will be split into two parts, separating the local and global aspects. As a teaser for the reader, we already mention that our contributions to the local theory will be of an intricate but technical nature, while our contributions to the global theory will be conceptual.

As mentioned above, the first local well-posedness result for dispersive equations relying on probabilistic methods was proven by Bourgain [5]. He considered the renormalized cubic nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u - u + \Delta u = :|u|^2 u:, & (t,x) \in \mathbb{R}\times\mathbb{T}^2, \\ u|_{t=0} = \phi. \end{cases} \tag{1.5}$$

The additional $-u$-term has been introduced for convenience, but can be easily removed through a gauge transformation. The random initial data ϕ is drawn from the corresponding Gibbs measure, which coincides with the (complex) Φ^4_2-model. Since the Φ^4_2-model is absolutely continuous with respect to the Gaussian free field and the local theory does not rely on the invariance of the Gibbs measure, we can represent ϕ through the random Fourier series

$$\phi = \sum_{n\in\mathbb{Z}^2} \frac{g_n}{\langle n\rangle} e^{i\langle n,x\rangle}. \tag{1.6}$$

Here, $\langle n\rangle \stackrel{\text{def}}{=} \sqrt{1+|n|^2}$ and $(g_n)_{n\in\mathbb{Z}^2}$ is a sequence of independent and standard complex-valued Gaussians. The independence of the Fourier coefficients, and more generally the simple structure of (1.6), is an essential ingredient for many arguments in [5]. A direct calculation shows that almost surely $\phi \in H^s(\mathbb{T}^2)\setminus L^2(\mathbb{T}^3)$ for all $s<0$. Since (1.5) is mass-critical, ϕ lives below the (deterministic) critical regularity. To overcome this obstruction, Bourgain decomposed the solution by writing

$$u(t) = e^{it(-1+\Delta)}\phi + v(t).$$

This decomposition is commonly referred to as Bourgain's trick, but is also known in the stochastic PDE literature as the Da Prato–Debussche trick [20]. Using this decomposition, we see that the nonlinear remainder v satisfies the evolution equation

$$i\partial_t v - v + \Delta v = :|e^{it(-1+\Delta)}\phi + v|^2(e^{it(-1+\Delta)}\phi + v):, \quad (t,x) \in \mathbb{R} \times \mathbb{T}^2.$$

Through a combination of probabilistic and PDE arguments, Bourgain proved that the Duhamel integral

$$I[:|e^{it(-1+\Delta)}\phi|^2 e^{it(-1+\Delta)}\phi:]$$

lives at spatial regularity $1/2-$ (see also [19]). This opens the door to a contraction argument for v at a positive (and hence subcritical) regularity. The contraction argument requires further ingredients from random matrix theory to handle mixed terms, but can in fact be closed. We emphasize that the nonlinear remainder v is treated purely deterministically and is not shown to exhibit any random structure.

We now discuss the more recent work of Gubinelli, Koch, and Oh [37], which covers the stochastic wave equation

$$\begin{cases} -\partial_t^2 u - u + \Delta u = :u^2: + \xi, & (t,x) \in \mathbb{R} \times \mathbb{T}^3, \\ u[0] = 0. \end{cases}$$

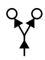
Here, ξ denotes space-time white noise. Inspired by a (higher-order version of) Bourgain's trick, we decompose

$u =$  $+$  $+ v.$

The linear stochastic object  solves the forced wave equation

$(-\partial_t^2 - 1 + \Delta)$  $= \xi.$

The black dot represents the stochastic noise ξ and the arrow represents the Duhamel integral. An elementary arguments shows that  has spatial regularity $-1/2-$. The quadratic stochastic object  is the solution of the forced wave equation

$(-\partial_t^2 - 1 + \Delta)$  $= :($$)^2:.$

Based on similar arguments for stochastic heat equations, one may expect that  has spatial regularity $2 \cdot (-1/2-) + 1 = 0-$, where the gain of one spatial derivative comes from the Fourier multiplier $\langle \nabla \rangle^{-1}$ in the Duhamel integral. Using multi-linear dispersive estimates, however, Gubinelli, Koch, and Oh proved that  has spatial regularity $1/2-$.

Using the definition of our stochastic objects, we obtain the evolution equation

$$(-\partial_t^2 - 1 + \Delta)v = 2\Big(\overset{\circ\,\circ}{\curlyvee} + v\Big) \cdot \overset{\circ}{\uparrow} + \Big(\overset{\circ\,\circ}{\curlyvee} + v\Big)^2$$

for the nonlinear remainder v. In the following discussion, we let $\textcircled{$\ll$}$ and $\textcircled{$=$}$ be the low×high and high×high paraproducts from Definition 2.1. Due to low×high interactions such as $v \,\textcircled{$\ll$}\, \overset{\circ}{\uparrow}$, we expect v to have spatial regularity at most $(-1/2-) + 1 = 1/2-$. We emphasize that, unlike high×high to high interactions, the low×high interactions are not affected by multi-linear dispersive effects. However, this implies that the spatial regularities of v and $\overset{\circ}{\uparrow}$ do not add up to a positive number, which means that the high×high term $v \,\textcircled{$=$}\, \overset{\circ}{\uparrow}$ cannot even be defined (without additional information on v). This problem cannot be removed through a direct higher-order expansion of u and persists through all orders of the Picard iteration scheme. Instead, Gubinelli, Koch, and Oh [37] utilize ideas from the paracontrolled calculus for singular stochastic PDEs [36]. We write $v = X + Y$, where X and Y solve

$$(-\partial_t^2 - 1 + \Delta)X = 2\Big(\overset{\circ\,\circ}{\curlyvee} + X + Y\Big) \,\textcircled{$\ll$}\, \overset{\circ}{\uparrow} \tag{1.7}$$

and

$$(-\partial_t^2 - 1 + \Delta)Y = 2\Big(\overset{\circ\,\circ}{\curlyvee} + X + Y\Big) \,\textcircled{$\geq$}\, \overset{\circ}{\uparrow} + \Big(\overset{\circ\,\circ}{\curlyvee} + X + Y\Big)^2. \tag{1.8}$$

The paracontrolled component X only has spatial regularity $1/2-$, but exhibits a random structure. In the analysis of the high×high interactions $X \,\textcircled{$=$}\, \overset{\circ}{\uparrow}$, this random structure can be exploited by replacing X with the Duhamel integral of the right-hand side in (1.7). Since this leads to a double Duhamel integral in the expression for Y, this approach is often called the "double Duhamel trick". In contrast to X, Y lives at a higher spatial regularity and can be controlled through deterministic arguments. The local theory in this paper will follow a similar approach, but relies on more intricate estimates, which will be further discussed below.

After this discussion of the local theory, we now turn to the global theory. We discuss Bourgain's globalization argument [4], which uses the invariance of the truncated Gibbs measures as a substitute for a conservation law. We first recall the definition of the different modes of convergence for a sequence of probability measures, which will be needed below.

Definition (Convergence of measures). Let $\mathscr{H}$ be a Hilbert space and let $B(\mathscr{H})$ be the Borel σ-algebra on $\mathscr{H}$. Furthermore, let $(\mu_N)_{N\geq 1}$ and μ be Borel probability measures on $\mathscr{H}$. We say that

(i) μ_N *converges in total variation* to μ if

$$\lim_{N\to\infty} \sup_{A\in B(\mathscr{H})} |\mu(A) - \mu_N(A)| = 0,$$

(ii) μ_N *converges strongly* to μ if

$$\lim_{N\to\infty} \mu_N(A) = \mu(A) \quad \text{for all } A \in B(\mathscr{H}),$$

(iii) μ_N *converges weakly* to μ if

$$\lim_{N\to\infty} \mu_N(A) = \mu(A) \quad \text{for all } A \in B(\mathscr{H}) \text{ satisfying } \mu(\partial A) = 0.$$

To isolate the key features of the argument, we switch to an abstract setting. Let $\mathscr{H}$ be a Hilbert space and let $\Phi_N : \mathbb{R} \times \mathscr{H} \to \mathscr{H}$ be a sequence of jointly continuous flow maps. Let μ_N be a sequence of Borel probability measures on $\mathscr{H}$. Most importantly, we assume that μ_N is invariant under Φ_N for all N, i.e.,

$$\mu_N(\Phi_N(t)^{-1}A) = \mu_N(A) \quad \text{for all } t \in \mathbb{R} \text{ and } A \in B(\mathscr{H}).$$

In our setting, Φ_N will be the flow for a frequency-truncated nonlinear wave equation and μ_N will be the corresponding truncated Gibbs measure. Our main interest lies in the removal of the truncation, i.e., the limit of the dynamics Φ_N and measure μ_N as N tends to infinity. Let μ be a limit of the sequence μ_N, where the mode of convergence will be specified below. In order to construct the limiting dynamics on the support of μ, we need uniform bounds on Φ_N on the support of μ. At the very least, we require an estimate of the form

$$\limsup_{N\to\infty} \mu\Big(\sup_{t\in[0,1]} \|\Phi_N(t)\phi\|_{\mathscr{H}} \le \epsilon^{-1}\Big) \ge 1 - o_\epsilon(1), \tag{1.9}$$

where $0 < \epsilon < 1$ and o is the small Landau symbol. Bourgain's globalization argument [4] proves (1.9) in two steps.

In a first measure-theoretic part, we use the inequality

$$\Big|\mu\Big(\sup_{t\in[0,1]} \|\Phi_N(t)\phi\|_{\mathscr{H}} \le \epsilon^{-1}\Big) - \mu_N\Big(\sup_{t\in[0,1]} \|\Phi_N(t)\phi\|_{\mathscr{H}} \le \epsilon^{-1}\Big)\Big| \le \sup_{A\in B(\mathscr{H})} |\mu(A) - \mu_N(A)|.$$

As long as μ_N converges in *total variation* to μ, we can reduce (1.9) to

$$\limsup_{N\to\infty} \mu_N\Big(\sup_{t\in[0,1]} \|\Phi_N(t)\phi\|_{\mathscr{H}} \le \epsilon^{-1}\Big) \ge 1 - o_\epsilon(1), \tag{1.10}$$

In a second dynamical part, we use the invariance of μ_N under Φ_N and the probabilistic local well-posedness. Let $J \ge 1$ be a large integer and define the step size $\tau = J^{-1}$. Then

$$\begin{aligned}\mu_N\Big(\sup_{t\in[0,1]} \|\Phi_N(t)\phi\|_{\mathscr{H}} > \epsilon^{-1}\Big) &\le \sum_{j=0}^{J-1} \mu_N\Big(\sup_{t\in[j\tau,(j+1)\tau]} \|\Phi_N(t)\phi\|_{\mathscr{H}} > \epsilon^{-1}\Big) \\ &= \sum_{j=0}^{J-1} \mu_N\Big(\sup_{t\in[0,\tau]} \|\Phi_N(t)\Phi_N(j\tau)\phi\|_{\mathscr{H}} > \epsilon^{-1}\Big).\end{aligned}$$

Using the invariance of μ_N under $\Phi_N(j\tau)$, we obtain

$$\mu_N\Big(\sup_{t\in[0,1]} \|\Phi_N(t)\phi\|_{\mathscr{H}} > \epsilon^{-1}\Big) \le \tau^{-1}\mu_N\Big(\sup_{t\in[0,\tau]} \|\Phi_N(t)\phi\|_{\mathscr{H}} > \epsilon^{-1}\Big). \tag{1.11}$$

The right-hand side of (1.11) can then be controlled through an appropriate choice of τ and the local theory (as well as tail estimates for μ_N).

In (this sketch of) Bourgain's globalization argument, the convergence in total variation played an essential role. In all previous results on the invariance of (defocusing) Gibbs measures [4–6, 28, 55, 58, 75], the truncated Gibbs measures converge in total variation, so that this assumption does not pose any problems. In our case, however, the truncated Gibbs measures μ_N only converge weakly to the Gibbs measure μ. The weak mode of convergence is related to the singularity of the Gibbs measure μ with respect to the Gaussian free field g, which necessitates softer arguments in the construction of μ. Using the weak convergence of μ_N to μ, we can only reduce (1.9) to

$$\limsup_{N\to\infty}\Big[\limsup_{M\to\infty}\mu_M\Big(\sup_{t\in[0,1]}\|\Phi_N(t)\phi\|_{\mathscr{H}}\leq\epsilon^{-1}\Big)\Big]\geq 1-o_\epsilon(1), \tag{1.12}$$

In (1.12), we will typically have $M>N$, and hence we cannot (directly) use the invariance of the truncated Gibbs measures.

In [50], Nahmod, Oh, Rey-Bellet, and Staffilani prove the invariance of a Wiener measure for the periodic derivative nonlinear Schrödinger equation. The truncated Wiener measures in [50] are defined using a frequency-truncation not only in the interaction but also in the Gaussian free field (cf. [50, (5.13)]. As a consequence, the truncated Wiener measures only converge weakly (cf. [50, Proposition 5.13]). In order to prove (1.12), the authors rely on the (quantitative) mutual absolute continuity of the (truncated) Wiener measure with respect to the (truncated) Gaussian free field (cf. [50, (6.7)]). Unfortunately, the singularity of the Gibbs measure in this work (as stated in Theorem 1.1) prevents us from using a similar approach.

1.1. Main results and methods

Before we can state our main results, we need to define the renormalized and frequency-truncated Hamiltonians, wave equations, and Gibbs measures. For any dyadic $N\geq 1$, we define the renormalized and frequency-truncated potential energy by

$$\begin{aligned}\frac{1}{4}\int_{\mathbb{T}^3}&{:}(V*(P_{\leq N}\phi)^2)(P_{\leq N}\phi)^2{:}\,\mathrm{d}x\\ &\overset{\text{def}}{=}\frac{1}{4}\int_{\mathbb{T}^3}\big[(V*(P_{\leq N}\phi)^2)(P_{\leq N}\phi)^2-2a_N(P_{\leq N}\phi)^2\\ &\qquad -4(\mathcal{M}_N P_{\leq N}\phi)P_{\leq N}\phi+\widehat{V}(0)a_N^2+2b_N\big]\,\mathrm{d}x+c_N.\end{aligned}$$

Here, the renormalization constants a_N, b_N, c_N are given by Definition 2.6, Definition 2.8, and Proposition 3.2 in the first paper of this series [12], but their precise values are not needed in this paper. The renormalization multiplier $\mathcal{M}_N$ is defined by

$$\widehat{\mathcal{M}_N f}(n)\overset{\text{def}}{=}\Big(\sum_{k\in\mathbb{Z}^3}\frac{\widehat{V}(n+k)}{\langle k\rangle^2}\rho_N(k)^2\Big)\widehat{f}(n), \tag{1.13}$$

where ρ_N is a truncation to frequencies of size $\lesssim N$. The Hamiltonian H_N is then defined as

$$H_N[\phi_0,\phi_1] \overset{\text{def}}{=} \frac{1}{2}(\|\phi_0\|_{L^2}^2 + \|\langle\nabla\rangle\phi_0\|_{L^2}^2 + \|\phi_1\|_{L^2}^2) + \frac{1}{4}\int_{\mathbb{T}^3} :(V * (P_{\leq N}\phi)^2)(P_{\leq N}\phi)^2: \mathrm{d}x. \tag{1.14}$$

The renormalized and frequency-truncated nonlinear wave equation corresponding to H_N is given by

$$\begin{cases} (-\partial_t^2 - 1 + \Delta)u = P_{\leq N}\big(:(V * (P_{\leq N}u)^2)P_{\leq N}u:\big), & (t,x) \in \mathbb{R}\times\mathbb{T}^3, \\ u|_{t=0} = \phi_0, \quad \partial_t u|_{t=0} = \phi_1, \end{cases} \tag{1.15}$$

where the renormalized nonlinearity is given by

$$:(V * (P_{\leq N}u)^2)P_{\leq N}u: \overset{\text{def}}{=} (V * (P_{\leq N}u)^2)P_{\leq N}u - a_N\widehat{V}(0)P_{\leq N}u - 2\mathcal{M}_N P_{\leq N}u. \tag{1.16}$$

We remark that frequencies much larger than N are not affected by the nonlinearity (1.16). As a result, the nonlinear component of the solution of (1.16) is always smooth. For a fixed $N \geq 1$, the coercivity of H_N implies the global well-posedness of (1.15). We also define the renormalized square

$$:(P_{\leq N}u)^2: \overset{\text{def}}{=} (P_{\leq N}u)^2 - a_N, \tag{1.17}$$

which will simplify the notation below. The Gibbs measure $\mu_N^{\otimes}$ corresponding to H_N is given by $\mu_N^{\otimes} = \mu_N \otimes (\langle\nabla\rangle)_{\#}g$, where μ_N is defined in [12, (1.10)] and $(\langle\nabla\rangle)_{\#}g$ is the pushforward of the three-dimensional Gaussian field (defined in the introduction of [12]) under $\langle\nabla\rangle$. Before we state the properties of the truncated Gibbs measures $\mu_N^{\otimes}$, we recall the assumptions on the interaction potential from the first paper of the series. In these assumptions, $0 < \beta < 3$ is a fixed parameter.

Assumptions A. We assume that the interaction potential V satisfies

(1) $V(x) = c_\beta |x|^{-(3-\beta)}$ for some $c_\beta > 0$ and all $x \in \mathbb{T}^3$ satisfying $\|x\| \leq 1/10$,

(2) $V(x) \gtrsim_\beta 1$ for all $x \in \mathbb{T}^3$,

(3) $V(x) = V(-x)$ for all $x \in \mathbb{T}^3$,

(4) V is smooth away from the origin.

The following properties of the Gibbs measures $\mu_N^{\otimes}$ are a direct consequence of [12, Theorem 1.1], which is phrased in terms of μ_N. For notational reasons related to the weak convergence instead of convergence in total variation, we use a second parameter M for the frequency-truncation. Our notation for the random variables, which is based on dots, will be discussed below the theorem.

Theorem 1.1 (Gibbs measures). *Let $\kappa > 0$ be a fixed positive parameter, let $0 < \beta < 3$ be a parameter, and let the interaction potential V be as in Assumptions A. Then the truncated Gibbs measures $(\mu_M^{\otimes})_{M\geq 1}$ weakly converge to a limiting measure $\mu_\infty^{\otimes}$ on $\mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$,* *which is called the Gibbs measure. If in addition* $0 < \beta < 1/2$*, then the Gibbs measure* $\mu_\infty^\otimes$ *is singular with respect to the Gaussian free field* $g^\otimes$*. Furthermore, there exists a sequence* $(\nu_M^\otimes)_{M\geq 1}$ *of reference measures on* $\mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$ *and an ambient probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *satisfying the following two properties:*

(1) (Absolute continuity and L^q-bounds) *The truncated Gibbs measure* $\mu_M^\otimes$ *is absolutely continuous with respect to the reference measure* $\nu_M^\otimes$*. More quantitatively, there exists a parameter* $q > 1$ *and a constant* $C \geq 1$ *independent of* M *such that*

$$\mu_M^\otimes(A) \leq C \nu_M^\otimes(A)^{1-1/q}$$

for all Borel sets $A \subseteq \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$.

(2) (Representation of $\nu_M^\otimes$) *Let* $\gamma = \min(1/2+\beta, 1)$*. Then there exist two random variables* $\bullet, \circ_{\mathbf{M}} \colon (\Omega, \mathcal{F}) \to \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$ *and a large integer* $k = k(\beta) \geq 1$ *such that, for all* $p \geq 2$*,*

$$\nu_M^\otimes = \mathrm{Law}_{\mathbb{P}}(\bullet + \circ_{\mathbf{M}}), \quad g^\otimes = \mathrm{Law}_{\mathbb{P}}(\bullet), \quad \big(\mathbb{E}_{\mathbb{P}} \|\circ_{\mathbf{M}}\|^p_{\mathscr{H}_x^{\gamma-\kappa}(\mathbb{T}^3)}\big)^{1/p} \leq p^{k/2}.$$

Remark 1.2. After the completion of this series, the author learned of independent work by Oh, Okamoto, and Tolomeo [55], which yields an analogue of Theorem 1.1. We refer to [12, Remark 1.2] for a more detailed comparison.

We will require that the ambient probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is rich enough to contain a family of independent Brownian motions, which is clear from the definition of $(\Omega, \mathcal{F}, \mathbb{P})$ in [12] and detailed in Section 4.5.

Let us further explain the notation in Theorem 1.1. We use dots to represent the random data, since they can be used as building blocks in more complicated stochastic objects. We already saw this graphical notation in our discussion of [37] and we refer the reader to [48] for a detailed discussion of similar diagrams. We use the blue dot $\bullet$ for the Gaussian random data, since it lives at low spatial regularities and is primarily viewed as a high-frequency term. We use the red dot $\circ_{\mathbf{M}}$ to denote the more regular component of the random data, since we primarily view it as a low-frequency term. Furthermore, the blue dot $\bullet$ is filled while the red dot $\circ_{\mathbf{M}}$ is not filled. The reason is that the manuscript should be accessible to colorblind readers and also readable as a black and white copy.

In the following, we often write $\blacklozenge$ for a generic element $\phi \in \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$. The purple diamond will be used as a building block for further stochastic objects. When working with the reference measure $\nu_M^\otimes$, we have

$$\mathrm{Law}_{\nu_M^\otimes}(\blacklozenge) = \mathrm{Law}_{\mathbb{P}}(\bullet + \circ_{\mathbf{M}}).$$

Naturally, we chose the color purple since it is a mixture of blue and red. The change in shape, i.e., from a dot to a diamond, is primarily made for colorblind readers. We also only use diamonds for intrinsic objects in $\mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$, while dots are used for objects defined on the ambient probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The significance of this distinction will be further discussed in Sections 2 and 3.

While Theorem 1.1 already contains the measure-theoretic results of this series, we now state the dynamical results.

Theorem 1.3 (Global well-posedness & invariance). *There exists a Borel-measurable set* $\mathcal{S} \subseteq \mathscr{H}^{-1/2-\kappa}(\mathbb{T}^3)$ *satisfying* $\mu_\infty^{\otimes}(\mathcal{S}) = 1$ *and such that the following two properties hold:*

(1) (Global well-posedness) *Let* Φ_N *be the flow of the renormalized and frequency-truncated wave equation* (1.15). *Then the limit*

$$\Phi_\infty[t]\blacklozenge \overset{\text{def}}{=} \lim_{N\to\infty} \Phi_N[t]\blacklozenge$$

exists in $\mathscr{H}^{-1/2-\kappa}(\mathbb{T}^3)$ *for all* $t \in \mathbb{R}$ *and* $\blacklozenge \in \mathcal{S}$.

(2) (Invariance) *The Gibbs measure* $\mu_\infty^{\otimes}$ *is invariant under* Φ_∞, *i.e., for all* $t \in \mathbb{R}$,

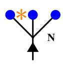
$$\Phi_\infty[t]_\# \mu_\infty^{\otimes} = \mu_\infty^{\otimes}.$$

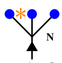
Remark 1.4. In the proof of Theorem 1.3, we restrict ourselves to the case $\beta \in (0, 1/2)$. The purpose of this restriction is purely notational. The same argument also works for $\beta \in [1/2, 3)$, as long as β in each estimate is replaced by $\min(\beta, 1/2)$.

Remark 1.5. While Theorem 1.3 shows that the limiting dynamics $\Phi_\infty[t]$ are well-defined, we have not been able to show that $\Phi_\infty[t]$ satisfies the group property. The author believes that the estimates in this paper (from Sections 5–8) are strong enough to prove the group property, but the stability theory (Sections 2.4 and 3.3) would need to be modified. Instead of working with a single flow $\Phi_N[t]$, one needs similar statements for the mixed flows $\Phi_{N_1}[t_1]\Phi_{N_2}[t_2]$. We refer the reader to [65] for a more detailed discussion of the group property and its relation to the recurrence properties of the flow.

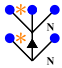
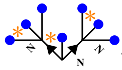
We now describe individual aspects of our argument. As in our discussion of the previous literature, we separate the local and global aspects. As mentioned above, our contributions to the local theory are of an intricate but technical nature, whereas our contributions to the global theory are conceptual.

In the local theory, we use the absolute continuity $\mu_M^{\otimes} \ll \nu_M^{\otimes}$ and the representation of $\nu_M^{\otimes}$ from Theorem 1.1. As a result, the reference measure ν_M serves the same purposes as the Gaussian free field in earlier results on invariant Gibbs measures. We then follow the paracontrolled approach of [37] and decompose the solution $u_N(t)$ of (1.15) as

$u_N =$  $+$  $+ X_N + Y_N,$ (1.18)

where the stochastic objects  and , the paracontrolled component X_N, and the smoother nonlinear remainder Y_N are defined in Section 2. The smoother component $\circ_{\mathbf{M}}$ in the representation of $\nu_M^{\otimes}$ will be placed inside Y_N. In comparison to [37], however, there is an increase in the complexity of the evolution equation for Y_N. We split the terms into four different categories, which correspond to the methods used in their estimates.

- *Stochastic objects:* These terms are explicit and include

 and .

In contrast to the previous literature, we use multiple stochastic integrals for the non-resonant/resonant decompositions, which significantly decreases the algebraic complexities. We also use counting estimates related to the dispersive symbol of the wave equation.

- *Random matrix terms:* The terms include

$(V *$  $)P_{\leq N} Y_N.$

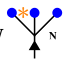
They will be controlled through a recent random matrix estimate of Deng, Nahmod, and Yue [30, Proposition 2.8], which is based on the moment method.

- *Contributions of paracontrolled terms:* These terms include

$V * (P_{\leq N}$  $\textcircled{=} P_{\leq N} X_N) P_{\leq N} Y_N.$

We use the double Duhamel trick to exploit stochastic cancellations between  and X_N. In our definition of X_N, we use the paradifferential operators $\textcircled{<}$ and $\boxed{\textcircled{<} \,\&\, \textcircled{<}}$ introduced in Section 2, which form a technical novelty.

- *Physical terms:* These terms include

$V * (P_{\leq N}$  $\cdot P_{\leq N} Y_N) P_{\leq N}$  and $(V * (P_{\leq N} Y_N)^2) P_{\leq N} Y_N.$

The first term should be viewed as a random operator in Y_N, but is mainly treated through physical-space arguments. We believe that our approach is of independent interest, since it provides an alternative to the more Fourier-analytic estimates in [5, 28, 30, 37]. The second term is treated deterministically and we rely on the refined Strichartz estimates of Klainerman and Tataru [45].

As we mentioned before, all stochastic objects have been based on $\bullet$, and the smoother component $\circ_{\mathrm{M}}$ is simply placed inside Y_N. This approach yields the convergence of the flows Φ_N on the support of $\mu_\infty^\otimes$ for a short time interval (see Corollary 2.12). The structural information in the decomposition (1.18), however, cannot (directly) be carried over to the support of $\mu_\infty^\otimes$, since $\bullet$ is only defined on the ambient probability space $(\Omega, \mathcal{F}, \mathbb{P})$. This defect will be addressed below, since the structural information is required for the global theory.

Remark 1.6. As was already mentioned in our overview of the literature, Deng, Nahmod, and Yue recently developed a theory of random tensors [30], which forms a comprehensive framework for the local theory of random dispersive equations. The theory of random tensors (and its precursor [28]) rely more intricately on the independence of the Fourier coefficients than the paracontrolled approach. Even under the reference measure $\nu_M^{\otimes}$, however, the random data $\blacklozenge = \bullet + \circ_{\mathbf{M}}$ has dependent Fourier coefficients. This presents a challenge for the theory of random tensors, which was already mentioned in [30, Section 9.1]. In addition, there are further technical problems related to the switch from Schrödinger to wave equations, which are described in Section 4.4. As a result, the author views the extension of the theory of random tensors to a local theory even for singular Gibbs measures and/or nonlinear wave equations as an interesting open problem.

After this discussion of the local theory, we turn to the global dynamics on the support of the Gibbs measure $\mu_\infty^{\otimes}$. As we have seen in our earlier discussion of Bourgain's globalization argument, its original version requires the convergence of the truncated Gibbs measures in total variation. Unfortunately, Theorem 1.1 only yields the weak convergence of the truncated Gibbs measures $\mu_M^{\otimes}$ to $\mu_\infty^{\otimes}$. We now give an informal description of our new globalization argument, but postpone a rigorous discussion until Section 3.

We let $T \geq 1$ be a large time, $B \geq 1$ be a large parameter describing the size of the evolution, $K \geq 1$ be a large frequency scale, and $\tau > 0$ be a small step size. For any $j \geq 1$, we let $\mathscr{E}_K(B, j\tau) \subseteq \mathscr{H}^{-1/2-\kappa}(\mathbb{T}^3)$ be the set of initial data $\blacklozenge$ such that, for all $t \in [0, j\tau]$ and $N \geq K$,

$$\Phi_N(t)\blacklozenge = \overset{\blacklozenge}{|}(t) + \overset{\blacklozenge\ast\blacklozenge\ \blacklozenge}{\curlyvee}_{N}(t) + w_N(t), \tag{1.19}$$

where w_N has size at most B in "structured high-regularity" norms. In our rigorous argument, B will depend on j, but we ignore this during our informal discussion. We also omit a smallness condition for the difference of $\Phi_N(t)\blacklozenge$ and $\Phi_K(t)\blacklozenge$. The goal is to prove by induction over $j \leq T/\tau$ that

$$\limsup_{M\to\infty} \mu_M^{\otimes}(\blacklozenge \in \mathscr{E}_K(B, j\tau))$$

is close to 1 as long as B, K, and τ are chosen appropriately. The proof relies on four separate ingredients:

(i) (*Structured local well-posedness*) This is the base case $j = 1$. Using our local theory, we only have to convert the stochastic objects in (1.18), which are based on $\bullet$, into stochastic objects based on $\blacklozenge$.

(ii) (*Structure and time-translation*) Using the induction hypothesis, we now assume that the probability $\mu_M^{\otimes}(\blacklozenge \in \mathscr{E}_K(B, (j-1)\tau))$ is close to 1. In order to increase the time interval, we let $\lozenge \overset{\text{def}}{=} \Phi_M[\tau]\blacklozenge$. Using the invariance of $\mu_M^{\otimes}$ under Φ_M, we obtain

$$\begin{aligned}\mu_M^{\otimes}(\lozenge \in \mathscr{E}_K(B, (j-1)\tau)) &= \mu_M^{\otimes}(\Phi_M[\tau]\blacklozenge \in \mathscr{E}_K(B, (j-1)\tau)) \\ &= \mu_M^{\otimes}(\blacklozenge \in \mathscr{E}_K(B, (j-1)\tau)),\end{aligned}$$

which is close to 1. After unpacking the definitions, we obtain information on the mixed flow $\Phi_N[t-\tau]\Phi_M[\tau]\blacklozenge$ for $t \in [\tau, j\tau]$. It therefore remains to analyze the difference between $\Phi_N[t-\tau]\Phi_M[\tau]\blacklozenge$ and $\Phi_N[t-\tau]\Phi_N[\tau]\blacklozenge$.

(iii) (*Structure and the cubic stochastic object*) The lowest regularity term in $\Phi_N(\tau)\blacklozenge - \Phi_M(\tau)\blacklozenge$ is given by a portion of the cubic stochastic object. In this step, we add the linear evolution of this portion to the mixed flow $\Phi_N[t-\tau]\Phi_M[\tau]\blacklozenge$, which yields a function $\widetilde{u}_N$. It is then shown that $\widetilde{u}_N(t)$ is an approximate solution of the nonlinear wave equation (1.15) for $t \in [\tau, j\tau]$.

(iv) (*Stability theory*) We develop a paracontrolled stability theory and construct a solution u_N close to the approximate solution $\widetilde{u}_N$, which also accounts for the remaining portion of $\Phi_N(\tau)\blacklozenge - \Phi_M(\tau)\blacklozenge$. Since our stability theory preserves the structure of $\widetilde{u}_N$, this yields (1.19) on the time interval $[\tau, j\tau]$. Since the base case already yields the desired structure on $[0, \tau]$, this completes the induction step.

As is evident from this sketch, the proof of global well-posedness is much more involved than in Bourgain's original setting [4,5]. While not perfectly accurate, the author finds the following comparison with the deterministic global theory of dispersive equations illustrative. Bourgain's globalization argument [4,5] is the probabilistic version of a deterministic global theory using a (subcritical) conservation law. The conservation law is replaced by the invariance, which implies that $t \mapsto \mu_N(\Phi_N(t)\phi \in \mathcal{E})$ is constant. In both cases, the global well-posedness is obtained by iterating the local well-posedness, but the estimates used in the local theory are no longer needed. In contrast, the new globalization argument is the probabilistic version of a deterministic global theory using almost conservation laws (cf. [16]). The place of the almost conserved quantities is taken by the functions $t \mapsto \mu_M(\Phi_N(t)\phi \in \mathcal{E})$, which should be close to a constant function. In addition, the proof of global well-posedness often intertwines the local estimates and the choice of the almost conserved quantities. For entirely different reasons, the similarity with almost conserved quantities also appears in the globalization argument of [50], which proves the invariance of a Wiener measure for the periodic derivative nonlinear Schrödinger equation. The truncated dynamics in [50, (3.1)] only approximately conserve the energy (cf. [50, Theorem 4.2]). Even with the same truncation parameter in the measure and the dynamics, the truncated Wiener measure is then only almost invariant (cf. [50, proof of Lemma 6.1]).

Our globalization argument for the nonlinear wave equation also differs from the globalization argument for the parabolic stochastic quantization equation as in [42]. While the invariant measure is singular in both situations, the dependence on the initial data in the parabolic setting is continuous even at spatial singularity $-1/2-$. As a result, it is possible to iterate the local theory over the time intervals $\{[(j-1)\tau, j\tau]\}_{j=1,\ldots,J}$ using only bounds in the $C^{-1/2-}(\mathbb{T}^3)$-norm. As can be seen from the sketch above, iterating the local theory for the nonlinear wave equation (1.15) requires more detailed information on the solution.

Once the global well-posedness has been proven, the proof of invariance is essentially the same as in [4].

Remark 1.7. A paper of this length creates both mathematical challenges and different options for the exposition. The author does not claim to have found the perfect solutions or made the best expository choice in every single instance. While we postpone a more detailed discussion to Remarks 1.6, 2.3, 3.4, 4.43, 8.2, and 9.11, the author wanted to make this point in a central location of the paper. The author hopes that this encourages the reader to think more about our result and related open problems.

1.2. Overview

Due to the excessive length of this paper, we include a few suggestions for the reader. We also display the (main) relationship between the sections in Figure 3.

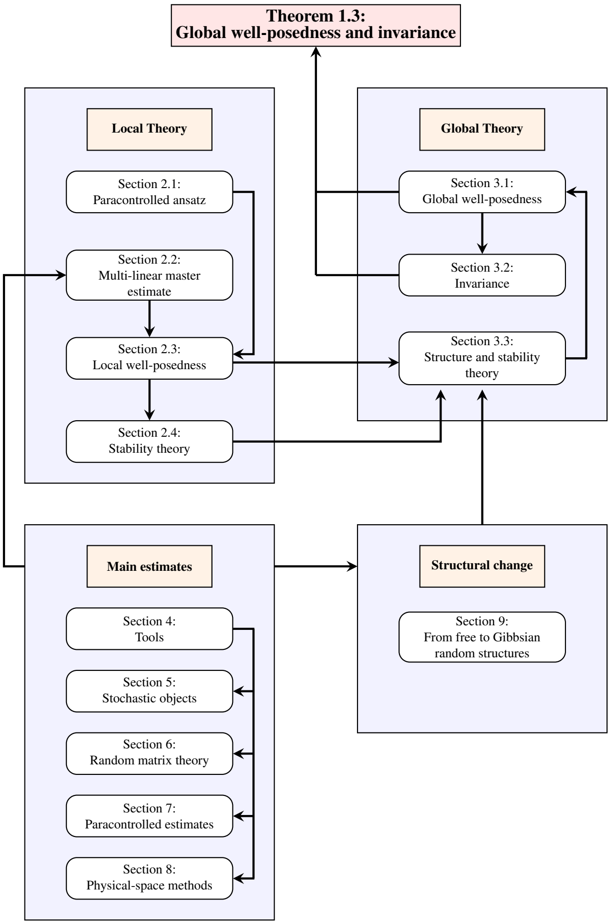


Fig. 3. This figure illustrates the main dependencies between the different sections. The heart of the paper lies in the local and global theory (Sections 2 and 3), which, as long as the reader believes certain estimates, can be read independently of the rest of the paper. A few minor dependencies between the different sections are not included in this illustration. For instance, basic properties of $\mathcal{X}^{s,b}$-spaces, which are recalled in Section 4, will also be used in Sections 2 and 3.

The local and global theory are described in Sections 2 and 3, respectively. These sections contain the main novelties of this paper and should be interesting to most readers. As long as the reader believes several estimates, these sections are also self-contained. We therefore encourage the expert to focus on these sections.

Section 4 contains a collection of tools from dispersive equations, harmonic analysis, and probability theory. The reader should be familiar with the content of each subsection before moving on, but the expert should be able to only skim most content.

Sections 5–8 contain the main technical aspects of this paper. They are concerned with separate terms in the evolution equation and rely on different methods. As a result, they can (essentially) be read independently.

In Section 9, we extend the multi-linear estimates from Sections 5–8, which have been phrased in terms of the Gaussian initial data $\bullet$, to random initial data $\blacklozenge$ drawn from the Gibbs measure. Each proof consists of a concatenation of previous results, and hence this section can safely be skipped on first reading.

1.3. Notation

We recall and introduce notation that will be used throughout the rest of the paper.

Dyadic numbers: Throughout this paper, we denote dyadic integers by K, L, M, and N. In limits or sums, such as $\lim_{M\to\infty}$ or $\sum_N$, we implicitly restrict ourselves to dyadic integers.

Parameters: We first introduce several parameters which are used in our function spaces, in the paradifferential operators, and our estimates. We fix

$$\epsilon > 0, \quad \delta_1, \delta_2 > 0, \quad \kappa > 0, \quad \eta, \eta' > 0, \quad b_+ > b > 1/2 > b_- > 0. \tag{1.20}$$

We use $\epsilon > 0$ in our paradifferential operators, $\kappa > 0$ to capture small losses in probabilistic estimates, $\eta, \eta' > 0$ to capture gains in the highest frequency scale, and $\delta_1, \delta_2, b_+, b, b_-$ in the definition of our function spaces. We impose the condition

$$1/2 - b_- \ll b - 1/2 \ll b_+ - 1/2 \ll \eta' \ll \eta \ll \kappa \ll \delta_2 \ll \epsilon \ll \delta_1. \tag{1.21}$$

In (1.21), the implicit constant in each "$\ll$" is allowed to depend on all parameters appearing to its right. We also define

$$s_1 = 1/2 - \delta_1 \quad \text{and} \quad s_2 = 1/2 + \delta_2.$$

In several statements of this paper, we will also use $0 < \zeta < 1$ and $C \geq 1$ as parameters. However, they may change their values between different lines and are allowed to depend on all parameters in (1.20).

Wave equation and flows: We denote the solution of the nonlinear wave equation (1.15) by $u_N(t)$. We also write

$$u_N[t] \overset{\text{def}}{=} (u_N(t), \partial_t u_N(t)),$$

which is standard in the literature on nonlinear wave equations. If $\blacklozenge \in \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$, we also write $\Phi_N(t)\blacklozenge$ and $\Phi_N[t]\blacklozenge$ for the solution with initial data $\blacklozenge$. When working with the flows $\Phi_N[t]$ and the Gibbs measures $\mu_M^{\otimes}$, we write $\Phi_N[t]_{\#}\mu_M^{\otimes}$ for the pushforward of $\mu_M^{\otimes}$ under $\Phi_N[t]$.

Furthermore, we denote the Duhamel integral operator of the wave equation by I. More precisely, we define

$$\mathrm{I}[F](t) \stackrel{\text{def}}{=} \int_0^t \frac{\sin((t-t')\langle\nabla\rangle)}{\langle\nabla\rangle} F(t')\,\mathrm{d}t'.$$

Fourier transform: With a slight abuse of notation, we write dx for the normalized Lebesgue measure on $\mathbb{T}^3 = \mathbb{R}^3/(2\pi\mathbb{Z})^3$, i.e., we require that

$$\int_{\mathbb{T}^3} 1\,\mathrm{d}x = 1.$$

We then define the Fourier transform of a function $f:\mathbb{T}^3 \to \mathbb{C}$ by

$$\widehat{f}(n) \stackrel{\text{def}}{=} \int_{\mathbb{T}^3} f(x)e^{-inx}\,\mathrm{d}x. \tag{1.22}$$

For any $k \in \mathbb{N}$ and any $n_1,\ldots,n_k \in \mathbb{Z}^3$, we define

$$n_{12\ldots k} \stackrel{\text{def}}{=} \sum_{j=1}^{k} n_j.$$

For example, $n_{12} = n_1 + n_2$ and $n_{123} = n_1 + n_2 + n_3$.

Interaction potential: For a given interaction potential V satisfying Assumptions A, we define

$$\widehat{V}_S(n_1,n_2,n_3) \stackrel{\text{def}}{=} \frac{1}{6}\sum_{\pi\in S_3} \widehat{V}(n_{\pi_1}+n_{\pi_2}).$$

Truncations and Littlewood–Paley operators: For each $t \geq 0$, we let $\rho_t:\mathbb{Z}^3 \to [0,1]$ be the same truncation to frequencies $n \in \mathbb{Z}^3$ satisfying $|n| \lesssim \langle t\rangle$ as in [12, Section 1.3]. For each dyadic $N \geq 1$, we define the Littlewood–Paley multiplier $P_{\leq N}$ by

$$\widehat{P_{\leq N} f}(n) = \rho_N(n)\widehat{f}(n).$$

We further set

$$P_1 f = P_{\leq 1} f \quad \text{and} \quad P_N f = P_{\leq N} f - P_{\leq N/2} f \quad \text{for all } N \geq 2.$$

The corresponding Fourier multipliers are denoted by

$$\chi(n) = \chi_1(n) = \rho_1(n) \quad \text{and} \quad \chi_N(n) = \rho_N(n) - \rho_{N/2}(n) \quad \text{for all } N \geq 2.$$

We also define fattened Littlewood–Paley multipliers by

$$\widetilde{P}_N = \sum_{N/16\leq K\leq 16K} P_K.$$

Function spaces: For any $s \in \mathbb{R}$, the $\mathcal{C}_x^s(\mathbb{T}^3)$-norm is defined as

$$\|f\|_{\mathcal{C}_x^s(\mathbb{T}^3)} \overset{\text{def}}{=} \sup_{N \geq 1} N^s \|P_N f\|_{L_x^\infty(\mathbb{T}^3)}. \tag{1.23}$$

We then define the corresponding space $\mathcal{C}_x^s(\mathbb{T}^3)$ by

$$\mathcal{C}_x^s(\mathbb{T}^3) \overset{\text{def}}{=} \Big\{ f\colon \mathbb{T}^3 \to \mathbb{R} \;\Big|\; \|f\|_{\mathcal{C}_x^s} < \infty,\ \lim_{N\to\infty} N^s \|P_N f\|_{L_x^\infty(\mathbb{T}^3)} = 0 \Big\}. \tag{1.24}$$

We let $H_x^s(\mathbb{T}^3)$ be the usual L^2-based Sobolev space. More precisely, for any $f\colon \mathbb{T}^3 \to \mathbb{C}$, we define the corresponding norm by

$$\|f\|_{H_x^s(\mathbb{T}^3)} \overset{\text{def}}{=} \|\langle n\rangle^s \widehat{f}(n)\|_{\ell_n^2(\mathbb{Z}^3)}.$$

Furthermore, we define $\mathcal{H}_x^s(\mathbb{T}^3) \overset{\text{def}}{=} H_x^s(\mathbb{T}^3) \times H_x^{s-1}(\mathbb{T}^3)$. In this paper, we will also use the Bourgain spaces $\mathfrak{X}^{s,b}(\mathcal{J})$ and the low-frequency modulation space $\mathcal{LM}(\mathcal{J})$, which are defined in Definitions 4.1 and 7.1, respectively.

2. Local theory

In this section, we show that the truncated and renormalized nonlinear wave equations

$$\begin{cases} (-\partial_t^2 - 1 + \Delta) u_N = P_{\leq N}\big(:(V * (P_{\leq N} u_N)^2) P_{\leq N} u_N:\big), \\ u_N[0] = \boldsymbol{\phi}, \end{cases} \tag{2.1}$$

are locally well-posed on the support of the Gibbs measures $\mu_M^{\otimes}$ uniformly in M. It is important in the definition of the limiting dynamics and the globalization argument that the truncation parameter N in the dynamics and the truncation parameter M in the Gibbs measure $\mu_M^{\otimes}$ are allowed to be different.

Due to the truncation, a soft argument based on the coercivity of the Hamiltonian shows that (2.1) is globally well-posed for a fixed truncation parameter N. We denote the corresponding flow by $\Phi_N(t)$.

2.1. *Paracontrolled ansatz*

We now introduce our paracontrolled approach. As discussed in the introduction, we will use a graphical notation for the several stochastic objects appearing in this paper. We denote the random initial data by $\blacklozenge$. In the local theory, we can work with the reference measure $\nu_M^{\otimes}$ and, more precisely, the representation of the reference measure with respect to the ambient measure $\mathbb{P}$.

Based on Theorem 1.1, we find that $\nu_M^{\otimes} = \mathrm{Law}_{\mathbb{P}}(\bullet + \circ_{\mathbf{M}})$, where $\bullet$ is the Gaussian low-regularity component and $\circ_{\mathbf{M}}$ is has regularity $\min(1/2 + \beta, 1)-$. Naturally, we chose the color purple for the random initial data $\blacklozenge$ since it is a mixture of the blue and red random initial data. We emphasize that $\bullet$ and $\circ_{\mathbf{M}}$ are probabilistically dependent! Fortunately, this does not introduce any major difficulties in our treatment of the wave equation with a Hartree nonlinearity. We believe, however, that the proof of the invariance of the Gibbs measure for both the cubic wave equation and the three-dimensional Schrödinger equation with cubic or Hartree nonlinearity will require a more detailed understanding of the relationship between $\bullet$ and $\circ_{\mathbf{M}}$. This additional information is provided in the first part of the series [12].

Before we introduce our stochastic and paracontrolled objects, we discuss the following question: Should we define our stochastic objects based on $\bullet$ or based on $\blacklozenge$? Due to the independence of the Fourier coefficient under $\mathbb{P}$ and its simple structure, it is much more convenient to work with $\bullet$. However, the decomposition $\blacklozenge = \bullet + \circ_{\mathbf{M}}$ of the samples of $\nu_{M}^{\otimes}$ is based on the ambient measure $\mathbb{P}$. It cannot be performed intrinsically on the samples of $\nu_{M}^{\otimes}$ and has *no meaning* for the Gibbs measure $\mu_{M}^{\otimes}$. In particular, if we want to examine the probability of an event under $\mu_{M}^{\otimes}$, we must phrase the event in terms of the full initial data $\blacklozenge$. Fortunately, there is a convenient solution to our conundrum: We first carry out most of our (local) analysis in terms of $\bullet$ and with respect to the ambient measure $\mathbb{P}$. Once all the estimates in terms of $\bullet$ are available, we can convert the stochastic objects and paracontrolled structures from $\bullet$ into $\blacklozenge$ (see Section 9). Then the absolute continuity of $\mu_{M}^{\otimes}$ with respect to the reference measure $\nu_{M}^{\otimes}$ allows us to obtain the same stochastic objects and paracontrolled structures on the support of the Gibbs measure $\mu_{M}^{\otimes}$.

We now begin with the construction of the stochastic objects and paracontrolled structures, which were briefly discussed in the introduction. We define $\overset{\bullet}{|}$ as the linear evolution of the random initial data $\bullet$. More precisely, $\overset{\bullet}{|}$ solves the evolution equation

$$(-\partial_t^2 - 1 + \Delta)\overset{\bullet}{|} = 0, \quad \overset{\bullet}{|}[0] = \bullet. \tag{2.2}$$

The black line in the stochastic object reflects the linear propagator of the wave equation. For future use, we define the frequency-truncated and renormalized square of $\overset{\bullet}{|}$ by

$$\overset{\bullet\ \bullet}{\vee}_{N} \overset{\text{def}}{=} :(P_{\leq N}\overset{\bullet}{|})^2:. \tag{2.3}$$

The multiplication is reflected by the joining of the two lines and the frequency-truncation is reflected in the subscript N. We then define the renormalized nonlinearity $\overset{\bullet * \bullet\ \bullet}{\Psi}_{N}$ by

$$\overset{\bullet * \bullet\ \bullet}{\Psi}_{N} \overset{\text{def}}{=} P_{\leq N}\big(:(V * (P_{\leq N}\overset{\bullet}{|})^2)(P_{\leq N}\overset{\bullet}{|}):\big). \tag{2.4}$$

The orange asterisk reflects the convolution with the interaction potential. The color orange has no significance and we only chose it for aesthetic reasons. As before, the nonlinearity is reflected in the joining of the three lines and the truncation parameter N in the nonlinearity appears as a subscript. Finally, we define the Duhamel integral of $\overset{\bullet * \bullet\ \bullet}{\Psi}_{N}$ by

$$(-\partial_t^2 - 1 + \Delta)\underset{\uparrow}{\overset{\bullet * \bullet\ \bullet}{\Psi}}_{N} = \overset{\bullet * \bullet\ \bullet}{\Psi}_{N}, \quad \underset{\uparrow}{\overset{\bullet * \bullet\ \bullet}{\Psi}}_{N}[0] = 0. \tag{2.5}$$

The line with an arrow reflects the integration in the Duhamel operator. In contrast to $\overset{\bullet}{|}$, we note that the distribution of $\underset{\uparrow}{\overset{\bullet * \bullet\ \bullet}{\Psi}}_{N}$ is not stationary in time. Naively, one may expect that $\curlyvee^{\ast}_{N}$ has spatial regularity $-1/2+\beta-$. Namely, one would expect spatial regularity $3\cdot(-1/2)-$ from the cube of the random initial data $\bullet$, a gain of one spatial derivative from the multiplier $\langle\nabla\rangle^{-1}$ in the Duhamel operator, and a gain of β derivatives from the convolution with the interaction potential. In Proposition 5.1, however, we will see that $\curlyvee^{\ast}_{N}$ actually has spatial regularity $\beta-$, which is half of a derivative better. The additional gain is a result of multi-linear dispersive effects. We now decompose our solution u_N by writing

$$u_N = \overset{\bullet}{\mid} + \curlyvee^{\ast}_{N} + w_N. \tag{2.6}$$

The remainder w_N has initial data $w_N[0] = \circ_{\mathbf{M}}$ and solves the forced nonlinear wave equation

$$
\begin{aligned}
&(-\partial_t^2 - 1 + \Delta) w_N \\
&\quad = P_{\leq N}\Big[:\!\Big(V * \Big(P_{\leq N}\big(\overset{\bullet}{\mid} + \curlyvee^{\ast}_{N} + w_N\big)\Big)^2\Big) P_{\leq N}\big(\overset{\bullet}{\mid} + \curlyvee^{\ast}_{N} + w_N\big)\!: - \vee^{\ast}_{N}\Big] \\
&\quad = P_{\leq N}\Big[2\Big(V * \Big(P_{\leq N}\overset{\bullet}{\mid}\cdot P_{\leq N}\big(\curlyvee^{\ast}_{N} + w_N\big)\Big)\Big) P_{\leq N}\overset{\bullet}{\mid} - \mathcal{M}_N P_{\leq N}\big(\curlyvee^{\ast}_{N} + w_N\big)\Big) && (2.7)\\
&\qquad + \Big(V * \Big(P_{\leq N}\big(\curlyvee^{\ast}_{N} + w_N\big)\Big)^2\Big) P_{\leq N}\overset{\bullet}{\mid} && (2.8)\\
&\qquad + 2V * \Big(P_{\leq N}\overset{\bullet}{\mid}\cdot P_{\leq N}\big(\curlyvee^{\ast}_{N} + w_N\big)\Big) P_{\leq N}\big(\curlyvee^{\ast}_{N} + w_N\big) && (2.9)\\
&\qquad + (V * \vee_{N}) P_{\leq N}\big(\curlyvee^{\ast}_{N} + w_N\big) && (2.10)\\
&\qquad + \Big(V * \Big(P_{\leq N}\big(\curlyvee^{\ast}_{N} + w_N\big)\Big)^2\Big) P_{\leq N}\big(\curlyvee^{\ast}_{N} + w_N\big)\Big]. && (2.11)
\end{aligned}
$$

If we intend to construct (or control) w_N via a "direct" contraction argument, we would need the following conditions on the regularity of w_N (uniformly in N):

(1) Due to the high×high→low interactions in factors such as $P_{\leq N}\overset{\bullet}{\mid}\cdot P_{\leq N} w_N$, the regularity of w_N needs to be greater than $1/2$.

(2) Due to "deterministic" nonlinear terms such as $(V * (P_{\leq N} w_N)^2) P_{\leq N} w_N$, the regularity of w_N needs to be greater than or equal to the deterministic critical regularity, which is given by $1/2-\beta$.

Clearly, the first regularity condition is more restrictive. Unfortunately, the contribution of the first two summands (2.7) and (2.8) has regularity at most $1/2-$. The low×low×high interaction gains one derivative from the multiplier $\langle\nabla\rangle^{-1}$ in the Duhamel operator, but does not benefit from the convolution with V and does not experience any multi-linear dispersive effects. Thus, we are "ϵ-away" from a working contraction argument. As was observed in [36, 37], the term responsible for the low regularity exhibits a paracontrolled structure. Even though $P_{\leq N}\overset{\bullet}{\mid}\cdot P_{\leq N} w_N$ is not well-defined for a general w_N at spatial regularity $1/2-$, we will see in Proposition 7.8 below that it is well-defined for a paracontrolled w_N at the same regularity! We therefore decompose the solution w_N into two components: a paracontrolled component X_N at regularity $1/2-$ and a smoother nonlinear remainder Y_N at a regularity greater than $1/2$.

Before we can define the decomposition, we need to introduce our paraproduct operators.

Definition 2.1 (Paraproduct operators). Let $\epsilon > 0$ be the fixed parameter from Section 1.3 and let $f, g, h\colon \mathbb{T}^3 \to \mathbb{R}$. We define the *low×high, high×high, and high×low paraproducts* by

$$
\begin{aligned}
f \mathbin{\textcircled{$<$}} g &\overset{\text{def}}{=} \sum_{N_1 \leq N_2/8} P_{N_1} f \cdot P_{N_2} g, \\
f \mathbin{\textcircled{$=$}} g &\overset{\text{def}}{=} \sum_{N_2/4 \leq N_1 \leq 4N_2} P_{N_1} f \cdot P_{N_2} g, \\
f \mathbin{\textcircled{$>$}} g &\overset{\text{def}}{=} \sum_{N_1 \geq 8N_2} P_{N_1} f \cdot P_{N_2} g.
\end{aligned}
$$

We also define

$$
f \mathbin{\textcircled{$\geq$}} g \overset{\text{def}}{=} f \mathbin{\textcircled{$>$}} g + f \mathbin{\textcircled{$=$}} g \quad \text{and} \quad f \mathbin{\textcircled{$\leq$}} g \overset{\text{def}}{=} f \mathbin{\textcircled{$<$}} g + f \mathbin{\textcircled{$=$}} g.
$$

In most of this paper, it will be convenient to replace "low" frequencies by "very low" frequencies. To this end, we define the bilinear operator

$$
f \mathbin{\textcircled{$\lessdot$}} g \overset{\text{def}}{=} \sum_{\substack{N_1, N_2: \\ N_1 \leq N_2^\epsilon}} P_{N_1} f \cdot P_{N_2} g \tag{2.12}
$$

and the trilinear operator

$$
\boxed{\textcircled{$\lessdot$} \,\&\, \textcircled{$\lessdot$}}(V * (fg)h) \overset{\text{def}}{=} \sum_{\substack{N_1, N_2, N_3: \\ N_1, N_2 \leq N_3^\epsilon}} V * (P_{N_1} f \cdot P_{N_2} g) P_{N_3} h. \tag{2.13}
$$

Furthermore, we define the negations of $\textcircled{$\lessdot$}$ and $\boxed{\textcircled{$\lessdot$} \,\&\, \textcircled{$\lessdot$}}$ by

$$
\begin{aligned}
&f \;(\neg \textcircled{$\lessdot$})\; g \overset{\text{def}}{=} fg - f \mathbin{\textcircled{$\lessdot$}} g, \\
&\big(\neg \boxed{\textcircled{$\lessdot$} \,\&\, \textcircled{$\lessdot$}}\big)(V * (fg)h) \overset{\text{def}}{=} V * (fg)h - \boxed{\textcircled{$\lessdot$} \,\&\, \textcircled{$\lessdot$}}(V * (fg)h).
\end{aligned}
$$

Remark 2.2. The notation "$\lessdot$" is seldom used in the mathematical literature, which is precisely the reason why we use it in Definition 2.1. Its meaning would otherwise easily be confused with projections to $N_1 \leq N_2$, $N_1 \lesssim N_2$, or $N_1 \ll N_2$, which are again more common, but less suitable in our situation than $N_1 \leq N_2^\epsilon$. Comparing our notation for the operators $\textcircled{$\lessdot$}$ and $\boxed{\textcircled{$\lessdot$} \,\&\, \textcircled{$\lessdot$}}$, it may seem more natural to write

$$
V * (fg) \boxed{\textcircled{$\lessdot$} \,\&\, \textcircled{$\lessdot$}} h
$$

instead of (2.13). We found, however, that the notation in (2.13) is much cleaner once it is combined with the stochastic objects. We also point out that the negation of ⧀ is not ⧁.

We are now ready to define X_N and Y_N. We define the paracontrolled component X_N by $X_N[0] = 0$ and

$$(-\partial_t^2 - 1 + \Delta)X_N = P_{\leq N}\Big[2\,\boxed{⧀ \,\&\, ⧀}\Big(V * \Big(P_{\leq N}\uparrow \cdot P_{\leq N}\big(\curlyvee_N + X_N\big)\Big) P_{\leq N}\uparrow\Big) \\ + 2\big(V * (P_{\leq N}\uparrow \cdot P_{\leq N}(Y_N)) ⧀ P_{\leq N}\uparrow\big) \\ + \Big(V * \Big(P_{\leq N}\big(\curlyvee_N + w_N\big)\Big)^2\Big) ⧀ P_{\leq N}\uparrow\Big]. \tag{2.14}$$

Remark 2.3. As far as the author is aware, the operator $\boxed{⧀ \,\&\, ⧀}$ has not been used in previous work on random dispersive equations. The reason for introducing the operator lies in the first term in (2.14), which contains $P_{\leq N}\uparrow \cdot P_{\leq N} X_N$. In order to define this term (uniformly in N), the spatial regularity of X_N alone is not sufficient. It is also difficult to use the structure of X_N, since this term appears in the evolution equation for X_N (and not for Y_N), and hence one may run into a circular argument. By using $\boxed{⧀ \,\&\, ⧀}$, however, this problem does not occur, since we can borrow a small amount of regularity from the third argument in $\boxed{⧀ \,\&\, ⧀}(V * (P_{\leq N}\uparrow \cdot P_{\leq N} X_N)\ P_{\leq N}\uparrow)$. We mention, however, that using $\boxed{⧀ \,\&\, ⧀}$ has a small drawback, which is explained in Remark 9.11.

We also did not include any component of $\mathcal{M}_N P_{\leq N} Y_N$ in the second term of (2.14). It turns out that the contribution coming from the ⧀-portion of the renormalization can be controlled at regularities bigger than $1/2$ and is therefore placed in the evolution equation for Y_N below.

As determined by our choice of X_N, the nonlinear remainder Y_N satisfies $Y_N[0] = \circ_{\mathbf{M}}$ and

$$(-\partial_t^2 - 1 + \Delta)Y_N \\ = 2P_{\leq N}\Big[(\neg\,\boxed{⧀ \,\&\, ⧀})\Big(V * \Big(P_{\leq N}\uparrow \cdot P_{\leq N}\big(\curlyvee_N + X_N\big)\Big) P_{\leq N}\uparrow\Big) \\ - \mathcal{M}_N P_{\leq N}\big(\curlyvee_N + X_N\big)\Big] \tag{2.15}$$

$$+ P_{\leq N}\Big[2\big(V * (P_{\leq N}\uparrow \cdot P_{\leq N}(Y_N))\ (\neg ⧀)\ P_{\leq N}\uparrow\big) - \mathcal{M}_N P_{\leq N}(Y_N) \tag{2.16}$$

$$+ \Big(V * \Big(P_{\leq N}\big(\curlyvee_N + w_N\big)\Big)^2\Big)\ (\neg ⧀)\ P_{\leq N}\uparrow \tag{2.17}$$

$$+ 2V * \Big(P_{\leq N}\uparrow \cdot P_{\leq N}\big(\curlyvee_N + w_N\big)\Big) P_{\leq N}\big(\curlyvee_N + w_N\big) \tag{2.18}$$

$$+ (V * \vee_N) P_{\leq N}\big(\curlyvee_N + w_N\big) \tag{2.19}$$

$$+ \Big(V * \Big(P_{\leq N}\big(\curlyvee_N + w_N\big)\Big)^2\Big) P_{\leq N}\big(\curlyvee_N + w_N\big)\Big]. \tag{2.20}$$

To facilitate the analysis in the body of this paper, we further organize the terms in the evolution equation for Y_N. We write

$$(-\partial_t^2 - 1 + \Delta)Y_N = \mathbf{So} + \mathbf{CPara} + \mathbf{RMT} + \mathbf{Phy}, \tag{2.21}$$

where the stochastic objects **So**, the contributions of the paracontrolled terms **CPara**, the random-matrix terms **RMT**, and the physical terms **Phy** are defined as follows:

We define the individual stochastic objects by

$$[\text{tree}] \overset{\text{def}}{=} \Big(V * \Big(P_{\leq N}[\text{tree}] \cdot P_{\leq N}\Big([\text{tree}]\Big)\Big) P_{\leq N}[\text{tree}]\Big) - \mathcal{M}_N P_{\leq N}\Big([\text{tree}]\Big), \tag{2.22}$$

$$\Big(\neg \boxed{\circledless \,\&\, \circledless}\Big)\, [\text{tree}] \overset{\text{def}}{=} \Big(\neg \boxed{\circledless \,\&\, \circledless}\Big)\Big(V * \Big(P_{\leq N}[\text{tree}] \cdot P_{\leq N}\Big([\text{tree}]\Big)\Big) P_{\leq N}[\text{tree}]\Big) - \mathcal{M}_N P_{\leq N}\Big([\text{tree}]\Big), \tag{2.23}$$

$$[\text{tree}] \overset{\text{def}}{=} (V * [\text{tree}]\,) P_{\leq N}[\text{tree}]\,, \tag{2.24}$$

$$[\text{tree}] \overset{\text{def}}{=} V * \Big(P_{\leq N}[\text{tree}] \cdot P_{\leq N}[\text{tree}]\Big) P_{\leq N}[\text{tree}]\,, \tag{2.25}$$

$$[\text{tree}] \overset{\text{def}}{=} \Big(V * \Big(P_{\leq N}[\text{tree}]\Big)^2\Big) P_{\leq N}[\text{tree}], \tag{2.26}$$

$$(\neg \circledless)\, [\text{tree}] \overset{\text{def}}{=} V * \Big(P_{\leq N}[\text{tree}]\Big)^2 \,(\neg \circledless)\; P_{\leq N}[\text{tree}]. \tag{2.27}$$

We then define

$$\begin{aligned}\mathbf{So} &= \mathbf{So}_N \\ &\overset{\text{def}}{=} P_{\leq N}\Big[2\,\Big(\neg \boxed{\circledless \,\&\, \circledless}\Big)\,[\text{tree}] + [\text{tree}] + (\neg \circledless)\,[\text{tree}] + 2\,[\text{tree}]\Big].\end{aligned} \tag{2.28}$$

In works on singular SPDEs, such as [48], the paradifferential operators are usually placed at the joints of the different lines. The advantage is that it works for arbitrary "trees" and can accommodate multiple paradifferential operators. Since this level of generality will not be needed here, we prefer our notation, since it is slightly easier to read.

We define

$$\begin{aligned}\mathbf{CPara} &= \mathbf{CPara}_N(X_N, w_N) \\ &\overset{\text{def}}{=} 2P_{\leq N}\Big[\Big(\Big(\neg \boxed{\circledless \,\&\, \circledless}\Big)\big(V * (P_{\leq N}[\text{tree}] \cdot P_{\leq N} X_N) P_{\leq N}[\text{tree}]\big) - \mathcal{M}_N P_{\leq N} X_N\Big)\Big] \\ &\quad + 2P_{\leq N}\big[V * (P_{\leq N}[\text{tree}] \circleddash P_{\leq N} X_N) P_{\leq N} w_N\big] \\ &\quad + 2P_{\leq N}\Big[V * (P_{\leq N}[\text{tree}] \circleddash P_{\leq N} X_N) P_{\leq N}[\text{tree}]\Big].\end{aligned} \tag{2.29}$$

In our analysis of **CPara**, we will use the double Duhamel trick, i.e., we will replace X_N by the Duhamel integral of the right-hand side in (2.14).

The random matrix term is defined as

$$
\begin{aligned}
\mathbf{RMT} &= \mathbf{RMT}_N(Y_N, w_N) \\
&\overset{\text{def}}{=} P_{\le N}[(V * \overset{\bullet\ \bullet}{\vee}_{N}) P_{\le N} w_N] && (2.30)\\
&\quad + 2P_{\le N}\big[\big(V * (P_{\le N}\overset{\bullet}{|} \cdot P_{\le N}(Y_N))\, (\neg \circledless)\, P_{\le N}\overset{\bullet}{|}\big) - \mathcal{M}_N P_{\le N}(Y_N)\big]. && (2.31)
\end{aligned}
$$

Our reason for calling (2.30) the random matrix term lies in the method used in its estimate. We will view the summands as random operators in w_N and Y_N, respectively, and estimate the operator norm using the moment method (as in [30, Proposition 2.8]).

Finally, we define the physical term by

$$
\begin{aligned}
\mathbf{Phy} &= \mathbf{Phy}_N(X_N, Y_N, w_N) \\
&\overset{\text{def}}{=} P_{\le N}\Big[2V * \Big(P_{\le N}\overset{\bullet}{|} \cdot P_{\le N} \Psi^{*}_{N}\Big) P_{\le N} w_N && (2.32)\\
&\quad + 2\Big(V * \Big(P_{\le N}\Psi^{*}_{N} \cdot P_{\le N} w_N\Big)\, (\neg \circledless)\, P_{\le N}\overset{\bullet}{|} && (2.33)\\
&\quad + 2V * (P_{\le N}\overset{\bullet}{|} \circledneq P_{\le N} w_N) P_{\le N} \Psi^{*}_{N} && (2.34)\\
&\quad + 2V * (P_{\le N}\overset{\bullet}{|} \circleddash P_{\le N} Y_N) P_{\le N} \Psi^{*}_{N} && (2.35)\\
&\quad + 2V * (P_{\le N}\overset{\bullet}{|} \circledneq P_{\le N} w_N) P_{\le N} w_N && (2.36)\\
&\quad + 2V * (P_{\le N}\overset{\bullet}{|} \circleddash P_{\le N} Y_N) P_{\le N} w_N && (2.37)\\
&\quad + (V * (P_{\le N} w_N)^2)\, (\neg \circledless)\, P_{\le N}\overset{\bullet}{|} && (2.38)\\
&\quad + \Big(V * \Big(P_{\le N}\Big(\Psi^{*}_{N} + w_N\Big)\Big)^2\Big) P_{\le N}\Big(\Psi^{*}_{N} + w_N\Big)\Big]. && (2.39)
\end{aligned}
$$

Similarly to **RMT**, we call **Phy** the physical term due to the methods used in its estimate. We point out, however, that (2.33) and (2.34) are "hybrid" terms and their estimates rely on both random matrix techniques and physical methods. In the estimates of the other terms in **Phy**, we also make use of the refined Strichartz estimates by Klainerman–Tataru [45].

2.2. *Multi-linear master estimate*

In this subsection, we combine all multi-linear estimates from Sections 5–8 into a single proposition, which we refer to as the multi-linear master estimate (Proposition 2.8). In particular, the multi-linear master estimate will include estimates of **So**, **CPara**, **RMT**, and **Phy**, even though the proofs of the individual estimates are quite different. Before we can state the multi-linear master estimate, however, we require additional notation. For the definition of the function spaces $\mathcal{X}^{s,b}$ and $\mathcal{LM}$, we refer to Definitions 4.1 and 7.1.

Definition 2.4 (Types). Let $\mathcal{J} \subseteq [0,\infty)$ be a bounded interval and let $\varphi\colon J \times \mathbb{T}^3 \to \mathbb{R}$. We say that φ is of *type*

- $\overset{\bullet}{|}$ if $\varphi = \overset{\bullet}{|}$,
- $\overset{\bullet\ast\bullet\bullet}{\Psi}$ if $\varphi = \overset{\bullet\ast\bullet\bullet}{\Psi}{}_{N}$ for some $N \geq 1$,
- w if $\|\varphi\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} \leq 1$ and $\sum_{L_1 \sim L_2} \|P_{L_1}\overset{\bullet}{|} \cdot P_{L_2} w\|_{L_t^2 H_x^{-4\delta_1}(\mathcal{J}\times\mathbb{T}^3)} \leq 1$,
- X if $\varphi = P_{\leq N}\, \mathrm{I}[1_{\mathcal{J}_0}\, \mathrm{PCtrl}(H, P_{\leq N}\overset{\bullet}{|})]$ for a dyadic integer $N \geq 1$, a subinterval $\mathcal{J}_0 \subseteq \mathcal{J}$, and a function $H \in \mathcal{LM}(\mathcal{J}_0)$ satisfying $\|H\|_{\mathcal{LM}(\mathcal{J}_0)} \leq 1$,
- Y if $\|\varphi\|_{\mathfrak{X}^{s_2,b}(\mathcal{J})} \leq 1$.

Let $\varphi_1, \varphi_2, \varphi_3\colon \mathcal{J} \times \mathbb{T}^3 \to \mathbb{R}$ and $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \in \{\overset{\bullet}{|}, \overset{\bullet\ast\bullet\bullet}{\Psi}, w, X, Y\}$. We write

$$(\varphi_1, \varphi_2) \overset{\text{type}}{=} (\mathcal{T}_1, \mathcal{T}_2)$$

if either φ_1 is of type $\mathcal{T}_1$ and φ_2 is of type $\mathcal{T}_2$ or φ_1 is of type $\mathcal{T}_2$ and φ_2 is of type $\mathcal{T}_1$. Furthermore, we write

$$(\varphi_1, \varphi_2; \varphi_3) \overset{\text{type}}{=} (\mathcal{T}_1, \mathcal{T}_2; \mathcal{T}_3)$$

if $(\varphi_1, \varphi_2) \overset{\text{type}}{=} (\mathcal{T}_1, \mathcal{T}_2)$ and φ_3 is of type $\mathcal{T}_3$.

Remark 2.5. The types w, X, and Y are designed for the functions w_N, X_N, and Y_N from Section 2.1. Our notation for the type of $(\varphi_1, \varphi_2; \varphi_3)$ respects the symmetry in the first two arguments of the nonlinearity $(V * (\varphi_1\varphi_2))\varphi_3$. We also mention that the types w and X implicitly depend on $\bullet$. In Section 9, we will therefore refer to the types w and X as $w^{\bullet}$ and $X^{\bullet}$, respectively.

In the next lemma, we show that functions of type X and Y are multiples of functions of type w. This allows us to prove several estimates for functions of type X and Y simultaneously.

Lemma 2.6. *Let $A \geq 1$, $T \geq 1$, and let $\zeta = \zeta(\epsilon, s_1, s_2, \kappa, \eta, \eta', b_+, b) > 0$ be sufficiently small. Then there exists a Borel set $\Theta_{\text{type}}(A,T) \subseteq \mathcal{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$ satisfying*

$$\mathbb{P}(\bullet \in \Theta^{\text{type}}_{\text{blue}}(A,T)) \geq 1 - \zeta^{-1}\exp(-\zeta A^{\zeta})$$

and such that the following holds for all $\bullet \in \Theta^{\text{type}}_{\text{blue}}(A,T)$: If $\varphi\colon \mathcal{J} \times \mathbb{T}^3 \to \mathbb{R}$ is of type X or Y, then $T^{-4}A^{-1}\varphi$ is of type w.

Proof. We treat the types X and Y separately. First, we assume that φ is of type X, and hence there exists a dyadic integer $N \geq 1$, a subinterval $\mathcal{J}_0 \subseteq \mathcal{J}$, and a function $H \in \mathcal{LM}(\mathcal{J}_0)$ satisfying $\|H\|_{\mathcal{LM}(\mathcal{J}_0)} \leq 1$ such that $\varphi = P_{\leq N}\, \mathrm{I}[1_{\mathcal{J}_0}\, \mathrm{PCtrl}(H, P_{\leq N}\overset{\bullet}{|})]$. Using the inhomogeneous Strichartz estimate (Lemma 4.9) and Lemma 7.3, we obtain

$$\begin{aligned}\|P_{\leq N} X\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} &\lesssim \|1_{\mathcal{J}_0}\, \mathrm{PCtrl}(H, P_{\leq N}\overset{\bullet}{|})\|_{L_t^{2b} H_x^{s_1-1}(\mathcal{J}\times\mathbb{T}^3)} \\ &\lesssim T\|H\|_{\mathcal{LM}(\mathcal{J})}\|\overset{\bullet}{|}\|_{L_t^{\infty} H_x^{s_1-1+8\epsilon}(\mathcal{J}\times\mathbb{T}^3)} \lesssim T\|\bullet\|_{\mathcal{H}_x^{-1/2-\kappa}(\mathbb{T}^3)}.\end{aligned}$$

This is bounded by TA on a set of acceptable probability. Using Proposition 7.8, we find that, on a set of acceptable probability

$$\sum_{L_1 \sim L_2} \| P_{L_1} \linear \cdot P_{L_2} \varphi \|_{L_t^2 H_x^{-4\delta_1}(\mathcal{J}\times\mathbb{T}^3)} \leq T^4 A \| H \|_{\mathscr{LM}(\mathcal{J}_0)} \leq T^4 A.$$

By combining both estimates, we see that $T^{-4}A^{-1}\varphi$ is of type w.

Second, we assume that φ is of type Y. Then we have $\|\varphi\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} \leq \|\varphi\|_{\mathfrak{X}^{s_2,b}(\mathcal{J})} \leq 1$. This implies

$$\begin{aligned} &\sum_{L_1 \sim L_2} \| P_{L_1} \linear \cdot P_{L_2} \varphi \|_{L_t^2 H_x^{-4\delta_1}(\mathcal{J}\times\mathbb{T}^3)} \\ &\qquad \lesssim T^{1/2} \sum_{L_1 \sim L_2} L_1^{\kappa-\delta_2} \| P_{L_1} \linear \|_{L_t^\infty \mathcal{C}_x^{-1/2-\kappa}(\mathcal{J}\times\mathbb{T}^3)} \| P_{L_2} \varphi \|_{L_t^\infty H_x^{s_2}(\mathcal{J}\times\mathbb{T}^3)} \\ &\qquad \lesssim T^{1/2} \| \linear \|_{L_t^\infty \mathcal{C}_x^{-1/2-\kappa}(\mathcal{J}\times\mathbb{T}^3)} \end{aligned}$$

As above, this is bounded by $T^{1/2}A$ on a set of acceptable probability. By combining both estimates, we see that $T^{-1/2}A^{-1}\varphi$ is of type w. ∎

In order to state the multi-linear master estimate, we need to introduce a multi-linear version of the renormalization in (1.16).

Definition 2.7 (Renormalization). Let $\mathcal{J}$ be a compact interval, let $\varphi_1, \varphi_2, \varphi_3$ be as in Definition 2.4, and let $N \geq 1$. Furthermore, assume that

$$(\varphi_1, \varphi_2; \varphi_3) \overset{\text{type}}{\neq} (\linear, \linear, \linear).$$

Then, we define the renormalized and frequency-truncated nonlinearity by

$$\begin{aligned} &{:}V * (P_{\leq N}\varphi_1 \cdot P_{\leq N}\varphi_2) P_{\leq N}\varphi_3{:} \\ &\quad \overset{\text{def}}{=} \begin{cases} (V * \quadratic_N) P_{\leq N}\varphi_3 & \text{if } (\varphi_1,\varphi_2) \overset{\text{type}}{=} (\linear, \linear), \\ V * (P_{\leq N}\linear \cdot P_{\leq N}\varphi_2) P_{\leq N}\linear - \mathcal{M}_N P_{\leq N}\varphi_2 & \text{if } (\varphi_1,\varphi_3) \overset{\text{type}}{=} (\linear, \linear), \\ V * (P_{\leq N}\varphi_1 \cdot P_{\leq N}\linear) P_{\leq N}\linear - \mathcal{M}_N P_{\leq N}\varphi_1 & \text{if } (\varphi_2,\varphi_3) \overset{\text{type}}{=} (\linear, \linear), \\ V * (P_{\leq N}\varphi_1 \cdot P_{\leq N}\varphi_2) P_{\leq N}\varphi_3 & \text{else}. \end{cases} \end{aligned} \tag{2.40}$$

If $(\varphi_1, \varphi_2) \overset{\text{type}}{\neq} (\linear, \linear)$, we define the action of the paradifferential operators $⧀$ and $\boxed{⧀ \,\&\, ⧀}$ on the renormalized and frequency-truncated nonlinearity by

$$\begin{aligned} {:}V * (P_{\leq N}\varphi_1 \cdot P_{\leq N}\varphi_2) ⧀ P_{\leq N}\varphi_3{:} &\overset{\text{def}}{=} V * (P_{\leq N}\varphi_1 \cdot P_{\leq N}\varphi_2) ⧀ P_{\leq N}\varphi_3, \\ \boxed{⧀ \,\&\, ⧀}\big({:}V * (P_{\leq N}\varphi_1 \cdot P_{\leq N}\varphi_2) P_{\leq N}\varphi_3{:}\big) &\overset{\text{def}}{=} \boxed{⧀ \,\&\, ⧀}\big(V * (P_{\leq N}\varphi_1 \cdot P_{\leq N}\varphi_2) P_{\leq N}\varphi_3\big), \end{aligned}$$

which does not involve a renormalization. We also define the negated paradifferential operators by

$$:V * (P_{\leq N}\varphi_1 \cdot P_{\leq N}\varphi_2)\ (\neg \olessthan)\ P_{\leq N}\varphi_3:$$
$$\overset{\text{def}}{=} :V * (P_{\leq N}\varphi_1 \cdot P_{\leq N}\varphi_2) P_{\leq N}\varphi_3: - :V * (P_{\leq N}\varphi_1 \cdot P_{\leq N}\varphi_2) \olessthan P_{\leq N}\varphi_3:,$$
$$(\neg \boxed{\olessthan \,\&\, \olessthan})(:V * (P_{\leq N}\varphi_1 \cdot P_{\leq N}\varphi_2) P_{\leq N}\varphi_3:)$$
$$\overset{\text{def}}{=} :V * (P_{\leq N}\varphi_1 \cdot P_{\leq N}\varphi_2) P_{\leq N}\varphi_3: - \boxed{\olessthan \,\&\, \olessthan}(:V * (P_{\leq N}\varphi_1 \cdot P_{\leq N}\varphi_2) P_{\leq N}\varphi_3:),$$

which contains the full renormalization.

Equipped with our notion of types and the renormalization, we can now state and prove the multi-linear master estimate.

Proposition 2.8 (Multi-linear master estimate). *Let $\zeta = \zeta(\epsilon, s_1, s_2, \kappa, \eta, \eta', b_+, b) > 0$ be sufficiently small, let $A \geq 1$, and let $T \geq 1$. Then there exists a Borel set $\Theta^{\mathrm{ms}}_{\mathrm{blue}}(A, T) \subseteq \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$ satisfying*

$$\mathbb{P}(\bullet \in \Theta^{\mathrm{ms}}_{\mathrm{blue}}(A, T)) \geq 1 - \zeta^{-1}\exp(-\zeta A^{\zeta}) \tag{2.41}$$

and such that for all $\bullet \in \Theta^{\mathrm{ms}}_{\mathrm{blue}}(A, T)$ the following hold:

Let $\mathcal{J} \subseteq [0, T]$ be an interval and let $N \geq 1$. Let $\varphi_1, \varphi_2, \varphi_3 \colon \mathcal{J} \times \mathbb{T}^3 \to \mathbb{R}$ be as in Definition 2.4 and let

$$(\varphi_1, \varphi_2; \varphi_3) \overset{\text{type}}{\neq} (\overset{\bullet}{|}, \overset{\bullet}{|}; \overset{\bullet}{|}), (\overset{\bullet}{|}, w; \overset{\bullet}{|}).$$

(i) *If $(\varphi_1, \varphi_2; \varphi_3) \overset{\text{type}}{=} (\overset{\bullet}{|}, \overset{\bullet * \bullet \bullet}{\curlyvee}; \overset{\bullet}{|}),\ (\overset{\bullet}{|}, X; \overset{\bullet}{|})$, then*

$$\big\| (\neg \boxed{\olessthan \,\&\, \olessthan})(:V * (P_{\leq N}\varphi_1 \cdot P_{\leq N}\varphi_2) P_{\leq N}\varphi_3:) \big\|_{\mathfrak{X}^{s_2-1, b_+-1}(\mathcal{J})} \leq T^{30} A.$$

(ii) *If $(\varphi_1, \varphi_2; \varphi_3) \overset{\text{type}}{=} (\overset{\bullet}{|}, Y; \overset{\bullet}{|})$ or $\varphi_1, \varphi_2 \overset{\text{type}}{\neq} \overset{\bullet}{|}$ and $\varphi_3 \overset{\text{type}}{=} \overset{\bullet}{|}$, then*

$$\| :V * (P_{\leq N}\varphi_1 \cdot P_{\leq N}\varphi_2)\ (\neg \olessthan)\ P_{\leq N}\varphi_3: \|_{\mathfrak{X}^{s_2-1, b_+-1}(\mathcal{J})} \leq T^{30} A.$$

(iii) *In all other cases,*

$$\| :V * (P_{\leq N}\varphi_1 \cdot P_{\leq N}\varphi_2) P_{\leq N}\varphi_3: \|_{\mathfrak{X}^{s_2-1, b_+-1}(\mathcal{J})} \leq T^{30} A.$$

Remark 2.9. The frequency-localized versions of each estimate in Proposition 2.8 gain an η'-power in the maximal frequency scale. Furthermore, functions of the type $\overset{\bullet * \bullet \bullet}{\curlyvee}$ can be replaced by $\overset{\bullet * \bullet \bullet}{\curlyvee}_{\tau}$ as defined in (3.4). For more details on these minor modifications, we refer the reader to the proof of the individual main estimates (Sections 5–8).

Proof of Proposition 2.8. It suffices to prove the estimates with A on the right-hand side replaced by CA^C, where $C = C(s_1, s_2, b, b_+, \epsilon)$. Then the desired estimate follows by replacing A with a small power of A and adjusting the constant ζ. In the following, we freely restrict to events with acceptable probabilities.

Proof of (i). If $(\varphi_1, \varphi_2; \varphi_3)$ has type

- $(\overset{\bullet}{\mid}, \overset{\bullet\ast\bullet\bullet}{\curlyvee}; \overset{\bullet}{\mid})$, we use Proposition 5.7,
- $(\overset{\bullet}{\mid}, X; \overset{\bullet}{\mid})$, we use Proposition 7.9.

Proof of (ii). If $(\varphi_1, \varphi_2; \varphi_3) \overset{\text{type}}{=} (\overset{\bullet}{\mid}, Y, \overset{\bullet}{\mid})$, this follows from Proposition 6.3. Using Lemma 2.6, we may assume in all remaining cases that φ_1 and φ_2 have type $\overset{\bullet\ast\bullet\bullet}{\curlyvee}$ or w, as long as we obtain the estimate with T^{18} instead of T^{30}. If $(\varphi_1, \varphi_2; \varphi_3)$ has type

- $(\overset{\bullet\ast\bullet\bullet}{\curlyvee}, \overset{\bullet\ast\bullet\bullet}{\curlyvee}; \overset{\bullet}{\mid})$, we use Lemma 7.4 and Proposition 5.10,
- $(w, \overset{\bullet\ast\bullet\bullet}{\curlyvee}; \overset{\bullet}{\mid})$, we use Lemma 7.6 and Proposition 8.12,
- $(w, w; \overset{\bullet}{\mid})$, we use Lemma 7.6 and Proposition 8.6.

Proof of (iii). Using Lemma 2.6, we may assume that all functions φ_j are of type $\overset{\bullet}{\mid}$, $\overset{\bullet\ast\bullet\bullet}{\curlyvee}$, or w, as long as we prove the estimate with T^{18} instead of T^{30}. If no factor is of type $\overset{\bullet}{\mid}$, the desired estimate follows from Propositions 5.1 and 8.10. The remaining cases can be estimated as follows: If $(\varphi_1, \varphi_2; \varphi_3)$ has type

- $(\overset{\bullet}{\mid}, \overset{\bullet}{\mid}; \overset{\bullet\ast\bullet\bullet}{\curlyvee})$, we use Proposition 5.8,
- $(\overset{\bullet}{\mid}, \overset{\bullet}{\mid}; w)$, we use Proposition 6.1,
- $(\overset{\bullet}{\mid}, \overset{\bullet\ast\bullet\bullet}{\curlyvee}; \overset{\bullet\ast\bullet\bullet}{\curlyvee})$, we use Proposition 5.10,
- $(\overset{\bullet}{\mid}, \overset{\bullet\ast\bullet\bullet}{\curlyvee}; w)$, we use Proposition 8.12,
- $(\overset{\bullet}{\mid}, w; \overset{\bullet\ast\bullet\bullet}{\curlyvee})$, we use Lemma 8.8 and Proposition 8.12,
- $(\overset{\bullet}{\mid}, w; w)$, we use Proposition 8.7 and Lemma 8.8. ■

2.3. *Local well-posedness*

In this subsection, we obtain our first local well-posedness result. It is phrased in terms of the ambient measure $\mathbb{P}$ and the random structure is based on the Gaussian initial data $\bullet$.

Proposition 2.10 (Structured local well-posedness with respect to the ambient measure). *Let $M \geq 1$, let $A \geq 1$, let $0 < \tau \leq 1$, and let $\zeta = \zeta(\epsilon, s_1, s_2, \kappa, \eta, \eta', b_+, b) > 0$ be sufficiently small. Denote by $\mathcal{L}_M^{\mathrm{amb}}(A, \tau)$ the event in the ambient space $(\Omega, \mathcal{F})$ defined by the following conditions:*

(i) *For any $N \geq 1$, the solution of* (2.1) *with initial data $\blacklozenge = \bullet + \circ_{\mathrm{M}}$ exists on $[0, \tau]$.*

(ii) *For all $N \geq 1$, there exist $w_N \in \mathcal{X}^{s_1,b}([0, \tau])$, $H_N \in \mathcal{LM}([0, \tau])$, and $Y_N \in \mathcal{X}^{s_2,b}([0, \tau])$ such that*

$$\Phi_N(t)\blacklozenge = \dot{|}(t) + \Psi^{\circledast}_{N}(t) + w_N(t),$$
$$w_N(t) = P_{\leq N}\,\mathrm{I}[\mathrm{PCtrl}(H_N, P_{\leq N}\dot{|})](t) + Y_N(t)$$

for all $t \in [0,\tau]$. *Furthermore, we have the bounds*

$$\|w_N\|_{\mathfrak{X}^{s_1,b}([0,\tau])}, \|H_N\|_{\mathfrak{LM}([0,\tau])}, \|Y_N\|_{\mathfrak{X}^{s_2,b}([0,\tau])} \leq A,$$
$$\sum_{L_1 \sim L_2} \|P_{L_1}\dot{|} \cdot P_{L_2} w_N\|_{L_t^2 H_x^{-4\delta_1}([0,\tau]\times\mathbb{T}^3)} \leq A.$$

(iii) *For all* $N, K \geq 1$,

$$\|\Phi_N[t]\blacklozenge - \Phi_K[t]\blacklozenge\|_{L_t^\infty \mathscr{H}_x^{\beta-\kappa}([0,\tau]\times\mathbb{T}^3)} \leq A\min(N,K)^{-\eta'}.$$

We further require that

$$\|H_N - H_K\|_{\mathfrak{LM}([0,\tau])}, \|Y_N - Y_K\|_{\mathfrak{X}^{s_2,b}([0,\tau])} \leq A\min(N,K)^{-\eta'}.$$

If $A\tau^{b_+-b} \leq 1$, *then* $\mathcal{L}_M^{\mathrm{amb}}(A,\tau)$ *has a high probability and*

$$\mathbb{P}(\mathcal{L}_M^{\mathrm{amb}}(A,\tau)) \geq 1 - \zeta^{-1}\exp(-\zeta A^{\zeta}). \tag{2.42}$$

Remark 2.11. The superscript "amb" in $\mathcal{L}_M^{\mathrm{amb}}(A,\tau)$ emphasizes that the event lives in the ambient probability space. The first item (i) is only stated for expository purposes. Indeed, since (i) is a soft statement and does not contain any uniformity in the frequency-truncation parameter, it follows from the global well-posedness of (2.1) (which is also not uniform in N). The interesting portions of the proposition are included in (ii) and (iii), which contain uniform structural information about the solution and allow us to locally define the limiting dynamics.

By combining Theorem 1.1 and Proposition 2.10, we easily obtain the local well-posedness of the renormalized nonlinear wave equation on the support of the Gibbs measure.

Corollary 2.12 (Local well-posedness for Gibbsian initial data). *Let* $0 < \tau < 1$ *and let* $\zeta = \zeta(\epsilon, s_1, s_2, \kappa, \eta, \eta', b_+, b) > 0$ *be sufficiently small. Then there exists a Borel set* $\mathcal{L}(\tau) \subseteq \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$ *such that* $\Phi_N[t]\blacklozenge$ *converges in* $C_t^0\mathscr{H}_x^{-1/2-\kappa}([0,\tau]\times\mathbb{T}^3)$ *as* $N \to \infty$ *and*

$$\mu_\infty^{\otimes}(\mathcal{L}(\tau)) \geq 1 - \zeta^{-1}\exp(-\zeta\tau^{-\zeta}). \tag{2.43}$$

Corollary 2.12 shows that the limiting dynamics $\Phi(t)\blacklozenge = \lim_{N\to\infty}\Phi_N(t)\blacklozenge$ are locally well-defined on the support of the Gibbs measure. However, it does not contain any structural information about the solution, which will be essential in the globalization argument (Section 3). The main difficulty, which was described in detail in Section 2.1, is that the free component of the initial data $\bullet$ is only defined on the ambient space. Nevertheless, in Proposition 3.3 below, we obtain a structured local well-posedness theorem in terms of $\blacklozenge$.

We first use the structured local well-posedness result for the ambient measure (Proposition 2.10) to prove the unstructured local well-posedness for Gibbsian random data (Corollary 2.12). Then we present the proof of Proposition 2.10.

Proof of Corollary 2.12. Let $M \geq 1$ and let A satisfy $A\tau^{b_+-b} \leq 1$. We define a closed set $\widetilde{\mathcal{L}}(A,\tau) \subseteq \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$ by requiring that $\blacklozenge \in \widetilde{\mathcal{L}}(A,\tau)$ if and only if

(a) for any $N \geq 1$, the solution of (2.1) with initial data $\blacklozenge$ exists on $[0,\tau]$,

(b) for all $N, K \geq 1$,

$$\|\Phi_N(t)\blacklozenge - \Phi_K(t)\blacklozenge\|_{L_t^\infty \mathscr{H}_x^{\beta-\kappa}([0,\tau]\times\mathbb{T}^3)} \leq A \min(N,K)^{-\eta'}.$$

It is clear from the definition that $\mathcal{L}(\tau) \subseteq \widetilde{\mathcal{L}}(A,\tau)$. We emphasize that $\widetilde{\mathcal{L}}(A,\tau)$ is defined intrinsically through $\blacklozenge$ and does not refer to the ambient probability space $(\Omega,\mathcal{F},\mathbb{P})$. From the definition of $\mathcal{L}_M^{\mathrm{amb}}(A,\tau)$ in Proposition 2.10, it follows that

$$\mathcal{L}_M^{\mathrm{amb}}(A,\tau) \subseteq \{\bullet + \circ_{\mathbf{M}} \in \widetilde{\mathcal{L}}(A,\tau)\}.$$

By using the representation of the reference measure in Theorem 1.1, we deduce that $\mathrm{Law}_{\mathbb{P}}(\bullet + \circ_{\mathbf{M}}) = \nu_M^{\otimes}$. This yields

$$\nu_M^{\otimes}(\widetilde{\mathcal{L}}(A,\tau)) = \mathbb{P}(\bullet + \circ_{\mathbf{M}} \in \widetilde{\mathcal{L}}(A,\tau)) \geq \mathbb{P}(\mathcal{L}_M^{\mathrm{amb}}(A,\tau)) \geq 1 - \zeta^{-1}\exp(-\zeta A^{\zeta}).$$

By using the quantitative version of the absolute continuity $\mu_M^{\otimes} \ll \nu_M^{\otimes}$ in Theorem 1.1, we obtain

$$\begin{aligned}\mu_M^{\otimes}(\mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3) \setminus \widetilde{\mathcal{L}}(A,\tau)) &\lesssim \nu_M^{\otimes}(\mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3) \setminus \widetilde{\mathcal{L}}(A,\tau))^{1-1/q} \\ &\lesssim \zeta^{-1}\exp(-\zeta(1-q^{-1})A^{\zeta}).\end{aligned}$$

After adjusting the value of ζ, this yields the desired estimate (2.43) with $\mu_\infty^{\otimes}$ replaced by $\mu_M^{\otimes}$. Since $\widetilde{\mathcal{L}}(A,\tau)$ is closed in $\mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$ and a subsequence of $\mu_M^{\otimes}$ weakly converges to $\mu_\infty^{\otimes}$, we obtain the same probabilistic estimate for the limiting measure $\mu_\infty^{\otimes}$. ■

Proof of Proposition 2.10. As discussed in Remark 2.11, (i) follows from a soft argument. We now turn to the proof of (ii), which is the heart of the proposition. We let $B = cA^c$, where $c = c(\epsilon, s_1, s_2, b_+, b)$ is a sufficiently small constant.

Using Theorem 1.1, Lemma 2.6, Proposition 2.8, and Proposition 5.1, we may restrict to the event

$$\begin{aligned}&\{\bullet \in \Theta_{\mathrm{blue}}^{\mathrm{ms}}(B,1)\} \cap \{\bullet \in \Theta_{\mathrm{blue}}^{\mathrm{type}}(B,1)\} \cap \{\|\,\dot{|}\,\|_{L_t^\infty \mathscr{C}_x^{-1/2-\kappa}([0,1]\times\mathbb{T}^3)} \leq B\} \\ &\quad \cap \Big\{\sup_N \Big\| \Psi_N \Big\|_{L_t^\infty \mathscr{C}_x^{\beta-\kappa}([0,1]\times\mathbb{T}^3)} \leq B\Big\} \cap \{\|\circ_{\mathbf{M}}\|_{\mathscr{H}_x^{1/2+\beta-\kappa}(\mathbb{T}^3)} \leq B\}. \qquad (2.44)\end{aligned}$$

We now define a map

$$\Gamma_N = (\Gamma_{N,X}, \Gamma_{N,Y}) \colon \mathfrak{X}^{s_1,b}([0,\tau]) \times \mathfrak{X}^{s_2,b}([0,\tau]) \to \mathfrak{X}^{s_1,b}([0,\tau]) \times \mathfrak{X}^{s_2,b}([0,\tau])$$

by

$$\Gamma_{N,X}(X_N,Y_N) \overset{\text{def}}{=} P_{\leq N}\,\mathrm{I}\Big[2\,\boxed{\circledless \,\&\, \circledless}\Big(V * \Big(P_{\leq N}\textcolor{blue}{\uparrow}\cdot P_{\leq N}\Big(\textcolor{blue}{\Psi}^{*}_{N} + X_N\Big)\Big)P_{\leq N}\textcolor{blue}{\uparrow}\Big)$$

$$+\,2\big(V * (P_{\leq N}\textcolor{blue}{\uparrow}\cdot P_{\leq N}(Y_N))\circledless P_{\leq N}\textcolor{blue}{\uparrow}\big) + \Big(V * \Big(P_{\leq N}\Big(\textcolor{blue}{\Psi}^{*}_{N} + w_N\Big)\Big)^2\Big)\circledless P_{\leq N}\textcolor{blue}{\uparrow}\Big]$$

and

$$\Gamma_{N,Y}(X_N,Y_N) \overset{\text{def}}{=} \textcolor{red}{\uparrow} + \mathrm{I}\big[\mathbf{So}_N + \mathbf{CPara}_N(\Gamma_{N,X}(X_N,Y_N), w_N) + \mathbf{RMT}_N(Y_N, w_N) + \mathbf{Phy}_N(X_N, Y_N, w_N)\big],$$

where $w_N = X_N + Y_N$. We emphasize our use of the double Duhamel trick, which is manifested in the argument $\Gamma_{N,X}(X_N,Y_N)$ of $\mathbf{CPara}_N$. Our goal is to show that Γ_N is a contraction on a ball in $\mathfrak{X}^{s_1,b}([0,\tau]) \times \mathfrak{X}^{s_2,b}([0,\tau])$, where the radius remains to be chosen.

Using Lemmas 7.4 and 7.6, it follows that there exists a (canonical) $H_N = H_N(X_N,Y_N)$ satisfying the identity

$$\Gamma_{N,X}(X_N,Y_N) = P_{\leq N}\,\mathrm{I}[\mathrm{PCtrl}(H_N, P_{\leq N}\textcolor{blue}{\uparrow})]$$

and the estimate

$$\|H_N\|_{\mathcal{LM}([0,\tau])} \lesssim B^2 + \|X_N\|^2_{\mathfrak{X}^{s_1,b}([0,\tau])} + \|Y_N\|^2_{\mathfrak{X}^{s_2,b}([0,\tau])}. \tag{2.45}$$

Using the energy estimate (Lemma 4.8), the inhomogeneous Strichartz estimate (Lemma 4.9), Lemma 7.3, and $s_1 - 1 + 8\epsilon < -1/2 - \kappa$, we obtain

$$\begin{aligned}\|\Gamma_{N,X}(X_N,Y_N)\|_{\mathfrak{X}^{s_1,b}([0,\tau])} &\lesssim \|\mathrm{PCtrl}(H_N, P_{\leq N}\textcolor{blue}{\uparrow})\|_{\mathfrak{X}^{s_1-1,b-1}([0,\tau])}\\ &\lesssim \|\mathrm{PCtrl}(H_N, P_{\leq N}\textcolor{blue}{\uparrow})\|_{L_t^{2b}H_x^{s_1-1}([0,\tau]\times\mathbb{T}^3)}\\ &\lesssim \tau^{\frac{1}{2b}}\|\mathrm{PCtrl}(H_N, P_{\leq N}\textcolor{blue}{\uparrow})\|_{L_t^{\infty}H_x^{s_1-1}([0,\tau]\times\mathbb{T}^3)}\\ &\lesssim \tau^{\frac{1}{2b}}\|H_N\|_{\mathcal{LM}([0,\tau])}\|\textcolor{blue}{\uparrow}\|_{L_t^{\infty}H_x^{s_1-1+8\epsilon}([0,\tau]\times\mathbb{T}^3)}\\ &\lesssim \tau^{\frac{1}{2b}}B(B^2 + \|X_N\|^2_{\mathfrak{X}^{s_1,b}([0,\tau])} + \|Y_N\|^2_{\mathfrak{X}^{s_2,b}([0,\tau])}).\end{aligned} \tag{2.46}$$

Using the multi-linear estimates from Proposition 2.8, which are available due to our restriction to the event (2.44), and the time-localization lemma (Lemma 4.3), we similarly obtain

$$\begin{aligned}&\|\Gamma_{N,Y}(X_N,Y_N)\|_{\mathfrak{X}^{s_2,b}([0,\tau])}\\ &\quad\lesssim \|\textcolor{red}{\uparrow}\|_{\mathfrak{X}^{s_2,b}([0,\tau])} + \|\mathbf{So} + \mathbf{CPara} + \mathbf{RMT} + \mathbf{Phy}\|_{\mathfrak{X}^{s_2-1,b-1}([0,\tau])}\\ &\quad\lesssim B + \tau^{b_+-b}\|\mathbf{So} + \mathbf{CPara} + \mathbf{RMT} + \mathbf{Phy}\|_{\mathfrak{X}^{s_2-1,b_+-1}([0,\tau])}\\ &\quad\lesssim B + \tau^{b_+-b}(B^3 + \|X_N\|^3_{\mathfrak{X}^{s_1,b}([0,\tau])} + \|Y_N\|^3_{\mathfrak{X}^{s_2,b}([0,\tau])}).\end{aligned} \tag{2.47}$$

By combining (2.46) and (2.47), we obtain, for a constant $C = C(\epsilon, s_1, s_2, b_+, b)$,

$$\begin{aligned} &\|\Gamma_N(X_N, Y_N)\|_{\mathfrak{X}^{s_1,b}([0,\tau])\times\mathfrak{X}^{s_2,b}([0,\tau])} \\ &\qquad \le CB + C\tau^{b_+-b}\big(B^3 + \|X_N\|^3_{\mathfrak{X}^{s_1,b}([0,\tau])} + \|Y_N\|^3_{\mathfrak{X}^{s_2,b}([0,\tau])}\big). \end{aligned} \tag{2.48}$$

Since $C^4\tau^{b_+-b}B^2 \le 1/100$, which follows from $\tau^{b_+-b}A \le 1$ and our choice of B, we see that Γ_N maps the ball in $\mathfrak{X}^{s_1,b}([0,\tau]) \times \mathfrak{X}^{s_2,b}([0,\tau])$ of radius $2CB$ to itself. A minor modification of the above argument also shows that Γ_N is a contraction, which implies the existence of a unique fixed point (X_N, Y_N) of Γ_N satisfying

$$\|X_N\|_{\mathfrak{X}^{s_1,b}([0,\tau])}, \|Y_N\|_{\mathfrak{X}^{s_2,b}([0,\tau])} \le 2CB. \tag{2.49}$$

Using (2.45), we obtain

$$X_N = P_{\le N}\,\mathrm{I}[\mathrm{PCtrl}(H_N, P_{\le N}\overset{\bullet}{|})]$$

with H_N satisfying $\|H_N\|_{\mathscr{LM}([0,\tau])} \lesssim B^2$. Finally, using the triangle inequality and the condition ${\color{blue}\bullet} \in \Theta^{\text{type}}_{\text{blue}}(B, 1)$ from (2.44), we find that $w_N = X_N + Y_N$ satisfies

$$\begin{aligned} &\|w_N\|_{\mathfrak{X}^{s_1,b}([0,\tau])} \le 4CB, \\ &\sum_{L_1\sim L_2} \|P_{L_1}\overset{\bullet}{|}\cdot P_{L_2}w_N\|_{L^2_t H^{-4\delta_1}_x([0,\tau])} \lesssim B^2. \end{aligned} \tag{2.50}$$

Since $B = cA^c$, (2.49) and (2.50) yield the desired estimates in (ii).

We now turn to (iii). This is a notationally extremely tedious but mathematically minor modification of the arguments leading to (ii). Similar modifications are usually omitted in the literature and we only outline the argument. In the frequency-localized versions of our estimates leading to (ii), we always had an additional decaying factor $N_{\max}^{-\eta'}$, where $N_{\max}$ was the maximal frequency scale (see Remark 2.9 and Sections 5–8). So far, this was only used to sum over all dyadic scales, but it also yields the smallness conditions in (iii). Indeed, one only has to apply the same estimates as above to the difference equation

$$(X_N - X_K, Y_N - Y_K) = \Gamma_N(X_N, Y_N) - \Gamma_K(X_K, Y_K). \qquad \blacksquare$$

2.4. *Stability theory*

In this subsection, we prove a stability estimate (Proposition 2.14) on large time intervals. Strictly speaking, the stability estimate is part of the global instead of the local theory, but the argument is closely related to the proof of local well-posedness (Proposition 2.10). While the stability estimate in this section is phrased in terms of ${\color{blue}\bullet}$, it can be used to obtain a similar estimate in terms of ${\color{purple}\blacklozenge}$ (Proposition 3.8). This second stability estimate will then be used in the globalization argument.

In order to state the stability result, we introduce the function space $\mathfrak{L}$, which captures the admissible perturbations of the initial data.

Definition 2.13 (Structured perturbations). Let $T, N, K \geq 1$ and $t_0 \in [0, T]$. For any $\blacklozenge \in \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$ and $Z[t_0] \in \mathscr{H}_x^{s_1}(\mathbb{T}^3)$, we define

$$\begin{aligned} \|Z[t_0]\|_{\mathfrak{Z}([0,T],\blacklozenge;t_0,N,K)} = \inf_{Z^{\circ},Z^{\square}} \max\Big(&\|Z^{\square}[t_0]\|_{\mathscr{H}_x^{s_1}(\mathbb{T}^3)}, \|Z^{\circ}[t_0]\|_{\mathscr{H}_x^{s_2}(\mathbb{T}^3)}, \\ &\sum_{L_1 \sim L_2} \|P_{L_1}\blacklozenge \cdot P_{L_2} Z\|_{L_t^2 H_x^{-4\delta_1}([0,T]\times\mathbb{T}^3)}, \\ &\|{:}V * (P_{\leq N}\blacklozenge \cdot P_{\leq N} Z^{\circ})\ (\neg \circledless)\ P_{\leq N}\blacklozenge{:}\|_{\mathfrak{X}^{s_2-1,b_+-1}([0,T])}, \\ &\|(\neg \boxed{\circledless \,\&\, \circledless})({:}V * (P_{\leq N}\blacklozenge \cdot P_{\leq N} Z^{\square})\ P_{\leq N}\blacklozenge{:})\|_{\mathfrak{X}^{s_2-1,b_+-1}([0,T])}\Big), \end{aligned}$$

where the infimum is taken over all $Z^{\square}[t_0] \in \mathscr{H}_x^{s_1}(\mathbb{T}^3)$ and $Z^{\circ}[t_0] \in \mathscr{H}_x^{s_2}(\mathbb{T}^3)$ satisfying the identity $Z[t_0] = Z^{\square}[t_0] + Z^{\circ}[t_0]$ and the Fourier support condition $\operatorname{supp} \widehat{Z^{\square}[t_0]}(n) \subseteq \{n \in \mathbb{Z}^3 : |n| \leq 8\max(N, K)\}$. Furthermore, we wrote $Z^{\square}$, Z°, and Z for the corresponding solutions to the linear wave equation.

The notation Z_N° and $Z_N^{\square}$ is motivated by the paradifferential operators used in their treatment. The contributions of Z_N° and $Z_N^{\square}$ are estimated using $\circledless$ and $\boxed{\circledless \,\&\, \circledless}$, respectively.

It is clear that, for fixed parameters T, t_0, N, and K, the maximum is jointly continuous in $Z^{\circ}[t_0] \in \mathscr{H}_x^{s_1}(\mathbb{T}^3)$ (satisfying the frequency-support condition), $Z^{\square}[t_0] \in \mathscr{H}_x^{s_2}(\mathbb{T}^3)$, and $\blacklozenge \in \mathscr{H}_x^{-1/2-\kappa}$. This is the primary reason for including the frequency support condition, since the sum in L_1 and L_2 would not otherwise be continuous in $Z^{\circ}[t_0]$. In particular, the norm $\|Z[t_0]\|_{\mathfrak{Z}([0,T],\blacklozenge;t_0,N,K)}$ is Borel-measurable in $Z[t_0] \in \mathscr{H}_x^{s_2}(\mathbb{T}^3)$ and $\blacklozenge \in \mathscr{H}_x^{-1/2-\kappa}$.

Proposition 2.14 (Stability estimate). *Let $T, A \geq 1$, and let $\zeta = \zeta(\epsilon, s_1, s_2, \kappa, \eta, \eta', b_+, b) > 0$ be sufficiently small. There exists a constant $C = C(\epsilon, s_1, s_2, b_+, b_-)$ and a Borel set $\Theta_{\text{blue}}^{\text{stab}}(A, T) \subseteq \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$ satisfying*

$$\mathbb{P}(\bullet \in \Theta_{\text{blue}}^{\text{stab}}(A, T)) \geq 1 - \zeta^{-1}\exp(-\zeta A^{\zeta})$$

such that the following holds for all $\bullet \in \Theta_{\text{blue}}^{\text{stab}}(A, T)$:

Let $N, B \geq 1$ and $0 < \theta < 1$, let $\mathcal{J} \subseteq [0, T]$ be a compact interval, and $t_0 \overset{\text{def}}{=} \min \mathcal{J}$. Let $\widetilde{u}_N : \mathcal{J} \times \mathbb{T}^3 \to \mathbb{R}$ be an approximate solution of (2.1) satisfying the following assumptions:

(A1) (Structure) *We have the decomposition*

$$\widetilde{u}_N = \bullet + \text{Y}_{N} + \widetilde{w}_N.$$

(A2) (Global bounds) *We have*

$$\|\widetilde{w}_N\|_{\mathfrak{X}^{s_1,b}(J)} \leq B \quad \text{and} \quad \sum_{L_1 \sim L_2} \|P_{L_1}\bullet \cdot P_{L_2}\widetilde{w}_N\|_{L_t^2 H_x^{-4\delta_1}(\mathcal{J}\times\mathbb{T}^3)} \leq B.$$

(A3) (Approximate solution) *There exist* $H_N \in \mathscr{LM}(\mathcal{J})$ *and* $F_N \in \mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})$ *satisfying the identity*

$$\begin{aligned}(-\partial_t^2 - 1 + \Delta)\widetilde{u}_N &= P_{\le N} :(V * (P_{\le N}\widetilde{u}_N)^2)P_{\le N}\widetilde{u}_N: \\ &\quad - P_{\le N}\,\mathrm{PCtrl}(H_N, P_{\le N}\lin) - F_N\end{aligned}$$

and the estimates

$$\|H_N\|_{\mathscr{LM}(\mathcal{J})} \le \theta \quad \textit{and} \quad \|F_N\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})} \le \theta.$$

Furthermore, let $Z_N[t_0] \in H_x^{s_1}(\mathbb{T}^3)$ *be a perturbation satisfying the following assumption.*

(A4) (Structured perturbation) *There exists a* $K \ge 1$ *such that*

$$\|Z[t_0]\|_{\mathfrak{Z}(\mathcal{J},\lin;t_0,N,K)} \le \theta.$$

Finally, assume that

(A5) (Parameter condition) $C\exp(C(A+B)^{\frac{2}{b_+-b}}T^{\frac{40}{b_+-b}})\theta \le 1.$

Then there exists a solution $u_N\colon \mathcal{J}\times\mathbb{T}^3 \to \mathbb{R}$ *of* (2.1) *satisfying the initial value condition* $u_N[t_0] = \widetilde{u}_N[t_0] + Z_N[t_0]$ *and the following conclusions:*

(C1) (Preserved structure) *We have the decomposition*

$$u_N = \lin + \cubicN + w_N.$$

(C2) (Closeness) *The difference* $u_N - \widetilde{u}_N = w_N - \widetilde{w}_N$ *satisfies*

$$\|u_N - \widetilde{u}_N\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} \le C\exp\big(C(A+B)^{\frac{2}{b_+-b}}T^{\frac{40}{b_+-b}}\big)\theta,$$

$$\sum_{L_1\sim L_2}\|P_{L_1}\lin\cdot P_{L_2}(u_N-\widetilde{u}_N)\|_{L_t^2H_x^{-4\delta_1}(\mathcal{J}\times\mathbb{T}^3)} \le C\exp\big(C(A+B)^{\frac{2}{b_+-b}}T^{\frac{40}{b_+-b}}\big)\theta.$$

(C3) (Preserved global bounds) *We have*

$$\|w_N\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} \le B_\theta \quad \textit{and} \quad \sum_{L_1\sim L_2}\|P_{L_1}\lin\cdot P_{L_2}w_N\|_{L_t^2H_x^{-4\delta_1}(\mathcal{J}\times\mathbb{T}^3)} \le B_\theta,$$

where $B_\theta \overset{\text{def}}{=} B + C\exp(C(A+B)^{\frac{2}{b_+-b}}T^{\frac{40}{b_+-b}})\theta.$

As mentioned above, the proof of Proposition 2.14 is close to the proof of local well-posedness. The most important additional ingredient is a Gronwall-type argument in $\mathfrak{X}^{s,b}$-spaces, which is slightly technical due to their nonlocal nature in the time variable.

Proof of Proposition 2.14. Let $N, B, \theta, \mathcal{J}, t_0, \widetilde{u}_N, \widetilde{w}_N, H_N, F_N, Z_N, Z_N^{\square}$, and Z_N° be as in the statement of the proposition and assume that (A1)–(A5) are satisfied. We make the Ansatz

$$u_N(t) = \widetilde{u}_N(t) + v_N(t) + Z_N(t),$$

where the nonlinear component $v_N(t)$ will be decomposed into a paracontrolled and a smoother component below. Based on the condition $u_N[t_0] = \widetilde{u}_N[t_0] + Z_N[t_0]$, we require that $v_N[t_0] = 0$. Using assumption (A3) and the fact that Z_N solves the linear wave equation, we obtain the evolution equation

$$
\begin{aligned}
(-\partial_t^2 - 1 + \Delta) v_N = {} & P_{\leq N} :\!\big(V * (P_{\leq N}(\widetilde{u}_N + v_N + Z_N)^2)\big) P_{\leq N}(\widetilde{u}_N + v_N + Z_N)\!: \\
& - P_{\leq N} :\!(V * (P_{\leq N}\widetilde{u}_N)^2) P_{\leq N}\widetilde{u}_N\!: \\
& + P_{\leq N} \operatorname{PCtrl}(H_N, P_{\leq N} \lollipop) + F_N.
\end{aligned}
$$

Inserting the structural assumption (A1) and using the binomial formula, we obtain

$$
\begin{aligned}
&(-\partial_t^2 - 1 + \Delta) v_N \\
&\quad = P_{\leq N}\Big[2 :\! V * \Big(P_{\leq N}\big(\lollipop + \ThreeStarN + w_N \big) \cdot P_{\leq N}(v_N + Z_N) \Big) P_{\leq N} \lollipop\!: \\
&\qquad + V * \big((P_{\leq N}(v_N + Z_N))^2 \big) P_{\leq N} \lollipop \\
&\qquad + \operatorname{PCtrl}(H_N, P_{\leq N} \lollipop) \\
&\qquad + 2V * \Big(P_{\leq N}\big(\lollipop + \ThreeStarN + w_N \big) \cdot P_{\leq N}(v_N + Z_N) \Big) P_{\leq N}\big(\ThreeStarN + w_N \big) \\
&\qquad + P_{\leq N}\big(V * (:\! P_{\leq N}(\widetilde{u}_N + v_N + Z_N)^2 \!:) \big) P_{\leq N}(v_N + Z_N) \Big] + F_N.
\end{aligned}
$$

We then decompose $v_N = X_N + Y_N$, where X_N is the paracontrolled component and Y_N is the smoother component. Since $v_N[t_0] = 0$, we impose the initial value conditions $X_N[t_0] = 0$ and $Y_N[t_0] = 0$. Similarly to Section 2.1, we define X_N and Y_N through the evolution equations

$$
\begin{aligned}
(-\partial_t^2 - 1 + \Delta) X_N = {} & P_{\leq N}\Big[2 \boxed{\olessthan \,\&\, \olessthan} \big(V * (P_{\leq N} \lollipop \cdot P_{\leq N}(X_N + Z_N^{\square}))\, P_{\leq N} \lollipop \big) \\
& + 2V * (P_{\leq N} \lollipop \cdot P_{\leq N}(Y_N + Z_N^{\circ})) \olessthan P_{\leq N} \lollipop \\
& + 2V * \Big(P_{\leq N}\big(\ThreeStarN + w_N \big) \cdot P_{\leq N}(X_N + Y_N + Z_N) \Big) \olessthan P_{\leq N} \lollipop \\
& + V * \big((P_{\leq N}(v_N + Z_N))^2 \big) \olessthan P_{\leq N} \lollipop + \operatorname{PCtrl}(H_N, P_{\leq N} \lollipop) \Big]
\end{aligned}
\tag{2.51}
$$

and

$$
\begin{aligned}
&(-\partial_t^2 - 1 + \Delta) Y_N = P_{\leq N}\Big[2 \big(\neg \boxed{\olessthan \,\&\, \olessthan} \big) \big(:\! V * (P_{\leq N} \lollipop \cdot P_{\leq N}(X_N + Z_N^{\square}))\, P_{\leq N} \lollipop \!: \big) \\
&\quad + 2 :\! V * (P_{\leq N} \lollipop \cdot P_{\leq N}(Y_N + Z_N^{\circ})) \, (\neg \olessthan)\, P_{\leq N} \lollipop\!: \\
&\quad + 2\, V * \Big(P_{\leq N}\big(\ThreeStarN + w_N \big) \cdot P_{\leq N}(v_N + Z_N) \Big) (\neg \olessthan)\, P_{\leq N} \lollipop \\
&\quad + V * \big((P_{\leq N}(v_N + Z_N))^2 \big) (\neg \olessthan)\, P_{\leq N} \lollipop \\
&\quad + 2V * \Big(P_{\leq N}\big(\lollipop + \ThreeStarN + w_N \big) \cdot P_{\leq N}(v_N + Z_N) \Big) P_{\leq N}\big(\ThreeStarN + w_N \big) \\
&\quad + \big(V * (:\! P_{\leq N}(\widetilde{u}_N + v_N + Z_N)^2 \!:) \big) P_{\leq N}(v_N + Z_N) \Big] + F_N.
\end{aligned}
\tag{2.52}
$$

Since the nonlinearity in (2.51) and (2.52) is frequency-truncated, a soft argument yields the local existence and uniqueness of X_N and Y_N in $C_t^0\mathscr{H}_x^{s_1}$ and $C_t^0\mathscr{H}_x^{s_2}$, respectively. Since $\mathfrak{X}^{s,b}(\mathcal{J})$ embeds into $C_t^0\mathscr{H}_x^s(\mathcal{J}\times\mathbb{T}^3)$ for all $s\in\mathbb{R}$, the solutions exist as long as the restricted $\mathfrak{X}^{s_1,b}$- and $\mathfrak{X}^{s_2,b}$-norms stay bounded.

In order to prove that X_N and Y_N exist on the full interval $\mathcal{J}$ and satisfy the desired bounds, we let T_* be the maximal time of existence of X_N and Y_N on $\mathcal{J}$. We now proceed through a Gronwall-type argument in $\mathfrak{X}^{s,b}$-spaces. We first define

$$f_N\colon [t_0,T^*)\to[0,\infty),\quad t\mapsto \|X_N\|_{\mathfrak{X}^{s_1,b}([t_0,t])}+\|Y_N\|_{\mathfrak{X}^{s_2,b}([t_0,t])}.$$

We emphasize that we neither rely on nor prove the continuity of f_N. By Lemmas 4.4 and 4.8, there exists an implicit constant $C_{\mathrm{En}}=C_{\mathrm{En}}(s_1,s_2,b)$ such that

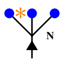
$$\begin{aligned}g_N(t)\overset{\mathrm{def}}{=}C_{\mathrm{En}}\big(&\|1_{[t_0,t]}(-\partial_t^2-1+\Delta)X_N\|_{\mathfrak{X}^{s_1-1,b-1}(\mathbb{R})}\\&+\|1_{[t_0,t]}(-\partial_t^2-1+\Delta)Y_N\|_{\mathfrak{X}^{s_2-1,b-1}(\mathbb{R})}\big)\end{aligned}$$

satisfies $f_N(t)\le g_N(t)$ for all $t\in[t_0,T_*)$. Due to Lemma 4.4, $g_N(t)$ is continuous. Now, let $\tau>0$ be a step size which remains to be chosen and assume that $t,t'\in[t_0,T_*)$ satisfy $t\le t'\le t+\tau$. Using Lemma 4.3, we find that for an implicit constant $C=C(s_1,s_2,b,b_+)$,

$$\begin{aligned}&g_N(t')\\&\le C_{\mathrm{En}}\big(\|1_{[t_0,t]}(-\partial_t^2-1+\Delta)X_N\|_{\mathfrak{X}^{s_1-1,b-1}(\mathbb{R})}+\|1_{[t_0,t]}(-\partial_t^2-1+\Delta)Y_N\|_{\mathfrak{X}^{s_2-1,b-1}(\mathbb{R})}\big)\\&\quad+C_{\mathrm{En}}\big(\|1_{(t,t']}(-\partial_t^2-1+\Delta)X_N\|_{\mathfrak{X}^{s_1-1,b-1}(\mathbb{R})}+\|1_{(t,t']}(-\partial_t^2-1+\Delta)Y_N\|_{\mathfrak{X}^{s_2-1,b-1}(\mathbb{R})}\big)\\&\le g_N(t)+C\tau^{b_+-b}\big(\|(-\partial_t^2-1+\Delta)X_N\|_{\mathfrak{X}^{s_1-1,b_+-1}((t,t'])}\\&\qquad+\|(-\partial_t^2-1+\Delta)Y_N\|_{\mathfrak{X}^{s_2-1,b_+-1}((t,t'])}\big)\\&\le g_N(t)+C\tau^{b_+-b}\big(\|(-\partial_t^2-1+\Delta)X_N\|_{\mathfrak{X}^{s_1-1,b_+-1}([t_0,t'])}\\&\qquad+\|(-\partial_t^2-1+\Delta)Y_N\|_{\mathfrak{X}^{s_2-1,b_+-1}([t_0,t'])}\big).\end{aligned}$$

Similarly to the proof of local well-posedness (Proposition 2.10), we can use Lemma 2.6, Proposition 2.8, and Proposition 5.1 to restrict to the event

$\{\bullet\in\Theta^{\mathrm{ms}}_{\mathrm{blue}}(A,T)\}\cap\{\bullet\in\Theta^{\mathrm{type}}_{\mathrm{blue}}(A,T)\}\cap\{\|$  $\|_{L_t^\infty\mathscr{C}_x^{-1/2-\kappa}([0,1]\times\mathbb{T}^3)}\le A\}$

$\cap\Big\{\sup_N\Big\|$  $\Big\|_{L_t^\infty\mathscr{C}_x^{\beta-\kappa}([0,1]\times\mathbb{T}^3)}\le T^3A\Big\}.$ (2.53)

By combining assumptions (A2)–(A4), and the multi-linear master estimate, a similar argument to the proof of Proposition 2.10 yields

$$\begin{aligned}\tau^{b_+-b}\big(\|(-\partial_t^2-1+\Delta)X_N\|_{\mathfrak{X}^{s_1-1,b_+-1}([t_0,t'])}+\|(-\partial_t^2-1+\Delta)Y_N\|_{\mathfrak{X}^{s_2-1,b_+-1}([t_0,t'])}\big)\\\lesssim T^{30}\tau^{b_+-b}((A+B)^2+f_N(t')^2)(\theta+f_N(t')).\end{aligned}$$

Altogether, we have proven for all $t, t' \in [t_0, T_*)$ satisfying $t \leq t' \leq t + \tau$ the estimate

$$f(t') \leq g(t') \leq g(t) + CT^{30}\tau^{b_+ - b}((A+B)^2 + f_N(t')^2)(\theta + f_N(t')).$$

Using $g(t_0) = 0$ and a continuity argument (Lemma 4.13), iterating the resulting bounds, and assuming the conditions

$$C(A+B)^2 e^{T/\tau}\theta \leq 1/2 \quad \text{and} \quad 2CT^{30}\tau^{b_+ - b}((A+B)^2 + 6) \leq 1/4, \tag{2.54}$$

we obtain

$$\sup_{t\in[t_0,T_*)} f(t) \leq \sup_{t\in[t_0,T_*)} g(t) \leq C(A+B)^2 e^{T/\tau}\theta. \tag{2.55}$$

Using the case of equality in the second condition in (2.54) as a definition for τ, the first condition follows from assumption (A5). Recalling the definition of f, we obtain

$$\sup_{t\in[t_0,T_*)} (\|X_N\|_{\mathfrak{X}^{s_1,b}([t_0,t])} + \|Y_N\|_{\mathfrak{X}^{s_2,b}([t_0,t])}) \leq C \exp\big(C(A+B)^{\frac{2}{b_+ - b}} T^{\frac{40}{b_+ - b}}\big).$$

This estimate rules out finite-time blowup on $\mathcal{J}$ and implies that $T_* = \sup \mathcal{J}$. Together with a soft argument, which is based on the integral equation for X_N and Y_N as well as the time-localization lemma (Lemma 4.3), we obtain

$$\|X_N\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} + \|Y_N\|_{\mathfrak{X}^{s_2,b}(\mathcal{J})} \leq C \exp\big(C(A+B)^{\frac{2}{b_+ - b}} T^{\frac{40}{b_+ - b}}\big). \tag{2.56}$$

With this uniform estimate in hand, we can now easily obtain the desired conclusions (C1)–(C3). In order to obtain (C1), we (are forced to) choose

$$w_N = \widetilde{w}_N + X_N + Y_N + Z_N.$$

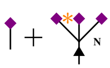
The conclusions (C2) and (C3) follow from (A4), (2.56), and the condition $\bullet \in \Theta^{\text{type}}_{\text{blue}}(A, T)$ in our event (2.53). ∎

3. Global theory

In this section, we prove the global well-posedness of the renormalized nonlinear wave equation and the invariance of the Gibbs measure. As mentioned in the introduction, the heart of this section is a new form of Bourgain's globalization argument. In Section 3.1, we prove the global well-posedness for Gibbsian initial data. We focus on the overall strategy and postpone several individual steps to Section 3.3 below. In Section 3.2, we prove the invariance of the Gibbs measure. Using the global well-posedness from Section 3.1, the proof of invariance is similar to that in Bourgain's seminal paper [4].

3.1. Global well-posedness

We now prove the (quantitative) global well-posedness of the renormalized nonlinear wave equation for Gibbsian initial data. In particular, we show that the structure

$$\Phi_N[t]\blacklozenge =$$

$$+\, w_N$$

from the local theory (see Proposition 3.3) is preserved by the global theory. Here, the linear and cubic stochastic objects are defined exactly as in (2.2) and (2.5), but with $\bullet$ replaced by $\blacklozenge$.

Proposition 3.1 (Global well-posedness). *Let $A, T \geq 1$, let $C = C(\epsilon, s_1, s_2, \kappa, \eta, \eta', b_+, b) \geq 1$ be sufficiently large, and let $\zeta = \zeta(\epsilon, s_1, s_2, \kappa, \eta, \eta', b_+, b) > 0$ be sufficiently small. Assume that $B, D \geq 1$ satisfy*

$$B \geq B(A,T) \overset{\text{def}}{=} C \exp(C(A+T)^C) \quad \textit{and} \quad D \geq D(A,T) \overset{\text{def}}{=} C \exp(\exp(C(A+T)^C)). \tag{3.1}$$

Furthermore, let $K \geq 1$ satisfy the condition

$$C \exp(C(A+B+T)^C) K^{-\eta'} \leq 1. \tag{3.2}$$

Then the Borel set

$$\begin{aligned} \mathcal{E}_K(B,D,T) = \bigcap_{N \geq K} \Big(\Big\{ \blacklozenge \in \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3) \colon w_N(t) = \Phi_N(t)\blacklozenge - \overset{\blacklozenge}{\uparrow} - \overset{\blacklozenge * \blacklozenge \blacklozenge}{\Psi}_N \textit{ satisfies} \\ \|w_N\|_{\mathfrak{X}^{s_1,b}([0,T])} \leq B \textit{ and } \sum_{L_1 \sim L_2} \|P_{L_1} \overset{\blacklozenge}{\uparrow} \cdot P_{L_2} w_N\|_{L_t^2 \mathscr{H}_x^{-4\delta_1}([0,T]\times\mathbb{T}^3)} \leq B \Big\} \\ \cap \{ \blacklozenge \in \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3) \colon \|\Phi_N[t]\blacklozenge - \Phi_K[t]\blacklozenge\|_{C_t^0 \mathscr{H}_x^{\beta-\kappa}([0,T]\times\mathbb{T}^3)} \leq D K^{-\eta'} \} \Big) \end{aligned}$$

satisfies the estimate

$$\inf_{M \geq K} \mu_M^{\otimes}(\mathcal{E}_K(B,D,T)) \geq 1 - T\zeta^{-1} \exp(-\zeta A^{\zeta}). \tag{3.3}$$

In the proof below, we need two modifications of the cubic stochastic object. We define

$$\overset{\blacklozenge * \blacklozenge \blacklozenge}{\Psi}_{N,\tau} \overset{\text{def}}{=} \mathrm{I}[1_{[0,\tau]}(t) \overset{\blacklozenge * \blacklozenge \blacklozenge}{\vee}_N] \quad \text{and} \quad \overset{\blacklozenge * \blacklozenge \blacklozenge}{\Psi}_{N\setminus M,\tau} \overset{\text{def}}{=} \mathrm{I}[1_{[0,\tau]}(t)(\overset{\blacklozenge * \blacklozenge \blacklozenge}{\vee}_N - \overset{\blacklozenge * \blacklozenge \blacklozenge}{\vee}_M)]. \tag{3.4}$$

Proof of Proposition 3.1. We encourage the reader to review the informal discussion of the argument in the introduction before delving into the details of this proof.

Let $\tau \in (0,1)$ be such that $1/2 \leq A\tau^{b_+-b} \leq 1$ and $J \overset{\text{def}}{=} T/\tau \in \mathbb{N}$. We let B_j, D_j, where $1 \leq j \leq J$, be increasing sequences which remain to be chosen. We will prove below that our choice satisfies $B_j \leq B$ and $D_j \leq D$ for all $1 \leq j \leq J$. We then have

$$\begin{aligned} \mathcal{E}_K(B_j,D_j,j\tau) = \bigcap_{N \geq K} \Big(\Big\{ \blacklozenge \in \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3) \colon w_N(t) = \Phi_N(t)\blacklozenge - \overset{\blacklozenge}{\uparrow} - \overset{\blacklozenge * \blacklozenge \blacklozenge}{\Psi}_N \text{ satisfies} \\ \|w_N\|_{\mathfrak{X}^{s_1,b}([0,j\tau])} \leq B_j \text{ and } \sum_{L_1 \sim L_2} \|P_{L_1} \overset{\blacklozenge}{\uparrow} \cdot P_{L_2} w_N\|_{L_t^2 \mathscr{H}_x^{-4\delta_1}([0,j\tau]\times\mathbb{T}^3)} \leq B_j \Big\} \\ \cap \{ \blacklozenge \in \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3) \colon \|\Phi_N[t]\blacklozenge - \Phi_K[t]\blacklozenge\|_{C_t^0 \mathscr{H}_x^{\beta-\kappa}([0,j\tau]\times\mathbb{T}^3)} \leq D_j K^{-\eta'} \} \Big). \end{aligned}$$

We now claim that for all $M \geq K$, under certain constraints on the sequences B_j and D_j detailed below,

$$\mu_M^{\otimes}(\mathcal{E}_K(B_1, D_1, \tau)) \geq 1 - \zeta^{-1}\exp(\zeta A^{\zeta}) \tag{3.5}$$

and

$$\mu_M^{\otimes}(\mathcal{E}_K(B_j, D_j, j\tau)) \geq \mu_M^{\otimes}\big(\mathcal{E}_K(B_{j-1}, D_{j-1}, (j-1)\tau)\big) - \zeta^{-1}\exp(\zeta A^{\zeta}). \tag{3.6}$$

We refer to (3.5) as the base case and to (3.6) as the induction step. We split the rest of the argument into several steps.

Step 1: The base case (3.5). We set $B_1 \overset{\text{def}}{=} A$ and $D_1 \overset{\text{def}}{=} A$. If $\mathcal{L}(A,\tau)$ is as in Proposition 3.3, we obtain $\mathcal{L}(A,\tau) \subseteq \mathcal{E}_K(B_1, D_1, \tau)$. This implies

$$\mu_M^{\otimes}(\mathcal{E}_K(B_1, D_1, \tau)) \geq \mu_M^{\otimes}(\mathcal{L}(A,\tau)) \geq 1 - \zeta^{-1}\exp(\zeta A^{\zeta}).$$

Step 2: The induction step (3.6). We first restrict to the event

$$\mathcal{S}^{\text{gwp}}(A,T,\tau) \overset{\text{def}}{=} \mathcal{L}(A,\tau) \cap \mathcal{L}(A,2\tau) \cap \mathcal{S}^{\text{time}}(A,T,\tau) \cap \mathcal{S}^{\text{cub}}(A,T,\tau) \cap \mathcal{S}^{\text{stab}}(A,T,\tau). \tag{3.7}$$

Using Propositions 3.3, 3.5, 3.7, and 3.8, which also contain the definitions of the sets in (3.7), we obtain

$$\mu_M^{\otimes}(\mathcal{S}^{\text{gwp}}(A,T,\tau)) \geq 1 - \zeta^{-1}\exp(\zeta A^{\zeta}).$$

Using the invariance of $\mu_M^{\otimes}$ under Φ_M, we also obtain

$$\mu_M^{\otimes}\big(\Phi_M[\tau]^{-1}\mathcal{E}_K(B_{j-1}, D_{j-1}, (j-1)\tau)\big) = \mu_M^{\otimes}\big(\mathcal{E}_K(B_{j-1}, D_{j-1}, (j-1)\tau)\big).$$

In order to obtain the probabilistic estimate (3.6), it therefore suffices to prove the inclusion

$$\mathcal{S}^{\text{gwp}}(A,T,\tau) \cap \Phi_M[\tau]^{-1}\mathcal{E}_K(B_{j-1}, D_{j-1}, (j-1)\tau) \subseteq \mathcal{E}_K(B_j, D_j, j\tau). \tag{3.8}$$

For the rest of this proof, we assume that $\blacklozenge \in \mathcal{S}^{\text{gwp}}(A,T,\tau) \cap \Phi_M[\tau]^{-1}\mathcal{E}_K(B_{j-1}, D_{j-1}, (j-1)\tau)$ and $N, M \geq K$. To clarify the structure of the proof, we divide our argument into further substeps.

Step 2.1: Time translation. We rephrase the condition $\lozenge = \Phi_M[\tau]\blacklozenge \in \mathcal{E}_K(B_{j-1}, D_{j-1}, (j-1)\tau)$ in terms of $\blacklozenge$.

Since $\lozenge \in \mathcal{E}_K(B_{j-1}, D_{j-1}, (j-1)\tau)$, we deduce for all $t \in [\tau, j\tau]$ that

$$\Phi_N(t-\tau)\Phi_M[\tau]\blacklozenge = \Phi_N(t-\tau)\lozenge = \linear(t-\tau) + \cubic_N(t-\tau) + w_{N,M}^{\text{grn}}(t-\tau),$$

where $w_{N,M}^{\text{grn}} \colon [0,(j-1)\tau] \times \mathbb{T}^3 \to \mathbb{R}$ satisfies

$$\begin{aligned} &\|w_{N,M}^{\text{grn}}\|_{\mathcal{X}^{s_1,b}([0,(j-1)\tau])} \leq B_{j-1}, \\ &\sum_{L_1 \sim L_2} \|P_{L_1}\linear \cdot P_{L_2} w_{N,M}^{\text{grn}}\|_{L_t^2 H_x^{-4\delta_1}([0,(j-1)\tau]\times\mathbb{T}^3)} \leq B_{j-1}. \end{aligned}$$

The superscript “grn” emphasizes that $w^{\mathrm{grn}}_{N,M}$ appears in the structure involving $\lozenge$. Furthermore, we also have

$$\|\Phi_N[t-\tau]\Phi_M[\tau]\blacklozenge - \Phi_K[t-\tau]\Phi_M[\tau]\blacklozenge\|_{C_t^0\mathcal{H}_x^{\beta-\kappa}([\tau,j\tau]\times\mathbb{T}^3)} \le D_{j-1}K^{-\eta'}. \tag{3.9}$$

Since $\blacklozenge \in \mathcal{S}^{\mathrm{time}}(A,T,\tau)$ (as in Proposition 3.5), it follows for all $t\in[\tau,j\tau]$ that

$$\Phi_N(t-\tau)\Phi_M[\tau]\blacklozenge = \overset{\blacklozenge}{|}(t) + \overset{\blacklozenge\ast\blacklozenge\blacklozenge}{\curlyvee}_{\scriptscriptstyle N}(t) - \overset{\blacklozenge\ast\blacklozenge\blacklozenge}{\curlyvee}_{\scriptscriptstyle N\setminus M,\,\tau}(t) + w_{N,M}(t), \tag{3.10}$$

where $w_{N,M}\colon[\tau,j\tau]\times\mathbb{T}^3\to\mathbb{R}$ satisfies

$$\|w_{N,M}\|_{\mathfrak{X}^{s_1,b}([\tau,j\tau])}, \sum_{L_1\sim L_2}\|P_{L_1}\overset{\blacklozenge}{|}\cdot P_{L_2}w_{N,M}\|_{L_t^2H_x^{-4\delta_1}([\tau,j\tau]\times\mathbb{T}^3)} \le T^{\alpha}AB_{j-1}. \tag{3.11}$$

Our next goal is to replace $\Phi_M[\tau]$ in (3.10) by $\Phi_N[\tau]$, which is done in Steps 2.2 and 2.3.

Step 2.2: The cubic stochastic object. In this step, we correct the structure of $\Phi_N(t-\tau)\Phi_M[\tau]\blacklozenge$, as stated in (3.10), by adding the “partial” cubic stochastic object.

We define $\widetilde{u}_N\colon[\tau,j\tau]\times\mathbb{T}^3\to\mathbb{R}$ by

$$\widetilde{u}_N(t) = \Phi_N(t-\tau)\Phi_M[\tau]\blacklozenge + \overset{\blacklozenge\ast\blacklozenge\blacklozenge}{\curlyvee}_{\scriptscriptstyle N\setminus M,\,\tau}(t) = \overset{\blacklozenge}{|}(t) + \overset{\blacklozenge\ast\blacklozenge\blacklozenge}{\curlyvee}_{\scriptscriptstyle N}(t) + w_{N,M}(t). \tag{3.12}$$

While $\widetilde{u}_N$ depends on M, this is not reflected in our notation. The reason is that, as will be shown below, $\widetilde{u}_N$ is a close approximation of $u_N(t)=\Phi_N(t)\blacklozenge$, which does not directly depend on M. In order to match the notation of $\widetilde{u}_N$, we also define $\widetilde{w}_N = w_{N,M}$, which leads to

$$\widetilde{u}_N(t) = \overset{\blacklozenge}{|}(t) + \overset{\blacklozenge\ast\blacklozenge\blacklozenge}{\curlyvee}_{\scriptscriptstyle N}(t) + \widetilde{w}_N(t).$$

Using $\blacklozenge\in\mathcal{S}^{\mathrm{cub}}(A,T,\tau)$ (as in Proposition 3.7), it follows that there are $H_N\in\mathcal{LM}([\tau,j\tau])$ and $F_N\in\mathfrak{X}^{s_2-1,b_+-1}([\tau,j\tau])$ satisfying the identity

$$\begin{aligned}(-\partial_t^2-1+\Delta)\widetilde{u}_N - P_{\le N}:&(V*(P_{\le N}\widetilde{u}_N)^2)P_{\le N}\widetilde{u}_N:\\ &= -P_{\le N}\,\mathrm{PCtrl}(H_N, P_{\le N}\overset{\blacklozenge}{|}) - F_N\end{aligned} \tag{3.13}$$

and the estimate

$$\|H_N\|_{\mathcal{LM}([\tau,j\tau])}, \|F_N\|_{\mathfrak{X}^{s_2-1,b_+-1}([\tau,j\tau])} \le T^{4\alpha}A^4B_{j-1}^3K^{-\eta'}. \tag{3.14}$$

Thus, $\widetilde{u}_N$ is an approximate solution to the nonlinear wave equation on $[\tau,j\tau]\times\mathbb{T}^3$. Furthermore,

$$\|\widetilde{u}_N[t] - \Phi_N[t-\tau]\Phi_M[\tau]\blacklozenge\|_{C_t^0\mathcal{H}_x^{\beta-\kappa}([\tau,j\tau]\times\mathbb{T}^3)} \le T^{4\alpha}A^4B_{j-1}^3K^{-\eta'}. \tag{3.15}$$

Step 2.3: Stability estimate. In this step, we turn the approximate solution $\widetilde{u}_N$ into an honest solution and fully correct the initial data at $t=\tau$.

We now verify assumptions (A1)–(A5) in Proposition 3.8, where we replace B by $T^{\alpha}AB_{j-1}$ and set $\theta = T^{4\alpha}A^4B_{j-1}^3K^{-\eta'}$. Assumption (A1) holds with $\widetilde{w}_N = w_{N,M}$ due to (3.12); (A2) coincides with the bounds (3.11); and (A3) coincides with (3.13) and (3.14).

For (A4), we rely on $\blacklozenge \in \mathcal{L}(A,\tau)$ (as in Proposition 3.3). First, we have

$$\widetilde{u}_N[\tau] = \Phi_M[\tau]\blacklozenge + \curlyvee_{N\setminus M}^{\tau}[\tau] = \uparrow[\tau] + \curlyvee_{M}[\tau] + \curlyvee_{N\setminus M}^{\tau}[\tau] + w_M[\tau]$$

$$= \uparrow[\tau] + \curlyvee_{N}[\tau] + w_M[\tau].$$

Second,

$$\Phi_N[\tau]\blacklozenge = \uparrow[\tau] + \curlyvee_{N}[\tau] + w_N[\tau].$$

Using (IV) of Proposition 3.3, this implies that $Z_N[\tau] \overset{\text{def}}{=} \Phi_N[\tau] - \widetilde{u}_N[\tau]$ satisfies

$$\|Z_N[\tau]\|_{\mathfrak{X}([0,T],\uparrow;\tau,N,M)} \le AT^{\alpha}K^{-\eta'}, \tag{3.16}$$

which yields (A4). Finally, as long as $B_j \le B$, (A5) follows from the parameter condition (3.2). Thus, assumptions (A1)–(A5) in Proposition 3.8 hold. Since $\blacklozenge \in \mathcal{S}^{\text{stab}}(A,T,\tau)$, we conclude for all $t \in [\tau, j\tau]$ that

$$\Phi_N(t)\blacklozenge = \uparrow(t) + \curlyvee_{N}(t) + w_N(t), \tag{3.17}$$

where the nonlinear component w_N satisfies

$$\|w_N\|_{\mathfrak{X}^{s_1,b}([\tau,j\tau]\times\mathbb{T}^3)}, \sum_{L_1\sim L_2} \|P_{L_1}\uparrow \cdot P_{L_2}w_{N,M}\|_{L_t^2H_x^{-4\delta_1}([\tau,j\tau]\times\mathbb{T}^3)} \le T^{\alpha}AB_{j-1} + 1 \le 2T^{\alpha}AB_{j-1}. \tag{3.18}$$

Furthermore,

$$\|\Phi_N[t]\blacklozenge - \widetilde{u}_N[t]\|_{C_t^0\mathscr{H}_x^{\beta-\kappa}([\tau,j\tau]\times\mathbb{T}^3)} \le C\exp(C(A+B_{j-1}+T)^C)K^{-\eta'}. \tag{3.19}$$

By combining (3.9), (3.15), and (3.19), we obtain

$$\|\Phi_N[t]\blacklozenge - \Phi_K[t-\tau]\Phi_M[\tau]\blacklozenge\|_{C_t^0\mathscr{H}_x^{\beta-\kappa}([\tau,j\tau]\times\mathbb{T}^3)} \le \big(D_{j-1} + T^{4\alpha}A^4B_{j-1}^3 + C\exp(C(A+B_{j-1}+T)^C)\big)K^{-\eta'}. \tag{3.20}$$

By combining the general case $N \ge K$ in (3.20) with the special case $N = K$, using the triangle inequality, and increasing C if necessary, we also obtain

$$\|\Phi_N[t]\blacklozenge - \Phi_K[t]\blacklozenge\|_{C_t^0\mathscr{H}_x^{\beta-\kappa}([\tau,j\tau]\times\mathbb{T}^3)} \le \big(2D_{j-1} + C\exp(C(A+B_{j-1}+T)^C)\big)K^{-\eta'}. \tag{3.21}$$

Step 2.4: Gluing. In this step, we "glue" together our information on $[0, 2\tau]$ (from local well-posedness) and $[\tau, j\tau]$ (from the previous step).

Since $\blacklozenge \in \mathcal{L}(A, 2\tau)$ (as in Proposition 3.3), the function w_N uniquely determined by

$$\Phi_N(t)\,\blacklozenge = \overset{\blacklozenge}{|}(t) + \overset{\blacklozenge\ast\blacklozenge\blacklozenge}{\curlyvee}_{N}(t) + w_N(t)$$

satisfies

$$\|w_N\|_{\mathcal{X}^{s_1,b}([0,2\tau]\times\mathbb{T}^3)} \leq A, \qquad \sum_{L_1\sim L_2} \|P_{L_1}\overset{\blacklozenge}{|}\cdot P_{L_2} w_N\|_{L_t^2 H_x^{-4\delta_1}([0,2\tau]\times\mathbb{T}^3)} \leq A.$$

Furthermore,

$$\|\Phi_N[t]\,\blacklozenge - \Phi_K[t]\,\blacklozenge\|_{C_t^0\mathcal{H}_x^{\beta-\kappa}([0,2\tau]\times\mathbb{T}^3)} \leq AK^{-\eta'}.$$

Together with (3.18), (3.21), and the gluing lemma (Lemma 4.5), which is only needed for the frequency-based $\mathcal{X}^{s_1,b}$-space, we obtain

$$\begin{aligned} \|w_N\|_{\mathcal{X}^{s_1,b}([0,j\tau]\times\mathbb{T}^3)}, \sum_{L_1\sim L_2} \|P_{L_1}\overset{\blacklozenge}{|}\cdot P_{L_2} w_N\|_{L_t^2 H_x^{-4\delta_1}([0,j\tau]\times\mathbb{T}^3)} \\ \leq C\tau^{1/2-b}T^{\alpha}AB_{j-1}, \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} \|\Phi_N[t]\,\blacklozenge - \Phi_K[t]\,\blacklozenge\|_{C_t^0\mathcal{H}_x^{\beta-\kappa}([0,j\tau]\times\mathbb{T}^3)} \\ \leq \big(2D_{j-1} + C\exp(C(A+B_{j-1}+T)^C)\big)K^{-\eta'}. \end{aligned} \tag{3.23}$$

Step 2.5: Choosing B_j and D_j. Based on (3.22) and (3.23), we now define

$$B_j \overset{\text{def}}{=} C\tau^{1/2-b}T^{\alpha}AB_{j-1} \quad \text{and} \quad D_j \overset{\text{def}}{=} 2D_{j-1} + C\exp(C(A+B_{j-1}+T)^C).$$

Step 3: Finishing up. We recall that $1/2 \leq A\tau^{b_+-b} \leq 1$, $J = T/\tau \sim TA^{\frac{1}{b_+-b}}$, $B_1 = A$, and $D_1 = A$. After increasing C if necessary, we obtain

$$B_J \leq C\exp(C(A+T)^C) \leq B \quad \text{and} \quad D_J \leq C\exp(C(A+B_J+T)^C) \leq D. \tag{3.24}$$

This implies $\mathcal{E}(B_J, D_J, J\tau) \subseteq \mathcal{E}_K(B, D, T)$. By iterating (3.6) and using the base case (3.5), we obtain (after decreasing ζ)

$$\mu_M^{\otimes}(\mathcal{E}_K(B,D,T)) \geq \mu_M^{\otimes}(\mathcal{E}_K(B_J,D_J,J\tau)) \geq 1 - T\zeta^{-1}\exp(-\zeta A^{\zeta}).$$

This completes the proof. ∎

In Proposition 3.1, we obtained a quantitative global well-posedness result. In particular, we obtained (almost) explicit bounds on the growth of w_N, which are of independent interest. In the proof of Theorem 1.3, however, a softer statement is sufficient, which we isolate in Corollary 3.2 below.

Corollary 3.2. *Let $T, K \geq 1$ and $\theta > 0$. Define a closed subset of $\mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$ by*

$$\mathcal{S}_K(T,\theta) \overset{\text{def}}{=} \Big\{ \blacklozenge \in \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3) \colon \sup_{N_1,N_2 \geq K} \|\Phi_{N_1}[t]\blacklozenge - \Phi_{N_2}[t]\blacklozenge\|_{C_t^0\mathscr{H}_x^{\beta-\kappa}([-T,T]\times\mathbb{T}^3)} \leq \theta \Big\}. \tag{3.25}$$

Furthermore, we define the event

$$\mathcal{S} \overset{\text{def}}{=} \bigcap_{T\in\mathbb{N}} \bigcap_{\theta\in\mathbb{Q}_{>0}} \bigcup_{K\geq 1} \mathcal{S}_K(T,\theta). \tag{3.26}$$

Then

$$\lim_{K,M\to\infty} \mu_M^{\otimes}(\mathcal{S}_K(T,\theta)) = 1 \quad \textit{and} \quad \mu_\infty^{\otimes}(\mathcal{S}) = 1. \tag{3.27}$$

Proof. We first prove the limit identity. Using the time-reflection symmetry, it suffices to prove the statement with $\mathcal{S}_K(T,\theta)$ replaced by

$$\mathcal{S}_K^{+}(T,\theta) \overset{\text{def}}{=} \Big\{ \blacklozenge \in \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3) \colon \sup_{N_1,N_2 \geq K} \|\Phi_{N_1}[t]\blacklozenge - \Phi_{N_2}[t]\blacklozenge\|_{C_t^0\mathscr{H}_x^{\beta-\kappa}([0,T]\times\mathbb{T}^3)} \leq \theta \Big\}.$$

For any fixed $T, A, B, D \geq 1$ satisfying (3.1) and $\theta > 0$, and for all sufficiently large $K, L \geq 1$ satisfying $K \geq L$, we have

$$\mathcal{S}_K^{+}(T,\theta) \supseteq \mathcal{E}_L(B,D,T),$$

where $\mathcal{E}_L(B,D,T)$ is as in Proposition 3.1. Thus,

$$\lim_{K,M\to\infty} \mu_M^{\otimes}(\mathcal{S}_K(T,\theta)) \geq \liminf_{M\to\infty} \mu_M^{\otimes}(\mathcal{E}_L(B,D,T)) \geq 1 - \zeta^{-1} T \exp(\zeta A^{\zeta}).$$

After letting $A \to \infty$, this yields the first identity in (3.27).

Using Theorem 1.1, we find that a subsequence of $\mu_M^{\otimes}$ converges weakly to $\mu_\infty^{\otimes}$. Since $\mathcal{S}_K(T,\theta)$ is closed, this implies

$$1 = \lim_{K,M\to\infty} \mu_M^{\otimes}(\mathcal{S}_K(T,\theta)) \leq \liminf_{K\to\infty} \mu_\infty^{\otimes}(\mathcal{S}_K(T,\theta)) \leq \mu_\infty^{\otimes}\Big(\bigcup_{K\geq 1} \mathcal{S}_K(T,\theta)\Big).$$

This yields the second identity in (3.27). ∎

3.2. *Invariance*

In this subsection, we complete the proof of Theorem 1.3. The global well-posedness follows from Corollary 3.2 and it remains to prove the invariance. Our argument closely resembles the proof of invariance for the one-dimensional nonlinear Schrödinger equation by Bourgain [4]. The only difference is that we work with the expectation of test functions instead of probabilities of sets, since they are more convenient for weakly convergent measures.

Proof of Theorem 1.3. The global well-posedness follows directly from Corollary 3.2. Thus, it remains to prove the invariance of the Gibbs measure $\mu_\infty^\otimes$.

Let $t \in \mathbb{R}$ be arbitrary. In order to prove that $\Phi_\infty[t]_\# \mu_\infty^\otimes = \mu_\infty^\otimes$, it suffices to prove for all bounded Lipschitz functions $f : \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3) \to \mathbb{R}$ that

$$\mathbb{E}_{\mu_\infty^\otimes}[f(\Phi_\infty[t]\,\blacklozenge)] = \mathbb{E}_{\mu_\infty^\otimes}[f(\blacklozenge)]. \tag{3.28}$$

We first rewrite the left-hand side of (3.28). Using the global well-posedness and dominated convergence, we have

$$\mathbb{E}_{\mu_\infty^\otimes}[f(\Phi_\infty[t]\,\blacklozenge)] = \lim_{N\to\infty} \mathbb{E}_{\mu_\infty^\otimes}[f(\Phi_N[t]\,\blacklozenge)].$$

Using the weak convergence of $\mu_M^\otimes$ to $\mu_\infty^\otimes$ (from Theorem 1.1) and the continuity of $\Phi_N[t]$ (for a fixed N), we have

$$\lim_{N\to\infty} \mathbb{E}_{\mu_\infty^\otimes}[f(\Phi_N[t]\,\blacklozenge)] = \lim_{N\to\infty}\Big(\lim_{M\to\infty} \mathbb{E}_{\mu_M^\otimes}[f(\Phi_N[t]\,\blacklozenge)]\Big).$$

We now turn to the right-hand side of (3.28). Using the weak convergence of $\mu_M^\otimes$ to $\mu_\infty^\otimes$ and the invariance of $\mu_M^\otimes$ under $\Phi_M[t]$, we obtain

$$\mathbb{E}_{\mu_\infty^\otimes}[f(\blacklozenge)] = \lim_{M\to\infty} \mathbb{E}_{\mu_M^\otimes}[f(\blacklozenge)] = \lim_{M\to\infty} \mathbb{E}_{\mu_M^\otimes}[f(\Phi_M[t]\,\blacklozenge)].$$

Combining the last three identities, we can reduce (3.28) to

$$\limsup_{N,M\to\infty} \big|\mathbb{E}_{\mu_M^\otimes}[f(\Phi_N[t]\,\blacklozenge)] - \mathbb{E}_{\mu_M^\otimes}[f(\Phi_M[t]\,\blacklozenge)]\big| = 0. \tag{3.29}$$

We now let $T \geq 1$ be such that $t \in [-T, T]$, let $\theta > 0$, and let $K \geq 1$. We also let $\mathcal{S}_K(T,\theta)$ be as in Corollary 3.2. Then

$$\begin{aligned}
&\limsup_{N,M\to\infty} \big|\mathbb{E}_{\mu_M^\otimes}[f(\Phi_N[t]\,\blacklozenge)] - \mathbb{E}_{\mu_M^\otimes}[f(\Phi_M[t]\,\blacklozenge)]\big| \\
&\quad\leq \sup_{N,M\geq K} \big|\mathbb{E}_{\mu_M^\otimes}[f(\Phi_N[t]\,\blacklozenge)] - \mathbb{E}_{\mu_M^\otimes}[f(\Phi_M[t]\,\blacklozenge)]\big| \\
&\quad\leq \sup_{N,M\geq K} \mathbb{E}_{\mu_M^\otimes}\big[1\{\blacklozenge \in \mathcal{S}_K(T,\theta)\}\big|f(\Phi_N[t]\,\blacklozenge) - f(\Phi_M[t]\,\blacklozenge)\big|\big] \\
&\qquad + \sup_{N,M\geq K} \mathbb{E}_{\mu_M^\otimes}\big[1\{\blacklozenge \notin \mathcal{S}_K(T,\theta)\}\big|f(\Phi_N[t]\,\blacklozenge) - f(\Phi_M[t]\,\blacklozenge)\big|\big] \\
&\quad\leq \mathrm{Lip}(f)\cdot\theta + 2\|f\|_\infty \sup_{M\geq K} \mu_M^\otimes(\mathscr{H}_x^{-1/2-\kappa} \setminus \mathcal{S}_K(T,\theta)).
\end{aligned}$$

In the last line, $\mathrm{Lip}(f)$ is the Lipschitz constant of f and $\|f\|_\infty$ is the supremum of f. Using Corollary 3.2, we obtain the estimate (3.29) by first letting $K\to\infty$ and then letting $\theta \to 0$. ∎

3.3. Structure and stability theory

In this subsection, we provide the ingredients used in the proof of global well-posedness (Proposition 3.1). As described in the introduction, we will further split this subsection into four parts.

3.3.1. Structured local well-posedness. In Proposition 2.10, we obtained a structured local well-posedness result in terms of $\bullet$ and $\mathbb{P}$. In Corollary 2.12, we already used Proposition 2.10 to prove the local existence of the limiting dynamics on the support of the Gibbs measure $\mu_\infty^{\otimes}$, but did not obtain any structural information on the solution. We now remedy this defect, and obtain a structured local well-posedness result even on the support of the Gibbs measure.

The statement of the proposition differs slightly from the earlier Proposition 2.10 for two reasons: First, we formulate the result closer to the assumptions in the stability theory (Propositions 2.14 and 3.8), which is useful in the globalization argument. Second, using the organization of this paper, it would be cumbersome to define the paracontrolled component of $\Phi_N(t)\blacklozenge$ intrinsically through $\blacklozenge$, i.e., without relying on the ambient objects.

Proposition 3.3 (Structured local well-posedness with respect to the Gibbs measure). *Let $A \geq 1$, let $0 < \tau < 1$, let $\alpha > 0$ be a sufficiently large absolute constant, and let $\zeta = \zeta(\epsilon, s_1, s_2, \kappa, \eta, \eta', b_+, b) > 0$ be sufficiently small. Denote by $\blacklozenge$ a generic element of $\mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$ and by $\mathscr{L}(A,\tau)$ the Borel subset of $\mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$ defined by the following conditions:*

(I) *For any $N \geq 1$, the solution of (2.1) with initial data $\blacklozenge$ exists on $[-\tau, \tau]$.*

(II) *For all $N \geq 1$, there exists (a unique) $w_N \in \mathfrak{X}^{s_1,b}([0,\tau])$ such that*

$$\Phi_N(t)\blacklozenge = \overset{\blacklozenge}{|}(t) + \overset{\blacklozenge\ast\blacklozenge\blacklozenge}{\curlyvee}_{N}(t) + w_N(t).$$

Furthermore,

$$\|w_N\|_{\mathfrak{X}^{s_1,b}([0,\tau])} \leq A \quad \textit{and} \quad \sum_{L_1 \sim L_2} \|P_{L_1}\overset{\blacklozenge}{|} \cdot P_{L_2} w_N\|_{L_t^2 H_x^{-4\delta_1}([0,\tau]\times\mathbb{T}^3)} \leq A.$$

(III) *For all $N, K \geq 1$,*

$$\|\Phi_N[t]\blacklozenge - \Phi_K[t]\blacklozenge\|_{C_t^0\mathscr{H}_x^{\beta-\kappa}([0,\tau]\times\mathbb{T}^3)} \leq A\min(N,K)^{-\eta'}.$$

(IV) *For all $N, K, T \geq 1$,*

$$\|w_K[\tau]\|_{\mathfrak{X}([0,T],\overset{\blacklozenge}{|};\tau,N,K)} \leq AT^{\alpha},$$
$$\|w_N[\tau] - w_K[\tau]\|_{\mathfrak{X}([0,T],\overset{\blacklozenge}{|};\tau,N,K)} \leq AT^{\alpha}\min(N,K)^{-\eta'}.$$

If $A\tau^{b_+-b} \leq 1$, then $\mathscr{L}(A,\tau)$ has high probability under $\mu_M^{\otimes}$ for all $M \geq 1$, and

$$\mu_M^{\otimes}(\mathscr{L}(A,\tau)) \geq 1 - \zeta^{-1}\exp(-\zeta A^{\zeta}). \tag{3.30}$$

Remark 3.4. Since we prove multi-linear estimates for $\blacklozenge$ instead of $\bullet$ in Section 9, a different incarnation of this paper may omit Proposition 2.10 and instead prove Proposition 3.3 directly. The author believes that our approach illustrates an interesting conceptual point: The singularity of the Gibbs measure does not enter heavily into the construction of the local limiting dynamics (see Corollary 2.12), but does affect the global theory. We believe, however, that this would be different for the cubic nonlinear wave equation. The reason is an additional renormalization in the construction of the Φ^4_3-model (see e.g. [1, Lemma 5: Step 3]).

We recall that the $\mathfrak{X}$-norm appearing in (IV) is defined in Definition 2.13.

Proof of Proposition 3.3. By using Theorem 1.1 and adjusting the value of ζ, it suffices to prove the probabilistic estimate (3.30) with the Gibbs measure $\mu_M^{\otimes}$ replaced by the reference measure $\nu_M^{\otimes}$. Using the representation of the reference measure from Theorem 1.1, we have

$$\nu_M^{\otimes} = \mathrm{Law}_{\mathbb{P}}(\bullet + \circ_{\mathbf{M}}).$$

By applying this identity to the Borel set $\mathcal{L}(A,\tau)$, we obtain

$$\nu_M^{\otimes}(\mathcal{L}(A,\tau)) = \mathbb{P}(\bullet + \circ_{\mathbf{M}} \in \mathcal{L}(A,\tau)).$$

Let $B = cA^c \leq A$, where $c = c(\epsilon, s_1, s_2, \kappa, \eta, \eta', b_+, b) > 0$ is sufficiently small. Let $\mathcal{L}_M^{\mathrm{amb}}(B,\tau) \subseteq \Omega$ be as in Proposition 2.10. We now show that

$$\mathbb{P}\big(\{\bullet + \circ_{\mathbf{M}} \notin \mathcal{L}(A,\tau)\} \cap \mathcal{L}_M^{\mathrm{amb}}(B,\tau)\big) \leq \tfrac{1}{2}\zeta^{-1}\exp(-\zeta A^{\zeta}). \tag{3.31}$$

Property (i) in Proposition 2.10 directly implies its counterpart. The main part of the argument lies in proving (II). Instead of (II), we currently only have the property

(ii) For all $N \geq 1$, there exist $w'_N \in \mathfrak{X}^{s_1,b}([0,\tau])$, $H'_N \in \mathcal{LM}([0,\tau])$, and $Y'_N \in \mathfrak{X}^{s_2,b}([0,\tau])$ such that for all $t \in [0,\tau]$,

$$\Phi_N(t)\blacklozenge = \overset{\bullet}{|}(t) + \overset{\bullet * \bullet\, \bullet}{\curlyvee}_{N}(t) + w'_N(t),$$
$$w'_N(t) = P_{\leq N}\, \mathrm{I}[\mathrm{PCtrl}(H'_N, P_{\leq N}\overset{\bullet}{|})](t) + Y'_N(t).$$

Furthermore, we have the bounds

$$\|w'_N\|_{\mathfrak{X}^{s_1,b}([0,\tau])}, \|H'_N\|_{\mathcal{LM}([0,\tau])}, \|Y'_N\|_{\mathfrak{X}^{s_2,b}([0,\tau])} \leq B,$$
$$\sum_{L_1 \sim L_2} \|P_{L_1}\overset{\bullet}{|} \cdot P_{L_2} w'_N\|_{L_t^2 H_x^{-4\delta_1}([0,\tau]\times\mathbb{T}^3)} \leq B.$$

Comparing (ii) and (II) forces us to take

$$w_N = \overset{\bullet}{|} - \overset{\blacklozenge}{|} + \overset{\bullet * \bullet\, \bullet}{\curlyvee}_{N} - \overset{\blacklozenge * \blacklozenge\, \blacklozenge}{\curlyvee}_{N} + w'_N. \tag{3.32}$$

We now have to prove that the right-hand side of (3.32) satisfies the estimates in (II). Due to the decomposition $\blacklozenge = \bullet + \circ_{\mathbf{M}}$, we have

$$\overset{\bullet}{|} - \overset{\blacklozenge}{|} = -\overset{\circ}{|}.$$

Using Theorem 1.1, we see that outside a set of probability $\lesssim \exp(-cB^{2/k})$ under $\mathbb{P}$,

$$\| \uparrow^{\circ} \|_{\mathfrak{X}^{s_2,b}([0,\tau])} \lesssim B.$$

Using Proposition 9.1, we find that outside an event with probability $\leq \zeta^{-1}\exp(-\zeta B^{\zeta})$ under $\mathbb{P}$,

$$\Upsilon^{\bullet\ast\bullet\bullet}_{N} = \Upsilon^{\blacklozenge\ast\blacklozenge\blacklozenge}_{N} + I\big[\mathrm{PCtrl}(H_N^{(3)}, P_{\leq N}\uparrow^{\blacklozenge})\big] + Y_N^{(3)}, \tag{3.33}$$

where $H_N^{(3)} \in \mathscr{LM}([0,\tau])$ and $Y_N^{(3)} \in \mathfrak{X}^{s_2,b}([0,\tau])$ satisfy

$$\| H_N^{(3)} \|_{\mathscr{LM}([0,\tau])} \leq B \quad \text{and} \quad \| Y_N^{(3)} \|_{\mathfrak{X}^{s_2,b}([0,\tau])} \leq B.$$

To ease the reader's mind, we mention that the proof of Proposition 9.1 is based on the algebraic identity

$$\Upsilon^{\blacklozenge\ast\blacklozenge\blacklozenge}_{N} = \Upsilon^{\bullet\ast\bullet\bullet}_{N} + 2\,\Upsilon^{\bullet\ast\circ\bullet}_{N} + \Upsilon^{\circ\ast\circ\bullet}_{N} + \Upsilon^{\bullet\ast\bullet\circ}_{N} + 2\,\Upsilon^{\bullet\ast\circ\circ}_{N} + \Upsilon^{\circ\ast\circ\circ}_{N},$$

which uses mixed cubic stochastic objects. Finally, we have

$$\begin{aligned} w'_N &= P_{\leq N}\,\mathrm{I}[\mathrm{PCtrl}(H'_N, P_{\leq N}\uparrow^{\bullet})] + Y'_N \\ &= P_{\leq N}\,\mathrm{I}[\mathrm{PCtrl}(H'_N, P_{\leq N}\uparrow^{\blacklozenge})] - P_{\leq N}\,\mathrm{I}[\mathrm{PCtrl}(H'_N, P_{\leq N}\uparrow^{\circ})] + Y'_N. \end{aligned}$$

Using the inhomogeneous Strichartz estimate (Lemma 4.9) and Lemma 7.3, we find that

$$\begin{aligned} \| P_{\leq N}\,\mathrm{I}[\mathrm{PCtrl}(H'_N, P_{\leq N}\uparrow^{\circ})] \|_{\mathfrak{X}^{s_2,b}([0,\tau])} &\lesssim \| \mathrm{PCtrl}(H'_N, P_{\leq N}\uparrow^{\circ}) \|_{L_t^\infty H_x^{s_2-1}([0,\tau]\times\mathbb{T}^3)} \\ &\lesssim \| H'_N \|_{\mathscr{LM}([0,\tau])} \| \uparrow^{\circ} \|_{L_t^\infty H_x^{s_2-1+8\epsilon}([0,\tau]\times\mathbb{T}^3)} \lesssim B^2. \end{aligned}$$

Thus,

$$w_N = P_{\leq N}\,\mathrm{I}[\mathrm{PCtrl}(H_N, P_{\leq N}\uparrow^{\blacklozenge})] + Y_N, \tag{3.34}$$

where

$$H_N = H'_N + H_N^{(3)} \quad \text{and} \quad Y_N = Y'_N - \uparrow^{\circ} + Y_N^{(3)} - P_{\leq N}\,\mathrm{I}[\mathrm{PCtrl}(H'_N, P_{\leq N}\uparrow^{\circ})]$$

satisfy $\| H_N \|_{\mathscr{LM}([0,\tau])}, \| Y_N \|_{\mathfrak{X}^{s_2,b}([0,\tau])} \lesssim B^2$. Using Lemma 9.8, we also obtain

$$\| w_N \|_{\mathfrak{X}^{s_1,b}([0,\tau])} \lesssim B^5 \quad \text{and} \quad \sum_{L_1\sim L_2} \| P_{L_1}\uparrow^{\blacklozenge} \cdot P_{L_2} w_N \|_{L_t^2 H_x^{-4\delta_1}([0,\tau]\times\mathbb{T}^3)} \lesssim B^5.$$

Inserting our choice of B, this completes the proof of (II).

Statement (III) directly follows from (iii) of Proposition 2.10. It now remains to prove (IV). We emphasize that $T \geq 1$ is arbitrary, which will be useful in the stability theory below. We focus on the estimate for the difference, since the proof of the estimate for $w_K[\tau]$ is easier (but similar). Using Lemma 9.9, we may restrict to $\bullet \in \Theta^{\mathrm{sp}}_{\mathrm{blue}}(B,T)$ and

$\circ \in \Theta^{\mathrm{sp}}_{\mathrm{red}}(B,T)$. Then we can replace the estimates in $\mathfrak{X}([0,T], \blacklozenge_{\text{purple}}; t_0, N, K)$ by estimates in $\mathfrak{X}([0,T], \blacklozenge_{\text{blue}}; t_0, N, K)$. After rearranging (3.34), we have

$$w_N = P_{\leq N}\, \mathrm{I}[\mathrm{PCtrl}(H_N' + H_N^{(3)}, P_{\leq N} \blacklozenge_{\text{blue}})] + Y_N' - \circ + Y_N^{(3)} + P_{\leq N}\, \mathrm{I}[\mathrm{PCtrl}(H_N^{(3)}, P_{\leq N} \circ)].$$

Thus, we obtain

$$w_N[\tau] - w_K[\tau] = Z^{\Box}_{N,K}[\tau] + Z^{\circ}_{N,K}[\tau],$$

where

$$\begin{aligned} Z^{\Box}_{N,K}[\tau] &\overset{\text{def}}{=} P_{\leq N}\, \mathrm{I}[\mathrm{PCtrl}(H_N' + H_N^{(3)}, P_{\leq N} \blacklozenge_{\text{blue}})][\tau] \\ &\quad - P_{\leq K}\, \mathrm{I}[\mathrm{PCtrl}(H_K' + H_K^{(3)}, P_{\leq K} \blacklozenge_{\text{blue}})][\tau] \end{aligned}$$

and

$$\begin{aligned} Z^{\circ}_{N,K}[\tau] &\overset{\text{def}}{=} Y_N' - Y_K' + Y_N^{(3)} - Y_K^{(3)} \\ &\quad + P_{\leq N}\, \mathrm{I}[\mathrm{PCtrl}(H_N^{(3)}, P_{\leq N} \circ)] - P_{\leq K}\, \mathrm{I}[\mathrm{PCtrl}(H_K^{(3)}, P_{\leq K} \circ)]. \end{aligned}$$

The desired estimate then follows from the frequency-localized version of the multi-linear master estimate (Proposition 2.8), (iii) in Proposition 2.10, and Proposition 9.1. ■

3.3.2. Structure and time translation. In the globalization argument, we use the invariance of the truncated Gibbs measures under the truncated flows to transform our bounds from the time interval $[0,(j-1)\tau]$ to $[\tau, j\tau]$. As the reader saw in the proof of Proposition 3.1, however, the structural bounds are now phrased in terms of $\lozenge = \Phi_M[\tau] \blacklozenge$. The next proposition translates the structural bounds back into $\blacklozenge$.

Proposition 3.5 (Structure and time-translation). *Let $A, T \geq 1$, let $0 < \tau \leq 1$, let $j \in \mathbb{N}$ satisfy $j\tau \leq T$, let $\alpha > 0$ be a sufficiently large absolute constant, and let $\zeta = \zeta(\epsilon, s_1, s_2, \kappa, \eta, \eta', b_+, b) > 0$ be sufficiently small. There exists a Borel set $\mathcal{S}^{\mathrm{time}}(A,T,\tau) \subseteq \mathcal{L}(A,\tau)$ satisfying*

$$\mu_M^{\otimes}(\mathcal{S}^{\mathrm{time}}(A,T,\tau)) \geq 1 - \zeta^{-1}\exp(-\zeta A^{\zeta}) \tag{3.35}$$

for all $M \geq 1$ and such that the following holds for all $\blacklozenge \in \mathcal{S}^{\mathrm{time}}(A,T,\tau)$:

Let $N, K, B \geq 1$, and define $\lozenge = \Phi_K[\tau]\blacklozenge$. Let $w^{\mathrm{grn}}_{N,K} \in \mathfrak{X}^{s_1,b}([0,(j-1)\tau])$ satisfy

(A1) (Global structured bounds in $\lozenge$)

$$\begin{aligned} &\|w^{\mathrm{grn}}_{N,K}\|_{\mathfrak{X}^{s_1,b}([0,(j-1)\tau])} \leq B, \\ &\sum_{L_1 \sim L_2} \|P_{L_1} \lozenge \cdot P_{L_2} w^{\mathrm{grn}}_{N,K}\|_{L_t^2 H_x^{-4\delta_1}([0,(j-1)\tau]\times\mathbb{T}^3)} \leq B. \end{aligned}$$

Define $w_{N,K} \colon [\tau, j\tau] \times \mathbb{T}^3 \to \mathbb{R}$ through the identity

$$\lozenge(t-\tau) + (\lozenge\lozenge\lozenge)_N(t-\tau) + w^{\mathrm{grn}}_{N,K}(t-\tau) = \blacklozenge(t) + (\blacklozenge\blacklozenge\blacklozenge)_N(t) - (\blacklozenge\blacklozenge\blacklozenge)_{N\setminus K,\tau}(t) + w_{N,K}(t). \tag{3.36}$$

Then we obtain the following conclusion regarding $w_{N,K}$:

(C1) *Incomplete structured global bounds in* $\blacklozenge$*:*

$$\begin{aligned}&\|w_{N,K}\|_{\mathfrak{X}^{s_1,b}([\tau,j\tau])} \leq T^{\alpha} AB,\\ &\sum_{L_1\sim L_2} \|P_{L_1}\overset{\blacklozenge}{|}\cdot P_{L_2} w_{N,K}\|_{L_t^2 H_x^{-4\delta_1}([\tau,j\tau]\times\mathbb{T}^3)} \leq T^{\alpha} AB.\end{aligned}$$

Remark 3.6. The superscript "grn" in $w_{N,K}^{\mathrm{grn}}$ stands for "green", which is motivated by the identity (3.36). We refer in the conclusion to "incomplete structured global bounds" since the right-hand side in (3.36) does not yet have the desired form. The partial cubic stochastic object

$$\overset{\blacklozenge\ast\blacklozenge\blacklozenge}{\underset{\uparrow\,\tau}{\curlyvee}}_{N\setminus K}$$

is subtracted from it and hence we regard the structure as incomplete.

Proof of Proposition 3.5. Before we turn to the analytical and probabilistic estimates, we discuss the definition and Borel-measurability of $\mathcal{S}^{\mathrm{time}}(A,T,\tau)$. We let $\mathcal{S}^{\mathrm{time}}(A,T,\tau)$ be the intersection of $\mathcal{L}(A,\tau)$ with the set of $\blacklozenge \in \mathscr{H}_x^{-1/2-\kappa}$ satisfying the implication (A1)→(C1) for all N, K, B, and $w_{N,K}^{\mathrm{grn}}$. For fixed parameters and a fixed function $w_{N,K}^{\mathrm{grn}}$, the set of $\blacklozenge \in \mathscr{H}_x^{-1/2-\kappa}$ satisfying (A1) and/or (C1) is closed and hence Borel-measurable. Using a separability argument, it suffices to require the implication (A1)→(C1) for countably many $w_{N,K}^{\mathrm{grn}}$, which yields the measurability of $\mathcal{S}^{\mathrm{time}}(A,T,\tau)$.

We now turn to the analytical and probabilistic estimates. If $\blacklozenge \in \mathcal{L}(A,\tau)$, it follows from (II) and (IV) of Proposition 3.3 that

$$\lozenge = \overset{\blacklozenge}{|}[\tau] + \overset{\blacklozenge\ast\blacklozenge\blacklozenge}{\underset{\uparrow}{\curlyvee}}_{K}[\tau] + Z_K[\tau],$$

where the remainder $Z_K[\tau]$ satisfies

$$\|Z_K[\tau]\|_{\mathfrak{X}([0,T],\overset{\blacklozenge}{|};\tau,N,K)} \leq A T^{\alpha}.$$

By applying the linear propagator to $\lozenge$ we obtain, for all $t \geq \tau$,

$$\overset{\lozenge}{|}(t-\tau) = \overset{\blacklozenge}{|}(t) + \overset{\blacklozenge\ast\blacklozenge\blacklozenge}{\underset{\uparrow\,\tau}{\curlyvee}}_{K}(t) + Z_K(t), \tag{3.37}$$

where we recall from (3.4) that

$$\overset{\blacklozenge\ast\blacklozenge\blacklozenge}{\underset{\uparrow\,\tau}{\curlyvee}}_{K}(t) = \mathrm{I}[1_{[0,\tau]}\overset{\blacklozenge\ast\blacklozenge\blacklozenge}{\curlyvee}_{K}](t).$$

Regarding the cubic stochastic object, we have

$$\begin{aligned}\overset{\lozenge\ast\lozenge\lozenge}{\underset{\uparrow}{\curlyvee}}_{N}(t-\tau) &= \mathrm{I}[1_{[\tau,\infty)}\overset{\lozenge\ast\lozenge\lozenge}{\curlyvee}_{N}(\cdot-\tau)](t)\\ &= \mathrm{I}[1_{[\tau,\infty)}\overset{\blacklozenge\ast\blacklozenge\blacklozenge}{\curlyvee}_{N}](t) + \mathrm{I}[1_{[\tau,\infty)}(\overset{\lozenge\ast\lozenge\lozenge}{\curlyvee}_{N}(\cdot-\tau) - \overset{\blacklozenge\ast\blacklozenge\blacklozenge}{\curlyvee}_{N}(\cdot))](t)\end{aligned} \tag{3.38}$$

Combining the algebraic identity

$$\mathrm{I}[1_{[0,\tau]} \mathbin{\text{(cubic tree)}}_{K}](t) + \mathrm{I}[1_{[\tau,\infty)} \mathbin{\text{(cubic tree)}}_{N}](t) = \text{(cubic object)}_{N}(t) - \text{(cubic object)}_{N\setminus K,\tau}(t)$$

with (3.37) and (3.38), it follows that

$$w_{N,K}(t) = w^{\mathrm{grn}}_{N,K}(t) + Z_K(t) + \mathrm{I}[1_{[\tau,\infty)}(\text{(cubic tree, open)}_{N}(\cdot - \tau) - \text{(cubic tree)}_{N}(\cdot))](t). \tag{3.39}$$

Equipped with the identity (3.39) for $w_{N,K}$, it remains to prove (C1) on an event satisfying (3.35). The second and third summands in (3.39) can be treated using Lemma 9.8, Proposition 9.12 (combined with (3.37)), and Lemma 9.13. Thus, it remains to prove (C1) for the first summand in (3.39). Using (3.37), we have

$$\begin{aligned}
&\sum_{L_1 \sim L_2} \| P_{L_1} \text{(linear object)}(t) \cdot P_{L_2} w^{\mathrm{grn}}_{N,K}(t-\tau) \|_{L_t^2 H_x^{-4\delta_1}([\tau, j\tau]\times \mathbb{T}^3)} \\
&\quad \leq \sum_{L_1 \sim L_2} \| P_{L_1} \text{(linear object, open)}(t) \cdot P_{L_2} w^{\mathrm{grn}}_{N,K}(t) \|_{L_t^2 H_x^{-4\delta_1}([0,(j-1)\tau]\times \mathbb{T}^3)} && (3.40) \\
&\qquad + \sum_{L_1 \sim L_2} \Big\| P_{L_1} \text{(cubic object)}_{K,\tau}(t) \cdot P_{L_2} w^{\mathrm{grn}}_{N,K}(t-\tau) \Big\|_{L_t^2 H_x^{-4\delta_1}([\tau, j\tau]\times \mathbb{T}^3)} && (3.41) \\
&\qquad + \sum_{L_1 \sim L_2} \| P_{L_1} Z_K(t) \cdot P_{L_2} w^{\mathrm{grn}}_{N,K}(t-\tau) \|_{L_t^2 H_x^{-4\delta_1}([\tau, j\tau]\times \mathbb{T}^3)}. && (3.42)
\end{aligned}$$

The first term (3.40) can be bounded using assumption (A1); the second term (3.41) is bounded by Corollary 9.3; and the third term (3.42) is bounded by Lemma 8.8. ■

3.3.3. Structure and the cubic stochastic object. In Proposition 3.5 above, the right-hand side of (3.36) does not have the desired structure. In the next proposition, we will show that adding the "partial" cubic stochastic object $\text{(cubic object)}_{N\setminus K,\tau}$ only leads to a small error in the nonlinear wave equation.

Proposition 3.7 (Structure and the cubic stochastic object). *Let $T, A \geq 1$, let $0 < \tau < 1$, let $\alpha > 0$ be a sufficiently large absolute constant, and let $\zeta = \zeta(\epsilon, s_1, s_2, \kappa, \eta, \eta', b_+, b) > 0$ be sufficiently small. Then there exists a Borel set $\mathcal{S}^{\mathrm{cub}}(A, T, \tau) \subseteq \mathcal{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$ satisfying*

$$\mu_M^{\otimes}(\mathcal{S}^{\mathrm{cub}}(A, T, \tau)) \geq 1 - \zeta^{-1} \exp(-\zeta A^{\zeta})$$

for all $M \geq 1$ and such that the following holds for all $\blacklozenge \in \mathcal{S}^{\mathrm{cub}}(A, T, \tau)$:

Let $N, K, B \geq 1$, let $j \in \mathbb{N}$, let $\mathcal{J} = [\tau, j\tau] \subseteq [0, T]$, and let $u_{N,K} \colon \mathcal{J} \times \mathbb{T}^3 \to \mathbb{R}$. Furthermore, we make the following assumptions:

(A1) (Incomplete structure) *There exists a $w_{N,K}(t) \in \mathcal{X}^{s_1,b}(\mathcal{J})$ satisfying for all $t \in \mathcal{J}$ the identity*

$$u_{N,K}(t) = \text{(linear object)}(t) + \text{(cubic object)}_{N}(t) - \text{(cubic object)}_{N\setminus K,\tau}(t) + w_{N,K}(t).$$

(A2) (Incomplete structured global bounds)

$$\|w_{N,K}\|_{\mathcal{X}^{s_1,b}(\mathcal{J})} \leq B \quad \textit{and} \quad \sum_{L_1\sim L_2} \|P_{L_1} \overset{\blacklozenge}{|} \cdot P_{L_2} w_{N,K}\|_{L_t^2 H_x^{-4\delta_1}(\mathcal{J}\times\mathbb{T}^3)} \leq B.$$

Define a function $\widetilde{u}_N : \mathcal{J} \times \mathbb{T}^3 \to \mathbb{R}$ *by*

$$\widetilde{u}_N(t) = u_{N,K}(t) + \overset{\blacklozenge\ast\blacklozenge\blacklozenge}{\Upsilon}_{N\setminus K}^{\tau}(t).$$

Then $\widetilde{u}_N$ *satisfies the following three properties:*

(C1) (Structure) *For all* $t \in \mathcal{J}$,

$$\widetilde{u}_N(t) = \overset{\blacklozenge}{|}(t) + \overset{\blacklozenge\ast\blacklozenge\blacklozenge}{\Upsilon}_{N}(t) + \widetilde{w}_N(t), \quad \textit{where} \quad \widetilde{w}_N = w_{N,K}.$$

(C2) (Approximate solution) *There exist* $H_N \in \mathcal{LM}(\mathcal{J})$ *and* $F_N \in \mathcal{X}^{s_2-1,b_+-1}(\mathcal{J})$ *satisfying*

$$\begin{aligned}(-\partial_t^2 - 1 + \Delta)\widetilde{u}_N &- P_{\leq N} :(V * (P_{\leq N}\widetilde{u}_N)^2) P_{\leq N}\, \widetilde{u}_N: \\ &= (-\partial_t^2 - 1 + \Delta)u_{N,K} - P_{\leq N} :(V * (P_{\leq N}u_{N,K})^2) P_{\leq N} u_{N,K}: \\ &\quad - P_{\leq N}\, \mathrm{PCtrl}(H_N, P_{\leq N} \overset{\blacklozenge}{|}) - F_N\end{aligned}$$

and

$$\|H_N\|_{\mathcal{LM}(\mathcal{J})}, \|F_N\|_{\mathcal{X}^{s_2-1,b_+-1}(\mathcal{J})} < T^{\alpha} A B^3 \min(N,K)^{-\eta'}.$$

(C3) (Closeness)

$$\|\widetilde{u}_N[t] - u_{N,K}[t]\|_{C_t^0\mathcal{H}_x^{\beta-\kappa}(\mathcal{J}\times\mathbb{T}^3)} < T^{\alpha} A B^3 \min(N,K)^{-\eta'}.$$

Proof. We simply choose $\mathcal{S}^{\mathrm{cub}}(A,T,\tau)$ as the set of all $\blacklozenge \in \mathcal{H}_x^{-1/2-\kappa}$ where the implication (A1)&(A2) →(C1)&(C2)&(C3) holds for all N, K, B, j, and $w_{N,K}$. As in the proof of Proposition 3.5, a separability argument yields the Borel measurability of $\mathcal{S}^{\mathrm{cub}}(A,T,\tau)$. We now show that $\mathcal{S}^{\mathrm{cub}}(A,T,\tau)$ satisfies the desired probabilistic estimate. The first conclusion (C1) follows directly from the definition of $\widetilde{u}_N$. We now turn to the second conclusion, which is the main part of the argument. First, we recall that $\overset{\blacklozenge\ast\blacklozenge\blacklozenge}{\Upsilon}_{N\setminus K}^{\tau}$ solves the linear wave equation on $\mathcal{J} = [\tau, j\tau]$. Together with the definition of $\widetilde{u}_N$, this implies

$$\begin{aligned}&(-\partial_t^2 - 1 + \Delta)\widetilde{u}_N - P_{\leq N} :(V * (P_{\leq N}\widetilde{u}_N)^2) P_{\leq N}\widetilde{u}_N: \\ &\qquad - \big((-\partial_t^2 - 1 + \Delta)u_{N,K} - P_{\leq N} :(V * (P_{\leq N}u_{N,K})^2) P_{\leq N} u_{N,K}:\big) \\ &\quad = P_{\leq N} :\Big(V * \Big(P_{\leq N}u_{N,K} + P_{\leq N} \overset{\blacklozenge\ast\blacklozenge\blacklozenge}{\Upsilon}_{N\setminus K}^{\tau}\Big)^2\Big) P_{\leq N}\Big(u_{N,K} + \overset{\blacklozenge\ast\blacklozenge\blacklozenge}{\Upsilon}_{N\setminus K}^{\tau}\Big): \\ &\qquad - P_{\leq N} :(V * (P_{\leq N}u_{N,K})^2) P_{\leq N} u_{N,K}:.\end{aligned}$$

We emphasize that in the cubic stochastic object $\Psi^{N\setminus K}_{\tau}$, the linear evolution $\uparrow$ enters at a frequency $\gtrsim \min(N, K)$ in at least one of the arguments. Using the frequency-localized version of the multi-linear master estimate for Gibbsian initial data (Proposition 9.12), we obtain the conclusion (C2).

Finally, (C3) directly follows from the frequency-localized version of Proposition 9.1.

∎

3.3.4. Stability theory. The last ingredient for the globalization argument is a stability estimate. The proof will rely on our previous stability estimate for Gaussian random data from Proposition 2.14. As a result, the argument closely resembles a similar step in the local theory, where we proved Proposition 3.3 through Proposition 2.10.

Proposition 3.8 (Stability estimate). *Let $T, A \geq 1$, let $0 < \tau \leq 1$, and let $\zeta = \zeta(\epsilon, s_1, s_2, \kappa, \eta, \eta', b_+, b) > 0$ be sufficiently small. There exists a constant $C = C(\epsilon, s_1, s_2, b_+, b_-)$ and a Borel set $\mathcal{S}^{\text{stab}}(A, T, \tau) \subseteq \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$ satisfying*

$$\mu_M^{\otimes}(\blacklozenge \in \mathcal{S}^{\text{stab}}_{\text{blue}}(A, T, \tau)) \geq 1 - \zeta^{-1} \exp(-\zeta A^{\zeta}) \tag{3.43}$$

such that the following holds for all $\blacklozenge \in \mathcal{S}^{\text{stab}}(A, T, \tau)$:

Let $N, B \geq 1$, $0 < \theta < 1$, and let $\mathcal{J} = [t_0, t_1] \subseteq [0, T]$, where $t_0, t_1 \in \tau\mathbb{Z}$. Let $\tilde{u}_N \colon J \times \mathbb{T}^3 \to \mathbb{R}$ be an approximate solution of (2.1) satisfying the following assumptions:

(A1) (Structure) *We have the decomposition*

$$\tilde{u}_N = \uparrow + \Psi^{N} + \tilde{w}_N.$$

(A2) (Global bounds)

$$\|\tilde{w}_N\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} \leq B \quad \text{and} \quad \sum_{L_1 \sim L_2} \|P_{L_1} \uparrow \cdot P_{L_2} \tilde{w}_N\|_{L_t^2 H_x^{-4\delta_1}(\mathcal{J} \times \mathbb{T}^3)} \leq B.$$

(A3) (Approximate solution) *There exist $H_N \in \mathscr{LM}(\mathcal{J})$ and $F_N \in \mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})$ satisfying the identity*

$$\begin{aligned}(-\partial_t^2 - 1 + \Delta)\tilde{u}_N = {} & P_{\leq N} :\!(V * (P_{\leq N}\tilde{u}_N)^2) P_{\leq N}\tilde{u}_N\!: \\ & - P_{\leq N} \operatorname{PCtrl}(H_N, P_{\leq N} \uparrow) - F_N\end{aligned}$$

and the estimates

$$\|H_N\|_{\mathscr{LM}(\mathcal{J})} < \theta \quad \text{and} \quad \|F_N\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})} < \theta.$$

Furthermore, let $Z_N[t_0] \in H_x^{s_1}(\mathbb{T}^3)$ be a perturbation satisfying the following assumption:

(A4) (Structured perturbation) *There exists a $K \geq 1$ such that*

$$\|Z[t_0]\|_{\mathfrak{Z}(\mathcal{J}, \uparrow; t_0, N, K)} \leq \theta.$$

Finally, assume that

(A5) (Parameter condition) $C\exp(C(A+B+T)^C)\theta \leq 1$.

Then there exists a solution $u_N\colon J\times\mathbb{T}^3\to\mathbb{R}$ *of* (2.1) *satisfying the initial value condition* $u_N[t_0]=\widetilde{u}_N[t_0]+Z_N[t_0]$ *and the following conclusions:*

(C1) (Preserved structure) *We have the decomposition*

$$u_N = \overset{\blacklozenge}{\mid} + \overset{\blacklozenge\ast\blacklozenge\;\blacklozenge}{\curlyvee}_{N} + w_N.$$

(C2) (Closeness) *The difference* $u_N-\widetilde{u}_N = w_N-\widetilde{w}_N$ *satisfies*

$$\|u_N-\widetilde{u}_N\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} \leq C\exp(C(A+B+T)^C)\theta,$$
$$\sum_{L_1\sim L_2}\|P_{L_1}\overset{\blacklozenge}{\mid}\cdot P_{L_2}(u_N-\widetilde{u}_N)\|_{L_t^2H_x^{-4\delta_1}(\mathcal{J}\times\mathbb{T}^3)} \leq C\exp(C(A+B+T)^C)\theta.$$

(C3) (Preserved global bounds)

$$\|w_N\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})}\leq B_\theta \quad \textit{and} \quad \sum_{L_1\sim L_2}\|P_{L_1}\overset{\blacklozenge}{\mid}\cdot P_{L_2}w_N\|_{L_t^2H_x^{-4\delta_1}(\mathcal{J}\times\mathbb{T}^3)}\leq B_\theta,$$

where $B_\theta \overset{\text{def}}{=} B + C\exp(C(A+B+T)^C)\theta$.

Proof. As in the proof of Proposition 3.7, we can define $\mathcal{S}^{\text{stab}}(A,T,\tau)$ through the implications (A1)–(A5) $\to$ (C1)–(C3) and prove its measurability using a separability argument.

It remains to prove the probabilistic estimate (3.43). Using Theorem 1.1, it suffices to prove that

$$\mathbb{P}(\bullet + \circ_{\mathbf{M}} \in \mathcal{S}^{\text{stab}}(A,T,\tau)) \geq 1-\zeta^{-1}\exp(\zeta A^{\zeta}).$$

Using Lemma 2.6, Corollary 9.3, Proposition 2.14, Lemma 2.6, and Lemma 9.9, which also contain the definitions of the sites below, we may restrict to the event

$$\{\bullet\in\Theta^{\text{type}}_{\text{blue}}(A,T)\cap\Theta^{\text{stab}}_{\text{blue}}(A,T)\cap\Theta^{\text{cub}}_{\text{blue}}(A,T)\cap\Theta^{\text{sp}}_{\text{blue}}(A,T)\}$$
$$\cap\{\circ_{\mathbf{M}}\in\Theta^{\text{type}}_{\text{red}}(A,T)\cap\Theta^{\text{sp}}_{\text{red}}(A,T)\}. \tag{3.44}$$

Our goal is to use Proposition 2.14 (with slightly adjusted parameters). To this end, we need to convert assumptions (A1)–(A5) involving $\blacklozenge$ into similar statements based on $\bullet$. We let $D>0$ be a large implicit (but absolute) constant, which may change its value between different lines. We now let $N, B, \theta, \mathcal{J}, \widetilde{u}_N, \widetilde{w}_N, H_N, F_N$, and $Z_N[t_0]$ be as in (A1)–(A5). We then define $w_N[\blacklozenge\to\bullet]$ by

$$\overset{\blacklozenge}{\mid} + \overset{\blacklozenge\ast\blacklozenge\;\blacklozenge}{\curlyvee}_{N} = \overset{\bullet}{\mid} + \overset{\bullet\ast\bullet\;\bullet}{\curlyvee}_{N} + w_N[\blacklozenge\to\bullet],$$

which implies

$$\widetilde{u}_N = \overset{\bullet}{\mid} + \overset{\bullet\ast\bullet\;\bullet}{\curlyvee}_{N} + w_N[\blacklozenge\to\bullet] + \widetilde{w}_N.$$

Using Corollary 9.3 and Lemma 9.7, we obtain

$$\|\widetilde{w}_N\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} \leq B \quad \text{and} \quad \sum_{L_1 \sim L_2} \|P_{L_1} \overset{\bullet}{|} \cdot P_{L_2} \widetilde{w}_N\|_{L_t^2 H_x^{-4\delta_1}(\mathcal{J} \times \mathbb{T}^3)} \leq T^\alpha A^D B$$

as well as

$$\begin{aligned} &\|w_N[\blacklozenge \to \bullet]\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} \leq T^\alpha A^D, \\ &\sum_{L_1 \sim L_2} \|P_{L_1} \overset{\bullet}{|} \cdot P_{L_2} w_N[\blacklozenge \to \bullet]\|_{L_t^2 H_x^{-4\delta_1}(\mathcal{J} \times \mathbb{T}^3)} \leq T^\alpha A^D. \end{aligned}$$

Thus, (A2) in Proposition 2.14 is satisfied with $B' = 2T^\alpha A^D B$. A similar argument based on Lemmas 9.7 and 9.9 also yields (A3) and (A4) in Proposition 2.14 with $\theta' = 2T^\alpha A^D B$. Furthermore, the stronger assumption (A5) in this proposition implies (as long as C is sufficiently large) that

$$C \exp\Big(C(A + B')^{\frac{2}{b_+ - b}} T^{\frac{40}{b_+ - b}}\Big)\theta' \leq 1.$$

Thus, Proposition 2.14 implies that

$$\begin{aligned} &\|u_N - \widetilde{u}_N\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} \leq C \exp\Big(C(A + B')^{\frac{2}{b_+ - b}} T^{\frac{40}{b_+ - b}}\Big)\theta', \\ &\sum_{L_1 \sim L_2} \|P_{L_1} \overset{\bullet}{|} \cdot P_{L_2}(u_N - \widetilde{u}_N)\|_{L_t^2 H_x^{-4\delta_1}(\mathcal{J} \times \mathbb{T}^3)} \leq C \exp\Big(C(A + B')^{\frac{2}{b_+ - b}} T^{\frac{40}{b_+ - b}}\Big)\theta'. \end{aligned}$$

Arguing as above to replace $\overset{\bullet}{|}$ by $\overset{\blacklozenge}{|}$ proves conclusion (C2). Conclusion (C3) then follows from the triangle inequality and assumption (A2). ∎

4. Ingredients, tools, and methods

In this section we provide tools that will be used throughout the rest of this paper. In order to make this section accessible to readers with a primary background in either dispersive or stochastic partial differential equations, our exposition will be detailed. We encourage the reader to skip sections covering areas of their expertise.

In Section 4.1, we cover $\mathfrak{X}^{s,b}$-spaces, which are also called Bourgain spaces. The $\mathfrak{X}^{s,b}$-spaces will allow us to utilize multi-linear dispersive effects. In Section 4.2, we present a continuity argument. In Section 4.3, we prove an oscillatory sum estimate for a series involving the sine function. While the proof is standard, its relevance to dispersive equations is surprising and the cancellation was first used by Gubinelli, Koch, and Oh [37]. In Section 4.4, we state several counting estimates related to the dispersive symbol of the wave equation. The counting estimates play an important role in the estimates of our stochastic objects. In Section 4.5, we recall elementary properties of Gaussian processes, which have been heavily used in the first part of the series [12]. In Section 4.6, we provide background regarding multiple stochastic integrals. This section has an algebraic flavor and the multiple stochastic integrals will be used to separate the nonresonant and resonant components of our stochastic objects. In Section 4.7, we discuss Gaussian hypercontractivity and its implications for random matrices.

4.1. Bourgain spaces and transference principles

In this subsection, we recall the definitions and elementary properties of $\mathcal{X}^{s,b}$-spaces, which are often also called Bourgain spaces. Heuristically, $\mathcal{X}^{s,b}$-spaces contain space-time functions u which behave like solutions to the linear wave equation. This principle will be made more precise through the transference principles below. We refer the reader to [67, Section 2.6] and [33, Section 3.3] for a more detailed introduction.

Definition 4.1 ($\mathcal{X}^{s,b}$-spaces). For any $s, b \in \mathbb{R}$ and $u: \mathbb{R} \times \mathbb{T}^3 \to \mathbb{R}$, we define the $\mathcal{X}^{s,b}$-*norm* by

$$\|u\|_{\mathcal{X}^{s,b}} \stackrel{\text{def}}{=} \|\langle n\rangle^s \langle |\lambda| - \langle n\rangle\rangle^b \widehat{u}(\lambda, n)\|_{L^2_\lambda \ell^2_n(\mathbb{R}\times\mathbb{Z}^3)}. \tag{4.1}$$

If $\mathcal{J} \subseteq \mathbb{R}$ is any interval, we define the restricted norm by

$$\|u\|_{\mathcal{X}^{s,b}(\mathcal{J})} \stackrel{\text{def}}{=} \inf\{\|v\|_{\mathcal{X}^{s,b}} : v(t,x)|_{\mathcal{J}} = u\}. \tag{4.2}$$

We denote the corresponding function spaces by $\mathcal{X}^{s,b}$ and $\mathcal{X}^{s,b}(\mathcal{J})$, respectively.

In (4.1), we could have used the symbol $\langle |\lambda| - |n|\rangle$ instead of $\langle |\lambda| - \langle n\rangle\rangle$. Since $\langle n\rangle = |n| + \mathcal{O}(1)$, this would yield an equivalent definition. Our first lemma shows the connection between the $\mathcal{X}^{s,b}$-spaces and the half-wave operators.

Lemma 4.2 (Characterization of $\mathcal{X}^{s,b}$). *Let $s, b \in \mathbb{R}$ and let $u: \mathbb{R} \times \mathbb{T}^3 \to \mathbb{R}$. Then*

$$\|u\|_{\mathcal{X}^{s,b}(\mathbb{R})} \lesssim \min_{\pm} \|\langle\nabla\rangle^s \exp(\mp it\langle\nabla\rangle)u\|_{L^2_x H^b_t(\mathbb{T}^3\times\mathbb{R})}. \tag{4.3}$$

Furthermore,

$$\|u\|_{\mathcal{X}^{s,b}(\mathbb{R})} \sim \min_{\substack{u_+, u_- \in \mathcal{X}^{s,b}(\mathbb{R}): \\ u = u_+ + u_-}} \max_{\pm} \|\langle\nabla\rangle^s \exp(\mp it\langle\nabla\rangle)u_\pm\|_{L^2_x H^b_t(\mathbb{T}^3\times\mathbb{R})}. \tag{4.4}$$

Proof. Using Plancherel's identity, we find that

$$\|\langle\nabla\rangle^s \exp(\mp it\langle\nabla\rangle)u\|_{L^2_x H^b_t(\mathbb{T}^3\times\mathbb{R})} = \|\langle n\rangle^s \langle \pm\lambda - \langle n\rangle\rangle^b \widehat{u}(\lambda, n)\|_{L^2_\lambda \ell^2_n(\mathbb{R}\times\mathbb{Z}^3)}.$$

The first estimate (4.3) then follows from $||\lambda| - \langle n\rangle| \leq |\pm\lambda - \langle n\rangle|$. The inequality "$\lesssim$" in (4.4) follows from the triangle inequality and (4.3). The inequality "$\gtrsim$" follows by defining $u_\pm$ via

$$\widehat{u}(\lambda, n) = 1\{\pm\lambda \geq 0\} \cdot \widehat{u}(\lambda, n).$$ ∎

Our next lemma plays an important role in the local theory. It yields the required smallness of the nonlinearity on a small time interval.

Lemma 4.3 (Time-localization lemma). *Let $-1/2 < b_1 \leq b_2 < 1/2$ and let $1/2 < b < 1$. Let $\psi \in \mathcal{S}(\mathbb{R})$ be a Schwartz function and let $0 < \tau \leq 1$. Then for all $F \in \mathcal{X}^{s,b_2}(\mathbb{R})$,*

$$\begin{aligned} \|\psi(t/\tau)F\|_{\mathcal{X}^{s,b_1}(\mathbb{R})} &\lesssim \tau^{b_2-b_1}\|F\|_{\mathcal{X}^{s,b_2}(\mathbb{R})}, \\ \|F\|_{\mathcal{X}^{s,b_1}([0,\tau])} &\lesssim \tau^{b_2-b_1}\|F\|_{\mathcal{X}^{s,b_2}([0,\tau])}. \end{aligned} \tag{4.5}$$

Furthermore, for all $u \in \mathcal{X}^{s,b}(\mathbb{R})$,

$$\|\psi(t/\tau)u\|_{\mathcal{X}^{s,b}(\mathbb{R})} \lesssim \tau^{1/2-b}\|u\|_{\mathcal{X}^{s,b}(\mathbb{R})}. \tag{4.6}$$

A proof of Lemma 4.3 or a similar result can be found in many textbooks on dispersive PDE, such as [67, Section 2.6] or [33, Section 3.3]. Since the second estimate (4.6) is not usually found in the literature, we present a self-contained proof.

Proof of Lemma 4.3. By using duality and a composition, we may assume that $0 \le b_1 \le b_2 < 1/2$. Let $F_+, F_- \in \mathcal{X}^{s,b_2}(\mathbb{R})$ satisfying $F = F_+ + F_-$. Using Lemma 4.2, we obtain

$$\|\psi(t/\tau)F\|_{\mathcal{X}^{s,b_1}(\mathbb{R})} \lesssim \max_{\pm} \|\psi(t/\tau)\langle\nabla\rangle^s \exp(\mp it\langle\nabla\rangle)F_\pm\|_{L_x^2 H_t^{b_1}(\mathbb{T}^3\times\mathbb{R})}. \tag{4.7}$$

Using interpolation between $b_1 = 0$ and $b_1 = b_2$ as well as the fractional product rule (or a simple paraproduct estimate), one gets for all $f \in H_t^{b_2}(\mathbb{R})$ the estimate

$$\|\psi(t/\tau)f\|_{H_t^{b_1}(\mathbb{R})} \lesssim \tau^{b_2-b_1}\|f\|_{H_t^{b_2}(\mathbb{R})}. \tag{4.8}$$

Combining (4.7) and (4.8) yields the first estimate in (4.5). The second estimate in (4.7) then follows from the first estimate and the definition of the restricted norms. Finally, the second estimate (4.8) follows from the same argument, except that (4.8) is replaced by

$$\|\psi(t/\tau)f\|_{H_t^b(\mathbb{R})} \lesssim \|\psi(t/\tau)\|_{H_t^b(\mathbb{R})}\|f\|_{H_t^b(\mathbb{R})} \lesssim \tau^{1/2-b}\|f\|_{H_t^b(\mathbb{R})}, \tag{4.9}$$

which follows from the algebra property of $H_t^b(\mathbb{R})$. ∎

Lemma 4.4 (Restricted norms and continuity). *Let $s \in \mathbb{R}$ and let $-1/2 < b' < 1/2$. Then, for any interval $\mathcal{J} \subseteq \mathbb{R}$ and any $F \in \mathcal{X}^{s,b'}(\mathbb{R})$,*

$$\|1_{\mathcal{J}}F\|_{\mathcal{X}^{s,b'}(\mathbb{R})} \lesssim \|F\|_{\mathcal{X}^{s,b'}(\mathbb{R})}. \tag{4.10}$$

Furthermore, if $G \in \mathcal{X}^{s,b'}(\mathcal{J})$, then

$$\|G\|_{\mathcal{X}^{s,b'}(\mathcal{J})} \sim \|1_{\mathcal{J}}G\|_{\mathcal{X}^{s,b'}(\mathbb{R})} \tag{4.11}$$

Finally, if $t_0 \overset{\text{def}}{=} \inf \mathcal{J}$, then the map

$$\mathcal{J} \ni t \mapsto \|1_{[t_0,t]}G\|_{\mathcal{X}^{s,b'}(\mathbb{R})} \tag{4.12}$$

is continuous.

Proof. We begin with the proof of (4.10). By using a reduction as in the proof of Lemma 4.3, it suffices to prove that

$$\|1_{\mathcal{J}}g(t)\|_{\mathcal{H}_t^{b'}(\mathbb{R})} \lesssim \|g(t)\|_{\mathcal{H}_t^{b'}(\mathbb{R})}. \tag{4.13}$$

By writing $1_{\mathcal{J}}$ as a superposition of different indicator functions, it suffices to prove the estimate for $(-\infty, a)$ and (a, ∞), where $a \in \mathbb{R}$, instead of $\mathcal{J}$. Using the time-reflection and time-translation symmetry of $H_t^{b'}(\mathbb{R})$, it suffices to prove the estimate for $\mathcal{J}$ replaced by $(0,\infty)$. Thus, it remains to prove

$$\|1_{(0,\infty)} g(t)\|_{\mathcal{H}_t^{b'}(\mathbb{R})} \lesssim \|g(t)\|_{\mathcal{H}_t^{b'}(\mathbb{R})}. \tag{4.14}$$

This follows from (a modification of) the fractional product rule or a simple paraproduct estimate.

We now turn to the proof of (4.11). By the definition of the restricted norms, we clearly have the upper bound $\|G\|_{\mathfrak{X}^{s,b'}(\mathcal{J})} \lesssim \|1_{\mathcal{J}} G\|_{\mathfrak{X}^{s,b'}(\mathbb{R})}$. Now, let $\widetilde{G} \in \mathfrak{X}^{s,b}(\mathbb{R})$ satisfy $\widetilde{G}|_{\mathcal{J}} = G$. Using (4.10), we obtain

$$\|1_{\mathcal{J}} G\|_{\mathfrak{X}^{s,b'}(\mathbb{R})} = \|1_{\mathcal{J}} \widetilde{G}\|_{\mathfrak{X}^{s,b'}(\mathbb{R})} \lesssim \|\widetilde{G}\|_{\mathfrak{X}^{s,b'}(\mathbb{R})}.$$

After taking the infimum in $\widetilde{G}$, this yields the other lower bound in (4.11).

Finally, we prove the continuity of (4.12). By a density argument, it suffices to take $G \in \mathfrak{X}^{s,1/2}(\mathbb{R})$. For any $0 < \delta < 1/2 - b$ and $t_1, t_2 \in \mathcal{J}$, we deduce from Lemma 4.3 that

$$\Big| \|1_{[t_0,t_1]} G\|_{\mathfrak{X}^{s,b'}(\mathbb{R})} - \|1_{[t_0,t_2]} G\|_{\mathfrak{X}^{s,b'}(\mathbb{R})} \Big| \leq \|1_{(t_1,t_2]} G\|_{\mathfrak{X}^{s,b'}(\mathbb{R})} \lesssim |t_1 - t_2|^{\delta} \|G\|_{\mathfrak{X}^{s,1/2}(\mathbb{R})}.$$

This implies the Hölder continuity. ■

The next gluing lemma will be used to combine $\mathfrak{X}^{s,b}$-bounds on different intervals. While such a result is trivial for purely physical function spaces, such as $L_t^q L_x^p$, it is slightly more complicated for the $\mathfrak{X}^{s,b}$-spaces, since they rely on the time-frequency variable.

Lemma 4.5 (Gluing lemma). *Let $s \in \mathbb{R}$, let $-1/2 < b' < 1/2$, let $1/2 < b < 1$, and let $\mathcal{J}, \mathcal{J}_1, \mathcal{J}_2$ be bounded intervals satisfying $\mathcal{J}_1 \cap \mathcal{J}_2 \neq \emptyset$. Then, for all $F\colon (\mathcal{J}_1 \cup \mathcal{J}_2) \times \mathbb{T}^3 \to \mathbb{R}$,*

$$\|F\|_{\mathfrak{X}^{s,b'}(\mathcal{J}_1 \cup \mathcal{J}_2)} \lesssim \|F\|_{\mathfrak{X}^{s,b'}(\mathcal{J}_1)} + \|F\|_{\mathfrak{X}^{s,b'}(\mathcal{J}_2)}. \tag{4.15}$$

Furthermore, let $\tau \overset{\text{def}}{=} |\mathcal{J}_1 \cap \mathcal{J}_2|$. Then for all $u\colon (\mathcal{J}_1 \cup \mathcal{J}_2) \times \mathbb{T}^3 \to \mathbb{R}$,

$$\|u\|_{\mathfrak{X}^{s,b}(\mathcal{J}_1 \cup \mathcal{J}_2)} \lesssim \tau^{1/2-b} \big(\|u\|_{\mathfrak{X}^{s,b}(\mathcal{J}_1)} + \|u\|_{\mathfrak{X}^{s,b}(\mathcal{J}_2)} \big). \tag{4.16}$$

Proof. We begin with the proof of (4.15). Using Lemma 4.4, we have

$$\begin{aligned} \|F\|_{\mathfrak{X}^{s,b'}(\mathcal{J}_1 \cup \mathcal{J}_2)} &\lesssim \|1_{\mathcal{J}_1 \cup \mathcal{J}_2} F\|_{\mathfrak{X}^{s,b'}(\mathbb{R})} \lesssim \|1_{\mathcal{J}_1} F\|_{\mathfrak{X}^{s,b'}(\mathbb{R})} + \|1_{\mathcal{J}_2 \backslash \mathcal{J}_1} F\|_{\mathfrak{X}^{s,b'}(\mathbb{R})} \\ &\lesssim \|F\|_{\mathfrak{X}^{s,b'}(\mathcal{J}_1)} + \|F\|_{\mathfrak{X}^{s,b'}(\mathcal{J}_2 \backslash \mathcal{J}_1)} \lesssim \|F\|_{\mathfrak{X}^{s,b'}(\mathcal{J}_1)} + \|F\|_{\mathfrak{X}^{s,b'}(\mathcal{J}_2)}. \end{aligned}$$

The proof of the second estimate (4.16) is similar. Instead of working with an actual indicator function, we use a smooth cut-off function on the spatial scale $\sim \tau$ and a variant of (4.9) instead of (4.14). ■

Our last two lemmas were concerned with the behavior of $\mathfrak{X}^{s,b}$-spaces over small or overlapping time intervals. In this respect, the $\mathfrak{X}^{s,b}$-spaces are more complicated than purely physical function spaces. We now turn to transference principles, which do not have a direct analog in purely physical function spaces.

Lemma 4.6 (Linear transference principle; cf. [67, Lemma 2.9]). *Let $b > 1/2$, let $s \in \mathbb{R}$, and assume that the norm $\|\cdot\|_Y$ satisfies*

$$\|e^{i\alpha t} e^{\pm it\langle\nabla\rangle} u_0\|_Y \leq C\|u_0\|_{H_x^s} \tag{4.17}$$

for all $\alpha \in \mathbb{R}$ and all $u_0 \in H_x^s$. Then, for all $u \in \mathfrak{X}^{s,b}$,

$$\|u\|_Y \lesssim C\|u\|_{\mathfrak{X}^{s,b}}. \tag{4.18}$$

The linear transference principle allows us to reduce linear estimates for functions in $\mathfrak{X}^{s,b}$-spaces to estimates for the half-wave operators.

Corollary 4.7. *For any $b > 1/2$, $s \in \mathbb{R}$, any $4 \leq p \leq \infty$, any compact interval $J \subseteq \mathbb{R}$, and any $u\colon J \times \mathbb{T}^3 \to \mathbb{C}$, we have*

$$\|u[t]\|_{C_t^0 \mathcal{H}_x^s(J\times\mathbb{T}^3)} \lesssim \|u\|_{\mathfrak{X}^{s,b}(J)}, \tag{4.19}$$

$$\|\langle\nabla\rangle^{s+4/p-3/2} u(t)\|_{L_t^p L_x^p(J\times\mathbb{T}^3)} \lesssim (1+|J|)^{1/p}\|u\|_{\mathfrak{X}^{s,b}(J)}, \tag{4.20}$$

$$\|\langle\nabla\rangle^{s-1-} u(t)\|_{L_t^2 L_x^\infty(J\times\mathbb{T}^3)} \lesssim (1+|J|)^{1/2}\|u\|_{\mathfrak{X}^{s,b}(J)}. \tag{4.21}$$

The corollary follows directly from the linear transference principle (Lemma 4.6) and the Strichartz estimates for the linear wave equation.

The next lemma is the most basic ingredient for any contraction argument based on $\mathfrak{X}^{s,b}$-spaces.

Lemma 4.8 (Energy estimate; cf. [67, Lemma 2.12] and [33, Lemma 3.2]). *Let $1/2 < b < 1$, let $s \in \mathbb{R}$, let $\mathcal{J} \subseteq \mathbb{R}$ be a compact interval, let $t_0 \in \mathcal{J}$, and let*

$$(-\partial_t^2 - 1 + \Delta)u = F. \tag{4.22}$$

Then

$$\|u\|_{\mathfrak{X}^{s,b}(\mathcal{J})} \lesssim (1+|\mathcal{J}|)^2(\|u[t_0]\|_{\mathcal{H}_x^s} + \|F\|_{\mathfrak{X}^{s-1,b-1}(\mathcal{J})}). \tag{4.23}$$

The statement of Lemma 4.8 in [33, 67] only includes intervals of size ~ 1. The more general version follows by using the triangle inequality, iterating the bound on unit intervals, and applying (4.19). The square in the pre-factor can likely be improved but is inessential in our argument, since the stability theory already loses exponential factors in the final time T.

The most important terms in the nonlinearity can only be estimated through multilinear dispersive effects and hence require a direct analysis of the $\mathfrak{X}^{s-1,b-1}$-norm. However, several minor terms can be estimated more easily through physical methods. In order to pass back from the frequency-based $\mathfrak{X}^{s-1,b-1}$-space to purely physical spaces, we provide the following inhomogeneous Strichartz estimate.

Lemma 4.9 (Inhomogeneous Strichartz estimate in $\mathfrak{X}^{s,b}$-spaces). *Let $1/2 < b < 1$, let $s \in \mathbb{R}$, let $\mathcal{J} \subseteq \mathbb{R}$ be a compact interval, and let $F\colon \mathcal{J} \times \mathbb{T}^3 \to \mathbb{R}$. Then*

$$\|F\|_{\mathfrak{X}^{s-1,b-1}(\mathcal{J})} \lesssim \|F\|_{L_t^{2b} H_x^{s-1}(\mathcal{J}\times\mathbb{T}^3)}, \tag{4.24}$$

$$\|F\|_{\mathfrak{X}^{s-1,b-1}(\mathcal{J})} \lesssim (1+|\mathcal{J}|)\|\langle\nabla\rangle^{s-\frac{1}{2}+\frac{2b-1}{b}s} F\|_{L_t^{4/3} L_x^{4/3}(\mathcal{J}\times\mathbb{T}^3)}. \tag{4.25}$$

Remark 4.10. For $0 \leq s \leq 1$, we will often simplify the right-hand side of (4.24) by using the fact that

$$\frac{2b-1}{b} s \leq 4(b-1/2).$$

Proof of Lemma 4.9. We first prove (4.24). Using (4.19) and duality, we have

$$\|F\|_{\mathfrak{X}^{s-1,-b}(\mathcal{J})} \lesssim \|F\|_{L_t^1 H_x^{s-1}(\mathcal{J}\times\mathbb{T}^3)}.$$

By Plancherel, we also have

$$\|F\|_{\mathfrak{X}^{s-1,0}(\mathcal{J})} \lesssim \|F\|_{L_t^2 H_x^{s-1}(\mathcal{J}\times\mathbb{T}^3)}.$$

Using interpolation, this implies (4.24). The proof of the second estimate (4.25) is similar and relies on duality, (4.20), Plancherel, and interpolation. ■

When utilizing multi-linear dispersive effects, we will often use the following lemma to estimate the $\mathfrak{X}^{s-1,b_- -1}$-norm.

Lemma 4.11. *Let $s \in \mathbb{R}$ and let $T \geq 1$. Let $\mathcal{A}$ be a finite index set and let $(n_\alpha)_{\alpha\in\mathcal{A}} \subseteq \mathbb{Z}^3$, $(\theta_\alpha)_{\alpha\in\mathcal{A}} \subseteq \mathbb{R}$, and $(c_\alpha)_{\alpha\in\mathcal{A}} \subseteq \mathbb{C}$. Define*

$$F(t,x) \stackrel{\text{def}}{=} \sum_{\alpha\in\mathcal{A}} c_\alpha \exp(i\langle n_\alpha, x\rangle + it\theta_\alpha). \tag{4.26}$$

Then

$$\begin{aligned} &\|F\|_{\mathfrak{X}^{s-1,b_- -1}([0,T])} \\ &\lesssim T \max_{\pm} \Big\| \langle\lambda\rangle^{b_- -1} \langle n\rangle^{s-1} \sum_{\alpha\in\mathcal{A}} 1\{n = n_\alpha\}\, c_\alpha\, \widehat{\chi}(T(\lambda \mp \langle n\rangle - \theta_\alpha)) \Big\|_{L_\lambda^2 \ell_n^2(\mathbb{R}\times\mathbb{Z}^3)}. \end{aligned} \tag{4.27}$$

Proof. For any $G\colon \mathbb{R}\times\mathbb{T}^3 \to \mathcal{C}$, we have

$$\begin{aligned} \|G\|_{\mathfrak{X}^{s-1,b_- -1}(\mathbb{R})} &= \|\langle|\lambda| - \langle n\rangle\rangle^{b_- -1} \langle n\rangle^{s-1} \widehat{G}(\lambda,n)\|_{L_\lambda^2\ell_n^2(\mathbb{R}\times\mathbb{Z}^3)} \\ &\lesssim \max_{\pm} \|\langle\lambda \pm \langle n\rangle\rangle^{b_- -1} \langle n\rangle^{s-1} \widehat{G}(\lambda,n)\|_{L_\lambda^2\ell_n^2(\mathbb{R}\times\mathbb{Z}^3)} \\ &= \max_{\pm} \|\langle\lambda\rangle^{b_- -1} \langle n\rangle^{s-1} \widehat{G}(\lambda\mp\langle n\rangle,n)\|_{L_\lambda^2\ell_n^2(\mathbb{R}\times\mathbb{Z}^3)}. \end{aligned}$$

We then apply this inequality to $G(t,x) = \chi(t/T)F(t,x)$. ■

Finally, we present an estimate for the Fourier transform of a (localized) time integral.

Lemma 4.12. *Let $T \geq 1$ and let $\lambda, \lambda_1, \lambda_2 \in \mathbb{R}$. Then*

$$\begin{aligned} &\left| \mathcal{F}_t\Big(\chi(t/T)\exp(i\lambda_1 t) \int_0^t \exp(i\lambda_2 t')\,\mathrm{d}t' \Big)(\lambda) \right| \\ &\qquad\qquad \lesssim T^2 (\langle\lambda-\lambda_1-\lambda_2\rangle^{-10} + \langle\lambda-\lambda_1\rangle^{-10}) \langle\lambda_2\rangle^{-1}. \end{aligned} \tag{4.28}$$

Furthermore, if $\mathcal{J} \subseteq [0, T]$ is an interval, then

$$\left| \mathcal{F}_t \left(\chi(t/T) \exp(i\lambda_1 t) \int_0^t 1_{\mathcal{J}}(t') \exp(i\lambda_2 t') \, \mathrm{d}t' \right)(\lambda) \right| \lesssim T^2 (\langle \lambda - \lambda_1 - \lambda_2 \rangle^{-1} + \langle \lambda - \lambda_1 \rangle^{-1}) \langle \lambda_2 \rangle^{-1}. \quad (4.29)$$

Proof. We first prove (4.28). A direct calculation yields

$$\mathcal{F}_t \left(\chi(t/T) \exp(i\lambda_1 t) \int_0^t \exp(i\lambda_2 t') \, \mathrm{d}t' \right)(\lambda) = \frac{T}{i\lambda_2} \big(\widehat{\chi}(T(\lambda - \lambda_1 - \lambda_2)) - \widehat{\chi}(T(\lambda - \lambda_1)) \big). \quad (4.30)$$

For $|\lambda_2| \gtrsim 1$, the estimate follows from the decay of $\widehat{\chi}$. For $|\lambda_2| \lesssim 1$, the estimate follows from the fundamental theorem of calculus and the decay of $\widehat{\chi}'$. We have also used $T \geq 1$, which implies that $\langle T \cdot \rangle^{-10} \lesssim \langle \cdot \rangle^{-10}$.

We now turn to (4.29). Since the restriction to $\mathcal{J}$ only appears in the integral, we can replace $\mathcal{J}$ by its closure. We now let $\mathcal{J} = [t_-, t_+] \subseteq [0, T]$. By integrating the exponential, we have

$$\int_0^t 1_{\mathcal{J}}(t') \exp(i\lambda_2 t') \, \mathrm{d}t' = \frac{1}{i\lambda_2} \big(\exp(i\lambda_2 (t \wedge t_+)) - \exp(i\lambda_2 (t \wedge t_-)) \big),$$

where $x \wedge y$ denotes the minimum of x and y. This implies

$$\mathcal{F}_t \left(\chi(t/T) \exp(i\lambda_1 t) \int_0^t 1_{\mathcal{J}}(t') \exp(i\lambda_2 t') \, \mathrm{d}t' \right)(\lambda) = \frac{1}{i\lambda_2} \int_{\mathbb{R}} \chi(t/T) \exp(i(\lambda + \lambda_1)t) (\exp(i\lambda_2 (t \wedge t_+)) - \exp(i\lambda_2 (t \wedge t_-))) \, \mathrm{d}t.$$

The estimate then follows by distinguishing the cases $|\lambda_1| \lesssim 1$, $|\lambda_1| \gg 1 \gtrsim |\lambda_2|$, and $|\lambda_1|, |\lambda_2| \gg 1$, together with the triangle inequality and a simple integration by parts. ■

4.2. *Continuity argument*

In this short subsection, we present a modification of the standard continuity argument. The modification is a result of the possible discontinuity of $[0, T] \ni t \mapsto \|u\|_{\mathfrak{X}^{s,b}([0,t])}$, where $u \in \mathfrak{X}^{s,b}([0, T])$ and $b > 1/2$. As a replacement, we will rely on the continuity statement in Lemma 4.4. A different approach to this problem was obtained in [66, Theorem 3], which yields the quasi-continuity, and may even yield the continuity (see the discussion in [66, Section 12]).

Lemma 4.13 (Continuity argument). *Let $\mathcal{J} = [t_0, t_1)$, let $f \colon \mathcal{J} \to [0, \infty)$ be a nonnegative function, and let $g \colon \mathcal{J} \to [0, \infty)$ be a continuous, nonnegative function. Let $A \geq 1$, $0 < \theta, \delta < 1$, and assume that*

$$f(t) \leq g(t) \leq g(t_0) + \delta(A^2 + f(t)^2)(f(t) + \theta) \quad (4.31)$$

for all $t \in [t_0, t_1)$. Furthermore, assume that

$$g(t_0) + \delta^2 A\theta \leq 1 \quad \text{and} \quad \delta(A^2 + 6) \leq 1/4. \quad (4.32)$$

Then

$$f(t) \le g(t) \le 2(g(t_0) + \delta A^2 \theta) \quad \textit{for all } t \in [t_0, t_1).$$

Proof. The estimate (4.31) implies that

$$g(t) \le g(t_0) + \delta(A^2 + g(t)^2)(g(t) + \theta)$$

for all $t \in [t_0, t_1)$. Using the condition (4.32), we also have

$$g(t_0) + \delta(A^2 + 4(g(t_0) + \delta A^2\theta)^2)(g(t_0) + \delta A^2 \theta + \theta) \le \tfrac{3}{2}(g(t_0) + \delta A^2 \theta).$$

Using the standard continuity method (see e.g. [67, Section 1.3]) implies

$$g(t) \le 2(g(t_0) + \delta A^2 \theta) \quad \text{for all } t \in [t_0, t_1). \qquad \blacksquare$$

4.3. *Sine-cancellation lemma*

In this subsection, we prove an oscillatory sum estimate which critically relies on the fact that the sine function is odd. The same cancellation was exploited in earlier work of Gubinelli–Koch–Oh [37, Section 4] and we present a slight generalization of their argument.

Lemma 4.14. *Let $f: \mathbb{R} \times \mathbb{R} \times \mathbb{Z}^3 \to \mathbb{C}$, $a \in \mathbb{Z}^3$, $T \ge 1$, let $\mathcal{J} \subseteq [0, T]$ be an interval, and let $A, N \ge 1$. Assume that $|a| \lesssim A \ll N$. Furthermore, assume that for all $|t|, |t'| \le T$,*

$$|f(t,t',n)| \le A\langle n\rangle^{-3}, \quad |f(t,t',n) - f(t,t',-n)| \le A\langle n\rangle^{-4}, \quad |\partial_{t'} f(t,t',n)| \le A\langle n\rangle^{-4}.$$

Then

$$\sup_{\lambda \in \mathbb{R}} \sup_{|t| \le T} \Big| \sum_{n \in \mathbb{Z}^3} \chi_N(n) \int_0^t 1_{\mathcal{J}}(t') \sin((t-t')\langle a+n\rangle) \cos((t-t')\langle n\rangle) \exp(i\lambda t') f(t,t',n)\, \mathrm{d}t' \Big| \lesssim T^2 A^3 \log(2+N) N^{-1}. \tag{4.33}$$

The dependence on A is not essential and can likely be improved. In all our applications of this lemma, A is negligible compared to N. We emphasize that the estimate fails if we only assume that $|f(t,t',n)| \le A\langle n\rangle^{-3}$. Indeed, after removing the truncation χ_N, the corresponding sum could diverge logarithmically.

Proof of Lemma 4.14. Using trigonometric identities, we have

$$\begin{aligned} 2 &\sum_{n \in \mathbb{Z}^3} \chi_N(n) \int_0^t 1_{\mathcal{J}}(t') \sin((t-t')\langle a+n\rangle) \cos((t-t')\langle n\rangle) \exp(i\lambda t') f(t,t',n)\, \mathrm{d}t' \\ &= \sum_{n \in \mathbb{Z}^3} \chi_N(n) \int_0^t 1_{\mathcal{J}}(t') \sin\big((t-t')(\langle a+n\rangle - \langle n\rangle)\big) \exp(i\lambda t') f(t,t',n)\, \mathrm{d}t' \qquad (4.34) \\ &\quad + \sum_{n \in \mathbb{Z}^3} \chi_N(n) \int_0^t 1_{\mathcal{J}}(t') \sin\big((t-t')(\langle a+n\rangle + \langle n\rangle)\big) \exp(i\lambda t') f(t,t',n)\, \mathrm{d}t'. \qquad (4.35) \end{aligned}$$

We estimate the terms (4.34) and (4.35) separately. We begin with (4.34), which is more difficult. Since $|\langle a+n\rangle - \langle n\rangle| \lesssim A$, we do not expect to gain in N through the integration in t'. Instead, we utilize a pointwise cancellation. By using the symmetry $n \leftrightarrow -n$ in the summation, we obtain

$$\begin{aligned}
&2\Big|\sum_{n\in\mathbb{Z}^3} \chi_N(n)\sin\big((t-t')(\langle a+n\rangle - \langle n\rangle)\big) f(t,t',n)\Big| \\
&= \Big|\sum_{n\in\mathbb{Z}^3} \chi_N(n)\Big(\sin\big((t-t')(\langle n+a\rangle - \langle n\rangle)\big) f(t,t',n) \\
&\qquad\qquad + \sin\big((t-t')(\langle n-a\rangle - \langle n\rangle)\big) f(t,t',-n)\Big)\Big| \\
&\lesssim \sum_{n\in\mathbb{Z}^3} \chi_N(n)\big|\sin\big((t-t')(\langle n+a\rangle - \langle n\rangle)\big) + \sin\big((t-t')(\langle n-a\rangle - \langle n\rangle)\big)\big|\cdot|f(t,t',n)| \\
&\quad + \sum_{n\in\mathbb{Z}^3} \chi_N(n)|f(t,t',n) - f(t,t',-n)|.
\end{aligned}$$

By the assumptions on f, the second summand is easily bounded by AN^{-1}. We now concentrate on the first summand. Using a Taylor expansion, we find that

$$\langle n \pm a\rangle - \langle n\rangle = \pm\frac{n\cdot a}{\langle n\rangle} + \mathcal{O}(A^2N^{-1}). \tag{4.36}$$

Using the fact that the sine function is odd, we obtain

$$\begin{aligned}
&\big|\sin\big((t-t')(\langle n+a\rangle - \langle n\rangle)\big) + \sin\big((t-t')(\langle n-a\rangle - \langle n\rangle)\big)\big| \\
&\qquad = \big|\sin\big((t-t')(\langle n+a\rangle - \langle n\rangle)\big) - \sin\big(-(t-t')(\langle n-a\rangle - \langle n\rangle)\big)\big| \\
&\qquad \le T|\langle n+a\rangle - \langle n\rangle + \langle n-a\rangle - \langle n\rangle| \lesssim TA^2N^{-1}.
\end{aligned}$$

Putting both estimates together and integrating in t', we see that the first term (4.34) is bounded by $T^2A^3N^{-1}$, which is acceptable.

We now turn to the estimate of (4.35). Since $\langle n+a\rangle + \langle n\rangle \gtrsim N$, we expect to gain a factor of N through integration by parts. We have

$$\begin{aligned}
&\Big|\sum_{n\in\mathbb{Z}^3} \chi_N(n)\int_0^t 1_{\mathcal{J}}(t')\sin\big((t-t')(\langle a+n\rangle + \langle n\rangle)\big)\exp(i\lambda t') f(t,t',n)\,\mathrm{d}t'\Big| \\
&\lesssim \max_{\pm}\Big|\sum_{n\in\mathbb{Z}^3} \chi_N(n)\int_0^t 1_{\mathcal{J}}(t')\exp\big(i\lambda t' \pm it'(\langle a+n\rangle + \langle n\rangle)\big) f(t,t',n)\,\mathrm{d}t'\Big| \\
&\lesssim \max_{\pm}\sum_{n\in\mathbb{Z}^3} \chi_N(n)\frac{1}{1+|\langle a+n\rangle+\langle n\rangle\pm\lambda|}\Big(\sup_{0\le t'\le t}|f(t,t',n)| + T\sup_{0\le t'\le t}|\partial_{t'}f(t,t',n)|\Big) \\
&\lesssim TAN^{-3}\max_{\pm}\sum_{n\in\mathbb{Z}^3}\chi_N(n)\frac{1}{1+|\langle a+n\rangle+\langle n\rangle\pm\lambda|}.
\end{aligned}$$

In order to finish the estimate, it only remains to prove that

$$\sum_{n\in\mathbb{Z}^3}\chi_N(n)\frac{1}{1+|\langle a+n\rangle+\langle n\rangle\pm\lambda|} \lesssim \log(2+N)N^2.$$

Since the function $x \mapsto \langle x \rangle$ is 1-Lipschitz, we can estimate the sum by an integral and obtain

$$\sum_{n \in \mathbb{Z}^3} \chi_N(n) \frac{1}{1 + |\langle a+n \rangle + \langle n \rangle \pm \lambda|} \lesssim \int_{\mathbb{R}^3} 1\{|\xi| \sim N\} \frac{1}{1 + |\langle \xi + a \rangle + \langle \xi \rangle \pm \lambda|} \mathrm{d}\xi.$$

Due to the rotation invariance of the Lebesgue measure, we can then reduce to $a = (0, 0, |a|)$. To estimate the integral, we first switch into polar coordinates (r, θ, φ). Since $A \ll N$, we see for fixed angles θ and φ that $r \mapsto \langle \xi + a \rangle + \langle \xi \rangle$ is bi-Lipschitz on $r \sim N$. After a further change of variables, this yields

$$\begin{aligned} \int_{\mathbb{R}^3} 1\{|\xi| \sim N\} \frac{1}{1 + |\langle \xi + a \rangle + \langle \xi \rangle \pm \lambda|} \mathrm{d}\xi &\lesssim N^2 \int_0^\infty 1\{r \sim N\} \frac{1}{1 + |r \pm \lambda|} \mathrm{d}r \\ &\lesssim N^2 \log(2 + N). \end{aligned}$$ ∎

4.4. *Counting estimates*

In this subsection, we record several counting estimates. The counting estimates are the most technical part of our treatment of **So**, **CPara**, and **RMT**. Fortunately, they can be used as a black-box, and we encourage the reader to only skim this section during first reading.

Before we state our counting estimates, we discuss the main ingredients and the differences between the nonlinear wave and Schrödinger equations. In contrast to the counting estimates for the nonlinear Schrödinger equation, the counting estimates for the wave equation require no analytic number theory. The reason is that the mapping $n \mapsto \langle n \rangle$ is globally 1-Lipschitz, whereas the Lipschitz constant of $n \mapsto |n|^2$ grows linearly. This allows us to reduce all (discrete) counting estimates to estimates of the volume of (continuous) sets. More specifically, we will use the fact that the intersection of (most) thin annuli has a smaller volume than the individual annuli.

Another difference between the wave and Schrödinger equation is related to the symmetries of the equation. The Schrödinger equation enjoys the Galilean symmetry, which is useful in obtaining "shifted" versions of several estimates. For instance, it implies that frequency-localized Strichartz estimates for the Schrödinger equation are the same for cubes centered either at or away from the origin. On the frequency side, it is related to the Galilean transform

$$(n, \lambda) \mapsto (n - a, \lambda - 2a \cdot n + |a|^2),$$

which preserves the discrete paraboloid and plays an important role in decoupling theory (cf. [25, Section 4]). It often allows us to replace conditions such as $|n| \sim N$ in counting estimates by the more general restriction $|n - a| \sim N$ for some fixed $a \in \mathbb{Z}^3$. In contrast, the Lorentzian symmetry of the wave equation on Euclidean space does not even preserve the periodicity of $u \colon \mathbb{R} \times \mathbb{T}^3 \to \mathbb{R}$. As illustrated by the Klainerman–Tataru–Strichartz estimates (cf. [45] and Lemma 8.1), the frequency-shifted Strichartz estimates are more complicated for the wave equation than for the Schrödinger equation. As will be clear from this section, similar difficulties arise in the counting estimates.

The last difference between the Schrödinger and wave equation we mention here is a result of the multiplier $\langle\nabla\rangle^{-1}$ in the Duhamel integral for the wave equation. Together with multi-linear dispersive effects, we therefore obtain two separate smoothing effects in the nonlinear wave equation, which are related to the elliptic symbol $\langle n\rangle$ and the dispersive symbol $\langle|\lambda| - |n|\rangle$. In contrast, the Schrödinger equation only exhibits a single smoothing effect related to the dispersive symbol $\lambda - |n|^2$. In most situations, we expect that the combined smoothing effects in the wave equation are stronger than the single smoothing effect in the Schrödinger equation. However, it may be more difficult to capture the combined smoothing effect in a single proposition, as has been done in [30, Proposition 4.9] for the Schrödinger equation.

In Section 4.4.1, we prove basic counting estimates which form the foundation of the rest of this section. In Sections 4.4.2–4.4.7, we state several cubic, quartic, quintic, and septic counting estimates. In order not to interrupt the flow of the main argument, we placed their (standard) proofs in the appendix. In Section 4.4.8, we present estimates for the operator norm of (deterministic) tensors. The tensor estimates are not (yet) standard in the literature on random dispersive equations, so we include their proofs in the body of the paper.

4.4.1. Basic counting estimates.

Lemma 4.15 (Basic counting lemma). *Let $a \in \mathbb{Z}^3$, let $A, N \geq 1$, and assume that $|a| \sim A$. Then*

$$\sup_{m\in\mathbb{Z}} \#\{n \in \mathbb{Z}^3 : |n| \sim N,\ |\langle a+n\rangle \pm \langle n\rangle - m| \lesssim 1\} \lesssim \min(A,N)^{-1} N^3. \tag{4.37}$$

We emphasize that the upper bound in (4.37) cannot be improved to N^2. The reason is that $|\langle a+n\rangle - \langle n\rangle| \lesssim A$, which implies that

$$\sup_{m\in\mathbb{Z}} \#\{n \in \mathbb{Z}^3 : |n| \sim N,\ |\langle a+n\rangle - \langle n\rangle - m| \lesssim 1\} \gtrsim A^{-1} N^3.$$

As already mentioned above, the main step in the proof converts the discrete estimate (4.37) into a continuous analogue. After this reduction, the estimate boils down to multivariable calculus.

Proof of Lemma 4.15. Since $\langle\xi\rangle = |\xi| + \mathcal{O}(1)$, we may replace $\langle\cdot\rangle$ in (4.37) by $|\cdot|$ after increasing the implicit constant. Furthermore, since $\xi \mapsto \langle\xi + a\rangle \pm \langle\xi\rangle$ is globally Lipschitz, we see that the 1-neighborhood of the set on the left-hand side of (4.37) is contained in

$$\big\{\xi \in \mathbb{R}^3 : |\xi| \sim N,\ \big||a+\xi| \pm |\xi| - m\big| \lesssim 1\big\}.$$

Since the integer vectors are 1-separated, it follows that

$$\begin{aligned}\#\big\{n \in \mathbb{Z}^3 &: |n| \sim N,\ \big||a+n| \pm |n| - m\big| \lesssim 1\big\}\\ &\lesssim \mathrm{Leb}\big(\big\{\xi \in \mathbb{R}^3 : |\xi| \sim N,\ \big||a+\xi| \pm |\xi| - m\big| \lesssim 1\big\}\big).\end{aligned}$$

We now decompose

$$\begin{aligned}
&\mathrm{Leb}\big(\{\xi \in \mathbb{R}^3 \colon |\xi| \sim N,\ \big||a+\xi| \pm |\xi| - m\big| \lesssim 1\}\big) \\
&\quad \lesssim \sum_{\substack{m_1,m_2\in\mathbb{Z}:\\ |m_1\pm m_2-m|\lesssim 1}} \mathrm{Leb}\big(\{\xi \in \mathbb{R}^3 \colon |\xi| \sim N,\ |a+\xi| = m_1 + \mathcal{O}(1),\ |\xi| = m_2 + \mathcal{O}(1)\}\big) \\
&\quad \lesssim N \sup_{m_1,m_2\in\mathbb{Z}} \mathrm{Leb}\big(\{\xi \in \mathbb{R}^3 \colon |\xi| \sim N, |a+\xi| = m_1 + \mathcal{O}(1),\ |\xi| = m_2 + \mathcal{O}(1)\}\big).
\end{aligned}$$

In the last line, we use the fact that there are at most $\sim N$ non-trivial choices of m_2. Once m_2 is fixed, the condition $|m_1 \pm m_2 - m| \lesssim 1$ implies that there are at most ~ 1 non-trivial choices for m_1. Thus, it remains to prove for $|m_1| \lesssim \max(A, N)$ and $|m_2| \sim N$ that

$$\mathrm{Leb}\big(\{\xi \in \mathbb{R}^3 \colon |\xi| \sim N,\ |a+\xi| = m_1 + \mathcal{O}(1),\ |\xi| = m_2 + \mathcal{O}(1)\}\big) \lesssim \min(A, N)^{-1} N^2. \tag{4.38}$$

Using the rotation invariance of the Lebesgue measure, we may assume that $a = |a|e_3$, i.e., a points in the direction of the z-axis. By switching to polar coordinates, we obtain

$$\begin{aligned}
&\mathrm{Leb}\big(\{\xi \in \mathbb{R}^3 \colon |\xi| \sim N,\ |a+\xi| = m_1 + \mathcal{O}(1),\ |\xi| = m_2 + \mathcal{O}(1)\}\big) \\
&\lesssim N^2 \int_0^\infty \int_0^\pi 1\{r = m_2 + \mathcal{O}(1)\} 1\{\sqrt{|a|^2 + 2r|a|\cos(\theta) + r^2} = m_1 + \mathcal{O}(1)\} \sin(\theta)\, \mathrm{d}\theta\, \mathrm{d}r.
\end{aligned}$$

The condition $\sqrt{|a|^2 + 2r|a|\cos(\theta) + r^2} = m_1 + \mathcal{O}(1)$ together with $|m_1| \lesssim \max(A, N)$ implies that

$$\cos(\theta) = 1 - \frac{(|a|+r)^2}{2|a|r} + \frac{m_1^2}{2|a|r} + \mathcal{O}(\max(A, N) A^{-1} N^{-1}). \tag{4.39}$$

For a fixed r, this shows that $\cos(\theta)$ is contained in an interval of size $\sim \min(A, N)^{-1}$. After a change of variables from θ to $\cos(\theta)$, this yields

$$\begin{aligned}
&N^2 \int_0^\infty \int_0^\pi 1\{r = m_2 + \mathcal{O}(1)\} 1\{\sqrt{|a|^2 + 2r|a|\cos(\theta) + r^2} = m_1 + \mathcal{O}(1)\} \sin(\theta)\, \mathrm{d}\theta\, \mathrm{d}r \\
&\qquad \lesssim \min(A, N)^{-1} N^2 \int_0^\infty 1\{r = m_2 + \mathcal{O}(1)\}\, \mathrm{d}r \lesssim \min(A, N)^{-1} N^2.
\end{aligned} \tag{4.40}$$

■

Remark 4.16. Our proof of the basic counting lemma (Lemma 4.15) easily generalizes to spatial dimensions $d \geq 3$. In two spatial dimensions, however, only weaker estimates are available. The reason lies in the absence of the sine-function in the area element for polar coordinates, which breaks (4.40). From a PDE perspective, the parallel interactions in two-dimensional wave equations are stronger than the planar interactions in three-dimensional wave equations. Ultimately, this requires a modification in the probabilistic scaling heuristic and we encourage the reader to compare [28, Section 1.3.2] and [53, Proposition 1.5].

We now present a minor modification of the basic counting lemma (Lemma 4.15). The condition $|n| \sim N$ is augmented by $|n+a| \sim B$. We emphasize that the vector $a \in \mathbb{Z}^3$ in this constraint is the same as in the dispersive symbol.

Lemma 4.17 ("Two-ball" basic counting lemma). *Let $N, A, B \geq 1$. Let $a \in \mathbb{Z}^3$ satisfy $|a| \sim A$. Then*

$$\sup_{m \in \mathbb{Z}} \#\{n \in \mathbb{Z}^3 \colon |n| \sim N,\ |n+a| \sim B,\ |\langle a+n\rangle \pm \langle n\rangle - m| \lesssim 1\} \lesssim \min(A, B, N)^{-1} \min(B, N)^3. \tag{4.41}$$

Proof. Using the basic counting lemma (Lemma 4.15), we have

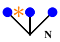
$$\begin{aligned} &\sup_{m \in \mathbb{Z}} \#\{n \in \mathbb{Z}^3 \colon |n| \sim N,\ |n+a| \sim B,\ |\langle a+n\rangle \pm \langle n\rangle - m| \lesssim 1\} \\ &\quad \leq \sup_{m \in \mathbb{Z}} \#\{n \in \mathbb{Z}^3 \colon |n| \sim N,\ |\langle a+n\rangle \pm \langle n\rangle - m| \lesssim 1\} \lesssim \min(A, N)^{-1} N^3. \end{aligned}$$

After a change of variables $b \overset{\text{def}}{=} n + a$, we obtain similarly

$$\sup_{m \in \mathbb{Z}} \#\{n \in \mathbb{Z}^3 \colon |n| \sim N, |n+a| \sim B, |\langle a+n\rangle \pm \langle n\rangle - m| \lesssim 1\} \lesssim \min(A, B)^{-1} B^3.$$

By combining both estimates we obtain (4.41). ∎

4.4.2. Cubic counting estimate. As mentioned at the beginning of this section, we only discuss and state the remaining counting estimates, but postpone the proofs until the appendix.

The cubic counting estimates play an important role in our analysis of the nonlinearity . In the following, we use max, med, and min for the maximum, median, and minimum of three frequency scales. Also, we assume $\pm_{123}, \pm_1, \pm_2, \pm_3 \in \{+, -\}$ are given signs and we define the phase function

$$\varphi(n_1, n_2, n_3) \overset{\text{def}}{=} \pm_{123}\langle n_{123}\rangle \pm_1 \langle n_1\rangle \pm_2 \langle n_2\rangle \pm_3 \langle n_3\rangle. \tag{4.42}$$

Proposition 4.18 (Main cubic counting estimate). *Let $N_1, N_2, N_3, N_{12}, N_{123} \geq 1$ and let $m \in \mathbb{Z}$. Then we have the following counting estimates:*

(i) *In the variables n_1, n_2, n_3, we have*

$$\#\{(n_1, n_2, n_3) \colon |n_1| \sim N_1,\ |n_2| \sim N_2,\ |n_3| \sim N_3,\ |\varphi - m| \leq 1\} \lesssim \mathrm{med}(N_1, N_2, N_3)^{-1} (N_1 N_2 N_3)^3,$$

(ii) *In the variables n_{123}, n_1, n_2, we have*

$$\#\{(n_{123}, n_1, n_2) \colon |n_{123}| \sim N_{123},\ |n_1| \sim N_1,\ |n_2| \sim N_2,\ |\varphi - m| \leq 1\} \lesssim \mathrm{med}(N_{123}, N_1, N_2)^{-1} (N_{123} N_1 N_2)^3.$$

(iii) *In the variables n_{123}, n_{12}, n_1, we have*

$$\#\{(n_{123}, n_{12}, n_1) \colon |n_{123}| \sim N_{123},\ |n_{12}| \sim N_{12},\ |n_1| \sim N_1,\ |\varphi - m| \le 1\} \lesssim \min(N_{12}, \max(N_{123}, N_1))^{-1} (N_{123} N_{12} N_1)^3.$$

(iv) *In the variables n_{12}, n_1, n_3, we have*

$$\#\{(n_{12}, n_1, n_3) \colon |n_{12}| \sim N_{12},\ |n_1| \sim N_1,\ |n_3| \sim N_3,\ |\varphi - m| \le 1\} \lesssim \min(N_{12}, \max(N_1, N_3))^{-1} (N_{12} N_1 N_3)^3.$$

Remark 4.19. The four estimates in Proposition 4.18 are sharp. In our analysis of the cubic nonlinearity, the frequencies n_1, n_2, and n_3 represent the frequencies of the three individual factors. The frequency n_{12} appears through the convolution with the interaction potential V. Finally, the frequency n_{123}, which is the frequency of the full nonlinearity, appears through the multiplier $\langle \nabla \rangle^{-1}$ in the Duhamel integral and in estimates of the H_x^s- and $\mathfrak{X}^{s,b}$-norms.

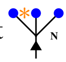
Since we postpone the proof, let us ease the reader's mind with the heuristic argument behind (i). Without the restriction due to the phase φ, the combined frequency variables (n_1, n_2, n_3) live in a set of cardinality $(N_1 N_2 N_3)^3$. As long as the level sets of φ have comparable cardinalities, we expect to gain a factor corresponding to the possible values of φ on the set $\{(n_1, n_2, n_3) \colon |n_1| \sim N_1,\ |n_2| \sim N_2,\ |n_3| \sim N_3\}$. Since φ is globally Lipschitz, one may ideally hope for a gain of the form $\max(N_1, N_2, N_3)$. Unfortunately, since

$$|\langle n_{123} \rangle - \langle n_1 \rangle + \langle n_2 \rangle + \langle n_3 \rangle| \lesssim \max(N_2, N_3), \tag{4.43}$$

the high×low×low interactions rule out a gain in $\max(N_1, N_2, N_3)$. As it turns out, however, our basic counting estimate allows us to obtain a gain of the form $\operatorname{med}(N_1, N_2, N_3)$, which is consistent with (4.43).

Proposition 4.20 (Cubic sum estimate). *Let $0 < s \le 1/2$, $0 \le \gamma < s + 1/2$, and let $N_1, N_2, N_3 \ge 1$. Let the phase function φ be as in (4.42). Then*

$$\sup_{m \in \mathbb{Z}} \sum_{n_1, n_2, n_3 \in \mathbb{Z}^3} \Big[\Big(\prod_{j=1}^{3} \chi_{N_j}(n_j) \Big) \langle n_{123} \rangle^{2(s-1)} \langle n_{12} \rangle^{-2\gamma} \Big(\prod_{j=1}^{3} \langle n_j \rangle^{-2} \Big) 1\{|\varphi - m| \le 1\} \Big] \lesssim \max(N_1, N_2, N_3)^{2(s-\gamma)} + \max(N_1, N_2)^{1-2\gamma} \max(N_1, N_2, N_3)^{2s-1}. \tag{4.44}$$

Remark 4.21. Proposition 4.20 plays an essential role in proving that  has regularity $\beta-$. In that argument, we will simply set $\gamma = \beta$.

4.4.3. Cubic sup-counting estimates. We now present cubic counting estimates involving suprema, which will be used in the proof of the tensor estimates in Section 4.4.8. In turn, the tensor estimates will then be used to prove the random matrix estimates in Section 6.

Lemma 4.22 (Cubic sup-counting estimates). *Let the phase function φ be as in (4.42).*

(i) *Taking the supremum in n and counting n_1, n_2, n_3, we have*

$$\sup_{n\in\mathbb{Z}^3} \#\{(n_1,n_2,n_3)\colon |n_1|\sim N_1,\ |n_2|\sim N_2,\ |n_3|\sim N_3,\ n=n_{123},\ |\varphi-m|\leq 1\} \lesssim \operatorname{med}(N_1,N_2,N_3)^3 \min(N_1,N_2,N_3)^2.$$

(ii) *Taking the supremum in n_2 and counting n, n_1, n_3, we have*

$$\sup_{n_2\in\mathbb{Z}^3} \#\{(n,n_1,n_3)\colon |n|\sim N_{123},\ |n_1|\sim N_1,\ |n_3|\sim N_3,\ n=n_{123},\ |\varphi-m|\leq 1\} \lesssim \operatorname{med}(N_{123},N_1,N_3)^3 \min(N_{123},N_1,N_3)^2.$$

(iii) *Taking the supremum in n and counting n_1, n_{12}, n_3, we have*

$$\sup_{n\in\mathbb{Z}^3} \#\{(n_{12},n_2,n_3)\colon |n_{12}|\sim N_{12},\ |n_2|\sim N_2,\ |n_3|\sim N_3,\ n=n_{123},\ |\varphi-m|\leq 1\} \lesssim \min(N_{12},N_1)^{-1}(N_{12}N_2)^3.$$

(iv) *Taking the supremum in n_3 and counting n, n_{12}, n_2, we have*

$$\sup_{n\in\mathbb{Z}^3} \#\{(n,n_{12},n_2)\colon |n|\sim N_{123},\ |n_{12}|\sim N_{12},\ |n_2|\sim N_2,\ n=n_{123},\ |\varphi-m|\leq 1\} \lesssim \min(N_{12},N_1)^{-1}(N_{12}N_2)^3.$$

4.4.4. Paracontrolled cubic counting estimate. We now present our final cubic counting estimate. It will be used to control

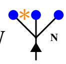
$$\big(\neg\, \boxed{\textcircled{\scriptsize<} \,\&\, \textcircled{\scriptsize<}}\big)\!:\!\big(V * (P_{\leq N}\overset{\bullet}{|} \cdot P_{\leq N} X_N) P_{\leq N}\overset{\bullet}{|}\big)\!:,$$

which appears in **CPara**.

Lemma 4.23 (Paracontrolled cubic sum estimate). *Let $N_{123}, N_1, N_2, N_3 \geq 1$ and $m\in\mathbb{Z}$. Let the phase function φ be as in (4.42). Then for all $0<\gamma<\beta$,*

$$\sup_{\substack{n_2\in\mathbb{Z}^3:\\ |n_2|\sim N_2}} \sum_{n_1,n_3\in\mathbb{Z}^3} \Big(\prod_{j=1,3} 1\{|n_j|\sim N_j\}\Big) \langle n_{123}\rangle^{2(s_2-1)} \langle n_{12}\rangle^{-2\beta} \langle n_1\rangle^{-2} \langle n_3\rangle^{-2}\, 1\{|\varphi-m|\leq 1\} \lesssim \max(N_1,N_2,N_3)^{2\delta_2} N_1^{-2\gamma} N_2^{2\gamma}. \tag{4.45}$$

4.4.5. Quartic counting estimates. Our expansion of the solution u_N and **So** only contains cubic, quintic, and septic stochastic objects. The quartic counting estimates will be used to control products such as

$P_{\leq N}\overset{\bullet}{|} \cdot P_{\leq N}$  , which occur as factors in the physical term **Phy**. We present two estimates which control the nonresonant (Lemma 4.24) and resonant portions (Lemma 4.26) of the product, respectively. On our way to the resonant estimate, we also prove the basic resonance estimate (Lemma 4.25).

Lemma 4.24 (Nonresonant quartic sum estimate). *Let $s < -1/2 - \eta$ and let $N_1, N_2, N_3, N_4 \geq 1$. Let the phase function φ be as in (4.42). Then*

$$\begin{aligned}\sup_{m\in\mathbb{Z}} \sum_{n_1,n_2,n_3,n_4\in\mathbb{Z}^3} \Big(\prod_{j=1}^{4} 1\{|n_j|\sim N_j\}\Big)\langle n_{1234}\rangle^{2s}\langle n_{123}\rangle^{-2}|\widehat{V}_S(n_1,n_2,n_3)|^2 \\ \times\Big(\prod_{j=1}^{4}\langle n_j\rangle^{-2}\Big)1\{|\varphi-m|\leq 1\} \\ \lesssim \max(N_1,N_2,N_3)^{-2\beta+2\eta}N_4^{-2\eta}.\end{aligned}$$

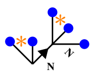
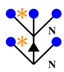
Lemma 4.25 (Basic resonance estimate). *Let $n_1, n_2 \in \mathbb{Z}^3$ be arbitrary and let $N_3 \geq 1$. Let the phase function φ be as in (4.42). Then*

$$\begin{aligned}\sum_{m\in\mathbb{Z}}\sum_{n_3\in\mathbb{Z}^3}\langle m\rangle^{-1}1\{|n_3|\sim N_3\}\langle n_{123}\rangle^{-1}\langle n_3\rangle^{-2}1\{|\varphi-m|\leq 1\} \\ \lesssim \log(2+N_3)\langle n_{12}\rangle^{-1}.\end{aligned} \tag{4.46}$$

Lemma 4.26 (Resonant quartic sum estimate). *Let $N_1, N_2, N_3 \geq 1$ and let $-1/2 < s < 0$. Let the phase function φ be as in (4.42). Then*

$$\begin{aligned}\sum_{n_1,n_2\in\mathbb{Z}^3}\Big[\Big(\prod_{j=1}^{2}1\{|n_j|\sim N_j\}\langle n_{12}\rangle^{2s}\langle n_1\rangle^{-2}\langle n_2\rangle^{-2} \\ \times\Big(\sum_{m\in\mathbb{Z}}\sum_{n_3\in\mathbb{Z}^3}\langle m\rangle^{-1}1\{|n_3|\sim N_3\}\langle n_{123}\rangle^{-1}\langle n_3\rangle^{-2}1\{|\varphi-m|\leq 1\}\Big)^2\Big] \\ \lesssim \log(2+N_3)^2\max(N_1,N_2)^{2s}.\end{aligned}$$

4.4.6. Quintic counting estimates. In order to estimate the quintic stochastic objects

 and ,

we require quintic sum estimates. Even at the quintic level, we need to make full use of dispersive effects. This is in contrast to the septic counting effects, which only rely on dispersive effects for cubic sub-objects but do not require dispersive effects at the full septic level.

We present three separate quintic sum estimates, which correspond to zero, one, or two probabilistic resonances.

Lemma 4.27 (Nonresonant quintic sum estimate). *Let* $s \leq 1/2 - 2\eta$ *and* $N_1, N_2, N_3, N_4, N_5 \geq 1$. *Furthermore, define three phase functions by*

$$\psi(n_3, n_4, n_5) \stackrel{\text{def}}{=} \pm_{345}\langle n_{345}\rangle \pm_3 \langle n_3\rangle \pm_4 \langle n_4\rangle \pm_5 \langle n_5\rangle,$$
$$\varphi(n_1, \ldots, n_5) \stackrel{\text{def}}{=} \pm_{12345}\langle n_{12345}\rangle \pm_{345} \langle n_{345}\rangle \pm_1 \langle n_1\rangle \pm_2 \langle n_2\rangle,$$
$$\widetilde{\varphi}(n_1, \ldots, n_5) \stackrel{\text{def}}{=} \pm_{12345}\langle n_{12345}\rangle \mp_{345} \langle n_{345}\rangle + \sum_{j=1}^{5} (\pm_j)\langle n_j\rangle.$$

Then

$$\begin{aligned}\sup_{m,m' \in \mathbb{Z}} \sum_{n_1,\ldots,n_5 \in \mathbb{Z}^3} \Big[\Big(\prod_{j=1}^{5} 1\{|n_j| \sim N_j\}\Big) \langle n_{12345}\rangle^{2(s-1)} \langle n_{1345}\rangle^{-2\beta} \langle n_{345}\rangle^{-2} \langle n_{34}\rangle^{-2\beta} \\ \times \Big(\prod_{j=1}^{5} \langle n_j\rangle^{-2}\Big) 1\{|\psi - m| \leq 1\} \cdot (1\{|\varphi - m'| \leq 1\} + 1\{|\widetilde{\varphi} - m'| \leq 1\}) \Big] \\ \lesssim \max(N_1, N_3, N_4, N_5)^{-2\beta + 4\eta} N_2^{-2\eta}.\end{aligned}$$

Lemma 4.28 (Single-resonance quintic sum estimate). *Let* $n_4, n_5 \in \mathbb{Z}^3$, $N_{45} \geq 1$, *and* $|n_{45}| \sim N_{45}$. *Furthermore, let* $\pm_3 \in \{+, -\}$. *Then*

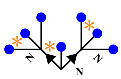
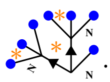
$$\sup_{m \in \mathbb{Z}^3} \sum_{n_3 \in \mathbb{Z}^3} \big[1\{|n_3| \sim N_3\} \langle n_{345}\rangle^{-1} \langle n_3\rangle^{-2} 1\{\langle n_{345}\rangle \pm_3 \langle n_3\rangle \in [m, m+1)\} \big] \lesssim N_{45}^{-1}.$$

After renaming the variables, Lemma 4.28 is essentially the same as Lemma 4.25. Our reason for restating Lemma 4.28 is to make it easier for the reader to refer back to this section.

Lemma 4.29 (Double-resonance quintic sum estimate). *Let* N_3, N_4, $N_5 \geq 1$ *and let* $\pm_3, \pm_4, \pm_5 \in \{+, -\}$. *Then*

$$\begin{aligned}\sup_{m \in \mathbb{Z}^3} \sup_{|n_5| \sim N_5} \sum_{n_3, n_4 \in \mathbb{Z}^3} \Big[\Big(\prod_{j=3}^{4} 1\{|n_j| \sim N_j\}\Big) \langle n_{345}\rangle^{-1} \langle n_{45}\rangle^{-\beta} \langle n_3\rangle^{-2} \langle n_4\rangle^{-2} \\ \times 1\{\langle n_{345}\rangle \pm_3 \langle n_3\rangle \pm_4 \langle n_4\rangle \pm_5 \langle n_5\rangle \in [m, m+1)\} \Big] \\ \lesssim \max(N_4, N_5)^{-\beta + \eta}. \qquad (4.47)\end{aligned}$$

4.4.7. Septic counting estimates. In order to state our septic counting estimates, we need to introduce pairings, where our definition is motivated by a similar notion in [28, Section 1.9]. The pairings are designed to capture the resonances in the septic stochastic objects

 and .

Definition 4.30 (Pairings). Let $J \geq 1$. We call a relation $\mathscr{P} \subseteq \{1, \dots, J\}^2$ a *pairing* if

(i) $\mathscr{P}$ is anti-reflexive, i.e., $(j, j) \notin \mathscr{P}$ for all $1 \leq j \leq J$,

(ii) $\mathscr{P}$ is symmetric, i.e., $(i, j) \in \mathscr{P}$ if and only if $(j, i) \in \mathscr{P}$,

(iii) $\mathscr{P}$ is univalent, i.e., for each $1 \leq i \leq J$, $(i, j) \in \mathscr{P}$ for at most one $1 \leq j \leq J$.

If $(i, j) \in \mathscr{P}$, the tuple (i, j) is called a *pair* (or $\mathscr{P}$-*pair*). If $1 \leq j \leq J$ is contained in a pair, we call j *paired* (or $\mathscr{P}$-*paired*). With a slight abuse of notation, we also write $j \in \mathscr{P}$ if j is paired. If j is not paired, we also say that j is *unpaired* and write $j \notin \mathscr{P}$.

Furthermore, let $\mathcal{A} = (\mathcal{A}_l)_{l=1,\dots,L}$ be a partition of $\{1, \dots, J\}$. We say that $\mathscr{P}$ *respects* $\mathcal{A}$ if $i, j \in \mathcal{A}_l$ for some $1 \leq l \leq L$ implies that $(i, j) \notin \mathscr{P}$. In other words, $\mathscr{P}$ does not pair elements of the same set inside the partition.

Finally, we call a vector $(n_1, \dots, n_J) \in (\mathbb{Z}^3)^J$ of frequencies *admissible* (or $\mathscr{P}$-*admissible*) if $(i, j) \in \mathscr{P}$ implies that $n_i = -n_j$.

Using Definition 4.30, we can now state the septic sum estimate.

Lemma 4.31 (Septic sum estimate). *Let* $1/2 < s < 1$ *and let* $N_{1234567}, N_{1234}, N_{567}, N_4 \geq 1$. *For any* $\pm_1, \pm_2, \pm_3 \in \{+, -\}$, *define the phase*

$$\varphi(n_j, \pm_j : 1 \leq j \leq 3) \overset{\text{def}}{=} \langle n_{123} \rangle \pm_1 \langle n_1 \rangle \pm_2 \langle n_2 \rangle \pm_3 \langle n_3 \rangle.$$

Furthermore, define

$$\begin{aligned} &\Phi(n_1, n_2, n_3) \\ &\quad = \sum_{\pm_1, \pm_2, \pm_3} \sum_{m \in \mathbb{Z}} \langle m \rangle^{-1} |\widehat{V}_S(n_1, n_2, n_3)| \langle n_{123} \rangle^{-1} \Big(\prod_{j=1}^{3} \langle n_j \rangle^{-1} \Big) 1\{|\varphi - m| \leq 1\}. \end{aligned}$$

Finally, let $\mathscr{P}$ *be a pairing of* $\{1, \dots, 7\}$ *which respects the partition* $\{1, 2, 3\}, \{4\}, \{5, 6, 7\}$ *and define the nonresonant frequency* $n_{\text{nr}} \in \mathbb{Z}^3$ *by*

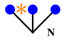
$$n_{\text{nr}} \overset{\text{def}}{=} \sum_{j \notin \mathscr{P}} n_j.$$

Then

$$\begin{aligned} \sum_{(n_j)_{j \notin \mathscr{P}}} \langle n_{\text{nr}} \rangle^{2(s-1)} \Big(\sum_{(n_j)_{j \in \mathscr{P}}}^{*} &1\{|n_{1234567}| \sim N_{1234567}\} 1\{|n_{1234}| \sim N_{1234}\} \\ &\times 1\{|n_{567}| \sim N_{567}\} 1\{|n_4| \sim N_4\} \\ &\times |\widehat{V}(n_{1234})| \Phi(n_1, n_2, n_3) \langle n_4 \rangle^{-1} \Phi(n_5, n_6, n_7) \Big)^2 \\ &\lesssim \log(2 + N_4)^2 \big(N_{1234567}^{2(s-1/2)} N_{567}^{-2(\beta-\eta)} + N_{1234567}^{-2(1-s-\eta)} \big) N_{1234}^{-2\beta}, \end{aligned}$$

where $\sum^{*}_{(n_j)_{j \in \mathscr{P}}}$ *denotes the sum over admissible frequencies.*

While the septic sum estimate (Lemma 4.31) may appear complicated, its proof is much easier than the cubic sum estimate (Lemma 4.20) or the quintic sum estimate (Lemma 4.27). The reason is that we do not rely on dispersive effects at the (full) septic level, and only use the dispersive effects in the cubic stochastic sub-objects.

4.4.8. Tensor estimates. The counting estimates from Sections 4.4.2–4.4.7 will be combined with Wiener chaos estimates to control stochastic objects such as . The estimates of the random matrix terms will follow a similar spirit. However, the Wiener chaos estimates will be replaced by the moment method (see Proposition 4.50) and the counting estimates will be replaced by deterministic tensor estimates. The tensor estimates, which partially rely on the counting estimates, are the main goal of this subsection.

We first recall the tensor notation from [30, Section 2.1].

Definition 4.32 (Tensors and tensor norms). Let $\mathcal{J} \subseteq \mathbb{N}_0$ be a finite set. A *tensor* $h = h_{n_\mathcal{J}}$ is a function from $(\mathbb{Z}^3)^{|\mathcal{J}|}$ into $\mathbb{C}$, where the input variables are given by $n_\mathcal{J}$. A *partition* of $\mathcal{J}$ is a pair of sets $(\mathcal{A}, \mathcal{B})$ such that $\mathcal{A} \cup \mathcal{B} = \mathcal{J}$ and $\mathcal{A} \cap \mathcal{B} = \emptyset$. For any partition $(\mathcal{A}, \mathcal{B})$, we define the tensor norm

$$\|h\|^2_{n_\mathcal{A} \to n_\mathcal{B}} = \sup\Big\{ \sum_{n_\mathcal{B}} \Big| \sum_{n_\mathcal{A}} h_{n_\mathcal{J}} z_{n_\mathcal{A}} \Big|^2 : \sum_{n_\mathcal{A}} |z_{n_\mathcal{A}}|^2 = 1 \Big\}. \tag{4.48}$$

For example, if $h = h_{nn_1n_2n_3}$, then

$$\|h\|^2_{n_1n_2n_3 \to n} = \sup\Big\{ \sum_{n \in \mathbb{Z}^3} \Big| \sum_{n_1,n_2,n_3 \in \mathbb{Z}^3} h_{nn_1n_2n_3} z_{n_1n_2n_3} \Big|^2 : \sum_{n_1,n_2,n_3 \in \mathbb{Z}^3} |z_{n_1n_2n_3}|^2 = 1 \Big\}.$$

Lemma 4.33 (First deterministic tensor estimate). *Let $s < 1/2 + \beta - 2\delta_1 - 6\eta$, $m \in \mathbb{Z}$, $N_1, N_2, N_3, N_{12}, N_{123} \geq 1$, and $\pm_1, \pm_2, \pm_3, \pm_{123} \in \{+,-\}$. Let the phase function φ be as in* (4.42)*. and define the truncated tensor h by*

$$\begin{aligned} h_{nn_1n_2n_3} \overset{\text{def}}{=} \chi_{N_{123}}(N_{123}) \chi_{N_{12}}(n_{12}) \Big(\prod_{j=1}^{3} \rho_{\leq N}(n_j) \chi_{N_j}(n_j) \Big) \\ 1\{n = n_{123}\} 1\{|\varphi - m| \leq 1\} \langle n \rangle^{s-1} \widehat{V}(n_{12}) \langle n_1 \rangle^{-1} \langle n_2 \rangle^{-1} \langle n_3 \rangle^{-s_1}. \end{aligned} \tag{4.49}$$

Then

$$\begin{aligned} \max(\|h\|_{n_1n_2n_3 \to n}, \|h\|_{n_3 \to nn_1n_2}, \|h\|_{n_1n_3 \to nn_2}, \|h\|_{n_2n_3 \to nn_1}) \\ \lesssim \max(N_1, N_2, N_3)^{-\eta}. \end{aligned} \tag{4.50}$$

Remark 4.34. The first deterministic tensor estimate (Lemma 4.33) is the main ingredient in the estimate of

$w_N \mapsto (V *$  $) P_{\leq N} w_N,$

which is the first term in **RMT**. In contrast to the second tensor estimate below, we only impose $s < 1/2 + \beta$ instead of $s < 1/2$ (up to small corrections). The reason is that both instances of  are part of the convolution with V.

Proof of Lemma 4.33. The main ingredients are Schur's test and the sup-counting estimate (Lemma 4.22).

Step 1: $\|h\|_{n_1 n_2 n_3 \to n}$. Due to the symmetry $n_1 \leftrightarrow n_2$, we may assume that $N_1 \geq N_2$. Using Schur's test, we have

$$\begin{aligned}
&\|h\|^2_{n_1 n_2 n_3 \to n} \\
&\lesssim N_{123}^{2(s-1)} N_{12}^{-2\beta} N_1^{-2} N_2^{-2} N_3^{-2s_1} \sup_{n \in \mathbb{Z}^3} \sum_{n_1, n_2, n_3 \in \mathbb{Z}^3} \Big(\prod_{j=1}^{3} 1\{|n_j| \sim N_k\} \Big) \\
&\qquad \times 1\{|n_{12}| \sim N_{12}|\} 1\{|n| \sim N_{123}\} 1\{n = n_{123}\} 1\{|\varphi - m| \leq 1\} \\
&\quad \times \sup_{n_1, n_2, n_3 \in \mathbb{Z}^3} \sum_{n \in \mathbb{Z}^3} \Big(\prod_{j=1}^{3} 1\{|n_j| \sim N_k\} \Big) \\
&\qquad \times 1\{|n_{12}| \sim N_{12}|\} 1\{|n| \sim N_{123}\} 1\{n = n_{123}\} 1\{|\varphi - m| \leq 1\}.
\end{aligned}$$

Since n is uniquely determined by n_1, n_2, and n_3, the last factor can easily be bounded by 1. By using Lemma 4.22 (iii) and $\max(N_{12}, N_2) \lesssim \max(N_1, N_2) = N_1$, we obtain

$$\begin{aligned}
\|h\|^2_{n_1 n_2 n_3 \to n} &\lesssim N_{123}^{2(s-1)} N_{12}^{-2\beta} N_1^{-2} N_2^{-2} N_3^{-2s_1} \max(N_{12}, N_2) N_{12}^2 N_2^2 \\
&\lesssim N_{123}^{2(s-1)} N_{12}^{2-2\beta} N_1^{-1} N_3^{-2s_1} \lesssim N_{123}^{2(s-1)} N_{12}^{1-2\beta+2\eta} N_1^{-2\eta} N_3^{-2s_1}.
\end{aligned}$$

Furthermore, $N_{12} \lesssim \max(N_{123}, N_3) \lesssim N_{123} \cdot N_3$. Inserting this into the last inequality yields

$$\|h\|^2_{n_1 n_2 n_3 \to n} \lesssim N_{123}^{2s-1-2\beta+2\eta} N_1^{-2\eta} N_3^{1-2s_1-2\beta+2\eta} \lesssim (N_1 N_3)^{-2\eta}.$$

Step 2: $\|h\|_{n_3 \to n_1 n_2 n}$. The argument follows Step 1 nearly verbatim, except that we use Lemma 4.22 (iv).

Step 3: $\|h\|_{n_1 n_3 \to n_2 n}$. In this step, we ignore the dispersive effects, i.e., we simply bound

$$1\{|\varphi - m| \leq 1\} \leq 1.$$

By increasing s if necessary, we may assume $s \geq 1/2$. Using Schur's test and a simple volume argument, we find that

$$\begin{aligned}
&\|h\|^2_{n_1 n_3 \to n_2 n} \lesssim N_{123}^{2(s-1)} N_{12}^{-2\beta} N_1^{-2} N_2^{-2} N_3^{-2s_1} \\
&\quad \times \sup_{n, n_2 \in \mathbb{Z}^3} \sum_{n_1, n_3 \in \mathbb{Z}^3} \Big(\prod_{j=1}^{3} 1\{|n_j| \sim N_k\} \Big) 1\{|n_{12}| \sim N_{12}|\} 1\{|n| \sim N_{123}\} 1\{n = n_{123}\} \\
&\quad \times \sup_{n_1, n_3 \in \mathbb{Z}^3} \sum_{n_2, n \in \mathbb{Z}^3} \Big(\prod_{j=1}^{3} 1\{|n_j| \sim N_k\} \Big) 1\{|n_{12}| \sim N_{12}|\} 1\{|n| \sim N_{123}\} 1\{n = n_{123}\} \\
&\lesssim N_{123}^{2(s-1)} N_{12}^{-2\beta} N_1^{-2} N_2^{-2} N_3^{-2s_1} \min(N_1, N_{12}, N_3)^3 \min(N_2, N_{12}, N_{123})^3 \\
&\lesssim N_{123}^{2(s-1)} N_{12}^{-2\beta} N_1^{-2} N_2^{-2} N_3^{-2s_1} N_1^{2-2\eta} N_{12}^{1+4\eta-2s_1} N_3^{2s_1-2\eta} N_2^{2-2\eta} N_{12}^{2s-1+2\eta} N_{123}^{2(s-1)} \\
&\lesssim N_{12}^{2s-1-2\beta+2\delta_1+6\eta} (N_1 N_2 N_3)^{-2\eta}.
\end{aligned}$$

In the second last inequality, we have used $s \geq 1/2$. Since $2s - 1 - 2\beta + 2\delta_1 + 6\eta \leq 0$, this is acceptable.

Step 4: $\|h\|_{n_2 n_3 \to n_1 n}$. Due to the symmetry $n_1 \leftrightarrow n_2$, the estimate follows from Step 3. ∎

We now turn to the second tensor estimate.

Lemma 4.35 (Second deterministic tensor estimate). *Let $s < 1/2 - \eta$, let $m \in \mathbb{Z}$, and let $N_1, N_2, N_3, N_{12}, N_{123} \geq 1$. Let the phase function φ be as in (4.42) be as in and define the truncated tensor h by*

$$h_{nn_1n_2n_3} \overset{\text{def}}{=} \chi_{N_{123}}(N_{123})\chi_{N_{12}}(n_{12})\Big(\prod_{j=1}^{3} \rho_{\leq N}(n_j)\chi_{N_j}(n_j)\Big) \times 1\{n = n_{123}\}1\{|\varphi - m| \leq 1\}\langle n\rangle^{s-1}\widehat{V}(n_{12})\langle n_1\rangle^{-1}\langle n_2\rangle^{-s_2}\langle n_3\rangle^{-1}. \tag{4.51}$$

Then

$$\max(\|h\|_{n_1n_2n_3\to n}, \|h\|_{n_2\to nn_1n_3}, \|h\|_{n_2n_3\to nn_1}, \|h\|_{n_1n_2\to nn_3}) \lesssim N_{12}^{-\beta}\max(N_1, N_2, N_3)^{-\eta}. \tag{4.52}$$

Remark 4.36. Lemma 4.35 is the main ingredient in the estimate of

$$Y_N \mapsto \,:V * (P_{\leq N}\,\overset{\bullet}{|} \cdot P_{\leq N}(Y_N))\,(\neg\!\circledless)\,P_{\leq N}\,\overset{\bullet}{|}:,$$

which is the second term in **RMT**.

Proof of Lemma 4.35. The argument is similar to the proof of Lemma 4.33.

Step 1: $\|h\|_{n_1n_2n_3\to n}$. Using Schur's test, we see that

$$\begin{aligned}\|h\|^2_{n_1n_2n_3\to n} \lesssim\, & N_{123}^{2(s-1)}N_{12}^{-2\beta}N_1^{-2}N_2^{-2s_2}N_3^{-2} \\ & \times \sup_{|n|\sim N_{123}} \sum_{n_1,n_2,n_3\in\mathbb{Z}^3} \Big(\prod_{j=1}^{3} 1\{|n_j| \sim N_k\}\Big)1\{n = n_{123}\}1\{|\varphi - m| \leq 1\} \\ & \times \sup_{n_1,n_2,n_3\in\mathbb{Z}^3} \sum_{n\in\mathbb{Z}^3} \Big(\prod_{j=1}^{3} 1\{|n_j| \sim N_k\}\Big)1\{n = n_{123}\}1\{|\varphi - m| \leq 1\}.\end{aligned}$$

The last factor is easily bounded by 1, since n is uniquely determined by n_1, n_2, and n_3. By using Lemma 4.22 (i) and $s_2 \leq 1$, we obtain

$$\begin{aligned}\|h\|^2_{n_1n_2n_3\to n} &\lesssim N_{123}^{2(s-1)}N_{12}^{-2\beta}\operatorname{med}(N_1,N_2,N_3)^3\min(N_1,N_2,N_3)^2N_1^{-2}N_2^{-2s_2}N_3^{-2} \\ &\lesssim N_{123}^{2(s-1)}N_{12}^{-2\beta}\max(N_1,N_2,N_3)^{-2s_2}\operatorname{med}(N_1,N_2,N_3)^{3-2}\min(N_1,N_2,N_3)^{2-2} \\ &\lesssim N_{123}^{2(s-1)}N_{12}^{-2\beta}\max(N_1,N_2,N_3)^{1-2s_2}.\end{aligned}$$

This is acceptable since $s \leq 1$ and $\eta \ll \delta_2$.

Step 2: $\|h\|_{n_2\to n_1n_3n}$. This argument is similar to Step 1, but the roles of n_2 and n are reversed. Using Schur's test, we obtain

$$\begin{aligned}\|h\|^2_{n_2\to n_1n_3n} &\lesssim N_{123}^{2(s-1)}N_{12}^{-2\beta}N_1^{-2}N_2^{-2s_2}N_3^{-2}\\ &\times \sup_{|n_2|\sim N_2}\sum_{n_1,n_3,n\in\mathbb{Z}^3}\Big(\prod_{j=1}^3 1\{|n_j|\sim N_k\}\Big)1\{|n|\sim N_{123}\}1\{n=n_{123}\}1\{|\varphi-m|\le 1\}\\ &\times \sup_{n_1,n_3,n\in\mathbb{Z}^3}\sum_{n_2\in\mathbb{Z}^3}\Big(\prod_{j=1}^3 1\{|n_j|\sim N_k\}\Big)1\{|n|\sim N_{123}\}1\{n=n_{123}\}1\{|\varphi-m|\le 1\}.\end{aligned}$$

As before, the last factor is easily bounded by 1. By using Lemma 4.22 (ii) and $2(s-1) \geq -2$, we obtain

$$\begin{aligned}\|h\|^2_{n_2\to n_1n_3n} &\lesssim N_{123}^{2(s-1)}N_{12}^{-2\beta}\operatorname{med}(N_{123},N_1,N_3)^3\min(N_{123},N_1,N_3)^2N_1^{-2}N_2^{-2s_2}N_3^{-2}\\ &\lesssim N_{12}^{-2\beta}N_2^{-2s_2}\max(N_{123},N_1,N_3)^{2s-1}\\ &\lesssim N_{12}^{-2\beta}N_2^{-2s_2}\max(N_1,N_2,N_3)^{-2\eta}.\end{aligned}$$

In the last line, we have used $s < 1/2-\eta$.

Step 3: $\|h\|_{n_1n_2\to n_3n}$. In this step, we ignore the dispersive effects, i.e., we simply bound

$$1\{|\varphi-m|\le 1\}\le 1.$$

Using Schur's test and a simple volume bound, we obtain

$$\begin{aligned}\|h\|^2_{n_1n_2\to n_3n} &\lesssim N_{123}^{2(s-1)}N_{12}^{-2\beta}N_1^{-2}N_2^{-2s_2}N_3^{-2}\\ &\quad\times \sup_{n_3,n\in\mathbb{Z}^3}\sum_{n_1,n_2\in\mathbb{Z}^3}\Big(\prod_{j=1}^3 1\{|n_j|\sim N_k\}\Big)1\{|n|\sim N_{123}\}1\{n=n_{123}\}\\ &\quad\times \sup_{n_1,n_2\in\mathbb{Z}^3}\sum_{n_3,n\in\mathbb{Z}^3}\Big(\prod_{j=1}^3 1\{|n_j|\sim N_k\}\Big)1\{|n|\sim N_{123}\}1\{n=n_{123}\}\\ &\lesssim N_{123}^{2(s-1)}N_{12}^{-2\beta}N_1^{-2}N_2^{-2s_2}N_3^{-2}\min(N_1,N_2)^3\min(N_3,N_{123})^3\\ &\lesssim N_{123}^{2(s-1)}N_{12}^{-2\beta}N_1^{-2}N_2^{-2s_2}N_3^{-2}N_1^{2-2\eta}N_2^{1+2\eta}N_3^{2-2\eta}N_{123}^{1+2\eta}\\ &\lesssim N_{12}^{-2\beta}\max(N_1,N_2,N_3)^{-2\eta}.\end{aligned}$$

Step 4: $\|h\|_{n_2n_3\to n_1n}$. Arguing exactly as in Step 3, we obtain

$$\begin{aligned}\|h\|^2_{n_2n_3\to n_1n} &\lesssim N_{123}^{2(s-1)}N_{12}^{-2\beta}N_1^{-2}N_2^{-2s_2}N_3^{-2}\min(N_2,N_3)^3\min(N_1,N_{123})^3\\ &\lesssim N_{12}^{-2\beta}\max(N_1,N_2,N_3)^{-2\eta}.\end{aligned}$$

∎

4.5. *Gaussian processes*

We briefly review the notation from the stochastic control perspective of the first paper in this series [12], which was used in the proof of Theorem 1.1. In comparison with the first part of this series, however, we change the notation for the stochastic time variable. We use $\mathscr{s}$, a calligraphic "s", to denote the time variable in the stochastic control perspective. While the chosen font in $\mathscr{s}$ may be slightly unusual, we hope that this prevents any confusion with the time variable t in the nonlinear wave equation.

We let $(B_{\mathscr{s}}^n)_{n\in\mathbb{Z}^3\setminus\{0\}}$ be a sequence of standard complex Brownian motions such that $B_{\mathscr{s}}^{-n} = \overline{B_{\mathscr{s}}^n}$ and $B_{\mathscr{s}}^n$, $B_{\mathscr{s}}^m$ are independent for $n \neq \pm m$. We let $B_{\mathscr{s}}^0$ be a standard real-valued Brownian motion independent of $(B_{\mathscr{s}}^n)_{n\in\mathbb{Z}^3\setminus\{0\}}$. Furthermore, we let $B_{\mathscr{s}}(\cdot)$ be the Gaussian process with Fourier coefficients $(B_{\mathscr{s}}^n)_{n\in\mathbb{Z}^3}$, i.e.,

$$B_{\mathscr{s}}(x) \overset{\text{def}}{=} \sum_{n\in\mathbb{Z}^3} e^{i\langle n,x\rangle} B_{\mathscr{s}}^n.$$

For every $\mathscr{s} \geq 0$, the Gaussian process formally satisfies $\mathbb{E}[B_{\mathscr{s}}(x)B_{\mathscr{s}}(y)] = \mathscr{s} \cdot \delta(x-y)$ and hence $B_{\mathscr{s}}(\cdot)$ is a scalar multiple of spatial white noise. We also let $(\mathcal{F}_{\mathscr{s}})_{\mathscr{s}\geq 0}$ be the filtration corresponding to the family $(B_{\mathscr{s}}^n)_{\mathscr{s}\geq 0}$ of Gaussian processes.

The Gaussian free field g, however, has covariance $(1-\Delta)^{-1}$. To this end, we now introduce the Gaussian process $W_{\mathscr{s}}(x)$. We let $\sigma_{\mathscr{s}}(\xi) = (\frac{\mathrm{d}}{\mathrm{d}\mathscr{s}}\rho_{\mathscr{s}}(\xi)^2)^{1/2}$, where $\rho_{\mathscr{s}}$ is the frequency truncation from Section 1.3. For any $n \in \mathbb{Z}^3$, we then define

$$W_{\mathscr{s}}^n \overset{\text{def}}{=} \int_0^{\mathscr{s}} \frac{\sigma_{\mathscr{s}'}(n)}{\langle n\rangle} \,\mathrm{d}B_{\mathscr{s}'}^n\,. \tag{4.53}$$

We note that $W_{\mathscr{s}}^n$ is a complex Gaussian random variable with variance $\rho_{\mathscr{s}}(n)^2/\langle n\rangle^2$. We finally set

$$W_{\mathscr{s}}(x) \overset{\text{def}}{=} \sum_{n\in\mathbb{Z}^3} e^{i\langle n,x\rangle} W_{\mathscr{s}}^n. \tag{4.54}$$

Since the Gaussian random data $\bullet \in \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$ in Theorem 1.1 is a tuple of the initial data and initial velocity, we now let $(B^{\cos}, W^{\cos})$ and $(B^{\sin}, W^{\sin})$ be two independent copies of (B, W). Using this notation, we then take

$$\bullet = (W_\infty^{\cos}(x), \langle\nabla\rangle W_\infty^{\sin}(x)). \tag{4.55}$$

Using (4.55), we can represent the linear evolution as

$$\overset{\bullet}{|}(t) = \cos(t\langle\nabla\rangle)W_\infty^{\cos} + \sin(t\langle\nabla\rangle)W_\infty^{\sin},$$

which also motivates our notation.

4.6. *Multiple stochastic integrals*

In this section, we recall several definitions and results related to multiple stochastic integrals. A similar but shorter section already appeared in the appendix of the first paper of this series [12]. More detailed introductions can be found in the excellent textbook [51] and lecture notes [47]. The usefulness of this section is best illustrated by Proposition 4.44 below.

We define a Borel measure λ on $\mathbb{R}_{\geq 0} \times \mathbb{Z}^3$ by

$$d\lambda(\mathfrak{s}, n) = \frac{\sigma_{\mathfrak{s}}(n)^2}{\langle n \rangle^2} \, d\mathfrak{s} \, dn,$$

where $d\mathfrak{s}$ is the Lebesgue measure and dn is the counting measure on $\mathbb{Z}^3$. We define the corresponding inner product by

$$\langle f, g \rangle = \sum_{n \in \mathbb{Z}^3} \int_0^\infty f(\mathfrak{s}, n) \overline{g(\mathfrak{s}, n)} \frac{\sigma_{\mathfrak{s}}(n)^2}{\langle n \rangle^2} \, d\mathfrak{s}. \tag{4.56}$$

For any $f \in L^2(\mathbb{R}_{\geq 0} \times \mathbb{Z}^3, d\lambda)$, we define

$$W[f] = \sum_{n \in \mathbb{Z}^3} \int_0^\infty f(\mathfrak{s}, n) \, dW_{\mathfrak{s}}^n.$$

The inner integral can be understood as an Itô integral. Then, we can identify W with the family of complex-valued Gaussian random variables

$$W = \{W[f] \colon f \in L^2(\mathbb{R}_{\geq 0} \times \mathbb{Z}^3, d\lambda)\}.$$

For any $f \in L^2(\mathbb{R}_{\geq 0} \times \mathbb{Z}^3, d\lambda)$, we define the reflection operator $\mathcal{R}$ by

$$\mathcal{R} f(\mathfrak{s}, n) \overset{\text{def}}{=} \overline{f(\mathfrak{s}, -n)}.$$

Clearly, $\mathcal{R}$ is a real-linear isometry. A short calculation using Itô's isometry yields

$$\mathbb{E}\big[W[f]\overline{W[g]}\big] = \langle f, g \rangle \quad \text{and} \quad \mathbb{E}\big[W[f]W[g]\big] = \langle f, \mathcal{R} g \rangle.$$

Since this will be important below, we note that the second identity reads

$$\mathbb{E}\big[W[f]W[g]\big] = \sum_{n \in \mathbb{Z}^3} \int_0^\infty f(\mathfrak{s}, n) g(\mathfrak{s}, -n) \frac{\sigma_{\mathfrak{s}}(n)^2}{\langle n \rangle^2} \, d\mathfrak{s}. \tag{4.57}$$

To emphasize the integral character of $W[f]$, we now write

$$\mathcal{I}_1[f] \overset{\text{def}}{=} W[f].$$

In this notation, it becomes evident that we have been working with single-variable stochastic calculus. In order to express the resonances in our stochastic objects, it is more natural to work with multi-variable stochastic calculus. For $k \geq 1$, we define the measure λ_k on $(\mathbb{R}_{\geq 0} \times \mathbb{Z}^3)^k$ by

$$\lambda_k \overset{\text{def}}{=} \lambda \otimes \cdots \otimes \lambda.$$

To simplify the notation, we set $\mathcal{H}_k \overset{\text{def}}{=} L^2((\mathbb{R} \times \mathbb{Z}^3)^k, d\lambda_k)$. For any $f \in \mathcal{H}_k$, the multiple stochastic integral $\mathcal{I}_k[f]$ can then be constructed as in [51, Section 1.1.2]. We only recall the basic ingredients and refer to [51] for more details.

We denote by $\mathcal{E}_k$ the set of elementary functions of the form

$$f(\mathfrak{s}_1, n_1, \dots, \mathfrak{s}_k, n_k) = \sum_{\substack{l_1,\dots,l_k \in \\ \{\pm 1,\dots,\pm L\}}} a_{l_1,\dots,l_k} 1_{A_{l_1}\times\dots\times A_{l_k}}(\mathfrak{s}_1, n_1, \dots, \mathfrak{s}_k, n_k).$$

Here, $\{A_1, A_{-1}, \dots, A_L, A_{-L}\}$ is a regular system (cf. [47, Chapter 4]), i.e.,

$$A_{-l} = \{(\mathfrak{s}, -n) \colon (\mathfrak{s}, n) \in A_l\}$$

for all $1 \leq l \leq L$ and $A_{l_1} \cap A_{l_2} = \emptyset$ for all $l_1 \neq l_2$. Furthermore, we impose the condition $a_{l_1,\dots,l_k} = 0$ if $l_{k_1} = \pm l_{k_2}$ for some $k_1 \neq k_2$. For an elementary function, we define the multiple stochastic integral by

$$\mathcal{I}_k[f] \overset{\text{def}}{=} \sum_{\substack{l_1,\dots,l_k \in \\ \{\pm 1,\dots,\pm L\}}} a_{l_1,\dots,l_k} \prod_{j=1}^{k} W[A_{l_j}]. \tag{4.58}$$

Furthermore, we define the symmetrization of f by

$$\widetilde{f}(\mathfrak{s}_1, n_1, \dots, \mathfrak{s}_k, n_k) = \frac{1}{k!} \sum_{\pi \in S_k} f(\mathfrak{s}_{\pi(1)}, n_{\pi(1)}, \dots, \mathfrak{s}_{\pi(k)}, n_{\pi(k)}). \tag{4.59}$$

Lemma 4.37 (Basic properties). *For any $k, l \geq 1$, $f \in \mathcal{E}_k$, and $g \in \mathcal{E}_l$, we have:*

(i) *$\mathcal{I}_k$ is linear.*

(ii) *The integral is invariant under symmetrization, i.e., $\mathcal{I}_k[f] = \mathcal{I}_k[\widetilde{f}]$.*

(iii) *We have the Itô-isometry formula*

$$\mathbb{E}\big[\mathcal{I}_k[f] \cdot \overline{\mathcal{I}_l[g]}\big] = \delta_{kl} k! \int \widetilde{f}\, \overline{\widetilde{g}} \, \mathrm{d}\lambda_k.$$

(iv) *We have the formula for the expectation*

$$\begin{aligned} &\mathbb{E}\big[\mathcal{I}_k[f] \cdot \mathcal{I}_l[g]\big] \\ &\quad = \delta_{kl} k! \sum_{n_1,\dots,n_k} \int_0^\infty \dots \int_0^\infty \widetilde{f}(\mathfrak{s}_1, n_1, \dots, \mathfrak{s}_k, n_k) \cdot \widetilde{g}(\mathfrak{s}_1, -n_1, \dots, \mathfrak{s}_k, -n_k) \\ &\qquad \times \bigg(\prod_{j=1}^{k} \frac{\sigma_{\mathfrak{s}_j}^2(n_j)}{\langle n_j \rangle^2}\bigg) \mathrm{d}\mathfrak{s}_k \dots \mathrm{d}\mathfrak{s}_1. \end{aligned}$$

Proof. Up to minor modifications, the proof can be found in [51, p. 9] or [47, Chapter 4]. ∎

Using a density argument (see e.g. [51, p. 10] or [47, Lemma 4.1]), we can extend $\mathcal{I}_k$ from elementary functions to $\mathcal{H}_k$. In particular, for any fixed $m_1, \dots, m_k \in \mathbb{Z}^3$, we have that

$$\prod_{j=1}^{k} \delta_{n_j = m_j} \in \mathcal{H}_k$$

and we can write

$$\int_{[0,\infty)^k} \mathrm{d}W^{m_k}_{\mathfrak{s}_k} \dots \mathrm{d}W^{m_1}_{\mathfrak{s}_1} \overset{\text{def}}{=} \mathcal{I}_k\Big[\prod_{j=1}^k \delta_{n_j=m_j}\Big]. \tag{4.60}$$

We vehemently emphasize that the stochastic integral (4.60) does not coincide with the product $\prod_{j=1}^k W^{m_j}_\infty$. Instead, as will be clear from the product formula (Lemma 4.40) below, the stochastic integral (4.60) only contains the nonresonant portion of this product.

If $f = f(n_1, \dots, n_k)$ does not depend on the stochastic time variables $\mathfrak{s}_1, \dots, \mathfrak{s}_k$, the linearity of the multiple stochastic integral $\mathcal{I}_k$ and (4.60) naturally imply that

$$\mathcal{I}_k[f] = \sum_{n_1,\dots,n_k \in \mathbb{Z}^3} f(n_1, \dots, n_k) \int_{[0,\infty)^k} \mathrm{d}W^{n_k}_{\mathfrak{s}_k} \dots \mathrm{d}W^{n_1}_{\mathfrak{s}_1}. \tag{4.61}$$

Using Lemma 4.37 (iii), it follows that

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_{[0,\infty)^k} \mathrm{d}W^{n_k}_{\mathfrak{s}_k} \dots \mathrm{d}W^{n_1}_{\mathfrak{s}_1}\Big) \cdot \overline{\Big(\int_{[0,\infty)^k} \mathrm{d}W^{m_k}_{\mathfrak{s}_k} \dots \mathrm{d}W^{m_1}_{\mathfrak{s}_1}\Big)}\Big] \\
&= \frac{1}{k!}\Big(\sum_{\pi,\pi' \in S_k} 1\{n_{\pi(j)} = m_{\pi'(j)} \text{ for all } 1 \le j \le k\}\Big) \int_{[0,\infty)^k} \prod_{j=1}^m \frac{\sigma_{\mathfrak{s}_j}(n_j)^2}{\langle n_j \rangle^2} \mathrm{d}\mathfrak{s}_k \dots \mathrm{d}\mathfrak{s}_1 \\
&= \Big(\sum_{\pi \in S_k} 1\{n_{\pi(j)} = m_j \text{ for all } 1 \le j \le k\}\Big) \prod_{j=1}^m \langle n_j \rangle^{-2}.
\end{aligned}$$

Up to permutations, the family of multiple stochastic integrals (4.60) is therefore orthogonal. Naturally, a similar formula holds without the complex conjugate. More generally, if f depends on the stochastic time variables $\mathfrak{s}_1, \dots, \mathfrak{s}_k$, we have

$$\mathcal{I}_k[f] = \sum_{n_1,\dots,n_k \in \mathbb{Z}^3} \int_{[0,\infty)^k} f(\mathfrak{s}_1, n_1, \dots, \mathfrak{s}_k, n_k)\, \mathrm{d}W^{n_k}_{\mathfrak{s}_k} \dots \mathrm{d}W^{n_1}_{\mathfrak{s}_1}. \tag{4.62}$$

Here, the summands on the right-hand side are understood as multiple stochastic integrals with fixed $n_1, \dots, n_k$ (by inserting an indicator as in (4.60)). As is shown in the next lemma, this notation is consistent with iterated Itô integrals.

Lemma 4.38. *Let $k \ge 1$ and let $f \in \mathcal{H}_k$ be symmetric. Then*

$$\mathcal{I}_k[f] = k! \sum_{n_1,\dots,n_k \in \mathbb{Z}^3} \int_0^\infty \int_0^{\mathfrak{s}_1} \dots \int_0^{\mathfrak{s}_{k-1}} f(\mathfrak{s}_1, n_1, \dots, \mathfrak{s}_k, n_k)\, \mathrm{d}W^{n_k}_{\mathfrak{s}_k} \dots \mathrm{d}W^{n_1}_{\mathfrak{s}_1}, \tag{4.63}$$

where the right-hand side is understood as an iterated Itô integral.

This follows from the discussion of [51, (1.27)]. As a consequence of this lemma, we could also work with iterated Itô integrals instead of multiple stochastic integrals. While the iterated Itô integrals are more natural whenever martingale properties are utilized, the multiple stochastic integrals have a much simpler product formula, which simplifies many of our computations.

Before we can state the product formula, we need to define the contraction.

Definition 4.39 (Contraction). Let $k, l \geq 1$, let $f \in \mathcal{H}_k$, and let $g \in \mathcal{H}_l$. For any $0 \leq r \leq \min(k,l)$, we define the contraction of r indices by

$$\begin{aligned}(f \otimes_r g)&(s_1, n_1, \ldots, s_{k+l-2r}, n_{k+l-2r}) \\ &\stackrel{\text{def}}{=} \sum_{m_1,\ldots,m_r \in \mathbb{Z}^3} \int_0^\infty \cdots \int_0^\infty \Big[f(s_1, n_1, \ldots, s_{k-r}, n_{k-r}, r_1, m_1, \ldots, r_r, m_r) \\ &\quad \times g(s_{k+1-r}, n_{k+1-r}, \ldots, s_{k+l-2r}, n_{k+l-2r}, r_1, -m_1, \ldots, r_r, -m_r) \\ &\quad \times \prod_{j=1}^{k} \frac{\sigma_{r_j}(m_j)^2}{\langle m_j \rangle^2} \Big] \mathrm{d}r_r \ldots \mathrm{d}r_1.\end{aligned}$$

We note that even if $f \in \mathcal{H}_k$ and $g \in \mathcal{H}_l$ are both symmetric, the contraction $f \otimes_r g$ may not be symmetric. The reader should note the similarity of the contraction with the formula for the expectation in Lemma 4.37 (iv), which is no coincidence. If $f, g \in \mathcal{H}_1$, then

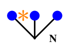
$$\mathbb{E}\big[\mathcal{I}_1[f] \cdot \mathcal{I}_1[g]\big] = f \otimes_1 g. \tag{4.64}$$

Thus, $f \otimes_1 g$ describes the (full) resonance portion of the product $\mathcal{I}_1[f] \cdot \mathcal{I}_1[g]$. The product formula is a (major) generalization of this simple fact.

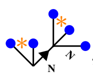
Lemma 4.40 (Product formula for multiple stochastic integrals; cf. [51, Prop. 1.1.3]). *Let $k, l \geq 1$ and let $f \in \mathcal{H}_k$ and $g \in \mathcal{H}_l$ be symmetric. Then*

$$\mathcal{I}_k[f] \cdot \mathcal{I}_l[g] = \sum_{r=0}^{\min(k,l)} r! \binom{k}{r} \binom{l}{r} \mathcal{I}_{k+l-2r}[f \otimes_r g]. \tag{4.65}$$

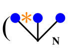
Using the product formula (Lemma 4.40), we can compute the nonresonant, partially resonant, and fully resonant portions of products such as

$(P_{\leq N}$  $)(t,x) \cdot (P_{\leq N}$  $)(t,x)$ and  $(t,x) \cdot$  (t,x).

Once the Duhamel operator occurs in the expression, however, we also need to consider two different physical times t and t'. For instance, in our estimate of the quintic stochastic object

 ,

we need to control

$(V *$  $(t,x)) \cdot \big(P_{\leq N} \sin((t-t')\langle\nabla\rangle)\langle\nabla\rangle^{-1}\big)($  $(t',x))$.

In order to consider two different physical times t and t', we need to consider multiple stochastic integrals with respect to two different (correlated) Gaussian processes, which we abstractly denote by W^a and W^b. We will assume that $\mathrm{Law}_{\mathbb{P}}(W^a) = \mathrm{Law}_{\mathbb{P}}(W^b) = \mathrm{Law}_{\mathbb{P}}(W)$. Regarding the relationship between W^a and W^b, we assume that $W^{a,n}$ and $W^{b,m}$ are independent for $m \neq \pm n$. Furthermore, let $\mathfrak{C}\colon \mathbb{Z}^3 \to [-1,1]$ be an even function. We assume that

$$\mathbb{E}[W^{(a),n}_{\mathfrak{s}_1} W^{(b),m}_{\mathfrak{s}_2}] = \delta_{n=-m}\, \mathfrak{C}(n) \int_0^{\mathfrak{s}_1 \wedge \mathfrak{s}_2} \frac{\sigma_{\mathfrak{s}}(n)^2}{\langle n \rangle^2} \,\mathrm{d}\mathfrak{s}, \tag{4.66}$$

$$\mathbb{E}[W^{(a),n}_{\mathfrak{s}_1} \overline{W^{(b),m}_{\mathfrak{s}_2}}] = \delta_{n=m}\, \mathfrak{C}(n) \int_0^{\mathfrak{s}_1 \wedge \mathfrak{s}_2} \frac{\sigma_{\mathfrak{s}}(n)^2}{\langle n \rangle^2} \,\mathrm{d}\mathfrak{s}. \tag{4.67}$$

Thus, $\mathfrak{C}$ is the (appropriately normalized) correlation of W^a and W^b. We can then set up the theory of multiple stochastic integrals with respect to a mixture of W^a and W^b as before. In order to fit this theory into the same framework as in [51], one only has to replace $\mathbb{R} \times \mathbb{Z}^3$ by $\mathbb{R} \times \mathbb{Z}^3 \times \{a,b\}$. A short calculation shows for any bounded and compactly supported $f, g\colon \mathbb{R} \times \mathbb{Z}^3 \times \{a,b\} \to \mathbb{C}$ that

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\sum_{\iota=a,b} \sum_{n\in\mathbb{Z}^3} \int_0^\infty f(\mathfrak{s},n,\iota)\,\mathrm{d}W^{(\iota),n}_{\mathfrak{s}}\Big)\Big(\sum_{\iota=a,b} \sum_{n\in\mathbb{Z}^3} \int_0^\infty g(\mathfrak{s},n,\iota)\,\mathrm{d}W^{(\iota),n}_{\mathfrak{s}}\Big)\Big] \\
&= \sum_{\iota,\iota'=a,b} \sum_{n\in\mathbb{Z}^3} (1\{\iota=\iota'\} + \mathfrak{C}(n) 1\{\iota\neq\iota'\}) \int_0^\infty f(\mathfrak{s},n,\iota)\cdot g(\mathfrak{s},-n,\iota') \frac{\sigma_{\mathfrak{s}}(n)^2}{\langle n \rangle^2}\,\mathrm{d}\mathfrak{s}.
\end{aligned} \tag{4.68}$$

and

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\sum_{\iota=a,b} \sum_{n\in\mathbb{Z}^3} \int_0^\infty f(\mathfrak{s},n,\iota)\,\mathrm{d}W^{(\iota),n}_{\mathfrak{s}}\Big)\overline{\Big(\sum_{\iota=a,b} \sum_{n\in\mathbb{Z}^3} \int_0^\infty g(\mathfrak{s},n,\iota)\,\mathrm{d}W^{(\iota),n}_{\mathfrak{s}}\Big)}\Big] \\
&= \sum_{\iota,\iota'=a,b} \sum_{n\in\mathbb{Z}^3} (1\{\iota=\iota'\} + \mathfrak{C}(n) 1\{\iota\neq\iota'\}) \int_0^\infty f(\mathfrak{s},n,\iota)\cdot \overline{g(\mathfrak{s},n,\iota')} \frac{\sigma_{\mathfrak{s}}(n)^2}{\langle n \rangle^2}\,\mathrm{d}\mathfrak{s}.
\end{aligned} \tag{4.69}$$

The sesquilinear form in (4.69), viewed as a function f and g, is no longer positive definite. For instance, if $W^{(a)} = -W^{(b)}$, and hence $\mathfrak{C} = -1$, $f = g$, and $f(\mathfrak{s},n,a) = f(\mathfrak{s},n,b)$ for all $\mathfrak{s} \in \mathbb{R}_{\geq 0}$ and $n \in \mathbb{Z}^3$, it vanishes identically. Nevertheless, due to the condition $|\mathfrak{C}| \leq 1$ imposed on the correlation function $\mathfrak{C}$, it is bounded by (a scalar multiple of) the inner product

$$\sum_{\iota=a,b} \sum_{n\in\mathbb{Z}^3} \int_0^\infty f(\mathfrak{s},n,\iota)\cdot \overline{g(\mathfrak{s},n,\iota)} \frac{\sigma_{\mathfrak{s}}(n)^2}{\langle n \rangle^2}\,\mathrm{d}\mathfrak{s}.$$

After defining a measure $\widetilde{\lambda}$ on $\mathbb{R} \times \mathbb{Z}^3 \times \{a,b\}$ by $\mathrm{d}\widetilde{\lambda} = \mathrm{d}\lambda\,\mathrm{d}\iota$, where $\mathrm{d}\iota$ is the integration with respect to the counting measure on $\{a,b\}$, this allows us to construct multiple stochastic integrals for functions in

$$L^2((\mathbb{R} \times \mathbb{Z}^3 \times \{a,b\})^k, \widetilde{\lambda}_k).$$

Similarly to (4.60), this allows us to define mixed multiple stochastic integrals such as

$$\int_{[0,\infty)^3} \mathrm{d}W^{(a),n_3}_{\mathfrak{s}_3}\, \mathrm{d}W^{(a),n_2}_{\mathfrak{s}_2}\, \mathrm{d}W^{(b),n_1}_{\mathfrak{s}_1}. \tag{4.70}$$

Unfortunately, the general theory now becomes notationally cumbersome. We therefore decided to only state the much simpler special case of the product formula needed in this paper.

Lemma 4.41 (Quadratic-cubic product formula). *Let $f\colon (\mathbb{Z}^3)^2 \to \mathbb{C}$ and $g\colon (\mathbb{Z}^3)^3 \to \mathbb{C}$. We assume that g is symmetric but do not require any symmetry of f. Then*

$$\begin{aligned}
&\Big(\sum_{n_1,n_2\in\mathbb{Z}^3} f(n_1,n_2) \int_{[0,\infty)^2} \mathrm{d}W^{(a),n_2}_{\mathfrak{s}_2}\, \mathrm{d}W^{(a),n_1}_{\mathfrak{s}_1} \Big) \\
&\qquad\qquad \times \Big(\sum_{n_3,n_4,n_5\in\mathbb{Z}^3} g(n_3,n_4,n_5) \int_{[0,\infty)^3} \mathrm{d}W^{(b),n_5}_{\mathfrak{s}_5}\, \mathrm{d}W^{(b),n_4}_{\mathfrak{s}_4}\, \mathrm{d}W^{(b),n_3}_{\mathfrak{s}_3} \Big) \\
&= \sum_{n_1,n_2,n_3,n_4,n_5\in\mathbb{Z}^3} f(n_1,n_2)g(n_3,n_4,n_5) \int_{[0,\infty)^5} \mathrm{d}W^{(b),n_5}_{\mathfrak{s}_5}\, \mathrm{d}W^{(b),n_4}_{\mathfrak{s}_4}\, \mathrm{d}W^{(b),n_3}_{\mathfrak{s}_3}\, \mathrm{d}W^{(a),n_2}_{\mathfrak{s}_2}\, \mathrm{d}W^{(a),n_1}_{\mathfrak{s}_1} \\
&\quad + 3 \sum_{n_2,n_4,n_5\in\mathbb{Z}^3} \Big(\sum_{n_1\in\mathbb{Z}^3} f(n_1,n_2) g(-n_1,n_4,n_5) \frac{\mathfrak{C}(n_1)}{\langle n_1\rangle^2} \Big) \int_{[0,\infty)^3} \mathrm{d}W^{(b),n_5}_{\mathfrak{s}_5}\, \mathrm{d}W^{(b),n_4}_{\mathfrak{s}_4}\, \mathrm{d}W^{(a),n_2}_{\mathfrak{s}_2} \\
&\quad + 3 \sum_{n_1,n_4,n_5\in\mathbb{Z}^3} \Big(\sum_{n_2\in\mathbb{Z}^3} f(n_1,n_2) g(-n_2,n_4,n_5) \frac{\mathfrak{C}(n_2)}{\langle n_2\rangle^2} \Big) \int_{[0,\infty)^3} \mathrm{d}W^{(b),n_5}_{\mathfrak{s}_5}\, \mathrm{d}W^{(b),n_4}_{\mathfrak{s}_4}\, \mathrm{d}W^{(a),n_1}_{\mathfrak{s}_1} \\
&\quad + 6 \sum_{n_5\in\mathbb{Z}^3} \Big(\sum_{n_1,n_2\in\mathbb{Z}^3} f(n_1,n_2) g(-n_1,-n_2,n_5) \frac{\mathfrak{C}(n_1)\mathfrak{C}(n_2)}{\langle n_1\rangle^2\langle n_2\rangle^2} \Big) \int_0^\infty \mathrm{d}W^{(b),n_5}_{\mathfrak{s}_5}.
\end{aligned}$$

Remark 4.42. Instead of working with the product $f(n_1,n_2)g(n_3,n_4,n_5)$, the formula has a natural extension to functions $h(n_1,\ldots,n_5)$ which are symmetric in n_3, n_4, and n_5. To this end, one only has to decompose

$$h(n_1,n_2,n_3,n_4,n_5) = \sum_{m_1,m_2\in\mathbb{Z}^3} 1\{(n_1,n_2)=(m_1,m_2)\} \cdot h(m_1,m_2,n_3,n_4,n_5).$$

We can then apply Lemma 4.41 to the individual summands.

Remark 4.43. While the formula in Lemma 4.41 is complicated, it is still an order of magnitude easier than working with products of Gaussians directly. If the reader is not convinced, we encourage him to work out (by hand) the corresponding resonant/nonresonant decomposition of

$$\begin{aligned}
&\Big(\sum_{n_1,n_2\in\mathbb{Z}^3} f(n_1,n_2) \Big(G^{(a)}_{n_1} \cdot G^{(a)}_{n_2} - \frac{\delta_{n_{12}=0}}{\langle n_1\rangle^2} \Big) \Big) \\
&\times \Big(\sum_{n_3,n_4,n_5\in\mathbb{Z}^3} g(n_3,n_4,n_5) \Big(G^{(b)}_{n_3} \cdot G^{(b)}_{n_4} \cdot G^{(b)}_{n_5} - \frac{\delta_{n_{34}=0}}{\langle n_3\rangle^2} G^{(b)}_{n_5} - \frac{\delta_{n_{35}=0}}{\langle n_3\rangle^2} G^{(b)}_{n_4} - \frac{\delta_{n_{45}=0}}{\langle n_4\rangle^2} G^{(b)}_{n_3} \Big) \Big),
\end{aligned}$$

where $G^{(\iota)} = W^{(\iota)}_\infty$ for $\iota = a, b$ are (correlated) families of Gaussian random variables.

After establishing the important definitions and properties of multiple stochastic integrals, it only remains to connect them with our stochastic objects. Let $W_{\mathfrak{z}}^{(\cos),n}$ and $W_{\mathfrak{z}}^{(\sin),n}$ be the Gaussian processes defined in Section 4.5. We recall that the linear evolution of the random initial data $\bullet$ is given by

$$\begin{aligned}\rotatebox[origin=c]{90}{$\multimap$}(t) &= \sum_{n\in\mathbb{Z}^3}\big(\cos(t\langle n\rangle)W_{\infty}^{(\cos),n} + \sin(t\langle n\rangle)W_{\infty}^{(\sin),n}\big)\exp(i\langle n,x\rangle)\\ &= \sum_{n\in\mathbb{Z}^3}\bigg(\int_0^\infty \mathrm{d}\big(\cos(t\langle n\rangle)W_{\mathfrak{z}}^{(\cos),n} + \sin(t\langle n\rangle)W_{\mathfrak{z}}^{(\sin),n}\big)\bigg)\exp(i\langle n,x\rangle). \qquad (4.71)\end{aligned}$$

In order to obtain a similar expression for the stochastic objects $\bullet\!\vee\!\bullet_{N}$ and $\bullet\!\overset{*}{\Psi}\!\bullet_{N}$, we define for any $k\geq 1$ and $n_1,\dots,n_k\in\mathbb{Z}^3$ the multiple stochastic integral

$$\begin{aligned}\mathcal{I}_k[t,n_1,\dots,n_k] \overset{\text{def}}{=} \int_{[0,\infty)^k} &\mathrm{d}\big(\cos(t\langle n_k\rangle)W_{\mathfrak{z}_k}^{(\cos),n_k} + \sin(t\langle n\rangle)W_{\mathfrak{z}_k}^{(\sin),n_k}\big)\dots\\ &\mathrm{d}\big(\cos(t\langle n_1\rangle)W_{\mathfrak{z}_1}^{(\cos),n_1} + \sin(t\langle n_1\rangle)W_{\mathfrak{z}_1}^{(\sin),n_1}\big). \qquad (4.72)\end{aligned}$$

In the proof of multi-linear dispersive estimates, it is essential to separate the time variable t from the randomness. To this end, we define the Gaussian processes

$$W_{\mathfrak{z}}^{(\pm),n} \overset{\text{def}}{=} W_{\mathfrak{z}}^{(\cos),n} \pm W_{\mathfrak{z}}^{(\sin),n}. \qquad (4.73)$$

Similarly to (4.72), we define for any $k\geq 1$, any $\pm_1,\dots,\pm_k\in\{+,-\}$, and any $n_1,\dots,n_k\in\mathbb{Z}^3$ the multiple stochastic integral

$$\mathcal{I}_k[n_j;\pm_j\colon 1\leq j\leq k] \overset{\text{def}}{=} \int_{[0,\infty)^k} \mathrm{d}W_{\mathfrak{z}_k}^{(\pm_k),n_k}\dots\mathrm{d}W_{\mathfrak{z}_1}^{(\pm_1),n_1}. \qquad (4.74)$$

It then follows that there exist coefficients $c\colon\{+,-\}^k\to\mathbb{C}$ depending only on the signs such that

$$\mathcal{I}_k[t,n_1,\dots,n_k] = \sum_{\pm_1,\dots,\pm_k} c(\pm_1,\dots,\pm_k)\Big(\prod_{j=1}^k \exp(\pm_j i t\langle n_j\rangle)\Big)\mathcal{I}_k[n_j;\pm_j\colon 1\leq j\leq k]. \qquad (4.75)$$

For convenience, we also define the normalized multiple stochastic integrals by

$$\widetilde{\mathcal{I}}_k[n_j;\pm_j\colon 1\leq j\leq k] = \Big(\prod_{j=1}^k\langle n_j\rangle\Big)\cdot\mathcal{I}_k[n_j;\pm_j\colon 1\leq j\leq k] \qquad (4.76)$$

We close this subsection with the following stochastic representation, which expresses the quadratic and cubic stochastic objects through multiple stochastic integrals.

Proposition 4.44. *Let $t\in\mathbb{R}$ and $N\geq 1$. Then, for all $n_1,n_2\in\mathbb{Z}^3$,*

$$\widehat{\rotatebox[origin=c]{90}{$\multimap$}}(t,n_1)\cdot\widehat{\rotatebox[origin=c]{90}{$\multimap$}}(t,n_2) - \frac{1}{\langle n_{12}\rangle^2}\delta_{n_{12}=0} = \mathcal{I}_2[t,n_1,n_2]. \qquad (4.77)$$

Furthermore,

$$(t,x) = \sum_{n_1,n_2\in\mathbb{Z}^3} \Big(\prod_{j=1}^{2} \rho_N(n_j)\Big) \mathcal{I}_2[t,n_1,n_2], \tag{4.78}$$

$$(t,x) = \sum_{n_1,n_2,n_3\in\mathbb{Z}^3} \Big(\prod_{j=1}^{3} \rho_N(n_j)\Big) \widehat{V}(n_{12})\, \mathcal{I}_3[t,n_1,n_2,n_3]. \tag{4.79}$$

Proof. This follows from [12, Lemma 2.5 and Proposition 2.9], Lemma 4.38, and the fact that the distribution of $(\mathfrak{s},n) \mapsto \cos(t\langle n\rangle) W_{\mathfrak{s}}^{(\cos),n} + \sin(t\langle n\rangle) W_{\mathfrak{s}}^{(\sin),n}$ is the same for all $t \in \mathbb{R}$. ∎

4.7. *Gaussian hypercontractivity and the moment method*

In this section, we first review Gaussian hypercontractivity and its consequences. To help the reader with a primary background in dispersive equations, let us first illustrate this phenomenon through a basic example. Let Z_σ be a Gaussian random variable with mean zero and variance σ^2. Using the exact formula for the moments of a Gaussian, we find for all $m \geq 1$ that

$$\mathbb{E}[Z_\sigma^2] = \sigma^2 \quad \text{and} \quad \mathbb{E}[Z_\sigma^{2m}] = \frac{(2m)!}{2^m m!} \cdot \sigma^{2m}.$$

A simple estimate now yields

$$(\mathbb{E}[Z_\sigma^{2m}])^{\frac{1}{2m}} \leq \Big(\frac{(2m)^{2m}}{2^m (m/e)^m}\Big)^{\frac{1}{2m}} \cdot \sigma = \sqrt{2em}\,(\mathbb{E}[Z_\sigma^2])^{1/2}.$$

Using Hölder's inequality, for all $p \geq 2$ we obtain

$$\|Z_\sigma\|_{L^p_\omega} \lesssim \sqrt{p}\, \|Z_\sigma\|_{L^2_\omega}. \tag{4.80}$$

Thus, higher L^p_ω-norms of Gaussians can be controlled through the lower L^2_ω-norm. The "hyper" in Gaussian hypercontractivity refers exactly to this gain of integrability. While (4.80) is not too interesting in itself, its significance lies in its generalizations to polynomials in infinitely many Gaussians! Furthermore, Gaussian hypercontractivity has connections with many different inequalities in analysis and probability theory, such as logarithmic Sobolev inequalities.

Our first proposition is also known as a Wiener chaos estimate. A version of this proposition can be found in [64, Theorem I.22] or [51, Theorem 1.4.1].

Proposition 4.45 (Gaussian hypercontractivity). *Let $k \geq 1$, let $\pm_1, \ldots, \pm_k \in \{+,-\}$, and let $a\colon (\mathbb{Z}^3)^k \to \mathbb{C}$ be a discrete function with finite support. Define the k-th order Gaussian chaos $\mathcal{G}_k$ by*

$$\mathcal{G}_k \overset{\text{def}}{=} \sum_{n_1,\ldots,n_k\in\mathbb{Z}^3} a(n_1,\ldots,n_k)\mathcal{I}_k[\pm_j, n_j\colon 1 \leq j \leq k]. \tag{4.81}$$

Then, for all $p \geq 2$,

$$\|\mathcal{G}_k\|_{L^p_\omega(\mathbb{P})} \lesssim p^{k/2} \|\mathcal{G}_k\|_{L^2_\omega(\mathbb{P})}. \tag{4.82}$$

Proposition 4.45 will play an important role in the estimates of stochastic objects such as . While Proposition 4.45 bounds the moments of the Gaussian chaos, the reader may prefer or be more familiar with a bound on probabilistic tails. As the next lemma shows, the two viewpoints are equivalent.

Lemma 4.46 (Moments and tails). *Let Z be a random variable and let $\gamma > 0$. Then the following properties are equivalent, where the parameters $K_1, K_2 > 0$ appearing below differ from each other by at most a constant factor depending only on γ.*

(1) *The tails of Z satisfy for all $\lambda \geq 0$ the inequality*

$$\mathbb{P}(|Z| \geq \lambda) \leq 2\exp(-(\lambda/K_1)^\gamma).$$

(2) *The moments of Z satisfy for all $p \geq 2$ the inequality*

$$\|Z\|_{L^p} \leq K_2 p^{1/\gamma}.$$

The lemma is an easy generalization of [73, Proposition 2.5.2 or Proposition 2.7.1]. As we have seen above, a Gaussian random variable corresponds to $\gamma = 2$. It is convenient to capture the size of K_2 in Lemma 4.46 (and hence K_1) through a norm.

Definition 4.47. Let $\gamma > 0$ and let Z be a random variable. We define

$$\|Z\|_{\Psi_\gamma} = \sup_{p \geq 2} p^{-1/\gamma} \|Z\|_{L^p_\omega}.$$

For more information regarding the Ψ_γ-norms, we refer the reader to the excellent textbook [73]. The next lemma shows that the Ψ_γ-norm is well-behaved under taking maxima of several random variables.

Lemma 4.48 (Maxima and the Ψ_γ-norm). *Let $\gamma > 0$, let $J \in \mathbb{N}$, and let $Z_1, \ldots, Z_J$ be random variables on the same probability space. Then*

$$\|\max(Z_1, \ldots, Z_J)\|_{\Psi_\gamma} \leq e \log(2+J)^{1/\gamma} \max_{j=1,\ldots,J} \|Z_j\|_{\Psi_\gamma}.$$

While this is only a minor generalization of [73, Exercise 2.5.10], we include the short proof.

Proof of Lemma 4.48. Let $p \geq 2$. For any $r \geq p$, it follows from the embedding $\ell^r_j \hookrightarrow \ell^\infty_j$ and Hölder's inequality that

$$\begin{aligned}\|\max(Z_1, \ldots, Z_J)\|_{L^p_\omega} &\leq \|Z_j\|_{L^p_\omega \ell^\infty_j} \leq \|Z_j\|_{L^p_\omega \ell^r_j} \leq \|Z_j\|_{L^r_\omega \ell^r_j} \\ &\leq J^{1/r} r^{1/\gamma} \max_{j=1,\ldots,J} \|Z_j\|_{\Psi_\gamma}.\end{aligned}$$

Then we choose $r = \log(2+J)p$, which yields the desired estimate. ■

We now turn to a combination of Gaussian hypercontractivity and the moment method, which will be essential to our treatment of the random matrix terms **RMT**. The following proposition, which is easy-to-use, general, and essentially sharp, was recently obtained by Deng, Nahmod, and Yue in [30, Proposition 2.8]. Before we state the estimate, we need the following definition, which relies on the tensor notation from Definition 4.32.

Definition 4.49 (Contracted random tensor). Let $\mathcal{J} \subseteq \mathbb{N}_0$, let $(\pm_j)_{j\in\mathcal{J}}$ be given, and let $N_{\max} \geq 1$. Let $h = h_{n_{\mathcal{J}}}$ be a tensor and assume that all vectors in the support of h satisfy $\|n_{\mathcal{J}}\| \leq N_{\max}$. Let $\mathcal{S} \subseteq \mathcal{J}$ and define $k \overset{\text{def}}{=} \#\mathcal{S}$. We then define the contracted random tensor $h_c = (h_c)_{n_{\mathcal{J}\backslash\mathcal{S}}}$ by

$$h_c(n_i : i \notin \mathcal{S}) \overset{\text{def}}{=} \sum_{(n_j)_{j\in\mathcal{S}}} h(n_{\mathcal{J}}) \cdot \widetilde{\mathcal{I}}_k[\pm_j, n_j : j \in \mathcal{S}], \tag{4.83}$$

where the normalized multiple stochastic integrals are as in (4.76).

In the next proposition, we use the tensor norms from Definition 4.32.

Proposition 4.50 ([30, Propositions 2.8, 4.14]). *Let $\mathcal{J}, \mathcal{S}, N_{\max}, h, h_c$, and k be as in Definition 4.49. Let $\mathcal{A}, \mathcal{B}$ be a partition of $\{1,\ldots,J\} \setminus \mathcal{S}$. Then, for all $p \geq 2$ and $\theta > 0$,*

$$\big\| \|h_c\|_{n_{\mathcal{A}}\to n_{\mathcal{B}}} \big\|_{L^p_\omega(\mathbb{P})} \lesssim_\theta N_{\max}^\theta \Big(\max_{\mathcal{X},\mathcal{Y}} \|h\|_{n_{\mathcal{X}}\to n_{\mathcal{Y}}} \Big) p^{k/2}, \tag{4.84}$$

where the maximum is taken over all sets $\mathcal{X}, \mathcal{Y}$ which satisfy $\mathcal{A} \subseteq \mathcal{X}$, $\mathcal{B} \subseteq \mathcal{Y}$, and form a partition of $\mathcal{J}$.

In [30], the proposition is stated in terms of nonresonant products of Gaussians instead of multiple stochastic integrals. Furthermore, the probabilistic estimate is stated in terms of the tail behavior instead of the moment growth. Both of these modifications can be obtained easily by replacing the large deviation estimate [30, Lemma 4.4] in the proof by Proposition 4.45.

We often simply refer to Proposition 4.50 as the moment method, since it is the main ingredient of the proof (cf. [30]). While the full generality of Proposition 4.50 is needed in [30], we will only rely on the following special case.

Example 4.51. Let $\pm_1, \pm_2 \in \{+,-\}$, let $h = h(n, n_1, n_2, n_3)$ be a tensor and assume that $\|(n, n_1, n_2, n_3)\| \lesssim N_{\max}$ on the support of h. Define the contracted random tensor h_c by

$$h_c(n, n_3) \overset{\text{def}}{=} \sum_{n_1,n_2\in\mathbb{Z}^3} h(n, n_1, n_2, n_3) \cdot \mathcal{I}_2[\pm_j, n_j : j = 1,2]. \tag{4.85}$$

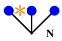
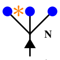
Then, for all $p \geq 2$ and $\theta > 0$,

$$\begin{aligned} &\big\| \|h_c\|_{n_3\to n} \big\|_{L^p_\omega} \\ &\quad \lesssim_\theta N_{\max}^\theta \max(\|h\|_{n_1n_2n_3\to n}, \|h\|_{n_3\to nn_1n_2}, \|h\|_{n_1n_3\to nn_2}, \|h\|_{n_2n_3\to nn_1}) \cdot p. \end{aligned}$$

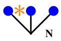
5. Explicit stochastic objects

In this section, we estimate the stochastic objects appearing in the expansion of u_N and in the evolution equations for X_N and Y_N. The analysis of explicit stochastic objects is necessary for both dispersive and parabolic equations. We refer the interested reader to the treatment of the cubic stochastic heat equation in [14, 41] and the quadratic stochastic wave equation in [37] for illustrative examples. While the algebraic aspects in dispersive and parabolic settings are similar, the analytic aspects are quite different. In the parabolic setting, the regularity of stochastic objects can be determined through simple "power-counting". In contrast, the optimal estimates in the dispersive setting require more complicated multi-linear dispersive estimates. We remind the reader that, as explained in Remark 1.4, we restrict ourselves to $0 < \beta < 1/2$.

5.1. Cubic stochastic objects

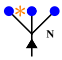
In this subsection, we analyze the cubic stochastic object  and the corresponding solution  to the forced wave equation. Ignoring the smoother component $\circ_{\mathbf{M}}$ of the initial data, they correspond to the first Picard iterate of (2.1).

Proposition 5.1 (Cubic stochastic objects). *Let $T \geq 1$ and let $s < \beta - \eta$. Then*

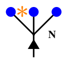


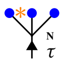
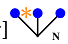
$$\Big\| \sup_{N\geq 1} \| \cdot \|_{\mathfrak{X}^{s-1,b_+-1}([0,T])} \Big\|_{L^p_\omega(\mathbb{P})} \lesssim T^2 p^{3/2}. \tag{5.1}$$

Furthermore,



$$\Big\| \sup_{N\geq 1} \Big\| \cdot \Big\|_{C_t^0 \mathscr{C}_x^s([0,T]\times\mathbb{T}^3)} \Big\|_{L^p_\omega(\mathbb{P})} \lesssim T^2 p^{3/2}. \tag{5.2}$$

In the frequency-localized version of (5.1) *and* (5.2)*, which is detailed in the proof, we gain an η'-power of the maximal frequency scale. Furthermore, we can replace*  *by*

 $= \mathrm{I}[1_{[0,\tau]}$  $]$.

Remark 5.2. We recall that the parameter T is important for the globalization argument, but does not enter into the local well-posedness theory. In order to achieve smallness on a short interval, we will instead use the time-localization lemma (Lemma 4.3) and $b_+ > b$.

Proof of Proposition 5.1. We first prove (5.1), which forms the main part of the argument. In the end, we follow a standard and short argument to show that (5.1), Gaussian hypercontractivity, and translation invariance imply (5.2). To simplify the notation, we set $N_{\max} = \max(N_1, N_2, N_3)$. In this argument, we rely on multiple stochastic integrals. Recalling the multiple stochastic integrals from (4.72) and the stochastic representation formula (Proposition 4.44), we have

$$
\begin{aligned}
\curlyvee_{N}(t,x) &= \sum_{n_1,n_2,n_3\in\mathbb{Z}^3} \rho_N(n_{123})\Big(\prod_{j=1}^{3}\rho_N(n_j)\Big)\widehat{V}(n_{12})\exp(i\langle n_{123},x\rangle)\mathcal{I}_3[t,n_1,n_2,n_3] \\
&= \sum_{\pm_1,\pm_2,\pm_3}\sum_{n_1,n_2,n_3\in\mathbb{Z}^3}\Big[c(\pm_j\colon 1\le j\le 3)\Big(\prod_{j=1}^{3}\rho_N(n_j)\Big)\widehat{V}(n_{12})\exp(i\langle n_{123},x\rangle) \\
&\qquad\times\Big(\prod_{j=1}^{3}\exp(\pm_j it\langle n_j\rangle)\Big)\mathcal{I}_3[\pm_j,n_j\colon 1\le j\le 3]\Big],
\end{aligned}
$$

where $c(\pm_j\colon 1\le j\le 3)$ are deterministic coefficients. Using a Littlewood–Paley decomposition, we obtain

$$
\curlyvee_{N} = \sum_{\pm_1,\pm_2,\pm_3}\sum_{N_1,N_2,N_3\ge 1} c(\pm_j\colon 1\le j\le 3)\curlyvee_{N}[\pm_j,N_j\colon 1\le j\le 3],
$$

where

$$
\begin{aligned}
\curlyvee_{N}[\pm_j,N_j\colon 1\le j\le 3](t,x) \overset{\text{def}}{=} \sum_{n_1,n_2,n_3\in\mathbb{Z}^3}\Big[\rho_N(n_{123})\Big(\prod_{j=1}^{3}\rho_N(n_j)\chi_{N_j}(n_j)\Big)\widehat{V}(n_{12}) \\
\times\exp(i\langle n_{123},x\rangle)\Big(\prod_{j=1}^{3}\exp(\pm_j it\langle n_j\rangle)\Big)\mathcal{I}_3[\pm_j,n_j\colon 1\le j\le 3]\Big].
\end{aligned}
$$

We estimate each dyadic block separately. We first prove the desired estimate for b_- instead of b_+ and then upgrade the estimate. Using Minkowski's integral inequality and Gaussian hypercontractivity (Proposition 4.45), we obtain

$$
\begin{aligned}
&\Big\|\big\|\curlyvee_{N}[\pm_j,N_j\colon 1\le j\le 3]\big\|_{\mathcal{X}^{s-1,b_--1}([0,T])}\Big\|_{L^p_\omega} \\
&\lesssim \max_{\pm_{123}}\Big\|\mathcal{F}_{t,x}\big(\chi(t/T)\curlyvee_{N}[\pm_j,N_j\colon 1\le j\le 3](t,x)\big)(\lambda\mp_{123}\langle n\rangle,n)\Big\|_{L^p_\omega L^2_\lambda \ell^2_n(\Omega\times\mathbb{R}\times\mathbb{T}^3)} \\
&\lesssim p^{3/2}\max_{\pm_{123}}\Big\|\mathcal{F}_{t,x}\big(\chi(t/T)\curlyvee_{N}[\pm_j,N_j\colon 1\le j\le 3](t,x)\big) \\
&\qquad\qquad\times(\lambda\mp_{123}\langle n\rangle,n)\Big\|_{L^2_\omega L^2_\lambda \ell^2_n(\Omega\times\mathbb{R}\times\mathbb{T}^3)}. \qquad (5.3)
\end{aligned}
$$

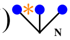
For a fixed sign $\pm_{123}$, we define the phase φ by

$$
\varphi(n_1,n_2,n_3) \overset{\text{def}}{=} \pm_{123}\langle n_{123}\rangle \pm_1 \langle n_1\rangle \pm_2 \langle n_2\rangle \pm_3 \langle n_3\rangle.
$$

Using the definition of φ, we can write the space-time Fourier transform of a dyadic piece in the cubic stochastic object $\chi(t/T)\curlyvee_{N}$ as

$$
\begin{aligned}
&\mathcal{F}_{t,x}\big(\chi(t/T)\curlyvee_{N}[\pm_j,N_j\colon 1\le j\le 3](t,x)\big)(\lambda\mp_{123}\langle n\rangle,n) \\
&\quad= T\sum_{n_1,n_2,n_3\in\mathbb{Z}^3}\Big[1\{n=n_{123}\}\rho_N(n_{123})\Big(\prod_{j=1}^{3}\rho_N(n_j)\chi_{N_j}(n_j)\Big)\widehat{V}(n_{12}) \\
&\qquad\qquad\times\widehat{\chi}(T(\lambda-\varphi(n_1,n_2,n_3)))\mathcal{I}_3[\pm_j,n_j\colon 1\le j\le 3]\Big]. \qquad (5.4)
\end{aligned}
$$

Using the orthogonality of the multiple stochastic integrals and the decay of $\widehat{\chi}$, we obtain

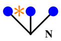
$\big\| \mathcal{F}_{t,x}\big(\chi(t/T)$  $[\pm_j, N_j : 1 \leq j \leq 3](t,x)\big)(\lambda \mp_{123} \langle n \rangle, n)\big\|^2_{L^2_\omega L^2_\lambda \ell^2_n(\Omega\times\mathbb{R}\times\mathbb{T}^3)}$

$$\lesssim T^2 N_1^{-2} N_2^{-2} N_3^{-2} \sum_{n_1,n_2,n_3\in\mathbb{Z}^3} \Big[\Big(\prod_{j=1}^{3} \chi_{N_j}(n_j)\Big)\langle n_{123}\rangle^{2(s-1)} |\widehat{V}(n_{12})|^2 \times \int_{\mathbb{R}} \mathrm{d}\lambda\, \langle\lambda\rangle^{2(b_--1)} |\widehat{\chi}(T(\lambda-\varphi(n_1,n_2,n_3)))|^2\Big]$$

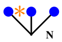
$$\lesssim T^2 N_1^{-2} N_2^{-2} N_3^{-2} \times \sum_{n_1,n_2,n_3\in\mathbb{Z}^3} \Big(\prod_{j=1}^{3} \chi_{N_j}(n_j)\Big)\langle n_{123}\rangle^{2(s-1)} |\widehat{V}(n_{12})|^2 \langle \varphi(n_1,n_2,n_3)\rangle^{2(b_--1)}$$

$$\lesssim T^2 N_1^{-2} N_2^{-2} N_3^{-2} \times \sup_{m\in\mathbb{Z}^3} \sum_{n_1,n_2,n_3\in\mathbb{Z}^3} \Big(\prod_{j=1}^{3} \chi_{N_j}(n_j)\Big)\langle n_{123}\rangle^{2(s-1)} |\widehat{V}(n_{12})|^2 1\{|\varphi - m| \leq 1\}.$$

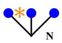
Combining this with (5.3) and using the cubic sum estimate (Proposition 4.20), we obtain

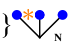
$\big\| \|$  $[\pm_j, N_j : 1 \leq j \leq 3]\|_{\mathfrak{X}^{s-1,b_--1}([0,T])} \big\|_{L^p_\omega} \lesssim T p^{3/2} N_{\max}^{s-\beta}.$

Since there are at most $\lesssim \log(10+N_{\max})$ nontrivial choices for N, we deduce from Lemma 4.48 that

$\big\| \sup_{N\geq 1} \|$  $[\pm_j, N_j : 1 \leq j \leq 3]\|_{\mathfrak{X}^{s-1,b_--1}([0,T])} \big\|_{L^p_\omega} \lesssim T \log\log(10+N_{\max})^2 N_{\max}^{s-\beta} p^{3/2}. \qquad (5.5)$

After summing over the dyadic scales, (5.5) almost implies (5.1) except that b_- needs to be replaced by b_+. To achieve this, we utilize the room of the estimate (5.5) in the maximal frequency scale. Using Plancherel's theorem, Minkowski's integral inequality, and Gaussian hypercontractivity, we have

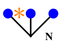
$\big\| \sup_{N\geq 1} \|$  $[\pm_j, N_j : 1 \leq j \leq 3]\|_{\mathfrak{X}^{0,0}([0,T])} \big\|_{L^p_\omega}$

$\lesssim \log\log(10+N_{\max})^2 \sup_N \|1\{0 \leq t \leq T\}$  $[\pm_j, N_j : 1 \leq j \leq 3]\|_{L^p_\omega L^2_t L^2_x}$

$$\lesssim T^{1/2} \log\log(10+N_{\max})^2 p^{3/2} \Big(\sum_{n_1,n_2,n_3\in\mathbb{Z}^3} \prod_{j=1}^{3} (\chi_{N_j}(n_j)\langle n_j\rangle^{-2})\Big)^{1/2}$$

$$\lesssim T^{1/2} \log\log(10+N_{\max})^2 N_{\max}^{3/2} p^{3/2}.$$

By interpolating this estimate with (5.5), we obtain

$\big\| \sup_{N\geq 1} \|$  $[\pm_j, N_j : 1 \leq j \leq 3]\|_{\mathfrak{X}^{s-1,b_+-1}([0,T])} \big\|_{L^p_\omega}$

$$\lesssim T \log\log(10+N_{\max})^2 N_{\max}^{s-\beta+4(b_+-b_-)} p^{3/2} \lesssim T N_{\max}^{s-\beta+5(b_+-b_-)} p^{3/2}. \qquad (5.6)$$

After summing over the dyadic scales, this finally yields (5.1). We prove the second estimate (5.2) using the (frequency-localized version of the) first estimate. We present the details of the (standard) argument, but skip similar steps in subsequent proofs. Using the energy estimate (Lemma 4.8) and the (frequency-localized version of the) first estimate (5.1), we obtain

$$\Big\| \sup_{N\geq 1} \Big\| \cubicstar_{N}[\pm_j, N_j \colon 1 \leq j \leq 3] \Big\|_{L_t^\infty H_x^s} \Big\|_{L_\omega^p} \lesssim (1+T) N_{\max}^{s-\beta+5(b_+-b_-)} p^{3/2}. \tag{5.7}$$

For any $2 \leq q \leq p$, from Sobolev embedding (in space-time), Minkowski's integral inequality, and Gaussian hypercontractivity we have

$$\begin{aligned} &\Big\| \cubicstar_{N}[\pm_j, N_j \colon 1 \leq j \leq 3] \Big\|_{L_\omega^p L_t^\infty \mathscr{C}_x^s} \\ &\quad \lesssim N_{\max}^{4/q} \Big\| \langle \nabla \rangle^s \cubicstar_{N}[\pm_j, N_j \colon 1 \leq j \leq 3] \Big\|_{L_\omega^p L_t^q L_x^q} \\ &\quad \lesssim N_{\max}^{4/q} \Big\| \langle \nabla \rangle^s \cubicstar_{N}[\pm_j, N_j \colon 1 \leq j \leq 3] \Big\|_{L_t^q L_x^q L_\omega^p} \\ &\quad \lesssim N_{\max}^{4/q} p^{3/2} \Big\| \langle \nabla \rangle^s \cubicstar_{N}[\pm_j, N_j \colon 1 \leq j \leq 3] \Big\|_{L_t^q L_x^q L_\omega^2}. \end{aligned} \tag{5.8}$$

For a fixed $t \in \mathbb{R}$, the distribution of $\langle \nabla \rangle^s \cubicstar_{N}[\pm_j, N_j \colon 1 \leq j \leq 3](t,x)$ is translation invariant. Thus, we can replace the L_x^q-norm in (5.8) by the L_x^2-norm. Using Minkowski's integral inequality and (5.7) then yields

$$\begin{aligned} &\Big\| \cubicstar_{N}[\pm_j, N_j \colon 1 \leq j \leq 3] \Big\|_{L_\omega^p L_t^\infty \mathscr{C}_x^s} \\ &\quad \lesssim N_{\max}^{4/q} p^{3/2} \Big\| \langle \nabla \rangle^s \cubicstar_{N}[\pm_j, N_j \colon 1 \leq j \leq 3] \Big\|_{L_\omega^2 L_t^q L_x^2} \\ &\quad \lesssim T^{1+1/q} N_{\max}^{s-\beta+5(b_+-b_-)+4/q} p^{3/2}. \end{aligned}$$

By choosing $q = q(b_+, b_-)$ sufficiently large and then summing over dyadic scales, this proves (5.2) for $p \gtrsim_{b_+,b_-} 1$. The smaller values of p can be handled by using Hölder's inequality in ω.

Finally, the statement for $\cubicstar_{N}$ replaced by $\cubicstar_{N,\tau}$ follows from the boundedness of $1_{[0,\tau]}(t)$ on $\mathcal{X}^{s_2-1,b_+-1}$, which was proven in Lemma 4.4. ∎

5.2. *Quartic stochastic objects*

The expansion $u_N = \linear + \cubicstar_{N} + w_N$ or the explicit stochastic objects in **So** only contain linear, cubic, quintic, or septic stochastic objects. However, the physical terms **Phy** contain terms such as

$$V * \Big(P_{\leq N} \linear \cdot P_{\leq N} \quartic_N \Big) P_{\leq N} w_N \quad \text{or} \quad V * \big(P_{\leq N} \linear \circledneq P_{\leq N} w_N \big) P_{\leq N} \quartic_N .$$

Since we treat $w_N \in \mathcal{X}^{s_1,b}$ using deterministic methods, they can be viewed as quartic expressions in the random initial data $\bullet$. Furthermore, due to the convolution with the interaction potential V in the second term, we also have to understand the product of $\linear$ and $\quartic_N$ at two different spatial points.

Proposition 5.3. *Let $N_{123}, N_4 \geq 1$. Then, for all $s < -1/2 - \eta$ and all $T \geq 1$,*

$$\begin{aligned} \Big\| \sup_{N \geq 1} \sup_{y \in \mathbb{T}^3} \Big\| \Big(P_{N_{123}} P_{\leq N} \quartic_N (t, x-y) \Big) \cdot P_{N_4} P_{\leq N} \linear (t,x) \Big\|_{C_t^0 \mathscr{C}_x^s([0,T] \times \mathbb{T}^3)} \Big\|_{L^p_\omega(\mathbb{P})} \\ \lesssim T^3 p^2 \max(N_{123}, N_4)^{-\eta/2} N_4^{\kappa}. \qquad (5.9) \end{aligned}$$

If $N_{123} \sim N_4$, then for all $s < -1/2 + \beta - 2\eta$,

$$\begin{aligned} \Big\| \sup_{N \geq 1} \sup_{y \in \mathbb{T}^3} \Big\| \Big(P_{N_{123}} P_{\leq N} \quartic_N (t, x-y) \Big) \cdot P_{N_4} P_{\leq N} \linear (t,x) \Big\|_{C_t^0 \mathscr{C}_x^s([0,T] \times \mathbb{T}^3)} \Big\|_{L^p_\omega(\mathbb{P})} \\ \lesssim T^3 p^2 N_4^{\kappa}. \qquad (5.10) \end{aligned}$$

Finally, without the shift in $y \in \mathbb{T}^3$, for $s < -1/2 - \eta$ we have

$$\begin{aligned} \Big\| \sup_{N \geq 1} \Big\| \Big(P_{N_{123}} P_{\leq N} \quartic_N (t, x) \Big) \cdot P_{N_4} P_{\leq N} \linear (t,x) \Big\|_{C_t^0 \mathscr{C}_x^s([0,T] \times \mathbb{T}^3)} \Big\|_{L^p_\omega(\mathbb{P})} \\ \lesssim T^3 p^2 \max(N_{123}, N_4)^{-\eta/10}. \qquad (5.11) \end{aligned}$$

Remark 5.4. In the fully frequency-localized version of Proposition 5.3, which is detailed in the proof, we gain an η'-power of the maximal frequency scale. As in Proposition 5.1, we may also replace $\quartic_N$ by $\quartic_{N,\tau} = \mathrm{I}[1_{[0,\tau]} \quarticnl_N]$.

Remark 5.5. We recall that η is much smaller than κ and hence the right-hand sides of (5.9) and (5.10) diverge as $N_4 \to \infty$. The third estimate (5.11) is quite delicate and requires the sine-cancellation lemma. A similar estimate is not available for the partially shifted process and it is likely that at least a logarithmic loss is necessary in (5.9) and (5.10) as N_4 tends to infinity.

Proof of Proposition 5.3. We prove (5.9) and (5.10) simultaneously. The third estimate (5.11) will mainly utilize the same estimates, but also requires the sine-cancellation lemma (Lemma 4.14). Using the representation based on multiple stochastic integrals (Proposition 4.44), we find that

$$
\begin{aligned}
&\Big(P_{N_{123}} P_{\leq N} \curlyvee_{\scriptscriptstyle N}(t, x-y)\Big) \cdot P_{N_4} P_{\leq N} \overset{\bullet}{|}(t,x) \\
&\quad= \sum_{N_1,N_2,N_3 \geq 1} \sum_{n_1,n_2,n_3,n_4 \in \mathbb{Z}^3} \rho_N(n_{123})^2 \chi_{N_{123}}(n_{123}) \Big(\prod_{j=1}^{4} \rho_N(n_j)\chi_{N_j}(n_j)\Big) \\
&\quad\quad \times \widehat{V}_S(n_1,n_2,n_3)\langle n_{123}\rangle^{-1} \exp(i\langle n_{1234}, x\rangle - i\langle n_{123}, y\rangle) \\
&\quad\quad \times \Big(\int_0^t \sin((t-t')\langle n_{123}\rangle)\, \mathcal{I}_3[t'; n_1,n_2,n_3] \cdot \mathcal{I}_1[t; n_4]\, \mathrm{d}t'\Big).
\end{aligned}
$$

Using the product formula for multiple stochastic integrals, we obtain

$$
\begin{aligned}
&\Big(P_{N_{123}} P_{\leq N} \curlyvee_{\scriptscriptstyle N}(t, x-y)\Big) \cdot P_{N_4} P_{\leq N} \overset{\bullet}{|}(t,x) \\
&\quad= \sum_{N_1,N_2,N_3 \geq 1} \mathcal{G}^{(4)}(t,x,y;N_*) + \sum_{N_1,N_2,N_3 \geq 1} \mathcal{G}^{(2)}(t,x,y;N_*),
\end{aligned}
$$

where the dependence on $N_{123}, N_1, N_2, N_3, N_4$ is indicated by N_* and the quartic and quadratic Gaussian chaoses are given by

$$
\begin{aligned}
&\mathcal{G}^{(4)}(t,x,y;N_*) \\
&= \sum_{\pm_1,\pm_2,\pm_3,\pm_4} \sum_{n_1,n_2,n_3,n_4 \in \mathbb{Z}^3} \Big[c(\pm_j : 1 \leq j \leq 4) \rho_N^2(n_{123})^2 \rho_N(n_4) \chi_{N_{123}}(n_{123}) \\
&\quad \times \Big(\prod_{j=1}^{4} \rho_{\leq N}(n_j)\chi_{N_j}(n_j)\Big) \widehat{V}_S(n_1,n_2,n_3)\langle n_{123}\rangle^{-1} \exp(i\langle n_{1234}, x\rangle - i\langle n_{123}, y\rangle) \\
&\quad \times \exp(\pm_4 i t \langle n_4\rangle) \Big(\int_0^t \sin((t-t')\langle n_{123}\rangle) \Big(\prod_{j=1}^{3} \exp(\pm_j i t' \langle n_j\rangle)\Big)\, \mathrm{d}t'\Big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \mathcal{I}_4(\pm_j, n_j : 1 \leq j \leq 4)\Big]
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathcal{G}^{(2)}(t,x,y;N_*) \\
&= 3 \sum_{\pm_1,\pm_2} \sum_{n_1,n_2 \in \mathbb{Z}^3} \Big[c(\pm_1,\pm_2) \Big(\prod_{j=1}^{2} \rho_N(n_j)\chi_{N_j}(n_j)\Big) \exp(i\langle n_{12}, x\rangle) \\
&\times \Big(\sum_{n_3 \in \mathbb{Z}^3} \Big[\rho_N(n_{123})^2 \rho_N(n_3)^2 \chi_{N_{123}}(n_{123}) \chi_{N_3}(n_3) \chi_{N_4}(n_3) \langle n_{123}\rangle^{-1} \langle n_3\rangle^{-2} \widehat{V}_S(n_1,n_2,n_3) \\
&\times \exp(-i\langle n_{123}, y\rangle) \int_0^t \sin((t-t')\langle n_{123}\rangle) \cos((t-t')\langle n_3\rangle) \prod_{j=1}^{2} \exp(\pm_j i t' \langle n_j\rangle))\, \mathrm{d}t' \Big]\Big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \mathcal{I}_2(\pm_j, n_j : j = 1,2)\Big].
\end{aligned}
$$

The quartic Gaussian chaos $\mathscr{G}^{(4)}$ and quadratic Gaussian chaoses $\mathscr{G}^{(2)}$ contain the resonant and nonresonant terms of the product, respectively. We estimate both terms separately.

The nonresonant term $\mathscr{G}^{(4)}$: We first let $s<-1/2-\eta$. Using Gaussian hypercontractivity and standard reductions (see e.g. the proof of Proposition 5.1), it suffices to estimate the $L_t^\infty L_\omega^2 H_x^s$-norm instead of the $L_\omega^p L_t^\infty \mathscr{C}_x^s$-norm. Let the phase function φ be as in (4.42). Using the orthogonality of the multiple stochastic integrals, for a fixed $t\in[0,T]$ we have

$$
\begin{aligned}
&\|\mathscr{G}^{(4)}(t,x,y;N_*)\|^2_{L^2_\omega H^s_x}\\
&\lesssim \sum_{\pm_1,\pm_2,\pm_3}\sum_{n_1,n_2,n_3,n_4\in\mathbb{Z}^3}\Big[\chi_{N_{123}}(n_{123})\Big(\prod_{j=1}^4\chi_{N_j}(n_j)\Big)|\widehat{V}_S(n_1,n_2,n_3)|^2\\
&\qquad\times\langle n_{1234}\rangle^{2s}\langle n_{123}\rangle^{-2}\Big(\prod_{j=1}^4\langle n_j\rangle^{-2}\Big)\Big|\int_0^t\sin((t-t')\langle n_{123}\rangle)\Big(\prod_{j=1}^3\exp(\pm_j it'\langle n_j\rangle)\Big)\,\mathrm{d}t'\Big|^2\Big]\\
&\lesssim (1+T)^2\sum_{\pm_1,\pm_2,\pm_3}\sum_{n_1,n_2,n_3,n_4\in\mathbb{Z}^3}\sum_{m\in\mathbb{Z}}\Big[\langle m\rangle^{-2}\chi_{N_{123}}(n_{123})\Big(\prod_{j=1}^4\chi_{N_j}(n_j)\Big)\\
&\qquad\times|\widehat{V}_S(n_1,n_2,n_3)|^2\langle n_{1234}\rangle^{2s}\langle n_{123}\rangle^{-2}\Big(\prod_{j=1}^4\langle n_j\rangle^{-2}\Big)1\{|\varphi-m|\le 1\}\Big]\\
&\lesssim T^2\sup_{m\in\mathbb{Z}}\sum_{\pm_1,\pm_2,\pm_3}\sum_{n_1,n_2,n_3,n_4\in\mathbb{Z}^3}\Big[\chi_{N_{123}}(n_{123})\Big(\prod_{j=1}^4\chi_{N_j}(n_j)\Big)|\widehat{V}_S(n_1,n_2,n_3)|^2\\
&\qquad\times\langle n_{1234}\rangle^{2s}\langle n_{123}\rangle^{-2}\Big(\prod_{j=1}^4\langle n_j\rangle^{-2}\Big)1\{|\varphi-m|\le 1\}\Big].
\end{aligned}
$$

Using the nonresonant quartic sum estimate (Lemma 4.24), it follows that

$$
\|\mathscr{G}^{(4)}(t,x,y;N_{123},N_1,N_2,N_3,N_4))\|^2_{L^2_\omega H^s_x}\lesssim T^2\max(N_1,N_2,N_3)^{-2\beta+2\eta}N_4^{-2\eta}.
$$

This yields (5.9) for the nonresonant component. If $N_{123}\sim N_4$, then

$$
\max(N_1,N_2,N_3)\gtrsim N_4,
$$

and hence we can raise the value of s by $\beta-\eta$. Thus, we also obtain (5.10) for the nonresonant component. Even when $y\neq 0$, our estimate for the nonresonant component does not exhibit any growth in N_4, and hence it also yields (5.11) for the nonresonant component.

The resonant term $\mathscr{G}^{(2)}$: This term exhibits a higher spatial regularity and we let $-1/2<s<0$. Using Gaussian hypercontractivity and standard reductions (see e.g. the proof of Proposition 5.1), it suffices to estimate the $L_t^\infty L_\omega^2 H_x^s$-norm instead of the $L_\omega^p L_t^\infty \mathscr{C}_x^s$-norm. Using the orthogonality of the multiple stochastic integrals, we have

$$
\begin{aligned}
&\|\mathscr{G}^{(2)}(t,x,y;N_*)\|^2_{L^2_\omega H^s_x} \\
&\lesssim \sum_{\pm_1,\pm_2} \sum_{n_1,n_2\in\mathbb{Z}^3} \bigg[\Big(\prod_{j=1}^{2} \chi_{N_j}(n_j)\Big) \langle n_{12}\rangle^{2s} \langle n_1\rangle^{-2} \langle n_2\rangle^{-2} \\
&\quad\times \bigg| \sum_{n_3\in\mathbb{Z}^3} \bigg[\rho_N(n_{123})^2 \rho_N(n_3)^2 \chi_{N_{123}}(n_{123}) \chi_{N_3}(n_3) \chi_{N_4}(n_3) \langle n_{123}\rangle^{-1} \langle n_3\rangle^{-2} \\
&\quad\times \widehat{V}_S(n_1,n_2,n_3) \exp(-i\langle n_{123}, y\rangle) \int_0^t \sin((t-t')\langle n_{123}\rangle) \cos((t-t')\langle n_3\rangle) \\
&\qquad\qquad \times \prod_{j=1}^{2} \exp(\pm_j i t' \langle n_j\rangle)\, \mathrm{d}t' \bigg]\bigg|^2\bigg]. \qquad (5.12)
\end{aligned}
$$

We now present two estimates of (5.12). The first estimate will yield (5.9) and (5.10). The second estimate is restricted to the case $y = 0$ and yields, combined with the first estimate, (5.11). After computing the integral in t' and decomposing according to the dispersive symbol, we deduce from Cauchy–Schwarz that

$$
\begin{aligned}
(5.12) &\lesssim T^2 1\{N_3 \sim N_4\} \sum_{n_1,n_2\in\mathbb{Z}^3} \bigg[\Big(\prod_{j=1}^{2} \chi_{N_j}(n_j)\Big) \langle n_{12}\rangle^{2s} \langle n_1\rangle^{-2} \langle n_2\rangle^{-2} \\
&\quad\times \Big(\sum_{m\in\mathbb{Z}} \sum_{n_3\in\mathbb{Z}^3} \langle m\rangle^{-1} \chi_{N_3}(n_3) |\widehat{V}(n_1,n_2,n_3)| \langle n_{123}\rangle^{-1} \langle n_3\rangle^{-2} 1\{|\varphi - m| \le 1\}\Big)^2\bigg].
\end{aligned}
$$

Using the resonant quartic sum estimate (Lemma 4.26), this implies that

$$
(5.12) \lesssim T^2 1\{N_3 \sim N_4\} \log(2+N_4)^2 \max(N_1,N_2)^{2s}.
$$

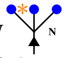
This clearly implies (5.9) and (5.10). Except for the logarithmic divergence in N_4 (and hence N_3), it also implies (5.11). We now need to restrict to $y = 0$ and we may assume that $N_1, N_2 \ll N_3$. For fixed $n_1, n_2 \in \mathbb{Z}^3$, we can apply the sine-cancellation lemma (Lemma 4.14) with $A = \max(N_1, N_2)$ and

$$
\begin{aligned}
f(t,t',n_3) &\overset{\text{def}}{=} \rho_N(n_{123})^2 \rho_N(n_3)^2 \chi_{N_{123}}(n_{123}) \chi_{N_3}(n_3) \chi_{N_4}(n_3) \\
&\quad\times \langle n_{123}\rangle^{-1} \langle n_3\rangle^{-2} \widehat{V}_S(n_1,n_2,n_3) \prod_{j=1}^{2} \exp(\pm_j i t' \langle n_j\rangle).
\end{aligned}
$$

This yields

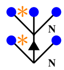
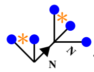
$$
\begin{aligned}
&(5.12)\big|_{y=0} \\
&\quad\lesssim T^4 1\{N_3 \sim N_4\} \max(N_1,N_2)^8 N_3^{-2} \sum_{n_1,n_2\in\mathbb{Z}^3} \langle n_{12}\rangle^{2s} \Big(\prod_{j=1}^{2} 1\{|n_j| \sim N_j\} \langle n_j\rangle^{-2}\Big) \\
&\quad\lesssim T^4 \max(N_1,N_2)^{10} N_3^{-2}.
\end{aligned}
$$

By combining our two estimates of $(5.12)|_{y=0}$ we arrive at (5.11). ∎

Remark 5.6. As we have seen in the proof of Proposition 5.3, the (probabilistic) resonant portion of $P_{\leq N}$  $\cdot\, P_{\leq N}$  has spatial regularity 0−, which is better than the sum of the individual spatial regularities. As a result, the probabilistic resonances between linear and cubic stochastic objects in Section 5.4 are relatively harmless.

5.3. *Quintic stochastic objects*

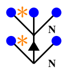
In this subsection, we control the quintic stochastic objects in **So**, i.e.,

$$(\neg\,\boxed{\textcircled{<}\,\&\,\textcircled{<}})$$  and .

Since **So** is part of the evolution equation for the smoother nonlinear remainder Y_N, the quintic stochastic objects have to be controlled at regularity $s_2 - 1$.

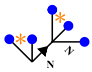
Proposition 5.7 (First quintic stochastic object). *For any $T \geq 1$ and any $p \geq 2$,*

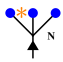
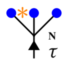
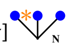
$$\Big\| \sup_{N\geq 1} \Big\| (\neg\,\boxed{\textcircled{<}\,\&\,\textcircled{<}})\ [\text{tree}] \Big\|_{\mathfrak{X}^{s_2-1,b_+-1}([0,T])} \Big\|_{L^p_\omega(\Omega)} \lesssim T^2 p^{5/2}. \tag{5.13}$$



Proposition 5.8 (Second quintic stochastic object). *For any $T \geq 1$ and any $p \geq 2$,*

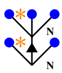
$$\Big\| \sup_{N\geq 1} \Big\| [\text{tree}] \Big\|_{\mathfrak{X}^{s_2-1,b_+-1}([0,T])} \Big\|_{L^p_\omega(\Omega)} \lesssim T^2 p^{5/2}. \tag{5.14}$$



Remark 5.9. In the frequency-localized versions of Propositions 5.7 and 5.8, which are detailed in the proof, we gain an η'-power of the maximal frequency scale. As in Proposition 5.1, we may also replace  by  $= \mathrm{I}[1_{[0,\tau]}$ $]$. We will not further comment on these minor modifications.

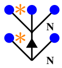
Proof of Proposition 5.7. Throughout the proof, we ignore the supremum in $N \geq 1$ and only prove a uniform estimate for a fixed N. Using the frequency-localized estimates below and the same argument as in the proof of Proposition 5.1, we can insert the supremum in N at the end of the proof.

We first obtain a representation of the quintic stochastic object using multiple stochastic integrals. Using (2.23) and Proposition 4.44, we have



$$\begin{aligned}
&(\neg\,\boxed{\textcircled{<}\,\&\,\textcircled{<}})\ [\text{tree}](t,x) \\
&= \sum_{\substack{N_{345},N_1,\ldots,N_5:\\ \max(N_1,N_{345})>N_2^\epsilon}} \sum_{n_1,\ldots,n_5\in\mathbb{Z}^3} \Bigg[\rho_N(n_{345})^2 \chi_{N_{345}}(n_{345}) \Big(\prod_{j=1}^{5} \rho_N(n_j)\chi_{N_j}(n_j) \Big) \\
&\qquad \times \widehat{V}(n_{1345})\widehat{V}_S(n_3,n_4,n_5)\langle n_{345}\rangle^{-1} \exp(i\langle n_{12345},x\rangle)\mathcal{I}_2[t,n_1,n_2] \\
&\qquad \times \Big(\int_0^t \sin((t-t')\langle n_{123}\rangle)\mathcal{I}_3[t',n_3,n_4,n_5]\,\mathrm{d}t' \Big) \Bigg].
\end{aligned}$$

Using the product formula for mixed multiple stochastic integrals (Proposition 4.44 and Lemma 4.41), we obtain

$$(\neg \boxed{\circledless \,\&\, \circledless})$$  $$(t,x) = \sum_{\substack{N_{345}, N_1, \ldots, N_5: \\ \max(N_1, N_{345}) > N_2^{\epsilon}}} (\mathscr{G}^{(5)} + \mathscr{G}^{(3)} + \widetilde{\mathscr{G}}^{(3)} + \mathscr{G}^{(1)})(t,x;N_*), \tag{5.15}$$

where the dependence on $N_{345}, N_1, \ldots, N_5$ is indicated by N_* and the quintic, cubic, and linear Gaussian chaoses are defined as follows. The quintic chaos is given by

$$\begin{aligned}
\mathscr{G}^{(5)}(t,x;N_*) \overset{\text{def}}{=} &\sum_{\pm_1,\ldots,\pm_5} c(\pm_j : 1 \leq j \leq 5) \\
&\times \sum_{n_1,\ldots,n_5 \in \mathbb{Z}^3} \Bigg[\rho_N(n_{345})^2 \chi_{N_{345}}(n_{345}) \Big(\prod_{j=1}^{5} \rho_N(n_j) \chi_{N_j}(n_j) \Big) \widehat{V}(n_{1345}) \\
&\times \widehat{V}_S(n_3, n_4, n_5) \langle n_{345} \rangle^{-1} \exp(i \langle n_{12345}, x \rangle) \Big(\prod_{j=1}^{2} \exp(\pm_j i t \langle n_j \rangle) \Big) \\
&\times \Big(\int_0^t \sin((t-t')\langle n_{123} \rangle) \prod_{j=3}^{5} \exp(\pm_j i t' \langle n_j \rangle) \, \mathrm{d}t' \Big) \mathcal{I}_5[\pm_j, n_j : 1 \leq j \leq 5] \Bigg].
\end{aligned}$$

The two cubic Gaussian chaoses are given by

$$\begin{aligned}
&\mathscr{G}^{(3)}(t,x;N_*) \overset{\text{def}}{=} \sum_{\pm_2,\pm_4,\pm_5} c(\pm_2, \pm_4, \pm_5) \\
&\times \sum_{n_2,n_4,n_5 \in \mathbb{Z}^3} \Bigg[\Big(\prod_{j=2,4,5} \rho_N(n_j) \chi_{N_j}(n_j) \Big) \widehat{V}(n_{45}) \exp(i \langle n_{245}, x \rangle) \\
&\times \sum_{n_3 \in \mathbb{Z}^3} \Big(\rho_N(n_3)^2 \rho_N(n_{345})^2 \chi_{N_{345}}(n_{345}) \chi_{N_1}(n_3) \chi_{N_3}(n_3) \\
&\times \widehat{V}_S(n_3, n_4, n_5) \langle n_{345} \rangle^{-1} \langle n_3 \rangle^{-2} \exp(\pm_2 i t \langle n_2 \rangle) \\
&\times \int_0^t \sin((t-t')\langle n_{345} \rangle) \cos((t-t')\langle n_3 \rangle) \prod_{j=4,5} \exp(\pm_j i t' \langle n_j \rangle) \, \mathrm{d}t' \Big) \mathcal{I}_3[\pm_j, n_j : j = 2,4,5] \Bigg]
\end{aligned}$$

and

$$\begin{aligned}
&\widetilde{\mathscr{G}}^{(3)}(t,x;N_*) \overset{\text{def}}{=} \sum_{\pm_1,\pm_4,\pm_5} c(\pm_1, \pm_4, \pm_5) \\
&\times \sum_{n_1,n_4,n_5 \in \mathbb{Z}^3} \Bigg[\Big(\prod_{j=1,4,5} \rho_N(n_j) \chi_{N_j}(n_j) \Big) \exp(i \langle n_{145}, x \rangle) \\
&\times \sum_{n_3 \in \mathbb{Z}^3} \Big(\rho_N(n_3)^2 \rho_N(n_{345})^2 \chi_{N_{345}}(n_{345}) \chi_{N_1}(n_3) \chi_{N_3}(n_3) \\
&\times \widehat{V}_S(n_3, n_4, n_5) \widehat{V}(n_{1345}) \langle n_{345} \rangle^{-1} \langle n_3 \rangle^{-2} \exp(\pm_1 i t \langle n_1 \rangle) \\
&\times \int_0^t \sin((t-t')\langle n_{345} \rangle) \cos((t-t')\langle n_3 \rangle) \prod_{j=4,5} \exp(\pm_j i t' \langle n_j \rangle) \, \mathrm{d}t' \Big) \mathcal{I}_3[\pm_j, n_j : j = 1,4,5] \Bigg].
\end{aligned}$$

Finally, the linear Gaussian chaos (or simply Gaussian) is given by

$$\begin{aligned}
\mathscr{G}^{(1)}(t,x;N_*) &\overset{\text{def}}{=} \sum_{\pm_5} c(\pm_5) \sum_{n_5\in\mathbb{Z}^3} \rho_N(n_5)\chi_{N_5}(n_5)\exp(i\langle n_5,x\rangle) \\
&\times \sum_{n_3,n_4\in\mathbb{Z}^4}\Big[\rho_N(n_{345})^2\rho_N(n_3)^2\rho_N(n_4)^2\chi_{N_{345}}(n_{345}) \\
&\times \chi_{N_1}(n_3)\chi_{N_3}(n_3)\chi_{N_2}(n_4)\chi_{N_4}(n_4)\widehat{V}_S(n_3,n_4,n_5)\widehat{V}(n_{45})\langle n_{345}\rangle^{-1}\langle n_3\rangle^{-2}\langle n_4\rangle^{-2} \\
&\times \int_0^t \sin((t-t')\langle n_{345}\rangle)\cos((t-t')\langle n_3\rangle)\cos((t-t')\langle n_4\rangle)\exp(\pm_5 it'\langle n_5\rangle)\,\mathrm{d}t'\Big] \\
&\hspace{25em}\times \mathcal{I}_1[\pm_5,n_5].
\end{aligned}$$

Each of the frequency-localized Gaussian chaoses in (5.15) is now estimated separately. We encourage the reader to concentrate on the estimates for $\mathscr{G}^{(5)}$ and $\mathscr{G}^{(1)}$, which already contain all ideas and ingredients.

The nonresonant term $\mathscr{G}^{(5)}$*:* Let $s=1/2-\eta$. We will first estimate the $\mathfrak{X}^{s-1,b_--1}$-norm of a dyadic piece and then use the condition $\max(N_1,N_{345})>N_2^{\epsilon}$ to increase the value of s. Using Gaussian hypercontractivity (Proposition 4.45), the orthogonality of multiple stochastic integrals, and Lemma 4.12, we obtain

$$\begin{aligned}
&\big\|\|\mathscr{G}^{(5)}(t,x;N_*)\|_{\mathfrak{X}^{s-1,b_--1}([0,T])}\big\|^2_{L^p_\omega} \\
&\lesssim \max_{\pm_{12345}}\big\|\|\langle\lambda\rangle^{b_--1}\langle n\rangle^{s-1}\mathcal{F}_{t,x}(\chi(t/T)\mathscr{G}^{(5)}(t,x;N_*))(\lambda\mp_{12345}\langle n\rangle,n)\|_{L^2_\lambda\ell^2_n(\mathbb{R}\times\mathbb{Z}^3)}\big\|^2_{L^p_\omega} \\
&\lesssim p^5\max_{\pm_{12345}}\big\|\|\langle\lambda\rangle^{b_--1}\langle n\rangle^{s-1}\mathcal{F}_{t,x}(\chi(t/T)\mathscr{G}^{(5)}(t,x;N_*))(\lambda\mp_{12345}\langle n\rangle,n)\|_{L^2_\lambda\ell^2_n(\mathbb{R}\times\mathbb{Z}^3)}\big\|^2_{L^2_\omega} \\
&\lesssim T^2p^5\max_{\substack{\pm_{12345},\pm_{345},\\ \pm_1,\dots,\pm_5}}\sum_{n_1,\dots,n_5\in\mathbb{Z}^3}\bigg[\chi_{N_{345}}(n_{345})\Big(\prod_{j=1}^5\chi_{N_j}(n_j)\Big)\langle n_{12345}\rangle^{2(s-1)}\langle n_{345}\rangle^{-2} \\
&\quad\times|\widehat{V}(n_{1345})|^2|\widehat{V}_S(n_3,n_4,n_5)|^2\Big(\prod_{j=1}^5\langle n_j\rangle^{-2}\Big) \\
&\quad\times(1+|\pm_{345}\langle n_{345}\rangle\pm_3\langle n_3\rangle\pm_4\langle n_4\rangle\pm_5\langle n_5\rangle|)^{-2} \\
&\times\int_{\mathbb{R}}\langle\lambda\rangle^{2(b_--1)}\Big(1+\min\Big(|\lambda-(\pm_{12345}\langle n_{12345}\rangle\pm_{345}\langle n_{345}\rangle\pm_1\langle n_1\rangle\pm_2\langle n_2\rangle)|, \\
&\qquad\Big|\lambda-\Big(\pm_{12345}\langle n_{12345}\rangle\mp_{345}\langle n_{345}\rangle+\sum_{j=1}^5(\pm_j)\langle n_j\rangle\Big)\Big|\Big)\Big)^{-2}\mathrm{d}\lambda\bigg].
\end{aligned}\tag{5.16}$$

To break down this long formula, we define the phase functions

$$\begin{aligned}
\psi(n_3,n_4,n_5) &\overset{\text{def}}{=} \pm_{345}\langle n_{345}\rangle\pm_3\langle n_3\rangle\pm_4\langle n_4\rangle\pm_5\langle n_5\rangle, \\
\varphi(n_1,\dots,n_5) &\overset{\text{def}}{=} \pm_{12345}\langle n_{12345}\rangle\pm_{345}\langle n_{345}\rangle\pm_1\langle n_1\rangle\pm_2\langle n_2\rangle, \\
\widetilde{\varphi}(n_1,\dots,n_5) &\overset{\text{def}}{=} \pm_{12345}\langle n_{12345}\rangle\mp_{345}\langle n_{345}\rangle+\sum_{j=1}^5(\pm_j)\langle n_j\rangle.
\end{aligned}$$

Integrating in λ and decomposing according to the value of the phases, we obtain

$$\begin{aligned}(5.16) &\lesssim T^2 p^5 \log(2+\max(N_1,\dots,N_5)) \\ &\times \max_{\substack{\pm_{12345},\pm_{345},\\ \pm_1,\dots,\pm_5}} \sup_{m,m'\in\mathbb{Z}} \sum_{n_1,\dots,n_5\in\mathbb{Z}^3} \Big[\chi_{N_{345}}(n_{345}) \\ &\times \Big(\prod_{j=1}^5 \chi_{N_j}(n_j)\Big)\langle n_{12345}\rangle^{2(s-1)}\langle n_{345}\rangle^{-2}|\widehat{V}(n_{1345})|^2|\widehat{V}_S(n_3,n_4,n_5)|^2\Big(\prod_{j=1}^5\langle n_j\rangle^{-2}\Big) \\ &\times 1\{|\psi-m|\le 1\}(1\{|\varphi-m'|\le 1\}+1\{|\widetilde{\varphi}-m'|\le 1\})\Big].\end{aligned}$$

Using the nonresonant quintic sum estimate (Lemma 4.27), we finally obtain

$$\big\|\|\mathscr{G}^{(5)}(t,x;N_*)\|_{\mathfrak{X}^{s-1,b_--1}([0,T])}\big\|_{L^p_\omega} \lesssim Tp^{5/2}\max(N_1,N_3,N_4,N_5)^{-\beta+\eta}N_2^{-\eta}. \quad (5.17)$$

Due to the operator $(\neg\,\boxed{\circledless\,\&\,\circledless})$, we have

$$\max(N_1,N_3,N_4,N_5) \gtrsim \max(N_1,N_2,N_3,N_4,N_5)^{\epsilon}.$$

Thus, (5.17) implies

$$\big\|\|\mathscr{G}^{(5)}(t,x;N_*)\|_{\mathfrak{X}^{s_2-1,b_--1}([0,T])}\big\|_{L^p_\omega} \lesssim Tp^{5/2}\max(N_1,N_2,N_3,N_4,N_5)^{\delta_2+3\eta-\epsilon\beta},$$

which is acceptable.

Single-resonance term $\mathscr{G}^{(3)}$*:* This term only yields a nontrivial contribution if $N_1\sim N_3$. In particular, $\max(N_1,N_{345})>N_2^{\epsilon}$ implies that $\max(N_3,N_4,N_5)\gtrsim N_2^{\epsilon}$. Using the inhomogeneous Strichartz estimate (Lemma 4.9) and Gaussian hypercontractivity, we have

$$\begin{aligned}\big\|\|\mathscr{G}^{(3)}(t,x;N_*)\|_{\mathfrak{X}^{s_2-1,b_--1}([0,T])}\big\|_{L^p_\omega} &\lesssim \big\|\|\mathscr{G}^{(3)}(t,x;N_*)\|_{L_t^{2b_+}H_x^{s_2-1}([0,T]\times\mathbb{T}^3)}\big\|_{L^p_\omega} \\ &\lesssim T^{1/2}\big\|\|\mathscr{G}^{(3)}(t,x;N_*)\|_{L_t^2H_x^{s_2-1}([0,T]\times\mathbb{T}^3)}\big\|_{L^p_\omega} \\ &\lesssim Tp^{3/2}\sup_{t\in[0,T]}\big\|\|\mathscr{G}^{(3)}(t,x;N_*)\|_{H_x^{s_2-1}(\mathbb{T}^3)}\big\|_{L^2_\omega}. \quad (5.18)\end{aligned}$$

Using the orthogonality of the multiple stochastic integrals, we have

$$\begin{aligned}&\sup_{t\in[0,T]}\big\|\|\mathscr{G}^{(3)}(t,x;N_*)\|_{H_x^{s_2-1}(\mathbb{T}^3)}\big\|_{L^2_\omega}^2 \lesssim N_{45}^{-2\beta}N_2^{-2}N_4^{-2}N_5^{-2} \\ &\times\sum_{n_2,n_4,n_5\in\mathbb{Z}^3}\chi_{N_{45}}(n_{45})\Big(\prod_{j=2,4,5}\chi_{N_j}(n_j)\Big)\langle n_{245}\rangle^{2(s_2-1)}\mathcal{S}(n_2,n_4,n_5;t,N_*)^2, \quad (5.19)\end{aligned}$$

where

$$\begin{aligned}\mathcal{S}(n_2,n_4,n_5;t,N_*) \overset{\text{def}}{=} \Big|\sum_{n_3\in\mathbb{Z}^3}\Big[&\rho_N(n_3)^2\rho_N(n_{345})^2\chi_{N_{345}}(n_{345})\chi_{N_1}(n_3)\chi_{N_3}(n_3) \\ &\times\widehat{V}_S(n_3,n_4,n_5)\langle n_{345}\rangle^{-1}\langle n_3\rangle^{-2}\exp(\pm_2 it\langle n_2\rangle) \\ &\times\int_0^t\sin((t-t')\langle n_{345}\rangle)\cos((t-t')\langle n_3\rangle)\prod_{j=4,5}\exp(\pm_j it'\langle n_j\rangle)\,\mathrm{d}t'\Big]\Big|.\end{aligned}$$

Define the phase function φ by

$$\varphi(n_3, n_4, n_5) \stackrel{\text{def}}{=} \langle n_{345}\rangle \pm_3 \langle n_3\rangle \pm_4 \langle n_4\rangle \pm_5 \langle n_5\rangle. \tag{5.20}$$

By calculating the integral, using the triangle inequality, expanding the square, and using Lemma 4.25, we obtain

$$\begin{aligned}
&\mathcal{S}(n_2, n_4, n_5; t, N_*)^2 \\
&\quad \lesssim T^2 \max_{\pm_3,\pm_4,\pm_5} \Big(\sum_{m\in\mathbb{Z}} \sum_{n_3\in\mathbb{Z}^3} \langle m\rangle^{-1} \chi_{N_3}(n_3) \langle n_{345}\rangle^{-1} \langle n_3\rangle^{-2} 1\{|\varphi - m| \leq 1\}\Big)^2 \\
&\quad \lesssim T^2 \log(2 + \max(N_3, N_4, N_5)) \\
&\qquad \times \Big(\max_{\pm_3,\pm_4,\pm_5} \sup_{m\in\mathbb{Z}} \sum_{n_3\in\mathbb{Z}^3} \chi_{N_3}(n_3) \langle n_{345}\rangle^{-1} \langle n_3\rangle^{-2} 1\{|\varphi - m| \leq 1\}\Big) \\
&\qquad \times \Big(\max_{\pm_3,\pm_4,\pm_5} \sum_{m\in\mathbb{Z}} \sum_{n_3\in\mathbb{Z}^3} \langle m\rangle^{-1} \chi_{N_3}(n_3) \langle n_{345}\rangle^{-1} \langle n_3\rangle^{-2} 1\{|\varphi - m| \leq 1\}\Big) \\
&\quad \lesssim T^2 \log(2 + \max(N_3, N_4, N_5)) \langle n_{45}\rangle^{-1} \\
&\qquad \times \max_{\pm_3,\pm_4,\pm_5} \sum_{m\in\mathbb{Z}} \sum_{n_3\in\mathbb{Z}^3} \langle m\rangle^{-1} \chi_{N_3}(n_3) \langle n_{345}\rangle^{-1} \langle n_3\rangle^{-2} 1\{|\varphi - m| \leq 1\}.
\end{aligned}$$

By inserting this into (5.19) and summing in $n_2 \in \mathbb{Z}^3$ first, we obtain

$$\begin{aligned}
&\sup_{t\in[0,T]} \big\| \|\mathscr{G}^{(3)}(t, x; N_*)\|_{H_x^{s_2-1}(\mathbb{T}^3)} \big\|_{L^2_\omega}^2 \\
&\lesssim T^2 \log(2 + \max(N_3, N_4, N_5)) \Big(\prod_{j=2}^{5} N_j^{-2}\Big) \\
&\quad \times \max_{\pm_3,\pm_4,\pm_5} \sum_{m\in\mathbb{Z}} \sum_{n_2,n_3,n_4,n_5\in\mathbb{Z}^3} \Big[\langle m\rangle^{-1} \Big(\prod_{j=2}^{5} \chi_{N_j}(n_j)\Big) \langle n_{245}\rangle^{2(s_2-1)} \\
&\qquad\qquad \times \langle n_{345}\rangle^{-1} \langle n_{45}\rangle^{-1-2\beta} 1\{|\varphi - m| \leq 1\}\Big] \\
&\lesssim T^2 \log(2 + \max(N_3, N_4, N_5)) N_2^{2s_2-1} \Big(\prod_{j=3}^{5} N_j^{-2}\Big) \\
&\quad \times \max_{\pm_3,\pm_4,\pm_5} \sum_{m\in\mathbb{Z}} \sum_{n_3,n_4,n_5\in\mathbb{Z}^3} \Big[\langle m\rangle^{-1} \Big(\prod_{j=3}^{5} \chi_{N_j}(n_j)\Big) \langle n_{345}\rangle^{-1} \langle n_{45}\rangle^{-1-2\beta} 1\{|\varphi - m| \leq 1\}\Big] \\
&\lesssim T^2 \log(2 + \max(N_3, N_4, N_5)) N_2^{2s_2-1} \max(N_4, N_5)^{-2\beta}.
\end{aligned}$$

In the last line, we have used the cubic sum estimate (Proposition 4.20). In total, this yields

$$\begin{aligned}
&\sup_{t\in[0,T]} \big\| \|\mathscr{G}^{(3)}(t, x; N_*)\|_{H_x^{s_2-1}(\mathbb{T}^3)} \big\|_{L^2_\omega} \\
&\qquad \lesssim T \log(2 + \max(N_3, N_4, N_5)) N_2^{s_2-1/2} \max(N_4, N_5)^{-\beta}.
\end{aligned} \tag{5.21}$$

Recalling that $\max(N_3, N_4, N_5) > N_2^{\epsilon}$, we are only missing decay in N_3. By using the sine-cancellation lemma (Lemma 4.14) to estimate $\mathcal{S}(n_2, n_4, n_5; t, N_*)$, we easily obtain

$$\sup_{t\in[0,T]} \big\| \|\mathcal{G}^{(3)}(t,x;N_*)\|_{H_x^{s_2-1}(\mathbb{T}^3)} \big\|_{L^2_\omega} \lesssim T^2 N_2^{s_2-1/2} \max(N_4,N_5)^5 N_3^{-1}. \tag{5.22}$$

Combining (5.21), (5.22), and the condition $\max(N_3, N_4, N_5) > N_2^{\epsilon}$, we obtain an acceptable estimate.

Single-resonance term $\widetilde{\mathcal{G}}^{(3)}$*:* This term can be controlled through similar (or simpler) arguments to those for $\mathcal{G}^{(3)}$ and we omit the details.

Double-resonance term $\mathcal{G}^{(1)}$*:* This term only yields a non-trivial contribution when $N_1 \sim N_3$ and $N_2 \sim N_4$. We note that the sum in $n_3 \in \mathbb{Z}^3$ may appear to diverge logarithmically (once the dyadic localization is removed). However, the sine-function in the Duhamel integral yields additional cancellation, which was first observed by Gubinelli, Koch, and Oh [37] and generalized slightly in Lemma 4.14.

Using the inhomogeneous Strichartz estimate (Lemma 4.9), it follows that

$$\begin{aligned}\|\mathcal{G}^{(1)}(t,x;N_*)\|_{\mathcal{X}^{s_2-1,b_--1}([0,T])} &\lesssim \|\mathcal{G}^{(1)}(t,x;N_*)\|_{L_t^{2b_+} H_x^{s_2-1}([0,T]\times\mathbb{T}^3)}\\ &\lesssim T^{1/2}\|\mathcal{G}^{(1)}(t,x;N_*)\|_{L_t^2 H_x^{s_2-1}([0,T]\times\mathbb{T}^3)}.\end{aligned}$$

Using Gaussian hypercontractivity (Proposition 4.45) and the orthogonality of multiple stochastic integrals, we obtain

$$\begin{aligned}T\big\| \|\mathcal{G}^{(1)}(t,x;N_*)\|_{L_t^2 H_x^{s_2-1}([0,T]\times\mathbb{T}^3)}\big\|^2_{L^p_\omega}\\ &\lesssim Tp \big\| \|\mathcal{G}^{(1)}(t,x;N_*)\|_{L_t^2 H_x^{s_2-1}([0,T]\times\mathbb{T}^3)}\big\|^2_{L^2_\omega}\\ &\lesssim T^2 p \sup_{t\in[0,T]} \sum_{n_5\in\mathbb{Z}^3} \chi_{N_5}(n_5)\langle n_5\rangle^{2(s_2-1)-2}\mathcal{S}(n_5;t,N_*)^2 \end{aligned}\tag{5.23}$$

where

$$\begin{aligned}&\mathcal{S}(n_5;t,N_*)\\ &\stackrel{\text{def}}{=} \Bigg| \sum_{n_3,n_4\in\mathbb{Z}^4} \Big[\rho_N(n_{345})^2 \rho_N(n_3)^2 \rho_N(n_4)^2 \chi_{N_{345}}(n_{345})\chi_{N_1}(n_3)\chi_{N_3}(n_3)\chi_{N_2}(n_4)\chi_{N_4}(n_4)\\ &\times \widehat{V}_S(n_3,n_4,n_5)\widehat{V}(n_{45})\langle n_{345}\rangle^{-1}\langle n_3\rangle^{-2}\langle n_4\rangle^{-2}\\ &\times \int_0^t \sin((t-t')\langle n_{345}\rangle)\cos((t-t')\langle n_3\rangle)\cos((t-t')\langle n_4\rangle)\exp(\pm_5 it'\langle n_5\rangle)\,\mathrm{d}t'\Big]\Bigg|.\end{aligned}$$

We now present two different estimates of $\mathcal{S}(n_5; t, N_*)$. The first (and main) estimate almost yields control over $\mathcal{G}^{(1)}$, but exhibits a logarithmic divergence in N_3. The second estimate exhibits polynomial growth in N_4 and N_5, but yields the desired decay in N_3.

Using $|\widehat{V}(n_{45})| \lesssim \langle n_{45}\rangle^{-\beta}$ and the crude estimate $|\widehat{V}_S(n_3,n_4,n_5)| \lesssim 1$, we obtain

$$
\begin{aligned}
&\mathcal{S}(n_5;t,N_*)\\
&\lesssim N_{345}^{-1} N_3^{-2} N_4^{-2} \sum_{n_3,n_4\in\mathbb{Z}^3} \Big[1\{|n_3|\sim N_3,\ |n_4|\sim N_4,\ |n_{345}|\sim N_{345}\} \langle n_{45}\rangle^{-\beta} \\
&\quad \times \Big| \int_0^t \sin((t-t')\langle n_{345}\rangle)\cos((t-t')\langle n_3\rangle)\cos((t-t')\langle n_4\rangle)\exp(\pm_5 i t'\langle n_5\rangle)\,\mathrm{d}t' \Big| \Big] \\
&\lesssim T\log(2+\max(N_3,N_4,N_5)) N_{345}^{-1} N_3^{-2} N_4^{-2} \\
&\quad \times \max_{\pm_3,\pm_4,\pm_5} \sup_{m\in\mathbb{Z}} \sum_{n_3,n_4\in\mathbb{Z}^3} \big[1\{|n_3|\sim N_3,\ |n_4|\sim N_4,\ |n_{345}|\sim N_{345}\}\langle n_{45}\rangle^{-\beta} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times 1\{|\varphi - m|\le 1\}\big],
\end{aligned}
$$

where the phase function φ is given by

$$\varphi(n_3,n_4,n_5) \overset{\text{def}}{=} \langle n_{345}\rangle \pm_3 \langle n_3\rangle \pm_4 \langle n_4\rangle \pm_5 \langle n_5\rangle.$$

Using the counting estimate from Lemma 4.29, it follows that

$$\mathcal{S}(n_5;t,N_*) \lesssim T\log(2+\max(N_3,N_4,N_5))\max(N_4,N_5)^{-\beta+\eta}. \tag{5.24}$$

Alternatively, it follows from the sine-cancellation lemma (Lemma 4.14) with $A = N_4^2 N_5^2$, say, that

$$\mathcal{S}(n_5;t,N_*) \lesssim T^2 N_3^{-1} N_4^5 N_5^2. \tag{5.25}$$

By combining (5.23)–(5.25), it follows that

$$
\begin{aligned}
&T^{1/2} \big\| \|\mathcal{G}^{(1)}(t,x;N_*)\|_{L_t^2 H_x^{s_2-1}([0,T]\times\mathbb{T}^3)} \big\|_{L_\omega^p} \\
&\qquad \lesssim T^3 p^{1/2}\log(2+\max(N_3,N_4,N_5)) N_5^{s_2-1/2}\min(N_4^{-\beta}, N_5^{-\beta}, N_3^{-1}N_4^5 N_5^5) \\
&\qquad \lesssim T^3 p^{1/2} N_5^{s_2-1/2-\beta+20\eta}\max(N_3,N_4,N_5)^{-\eta} \lesssim T^3 p^{1/2}\max(N_3,N_4,N_5)^{-\eta}.
\end{aligned}
$$

This contribution is acceptable. ∎

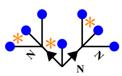
Proof of Proposition 5.8. This estimate is similar to (but easier than) Proposition 5.7 and we therefore omit the details. Instead of gaining additional regularity through the para-differential operator as in Proposition 5.8, we simply use the interaction potential V and the crude inequality

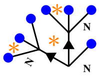
$$\langle n_{12}\rangle^{-2\beta} \lesssim \langle n_{12}\rangle^{-2\gamma} \lesssim \langle n_{12345}\rangle^{-2\gamma}\langle n_{345}\rangle^{2\gamma} \quad \text{for } 0<\gamma<\beta. \qquad ∎$$

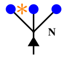
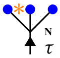
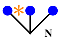
5.4. *Septic stochastic objects*

The next proposition controls the third and fourth term in **So**, i.e., in (2.28).

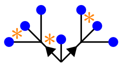
Proposition 5.10 (Septic stochastic objects). *Let $T, p \ge 1$. Then*

$$\Big\| \sup_{N\ge 1} \Big\| \quad$$  $$\quad \Big\|_{\mathfrak{X}^{s_2-1,b_+-1}([0,T])} \Big\|_{L_\omega^p(\mathbb{P})} \lesssim T^4 p^{7/2}, \tag{5.26}$$

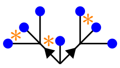
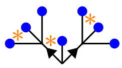
$\Big\| \sup_{N\geq 1} \Big\| (\neg \circledless)$  $\Big\|_{\mathfrak{X}^{s_2-1,b_+-1}([0,T])} \Big\|_{L^p_\omega(\mathbb{P})} \lesssim T^4 p^{7/2}.$ (5.27)

Remark 5.11. In the frequency-localized version of Proposition 5.10, we gain an η'-power of the maximal frequency scale. As in Proposition 5.1, we may also replace  by  $= \mathrm{I}[1_{[0,\tau]}$ $]$. We will not further comment on these minor modifications.

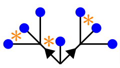
Proof of Proposition 5.10. We only prove (5.26); the second estimate (5.27) follows from similar (but slightly simpler) arguments. To simplify the notation, we formally set $N = \infty$. The same argument also yields the estimate for the supremum over N. Using the inhomogeneous Strichartz estimate (Lemma 4.9) and Gaussian hypercontractivity (Proposition 4.45), it suffices to prove that

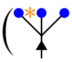
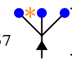
$\sup_{t\in[0,T]} \Big\| \Big\|$  $\Big\|_{H^{s_2-1}_x(\mathbb{T}^3)} \Big\|_{L^2_\omega(\mathbb{P})} \lesssim T^3.$ (5.28)

Using a Littlewood–Paley decomposition, we write

 $$= \sum_{N_{1234567}, N_{1234}, N_4, N_{567}}$$  $[N_{1234567}, N_{1234}, N_4, N_{567}]$,

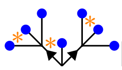
where

 $[N_{1234567}, N_{1234}, N_4, N_{567}]$

$\stackrel{\text{def}}{=} P_{N_{1234567}}\Big[(P_{N_{1234}} \widehat{V}) * \Big($  $\cdot P_{N_4}$  $\Big) P_{N_{567}}$  $\Big].$ (5.29)

We now present two separate estimates of (5.29). The first estimate, which is the main part of the argument, almost yields (5.28), but contains a logarithmic divergence in N_4. The second (short) estimate exhibits polynomial decay in N_4, and is only used to remove this logarithmic divergence.

Main estimate: Using the stochastic representation of the cubic nonlinearity (Proposition 4.44) and (4.76), we obtain

 $[N_{1234567}, N_{1234}, N_4, N_{567}]$

$$= \sum_{n_1,\ldots,n_7\in\mathbb{Z}^3} \sum_{\pm_1,\ldots,\pm_7} \Big[\chi_{N_{1234567}}(n_{1234567}) \chi_{N_{1234}}(n_{1234}) \chi_{N_4}(n_4) \chi_{N_{567}}(n_{567}) \widehat{V}(n_{1234})$$
$$\times \Phi(t, n_j, \pm_j : 1 \leq j \leq 3) e^{\pm i t \langle n_4\rangle} \frac{1}{\langle n_4 \rangle} \Phi(t, n_j, \pm_j : 5 \leq j \leq 7) \exp(i \langle n_{1234567}, x\rangle)$$
$$\times \widetilde{\mathcal{I}}_3[n_j, \pm_j : 1 \leq j \leq 3] \widetilde{\mathcal{I}}_1[n_4, \pm_4] \widetilde{\mathcal{I}}_3[n_j, \pm_j : 5 \leq j \leq 7] \Big]. \tag{5.30}$$

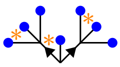
Here, the amplitude Φ is given by

$$\Phi(t, n_j, \pm_j : 1 \leq j \leq 3) \stackrel{\text{def}}{=} \langle n_{123}\rangle^{-1} \widehat{V}_S(n_1, n_2, n_3) \times \Big(\prod_{j=1}^{3} \langle n_j \rangle^{-1}\Big)\Big(\int_0^t \sin((t-t')\langle n_{123}\rangle) \prod_{j=1}^{3} \exp(\pm_j i t' \langle n_j\rangle)\, \mathrm{d}t'\Big).$$

Comparing with $\Phi(n_1, n_2, n_3)$ as in Lemma 4.31, we have

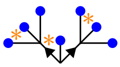
$$\sup_{t\in[0,T]} |\Phi(t, n_j, \pm_j : 1 \leq j \leq 3)| \lesssim T\Phi(n_1, n_2, n_3). \tag{5.31}$$

We now rely on the notation from Definition 4.30 and Lemma 4.31. Using the product formula for multiple stochastic integrals twice (Lemma 4.40), the orthogonality of multiple stochastic integrals, and (5.31), we obtain



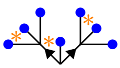
$$\sup_{t\in[0,T]} \Big\| \cdots [N_{1234567}, N_{1234}, N_4, N_{567}] \Big\|^2_{L^2_\omega H_x^{s_2-1}(\Omega\times\mathbb{T}^3)} \lesssim T^4 \sum_{\mathscr{P}} \sum_{(n_j)_{j\notin\mathscr{P}}} \langle n_{\mathrm{nr}}\rangle^{2(s_2-1)} \times \Big(\sum^{*}_{(n_j)_{j\in\mathscr{P}}} 1\{|n_{1234567}| \sim N_{1234567}\} 1\{|n_{1234}| \sim N_{1234}\} 1\{|n_{567}| \sim N_{567}\} \times 1\{|n_4| \sim N_4\} |\widehat{V}(n_{1234})| \Phi(n_1, n_2, n_3) \langle n_4\rangle^{-1} \Phi(n_5, n_6, n_7)\Big)^2.$$

The sum in $\mathscr{P}$ is taken over all pairings which respect to the partition $\{1,2,3\}, \{4\}, \{5,6,7\}$. For a similar argument, we refer the reader to [28, Lemma 4.1]. Using Lemma 4.31, it follows that



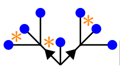
$$\sup_{t\in[0,T]} \Big\| \cdots [N_{1234567}, N_{1234}, N_4, N_{567}] \Big\|_{L^2_\omega H_x^{s_2-1}(\Omega\times\mathbb{T}^3)} \lesssim T^2 \log(2+N_4)(N_{1234567}^{(s_2-1/2)} N_{567}^{-(\beta-\eta)} + N_{1234567}^{-(1-s_2-\eta)}) N_{1234}^{-\beta}. \tag{5.32}$$

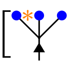
Since $N_{1234567} \lesssim \max(N_{1234}, N_{567})$ and $N_{1234567} \sim N_{567}$ if $N_{1234} \ll N_{567}$, we obtain

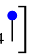


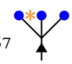
$$\sup_{t\in[0,T]} \Big\| \cdots [N_{1234567}, N_{1234}, N_4, N_{567}] \Big\|_{L^2_\omega H_x^{s_2-1}(\Omega\times\mathbb{T}^3)} \lesssim T^2 \log(2+N_4) \max(N_{1234567}, N_{1234}, N_{567})^{-(\beta-\eta-\delta_2)}. \tag{5.33}$$

Removing the logarithmic divergence in N_4: Using Proposition 5.1 and (5.11) from Proposition 5.3, we obtain



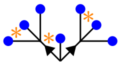






$$\sup_{t\in[0,T]} \Big\| \cdots [N_{1234567}, N_{1234}, N_4, N_{567}] \Big\|_{L^2_\omega H_x^{s_2-1}(\Omega\times\mathbb{T}^3)} \lesssim \Big\| P_{N_{1234}} \Big[\cdots \cdot P_{N_4} \cdots \Big] \Big\|_{L^4_\omega L^\infty_t L^2_x(\Omega\times[0,T]\times\mathbb{T}^3)} \Big\| P_{N_{567}} \cdots \Big\|_{L^4_\omega L^\infty_t L^\infty_x(\Omega\times[0,T]\times\mathbb{T}^3)} \lesssim T^5 N_{1234} N_4^{-\eta/10}. \tag{5.34}$$

Combining (5.33) and (5.34), we obtain



$$\sup_{t\in[0,T]} \Big\| [N_{1234567}, N_{1234}, N_4, N_{567}] \Big\|_{L^2_\omega H^{s_2-1}_x(\Omega\times\mathbb{T}^3)} \lesssim T^3 N_4^{-\eta^2} \max(N_{1234567}, N_{1234}, N_{567})^{-(\beta-\eta-2\delta_2)}. \tag{5.35}$$

Summing over the dyadic scales yields (5.26). ■

6. Random matrix theory estimates

In this section, we control the random matrix terms **RMT**. Techniques from random matrix theory, such as the moment method, were first applied to dispersive equations in Bourgain's seminal paper [5]. Over the last decade, they have become an indispensable tool in the study of dispersive PDE and we refer the interested reader to [6, 17, 27, 28, 34, 37, 63]. Very recently, Deng, Nahmod, and Yue [30, Proposition 2.8] obtained an easy-to-use, general, and essentially sharp random matrix estimate, which is proved using the moment method. We have previously recalled their estimate in Proposition 4.50. The proofs of Propositions 6.1 and 6.3 combine their random matrix estimate with the counting estimates in Section 4.4.

Proposition 6.1 (First RMT estimate). *Let $T, p \geq 1$. Then*

$$\Big\| \sup_{N\geq 1} \sup_{\mathcal{J}\subseteq[0,T]} \sup_{\|w\|_{\mathcal{X}^{s_1,b}(\mathcal{J})}\leq 1} \|(V * \vee_N) \cdot P_{\leq N} w\|_{\mathcal{X}^{s_2-1,b_+-1}(\mathcal{J})} \Big\|_{L^p_\omega(\mathbb{P})} \lesssim Tp. \tag{6.1}$$

Remark 6.2. This proposition controls the first term in **RMT**, i.e., in (2.30). In the frequency-localized version of (6.1), which is detailed in the proof, we gain an η'-power in the maximal frequency scale.

Proof of Proposition 6.1. The argument splits into two steps: First, we bring (6.1) into a random matrix form. Then, we prove a random matrix estimate using the moment method (Proposition 4.50).

Step 1: The random matrix form. By definition of the restricted norms,

$$\begin{aligned} &\sup_{\mathcal{J}\subseteq[0,T]} \sup_{\|w\|_{\mathcal{X}^{s_1,b}(\mathcal{J})}\leq 1} \|(V * \vee_N) \cdot P_{\leq N} w\|_{\mathcal{X}^{s_2-1,b_+-1}(\mathcal{J})} \\ &\qquad \leq \sup_{\|w\|_{\mathcal{X}^{s_1,b}(\mathbb{R})}\leq 1} \|\chi(t/T)(V * \vee_N) \cdot P_{\leq N} w\|_{\mathcal{X}^{s_2-1,b_+-1}(\mathbb{R})}. \end{aligned} \tag{6.2}$$

We bound the right-hand side of (6.2) with b_+ replaced by b_-. Using the frequency-localized estimate in the arguments below and a reduction similar to that in the proof of Proposition 5.1, we can then upgrade the value from b_- to b_+. Let $w \in \mathcal{X}^{s_1,b}(\mathbb{R})$ satisfy $\|w\|_{\mathcal{X}^{s_1,b}(\mathbb{R})} \leq 1$. We define $w_\pm \in \mathcal{X}^{s_1,b}(\mathbb{R})$ by

$$\widehat{w}_\pm(\lambda, n) \stackrel{\text{def}}{=} 1\{\pm\lambda \geq 0\}\widehat{w}(\lambda, n).$$

Then $w = w_+ + w_-$ and

$$\|w\|_{\mathfrak{X}^{s_1,b}(\mathbb{R})} \sim \max_{\pm} \|\langle n\rangle^{s_1}\langle\lambda\rangle^{b}\widehat{w}_{\pm}(\lambda \pm \langle n\rangle, n)\|_{L^2_\lambda \ell^2_n(\mathbb{R}\times\mathbb{T}^3)}.$$

Using this decomposition of w and the stochastic representation of the renormalized square, we deduce that the nonlinearity is given by

$$\begin{aligned}
&(V * \overset{\bullet\,\bullet}{\vee}_{N}) \cdot P_{\leq N} w \\
&= \sum_{\pm_1,\pm_2,\pm_3} \sum_{N_1,N_2,N_3} \sum_{n_1,n_2,n_3\in\mathbb{Z}^3} \Big[c(\pm_1,\pm_2)\Big(\prod_{j=1}^{3}\rho_N(n_j)\chi_{N_j}(n_j)\Big) \\
&\qquad \times \widehat{V}(n_{12})\mathcal{I}_2[\pm_j, n_j \colon j=1,2]\Big(\prod_{j=1}^{2}\exp(\pm_j it\langle n_j\rangle)\Big)\widehat{w}_{\pm_3}(t,n_3)\exp(i\langle n_{123},x\rangle)\Big] \\
&= \sum_{\pm_1,\pm_2,\pm_3} \sum_{N_1,N_2,N_3} \int_{\mathbb{R}} \mathrm{d}\lambda_3 \sum_{n_1,n_2,n_3\in\mathbb{Z}^3} \Big[c(\pm_1,\pm_2)\Big(\prod_{j=1}^{3}\rho_N(n_j)\chi_{N_j}(n_j)\Big) \\
&\qquad \times \widehat{V}(n_{12})\mathcal{I}_2[\pm_j, n_j \colon j=1,2] \\
&\qquad \times \exp(it\lambda_3)\Big(\prod_{j=1}^{3}\exp(\pm_j it\langle n_j\rangle)\Big)\widehat{w}_{\pm_3}(\lambda_3 \pm_3 \langle n_3\rangle, n_3)\exp(i\langle n_{123},x\rangle)\Big].
\end{aligned}$$

To simplify the notation, we define the phase function $\varphi\colon (\mathbb{Z}^3)^3 \to \mathbb{R}$ by

$$\varphi(n_1,n_2,n_3) = \pm_{123}\langle n_{123}\rangle \pm_1 \langle n_1\rangle \pm_2 \langle n_2\rangle \pm_3 \langle n_3\rangle. \tag{6.3}$$

The space-time Fourier transform of the time-truncated nonlinearity is therefore given by

$$\begin{aligned}
&\mathcal{F}\big(\chi(\cdot/T)(V * \overset{\bullet\,\bullet}{\vee}_{N}) \cdot P_{\leq N} w\big)(\lambda \pm_{123} \langle n\rangle, n) = T \sum_{\pm_1,\pm_2,\pm_3} \sum_{N_1,N_2,N_3} \int_{\mathbb{R}} \mathrm{d}\lambda_3 \\
&\quad \times \sum_{n_1,n_2,n_3\in\mathbb{Z}^3} \Big[c(\pm_1,\pm_2)1\{n = n_{123}\}\widehat{\chi}(T(\lambda - \lambda_3 - \varphi(n_1,n_2,n_3))) \\
&\quad \times \Big(\prod_{j=1}^{3}\rho_N(n_j)\chi_{N_j}(n_j)\Big)\widehat{V}(n_{12})\mathcal{I}_2[\pm_j, n_j \colon j=1,2]\widehat{w}_{\pm_3}(\lambda_3 \pm_3 \langle n_3\rangle, n_3)\Big].
\end{aligned} \tag{6.4}$$

To simplify the notation, we emphasize the dependence on the frequency scales N_1, N_2, N_3 by writing N_* and omit the dependence on $\pm_{123}, \pm_1, \pm_2, \pm_3$, and T from our notation. We define the tensor $h(n,n_1,n_2,n_3;\lambda,\lambda_3,N_*)$ by

$$\begin{aligned}
h(n,n_1,n_2,n_3;\lambda,\lambda_3,N_*) &\overset{\text{def}}{=} Tc(\pm_1,\pm_2)1\{n = n_{123}\}\widehat{\chi}\big(T(\lambda - \lambda_3 - \varphi(n_1,n_2,n_3))\big) \\
&\quad \times \Big(\prod_{j=1}^{3}\rho_N(n_j)\chi_{N_j}(n_j)\Big)\widehat{V}(n_{12})\langle n\rangle^{s_2-1}\langle n_1\rangle^{-1}\langle n_2\rangle^{-1}\langle n_3\rangle^{-s_1}.
\end{aligned} \tag{6.5}$$

Furthermore, we define the contracted random tensor $h_c(n, n_3; \lambda, \lambda_3)$ by

$$h_c(n, n_3; \lambda, \lambda_3, N_*) = \sum_{n_1, n_2 \in \mathbb{Z}^3} h(n, n_1, n_2, n_3; \lambda, \lambda_3, N_*) \cdot \widetilde{\mathcal{I}}_2[\pm_j, n_j \colon j = 1, 2]. \tag{6.6}$$

By combining our previous expression of the nonlinearity (6.4) with the definition (6.6), we obtain

$$\begin{aligned} \mathcal{F}\big(\chi(\cdot/T)(V * \overset{\bullet\,\bullet}{\vee}_{N}) \cdot P_{\leq N} w\big)(\lambda \pm_{123} \langle n \rangle, n) = \langle n \rangle^{-(s_2-1)} \sum_{\pm_1, \pm_2, \pm_3} \sum_{N_1, N_2, N_3} \int_{\mathbb{R}} \mathrm{d}\lambda_3 \\ \times \sum_{n_3 \in \mathbb{Z}^3} h_c(n, n_3; \lambda, \lambda_3, N_*) \langle n_3 \rangle^{s_1} \widehat{w}_{\pm_3}(\lambda_3 \pm_3 \langle n_3 \rangle, n_3). \end{aligned}$$

We estimate each combination of signs and each dyadic block separately. Using the tensor norms from Definition 4.32, the contribution to the $\mathfrak{X}^{s_2-1, b_- -1}$-norm is bounded by

$$\begin{aligned} &\Big\| \langle \lambda \rangle^{b_- -1} \int_{\mathbb{R}} \mathrm{d}\lambda_3 \sum_{n_3 \in \mathbb{Z}^3} h_c(n, n_3; \lambda, \lambda_3, N_*) \langle n_3 \rangle^{s_1} \widehat{w}_{\pm_3}(\lambda_3 \pm_3 \langle n_3 \rangle, n_3) \Big\|_{L^2_\lambda \ell^2_n(\mathbb{R} \times \mathbb{T}^3)} \\ &\qquad \lesssim \big\| \langle \lambda \rangle^{b_- -1} \langle \lambda_3 \rangle^{-b} \| h_c(n, n_3; \lambda, \lambda_3, N_*) \|_{n_3 \to n} \big\|_{L^2_\lambda L^2_{\lambda_3}(\mathbb{R} \times \mathbb{R})} \cdot \| w \|_{\mathfrak{X}^{s_1, b}(\mathbb{R})}. \end{aligned}$$

In order to control the operator norm in (6.2), it therefore remains to prove that

$$\begin{aligned} \Big\| \big\| \langle \lambda \rangle^{b_- -1} \langle \lambda_3 \rangle^{-b} \| h_c(n, n_3; \lambda, \lambda_3, N_*) \|_{n_3 \to n} \big\|_{L^2_\lambda L^2_{\lambda_3}(\mathbb{R} \times \mathbb{R})} \Big\|_{L^p_\omega(\mathbb{P})} \\ \lesssim T \max(N_1, N_2, N_3)^{-\eta/2} p. \end{aligned} \tag{6.7}$$

Step 2: Proof of the random matrix estimate (6.7). Using Minkowski's integral inequality, we have that

$$\begin{aligned} &\Big\| \big\| \langle \lambda \rangle^{b_- -1} \langle \lambda_3 \rangle^{-b} \| h_c(n, n_3; \lambda, \lambda_3, N_*) \|_{n_3 \to n} \big\|_{L^2_\lambda L^2_{\lambda_3}(\mathbb{R} \times \mathbb{R})} \Big\|_{L^p_\omega(\mathbb{P})} \\ &\quad \leq \Big\| \langle \lambda \rangle^{b_- -1} \langle \lambda_3 \rangle^{-b} \big\| \| h_c(n, n_3; \lambda, \lambda_3, N_*) \|_{n_3 \to n} \big\|_{L^p_\omega(\mathbb{P})} \Big\|_{L^2_\lambda L^2_{\lambda_3}(\mathbb{R} \times \mathbb{R})} \\ &\quad \leq \| \langle \lambda \rangle^{b_- -1} \langle \lambda_3 \rangle^{-b} \|_{L^2_\lambda L^2_{\lambda_3}(\mathbb{R} \times \mathbb{R})} \cdot \sup_{\lambda, \lambda_3 \in \mathbb{R}} \big\| \| h_c(n, n_3; \lambda, \lambda_3, N_*) \|_{n_3 \to n} \big\|_{L^p_\omega(\mathbb{P})} \\ &\quad \lesssim \sup_{\lambda, \lambda_3 \in \mathbb{R}} \big\| \| h_c(n, n_3; \lambda, \lambda_3, N_*) \|_{n_3 \to n} \big\|_{L^p_\omega(\mathbb{P})}. \end{aligned}$$

We emphasize that the supremum over $\lambda, \lambda_3 \in \mathbb{R}$ is outside the $L^p_\omega(\mathbb{P})$-norm. Using the moment method (Proposition 4.50), we get

$$\begin{aligned} &\sup_{\lambda, \lambda_3 \in \mathbb{R}} \big\| \| h_c(n, n_3; \lambda, \lambda_3, N_*) \|_{n_3 \to n} \big\|_{L^p_\omega(\mathbb{P})} \lesssim \max(N_1, N_2, N_3)^{\eta/2} \\ &\qquad \times \sup_{\lambda, \lambda_3 \in \mathbb{R}} \max\big(\| h(\cdot; \lambda, \lambda_3, N_*) \|_{n_1 n_2 n_3 \to n}, \| h(\cdot; \lambda, \lambda_3, N_*) \|_{n_3 \to n n_1 n_2}, \\ &\qquad\qquad \| h(\cdot; \lambda, \lambda_3, N_*) \|_{n_1 n_3 \to n n_2}, \| h(\cdot; \lambda, \lambda_3, N_*) \|_{n_2 n_3 \to n n_1} \big) p. \end{aligned}$$

In order to estimate the tensor norms of $h(\cdot;\lambda,\lambda_3,N_*)$, we further decompose it according to the value of the phase function φ. For any $m\in\mathbb{Z}$, we define

$$\begin{aligned}&\widetilde{h}(n,n_1,n_2,n_3;m,N_*)\\&\qquad\overset{\text{def}}{=}T1\{n=n_{123}\}1\{|\varphi(n_1,n_2,n_3))-m|\leq 1\}\\&\qquad\qquad\times\Big(\prod_{j=1}^{3}\rho_N(n_j)\chi_{N_j}(n_j)\Big)|\widehat{V}(n_{12})|\langle n\rangle^{s_2-1}\langle n_1\rangle^{-1}\langle n_2\rangle^{-1}\langle n_3\rangle^{-s_1}.\end{aligned}$$

Using the definition of h in (6.5) and the decay of $\widehat{\chi}$, we obtain

$$\begin{aligned}&|h(n,n_1,n_2,n_3;\lambda,\lambda_3,N_*)|\\&\qquad\lesssim\sum_{m\in\mathbb{Z}}|\widehat{\chi}(T(\lambda-\lambda_3-\varphi(n_1,n_2,n_3)))|1\{|\varphi(n_1,n_2,n_3)-m|\leq 1\}\\&\qquad\quad\times\widetilde{h}(n,n_1,n_2,n_3;m,N_*)\\&\qquad\lesssim\sum_{m\in\mathbb{Z}}\langle\lambda_3-\lambda-m\rangle^{-2}\widetilde{h}(n,n_1,n_2,n_3;m,N_*).\end{aligned}$$

Using the triangle inequality for the tensor norms and the first deterministic tensor estimate (Lemma 4.33), it follows that

$$\begin{aligned}&\max(N_1,N_2,N_3)^{\eta/2}\sup_{\lambda,\lambda_3\in\mathbb{R}}\max\big(\|h(\cdot;\lambda,\lambda_3,N_*)\|_{n_1n_2n_3\to n},\|h(\cdot;\lambda,\lambda_3,N_*)\|_{n_3\to nn_1n_2},\\&\qquad\qquad\|h(\cdot;\lambda,\lambda_3,N_*)\|_{n_1n_3\to nn_2},\|h(\cdot;\lambda,\lambda_3,N_*)\|_{n_2n_3\to nn_1}\big)\\&\lesssim\max(N_1,N_2,N_3)^{\eta/2}\sup_{m\in\mathbb{Z}}\max\big(\|\widetilde{h}(\cdot;m,N_*)\|_{n_1n_2n_3\to n},\|\widetilde{h}(\cdot;m,N_*)\|_{n_3\to nn_1n_2},\\&\qquad\qquad\|\widetilde{h}(\cdot;m,N_*)\|_{n_1n_3\to nn_2},\|\widetilde{h}(\cdot;m,N_*)\|_{n_2n_3\to nn_1}\big)\\&\lesssim T\max(N_1,N_2,N_3)^{-\eta/2}.\end{aligned}$$

■

Proposition 6.3 (Second RMT estimate). *Let $T,p\geq 1$. Then*

$$\Big\|\sup_{N\geq 1}\sup_{\mathcal{J}\subseteq[0,T]}\sup_{\|Y\|_{\mathfrak{X}^{s_2,b}(\mathcal{J})}\leq 1}\|{:}V*(P_{\leq N}\overset{\bullet}{|}\cdot P_{\leq N}Y)\,(\neg\circledless)\,P_{\leq N}\overset{\bullet}{|}{:}\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})}\Big\|_{L^p_\omega(\mathbb{P})}\lesssim Tp. \tag{6.8}$$

Remark 6.4. This proposition controls the second term in **RMT**, i.e., in (2.30). In the frequency-localized version of (6.8), which is detailed in the proof, we gain an η'-power in the maximal frequency scale.

Proof. Due to the operator $(\neg\circledless)$, the renormalization $\mathcal{M}_N P_{\leq N}Y$ does not just cancel the probabilistic resonances between the two factors of $\overset{\bullet}{|}$ in

$$V*(P_{\leq N}\overset{\bullet}{|}\cdot P_{\leq N}Y)\,(\neg\circledless)\,P_{\leq N}\overset{\bullet}{|}.$$

As a result, we need to decompose $\mathcal{M}_N = \mathcal{M}_N^{\olessthan} + \mathcal{M}_N^{\neg\olessthan}$, where the symbols corresponding to the multipliers are given by

$$m_N^{\olessthan}(n) \stackrel{\text{def}}{=} \sum_{L,K\colon L\leq K^\epsilon} \frac{\widehat{V}(n+k)}{\langle k\rangle^2} \chi_L(n+k)\chi_K(k)\rho_N(k)^2,$$

$$m_N^{\neg\olessthan}(n) \stackrel{\text{def}}{=} \sum_{L,K\colon L> K^\epsilon} \frac{\widehat{V}(n+k)}{\langle k\rangle^2} \chi_L(n+k)\chi_K(k)\rho_N(k)^2.$$

The random operator

$$V * (P_{\leq N} \stackrel{\bullet}{|} \cdot P_{\leq N} Y) \,(\neg\olessthan)\, P_{\leq N} \stackrel{\bullet}{|} - \mathcal{M}_N^{\neg\olessthan} P_{\leq N} Y$$

can then be controlled using the same argument as in the proof of Proposition 6.1, except that we use Lemma 4.35 instead of Lemma 4.33. Thus, it only remains to show that

$$\| \mathcal{M}_N^{\olessthan} P_{\leq N} Y \|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})} \lesssim T \|Y\|_{\mathfrak{X}^{s_2,b}(\mathcal{J})}. \tag{6.9}$$

The estimate (6.9) has a lot of room and can be established through the following simple argument. On the support of the summand in the definition of $m_N^{\olessthan}$, we have $|n+k| \lesssim |k|^\epsilon$. Using only the fact that $\widehat{V}$ is bounded, this implies that

$$|m_N^{\olessthan}(n)| \lesssim \sum_{K\geq 1}\sum_{k\in\mathbb{Z}^3} K^{-2} 1\{|n+k|\lesssim K^\epsilon\} \lesssim \sum_{K\geq 1} K^{-2+3\epsilon} \lesssim 1.$$

Thus, the symbol $m_N^{\olessthan}(n)$ is uniformly bounded and hence the corresponding multiplier $\mathcal{M}_N^{\olessthan}$ is bounded on each Sobolev space $H_x^s(\mathbb{T}^3)$. Using the Strichartz estimates (Corollary 4.7 and Lemma 4.9), we obtain

$$\begin{aligned} \| \mathcal{M}_N^{\olessthan} P_{\leq N} Y \|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})} &\lesssim \| \mathcal{M}_N^{\olessthan} P_{\leq N} Y \|_{L_t^{2b_+} H_x^{s_2-1}(\mathcal{J}\times\mathbb{T}^3)} \\ &\lesssim (1+|\mathcal{J}|)\|Y\|_{L_t^\infty H_x^{s_2-1}(\mathcal{J}\times\mathbb{T}^3)} \lesssim (1+|\mathcal{J}|)\|Y\|_{\mathfrak{X}^{s_2,b}(\mathcal{J})}. \qquad \blacksquare \end{aligned}$$

7. Paracontrolled estimates

The main goal of this section is to estimate the terms in **CPara**. We remind the reader that the paracontrolled approach to stochastic partial differential equations was introduced in the seminal paper of Gubinelli, Imkeller, and Perkowski [36] and first applied to dispersive equations by Gubinelli, Koch, and Oh [37].

The definitions of the low-frequency modulation space $\mathcal{LM}$ and the paracontrolled structure PCtrl given below follow similar ideas to the framework in [37].

Definition 7.1 (Low-frequency modulation space). Let $H = \{H(t,x;K)\}_{K\geq 1}$ be a family of space-time functions from $\mathbb{R}\times\mathbb{T}^3$ into $\mathbb{C}$ satisfying

$$\operatorname{supp}(\widehat{H}(t,x;K)) \subseteq \{k\in\mathbb{Z}^3\colon |k|\leq 8K^\epsilon\}. \tag{7.1}$$

We define the low-frequency modulation norm by

$$\|H\|_{\mathcal{LM}(\mathbb{R})} \stackrel{\text{def}}{=} \sup_{K\geq 1} K^{-4\epsilon} \|\widehat{H}(\lambda, k; K)\|_{\ell_k^\infty L_\lambda^1(\mathbb{Z}^3\times\mathbb{R})}. \tag{7.2}$$

We define the corresponding low-frequency modulation space $\mathcal{LM}(\mathbb{R})$ by

$$\mathcal{LM}(\mathbb{R}) = \{H : \|H\|_{\mathcal{LM}(\mathbb{R})} < \infty\}. \tag{7.3}$$

Furthermore, let $\mathcal{J} \subseteq \mathbb{R}$ be a time interval and let $H = \{H(t,x;K)\}_{K\geq 1}$ be a family of space-time functions from $\mathcal{J} \times \mathbb{T}^3$ into $\mathbb{R}$ satisfying (7.1). As in the definition of $\mathcal{X}^{s,b}$-spaces, we define the restricted norm by

$$\|H\|_{\mathcal{LM}(\mathcal{J})} = \inf\{\|H'\|_{\mathcal{LM}(\mathbb{R})} : H'(t) = H(t) \text{ for all } t \in \mathcal{J}\}. \tag{7.4}$$

The corresponding time-restricted low-frequency modulation space $\mathcal{LM}(\mathcal{J})$ can then be defined as in (7.2) after replacing the norm.

Definition 7.2 (Paracontrolled). Let $\mathcal{J} \subseteq \mathbb{R}$ be an interval, let $\phi: \mathcal{J} \times \mathbb{T}^3 \to \mathbb{C}$ be a distribution, and let H be as in Definition 7.1. Then we define

$$\mathrm{PCtrl}(H,\phi)(t,x) = \sum_{K\geq 1} H(t,x;K)(P_K\phi)(t,x). \tag{7.5}$$

If $H \in \mathcal{LM}(\mathbb{R})$, we have

$$\begin{aligned}&\mathrm{PCtrl}(H,\phi)(t,x)\\&= \sum_{K\geq 1}\sum_{k_1\in\mathbb{Z}^3}\int_{\mathbb{R}} \mathrm{d}\lambda_1\, \widehat{H}(\lambda_1,k_1;K)(\exp(i\lambda_1 t)\sum_{k_2\in\mathbb{Z}^3}\chi_K(k_2)\widehat{\phi}(t,k_2)\exp(i\langle k_{12},x\rangle)).\end{aligned} \tag{7.6}$$

The expression (7.6) will be used in all of our estimates involving PCtrl. The sum in k_1, the integral in λ_1, and the pre-factor $\widehat{H}(\lambda_1,k_1;K)$ will be inessential. The main step will consist of estimates for

$$\exp(i\lambda_1 t)\sum_{k_2\in\mathbb{Z}^3}\chi_K(k_2)\widehat{\phi}(t,k_2)\exp(i\langle k_{12},x\rangle),$$

which essentially behaves like $P_K\phi(t,x)$. For most purposes, the reader may simply think of $\mathrm{PCtrl}(H,\phi)$ as ϕ.

Lemma 7.3 (Basic mapping properties of PCtrl). *For any $s \in \mathbb{R}$, any interval $\mathcal{J} \subseteq \mathbb{R}$, any $\phi \in L_t^\infty H_x^s(\mathcal{J}\times\mathbb{T}^3)$, and any $H \in \mathcal{LM}(\mathcal{J})$, we have*

$$\|\mathrm{PCtrl}(H,\phi)\|_{L_t^\infty H_x^{s-8\epsilon}(\mathcal{J}\times\mathbb{T}^3)} \lesssim \|H\|_{\mathcal{LM}(\mathcal{J})}\|\phi\|_{L_t^\infty H_x^s(\mathcal{J}\times\mathbb{T}^3)}. \tag{7.7}$$

Proof. We treat each dyadic piece in PCtrl separately. Using the Fourier support condition (7.1), we have

$$\begin{aligned}
&\| H(t,x;K)(P_K\phi)(t,x)\|_{H_x^{s-8\epsilon}(\mathbb{T}^3)} \\
&\quad = \Big\| \sum_{k_1,k_2\in\mathbb{Z}^3} \chi_K(k_2)\widehat{H}(t,k_1;K)\widehat{\phi}(t,k_2)\exp(i\langle k_{12},x\rangle)\Big\|_{H_x^{s-8\epsilon}(\mathbb{T}^3)} \\
&\quad \lesssim \sum_{k_1\in\mathbb{Z}^3} |\widehat{H}(t,k_1;K)| \Big\| \sum_{k_2\in\mathbb{Z}^3} \chi_K(k_2)\widehat{\phi}(t,k_2)\exp(i\langle k_{12},x\rangle)\Big\|_{H_x^{s-8\epsilon}(\mathbb{T}^3)} \\
&\quad \lesssim K^{-8\epsilon}\Big(\sum_{k_1\in\mathbb{Z}^3} |\widehat{H}(t,k_1;K)|\Big)\|\phi(t)\|_{H_x^s(\mathbb{T}^3)} \lesssim K^{-\epsilon}\|H\|_{\mathcal{LM}(\mathcal{J})}\|\phi(t)\|_{H_x^s(\mathbb{T}^3)}.
\end{aligned}$$

The desired estimate follows after summing over K. ∎

In the next two lemmas, we show that the terms appearing in the evolution equation (2.14) for X_N fit into our paracontrolled framework.

Lemma 7.4. *Let $\mathcal{J} \subseteq \mathbb{R}$ be an interval and let $f, g \in \mathcal{X}^{-1,b}(\mathcal{J})$. Then there exists a (canonical) $H \in \mathcal{LM}(\mathcal{J})$ satisfying*

$$\boxed{\textcircled{<}\,\&\,\textcircled{<}}(V * (f\, g)\phi) = \mathrm{PCtrl}(H,\phi) \tag{7.8}$$

for all space-time distributions $\phi\colon \mathcal{J}\times\mathbb{T}^3 \to \mathbb{C}$. Furthermore

$$\|H\|_{\mathcal{LM}(\mathcal{J})} \lesssim \|f\|_{\mathcal{X}^{-1,b}(\mathcal{J})}\cdot\|g\|_{\mathcal{X}^{-1,b}(\mathcal{J})}. \tag{7.9}$$

Remark 7.5. Due to the overlaps in the support of the Littlewood–Paley multipliers χ_K, the low-frequency modulation $H \in \mathcal{LM}(\mathcal{J})$ is not quite unique. As will be clear from the proof, however, there is a canonical choice. This canonical choice is also bilinear in f and g.

Proof of Lemma 7.4. Using the definition of the restricted norms, it suffices to treat the case $\mathcal{J} = \mathbb{R}$. We have

$$\begin{aligned}
\boxed{\textcircled{<}\,\&\,\textcircled{<}}(V * (f\, g)\phi)(t,x) &= \sum_{\substack{N_1,N_2,K:\\ N_1,N_2\leq K^\epsilon}} \sum_{n_1,n_2,k\in\mathbb{Z}^3} \chi_{N_1}(n_1)\chi_{N_2}(n_2)\chi_K(k) \\
&\qquad \times \widehat{V}(n_{12})\widehat{f}(t,n_1)\widehat{g}(t,n_2)\widehat{\phi}(t,k)\exp(i\langle n_{12}+k,x\rangle) \\
&= \mathrm{PCtrl}(H,\phi)(t,x),
\end{aligned}$$

where

$$\widehat{H}(t,k_1;K) = \sum_{\substack{N_1,N_2:\\ N_1,N_2\leq K^\epsilon}} \sum_{\substack{n_1,n_2\in\mathbb{Z}^3:\\ n_{12}=k_1}} \chi_{N_1}(n_1)\chi_{N_2}(n_2)\widehat{V}(n_{12})\widehat{f}(t,n_1)\widehat{g}(t,n_2). \tag{7.10}$$

It therefore remains to show $H \in \mathcal{LM}(\mathbb{R})$ and the estimate (7.9). The Fourier support condition (7.1) is a consequence of the multiplier $\chi_{N_1}(n_1)\chi_{N_2}(n_2)$ in (7.10). To see the estimate (7.9), we first note that

$$\widehat{H}(\lambda,k_1;K) = \sum_{\substack{N_1,N_2:\\ N_1,N_2\leq K^\epsilon}} \sum_{\substack{n_1,n_2\in\mathbb{Z}^3:\\ n_{12}=k_1}} \chi_{N_1}(n_1)\chi_{N_2}(n_2)\widehat{V}(n_{12})\big(\widehat{f}(\cdot,n_1) * \widehat{g}(\cdot,n_2)\big)(\lambda).$$

Using Young's convolution inequality and Cauchy–Schwarz, we obtain

$$
\begin{aligned}
&\| \widehat{H}(\lambda, k_1; K) \|_{L^1_\lambda(\mathbb{R})} \\
&\lesssim \sum_{\substack{N_1, N_2: \\ N_1, N_2 \leq K^\epsilon}} \sum_{\substack{n_1, n_2 \in \mathbb{Z}^3: \\ n_{12} = k_1}} \chi_{N_1}(n_1) \chi_{N_2}(n_2) |\widehat{V}(n_{12})| \, \| \widehat{f}(\lambda, n_1) \|_{L^1_\lambda(\mathbb{R})} \| \widehat{g}(\lambda, n_2) \|_{L^1_\lambda(\mathbb{R})} \\
&\lesssim \sum_{\substack{n_1, n_2 \in \mathbb{Z}^3: \\ n_{12} = k_1}} 1\{ |n_1|, |n_2| \lesssim K^\epsilon \} \| \langle |\lambda| - \langle n_1 \rangle \rangle^b \widehat{f}(\lambda, n_1) \|_{L^2_\lambda(\mathbb{R})} \\
&\qquad \times \| \langle |\lambda| - \langle n_2 \rangle \rangle^b \widehat{g}(\lambda, n_2) \|_{L^2_\lambda(\mathbb{R})} \\
&\lesssim \Big(\sum_{n_1 \in \mathbb{Z}^3} 1\{ |n_1| \lesssim K^\epsilon \} \| \langle |\lambda| - \langle n_1 \rangle \rangle^b \widehat{f}(\lambda, n_1) \|^2_{L^2_\lambda(\mathbb{R})} \Big)^{1/2} \\
&\quad \times \Big(\sum_{n_2 \in \mathbb{Z}^3} 1\{ |n_2| \lesssim K^\epsilon \} \| \langle |\lambda| - \langle n_2 \rangle \rangle^b \widehat{g}(\lambda, n_2) \|^2_{L^2_\lambda(\mathbb{R})} \Big)^{1/2} \\
&\lesssim K^{-2\epsilon} \| f \|_{\mathfrak{X}^{-1,b}(\mathcal{J})} \cdot \| g \|_{\mathfrak{X}^{-1,b}(\mathcal{J})}.
\end{aligned}
$$

The desired estimate (7.9) now follows by taking the supremum in $K \geq 1$ and $k_1 \in \mathbb{Z}^3$. ■

Lemma 7.6. *Let $\mathcal{J} \subseteq \mathbb{R}$ be an interval, let $s \in [-1, 1]$, let $f \in \mathfrak{X}^{-s,b}(\mathcal{J})$, and let $g \in \mathfrak{X}^{s,b}$. Then there exists a (canonical) $H \in \mathcal{LM}(\mathcal{J})$ satisfying*

$$
V * (f \, g) \circledless \phi = \mathrm{PCtrl}(H, \phi) \tag{7.11}
$$

for all space-time distributions $\phi \colon J \times \mathbb{T}^3 \to \mathbb{C}$. Furthermore,

$$
\| H \|_{\mathcal{LM}(\mathcal{J})} \lesssim \| f \|_{\mathfrak{X}^{-s,b}(\mathcal{J})} \cdot \| g \|_{\mathfrak{X}^{s,b}(\mathcal{J})}. \tag{7.12}
$$

Remark 7.7. We emphasize that Lemma 7.6 fails if we replace the assumptions by $f, g \in \mathfrak{X}^{-1,b}(\mathcal{J})$ as in Lemma 7.4. The reason is that the product $f \cdot g$ inside the convolution with the interaction potential V is not even well-defined.

Proof. Proof of Lemma 7.6] The argument is similar to the proof of Lemma 7.4. As before, it suffices to treat the case $\mathcal{J} = \mathbb{R}$. A direct calculation yields the identity (7.11) with

$$
H(t, k_1; K) = \sum_{K_1 \leq K^\epsilon} \chi_{K_1}(k_1) \widehat{V}(k_1) \sum_{\substack{n_1, n_2 \in \mathbb{Z}^3: \\ n_{12} = k_1}} \widehat{f}(t, n_1) \widehat{g}(t, n_2). \tag{7.13}
$$

Using Young's convolution inequality and Cauchy–Schwarz, we obtain

$$
\| \widehat{H}(\lambda, k_1; K) \|_{L^1_\lambda(\mathbb{R})} \lesssim \sum_{\substack{n_1, n_2 \in \mathbb{Z}^3: \\ n_{12} = k_1}} \| \widehat{f}(\lambda, n_1) \|_{L^1_\lambda(\mathbb{R})} \| \widehat{g}(\lambda, n_2) \|_{L^1_\lambda(\mathbb{R})}
$$

$$\lesssim \Big(\sum_{n_1 \in \mathbb{Z}^3} \langle n_1 \rangle^{-2s} \| \langle |\lambda| - \langle n_1 \rangle \rangle^b \widehat{f}(\lambda, n_1) \|^2_{L^2_\lambda(\mathbb{R})} \Big)^{1/2}$$
$$\times \Big(\sum_{n_2 \in \mathbb{Z}^3} \langle n_2 - k_1 \rangle^{2s} \| \langle |\lambda| - \langle n_2 \rangle \rangle^b \widehat{g}(\lambda, n_2) \|^2_{L^2_\lambda(\mathbb{R})} \Big)^{1/2}.$$

Using $\langle n_2 - k_1 \rangle \lesssim \langle k_1 \rangle + \langle n_2 \rangle \lesssim K^\epsilon \langle n_2 \rangle$, we obtain the estimate (7.12). ∎

7.1. *Quadratic paracontrolled estimate*

In this subsection, we show that $P_{\leq N} X_N \circleddash P_{\leq N} \stackrel{\bullet}{|}$ is well-defined uniformly in N even though the sum of the individual spatial regularities is negative. Together with Lemma 8.8, this will control the second and third terms in **Phy**, i.e.,

$$V * (P_{\leq N} X_N \circleddash P_{\leq N} \stackrel{\bullet}{|}) \cdot P_{\leq N} \Psi_N \quad \text{and} \quad V * (P_{\leq N} X_N \circleddash P_{\leq N} \stackrel{\bullet}{|}) \cdot P_{\leq N} w_N .$$

Proposition 7.8 (Quadratic paracontrolled object). *Let $T \geq 1$. For any $s < -2\eta - 10\epsilon$ and $p \geq 2$, we have*

$$\sum_{L_1 \sim L_2} L_1^{2\eta} \Big\| \sup_{N \geq 1} \sup_{\mathcal{J} \subseteq [0,T]} \sup_{\|H\|_{\mathcal{LM}(\mathcal{J})} \leq 1} \| (P_{L_1} P_{\leq N} \mathrm{I})[1_{\mathcal{J}} \operatorname{PCtrl}(H, P_{\leq N} \stackrel{\bullet}{|})] \cdot P_{L_2} \stackrel{\bullet}{|} \|_{L^\infty_t \mathcal{C}^s_x([0,T] \times \mathbb{T}^3)} \Big\|_{L^p_\omega(\mathbb{P})} \lesssim T^3 p,$$

where the supremum over $\mathcal{J}$ is taken only over intervals.

Proof. The supremum in N can be handled through the decay in the frequency-localized version below and we omit it throughout the proof. Using the definition of the $\mathcal{LM}(\mathcal{J})$-norm, we may take the supremum over $H \in \mathcal{LM}(\mathbb{R})$ with norm bounded by 1. By inserting the expansion (7.6), we obtain

$$(P_{L_1} P_{\leq N} \mathrm{I})[1_{\mathcal{J}} \operatorname{PCtrl}(H, P_{\leq N} \stackrel{\bullet}{|})](t,x) \cdot P_{L_2} \stackrel{\bullet}{|}(t,x)$$
$$= \sum_{N_1} \sum_{k_1 \in \mathbb{Z}^3} \int_{\mathbb{R}} \mathrm{d}\lambda_1 \, \widehat{H}(\lambda_1, k_1; N_1)$$
$$\times \sum_{n_1, n_2 \in \mathbb{Z}^3} \Big[\rho_N(k_1 + n_1) \chi_{L_1}(n_1 + k_1) \rho_N(n_1) \chi_{N_1}(n_1) \chi_{L_2}(n_2)$$
$$\times \widehat{\stackrel{\bullet}{|}}(t, n_2) \Big(\int_0^t 1_{\mathcal{J}}(t') \frac{\sin((t-t')\langle k_1 + n_1 \rangle)}{\langle k_1 + n_1 \rangle} \exp(i \lambda_1 t') \widehat{\stackrel{\bullet}{|}}(t', n_1) \, \mathrm{d}t' \Big) \exp(i \langle n_{12} + k_1, x \rangle) \Big].$$

Due to the definition of $\mathcal{LM}$, we only obtain a non-trivial contribution if $N_1 \sim L_1 \sim L_2$. Using the triangle inequality, it follows that

$$\sup_{\mathcal{J}\subseteq[0,T]} \sup_{\|H\|_{\mathcal{LM}(\mathbb{R})}\leq 1} \|(P_{L_1} P_{\leq N}\,\mathrm{I})[1_{\mathcal{J}}\,\mathrm{PCtrl}(H, P_{\leq N}\lollipop)]\cdot P_{L_2}\lollipop\|_{L_t^\infty \mathcal{C}_x^s([0,T]\times\mathbb{T}^3)}$$

$$\lesssim \sum_{N_1} N_1^{7\epsilon} \sup_{\mathcal{J}\subseteq[0,T]} \sup_{\substack{k_1\in\mathbb{Z}^3:\\ |k_1|\leq 8N_1^\epsilon}} \sup_{\lambda_1\in\mathbb{R}} \Big\| \sum_{n_1,n_2\in\mathbb{Z}^3} \Big[\rho_N(k_1+n_1)\chi_{L_1}(n_1+k_1)\rho_N(n_1)$$
$$\times \chi_{N_1}(n_1)\chi_{L_2}(n_2)\exp(i\langle n_{12}+k_1, x\rangle)\widehat{\lollipop}(t,n_2)$$
$$\times \Big(\int_0^t 1_{\mathcal{J}}(t')\frac{\sin((t-t')\langle k_1+n_1\rangle)}{\langle k_1+n_1\rangle}\exp(i\lambda_1 t')\widehat{\lollipop}(t',n_1)\,\mathrm{d}t'\Big)\Big]\Big\|_{L_t^\infty\mathcal{C}_x^s([0,T]\times\mathbb{T}^3)}.$$

To obtain the desired estimate, it suffices to prove for all $N_1 \sim L_1 \sim L_2$ that

$$\Big\| \sup_{\mathcal{J}\subseteq[0,T]} \sup_{\substack{k_1\in\mathbb{Z}^3:\\ |k_1|\leq 8N_1^\epsilon}} \sup_{\lambda_1\in\mathbb{R}} \Big\| \sum_{n_1,n_2\in\mathbb{Z}^3} \Big[\rho_N(k_1+n_1)\chi_{L_1}(n_1+k_1)$$
$$\times \rho_N(n_1)\chi_{N_1}(n_1)\chi_{L_2}(n_2)\exp(i\langle n_{12}+k_1, x\rangle)\widehat{\lollipop}(t,n_2)$$
$$\times \Big(\int_0^t 1_{\mathcal{J}}(t')\frac{\sin((t-t')\langle k_1+n_1\rangle)}{\langle k_1+n_1\rangle}\exp(i\lambda_1 t')\widehat{\lollipop}(t',n_1)\,\mathrm{d}t'\Big)\Big]\Big\|_{L_t^\infty\mathcal{C}_x^s([0,T]\times\mathbb{T}^3)}\Big\|_{L_\omega^p(\Omega)}$$
$$\lesssim T^3 N_1^{-2\eta-9\epsilon}. \tag{7.14}$$

We claim that instead of (7.14), it suffices to prove the simpler estimate

$$\sup_{t\in[0,T]} \sup_{\mathcal{J}\subseteq[0,T]} \sup_{\substack{k_1\in\mathbb{Z}^3:\\ |k_1|\leq 8N_1^\epsilon}} \sup_{\lambda_1\in\mathbb{R}} \Big\| \sum_{n_1,n_2\in\mathbb{Z}^3} \Big[\rho_N(k_1+n_1)\chi_{L_1}(n_1+k_1)$$
$$\times \rho_N(n_1)\chi_{N_1}(n_1)\chi_{L_2}(n_2)\exp(i\langle n_{12}+k_1, x\rangle)\widehat{\lollipop}(t,n_2)$$
$$\times \Big(\int_0^t 1_{\mathcal{J}}(t')\frac{\sin((t-t')\langle k_1+n_1\rangle)}{\langle k_1+n_1\rangle}\exp(i\lambda_1 t')\widehat{\lollipop}(t',n_1)\,\mathrm{d}t'\Big)\Big]\Big\|_{L_\omega^2 H_x^s}$$
$$\lesssim T^2 N_1^{-2\eta-10\epsilon} p. \tag{7.15}$$

The reduction of (7.14) to (7.15) is standard and we only sketch the argument. The supremum in k_1 can easily be moved outside the moment by using Lemma 4.48 and accepting a logarithmic loss in N_1. To deal with the supremum in $\lambda_1 \in \mathbb{R}$, we treat two separate cases. Using the Lipschitz estimate $|\exp(i\lambda_1 t') - \exp(i\widetilde{\lambda}_1 t')| \lesssim |t'|\,|\lambda_1 - \widetilde{\lambda}_1|$, the supremum over $|\lambda_1| \lesssim N_1^{10}$ can easily be replaced by the supremum over a grid on $[-N_1^{10}, N_1^{10}]$ with mesh size $\sim N_1^{-10}$. The discrete supremum can then be moved outside the probabilistic moment using Lemma 4.48. For $|\lambda_1| \gtrsim N_1^{10}$, a simple integration by parts gains a factor of $|\lambda_1|^{-1}$ and we can proceed using crude estimates. The supremum over $t \in [0, T]$ and $\mathcal{J} \subseteq [0, T]$, which is parametrized by its two endpoints, can be moved outside the probabilistic moment using the first part of the argument for λ_1. Finally, Gaussian hypercontractivity allows us to replace $L_\omega^p \mathcal{C}_x^s$ by $L_\omega^2 H_x^s$.

We now turn to the proof of the simpler estimate (7.15). Using the product formula for multiple stochastic integrals, we have

$$
\begin{aligned}
&\sum_{n_1,n_2\in\mathbb{Z}^3}\Big[\rho_N(k_1+n_1)\chi_{L_1}(n_1+k_1)\rho_N(n_1)\chi_{N_1}(n_1)\chi_{L_2}(n_2)\exp(i\langle n_{12}+k_1,x\rangle)\\
&\qquad\times\widehat{\mid}(t,n_2)\Big(\int_0^t 1_{\mathcal{J}}(t')\frac{\sin((t-t')\langle k_1+n_1\rangle)}{\langle k_1+n_1\rangle}\exp(i\lambda_1 t')\widehat{\mid}(t',n_1)\,\mathrm{d}t'\Big)\Big]\\
&\qquad\qquad= \mathcal{G}^{(2)}(t,x)+\mathcal{G}^{(0)}(t,x),
\end{aligned}
$$

where the Gaussian chaoses $\mathcal{G}^{(2)}$ and $\mathcal{G}^{(0)}$ are given by

$$
\begin{aligned}
&\mathcal{G}^{(2)}(t,x)\\
&\quad\stackrel{\text{def}}{=}\sum_{\pm_1,\pm_2}\sum_{n_1,n_2\in\mathbb{Z}^3}\Big[c(\pm_1,\pm_2)\rho_N(k_1+n_1)\chi_{L_1}(n_1+k_1)\rho_N(n_1)\chi_{N_1}(n_1)\chi_{L_2}(n_2)\\
&\quad\times\Big(\int_0^t 1_{\mathcal{J}}(t')\frac{\sin((t-t')\langle k_1+n_1\rangle)}{\langle k_1+n_1\rangle}\exp(i\lambda_1 t'\pm_1 it'\langle n_1\rangle\pm_2 it\langle n_2\rangle)\,\mathrm{d}t'\Big)\\
&\quad\times\exp(i\langle n_{12}+k_1,x\rangle)\mathcal{I}_2[\pm_j,n_j\colon j=1,2]\Big],\\
&\mathcal{G}^{(0)}(t,x)\stackrel{\text{def}}{=}\exp(i\langle k_1,x\rangle)\sum_{n_1\in\mathbb{Z}^3}\Big[\rho_N(k_1+n_1)\chi_{L_1}(n_1+k_1)\rho_N(n_1)\\
&\qquad\times\chi_{N_1}(n_1)\chi_{L_2}(n_1)\frac{1}{\langle n_1+k_1\rangle\langle n_1\rangle^2}\\
&\qquad\times\Big(\int_0^t 1_{\mathcal{J}}(t')\sin((t-t')\langle k_1+n_1\rangle)\cos((t-t')\langle n_1\rangle)\exp(i\lambda_1 t')\,\mathrm{d}t'\Big)\Big].
\end{aligned}
$$

The quadratic Gaussian chaos $\mathcal{G}^{(2)}$ is the nonresonant part and the constant "Gaussian chaos" $\mathcal{G}^{(0)}$ is the resonant part. We now treat the two components separately.

Contribution of the quadratic Gaussian chaos $\mathcal{G}^{(2)}$: Using the orthogonality of multiple stochastic integrals and taking absolute values inside the t'-integral, we find that

$$
\begin{aligned}
&\|\mathcal{G}^{(2)}(t,x)\|^2_{L^2_\omega H^s_x(\Omega\times\mathbb{T}^3)}\\
&\quad\lesssim T^2\sum_{n_1,n_2\in\mathbb{Z}^3}\chi_{N_1}(n_1)\chi_{L_2}(n_2)\langle k_1+n_{12}\rangle^{2s}\langle k_1+n_1\rangle^{-2}\langle n_1\rangle^{-2}\langle n_2\rangle^{-2}\\
&\quad\lesssim T^2N_1^{-6}\sum_{n_1\in\mathbb{Z}^3}\chi_{N_1}(n_1)\chi_{L_2}(n_2)\langle k_1+n_{12}\rangle^{2s}\\
&\quad\lesssim T^2N_1^{-4\eta-20\epsilon},
\end{aligned}
$$

which is acceptable.

Contribution of the constant "Gaussian chaos" $\mathscr{G}^{(0)}$: Using the sine-cancellation lemma (Lemma 4.14), we have

$$
\begin{aligned}
&\|\mathscr{G}^{(0)}(t,x)\|_{H_x^s(\mathbb{T}^3)} \\
&\quad\lesssim \Bigg| \sum_{n_1\in\mathbb{Z}^3} \Bigg[\rho_N(k_1+n_1)\chi_{L_1}(n_1+k_1)\rho_N(n_1)\chi_{N_1}(n_1)\chi_{L_2}(n_1)\frac{1}{\langle n_1+k_1\rangle\langle n_1\rangle^2} \\
&\qquad\times \bigg(\int_0^t 1_{\mathcal{J}}(t')\sin((t-t')\langle k_1+n_1\rangle)\cos((t-t')\langle n_1\rangle)\exp(i\lambda_1 t')\,\mathrm{d}t' \bigg) \Bigg] \Bigg| \\
&\quad\lesssim N_1^{-1+3\epsilon},
\end{aligned}
$$

which is also acceptable. ∎

7.2. *Cubic paracontrolled estimate*

In this subsection, we control the cubic paracontrolled object, i.e., the first summand in the definition of **CPara** in (2.29).

Proposition 7.9. *Let $T\geq 1$. For any interval $\mathcal{J}\subseteq[0,T]$, any $\phi\colon[0,T]\times\mathbb{T}^3\to\mathbb{C}$, and $H\in\mathscr{LM}(\mathcal{J})$, define*

$$
\begin{aligned}
\mathrm{PCtrl}_N^{(3)}(H,\phi;\mathcal{J}) &\overset{\text{def}}{=} \Big(\neg\,\boxed{\textcircled{$<$}\,\&\,\textcircled{$<$}}\Big)\Big(V*\big((P_{\leq N}^2\,\mathrm{I})[1_{\mathcal{J}}\,\mathrm{PCtrl}(H,\phi)]\cdot\phi\big)\cdot\phi\Big) \\
&\qquad - \mathcal{M}_N\,P_{\leq N}^2\,\mathrm{I}[1_{\mathcal{J}}\,\mathrm{PCtrl}(H,\phi)].
\end{aligned}
$$

Then, for all $p\geq 2$,

$$
\Big\| \sup_{N\geq 1}\ \sup_{\mathcal{J}\subseteq[0,T]}\ \sup_{\|H\|_{\mathscr{LM}(\mathcal{J})}\leq 1} \|\mathrm{PCtrl}_N^{(3)}(H,P_{\leq N}\overset{\bullet}{|};\mathcal{J})\|_{\mathscr{X}^{s_2-1,b_+-1}([0,T])} \Big\|_{L_\omega^p(\mathbb{P})} \lesssim T^3 p^{3/2},
$$

where the supremum over $\mathcal{J}$ is only taken over intervals.

Remark 7.10. The notation $\mathrm{PCtrl}_N^{(3)}(H,P_{\leq N}\overset{\bullet}{|};\mathcal{J})$ will only be used in Proposition 7.9 and its proof. The frequency-localized version of Proposition 7.9 also gains an η'-power in the maximal frequency scale.

Proof of Proposition 7.9. As before, we ignore the supremum in N, which can be easily handled through the decay in the frequency-localized version below. Using the decay in the frequency-localized version and a crude estimate, we can also replace the $\mathscr{X}^{s_2,b_+-1}$-norm by the $\mathscr{X}^{s_2,b_--1}$-norm. Using the definition of the restricted norms, it suffices to consider $H\in\mathscr{LM}(\mathbb{R})$ with $\|H\|_{\mathscr{LM}(\mathbb{R})}\leq 1$. In order to use a Littlewood–Paley decomposition, we need to break up the multiplier $\mathcal{M}_N$. We define $\mathcal{M}_N[N_1,N_2,N_3]$ as the multiplier with symbol

$$
m_N[N_1,N_2,N_3](n_2) = \sum_{k\in\mathbb{Z}^3} \frac{\widehat{V}(k+n_2)}{\langle k\rangle^2}\rho_N(k)^2\chi_{N_1}(k)\chi_{N_2}(n_2)\chi_{N_3}(k). \tag{7.16}
$$

We note that $\mathcal{M}_N[N_1, N_2, N_3]$ is only nonzero when $N_1 \sim N_3$, and hence in particular when $N_1 > N_3^{\epsilon}$. We now face a notational nuisance: both PCtrl and $\boxed{\textcircled{$<$} \,\&\, \textcircled{$<$}}$ contain frequency projections. To handle this, we use N_2 and N_2' for the respective frequency scales, but encourage the reader to mentally set $N_2 = N_2'$. It then follows that

$$
\begin{aligned}
&\mathrm{PCtrl}^{(3)}_N(H, P_{\leq N}\overset{\bullet}{|}; \mathcal{J}) \\
&\quad= \sum_{\substack{N_1, N_2', N_3: \\ \max(N_1, N_2') > N_3^{\epsilon}}} \Big[V * \big(P_{N_1} P_{\leq N}\overset{\bullet}{|} \cdot P_{N_2'} P_{\leq N}^2 \, \mathrm{I}[1_{\mathcal{J}} \, \mathrm{PCtrl}(H, P_{\leq N}\overset{\bullet}{|})] \big) \cdot P_{N_3} P_{\leq N}\overset{\bullet}{|} \\
&\qquad\qquad - \mathcal{M}_N[N_1, N_2', N_3] P_{\leq N}^2 \, \mathrm{I}[1_{\mathcal{J}} \, \mathrm{PCtrl}(H, P_{\leq N}\overset{\bullet}{|})] \Big]. \qquad (7.17)
\end{aligned}
$$

Using the stochastic representation formula (4.77) in Proposition 4.44 and the expansion (7.6), we obtain

$$
\begin{aligned}
&\mathrm{PCtrl}^{(3)}_N(H, P_{\leq N}\overset{\bullet}{|}; \mathcal{J})(t,x) \\
&= \sum_{\substack{N_1, N_2, N_2', N_3: \\ \max(N_1, N_2') > N_3, \\ N_2 \sim N_2'}} \sum_{k_2 \in \mathbb{Z}^3} \int_{\mathbb{R}} \mathrm{d}\lambda_2 \, \widehat{H}(\lambda_2, k_2; N_2) \sum_{n_1, n_2, n_3 \in \mathbb{Z}^3} \Big[\rho_N(n_2+k_2)^2 \chi_{N_2'}(n_2+k_2) \\
&\qquad \times \Big(\prod_{j=1}^{3} \rho_N(n_j) \chi_{N_j}(n_j) \Big) \widehat{V}(n_{12} + k_2) \\
&\qquad \times \Big(\int_0^t 1_{\mathcal{J}}(t') \frac{\sin((t-t')\langle n_2 + k_2 \rangle)}{\langle n_2 + k_2 \rangle} \exp(i t' \lambda_2) \mathcal{I}_1[t', n_2] \, \mathrm{d}t' \Big) \\
&\qquad \times \exp(i \langle n_{123} + k_2, x \rangle) \mathcal{I}_2[t, n_1, n_3] \Big].
\end{aligned}
$$

Using the product formula for multiple stochastic integrals, we can decompose the inner sum over n_1, n_2, and n_3 as

$$
\begin{aligned}
&\sum_{n_1, n_2, n_3 \in \mathbb{Z}^3} \Big[\rho_N(n_2+k_2)^2 \chi_{N_2'}(n_2+k_2) \Big(\prod_{j=1}^{3} \rho_N(n_j) \chi_{N_j}(n_j) \Big) \widehat{V}(n_{12} + k_2) \mathcal{I}_2[t, n_1, n_3] \\
&\quad \times \Big(\int_0^t 1_{\mathcal{J}}(t') \frac{\sin((t-t')\langle n_2 + k_2 \rangle)}{\langle n_2 + k_2 \rangle} \exp(i t' \lambda_2) \mathcal{I}_1[t', n_2] \, \mathrm{d}t' \Big) \exp(i \langle n_{123} + k_2, x \rangle) \Big] \\
&\quad = \mathcal{G}^{(3)}(t, x; \lambda_2, k_2, \mathcal{J}, N_*) + \mathcal{G}^{(1)}(t, x; \lambda_2, k_2, \mathcal{J}, N_*) + \widetilde{\mathcal{G}}^{(1)}(t, x; \lambda_2, k_2, \mathcal{J}, N_*),
\end{aligned}
$$

where the cubic and linear Gaussian chaoses are given by

$$
\begin{aligned}
&\mathcal{G}^{(3)}(t,x) \\
&= \sum_{\pm_1, \pm_2, \pm_3} c(\pm_j : 1 \leq j \leq 3) \sum_{n_1, n_2, n_3 \in \mathbb{Z}^3} \Big[\rho_N(n_2+k_2)^2 \chi_{N_2'}(n_2+k_2) \Big(\prod_{j=1}^{3} \rho_N(n_j) \chi_{N_j}(n_j) \Big) \\
&\qquad \times \widehat{V}(n_{12}+k_2) \Big(\int_0^t 1_{\mathcal{J}}(t') \frac{\sin((t-t')\langle n_2+k_2 \rangle)}{\langle n_2+k_2 \rangle} \exp(i t' \lambda_2 \pm_2 i t' \langle n_2 \rangle) \, \mathrm{d}t' \Big) \\
&\qquad \times \exp(\pm_1 i t \langle n_1 \rangle \pm_3 i t \langle n_3 \rangle) \exp(i \langle n_{123} + k_2, x \rangle) \mathcal{I}_3[\pm_j, n_j : 1 \leq j \leq 3] \Big],
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{G}^{(1)}(t,x) \\
&= \sum_{n_3\in\mathbb{Z}^3} \rho_N(n_3)\chi_{N_3}(n_3)\exp(\langle n_3+k_2,x\rangle) \sum_{n_1\in\mathbb{Z}^3}\Big[\rho_N(n_2+k_2)^2\chi_{N_2'}(n_2+k_2)\rho_N(n_2)^2 \\
&\quad \times \chi_{N_1}(n_2)\chi_{N_2}(n_2)\Big(\int_0^t 1_{\mathcal{J}}(t')\sin((t-t')\langle n_2+k_2\rangle)\cos((t-t')\langle n_2\rangle)\exp(it'\lambda_2)\,\mathrm{d}t'\Big) \\
&\quad \times \widehat{V}(k_2)\langle n_2+k_2\rangle^{-1}\langle n_2\rangle^{-2}\Big]\,\mathcal{I}_1[t;n_3], \\
&\widetilde{\mathcal{G}}^{(1)}(t,x) \\
&= \sum_{n_1\in\mathbb{Z}^3} \rho_N(n_1)\chi_{N_1}(n_1)\exp(\langle n_1+k_2,x\rangle) \sum_{n_1\in\mathbb{Z}^3}\Big[\rho_N(n_2+k_2)^2\chi_{N_2'}(n_2+k_2)\rho_N(n_2)^2 \\
&\quad \times \chi_{N_2}(n_2)\chi_{N_3}(n_2)\Big(\int_0^t 1_{\mathcal{J}}(t')\sin((t-t')\langle n_2+k_2\rangle)\cos((t-t')\langle n_2\rangle)\exp(it'\lambda_2)\,\mathrm{d}t'\Big) \\
&\quad \times \widehat{V}(n_{12}+k_2)\langle n_2+k_2\rangle^{-1}\langle n_2\rangle^{-2}\Big]\,\mathcal{I}_1[t;n_1].
\end{aligned}
$$

We refer to $\mathcal{G}^{(3)}$ as the nonresonant term and to $\mathcal{G}^{(1)}$ and $\widetilde{\mathcal{G}}^{(1)}$ as the resonant terms. Using the triangle inequality and $\|H\|_{\mathcal{LM}(\mathbb{R})} \leq 1$, we obtain

$$
\begin{aligned}
&\|\operatorname{PCtrl}_N^{(3)}(H, P_{\leq N}\,;\mathcal{J})\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})} \\
&\lesssim \sum_{\substack{N_1,N_2,N_2',N_3:\\ \max(N_1,N_2')>N_3,\\ N_2\sim N_2'}} N_2^{7\epsilon} \sup_{\lambda_2\in\mathbb{R}} \sup_{\substack{k_2\in\mathbb{Z}^3:\\ |k_2|\lesssim N_2^{\epsilon}}} \Big(\|\mathcal{G}^{(3)}(\cdot;\lambda_2,k_2,\mathcal{J},N_*)\|_{\mathfrak{X}^{s_2-1,b_--1}([0,T])} \\
&\qquad + \|\mathcal{G}^{(1)}(\cdot;\lambda_2,k_2,\mathcal{J},N_*)\|_{\mathfrak{X}^{s_2-1,b_--1}([0,T])} \\
&\qquad + \|\widetilde{\mathcal{G}}^{(1)}(\cdot;\lambda_2,k_2,\mathcal{J},N_*)\|_{\mathfrak{X}^{s_2-1,b_--1}([0,T])}\Big).
\end{aligned}
$$

We now use Gaussian hypercontractivity and a reduction similar to the proof of Proposition 7.8 to move the supremum outside the probabilistic moments. Then, it remains to show for all frequency scales N_1, N_2, and N_3 satisfying $\max(N_1, N_2) > N_3^{\epsilon}$ that

$$
\begin{aligned}
&\sup_{\lambda_2\in\mathbb{R}} \sup_{\substack{k_2\in\mathbb{Z}^3:\\ |k_2|\lesssim N_2^{\epsilon}}} \Big\| \|\mathcal{G}^{(3)}(\cdot;\lambda_2,k_2,\mathcal{J},N_*)\|_{\mathfrak{X}^{s_2-1,b_--1}([0,T])} \\
&\qquad + \|\mathcal{G}^{(1)}(\cdot;\lambda_2,k_2,\mathcal{J},N_*)\|_{\mathfrak{X}^{s_2-1,b_--1}([0,T])} \\
&\qquad + \|\widetilde{\mathcal{G}}^{(1)}(\cdot;\lambda_2,k_2,\mathcal{J},N_*)\|_{\mathfrak{X}^{s_2-1,b_--1}([0,T])}\Big\|_{L^2_\omega} \\
&\qquad \lesssim T^2 \max(N_1,N_2,N_3)^{-\eta}.
\end{aligned}
$$

We treat the estimates for the nonresonant and resonant components separately.

Contribution of the nonresonant terms: To estimate the $\mathfrak{X}^{s_2-1,b_--1}$-norm, we calculate the space-time Fourier transform of $\chi(t/T)\mathcal{G}^{(3)}(t,x;\lambda_2,k_2,\mathcal{J},N_*)$. We have

$$\begin{aligned}
&\mathcal{F}_{t,x}\big(\chi(t/T)\mathcal{G}^{(3)}(t,x;\lambda_2,k_2,\mathcal{J},N_*)\big)(\lambda\mp\langle n\rangle,n) = \sum_{\pm_1,\pm_2,\pm_3} c(\pm_j\colon 1\leq j\leq 3)\\
&\quad\times \sum_{n_1,n_2,n_3\in\mathbb{Z}^3}\Bigg[1\{n=n_{123}+k_2\}\rho_N(n_2+k_2)^2\Big(\prod_{j=1}^{3}\rho_N(n_j)\chi_{N_j}(n_j)\Big)\\
&\quad\times \chi_{N_2'}(n_2+k_2)\widehat{V}(n_{12}+k_2)\mathcal{I}_3[\pm_j,n_j\colon 1\leq j\leq 3]\\
&\quad\times \mathcal{F}_t\Bigg(\exp(\pm_1 it\langle n_1\rangle \pm_3 it\langle n_3\rangle)\\
&\quad\times \int_0^t 1_{\mathcal{J}}(t')\frac{\sin((t-t')\langle n_2+k_2\rangle)}{\langle n_2+k_2\rangle}\exp(it'\lambda_2 \pm_2 it'\langle n_2\rangle)\,\mathrm{d}t'\Bigg)(\lambda\mp\langle n\rangle)\Bigg].
\end{aligned}$$

Using the orthogonality of the multiple stochastic integrals and Lemma 4.12 to estimate the Fourier transform of the time integral, we obtain

$$\begin{aligned}
&\big\|\|\mathcal{G}^{(3)}\|_{\mathfrak{X}^{s_2-1,b_--1}([0,T])}\big\|_{L^2_\omega}^2\\
&\lesssim \max_{\pm}\Big\|\langle\lambda\rangle^{b_--1}\langle n\rangle^{s_2-1}\\
&\quad\times \big\|\mathcal{F}_{t,x}\big(\chi(t/T)\mathcal{G}^{(3)}(t,x;\lambda_2,k_2,\mathcal{J},N_*)\big)(\lambda\mp\langle n\rangle,n)\big\|_{L^2_\lambda\ell^2_n(\mathbb{R}\times\mathbb{Z}^3)}\Big\|_{L^2_\omega}^2\\
&\lesssim T^4 \max_{\pm,\pm_1,\pm_2,\pm_3}\max_{\iota_2=-1,0,1}\int_{\mathbb{R}}\mathrm{d}\lambda\,\langle\lambda\rangle^{2(b_--1)}\\
&\quad\times \sum_{n_1,n_2,n_3\in\mathbb{Z}^3}\Big[\Big(\prod_{j=1}^{3}\chi_{N_j}(n_j)\Big)\langle n_{123}+k_2\rangle^{2(s_2-1)}\langle n_{12}+k_2\rangle^{-2\beta}\langle n_1\rangle^{-2}\langle n_2\rangle^{-4}\langle n_3\rangle^{-2}\\
&\quad\times\big(1+\big|\lambda-\lambda_3-(\pm\langle n_{123}+k_2\rangle \pm_1\langle n_1\rangle \pm_2\langle n_2\rangle+\iota_2\langle n_2+k_2\rangle\pm_3\langle n_3\rangle\big|\big)^{-2}\Big]\\
&\lesssim T^4 N_2^{-1+5\epsilon}\max_{\pm,\pm_1,\pm_3}\sup_{\substack{n_2\in\mathbb{Z}^3:\\ |n_2|\sim N_2}}\sup_{m\in\mathbb{Z}^3}\sum_{n_1,n_3\in\mathbb{Z}^3}\Big[\Big(\prod_{j=1,3}\chi_{N_j}(n_j)\Big)\langle n_{123}\rangle^{2(s_2-1)}\langle n_{12}\rangle^{-2\beta}\\
&\qquad\qquad\times\langle n_1\rangle^{-2}\langle n_3\rangle^{-2}1\{\pm\langle n_{123}\rangle\pm_1\langle n_1\rangle\pm_3\langle n_3\rangle\in[m,m+1)\}\Big]\\
&\lesssim T^4\max(N_1,N_2,N_3)^{2\delta_2}N_1^{-2\epsilon}N_2^{-1+7\epsilon}.
\end{aligned}$$

In the last line, we have used Lemma 4.23 with $\gamma=\epsilon$. Since $\max(N_1,N_2)>N_3^\epsilon$ and δ_2 is much smaller than ϵ^2, this contribution is acceptable.

Contribution of the resonant terms: We only estimate $\mathcal{G}^{(1)}$. Due to the factor $\widehat{V}(n_{12}+k_2)$, a simpler but similar argument also controls $\widetilde{\mathcal{G}}^{(1)}$.

Using the inhomogeneous Strichartz estimate (Lemma 4.9), we have

$$\begin{aligned}
\|\mathcal{G}^{(1)}\|_{\mathfrak{X}^{s_2-1,b_--1}([0,T])} &\lesssim \|\mathcal{G}^{(1)}\|_{L_t^{2b_+}H_x^{s_2-1}([0,T]\times\mathbb{T}^3)}\\
&\lesssim T^{1/2}\|\mathcal{G}^{(1)}\|_{L_t^2H_x^{s_2-1}([0,T]\times\mathbb{T}^3)}.
\end{aligned}$$

Using Fubini's theorem and the sine-cancellation lemma (Lemma 4.14) yields

$$\begin{aligned}
&\big\| \|\mathscr{G}^{(1)}\|_{\mathfrak{X}^{s_2-1,b_--1}([0,T])} \big\|_{L^2_\omega}^2 \lesssim T^2 \sup_{t\in[0,T]} \big\| \|\mathscr{G}^{(1)}\|_{H_x^{s_2-1}(\mathbb{T}^3)} \big\|_{L^2_\omega}^2 \\
&\lesssim T^2 \sum_{n_3\in\mathbb{Z}^3} \chi_{N_3}(n_3)\langle n_3+k_2\rangle^{2(s_2-1)}\langle n_3\rangle^{-2} \\
&\quad\times \Big| \sum_{n_2\in\mathbb{Z}^3} \rho_N(n_2+k_2)^2\rho_N(n_2)^2\chi_{N_1}(n_2)\chi_{N_2}(n_2)\chi_{N_2'}(n_2+k_2) \\
&\quad\times \langle n_2+k_2\rangle^{-1}\langle n_2\rangle^{-2}\Big(\int_0^t 1_{\mathcal{J}}(t')\sin((t-t')\langle n_2+k_2\rangle)\cos((t-t')\langle n_2\rangle)\exp(it'\lambda_2)\,\mathrm{d}t'\Big)\Big|^2 \\
&\lesssim T^4\, 1\{N_1\sim N_2\} N_1^{-2+6\epsilon}\langle k_2\rangle^{2(1-s_2)} \sum_{n_3\in\mathbb{Z}^3}\chi_{N_3}(n_3)\langle n_3\rangle^{2(s_2-1)}\langle n_3\rangle^{-2} \\
&\lesssim T^4\, 1\{N_1\sim N_2\} N_1^{-2+8\epsilon} N_3^{2\delta_2}.
\end{aligned}$$

Since $\max(N_1, N_2) \gtrsim N_3^\epsilon$ and δ_2 is much smaller than ϵ, this contribution is acceptable. ∎

8. Physical-space methods

In this section, we estimate the terms in **Phy**. The main ingredients are paraproduct decompositions and Strichartz estimates. In Section 8.1, we recall the refined Strichartz estimates for the wave equation by Klainerman and Tataru [45]. In Section 8.2, we use the Klainerman–Tataru–Strichartz estimate to control several terms in **Phy**. The remaining terms in **Phy** are estimated in Section 8.3, which also requires estimates on the quartic stochastic object from Section 5.2.

8.1. Klainerman–Tataru–Strichartz estimates

We first recall the refined (linear) Strichartz estimate from [45, (A.59)].

Lemma 8.1 (Klainerman–Tataru–Strichartz estimates). *Let $\mathcal{J}$ be a compact interval. Let Q be a box of sidelength $\sim M$ at a distance $\sim N$ from the origin. Let P_Q be the corresponding Fourier truncation operator and let $2 \le p, q < \infty$ satisfy the sharp wave-admissibility condition $1/q + 1/p = 1/2$. Then*

$$\|P_Q u\|_{L_t^q L_x^p(\mathcal{J}\times\mathbb{T}^3)} \lesssim (1+|\mathcal{J}|)^{1/q}\Big(\frac{M}{N}\Big)^{1/2-1/p} N^{3/2-1/q-3/p}\|P_Q u\|_{\mathfrak{X}^{0,b}(\mathcal{J})}. \tag{8.1}$$

Remark 8.2. The factor $N^{3/2-1/q-3/p}$ is the same as in the standard deterministic Strichartz estimate. The gain from the stronger localization in frequency space is described by the factor $(M/N)^{1/2-1/p}$. Naturally, there is no gain when $p = 2$.

We emphasize that (8.1) has a more complicated dependence on M and N than the corresponding result for the Schrödinger equation. In the Schrödinger setting, the frequency-localized Strichartz estimates for the operator P_Q and the standard Littlewood–Paley operators $P_{\leq M}$ are equivalent, which follows from the Galilean symmetry. This difference between the Schrödinger and wave equation already played a role in our counting estimates (Section 4.4).

Corollary 8.3. *Let $\mathcal{J}$ be a compact interval. Let Q be a box of sidelength $\sim M$ at a distance $\sim N$ from the origin. Let P_Q be the corresponding Fourier truncation operator and let $q \geq 4$. Then*

$$\|P_Q u\|_{L_t^q L_x^q(\mathcal{J}\times\mathbb{T}^3)} \lesssim (1+|\mathcal{J}|)^{1/q} M^{3/2-5/q} N^{1/q} \|P_Q u\|_{\mathfrak{X}^{0,b}(\mathcal{J})}. \tag{8.2}$$

Proof. This follows by combining Lemma 8.1 (with $q = p = 4$) and the Bernstein inequality

$$\|P_Q u\|_{L_t^\infty L_x^\infty(\mathcal{J}\times\mathbb{T}^3)} \lesssim M^{3/2} \|P_Q u\|_{\mathfrak{X}^{0,b}(\mathcal{J})}.$$ ■

We now state a bilinear version of the Klainerman–Tataru–Strichartz estimate, which is a consequence of Lemma 8.1 (cf. [45, Theorems 4 and 5]). However, since we only require a special case, we provide a self-contained proof.

Lemma 8.4 (Bilinear Klainerman–Tataru–Strichartz estimate). *Let $T \geq 1$, $q \geq 4$, let $\gamma < 3 - 10/q$ and let $N_1, N_2 \geq 1$. Then*

$$\|\langle\nabla\rangle^{-\gamma}(P_{N_1} f \cdot P_{N_2} g)\|_{L_t^{q/2} L_x^{q/2}([0,T]\times\mathbb{T}^3)} \lesssim T^{2/q} \max(N_1,N_2)^{3-2s_1-8/q-\gamma} \|f\|_{\mathfrak{X}^{s_1,b}([0,T])} \|g\|_{\mathfrak{X}^{s_1,b}([0,T])}.$$

In particular,

$$\sum_{N_1,N_2} \|P_{N_1} f \cdot P_{N_2} g\|_{L_t^2 H_x^{-4\delta_1}([0,T]\times\mathbb{T}^3)} \lesssim T^{1/2} \|f\|_{\mathfrak{X}^{s_1,b}([0,T])} \|g\|_{\mathfrak{X}^{s_1,b}([0,T])}.$$

Furthermore, if $N_{12} \geq 1$, then

$$\|(P_{N_{12}} V) * (P_{N_1} f \cdot P_{N_2} g)\|_{L_t^2 L_x^2([0,T]\times\mathbb{T}^3)} \lesssim T^{1/2} N_{12}^{1/2-\beta-2\delta_1} \max(N_1,N_2)^{-1/2+4\delta_1} \|f\|_{\mathfrak{X}^{s_1,b}([0,T])} \|g\|_{\mathfrak{X}^{s_1,b}([0,T])}.$$

Remark 8.5. Bilinear Strichartz estimates are also important in the random data theory for nonlinear Schrödinger equations in [2, 3]. In the proof of Proposition 8.10 below, we will only require the case $q = 4+$ and the reader may simply think of q as 4.

Proof of Lemma 8.4. We begin with the first estimate, which is the main part of the argument. Using the definition of the restricted $\mathfrak{X}^{s,b}$-spaces, we may replace $\|f\|_{\mathfrak{X}^{s_1,b}([0,T])}$ and $\|g\|_{\mathfrak{X}^{s_1,b}([0,T])}$ by $\|f\|_{\mathfrak{X}^{s_1,b}(\mathbb{R})}$ and $\|g\|_{\mathfrak{X}^{s_1,b}(\mathbb{R})}$, respectively. The proof relies on the linear Klainerman–Tataru–Strichartz estimate (Corollary 8.3) and box localization. We decompose

$$\|\langle\nabla\rangle^{-\gamma}(P_{N_1} f \cdot P_{N_2} g)\|_{L_t^{q/2} L_x^{q/2}([0,T]\times\mathbb{T}^3)} \lesssim \sum_{\substack{N_{12}:\\ N_{12}\lesssim\max(N_1,N_2)}} N_{12}^{-\gamma} \|P_{N_{12}}(P_{N_1} f \cdot P_{N_2} g)\|_{L_t^{q/2} L_x^{q/2}([0,T]\times\mathbb{T}^3)}.$$

If $N_1 \not\sim N_2$, then $N_{12} \sim \max(N_1, N_2)$ and the desired estimate follows from Hölder's inequality and the $L_t^q L_x^q$-estimate from Corollary 8.3 with $M \sim N$. Thus, it remains to treat the case $N_1 \sim N_2$. Let $\mathcal{Q} = \mathcal{Q}(N_1, N_{12})$ be a cover of the dyadic annulus at distance $\sim N_1$ by finitely overlapping cubes of diameter $\sim N_{12}$. From Fourier support considerations and Lemma 8.1, it follows that

$$
\begin{aligned}
&\| P_{N_{12}}(P_{N_1} f \cdot P_{N_2} g)\|_{L_t^{q/2} L_x^{q/2}([0,T]\times\mathbb{T}^3)} \\
&\quad\lesssim \sum_{\substack{Q_1,Q_2\in\mathcal{Q}:\\ d(Q_1,Q_2)\lesssim N_{12}}} \| P_{Q_1} P_{N_1} f \cdot P_{Q_2} P_{N_2} g\|_{L_t^{q/2} L_x^{q/2}([0,T]\times\mathbb{T}^3)} \\
&\quad\lesssim \sum_{\substack{Q_1,Q_2\in\mathcal{Q}:\\ d(Q_1,Q_2)\lesssim N_{12}}} \| P_{Q_1} P_{N_1} f \|_{L_t^{q} L_x^{q}([0,T]\times\mathbb{T}^3)} \| P_{Q_2} P_{N_2} g\|_{L_t^{q} L_x^{q}([0,T]\times\mathbb{T}^3)} \\
&\quad\lesssim T^{2/q} N_{12}^{3-10/q} N_1^{2/q-2s_1} \sum_{\substack{Q_1,Q_2\in\mathcal{Q}:\\ d(Q_1,Q_2)\lesssim N_{12}}} \| P_{Q_1} P_{N_1} f \|_{\mathfrak{X}^{s_1,b}(\mathbb{R})} \| P_{Q_2} P_{N_2} g\|_{\mathfrak{X}^{s_1,b}(\mathbb{R})} \\
&\quad\lesssim T^{2/q} N_{12}^{3-10/q} N_1^{2/q-2s_1} \Big(\sum_{\substack{Q_1,Q_2\in\mathcal{Q}:\\ d(Q_1,Q_2)\lesssim N_{12}}} \| P_{Q_1} P_{N_1} f \|_{\mathfrak{X}^{s_1,b}(\mathbb{R})}^2 \Big)^{1/2} \\
&\qquad \times \Big(\sum_{\substack{Q_1,Q_2\in\mathcal{Q}:\\ d(Q_1,Q_2)\lesssim N_{12}}} \| P_{Q_2} P_{N_2} g \|_{\mathfrak{X}^{s_1,b}(\mathbb{R})}^2 \Big)^{1/2} \\
&\quad\lesssim T^{2/q} N_{12}^{3-10/q} N_1^{2/q-2s_1} \| f\|_{\mathfrak{X}^{s_1,b}(\mathbb{R})} \| g\|_{\mathfrak{X}^{s_1,b}(\mathbb{R})}.
\end{aligned}
$$

The desired result then follows by using the upper bound $\gamma < 3 - 10/q$ and summing over N_{12}.

We now turn to the second estimate. After estimating

$$
\begin{aligned}
&\| (P_{N_{12}} V) * (P_{N_1} f \cdot P_{N_2} g)\|_{L_t^2 L_x^2([0,T]\times\mathbb{T}^3)} \\
&\qquad\qquad \lesssim N_{12}^{1/2-\beta-2\delta_1} \| \langle \nabla\rangle^{-1/2+2\delta_1} (P_{N_1} f \cdot P_{N_2} g)\|_{L_t^2 L_x^2([0,T]\times\mathbb{T}^3)},
\end{aligned}
$$

the result follows from the first estimate. ∎

8.2. *Physical terms*

In this subsection, we use the Klainerman–Tataru–Strichartz estimate and a paraproduct decomposition to control several terms in **Phy**.

Proposition 8.6. *Let $\mathcal{J}$ be a bounded interval and let $f, g \in \mathfrak{X}^{s_1,b}(\mathcal{J})$. Then*

$$
\begin{aligned}
&\sup_{N\geq 1} \| V * (P_{\leq N} f \cdot P_{\leq N} g) \, (\neg \textcircled{$<$}) \, P_{\leq N} \dot{\mid} \|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})} \\
&\qquad\qquad \lesssim (1+|\mathcal{J}|)^2 \| f\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} \| g\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} \| \dot{\mid} \|_{L_t^\infty \mathcal{C}_x^{-1/2-\kappa}(\mathcal{J}\times\mathbb{T}^3)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\sup_{N\geq 1} \Big\| \big(\neg \boxed{\textcircled{$<$} \,\&\, \textcircled{$<$}}\big) \big(V * (P_{\leq N} f \cdot P_{\leq N} g) \; P_{\leq N} \dot{\mid} \big) \Big\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})} \\
&\qquad\qquad \lesssim (1+|\mathcal{J}|)^2 \| f\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} \| g\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} \| \dot{\mid} \|_{L_t^\infty \mathcal{C}_x^{-1/2-\kappa}(\mathcal{J}\times\mathbb{T}^3)}.
\end{aligned}
$$

In the frequency-localized versions of the two estimates, which are detailed in the proof, we gain an η'-power in the maximal frequency scale.

Proof. After using a Littlewood–Paley decomposition, we obtain

$$
\begin{aligned}
&\| V * (P_{\leq N} f \cdot P_{\leq N} g)\, (\neg \textcircled{\scriptsize<})\, P_{\leq N} \overset{\bullet}{|} \|_{\mathcal{X}^{s_2-1,b_+-1}(\mathcal{J})} \\
&\qquad\qquad + \big\| \big(\neg\, \boxed{\textcircled{\scriptsize<} \,\&\, \textcircled{\scriptsize<}}\big) \big(V * (P_{\leq N} f \cdot P_{\leq N} g) P_{\leq N} \overset{\bullet}{|} \big) \big\|_{\mathcal{X}^{s_2-1,b_+-1}(\mathcal{J})} \\
&\lesssim \sum_{\substack{N_1,N_2,N_3,N_{12}: \\ \max(N_1,N_2) \gtrsim N_3^{\epsilon}}} \| (P_{N_{12}} V) * (P_{\leq N} P_{N_1} f \cdot P_{\leq N} P_{N_2} g) P_{\leq N} P_{N_3} \overset{\bullet}{|} \|_{\mathcal{X}^{s_2-1,b_+-1}(\mathcal{J})},
\end{aligned}
$$

where we have also used the fact that $N_{12} \lesssim \max(N_1, N_2)$. We estimate each dyadic piece separately and distinguish two cases:

Case 1: $N_{12} \nsim N_3$. Using the inhomogeneous Strichartz estimate (Lemma 4.9) and Lemma 8.4, we obtain

$$
\begin{aligned}
&\| (P_{N_{12}} V) * (P_{\leq N} P_{N_1} f \cdot P_{\leq N} P_{N_2} g) P_{\leq N} P_{N_3} \overset{\bullet}{|} \|_{\mathcal{X}^{s_2-1,b_+-1}(\mathcal{J})} \\
&\quad \lesssim \| (P_{N_{12}} V) * (P_{\leq N} P_{N_1} f \cdot P_{\leq N} P_{N_2} g) P_{\leq N} P_{N_3} \overset{\bullet}{|} \|_{L_t^{2b_+} H_x^{s_2-1}(\mathcal{J} \times \mathbb{T}^3)} \\
&\quad \lesssim (1 + |\mathcal{J}|)^{1/2} \max(N_{12}, N_3)^{s_2-1} \| (P_{N_{12}} V) * (P_{\leq N} P_{N_1} f \cdot P_{\leq N} P_{N_2} g) \|_{L_t^2 L_x^2(\mathcal{J} \times \mathbb{T}^3)} \\
&\qquad \times \| P_{\leq N} P_{N_3} \overset{\bullet}{|} \|_{L_t^\infty L_x^\infty(\mathcal{J} \times \mathbb{T}^3)} \\
&\quad \lesssim (1 + |\mathcal{J}|) \max(N_{12}, N_3)^{s_2-1} N_{12}^{1/2-\beta+2\delta_1} \max(N_1, N_2)^{-1/2+4\delta_1} N_3^{1/2+\kappa} \\
&\qquad \times \| f \|_{\mathcal{X}^{s_1,b}(\mathcal{J})} \| g \|_{\mathcal{X}^{s_1,b}(\mathcal{J})} \| \overset{\bullet}{|} \|_{L_t^\infty \mathcal{C}_x^{-1/2-\kappa}(\mathcal{J} \times \mathbb{T}^3)}.
\end{aligned}
$$

Since $\max(N_1, N_2) \geq N_3^{\epsilon}$, we can bound the pre-factor by

$$
\begin{aligned}
&\max(N_{12}, N_3)^{s_2-1} N_{12}^{1/2-\beta+2\delta_1} \max(N_1, N_2)^{-1/2+4\delta_1} N_3^{1/2+\kappa} \\
&\qquad\qquad \lesssim \max(N_1, N_2)^{-\beta+6\delta_1} N_3^{\delta_2+\kappa} \lesssim \max(N_1, N_2, N_3)^{-2\eta}.
\end{aligned}
$$

Case 2: $N_{12} \sim N_3$. By symmetry, we can assume that $N_1 \geq N_2$. Furthermore, we have $N_3 \sim N_{12} \lesssim N_1$. Using the inhomogeneous Strichartz estimate (Lemma 4.9), we obtain

$$
\begin{aligned}
&\| (P_{N_{12}} V) * (P_{\leq N} P_{N_1} f \cdot P_{\leq N} P_{N_2} g) P_{\leq N} P_{N_3} \overset{\bullet}{|} \|_{\mathcal{X}^{s_2-1,b_+-1}(\mathcal{J})} \\
&\qquad \lesssim (1 + |\mathcal{J}|) \big\| \langle \nabla \rangle^{s_2-1/2+4(b_+-1/2)} \big((P_{N_{12}} V) * (P_{\leq N} P_{N_1} f \\
&\qquad\qquad\qquad \cdot P_{\leq N} P_{N_2} g) P_{\leq N} P_{N_3} \overset{\bullet}{|} \big) \big\|_{L_t^{4/3} L_x^{4/3}(\mathcal{J} \times \mathbb{T}^3)} \\
&\qquad \lesssim (1 + |\mathcal{J}|)^{3/2} N_3^{s_2-1/2+4(b_+-1/2)-\beta} \\
&\qquad\quad \times \| P_{N_1} f \|_{L_t^\infty L_x^2(\mathcal{J} \times \mathbb{T}^3)} \| P_{N_2} g \|_{L_t^4 L_x^4(\mathcal{J} \times \mathbb{T}^3)} \| P_{N_3} \overset{\bullet}{|} \|_{L_t^\infty L_x^\infty(\mathcal{J} \times \mathbb{T}^3)} \\
&\qquad \lesssim (1 + |\mathcal{J}|)^2 N_1^{-s_1} N_2^{1/2-s_1} N_3^{s_2-1/2+4(b_+-1/2)-\beta+1/2+\kappa} \\
&\qquad\quad \times \| f \|_{\mathcal{X}^{s_1,b}(\mathcal{J})} \| g \|_{\mathcal{X}^{s_1,b}(\mathcal{J})} \| \overset{\bullet}{|} \|_{L_t^\infty \mathcal{C}_x^{-1/2-\kappa}(\mathcal{J} \times \mathbb{T}^3)}.
\end{aligned}
$$

Since $N_2, N_3 \geq 1$, the pre-factor can be bounded by

$$\begin{aligned} N_1^{-s_1} N_2^{1/2-s_1} N_3^{s_2-1/2+4(b_+-1/2)-\beta+1/2+\kappa} &\lesssim N_1^{1-2s_1+s_2-1/2+4(b_+-1/2)-\beta+\kappa} \\ &= N_1^{2\delta_1+\delta_2+4(b_+-1/2)+\kappa-\beta}, \end{aligned}$$

which is acceptable. ∎

Proposition 8.7. *Let $T \geq 1$, let $\mathcal{J} \subseteq [0, T]$ be an interval, and let $f, g \colon \mathcal{J} \times \mathbb{T}^3 \to \mathbb{R}$. Then*

$$\begin{aligned} &\sup_{N \geq 1} \| V * (P_{\leq N} \overset{\bullet}{\mid} \textcircled{$\neq$} P_{\leq N} f) P_{\leq N} g \|_{\mathcal{X}^{s_2-1, b_+-1}(\mathcal{J})} \\ &\qquad \lesssim (1 + |\mathcal{J}|)^2 \| \overset{\bullet}{\mid} \|_{L_t^\infty \mathcal{C}^{-1/2-\kappa}(\mathcal{J} \times \mathbb{T}^3)} \| f \|_{\mathcal{X}^{s_1, b}(\mathcal{J})} \| g \|_{\mathcal{X}^{s_1, b}(\mathcal{J})}. \end{aligned}$$

In the frequency-localized version of this estimate, which is detailed in the proof, we gain an η'-power in the maximal frequency scale.

Proof. By using a Littlewood–Paley decomposition and the definitions of $\textcircled{$\neq$}$, we have

$$V * (P_{\leq N} \overset{\bullet}{\mid} \textcircled{$\neq$} P_{\leq N} f) P_{\leq N} g = \sum_{\substack{N_1, N_2, N_3: \\ N_1 \not\sim N_2}} V * (P_{\leq N} P_{N_1} \overset{\bullet}{\mid} \cdot P_{\leq N} P_{N_2} f) P_{\leq N} P_{N_3} g.$$

We treat each dyadic block separately and distinguish two cases.

Case 1: $N_1 \gg N_2, N_3$. Using the inhomogeneous Strichartz estimate (Lemma 4.9), we have

$$\begin{aligned} &\| V * (P_{\leq N} P_{N_1} \overset{\bullet}{\mid} \cdot P_{\leq N} P_{N_2} f) P_{\leq N} P_{N_3} g \|_{\mathcal{X}^{s_2-1, b_+-1}(\mathcal{J})} \\ &\lesssim \| V * (P_{\leq N} P_{N_1} \overset{\bullet}{\mid} \cdot P_{\leq N} P_{N_2} f) P_{\leq N} P_{N_3} g \|_{L_t^{2b_+} H_x^{s_2-1}(\mathcal{J} \times \mathbb{T}^3)} \\ &\lesssim (1+|\mathcal{J}|)^{1/2} N_1^{s_2-1-\beta} \| P_{N_1} \overset{\bullet}{\mid} \|_{L_t^\infty L_x^\infty(\mathcal{J} \times \mathbb{T}^3)} \| P_{N_1} f \|_{L_t^4 L_x^4(\mathcal{J} \times \mathbb{T}^3)} \| P_{N_2} g \|_{L_t^4 L_x^4(\mathcal{J} \times \mathbb{T}^3)} \\ &\lesssim (1+|\mathcal{J}|) N_1^{s_2-1-\beta+1/2+\kappa} N_2^{1/2-s_1} N_3^{1/2-s_1} \| \overset{\bullet}{\mid} \|_{L_t^\infty \mathcal{C}_x^{-1/2-\kappa}(\mathcal{J} \times \mathbb{T}^3)} \| f \|_{\mathcal{X}^{s_1, b}(\mathcal{J})} \| g \|_{\mathcal{X}^{s_1, b}(\mathcal{J})}. \end{aligned}$$

Since $N_2, N_3 \ll N_1$, the pre-factor can be bounded by

$$N_1^{s_2-1-\beta+1/2+\kappa} N_2^{1/2-s_1} N_3^{1/2-s_1} \lesssim N_1^{2\delta_1+\delta_2+\kappa-\beta},$$

which is acceptable.

Case 2.a: $N_1 \ll N_2$, $N_3 \lesssim N_2$. Using the inhomogeneous Strichartz estimate (Lemma 4.9), we have

$$\begin{aligned} &\| V * (P_{\leq N} P_{N_1} \overset{\bullet}{\mid} \cdot P_{\leq N} P_{N_2} f) P_{\leq N} P_{N_3} g \|_{\mathcal{X}^{s_2-1, b_+-1}(\mathcal{J})} \\ &\lesssim (1+|\mathcal{J}|) \| \langle \nabla \rangle^{s_2-1/2+4(b_+-1)} \big(V * (P_{\leq N} P_{N_1} \overset{\bullet}{\mid} \cdot P_{\leq N} P_{N_2} f) P_{\leq N} P_{N_3} g \big) \|_{L_t^{4/3} L_x^{4/3}(\mathcal{J} \times \mathbb{T}^3)} \\ &\lesssim (1+|\mathcal{J}|) N_2^{s_2-1/2+4(b_+-1)} \\ &\quad \times \| V * (P_{\leq N} P_{N_1} \overset{\bullet}{\mid} \cdot P_{\leq N} P_{N_2} f) \|_{L_t^2 L_x^2(\mathcal{J} \times \mathbb{T}^3)} \| P_{\leq N} P_{N_3} g \|_{L_t^4 L_x^4(\mathcal{J} \times \mathbb{T}^3)} \end{aligned}$$

$$
\begin{aligned}
&\lesssim (1+|\mathcal{J}|) N_2^{s_2-1/2+4(b_+-1)-\beta} \|\overset{\bullet}{|}\|_{L_t^\infty L_x^\infty(\mathcal{J}\times\mathbb{T}^3)} \|P_{N_2} f\|_{L_t^2 L_x^2(\mathcal{J}\times\mathbb{T}^3)} \|P_{N_3} g\|_{L_t^4 L_x^4(\mathcal{J}\times\mathbb{T}^3)} \\
&\lesssim (1+|\mathcal{J}|)^2 N_1^{1/2+\kappa} N_2^{s_2-1/2+4(b_+-1)-\beta-s_1} N_3^{1/2-s_1} \\
&\quad \times \|\overset{\bullet}{|}\|_{L_t^\infty \mathcal{C}_x^{-1/2-\kappa}(\mathcal{J}\times\mathbb{T}^3)} \|f\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} \|g\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})}.
\end{aligned}
$$

The pre-factor can now be bounded as before.

Case 2.b: $N_1 \ll N_2$, $N_2 \ll N_3$. Using the inhomogeneous Strichartz estimate (Lemma 4.9), we have

$$
\begin{aligned}
&\|V * (P_{\leq N} P_{N_1} \overset{\bullet}{|} \cdot P_{\leq N} P_{N_2} f) P_{\leq N} P_{N_3} g\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})} \\
&\lesssim (1+|\mathcal{J}|) \|\langle\nabla\rangle^{s_2-1/2+4(b_+-1)} \big(V * (P_{\leq N} P_{N_1} \overset{\bullet}{|} \cdot P_{\leq N} P_{N_2} f) P_{\leq N} P_{N_3} g\big)\|_{L_t^{4/3} L_x^{4/3}(\mathcal{J}\times\mathbb{T}^3)} \\
&\lesssim (1+|\mathcal{J}|) \max(N_1, N_2)^{-\beta} N_3^{s_2-1/2+4(b_+-1)} \\
&\quad \times \|\overset{\bullet}{|}\|_{L_t^\infty L_x^\infty(\mathcal{J}\times\mathbb{T}^3)} \|P_{N_2} f\|_{L_t^4 L_x^4(\mathcal{J}\times\mathbb{T}^3)} \|P_{N_3} g\|_{L_t^2 L_x^2(\mathcal{J}\times\mathbb{T}^3)} \\
&\lesssim (1+|\mathcal{J}|)^2 \max(N_1, N_2)^{-\beta} N_1^{1/2+\kappa} N_2^{1/2-s_1} \\
&\quad \times N_3^{s_2-1/2+4(b_+-1)-s_1} \|\overset{\bullet}{|}\|_{L_t^\infty \mathcal{C}_x^{-1/2-\kappa}(\mathcal{J}\times\mathbb{T}^3)} \|f\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} \|g\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})}.
\end{aligned}
$$

The pre-factor can now be bounded by

$$
\begin{aligned}
&\max(N_1, N_2)^{-\beta} N_1^{1/2+\kappa} N_2^{1/2-s_1} N_3^{s_2-1/2+4(b_+-1)-s_1} \\
&\qquad \lesssim N_1^{1/2+\kappa-\beta} N_2^{\delta_1} N_3^{-1/2+\delta_1+\delta_2+4(b_+-1)} \lesssim N_3^{2\delta_1+\delta_2+\kappa-\beta},
\end{aligned}
$$

which is acceptable. ∎

Lemma 8.8 (Bilinear physical estimate). *Let $\mathcal{J} \subseteq \mathbb{R}$ be a bounded interval. If $\Psi, f : \mathcal{J} \times \mathbb{T}^3 \to \mathbb{C}$, then*

$$
\begin{aligned}
&\|(V * \Psi) f\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J}\times\mathbb{T}^3)} \\
&\qquad \lesssim (1+|\mathcal{J}|)^{3/2} \|\Psi\|_{L_t^2 H_x^{-4\delta_1}(\mathcal{J}\times\mathbb{T}^3)} \min(\|f\|_{L_t^\infty \mathcal{C}_x^{\beta-\kappa}(\mathcal{J}\times\mathbb{T}^3)}, \|f\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})}).
\end{aligned}
$$

In the frequency-localized version of this estimate we also gain an η'-power in the maximal frequency scale.

Lemma 8.8 can be combined with our bound on $\overset{\diamond}{|} \circleddash w_N$ in the stability theory (see Section 3.3). In the local theory, its primary application is isolated in the following corollary.

Corollary 8.9. *Let $\mathcal{J} \subseteq \mathbb{R}$ be a bounded interval and let $w, Y : \mathcal{J} \times \mathbb{T}^3 \to \mathbb{R}$. Then, uniformly in $N \geq 1$,*

$$
\begin{aligned}
&\Big\| V * (P_{\leq N} \overset{\bullet}{|} \circleddash P_{\leq N} Y) P_{\leq N} \curlyvee_N \Big\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})} \\
&\qquad \lesssim (1+|\mathcal{J}|)^2 \|\overset{\bullet}{|}\|_{L_t^\infty \mathcal{C}_x^{-1/2+\kappa}(\mathcal{J}\times\mathbb{T}^3)} \|Y\|_{\mathfrak{X}^{s_2,b}(\mathcal{J})} \Big\| \curlyvee_N \Big\|_{L_t^\infty \mathcal{C}_x^{\beta-\kappa}(\mathcal{J}\times\mathbb{T}^3)},
\end{aligned}
$$

$$\begin{aligned}&\|V * (P_{\leq N}\stackrel{\bullet}{|} \textcircled{=} P_{\leq N} Y) P_{\leq N} w\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})}\\ &\qquad\qquad \lesssim (1+|\mathcal{J}|)^2 \|\stackrel{\bullet}{|}\|_{L_t^\infty \mathcal{C}_x^{-1/2+\kappa}(\mathcal{J}\times\mathbb{T}^3)} \|Y\|_{\mathfrak{X}^{s_2,b}(\mathcal{J})} \|w\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})}.\end{aligned}$$

Proof. We have

$$\begin{aligned}\|P_{\leq N}\stackrel{\bullet}{|} \textcircled{=} P_{\leq N} Y\|_{L_t^2 H_x^{-4\delta_1}(\mathcal{J}\times\mathbb{T}^3)} &\lesssim |\mathcal{J}|^{1/2} \|\stackrel{\bullet}{|}\|_{L_t^\infty \mathcal{C}_x^{-1/2-\kappa}(\mathcal{J}\times\mathbb{T}^3)} \|Y\|_{L_t^\infty H_x^{s_2}(\mathcal{J}\times\mathbb{T}^3)}\\ &\lesssim |\mathcal{J}|^{1/2} \|\stackrel{\bullet}{|}\|_{L_t^\infty \mathcal{C}_x^{-1/2-\kappa}(\mathcal{J}\times\mathbb{T}^3)} \|Y\|_{\mathfrak{X}^{s_2,b}(\mathcal{J})}.\end{aligned}$$

Together with Lemma 8.9, this implies the corollary. ∎

Proof of Lemma 8.8. Let $0 \leq \theta \ll \beta$ remain to be chosen. Using the inhomogeneous Strichartz estimate and (a weaker version of) the fractional product rule, we have

$$\begin{aligned}&\|(V * \Psi) f\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J}\times\mathbb{T}^3)}\\ &\qquad \lesssim (1+|\mathcal{J}|) \|\langle\nabla\rangle^{s_2-1/2+4(b_+-1/2)}((V*\Psi)f)\|_{L_t^{4/3} L_x^{4/3}(\mathcal{J}\times\mathbb{T}^3)}\\ &\qquad \lesssim (1+|\mathcal{J}|) \|\langle\nabla\rangle^{s_2-1/2+4(b_+-1/2)}(V*\Psi)\|_{L_t^2 L_x^{4/2-\theta}(\mathcal{J}\times\mathbb{T}^3)}\\ &\qquad\quad \times \|\langle\nabla\rangle^{s_2-1/2+4(b_+-1/2)} f\|_{L_t^4 L_x^{\frac{4}{1+\theta}}(\mathcal{J}\times\mathbb{T}^3)}.\end{aligned}$$

Using Sobolev embedding, the first factor is bounded by

$$\begin{aligned}&\|\langle\nabla\rangle^{s_2-1/2+4(b_+-1/2)}(V*\Psi)\|_{L_t^2 L_x^{4/2-\theta}(\mathcal{J}\times\mathbb{T}^3)}\\ &\qquad \lesssim \|\langle\nabla\rangle^{s_2-1/2+4(b_+-1/2)+3\theta/4-\beta}\Psi\|_{L_t^2 L_x^2(\mathcal{J}\times\mathbb{T}^3)} \lesssim \|\Psi\|_{L_t^2 H_x^{-4\delta_1}(\mathcal{J}\times\mathbb{T}^3)}.\end{aligned}$$

Thus, it remains to present two different estimates of the second factor. By simply choosing $\theta = 0$, we see that

$$\|\langle\nabla\rangle^{s_2-1/2+4(b_+-1/2)} f\|_{L_t^4 L_x^4(\mathcal{J}\times\mathbb{T}^3)} \lesssim (1+|\mathcal{J}|)^{1/4} \|f\|_{L_t^\infty \mathcal{C}_x^{\beta-\kappa}(\mathcal{J}\times\mathbb{T}^3)},$$

which yields the first term in the minimum. Using Hölder's inequality in time and Strichartz estimates, we also have

$$\begin{aligned}&\|\langle\nabla\rangle^{s_2-1/2+4(b_+-1/2)} f\|_{L_t^4 L_x^{4/1+\theta}(\mathcal{J}\times\mathbb{T}^3)}\\ &\qquad \lesssim (1+|\mathcal{J}|)^{\theta/4} \|\langle\nabla\rangle^{s_2-1/2+4(b_+-1/2)} f\|_{L_t^{4/1-\theta} L_x^{4/1+\theta}(\mathcal{J}\times\mathbb{T}^3)}\\ &\qquad \lesssim (1+|\mathcal{J}|)^{1/4} \|f\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})}\end{aligned}$$

provided that

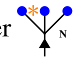
$$s_2 - \frac{1}{2} + 4\left(b_+ - \frac{1}{2}\right) + \frac{3}{2} - 1 - \theta/4 - 31 + \theta/4 \leq s_1.$$

The last condition can be satisfied by choosing $\theta = 4\delta_1$, which also satisfies $\theta \ll \beta$. ∎

Proposition 8.10. *Let $\mathcal{J} \subseteq \mathbb{R}$ be a bounded interval and let $f, g, h\colon J \times \mathbb{T}^3$. Then*

$$\sup_{N\geq 1} \|V * (P_{\leq N} f \cdot P_{\leq N} g) P_{\leq N} h\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})} \lesssim (1+|\mathcal{J}|)^2 \prod_{\varphi=f,g,h} \min(\|\varphi\|_{L_t^\infty \mathcal{C}_x^{\beta-\kappa}(\mathcal{J}\times\mathbb{T}^3)}, \|\varphi\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})}). \tag{8.3}$$

In the frequency-localized version of this estimate we also gain an η'-power in the maximal frequency scale.

Remark 8.11. In applications of Lemma 8.10, we will choose f, g, and h as either , which is contained in $L_t^\infty \mathcal{C}_x^{\beta-\kappa}$, or w_N, which is contained in $\mathfrak{X}^{s_1,b}$.

Proof of Proposition 8.10. Since the proof is relatively standard, we only present the argument when all functions f, g, and h are in the same space. The intermediate cases follow from a combination of our arguments below.

Estimate for $L_t^\infty \mathcal{C}_x^{\beta-\kappa}$: Using the inhomogeneous Strichartz estimate (Lemma 4.9) and $s_2 \leq 1$, we have

$$\begin{aligned}
&\|V * (P_{\leq N} f \cdot P_{\leq N} g) P_{\leq N} h\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})} \\
&\quad\lesssim \|V * (P_{\leq N} f \cdot P_{\leq N} g) P_{\leq N} h\|_{L_t^{2b_+} L_x^2(\mathcal{J}\times\mathbb{T}^3)} \\
&\quad\lesssim (1+|\mathcal{J}|) \prod_{\varphi=f,g,h} \|\varphi\|_{L_t^\infty L_x^\infty(\mathcal{J}\times\mathbb{T}^3)} \lesssim (1+|\mathcal{J}|) \prod_{\varphi=f,g,h} \|\varphi\|_{L_t^\infty \mathcal{C}_x^{\beta-\kappa}(\mathcal{J}\times\mathbb{T}^3)}.
\end{aligned}$$

Estimate for $\mathfrak{X}^{s_1,b}(\mathcal{J})$: Let $0 < \theta \ll 1$ remain to be chosen. Using the inhomogeneous Strichartz estimate (Lemma 4.9), we have

$$\begin{aligned}
&\|V * (P_{\leq N} f \cdot P_{\leq N} g) P_{\leq N} h\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})} \\
&\quad\lesssim (1+|\mathcal{J}|) \big\| \langle\nabla\rangle^{s_2-1/2+4(b_+-1/2)} \big(V * (P_{\leq N} f \cdot P_{\leq N} g) P_{\leq N} h\big) \big\|_{L_t^{4/3} L_x^{4/3}(\mathcal{J}\times\mathbb{T}^3)} \\
&\quad\lesssim (1+|\mathcal{J}|) \big\| \langle\nabla\rangle^{s_2-1/2+4(b_+-1/2)} \big(V * (P_{\leq N} f \cdot P_{\leq N} g)\big) \big\|_{L_t^{4/2-\theta} L_x^{4/2-\theta}(\mathcal{J}\times\mathbb{T}^3)} \\
&\qquad\times \|\langle\nabla\rangle^{s_2-1/2+4(b_+-1/2)} h\|_{L_t^{\frac{4}{1+\theta}} L_x^{\frac{4}{1+\theta}}(\mathcal{J}\times\mathbb{T}^3)}.
\end{aligned}$$

Using Lemma 8.4, the first term is bounded by $(1+|\mathcal{J}|)^{2-\theta/4} \|f\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} \|g\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})}$ as long as

$$2\delta_1 + \delta_2 + 4(b_+ - 1/2) + \theta < \beta. \tag{8.4}$$

Using Hölder's inequality in the time variable and the linear Strichartz estimate, we have

$$\begin{aligned}
&\|\langle\nabla\rangle^{s_2-1/2+4(b_+-1/2)} h\|_{L_t^{4/1+\theta} L_x^{4/1+\theta}(\mathcal{J}\times\mathbb{T}^3)} \\
&\quad\lesssim (1+|\mathcal{J}|)^{\theta/2} \|\langle\nabla\rangle^{s_2-1/2+4(b_+-1/2)} h\|_{L_t^{\frac{4}{1-\theta}} L_x^{\frac{4}{1+\theta}}(\mathcal{J}\times\mathbb{T}^3)} \\
&\quad\lesssim (1+|\mathcal{J}|)^{\frac{1+\theta}{4}} \|h\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})}
\end{aligned}$$

provided that

$$\theta/2 > \delta_1 + \delta_2 + 4(b_+ - 1/2). \tag{8.5}$$

In order to satisfy both conditions (8.4) and (8.5), we can choose $\theta = 4\delta_1$. ∎

8.3. Hybrid physical-RMT terms

In this subsection, we estimate the remaining terms in **Phy**. Our estimates will be phrased as bounds on the operator norm of certain random operators. In contrast to Propositions 6.1 and 6.3, however, we will not need the moment method (from [30]). Instead, we will rely on Strichartz estimates and the estimates for the quartic stochastic object from Section 5.2.

Proposition 8.12. *Let $T, p \geq 1$. Then*

$$\Big\| \sup_{N\geq 1} \sup_{\mathcal{J}\subseteq[0,T]} \sup_{\|w\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})}\leq 1} \Big\| V * \Big(P_{\leq N} \uparrow \cdot P_{\leq N} \Psi_N \Big) P_{\leq N} w \Big\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})} \Big\|_{L^p_\omega(\mathbb{P})} \lesssim T^3 p^2, \tag{8.6}$$

$$\Big\| \sup_{N\geq 1} \sup_{\mathcal{J}\subseteq[0,T]} \sup_{\|w\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})}\leq 1} \Big\| V * (P_{\leq N} \uparrow \circledneq P_{\leq N} w) P_{\leq N} \Psi_N \Big\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})} \Big\|_{L^p_\omega(\mathbb{P})} \lesssim T^3 p^2, \tag{8.7}$$

$$\Big\| \sup_{N\geq 1} \sup_{\mathcal{J}\subseteq[0,T]} \sup_{\|w\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})}\leq 1} \Big\| V * \Big(P_{\leq N} \Psi_N \cdot P_{\leq N} w \Big) (\neg \circledless)\, P_{\leq N} \uparrow \Big\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})} \Big\|_{L^p_\omega(\mathbb{P})} \lesssim T p^2. \tag{8.8}$$

Remark 8.13. In the frequency-localized versions of (8.6)–(8.8), we also gain an η'-power of the maximal frequency scale. Similarly to Proposition 5.3 and Remark 5.4, we may also replace Ψ_N by $\Psi_{N,\tau}$.

Proof of Proposition 8.12. We first prove (8.6), which is the easiest part. Using the inhomogeneous Strichartz estimate (Lemma 4.9), $s_2 - 1 < -s_1$, and the (dual of) the fractional product rule, we have

$$\begin{aligned} &\Big\| V * \Big(P_{\leq N} \uparrow \cdot P_{\leq N} \Psi_N \Big) P_{\leq N} w \Big\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})} \\ &\quad \lesssim \Big\| V * \Big(P_{\leq N} \uparrow \cdot P_{\leq N} \Psi_N \Big) P_{\leq N} w \Big\|_{L_t^{2b_+} H_x^{s_2-1}(\mathcal{J})} \\ &\quad \lesssim \Big\| V * \Big(P_{\leq N} \uparrow \cdot P_{\leq N} \Psi_N \Big) P_{\leq N} w \Big\|_{L_t^{2b_+} H_x^{-s_1}(\mathcal{J})} \\ &\quad \lesssim T \Big\| V * \Big(P_{\leq N} \uparrow \cdot P_{\leq N} \Psi_N \Big) \Big\|_{L_t^\infty \mathcal{C}_x^{-s_1+\eta}([0,T]\times\mathbb{T}^3)} \|w\|_{L_t^\infty H_x^{s_1}(\mathcal{J}\times\mathbb{T}^3)}. \end{aligned}$$

Using now (5.11) implies (8.6).

We now turn to (8.7) and (8.8), which are more difficult. The main step consists in the following estimate: For any $M_1, N_1, K_1, K_2 \geq 1$, we have

$$\begin{aligned}
\Bigg\| \sup_{N\geq 1} \sup_{t\in[0,T]} \sup_{\|f\|_{H_x^{s_1}}, \|g\|_{H_x^{s_1}} \leq 1} \Bigg| \int_{\mathbb{T}^3} V * \Big(P_{M_1} P_{\leq N} \TreeThreeStar_{N} \cdot P_{K_1} P_{\leq N} f \Big) \\
\times P_{N_1} P_{\leq N} \TreeOne \cdot P_{K_2} P_{\leq N} g \, \mathrm{d}x \Bigg| \Bigg\|_{L^p_\omega(\mathbb{P})} \\
\lesssim T^3 \max(K_1, K_2, N_1, M_1)^{-\eta} (1 + 1\{N_1 \sim K_2\} M_1^{-\beta+\kappa+\eta} K_1^{-s_1+\eta} N_1^{1/2+\kappa-s_1}) p^2.
\end{aligned} \tag{8.9}$$

For notational convenience, we now omit the multiplier $P_{\leq N}$. As will be evident from the proof, the same argument applies (uniformly in N) with the multiplier. The proof of (8.9) splits into two cases. The impatient reader may wish to skim ahead to Case 2.b, which contains the most interesting part of the argument.

Case 1: $M_1 \nsim N_1$. From Fourier support considerations, it follows that $\max(K_1, K_2) \gtrsim \max(N_1, M_1)$. Then, we estimate the integral in (8.9) by

$$\begin{aligned}
&\Big| \int_{\mathbb{T}^3} V * (P_{M_1} \TreeThreeStar_{N} \cdot P_{K_1} f) P_{N_1} \TreeOne \cdot P_{K_2} g \, \mathrm{d}x \Big| \\
&\lesssim \sum_{L \lesssim \max(N_1, K_2)} \Big| \int_{\mathbb{T}^3} (P_L V) * \Big(P_{M_1} \TreeThreeStar_{N} \cdot P_{K_1} f \Big) \cdot \widetilde{P}_L (P_{N_1} \TreeOne \cdot P_{K_2} g) \, \mathrm{d}x \Big| \\
&\lesssim \sum_{L \lesssim \max(N_1, K_2)} \|P_L V\|_{L^1_x} \Big\| P_{M_1} \TreeThreeStar_{N} \cdot P_{K_1} f \Big\|_{L^2_x} \|\widetilde{P}_L (P_{N_1} \TreeOne \cdot P_{K_2} g) \, \mathrm{d}x\|_{L^2_x} \\
&\lesssim M_1^{-\beta+\kappa} K_1^{-s_1} \Big\| \TreeThreeStar_{N} \Big\|_{\mathcal{C}_x^{\beta-\kappa}} \sum_{L \lesssim \max(N_1, K_2)} L^{-\beta} \|\widetilde{P}_L (P_{N_1} \TreeOne \cdot P_{K_2} g) \, \mathrm{d}x\|_{L^2_x}.
\end{aligned} \tag{8.10}$$

We now further split the argument into two subcases.

Case 1.a: $M_1 \nsim N_1$, $K_2 \nsim N_1$. Then we only obtain a nontrivial contribution if $L \sim \max(N_1, K_2)$. Using $\max(K_1, K_2) \gtrsim \max(M_1, N_1) \geq N_1$, we obtain

$$\begin{aligned}
(8.10) &\lesssim M_1^{-\beta+\kappa} K_1^{-s_1} \max(K_2, N_1)^{-\beta} K_2^{-s_1} N_1^{1/2+\kappa} \Big\| \TreeThreeStar_{N} \Big\|_{\mathcal{C}_x^{\beta-\kappa}} \| \TreeOne \|_{\mathcal{C}_x^{-1/2-\kappa}} \\
&\lesssim M_1^{-\beta+\kappa} K_1^{-\eta} K_2^{-\eta} N_1^{1/2+\kappa+\eta-\beta-s_1} \Big\| \TreeThreeStar_{N} \Big\|_{\mathcal{C}_x^{\beta-\kappa}} \| \TreeOne \|_{\mathcal{C}_x^{-1/2-\kappa}}.
\end{aligned}$$

The pre-factor is bounded by $(M_1 K_1 K_2 N_1)^{-\eta}$, which is acceptable.

Case 1.b: $M_1 \nsim N_1$, $K_2 \sim N_1$. In this case, the worst case corresponds to $L \sim 1$. Using only Hölder's inequality, we obtain

$$(8.10) \lesssim 1\{K_2 \sim N_1\} M_1^{-\beta+\kappa} K_1^{-s_1} N_1^{1/2+\kappa-s_1} \Big\| \TreeThreeStar_{N} \Big\|_{\mathcal{C}_x^{\beta-\kappa}} \| \TreeOne \|_{\mathcal{C}_x^{-1/2-\kappa}}.$$

This case is responsible for the second summand in (8.9).

Case 2: $M_1 \sim N_1$. This case is more delicate and requires the estimates on the quartic stochastic objects from Section 5.2. Inspired by the uncertainty principle, we decompose

$$\begin{aligned}
&\Big| \int_{\mathbb{T}^3} V * \Big(P_{M_1} \quarticN \cdot P_{K_1} f \Big) P_{N_1} \linear \cdot P_{K_2} g \, \mathrm{d}x \Big| \\
&\qquad \leq \Big| \int_{\mathbb{T}^3} (P_{\ll N_1} V) * \Big(P_{M_1} \quarticN \cdot P_{K_1} f \Big) P_{N_1} \linear \cdot P_{K_2} g \, \mathrm{d}x \Big| \\
&\qquad\quad + \Big| \int_{\mathbb{T}^3} (P_{\gtrsim N_1} V) * \Big(P_{M_1} \quarticN \cdot P_{K_1} f \Big) P_{N_1} \linear \cdot P_{K_2} g \, \mathrm{d}x \Big|.
\end{aligned}$$

We estimate both terms separately and hence divide the argument into two subcases.

Case 2.a: $M_1 \sim N_1$, *contribution of* $P_{\ll N_1} V$. For this term, we only obtain a non-trivial contribution if $K_1 \sim K_2 \sim N_1$. Using Hölder's inequality and Young's convolution inequality, we obtain

$$\begin{aligned}
&\Big| \int_{\mathbb{T}^3} (P_{\ll N_1} V) * \Big(P_{M_1} \quarticN \cdot P_{K_1} f \Big) P_{N_1} \linear \cdot P_{K_2} g \, \mathrm{d}x \Big| \\
&\quad \lesssim 1\{K_1 \sim K_2 \sim M_1 \sim N_1\} \| P_{\ll N_1} V \|_{L^1_x} \Big\| P_{M_1} \quarticN \Big\|_{L^\infty_x} \\
&\qquad \times \| P_{K_1} f \|_{L^2_x} \| P_{N_1} \linear \|_{L^\infty_x} \| P_{K_2} g \|_{L^2_x} \\
&\quad \lesssim 1\{K_1 \sim K_2 \sim M_1 \sim N_1\} N_1^{1/2+2\kappa-\beta-2s_1} \Big\| \quarticN \Big\|_{\mathscr{C}^{\beta-\kappa}_x} \| \linear \|_{\mathscr{C}^{-1/2-\kappa}_x}.
\end{aligned}$$

The pre-factor is easily bounded by (and generally much smaller than) $(M_1 K_1 K_2 N_1)^{-\eta}$.

Case 2.b: $M_1 \sim N_1$, *contribution of* $P_{\gtrsim N_1} V$. By expanding the convolution with the interaction potential, we obtain

$$\begin{aligned}
&\Big| \int_{\mathbb{T}^3} (P_{\gtrsim N_1} V) * \Big(P_{M_1} \quarticN \cdot P_{K_1} f \Big) P_{N_1} \linear \cdot P_{K_2} g \, \mathrm{d}x \Big| \\
&\leq \int_{\mathbb{T}^3} | P_{\gtrsim N_1} V(y) | \Big| \int_{\mathbb{T}^3} (P_{K_1} f(x-y) \cdot P_{K_2} g(x)) \\
&\qquad\qquad \cdot \Big(P_{M_1} \quarticN (t, x-y) \cdot P_{N_1} \linear (t,x) \Big) \mathrm{d}x \Big| \, \mathrm{d}y \\
&\quad \lesssim \| P_{\gtrsim N_1} V(y) \|_{L^1_y} \cdot \sup_{y \in \mathbb{T}^3} \| \langle \nabla_x \rangle^{1/2-\beta+2\kappa} (P_{K_1} f(x-y) \cdot P_{K_2} g(x)) \|_{L^1_x} \\
&\qquad \times \sup_{y \in \mathbb{T}^3} \Big\| P_{M_1} \quarticN (t, x-y) \cdot P_{N_1} \linear (t,x) \Big\|_{\mathscr{C}^{-1/2+\beta-\kappa}_x} \\
&\quad \lesssim N_1^{-\beta} K_1^{-\eta} K_2^{-\eta} \sup_{y \in \mathbb{T}^3} \Big\| P_{M_1} \quarticN (t, x-y) \cdot P_{N_1} \linear (t,x) \Big\|_{\mathscr{C}^{-1/2+\beta-\kappa}_x}.
\end{aligned}$$

By Proposition 5.3, this contribution is acceptable. We note that the pre-factor $N_1^{-\beta}$ is essential, since Proposition 5.3 is not uniformly bounded over all frequency scales.

By combining Cases 1 and 2, we have finished the proof of (8.9). It remains to show that (8.9) implies (8.7) and (8.8). To simplify the notation, we denote the expression inside the L^p_ω-norm in (8.9) by

$$\mathcal{A}(K_1, K_2, M_1, N_1) \overset{\mathrm{def}}{=} \sup_{t\in[0,T]} \sup_{\|f\|_{H_x^{s_1}}, \|g\|_{H_x^{s_1}} \leq 1} \left| \int_{\mathbb{T}^3} V * \left(P_{M_1} \Upsilon^{*}_{\mathrm{N}} \cdot P_{K_1} f \right) P_{N_1} \uparrow \cdot P_{K_2} g \, \mathrm{d}x \right|. \tag{8.11}$$

To see (8.7), we use the self-adjointness of V, duality, and $s_1 < 1 - s_2$, which leads to

$$\begin{aligned}
&\left\| V * (\uparrow \circledcirc_{\neq} w) \Upsilon^{*}_{\mathrm{N}} \right\|_{H_x^{s_2-1}} \\
&\quad\leq \sum_{K_1,K_2,M_1,N_1} 1\{K_2 \nsim N_1\} \left\| P_{K_1} \left(V * (P_{N_1} \uparrow \cdot P_{K_2} w) P_{M_1} \Upsilon^{*}_{\mathrm{N}} \right) \right\|_{H_x^{s_2-1}} \\
&\quad\lesssim \left(\sum_{K_1,K_2,M_1,N_1} 1\{K_2 \nsim N_1\} \mathcal{A}(K_1, K_2, M_1, N_1) \right) \|w\|_{H_x^{s_1}}.
\end{aligned}$$

Using the inhomogeneous Strichartz estimate and (8.9) now completes the argument.

Finally, we turn to (8.8). Using duality, we have

$$\begin{aligned}
&\left\| V * \left(\Upsilon^{*}_{\mathrm{N}} \cdot w \right) (\neg \circledcirc_{<}) \uparrow \right\|_{H_x^{s_2-1}} \\
&\quad\leq \sum_{K_1,K_2,M_1,N_1} 1\{\max(M_1, K_1) \geq N_1^{\epsilon}\} \left\| P_{K_2} \left(V * \left(P_{M_1} \Upsilon^{*}_{\mathrm{N}} \cdot P_{K_1} w \right) P_{N_1} \uparrow \right) \right\|_{H_x^{s_2-1}} \\
&\quad\lesssim \sum_{K_1,K_2,M_1,N_1} 1\{\max(M_1, K_1) \geq N_1^{\epsilon}\} K_2^{s_1+s_2-1} \\
&\qquad\times \left\| P_{K_2} \left(V * \left(P_{M_1} \Upsilon^{*}_{\mathrm{N}} \cdot P_{K_1} w \right) P_{N_1} \uparrow \right) \right\|_{H_x^{-s_1}} \\
&\quad\lesssim \sum_{K_1,K_2,M_1,N_1} 1\{\max(M_1, K_1) \geq N_1^{\epsilon}\} K_2^{s_1+s_2-1} \mathcal{A}(K_1, K_2, M_1, N_1) \|w\|_{H_x^{s_1}}.
\end{aligned}$$

We now note that $\max(M_1, K_1) \geq N_1^{\epsilon}$ implies

$$\begin{aligned}
&1\{N_1 \sim K_2\} M_1^{-\beta+\kappa+\eta} K_1^{-s_1+\eta} K_2^{s_1+s_2-1} N_1^{1/2+\kappa-s_1} \\
&\qquad\lesssim N_1^{-\epsilon \min(\beta-\kappa-\eta, 1/2-\delta_1-\eta)} N_1^{\kappa+\delta_2} \lesssim 1.
\end{aligned}$$

In the last inequality, we have used the parameter conditions (1.21). We also emphasize that the factor $K_2^{s_1+s_2-1}$ is essential for this inequality. Using the inhomogeneous Strichartz estimate and (8.9), we then obtain the desired estimate. ∎

9. From free to Gibbsian random structures

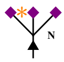
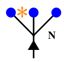
In the previous four sections, we proved several estimates for stochastic objects, random matrices, and paracontrolled structures based on $\bullet$. In Section 2, these estimates were used to prove the local convergence of the truncated dynamics as N tends to infinity. Unfortunately, the object $\bullet$ only exists on the ambient probability space and the global theory requires (intrinsic) estimates for $\blacklozenge$ with respect to the Gibbs measure. If the desired estimate does not rely on the invariance of $\mu_M^{\otimes}$ under the nonlinear flow, however, we can use Theorem 1.1 to replace the Gibbs measure $\mu_M^{\otimes}$ by the reference measure $\nu_M^{\otimes}$. In particular, this works for stochastic objects only depending on the linear evolution of $\blacklozenge$, such as  or . Once we are working with the reference measure $\nu_M^{\otimes}$, we can use the fact that

$$\nu_M^{\otimes} = \mathrm{Law}_{\mathbb{P}}(\bullet + \circ_{\mathbf{M}}).$$

Since $\circ_{\mathbf{M}}$ has spatial regularity $1/2+\beta-$, we expect that our estimates for $\bullet$ will imply the same estimates for $\blacklozenge$. As a result, this section contains no inherently new estimates and only combines our previous bounds.

9.1. *The Gibbsian cubic stochastic object*

This subsection should be seen as a warm-up for Section 9.2 below. We explore the relationship between the two cubic stochastic objects

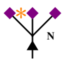
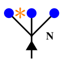
 and .

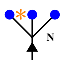
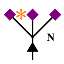
This is already sufficient for the structured local well-posedness in Proposition 3.3 on the support of the Gibbs measure. It will also be needed in the proof of several propositions and lemmas in Section 9.3 below.

Proposition 9.1. *Let $A, T \geq 1$, and let $\zeta = \zeta(\epsilon, s_1, s_2, \kappa, \eta, \eta', b_+, b) > 0$ be sufficiently small. There exist Borel sets $\Theta^{\mathrm{cub}}_{\mathrm{blue}}(A,T), \Theta^{\mathrm{cub}}_{\mathrm{red}}(A,T) \subseteq \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$ satisfying*

$$\mathbb{P}\Big(\bullet \in \Theta^{\mathrm{cub}}_{\mathrm{blue}}(A,T) \text{ and } \circ_{\mathbf{M}} \in \Theta^{\mathrm{cub}}_{\mathrm{red}}(A,T)\Big) \geq 1 - \zeta^{-1}\exp(\zeta A^{\zeta})$$

for all $M \geq 1$ and such that the following holds for all $\bullet \in \Theta^{\mathrm{cub}}_{\mathrm{blue}}(A,T)$ and $\circ_{\mathbf{M}} \in \Theta^{\mathrm{cub}}_{\mathrm{red}}(A,T)$: For all $N \geq 1$, there exist $H_N[\blacklozenge \to \bullet], H_N[\bullet \to \blacklozenge] \in \mathscr{LM}([0,T])$ and $Y_N[\blacklozenge \to \bullet], Y_N[\bullet \to \blacklozenge] \in \mathfrak{X}^{s_2,b}([0,T])$ satisfying the identities

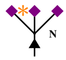
 $=$  $+ P_{\leq N}\,\mathrm{I}\big[\mathrm{PCtrl}(H_N[\blacklozenge \to \bullet], P_{\leq N}$ $)\big] + Y_N[\blacklozenge \to \bullet],$

 $=$  $+ P_{\leq N}\,\mathrm{I}\big[\mathrm{PCtrl}(H_N[\bullet \to \blacklozenge], P_{\leq N}$ $)\big] + Y_N[\bullet \to \blacklozenge].$

and the estimates

$$\|H_N[\bullet \to \blacklozenge]\|_{\mathscr{LM}([0,T])}, \|H_N[\blacklozenge \to \bullet]\|_{\mathscr{LM}([0,T])} \leq T^2 A,$$
$$\|Y_N[\bullet \to \blacklozenge]\|_{\mathfrak{X}^{s_2,b}([0,T])}, \|Y_N[\blacklozenge \to \bullet]\|_{\mathfrak{X}^{s_2,b}([0,T])} \leq T^3 A.$$

Furthermore, in the frequency-localized version of this estimate, we gain an η'-power of the maximal frequency scale.

Remark 9.2. The results in Proposition 9.1 do not yield a bound on  in $L_t^\infty \mathcal{C}_x^{\beta-\kappa}$, since $\mathcal{X}^{s_2,b}$ does not embed into $L_t^\infty \mathcal{C}_x^{\beta-\kappa}$ and we do not state any additional information on Y_N. However, such an estimate is possible and only requires the translation invariance of the law of $(\bullet, \circ_{\mathbf{M}})$, which is a consequence of [12, Theorem 1.4].

Before we start with the proof of Proposition 9.1, we prove the following corollary.

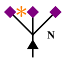
Corollary 9.3. *Let $A, T \geq 1$, let $\alpha > 0$ be a large absolute constant, and let $\zeta = \zeta(\epsilon, s_1, s_2, \kappa, \eta, \eta', b_+, b) > 0$ be sufficiently small. Then there exists a Borel set $\Theta_{\text{pur}}^{\text{bil}}(A,T) \subseteq \mathcal{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$ satisfying*

$$\mu_M^{\otimes}(\Theta_{\text{pur}}^{\text{bil}}(A,T)), \nu_M^{\otimes}(\Theta_{\text{pur}}^{\text{bil}}(A,T)) \geq 1 - \zeta^{-1}\exp(\zeta A^\zeta) \tag{9.1}$$

for all $M \geq 1$ and such that the following holds for all $\blacklozenge \in \Theta_{\text{pur}}^{\text{bil}}(A,T)$:

For all intervals $\mathcal{J} \subseteq [0,T]$ and $w \in \mathcal{X}^{s_1,b}(\mathcal{J})$,

$$\sum_{L_1,L_2} \Big\| P_{L_1} \text{[tree]} \cdot P_{L_2} w \Big\|_{L_t^2 H_x^{-4\delta_1}(\mathcal{J}\times\mathbb{T}^3)} \leq T^\alpha A \|w\|_{\mathcal{X}^{s_1,b}(\mathcal{J})}. \tag{9.2}$$

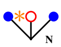


Proof. We simply define $\Theta_{\text{pur}}^{\text{bil}}(A,T)$ as the set of initial data $\blacklozenge \in \mathcal{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$ where (9.2) holds for a countable but dense subset of $\mathcal{X}^{s_1,b}(\mathbb{R})$, which is Borel-measurable, and it remains to prove the probabilistic estimate (9.1). Using Theorem 1.1, it suffices to prove

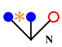
$$\mathbb{P}(\bullet + \circ_{\mathbf{M}} \in \Theta_{\text{pur}}^{\text{bil}}(A,T)) \geq 1 - \zeta^{-1}\exp(\zeta A^\zeta).$$

This follows directly from Proposition 5.1, Lemma 8.4, and Proposition 9.1. ∎

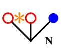
We now turn to the proof of Proposition 9.1. The argument relies on the multi-linearity of the stochastic objects in the initial data. In order to use the decomposition of $\blacklozenge$, we define mixed cubic stochastic objects. In Section 3.1, we defined stochastic objects in $\blacklozenge$ instead of $\bullet$, which had the exact same renormalization constants and multipliers. In the proof of Proposition 9.1, we also work with stochastic objects that contain a mixture of both $\bullet$ and $\circ = \circ_{\mathbf{M}}$. In this case, only factors of $\bullet$ require a renormalization. The renormalized mixed stochastic objects are then defined by



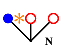
$$\stackrel{\text{def}}{=} P_{\leq N}[V * (P_{\leq N}\bullet \cdot P_{\leq N}\circ) \cdot P_{\leq N}\bullet - \mathcal{M}_N P_{\leq N}\circ],$$



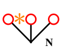
$$\stackrel{\text{def}}{=} P_{\leq N}[(V * \text{[tree]}) \cdot P_{\leq N}\circ],$$



$$\stackrel{\text{def}}{=} P_{\leq N}[V * (P_{\leq N}\circ \cdot P_{\leq N}\circ) \cdot P_{\leq N}\bullet],$$



$$\stackrel{\text{def}}{=} P_{\leq N}[V * (P_{\leq N}\bullet \cdot P_{\leq N}\circ) \cdot P_{\leq N}\circ],$$



$$\stackrel{\text{def}}{=} P_{\leq N}[V * (P_{\leq N}\circ \cdot P_{\leq N}\circ) \cdot P_{\leq N}\circ].$$

Furthermore, we define the solution to the nonlinear wave equation with forcing term tree_N by

$$(-\partial_t^2 - 1 + \Delta)\,\mathrm{tree}_N = \mathrm{tree}_N, \qquad \mathrm{tree}_N[0] = 0.$$

The solutions for the other forcing terms above are defined similarly. Using these definitions, we deduce the identity

$$\mathrm{tree}_N = \mathrm{tree}_N + 2\,\mathrm{tree}_N + \mathrm{tree}_N + \mathrm{tree}_N + 2\,\mathrm{tree}_N + \mathrm{tree}_N. \tag{9.3}$$

With this identity, the proof of Proposition 9.1 is now split into two lemmas.

Lemma 9.4. *Let $A, T \geq 1$, and let $\zeta = \zeta(\epsilon, s_1, s_2, \kappa, \eta, \eta', b_+, b) > 0$ be sufficiently small. Then there exist Borel sets $\Theta^{\mathrm{cub},(1)}_{\mathrm{blue}}(A,T), \Theta^{\mathrm{cub},(1)}_{\mathrm{red}}(A,T) \subseteq \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$ satisfying*

$$\mathbb{P}\big(\bullet \in \Theta^{\mathrm{cub},(1)}_{\mathrm{blue}}(A,T) \text{ and } \circ_{\mathbf{M}} \in \Theta^{\mathrm{cub},(1)}_{\mathrm{red}}(A,T)\big) \geq 1 - \zeta^{-1}\exp(\zeta A^{\zeta}) \tag{9.4}$$

for all $M \geq 1$ and such that the following holds for all $\bullet \in \Theta^{\mathrm{cub},(1)}_{\mathrm{blue}}(A,T)$ and $\circ_{\mathbf{M}} \in \Theta^{\mathrm{cub},(1)}_{\mathrm{red}}(A,T)$:

For all $N \geq 1$, there exists an $H_N \in \mathscr{LM}([0,T])$ satisfying the identity

$$2\,(\circledless)\,\mathrm{tree}_N + (\circledless)\,\mathrm{tree}_N = P_{\leq N}\,\mathrm{I}[\mathrm{PCtrl}(H_N, P_{\leq N}\,\mathrm{tree})] \tag{9.5}$$

and the estimate

$$\|H_N\|_{\mathscr{LM}([0,T])} \leq T^2 A.$$

Furthermore, the difference $H_N - H_K$ gains an η'-power of $\min(N,K)$.

Proof. From Lemma 7.6, it follows that there exists a (canonical) random variable $H_N : \Omega \to \mathscr{LM}([0,T])$ such that

$$2\,(\circledless)\,\mathrm{tree}_N + (\circledless)\,\mathrm{tree}_N = P_{\leq N}\,\mathrm{I}[\mathrm{PCtrl}(H_N, P_{\leq N}\,\mathrm{tree})]$$

and

$$\begin{aligned}\|H_N\|_{\mathscr{LM}[0,T]} &\lesssim \big(\|\mathrm{tree}\|_{\mathfrak{X}^{-s_2,b}([0,T])} + \|\mathrm{tree}\|_{\mathfrak{X}^{-s_2,b}([0,T])}\big)\|\mathrm{tree}\|_{\mathfrak{X}^{s_2,b}([0,T])} \\ &\lesssim T^2\big(\|\bullet\|_{\mathscr{H}_x^{-s_2}(\mathbb{T}^3)} + \|\circ\|_{\mathscr{H}_x^{s_2}(\mathbb{T}^3)}\big)\cdot\|\circ\|_{\mathscr{H}_x^{s_2}(\mathbb{T}^3)}\end{aligned}$$

The estimate for H_N then follows from elementary properties of $\bullet$ and the high-regularity bound for $\circ$ in Theorem 1.1. ■

Lemma 9.5. *Let $A, T \geq 1$, and let $\zeta = \zeta(\epsilon, s_1, s_2, \kappa, \eta, \eta', b_+, b) > 0$ be sufficiently small. Then there exist Borel sets $\Theta^{\mathrm{cub},(2)}_{\mathrm{blue}}(A,T), \Theta^{\mathrm{cub},(2)}_{\mathrm{red}}(A,T) \subseteq \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$ satisfying*

$$\mathbb{P}\big(\bullet \in \Theta^{\mathrm{cub},(2)}_{\mathrm{blue}}(A,T) \text{ and } \circ_{\mathbf{M}} \in \Theta^{\mathrm{cub},(2)}_{\mathrm{red}}(A,T)\big) \geq 1 - \zeta^{-1}\exp(\zeta A^{\zeta}) \tag{9.6}$$

for all $M \geq 1$ *and such that the following holds for all* $\bullet \in \Theta^{\mathrm{cub},(2)}_{\mathrm{blue}}(A, T)$ *and* $\circ_{\mathbf{M}} \in \Theta^{\mathrm{cub},(2)}_{\mathrm{red}}(A, T)$:

For all $N \geq 1$, *we have*

$$\max\Big(\Big\| (\neg ⓛ)\, Y_N[\bullet\overset{*}{\bullet}\bullet] \Big\|_{\mathfrak{X}^{s_2,b}([0,T])}, \Big\| (\neg ⓛ)\, Y_N[\circ\overset{*}{\circ}\bullet] \Big\|_{\mathfrak{X}^{s_2,b}([0,T])}, \Big\| Y_N[\bullet\overset{*}{\bullet}\circ] \Big\|_{\mathfrak{X}^{s_2,b}([0,T])}, \Big\| Y_N[\bullet\overset{*}{\circ}\circ] \Big\|_{\mathfrak{X}^{s_2,b}([0,T])}, \Big\| Y_N[\circ\overset{*}{\circ}\circ] \Big\|_{\mathfrak{X}^{s_2,b}([0,T])} \Big) \leq T^3 A.$$

Furthermore, the difference of the cubic stochastic objects with two parameters N *and* K *gains an* η'*-power of* $\min(N, K)$.

Proof. This follows from our previous estimates for $\bullet$ from Sections 5–8 and the high-regularity bound for $\circ$ in Theorem 1.1. More precisely, we estimate the $L^p_\omega \mathfrak{X}^{s_2,b}$-norm of

- $(\neg ⓛ)\, Y_N[\bullet\overset{*}{\bullet}\bullet]$ by $T^2 p^{(2+k)/2}$ through Proposition 6.3,
- $(\neg ⓛ)\, Y_N[\circ\overset{*}{\circ}\bullet]$ by $T^3 p^{(1+2k)/2}$ through Proposition 8.6,
- $Y_N[\bullet\overset{*}{\bullet}\circ]$ by $T^2 p^{2+k/2}$ through Proposition 6.1,
- $Y_N[\bullet\overset{*}{\circ}\circ]$ by $T^3 p^{1+2k/2}$ through Proposition 8.7 and Corollary 8.9,
- $Y_N[\circ\overset{*}{\circ}\circ]$ by $T p^{3k/2}$ through Proposition 8.10. ■

Proof of Proposition 9.1. The first algebraic identity and related estimates follow directly from (9.3) and Lemmas 9.4 and 9.5. By using $\uparrow^{\blacklozenge} - \uparrow^{\bullet} = \uparrow^{\circ}$ and the high regularity bound for $\circ$, we obtain the second identity and the related estimates from the first identity. ■

9.2. *Comparing random structures in Gibbsian and Gaussian initial data*

In Definition 2.4, we introduced the types of functions occurring in our multi-linear master estimate for $\bullet$ (Proposition 2.8). The types w and X in Definition 2.4 implicitly depend on $\bullet$ and, as already mentioned in Remark 2.5, we now refer to type w and X as type $w^{\bullet}$ and $X^{\bullet}$, respectively. We now introduce a similar notation for the generic initial data $\blacklozenge$. In order to orient the reader, we include an overview of the different types and their relationship in Figure 4.

Definition 9.6 (Purple types). Let $\mathcal{J} \subseteq [0, \infty)$ be a bounded interval and let $\varphi\colon J \times \mathbb{T}^3 \to \mathbb{R}$. We say that φ is of type

- $\uparrow^{\blacklozenge}$ if $\varphi = \uparrow^{\blacklozenge}$,
- $Y[\blacklozenge\overset{*}{\blacklozenge}\blacklozenge]$ if $\varphi = Y_N[\blacklozenge\overset{*}{\blacklozenge}\blacklozenge]$ for some $N \geq 1$,

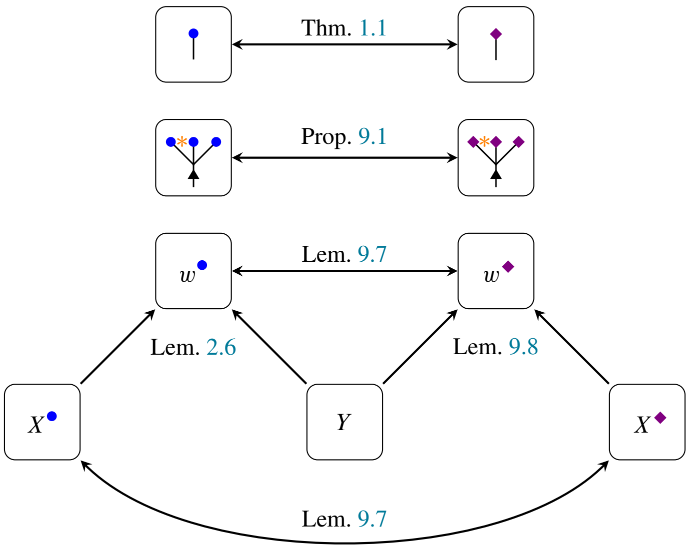


Fig. 4. We display the relationship between the different types of functions used in this paper. The equivalence "$\leftrightarrow$" means that both types agree modulo scalar multiples and/or terms further down in the hierarchy. The implication "$\rightarrow$" means that, up to scalar multiples, the left type forms a subclass of the right type.

- $w^{\blacklozenge}$ if $\|\varphi\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} \leq 1$ and $\sum_{L_1 \sim L_2} \| P_{L_1} \overset{\blacklozenge}{|} \cdot P_{L_2} w \|_{L_t^2 H_x^{-4\delta_1}(\mathcal{J} \times \mathbb{T}^3)} \leq 1$ for all $N \geq 1$,
- $X^{\blacklozenge}$ if $\varphi = P_{\leq N} \, \mathrm{I}[1_{\mathcal{J}_0} \, \mathrm{PCtrl}(H, P_{\leq N} \overset{\blacklozenge}{|})]$ for a dyadic integer $N \geq 1$, a subinterval $\mathcal{J}_0 \subseteq \mathcal{J}$, and a function $H \in \mathcal{LM}(\mathcal{J}_0)$ satisfying $\|H\|_{\mathcal{LM}(\mathcal{J}_0)} \leq 1$.

Since the type Y in Definition 2.4 does not depend on the stochastic object, its meaning remains unchanged. In Proposition 9.1, we have already seen that the types $\overset{\bullet \ast \bullet \ \bullet}{\curlyvee}$ and $\overset{\blacklozenge \ast \blacklozenge \ \blacklozenge}{\curlyvee}$ only differ by functions of type $X^{\bullet}$ and Y (or $X^{\blacklozenge}$ and Y). In the next lemma, we clarify the relationship between the types $w^{\bullet}$ and $w^{\blacklozenge}$ as well as $X^{\bullet}$ and $X^{\blacklozenge}$.

Lemma 9.7 (The equivalences $w^{\bullet} \leftrightarrow w^{\blacklozenge}$ and $X^{\bullet} \leftrightarrow X^{\blacklozenge}$). *Let $A, T \geq 1$, and let $\zeta = \zeta(\epsilon, s_1, s_2, \kappa, \eta, \eta', b_+, b) > 0$ be sufficiently small. Then there exists a Borel set $\Theta^{\text{type}}_{\text{red}}(A, T) \subseteq \mathcal{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$ such that*

$$\mathbb{P}(\circ_{\mathbf{M}} \in \Theta^{\text{type}}_{\text{red}}(A, T)) \geq 1 - \zeta^{-1} \exp(-\zeta A^{\zeta})$$

and such that the following holds for $\blacklozenge = \bullet + \circ_{\mathbf{M}}$:

- *The types $w^{\bullet}$ and $w^{\blacklozenge}$ are equivalent up to multiplication by a scalar $\lambda \in \mathbb{R}_{>0}$ satisfying $\lambda, \lambda^{-1} \leq T^2 A$.*
- *The types $X^{\bullet}$ and $X^{\blacklozenge}$ are equivalent up to addition/subtraction of a function in $\mathfrak{X}^{s_2,b}$ with norm $\leq TA$.*

Proof. We will prove the desired statement on the event

$$\Theta^{\text{type}}_{\text{red}}(A,T) = \{\phi \in \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3) \colon \|\phi\|_{\mathscr{H}_x^{1/2+\beta-\kappa}(\mathbb{T}^3)} \leq cA\},$$

where $c = c(\epsilon, s_1, s_2, b)$ is a small constant. Based on Theorem 1.1, this event has an acceptable probability.

We start with the statement regarding the types $w^{\bullet}$ and $w^{\blacklozenge}$. Let $\varphi \in \mathfrak{X}^{s_1,b}(\mathcal{J})$ satisfy $\|\varphi\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} \leq 1$, which holds for φ of type either $w^{\bullet}$ or $w^{\blacklozenge}$. For any $L \geq 1$, we have

$$\Big| \sum_{L_1 \sim L_2} \|P_{L_1} \overset{\blacklozenge}{|} \cdot P_{L_2}\varphi\|_{L_t^2 H_x^{-4\delta_1}(\mathcal{J}\times\mathbb{T}^3)} - \sum_{L_1 \sim L_2} \|P_{L_1} \overset{\bullet}{|} \cdot P_{L_2}\varphi\|_{L_t^2 H_x^{-4\delta_1}(\mathcal{J}\times\mathbb{T}^3)} \Big| \leq \sum_{L_1 \sim L_2} \|P_{L_1} \overset{\circ}{|} \cdot P_{L_2}\varphi\|_{L_t^2 H_x^{-4\delta_1}(\mathcal{J}\times\mathbb{T}^3)}.$$

By Lemma 8.4, it follows that

$$\begin{aligned} \sum_{L_1 \sim L_2} \|P_{L_1} \overset{\circ}{|} \cdot P_{L_2}\varphi\|_{L_t^2 H_x^{-4\delta_1}(\mathcal{J}\times\mathbb{T}^3)} &\lesssim T^{1/2} \Big(\sum_{L_1} L_1^{3-s_1-(1/2+\beta-\kappa)-2}\Big) \|\overset{\circ}{|}\|_{\mathfrak{X}^{1/2+\beta-\kappa,b}(\mathcal{J})} \|\varphi\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} \\ &\lesssim T^{3/2} \|\circ_{\mathbf{M}}\|_{\mathscr{H}_x^{1/2+\beta-\kappa}(\mathbb{T}^3)} \|\varphi\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} \leq \tfrac{1}{2} T^{3/2} A \|\varphi\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})}. \end{aligned}$$

This yields the stated equivalence of the types $w^{\bullet}$ and $w^{\blacklozenge}$.

We now turn to the statement regarding the types $X^{\bullet}$ and $X^{\blacklozenge}$. For any $H \in \mathcal{LM}(\mathcal{J})$,

$$\begin{aligned} &\|P_{\leq N}\, \mathrm{I}[1_{\mathcal{J}_0} \operatorname{PCtrl}(H, P_{\leq N} \overset{\blacklozenge}{|})] - P_{\leq N}\, \mathrm{I}[1_{\mathcal{J}_0} \operatorname{PCtrl}(H, P_{\leq N} \overset{\bullet}{|})]\|_{\mathfrak{X}^{s_2,b}(\mathcal{J})} \\ &\qquad\lesssim \|P_{\leq N}\, \mathrm{I}[1_{\mathcal{J}_0} \operatorname{PCtrl}(H, P_{\leq N} \overset{\circ}{|})]\|_{\mathfrak{X}^{s_2,b}(\mathcal{J})}. \end{aligned}$$

Using Lemmas 4.8, 4.9, and 7.3, we have

$$\begin{aligned} \|P_{\leq N}\, \mathrm{I}[1_{\mathcal{J}_0} \operatorname{PCtrl}(H, P_{\leq N} \overset{\circ}{|})]\|_{\mathfrak{X}^{s_2,b}(\mathcal{J})} &\lesssim T \|\operatorname{PCtrl}(H, P_{\leq N} \overset{\circ}{|})\|_{L_t^\infty H_x^{s_2-1}(\mathcal{J}_0\times\mathbb{T}^3)} \\ &\lesssim T \|H\|_{\mathcal{LM}(\mathcal{J}_0)} \|\overset{\circ}{|}\|_{L_t^\infty H_x^{s_2-1+8\epsilon}(\mathcal{J}\times\mathbb{T}^3)} \\ &\lesssim T \|\circ_{\mathbf{M}}\|_{\mathscr{H}_x^{1/2+\beta-\kappa}(\mathbb{T}^3)} \leq \tfrac{1}{2} TA. \end{aligned}$$

This yields the desired estimate. ■

Lemma 9.8 (The implication $X^{\blacklozenge}, Y \to w^{\blacklozenge}$). *Let $\zeta = \zeta(\epsilon, s_1, s_2, \kappa, \eta, \eta', b_+, b) > 0$ be sufficiently small, and let $A, T \geq 1$. Then there exists a Borel set $\Theta^{\text{type}}_{\text{pur}}(A,T) \subseteq \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$ satisfying*

$$\mu_M^{\otimes}(\Theta^{\text{type}}_{\text{pur}}(A,T)),\ \nu_M^{\otimes}(\Theta^{\text{type}}_{\text{pur}}(A,T)) \geq 1 - \zeta^{-1}\exp(\zeta A^{\zeta}) \tag{9.7}$$

for all $M \geq 1$ and such that the following holds for all $\blacklozenge \in \Theta^{\text{type}}_{\text{pur}}(A,T)$: If φ is of type $X^{\blacklozenge}$ or Y, the scalar multiple $T^{-7}A^{-1}\varphi$ is of type $w^{\blacklozenge}$.

Proof. Using a separability argument, we can define $\Theta^{\text{type}}_{\text{pur}}(A, T)$ through countably many bounds of the same form as in the definition of the type $w^{\textcolor{purple}{\blacklozenge}}$. We first note that, after adjusting ζ, we can replace A^{-1} in the conclusion by A^{-3}. Using Theorem 1.1, it suffices to prove that

$$\mathbb{P}(\textcolor{blue}{\bullet} + \textcolor{red}{\circ_{\mathbf{M}}} \in \Theta^{\text{type}}_{\text{pur}}(A, T)) \geq 1 - \zeta^{-1} \exp(\zeta A^{\zeta}).$$

Thus, we may restrict both $\textcolor{blue}{\bullet}$ and $\textcolor{red}{\circ_{\mathbf{M}}}$ to sets with acceptable probabilities under $\mathbb{P}$. After these preparations, we now start with the main part of the argument.

First, we let w be of type Y. Using Lemma 2.6, it follows that $T^{-4} A\varphi$ is of type $w^{\textcolor{blue}{\bullet}}$. Using Lemma 9.7, it follows that $T^{-6} A^{-2}\varphi$ is of type $w^{\textcolor{purple}{\blacklozenge}}$.

Now, let φ be of type $X^{\textcolor{purple}{\blacklozenge}}$. Using Lemma 9.7 and the first step in this proof, we can assume that φ is of type $X^{\textcolor{blue}{\bullet}}$. Using Lemma 2.6, $T^{-4} A^{-1}\varphi$ is of type $w^{\textcolor{blue}{\bullet}}$. Finally, using Lemma 9.7 again, we find that $T^{-6} A^{-2}\varphi$ is of type $w^{\textcolor{purple}{\blacklozenge}}$. ■

In Definition 2.13 above, we introduced the function $\mathfrak{X}$-norms, which are used to quantify structured perturbations of the initial data. We now prove the equivalence of the $\mathfrak{X}([0, T], \textcolor{purple}{\uparrow}; t_0, N, K)$- and $\mathfrak{X}([0, T], \textcolor{blue}{\uparrow}; t_0, N, K)$-norms, which is similar to the statements in Lemmas 9.7 and 9.8.

Lemma 9.9 (Equivalence of the blue and purple structured perturbations). *Let $A \geq 1$, let $\alpha > 0$ be a sufficiently large absolute constant, and let $\zeta = \zeta(\epsilon, s_1, s_2, \kappa, \eta, \eta', b_+, b) > 0$ be sufficiently small. Then there exist Borel sets $\Theta^{\text{sp}}_{\text{blue}}(A), \Theta^{\text{sp}}_{\text{red}}(A) \subseteq \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$ satisfying*

$$\mathbb{P}(\textcolor{blue}{\bullet} \in \Theta^{\text{sp}}_{\text{blue}}(A),\ \textcolor{red}{\circ_{\mathbf{M}}} \in \Theta^{\text{sp}}_{\text{red}}(A)) \geq 1 - \zeta^{-1} \exp(-\zeta A^{\zeta}). \tag{9.8}$$

and such that the following holds on this event:

For all $T \geq 1$, $t_0 \in [0, T]$, $N, K \geq 1$, and $Z[t_0] \in \mathscr{H}_x^{s_1}(\mathbb{T}^3)$, we have

$$\begin{aligned} &T^{-\alpha} A^{-1} \| Z[t_0] \|_{\mathfrak{X}([0,T],\textcolor{blue}{\uparrow};t_0,N,K)} \\ &\qquad \leq \| Z[t_0] \|_{\mathfrak{X}([0,T],\textcolor{purple}{\uparrow};t_0,N,K)} \leq T^{\alpha} A \| Z[t_0] \|_{\mathfrak{X}([0,T],\textcolor{blue}{\uparrow};t_0,N,K)}. \end{aligned} \tag{9.9}$$

Proof. It suffices to prove (9.9) for events $\Theta^{\text{sp}}_{\text{blue}}(A, T)$ and $\Theta^{\text{sp}}_{\text{red}}(A, T)$ satisfying the probabilistic estimate (9.8) as long as the lower bound in (9.8) does not depend on T. We can then simply take the intersection of $\Theta^{\text{sp}}_{\text{blue}}(T \cdot A, T)$ and $\Theta^{\text{sp}}_{\text{red}}(T \cdot A, T)$ over all integer times and increase α by 1.

After using Lemma 9.7 to compare the high×high interaction terms (involving $L_1 \sim L_2$), it remains to prove that

$$\begin{aligned} &\Big| \big\| (\neg \boxed{\circledless \,\&\, \circledless}) (:V * (P_{\leq N} \textcolor{purple}{\uparrow} \cdot P_{\leq N} Z^{\square}_N)\, P_{\leq N} \textcolor{purple}{\uparrow}:) \big\|_{\mathfrak{X}^{s_2-1, b_+-1}([0,T])} \\ &\qquad - \big\| (\neg \boxed{\circledless \,\&\, \circledless}) (:V * (P_{\leq N} \textcolor{blue}{\uparrow} \cdot P_{\leq N} Z^{\square}_N)\, P_{\leq N} \textcolor{blue}{\uparrow}:) \big\|_{\mathfrak{X}^{s_2-1, b_+-1}([0,T])} \Big| \\ &\qquad \lesssim T^{\alpha} A \Big(\| Z^{\square}[t_0] \|_{\mathscr{H}_x^{s_1}} + \sum_{L_1 \sim L_2} \| P_{L_1} \textcolor{blue}{\uparrow} \cdot P_{L_2} Z \|_{L_t^2 H_x^{-4\delta_1}([0,T] \times \mathbb{T}^3)} \Big) \end{aligned}$$

and

$$\Big| \| {:}V * (P_{\leq N} \overset{\blacklozenge}{\mid} \cdot P_{\leq N} Z_N^{\circ}) \, (\neg \textcircled{<}) \, P_{\leq N} \overset{\blacklozenge}{\mid}{:} \|_{\mathfrak{X}^{s_2-1,b_+-1}([0,T])}$$
$$- \| {:}V * (P_{\leq N} \overset{\bullet}{\mid} \cdot P_{\leq N} Z_N^{\circ}) \, (\neg \textcircled{<}) \, P_{\leq N} \overset{\bullet}{\mid}{:} \|_{\mathfrak{X}^{s_2-1,b_+-1}([0,T])} \Big|$$
$$\lesssim T^{\alpha} A \| Z^{\circ}[t_0] \|_{\mathscr{H}_x^{s_2}}.$$

Regarding the first estimate, we have

$$\Big| \Big\| \big(\neg \boxed{\textcircled{<} \, \& \, \textcircled{<}}\big) \big({:}V * (P_{\leq N} \overset{\blacklozenge}{\mid} \cdot P_{\leq N} Z_N^{\square}) \, P_{\leq N} \overset{\blacklozenge}{\mid}{:}\big) \Big\|_{\mathfrak{X}^{s_2-1,b_+-1}([0,T])}$$
$$- \Big\| \big(\neg \boxed{\textcircled{<} \, \& \, \textcircled{<}}\big) \big({:}V * (P_{\leq N} \overset{\bullet}{\mid} \cdot P_{\leq N} Z_N^{\square}) \, P_{\leq N} \overset{\bullet}{\mid}{:}\big) \Big\|_{\mathfrak{X}^{s_2-1,b_+-1}([0,T])} \Big|$$
$$\lesssim \Big\| \big(\neg \boxed{\textcircled{<} \, \& \, \textcircled{<}}\big) \big(V * (P_{\leq N} \overset{\circ}{\mid} \cdot P_{\leq N} Z_N^{\square}) \, P_{\leq N} \overset{\bullet}{\mid}\big) \Big\|_{\mathfrak{X}^{s_2-1,b_+-1}([0,T])} \tag{9.10}$$
$$+ \Big\| \big(\neg \boxed{\textcircled{<} \, \& \, \textcircled{<}}\big) \big(V * (P_{\leq N} \overset{\bullet}{\mid} \cdot P_{\leq N} Z_N^{\square}) \, P_{\leq N} \overset{\circ}{\mid}\big) \Big\|_{\mathfrak{X}^{s_2-1,b_+-1}([0,T])} \tag{9.11}$$
$$+ \Big\| \big(\neg \boxed{\textcircled{<} \, \& \, \textcircled{<}}\big) \big(V * (P_{\leq N} \overset{\circ}{\mid} \cdot P_{\leq N} Z_N^{\square}) \, P_{\leq N} \overset{\circ}{\mid}\big) \Big\|_{\mathfrak{X}^{s_2-1,b_+-1}([0,T])}. \tag{9.12}$$

We can then control

- (9.10) through Proposition 8.6,
- (9.11) through Proposition 8.7 and Lemma 8.8,
- (9.11) through Proposition 8.10.

The proof of the second estimate is similar, except that we use Corollary 8.9 instead of Lemma 8.8.

■

9.3. *Multi-linear master estimate for Gibbsian initial data*

In this subsection, we prove a version of the multi-linear master estimate for Gaussian data (Proposition 2.8) for the purple types (Definition 9.6) instead of the blue types (Definition 2.4). Since we will only need this estimate in Propositions 3.5 and 3.7, which do not involve contraction or continuity arguments, we can be less precise than in the multi-linear master estimate for Gaussian data and simply capture the size of the forcing term in the following norm.

Definition 9.10. Let $N \geq 1$, let $\mathcal{J} \subseteq \mathbb{R}$ be a compact interval, and let $R, \varphi \colon \mathcal{J} \times \mathbb{T}^3 \to \mathbb{R}$. Then we define

$$\|R\|_{\mathcal{NL}_N(J,\varphi)} \overset{\text{def}}{=} \inf \{ \|H\|_{\mathcal{LM}(\mathcal{J})} + \|F\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})} \colon$$
$$R = P_{\leq N} \, \mathrm{PCtrl}[H, P_{\leq N} \varphi] + F \text{ on } \mathcal{J} \times \mathbb{T}^3 \}.$$

Remark 9.11 (Drawback of $\boxed{\textcircled{<} \, \& \, \textcircled{<}}$). As mentioned above, the $\mathcal{NL}_N(J,\varphi)$-norm is less precise than our estimates in Section 2.1, since it does not give an explicit description of the low-frequency modulation H. This allows us to circumvent a technical problem which the author was unable to resolve. In Proposition 5.7, we proved that

$$\Big(\neg \boxed{\circledless \,\&\, \circledless}\Big)\Big(:V * \Big(P_{\leq N}\uparrow \cdot P_{\leq N}\,\text{tree}_N\Big)P_{\leq N}\uparrow:\Big)$$

lives in $\mathfrak{X}^{s_2-1,b_+-1}$. One may therefore expect that

$$\Big(\neg \boxed{\circledless \,\&\, \circledless}\Big)\Big(:V * \Big(P_{\leq N}\uparrow^{\blacklozenge} \cdot P_{\leq N}\,\text{tree}^{\blacklozenge}_N\Big)P_{\leq N}\uparrow^{\blacklozenge}:\Big)$$

also lives in $\mathfrak{X}^{s_2-1,b_+-1}$. However, after using Proposition 9.1, we would need an estimate for

$$\Big(\neg \boxed{\circledless \,\&\, \circledless}\Big)(:V * (P_{\leq N}\uparrow \cdot P_{\leq N} Y_N)P_{\leq N}\uparrow:)$$

in $\mathfrak{X}^{s_2-1,b_+-1}$. Unfortunately, this is not covered by Proposition 6.3. In fact, without any additional assumptions on Y_N other than bounds in $\mathfrak{X}^{s_2,b}$, the high$\times$high$\to$low interactions in $P_{\leq N}\uparrow \cdot P_{\leq N} Y_N$ rule out this estimate.

Equipped with the $\mathcal{NL}$-norm, we now turn to the master estimate for Gibbsian initial data.

Proposition 9.12 (Multi-linear master estimate for Gibbsian initial data). *Let $A, T \geq 1$, let $\alpha > 0$ be a sufficiently large absolute constant, and let $\zeta = \zeta(\epsilon, s_1, s_2, \kappa, \eta, \eta', b_+, b) > 0$ be sufficiently small. Then there exists a Borel set $\Theta^{\mathrm{ms}}_{\mathrm{pur}}(A,T) \subseteq \mathscr{H}_x^{-1/2-\kappa}$ satisfying*

$$\mu_M^{\otimes}(\blacklozenge \in \Theta^{\mathrm{ms}}_{\mathrm{pur}}(A,T)) \geq 1 - \zeta^{-1}\exp(-\zeta A^{\zeta}) \tag{9.13}$$

for all $M \geq 1$ and such that the following estimates hold for all $\blacklozenge \in \Theta^{\mathrm{ms}}_{\mathrm{pur}}(A,T)$:

Let $\mathcal{J} \subseteq [0,T]$ be an interval and let $N \geq 1$. Let $\varphi_1, \varphi_2, \varphi_3 \colon \mathcal{J} \times \mathbb{T}^3 \to \mathbb{R}$ be as in Definition 9.6 and let

$$(\varphi_1, \varphi_2; \varphi_3) \overset{\text{type}}{\neq} (\uparrow^{\blacklozenge}, \uparrow^{\blacklozenge}; \uparrow^{\blacklozenge}),\ (\uparrow^{\blacklozenge}, w^{\blacklozenge}; \uparrow^{\blacklozenge}).$$

(i) *If $\varphi_3 \overset{\text{type}}{=} \uparrow^{\blacklozenge}$, then*

$$\big\| P_{\leq N}\big(:V * (P_{\leq N}\varphi_1 \cdot P_{\leq N}\varphi_2)P_{\leq N}\varphi_3:\big)\big\|_{\mathcal{NL}_N(\mathcal{J}, P_{\leq N}\uparrow^{\blacklozenge})} \leq T^{\alpha} A.$$

(ii) *In all other cases,*

$$\|:V * (P_{\leq N}\varphi_1 \cdot P_{\leq N}\varphi_2)P_{\leq N}\varphi_3:\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})} \leq T^{\alpha} A.$$

Proof. While the proof requires no new ingredients, it relies on several earlier results. For the advantage of the reader, we break up the proof into several steps.

Step 1: Definition of $\Theta^{\mathrm{ms}}_{\mathrm{pur}}(A,T)$ and its Borel-measurability. Using the definition of the time-restricted norms, we see that the statement for all intervals $\mathcal{J} \subseteq [0,T]$ is equivalent to the statement for only $\mathcal{J} = [0,T]$. Thus, we may simply choose $\Theta^{\mathrm{ms}}_{\mathrm{pur}}(A,T)$ as the set where (i) and (ii) hold for all $N \geq 1$. To see that this leads to a Borel-measurable set, we note that both $\mathcal{LM}([0,T])$ and $\mathcal{X}^{s_2,b}([0,T])$ are separable. For a fixed $N \geq 1$, we also find that the functions

$$(\varphi_1, \varphi_2, \varphi_3) \mapsto \big\| P_{\leq N}\big(:V * (P_{\leq N}\varphi_1 \cdot P_{\leq N}\varphi_2) P_{\leq N}\varphi_3:\big) \big\|_{\mathcal{NL}_N(\mathcal{J}, P_{\leq N} \overset{\color{purple}\blacklozenge}{|})}$$

and

$$(\varphi_1, \varphi_2, \varphi_3) \mapsto \|:V * (P_{\leq N}\varphi_1 \cdot P_{\leq N}\varphi_2) P_{\leq N}\varphi_3:\|_{\mathcal{X}^{s_2-1,b_+-1}(\mathcal{J})}$$

are continuous with respect to the $C_t^0 H_x^{-1/2-\kappa}([0,T] \times \mathbb{T}^3)$-norm. Thus, we can represent $\Theta^{\mathrm{ms}}_{\mathrm{pur}}(A,T)$ through countably many constraints of the same form as in (i) and (ii), and hence as a countable intersection of closed sets. In particular, $\Theta^{\mathrm{ms}}_{\mathrm{pur}}(A,T)$ is Borel-measurable.

Step 2: Reductions. It therefore remains to show the probabilistic estimate (9.13). Using the absolute continuity and representation of the reference measures from Theorem 1.1, it suffices to prove that

$$\mathbb{P}\big({\color{blue}\bullet} + {\color{red}\circ_{\mathbf{M}}} \in \Theta^{\mathrm{ms}}_{\mathrm{pur}}(A,T)\big) \geq 1 - \zeta^{-1}\exp(-\zeta A^{\zeta})$$

for all $M \geq 1$. Furthermore, we can replace the upper bound $T^\alpha A$ in (i) and (ii) by $CT^\alpha A^C$, where $C = C(\epsilon, s_1, s_2, \kappa, \eta, \eta', b_+, b) \geq 1$. After the estimate has been proven, this can be repaired by adjusting A and ζ. Using Lemma 2.6, Proposition 2.8, Corollary 9.3, Lemma 9.7, and Lemma 9.8, we may restrict to the event

$$\begin{aligned}\{{\color{blue}\bullet} \in \Theta^{\mathrm{ms}}_{\mathrm{blue}}(A,T) \cap \Theta^{\mathrm{type}}_{\mathrm{blue}}(A,T) \cap \Theta^{\mathrm{cub}}_{\mathrm{blue}}(A,T)\} \cap \{{\color{red}\circ_{\mathbf{M}}} \in \Theta^{\mathrm{type}}_{\mathrm{red}}(A,T) \cap \Theta^{\mathrm{cub}}_{\mathrm{red}}(A,T)\}& \\ \cap \{{\color{blue}\bullet} + {\color{red}\circ_{\mathbf{M}}} \in \Theta^{\mathrm{type}}_{\mathrm{pur}}(A,T)\}&.\end{aligned}$$

Step 3: Multi-linear estimates. The estimates for $\varphi_3 \overset{\mathrm{type}}{\neq} \overset{\color{purple}\blacklozenge}{|}$ follow directly from the multi-linear master estimate for ${\color{blue}\bullet}$ and the equivalence of the types in Corollary 9.3 and Lemmas 9.7 and 9.8. It then remains to treat the case $\varphi_3 \overset{\mathrm{type}}{=} \overset{\color{purple}\blacklozenge}{|}$. We further separate the proof of the estimates into two cases.

Step 3.1: $\varphi_1, \varphi_2 \overset{\mathrm{type}}{\neq} \overset{\color{purple}\blacklozenge}{|}$. We first remind the reader that in this case the nonlinearity does not require a renormalization. We then decompose

$$\begin{aligned}&P_{\leq N}\big(V * (P_{\leq N}\varphi_1 \cdot P_{\leq N}\varphi_2) P_{\leq N} \overset{\color{purple}\blacklozenge}{|}\big)\\ &\qquad = P_{\leq N}\big(V * (P_{\leq N}\varphi_1 \cdot P_{\leq N}\varphi_2) \circledless P_{\leq N} \overset{\color{purple}\blacklozenge}{|}\big)\\ &\qquad\quad + P_{\leq N}\big(V * (P_{\leq N}\varphi_1 \cdot P_{\leq N}\varphi_2)\, (\neg \circledless)\, P_{\leq N} \overset{\color{blue}\bullet}{|}\big)\\ &\qquad\quad + P_{\leq N}\big(V * (P_{\leq N}\varphi_1 \cdot P_{\leq N}\varphi_2)\, (\neg \circledless)\, P_{\leq N} \overset{\color{red}\circ}{|}\big).\end{aligned}$$

Using Lemma 7.6, the first term is of the form $P_{\leq N}\,\mathrm{PCtrl}(H_N, P_{\leq N} \overset{\color{purple}\blacklozenge}{|})$ with $\|H_N\|_{\mathcal{LM}([0,T])} \lesssim T^\alpha A^2$. The second and third terms can be controlled through the multi-linear master estimate for Gaussian random data.

Step 3.2: $\varphi_1, \varphi_3 \overset{\text{type}}{=} \purplelinear$, $\varphi_2 \overset{\text{type}}{\neq} \purplelinear$. Using the equivalence of types (as in Corollary 9.3 and Lemma 9.7) together with the previous cases, it suffices to treat

$$\varphi_1, \varphi_3 \overset{\text{type}}{=} \bluelinear, \quad \varphi_2 \overset{\text{type}}{=} \bluecubic, \ X^{\bullet}, \ Y.$$

We decompose the nonlinearity

$$V * (P_{\leq N} \bluelinear \cdot P_{\leq N} \varphi_2) P_{\leq N} \bluelinear$$

using $\boxed{\circledless \,\&\, \circledless}$ if $\varphi_2 \overset{\text{type}}{=} \bluecubic, \ X^{\bullet}$ and using $\circledless$ if $\varphi_2 \overset{\text{type}}{=} Y$. Then the bound follows from the multi-linear master estimate for Gaussian initial data and Lemmas 7.4 and 7.6. ■

In Definition 2.13, we also introduced a structured perturbation of the initial data, which we briefly examined in Lemma 9.9 above. While the multi-linear estimate does not apply to the type $(\purplelinear, w^{\blacklozenge}; \purplelinear)$, we now obtain a multi-linear estimate if the second argument is a linear evolution with initial data as in Definition 2.13. Since the definition has been tailored towards this estimate, the proof will be easy and short.

Lemma 9.13 (Multi-linear estimate for the structured perturbation). *Let $A, T \geq 1$, let $\alpha > 0$ be a sufficiently large absolute constant, and let $\zeta = \zeta(\epsilon, s_1, s_2, \kappa, \eta, \eta', b_+, b) > 0$ be sufficiently small. Then there exists a Borel set $\Theta^{\mathrm{sp}}_{\mathrm{pur}}(A,T) \subseteq \mathscr{H}_x^{-1/2-\kappa}$ satisfying*

$$\mu_M^{\otimes}(\blacklozenge \in \Theta^{\mathrm{sp}}_{\mathrm{pur}}(A,T)) \geq 1 - \zeta^{-1} \exp(-\zeta A^{\zeta}) \tag{9.14}$$

for all $M \geq 1$ and such that the following estimates hold for all $\blacklozenge \in \Theta^{\mathrm{sp}}_{\mathrm{pur}}(A,T)$:

Let $N, K \geq 1$, let $t_0 \in [0,T]$, let $Z[t_0] \in \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$, and let $Z(t)$ be the corresponding solution to the linear wave equation. Then

$$\| P_{\leq N}[:V * (P_{\leq N} \purplelinear \cdot P_{\leq N} Z) P_{\leq N} \purplelinear :] \|_{\mathcal{N}\mathcal{L}_N([0,T], P_{\leq N} \purplelinear)} \leq T^{\alpha} A \| Z[t_0] \|_{\mathfrak{X}([0,T], \purplelinear; t_0, N, K)}.$$

Proof. Let $Z^{\square}[t_0]$ and $Z^{\circ}[t_0]$ be as in Definition 2.13. Then we can decompose

$$\begin{aligned} P_{\leq N}[:V * (P_{\leq N} \purplelinear \cdot P_{\leq N} Z) P_{\leq N} \purplelinear :] &= \boxed{\circledless \,\&\, \circledless}\big(P_{\leq N}[V * (P_{\leq N} \purplelinear \cdot P_{\leq N} Z^{\square}) P_{\leq N} \purplelinear]\big) \\ &+ \big(\neg \boxed{\circledless \,\&\, \circledless}\big)\big(P_{\leq N}[:V * (P_{\leq N} \purplelinear \cdot P_{\leq N} Z^{\square}) P_{\leq N} \purplelinear :]\big) \\ &+ P_{\leq N}[:V * (P_{\leq N} \purplelinear \cdot P_{\leq N} Z^{\circ}) \circledless P_{\leq N} \purplelinear :] \\ &+ P_{\leq N}[V * (P_{\leq N} \purplelinear \cdot P_{\leq N} Z^{\circ}) \, (\neg \circledless) \, P_{\leq N} \purplelinear]. \end{aligned}$$

The estimate then follows directly from Definition 2.13 and Lemmas 7.4 and 7.6. ■

Appendix A. Proofs of counting estimates

A.1. Cubic counting estimate

We start with the proof of the cubic counting estimate.

Proof of Proposition 4.18. We separately prove the four counting estimates (i)–(iv).

Proof of (i)*:* By symmetry, we can assume that $N_1 \geq N_2 \geq N_3$. Using the basic counting estimate to handle the sum over $n_2 \in \mathbb{Z}^3$, we obtain

$$\begin{aligned}\#\{(n_1,n_2,n_3)\colon |n_1|\sim N_1,\ |n_2|\sim N_2,\ |n_3|\sim N_3,\ |\varphi-m|\leq 1\}\\ \lesssim \sum_{n_1,n_3\in\mathbb{Z}^3}\Big(\prod_{j=1,3} 1\{|n_j|\sim N_j\}\Big)\min(\langle n_{13}\rangle, N_2)^{-1}N_2^3\\ \lesssim N_1^2N_2^3N_3^3+N_1^3N_2^2N_3^3\lesssim N_2^{-1}(N_1N_2N_3)^3,\end{aligned}$$

which is acceptable.

Proof of (ii)*:* We emphasize that n_{123} is viewed as a free variable. In the variables (n_{123}, n_1, n_2), the phase takes the form

$$\varphi = \pm_{123}\langle n_{123}\rangle \pm_1 \langle n_1\rangle \pm_2 \langle n_2\rangle \pm_3 \langle n_{123}-n_1-n_2\rangle.$$

After changing $(n_1, n_2) \to (-n_1, -n_2)$, we obtain the same form as in (i) and hence the desired estimate.

Proof of (iii)*:* In the variables (n_{123}, n_{12}, n_1), the phase takes the form

$$\varphi = \pm_{123}\langle n_{123}\rangle \pm_1 \langle n_1\rangle \pm_2 \langle n_{12}-n_1\rangle \pm_3 \langle n_{123}-n_{12}\rangle.$$

By first summing over n_1 and using the basic counting lemma, we gain a factor of $\min(N_1, N_{12})$. Alternatively, by first summing over n_{123} and using the basic counting lemma, we gain a factor of $\min(N_{123}, N_{12})$. By combining both estimates, we gain a factor of

$$\max\big(\min(N_1,N_{12}),\min(N_{123},N_{12})\big)=\min\big(N_{12},\max(N_{123},N_1)\big).$$

While not part of the proof, we also remark that

$$|\langle n_{123}\rangle+\langle n_1\rangle-\langle n_{12}-n_1\rangle-\langle n_{123}-n_{12}\rangle|\lesssim N_{12}.$$

This shows that we cannot gain a factor of the form $\mathrm{med}(N_{123}, N_{12}, N_1)$.

Proof of (iv)*:* In the variables (n_{12}, n_1, n_3), the phase takes the form

$$\varphi = \pm_{123}\langle n_{12}+n_3\rangle \pm_1 \langle n_1\rangle \pm_2 \langle n_{12}-n_1\rangle \pm_3 \langle n_3\rangle.$$

By first summing over n_1 and using the basic counting lemma, we gain a factor of $\min(N_{12}, N_1)$. Alternatively, by first summing over n_3 and using the basic counting lemma, we gain a factor of $\min(N_{12}, N_3)$. Combining both estimates completes the argument. The same obstruction as described in (iii) shows that the estimate is sharp. ∎

We now use the cubic counting estimate to prove the cubic sum estimate.

Proof of Proposition 4.20. Due to the symmetry $n_1 \leftrightarrow n_2$, we may assume that $N_1 \geq N_2$. To simplify the notation, we set

$$\begin{aligned}&\mathcal{C}(m) = \mathcal{C}(N_1, N_2, N_3, N_{12}, N_{123}, m)\\ &= \{(n_1,n_2,n_3) \in (\mathbb{Z}^3)^3 : |n_j| \sim N_j,\ 1 \leq j \leq 3,\ |n_{12}| \sim N_{12},\ |n_{123}| \sim N_{123},\ |\varphi - m| \leq 1\}.\end{aligned}$$

We then have

$$\begin{aligned}\sum_{n_1,n_2,n_3\in\mathbb{Z}^3} &\Big[\Big(\prod_{j=1}^{3} \chi_{N_j}(n_j)\Big)\langle n_{123}\rangle^{2(s-1)}\langle n_{12}\rangle^{-2\gamma}\Big(\prod_{j=1}^{3}\langle n_j\rangle^{-2}\Big)1\{|\varphi - m| \leq 1\}\Big]\\ &\lesssim \sum_{N_{123},N_{12}} N_{123}^{2(s-1)} N_{12}^{-2\gamma}\Big(\prod_{j=1}^{3} N_j^{-2}\Big)\#\mathcal{C}(m). \qquad \text{(A.1)}\end{aligned}$$

To obtain the optimal estimate, we unfortunately need to distinguish five cases, which we listed in Figure 5. Cases 1 and 2 distinguish between the high×high and high×low interactions in the first two factors. This distinction is necessary to utilize the gain in N_{12}. The subcases mostly deal with the relation between N_{12} and N_3, which is important to use the gain in N_{123}.

Case	$N_1 \leftrightarrow N_2$	$N_1 \leftrightarrow N_3$	$N_3 \leftrightarrow N_{12}$	Basic counting estimate
1.a	$N_1 \sim N_2$	$N_1 \ll N_3$		(iv)
1.b.i	$N_1 \sim N_2$	$N_1 \gtrsim N_3$	$N_3 \ll N_{12}$	(iv)
1.b.ii	$N_1 \sim N_2$	$N_1 \gtrsim N_3$	$N_3 \gtrsim N_{12}$	(iii)
2.a	$N_1 \gg N_2$	$N_1 \not\sim N_3$		(i)
2.b	$N_1 \gg N_2$	$N_1 \sim N_3$		(ii)

Fig. 5. Case distinction in the proof of Proposition 4.20.

Case 1.a: $N_1 \sim N_2$, $N_1 \ll N_3$. In this case, $N_{123} \sim N_3$. Using Proposition 4.18 (iv), the contribution is bounded by

$$\sum_{\substack{N_{12}:\\ N_{12}\lesssim N_1}} N_{12}^{-2\gamma} N_1^{-4} N_3^{2s-4}\#\mathcal{C}(m) \lesssim \sum_{\substack{N_{12}:\\ N_{12}\lesssim N_1}} N_{12}^{2-2\gamma} N_1^{-1} N_3^{2s-1} \lesssim N_1^{1-2\gamma} N_3^{2s-1},$$

which is acceptable. In estimating the sum, we have used the fact that $\gamma < 1$.

Case 1.b.i: $N_1 \sim N_2$, $N_1 \gtrsim N_3$, $N_3 \ll N_{12}$. In this case, $N_{123} \sim N_{12}$. Using Proposition 4.18 (iv), the contribution is bounded by

$$\begin{aligned}\sum_{\substack{N_{12}:\\ N_3\ll N_{12}\lesssim N_1}} N_{12}^{2s-2-2\gamma} N_1^{-4} N_3^{-2}\#\mathcal{C}(m) &\lesssim \sum_{\substack{N_{12}:\\ N_3\ll N_{12}\lesssim N_1}} N_{12}^{2s-2\gamma} N_1^{-1} N_3\\ &\lesssim \sum_{\substack{N_{12}:\\ N_{12}\lesssim N_1}} N_{12}^{2s-2\gamma+1} N_1^{-1} \lesssim N_1^{2(s-\gamma)},\end{aligned}$$

which is acceptable. In estimating the sum, we have used the fact that $\gamma < s + 1/2$.

Case 1.b.i: $N_1 \sim N_2$, $N_1 \gtrsim N_3$, $N_3 \gtrsim N_{12}$. We note that $N_{123} \lesssim \max(N_{12}, N_3) \lesssim N_3$. Using Proposition 4.18 (iii), the contribution is bounded by

$$\begin{aligned}
&\sum_{\substack{N_{12},N_{123}:\\ N_{12},N_{123}\lesssim N_3}} N_{123}^{2s-2} N_{12}^{-2\gamma} N_1^{-4} N_3^{-2} \#\mathscr{C}(m) \\
&\qquad\lesssim \sum_{\substack{N_{12},N_{123}:\\ N_{12},N_{123}\lesssim N_3}} \min(N_{123}, N_{12})^{-1} N_{123}^{2s+1} N_{12}^{3-2\gamma} N_1^{-1} N_3^{-2} \\
&\qquad\lesssim N_1^{-1} N_3^{2s-2\gamma+1} \lesssim N_1^{2(s-\gamma)},
\end{aligned}$$

which is acceptable. In the last inequality, we have used $\gamma < s + 1/2$ again.

Case 2.a: $N_1 \gg N_2$, $N_1 \nsim N_3$. In this case, $N_{12} \sim N_1$ and $N_{123} \sim \max(N_1, N_3)$. Using Proposition 4.18 (i), the contribution is bounded by

$$\begin{aligned}
&\max(N_1, N_3)^{2s-2} N_1^{-2-2\gamma} N_2^{-2} N_3^{-2} \#\mathscr{C}(m) \\
&\quad\lesssim \max(N_1, N_3)^{2s-2} \min(N_1, N_3)^{-1} N_1^{1-2\gamma} N_2 N_3 \\
&\quad\lesssim \max(N_1, N_3)^{2s-2} \min(N_1, N_3)^{-1} N_1^{2-2\gamma} N_3 = \max(N_1, N_3)^{2s-1} N_1^{1-2\gamma}.
\end{aligned}$$

The restriction $s \leq 1/2$ is not strictly necessary for the statement of the proposition, but ensures that the first factor does not grow in N_1 or N_3, which is essential in applications.

Case 2.a: $N_1 \gg N_2$, $N_1 \sim N_3$. In this case, $N_{12} \sim N_1$. Using Proposition 4.18 (ii), the contribution is bounded by

$$\sum_{\substack{N_{123}:\\ N_{123}\lesssim N_1}} N_{123}^{2s-2} N_1^{-4-2\gamma} N_2^{-2} \#\mathscr{C}(m) \lesssim \sum_{\substack{N_{123}:\\ N_{123}\lesssim N_1}} N_{123}^{2s} N_1^{-1-2\gamma} N_2 \lesssim N_1^{2s-2\gamma},$$

which is acceptable. In estimating the sum, we have used the fact that $s > 0$. ∎

A.2. *Cubic sup-counting estimates*

Proof of Lemma 4.22. We prove the four estimates separately.

Proof of (i)*:* By symmetry, we can assume without loss of generality that $N_1 \geq N_2 \geq N_3$. Using the basic counting estimate in $n_2 \in \mathbb{Z}^3$, we have

$$\begin{aligned}
&\#\{(n_1, n_2, n_3) \colon |n_1| \sim N_1,\ |n_2| \sim N_2,\ |n_3| \sim N_3,\ n = n_{123},\ |\varphi - m| \leq 1\} \\
&\quad\lesssim \#\{(n_2, n_3) \colon |n_2| \sim N_2,\ |n_3| \sim N_3,\ |\pm_{123} \langle n \rangle \pm_1 \langle n - n_{23} \rangle \pm_2 \langle n_2 \rangle \pm_3 \langle n_3 \rangle - m| \leq 1\} \\
&\quad\lesssim \sum_{n_3 \in \mathbb{Z}^3} 1\{|n_3| \sim N_3\} \min(\langle n - n_3 \rangle, N_2)^{-1} N_2^3 \lesssim N_2^3 N_3^2.
\end{aligned}$$

Proof of (ii)*:* The proof is essentially the same as the proof of (i) and we omit the details.

Proof of (iii)*:* Using the basic counting estimate in $n_2 \in \mathbb{Z}^3$, we have

$$
\begin{aligned}
&\#\{(n_{12}, n_2, n_3) \colon |n_{12}| \sim N_{12},\ |n_2| \sim N_2,\ |n_3| \sim N_3,\ n = n_{123},\ |\varphi - m| \leq 1\} \\
&\lesssim \#\{(n_{12}, n_2) \colon |n_{12}| \sim N_{12}, |n_2| \sim N_2, \\
&\qquad\qquad |\pm_{123} \langle n \rangle \pm_1 \langle n_{12} - n_2 \rangle \pm_2 \langle n_2 \rangle \pm_3 \langle n - n_{12} \rangle - m| \leq 1\} \\
&\lesssim \sum_{n_{12} \in \mathbb{Z}^3} 1\{|n_{12}| \sim N_{12}\} \min(N_{12}, N_2)^{-1} N_2^3 \lesssim \min(N_{12}, N_2)^{-1} N_{12}^3 N_2^3.
\end{aligned}
$$

Proof of (iv)*:* The proof is essentially the same as the proof of (iii) and we omit the details. ■

A.3. *Paracontrolled cubic counting estimates*

Proof of Lemma 4.23. To simplify the notation, we set $N_{\max} = \max(N_1, N_2, N_3)$. For $0 < \gamma < \beta$, we have

$$
\langle n_{12} \rangle^{-2\beta} \lesssim \langle n_{12} \rangle^{-2\gamma} \lesssim \langle n_1 \rangle^{-2\gamma} \langle n_2 \rangle^{2\gamma}.
$$

Together with Lemma 4.22 (ii), this yields

$$
\begin{aligned}
&\sum_{n_1, n_3 \in \mathbb{Z}^3} \Big(\prod_{j=1,3} 1\{|n_j| \sim N_j\} \Big) \langle n_{123} \rangle^{2(s_2-1)} \langle n_{12} \rangle^{-2\beta} \langle n_1 \rangle^{-2} \langle n_3 \rangle^{-2}\, 1\{|\varphi - m| \leq 1\} \\
&\quad\lesssim N_1^{-2-2\gamma} N_2^{2\gamma} N_3^{-2} \\
&\qquad \times \sum_{N_{123}} N_{123}^{2(s_2-1)} \#\{(n_1, n_3) \colon |n_{123}| \sim N_{123}, |n_1| \sim N_1, |n_3| \sim N_3, |\varphi - m| \leq 1\} \\
&\quad\lesssim N_1^{-2-2\gamma} N_2^{2\gamma} N_3^{-2} \sum_{\substack{N_{123}: \\ |N_{123}| \lesssim N_{\max}}} N_{123}^{2(s_2-1)} \operatorname{med}(N_{123}, N_1, N_3)^3 \min(N_{123}, N_1, N_3)^2.
\end{aligned}
$$

Since $\operatorname{med}(N_{123}, N_1, N_3)^3 \min(N_{123}, N_1, N_3)^2 \lesssim N_{123} N_1^2 N_3^2$, we obtain

$$
\begin{aligned}
&N_1^{-2-2\gamma} N_2^{2\gamma} N_3^{-2} \sum_{\substack{N_{123}: \\ |N_{123}| \lesssim N_{\max}}} N_{123}^{2(s_2-1)} \operatorname{med}(N_{123}, N_1, N_3)^3 \min(N_{123}, N_1, N_3)^2 \\
&\qquad\qquad \lesssim N_1^{-2\gamma} N_2^{2\gamma} \sum_{\substack{N_{123}: \\ |N_{123}| \lesssim N_{\max}}} N_{123}^{2s_2-1} \lesssim N_{\max}^{2\delta_2} N_1^{-2\gamma} N_2^{2\gamma}. \quad ■
\end{aligned}
$$

A.4. *Quartic counting estimate*

Proof of Lemma 4.24. Using the upper bound on s, we can first sum over $n_4 \in \mathbb{Z}^3$ to obtain

$$\sum_{n_1,n_2,n_3,n_4\in\mathbb{Z}^3} \Big(\prod_{j=1}^4 1\{|n_j|\sim N_j\}\Big)\langle n_{1234}\rangle^{2s}\langle n_{123}\rangle^{-2} \times |\widehat{V}_S(n_1,n_2,n_3)|^2\Big(\prod_{j=1}^4 \langle n_j\rangle^{-2}\Big)1\{|\varphi-m|\leq 1\}$$
$$\lesssim N_4^{-2\eta} \sum_{n_1,n_2,n_3\in\mathbb{Z}^3} \Big(\prod_{j=1}^3 1\{|n_j|\sim N_j\}\Big)\langle n_{123}\rangle^{-2} \times |\widehat{V}_S(n_1,n_2,n_3)|^2\Big(\prod_{j=1}^3 \langle n_j\rangle^{-2}\Big)1\{|\varphi-m|\leq 1\}.$$

The remaining sum over n_1, n_2, and n_3 can then be estimated using Proposition 4.20, which yields the desired estimate. ■

Having proved the nonresonant quartic sum estimate (Lemma 4.24), we now turn to the resonant quartic sum estimate. We begin with the basic resonance estimate (Lemma 4.25), which forms the main part of the proof.

Proof of Lemma 4.25. Since $n_1, n_2 \in \mathbb{Z}^3$ are fixed and the phase φ is globally Lipschitz, there are at most $\sim N_1$ nontrivial choices of $m \in \mathbb{Z}$. Due to the log-factor in (4.46), it suffices to prove

$$\sup_{m\in\mathbb{Z}} \sum_{n_3\in\mathbb{Z}^3} 1\{|n_3|\sim N_3\}\langle n_{123}\rangle^{-1}\langle n_3\rangle^{-2}1\{|\varphi-m|\leq 1\} \lesssim \langle n_{12}\rangle^{-1}.$$

By inserting an additional dyadic localization, we obtain

$$\sum_{n_3\in\mathbb{Z}^3} 1\{|n_3|\sim N_3\}\langle n_{123}\rangle^{-1}\langle n_3\rangle^{-2}1\{|\varphi-m|\leq 1\}$$
$$\leq N_3^{-2} \sum_{N_{123}\geq 1} N_{123}^{-1} \sum_{n_3\in\mathbb{Z}^3} 1\{|n_{123}|\sim N_{123}\}1\{|n_3|\sim N_3\}1\{|\varphi-m|\leq 1\}. \tag{A.2}$$

To simplify the notation, we write N_{12} for the dyadic scale of $n_{12}\in\mathbb{Z}^3$. Using Lemma 4.17, we have

$$N_{123}^{-1}N_3^{-2} \sum_{n_3\in\mathbb{Z}^3} 1\{|n_{123}|\sim N_{123}\}1\{|n_3|\sim N_3\}1\{|\varphi-m|\leq 1\}$$
$$\lesssim N_{123}^{-1}N_3^{-2}\min(N_{123},N_{12},N_3)^{-1}\min(N_{123},N_3)^3.$$

We now separate the contributions of the three cases $N_{123}\ll N_3$, $N_{123}\sim N_3$, $N_{123}\gg N_3$. In the following, we implicitly restrict the sum over N_{123} to values which are consistent with $|n_{123}|\sim N_{123}$, $|n_{12}|\sim N_{12}$, and $|n_3|\sim N_3$ for some $n_1,n_2,n_3\in\mathbb{Z}^3$.

If $N_{123}\ll N_3$, then $N_{12}\sim N_3$. Thus,

$$\sum_{N_{123}\ll N_3} N_{123}^{-1}N_3^{-2}\min(N_{123},N_{12},N_3)^{-1}\min(N_{123},N_3)^3$$
$$\lesssim 1\{N_{12}\sim N_3\} \sum_{N_{123}\ll N_3} N_{123}N_3^{-2} \lesssim N_{12}^{-1}.$$

If $N_{123} \sim N_3$, then $N_{12} \lesssim N_{123} \sim N_3$. Thus,

$$\sum_{N_{123}\sim N_3} N_{123}^{-1} N_3^{-2} \min(N_{123}, N_{12}, N_3)^{-1} \min(N_{123}, N_3)^3 \sim N_{12}^{-1}.$$

Finally, if $N_{123} \gg N_3$, then $N_{123} \sim N_{12} \gg N_3$. Thus,

$$\sum_{N_{123}\gg N_3} N_{123}^{-1} N_3^{-2} \min(N_{123}, N_{12}, N_3)^{-1} \min(N_{123}, N_3)^3 = N_{12}^{-1} N_3^{-2} N_3^{-1} N_3^3 \sim N_{12}^{-1}.$$

This completes the proof. ∎

The resonant quartic sum estimate (Lemma 4.26) is now an easy consequence of the basic resonance estimate (Lemma 4.25).

Proof of Lemma 4.26. Using Lemma 4.25, we have

$$\begin{aligned}
&\sum_{n_1,n_2\in\mathbb{Z}^3} \Bigg[\Big(\prod_{j=1}^{2} 1\{|n_j| \sim N_j\}\Big) \langle n_{12}\rangle^{2s} \langle n_1\rangle^{-2} \langle n_2\rangle^{-2} \\
&\qquad\qquad \times \Big(\sum_{m\in\mathbb{Z}} \sum_{n_3\in\mathbb{Z}^3} \langle m\rangle^{-1} 1\{|n_3| \sim N_3\} \langle n_{123}\rangle^{-1} \langle n_3\rangle^{-2} 1\{|\varphi - m| \leq 1\}\Big)^2\Bigg] \\
&\qquad \lesssim \log(2+N_3)^2 \sum_{n_1,n_2\in\mathbb{Z}^3} \Bigg[\Big(\prod_{j=1}^{2} 1\{|n_j| \sim N_j\}\Big) \langle n_{12}\rangle^{2s-2} \langle n_1\rangle^{-2} \langle n_2\rangle^{-2}\Bigg] \\
&\qquad \lesssim \log(2+N_3)^2 \max(N_1, N_2)^{2s}.
\end{aligned}$$

∎

A.5. *Quintic counting estimates*

Before we turn to the proof of the nonresonant quintic counting estimate, we isolate a helpful auxiliary lemma.

Lemma A.1 (Frequency scale estimate). *Let $N_1, N_2, N_{1345}, N_{12345}$ be frequency scales which can be achieved by frequencies $n_1, \dots, n_5 \in \mathbb{Z}^3$, i.e., satisfying*

$$1\{|n_1| \sim N_1\} \cdot 1\{|n_2| \sim N_2\} \cdot 1\{|n_{1345}| \sim N_{1345}\} \cdot 1\{|n_{12345}| \sim N_{12345}\} \not\equiv 0.$$

Then

$$\frac{\min(N_2, N_{12345})^2 \min(N_1, N_{1345})}{\min(N_{12345}, N_{1345}, N_2)} \lesssim N_2 \cdot N_{12345}.$$

Proof. By using the properties of min and max, we have

$$\begin{aligned}
\frac{\min(N_2, N_{12345}) \min(N_1, N_{1345})}{\min(N_{12345}, N_{1345}, N_2)} &\lesssim \frac{\min(N_2, N_{12345}) N_{1345}}{\min(N_{12345}, N_{1345}, N_2)} \\
&\lesssim \max(\min(N_2, N_{12345}), N_{1345}).
\end{aligned}$$

Since $N_{1345} \lesssim \max(N_2, N_{12345})$, this yields

$$\begin{aligned}
\frac{\min(N_2, N_{12345})^2 \min(N_1, N_{1345})}{\min(N_{12345}, N_{1345}, N_2)} &\lesssim \min(N_2, N_{12345}) \cdot \max(N_2, N_{12345}) \\
&= N_2 N_{12345}.
\end{aligned}$$

∎

Proof of Lemma 4.27. Let $m, m' \in \mathbb{Z}$ be arbitrary. We introduce N_{12345} and N_{1345} to further decompose according to the size of n_{12345} and n_{1345}. Using the two-ball basic counting lemma (Lemma 4.17) for the sum over $n_2 \in \mathbb{Z}^3$ and summing over $n_1 \in \mathbb{Z}^3$ directly, we obtain

$$\begin{aligned}
&\sum_{n_1,\ldots,n_5\in\mathbb{Z}^3}\Big[\Big(\prod_{j=1}^5 1\{|n_j|\sim N_j\}\Big)1\{|n_{12345}|\sim N_{12345}\}1\{|n_{1345}|\sim N_{1345}\}\\
&\qquad\qquad\times\langle n_{12345}\rangle^{2(s-1)}\langle n_{1345}\rangle^{-2\beta}\langle n_{345}\rangle^{-2}\langle n_{34}\rangle^{-2\beta}\Big(\prod_{j=1}^5\langle n_j\rangle^{-2}\Big)\\
&\qquad\qquad\times 1\{|\psi-m|\leq 1\}\cdot\big(1\{|\varphi-m'|\leq 1\}+1\{|\widetilde{\varphi}-m'|\leq 1\}\big)\Big]\\
&\lesssim N_{12345}^{2(s-1)}N_{1345}^{-2\beta}\min(N_{12345},N_{1345},N_2)^{-1}\min(N_2,N_{12345})^3\prod_{j=1}^5 N_j^{-2}\\
&\quad\times\sum_{n_1,n_3,n_4,n_5\in\mathbb{Z}^3}\Big(\prod_{j=1,3,4,5}1\{|n_j|\sim N_j\}\Big)1\{|n_{1345}|\sim N_{1345}\}\\
&\qquad\qquad\times\langle n_{345}\rangle^{-2}\langle n_{34}\rangle^{-2\beta}1\{|\psi-m|\leq 1\}\\
&\lesssim N_{12345}^{2(s-1)}N_{1345}^{-2\beta}\min(N_{12345},N_{1345},N_2)^{-1}\min(N_2,N_{12345})^3\min(N_1,N_{1345})^3\prod_{j=1}^5 N_j^{-2}\\
&\quad\times\sum_{n_3,n_4,n_5\in\mathbb{Z}^3}\Big(\prod_{j=3}^5 1\{|n_j|\sim N_j\}\Big)\langle n_{345}\rangle^{-2}\langle n_{34}\rangle^{-2\beta}1\{|\psi-m|\leq 1\}.
\end{aligned}$$

Using Proposition 4.20 with $s=0$ and $\gamma=\beta$ to bound the remaining sum over n_3, n_4, and n_5, we obtain a bound of the total contribution by

$$N_{12345}^{2(s-1)}(N_1N_2)^{-2}\frac{\min(N_2,N_{12345})^3\min(N_1,N_{1345})^3}{\min(N_{12345},N_{1345},N_2)}(N_{1345}\max(N_3,N_4,N_5))^{-2\beta}.$$

As long as the contribution is nontrivial, we find that $N_{1345}\max(N_3,N_4,N_5)\gtrsim\max(N_1,N_3,N_4,N_5)$. Thus, it remains to prove that

$$N_{12345}^{2(s-1)}(N_1N_2)^{-2}\frac{\min(N_2,N_{12345})^3\min(N_1,N_{1345})^3}{\min(N_{12345},N_{1345},N_2)}\lesssim N_2^{-2\eta},$$

which follows from a short calculation. Indeed, using Lemma A.1, we can estimate the left-hand side by

$$\begin{aligned}
&N_{12345}^{2(s-1)}(N_1N_2)^{-2}\frac{\min(N_2,N_{12345})^3\min(N_1,N_{1345})^3}{\min(N_{12345},N_{1345},N_2)}\\
&\qquad\lesssim N_{12345}^{2s-1}\min(N_2,N_{12345})\min(N_1,N_{1345})^2N_1^{-2}N_2^{-1}\lesssim N_{12345}^{2s-1+2\eta}N_2^{-2\eta}.
\end{aligned}$$

Due to our condition on s, this is acceptable. ■

We now prove the double-resonance quintic counting estimate.

Proof of Lemma 4.29. We also apply a dyadic localization to $|n_{345}| \sim N_{345}$ and $|n_{45}| \sim N_{45}$. By paying a factor of $\log(2+\max(N_4, N_5))^2$, it suffices to estimate the maximum over N_{345}, N_{45} instead of the sum. We do not require a logarithmic loss in N_3, since $N_3 \gg N_4, N_5$ implies that there are only ~ 1 nontrivial choices for N_{345}. We first sum over $n_3 \in \mathbb{Z}^3$ using the two-ball basic counting lemma (Lemma 4.17). We then sum over $n_4 \in \mathbb{Z}^3$ using only the dyadic constraint. This yields

$$\begin{aligned}
&N_3^{-2} N_4^{-2} \sup_{m \in \mathbb{Z}^3} \sup_{|n_5| \sim N_5} \sum_{n_3, n_4 \in \mathbb{Z}^3} \Big[\Big(\prod_{j=3}^{4} 1\{|n_j| \sim N_j\} \Big) 1\{|n_{345}| \sim N_{345}\} \\
&\quad \times 1\{|n_{45}| \sim N_{45}\} \langle n_{345} \rangle^{-1} \langle n_{45} \rangle^{-\beta} 1\{\langle n_{345} \rangle \pm_3 \langle n_3 \rangle \pm_4 \langle n_4 \rangle \pm_5 \langle n_5 \rangle \in [m, m+1)\} \Big] \\
&\qquad \lesssim \min(N_{345}, N_{45}, N_3)^{-1} \min(N_3, N_{345})^3 N_{345}^{-1} N_{45}^{-\beta} N_3^{-2} N_4^{-2} \\
&\qquad\quad \times \sum_{n_4 \in \mathbb{Z}^3} 1\{|n_4| \sim N_4\} 1\{|n_{345}| \sim N_{345}\} \\
&\qquad \lesssim \min(N_{345}, N_{45}, N_3)^{-1} \min(N_3, N_{345})^3 \min(N_4, N_{45})^3 N_{345}^{-1} N_{45}^{-\beta} N_3^{-2} N_4^{-2}.
\end{aligned}$$

Using a minor variant of Lemma A.1, this contribution is bounded by

$$\begin{aligned}
N_{45}^{-\beta} N_3^{-1} N_4^{-2} \min(N_3, N_{345}) \min(N_4, N_{45})^2 &\lesssim \max(N_4, N_{45})^{-\beta} \\
&\lesssim \max(N_4, N_5)^{-\beta}.
\end{aligned}$$

■

A.6. *Septic counting estimates*

Proof of Lemma 4.31. Using the decay of $\widehat{V}$, it suffices to prove

$$\begin{aligned}
&\sum_{(n_j)_{j \notin \mathcal{P}}} \langle n_{\mathrm{nr}} \rangle^{2(s-1)} \Big(\sum_{(n_j)_{j \in \mathcal{P}}}^{*} 1\{|n_{1234567}| \sim N_{1234567}\} 1\{|n_{567}| \sim N_{567}\} 1\{|n_4| \sim N_4\} \\
&\qquad\qquad \times \Phi(n_1, n_2, n_3) \langle n_4 \rangle^{-1} \Phi(n_5, n_6, n_7) \Big)^2 \\
&\qquad \lesssim \log(2+N_4)^2 \big(N_{1234567}^{2(s-1/2)} N_{567}^{-2(\beta-\eta)} + N_{1234567}^{-2(1-s+\eta)} \big).
\end{aligned} \tag{A.3}$$

The argument relies on two of our previous estimates. Using the cubic sum estimate (Proposition 4.20), we deduce that for all $N_{123} \geq 1$,

$$\sum_{n_1, n_2, n_3 \in \mathbb{Z}^3} 1\{|n_{123}| \sim N_{123}\} \Big(\prod_{j=1}^{3} \langle n_j \rangle^{\eta/3} \Big) \Phi(n_1, n_2, n_3)^2 \lesssim N_{123}^{-2(\beta-\eta)}. \tag{A.4}$$

Using the basic resonance estimate (Lemma 4.25), we find for all $N_3 \geq 1$ that

$$\sum_{n_3 \in \mathbb{Z}^3} 1\{|n_3| \sim N_3\} \langle n_3 \rangle^{-1} \Phi(n_1, n_2, n_3) \lesssim \log(2+N_3) \langle n_{12} \rangle^{-1} \langle n_1 \rangle^{-1} \langle n_2 \rangle^{-1}. \tag{A.5}$$

Using the symmetry of Φ, it remains to consider the following three cases.

Case 1: $j = 4$ is unpaired. By first using Cauchy–Schwarz, summing over n_4, and then using (A.4), we obtain

$$\begin{aligned}
&\sum_{(n_j)_{j\notin\mathcal{P}}} \langle n_{\mathrm{nr}}\rangle^{2(s-1)} \Big(\sum_{(n_j)_{j\in\mathcal{P}}}^{*} 1\{|n_{1234567}| \sim N_{1234567}\} 1\{|n_{567}| \sim N_{567}\} 1\{|n_4| \sim N_4\} \\
&\qquad\qquad \times \Phi(n_1,n_2,n_3)\langle n_4\rangle^{-1}\Phi(n_5,n_6,n_7)\Big)^2 \\
&\lesssim \sum_{(n_j)_{j\notin\mathcal{P}}} \Big[1\{|n_{\mathrm{nr}}| \sim N_{1234567}\}\langle n_{\mathrm{nr}}\rangle^{2(s-1)} \langle n_4\rangle^{-2} \Big(\sum_{(n_j)_{j\in\mathcal{P}}}^{*} \Phi(n_1,n_2,n_3)^2\Big) \\
&\qquad \times \Big(\sum_{(n_j)_{j\in\mathcal{P}}}^{*} 1\{n_{567} \sim N_{567}\}\Phi(n_5,n_6,n_7)^2\Big)\Big] \\
&\lesssim N_{1234567}^{2(s-1/2)} \sum_{(n_j)_{j\notin\mathcal{P}\wedge j\neq 4}} \Big(\sum_{(n_j)_{j\in\mathcal{P}}}^{*} \Phi(n_1,n_2,n_3)^2\Big) \\
&\qquad \times \Big(\sum_{(n_j)_{j\in\mathcal{P}}}^{*} 1\{n_{567} \sim N_{567}\}\Phi(n_5,n_6,n_7)^2\Big)\Big] \\
&= N_{1234567}^{2(s-1/2)} \Big(\sum_{n_1,n_2,n_3\in\mathbb{Z}^3} \Phi(n_1,n_2,n_3)^2\Big)\Big(\sum_{n_5,n_6,n_7\in\mathbb{Z}^3} 1\{n_{567} \sim N_{567}\}\Phi(n_5,n_6,n_7)^2\Big)\Big] \\
&\lesssim N_{1234567}^{2(s-1/2)} N_{567}^{-2(\beta-\eta)}.
\end{aligned}$$

This contribution is acceptable.

Case 2: $(3,4) \in \mathcal{P}$. We let $\mathcal{P}'$ be the pairing on $\{1,2,5,6,7\}$ obtained by removing the pair $(3,4)$ from $\mathcal{P}$. We also understand the condition $j \notin \mathcal{P}'$ as a subset of $\{1,2,5,6,7\}$. By first using (A.5) and then Cauchy–Schwarz, we have that

$$\begin{aligned}
&\sum_{(n_j)_{j\notin\mathcal{P}}} \langle n_{\mathrm{nr}}\rangle^{2(s-1)} \Big(\sum_{(n_j)_{j\in\mathcal{P}}}^{*} 1\{|n_{1234567}| \sim N_{1234567}\} 1\{|n_{567}| \sim N_{567}\} 1\{|n_4| \sim N_4\} \\
&\qquad\qquad \times \Phi(n_1,n_2,n_3)\langle n_4\rangle^{-1}\Phi(n_5,n_6,n_7)\Big)^2 \\
&\quad\lesssim \log(2+N_4)^2 N_{1234567}^{2(s-1+\eta)} \\
&\qquad \times \sum_{(n_j)_{j\notin\mathcal{P}'}} \langle n_{\mathrm{nr}}\rangle^{-2\eta} \Big(\sum_{(n_j)_{j\in\mathcal{P}'}}^{*} 1\{|n_{567}| \sim N_{567}\}\langle n_{12}\rangle^{-1}\langle n_1\rangle^{-1}\langle n_2\rangle^{-1}\Phi(n_5,n_6,n_7)\Big)^2 \\
&\quad\lesssim \log(2+N_4)^2 N_{1234567}^{2(s-1+\eta)} \\
&\qquad \times \sum_{(n_j)_{j\notin\mathcal{P}'}} \Big[\Big(\sum_{(n_j)_{j\in\mathcal{P}'}}^{*} \langle n_{\mathrm{nr}}\rangle^{-2\eta} \Big(\prod_{j\in\mathcal{P}'} \langle n_j\rangle^{-\eta/6}\Big) \langle n_{12}\rangle^{-2}\langle n_1\rangle^{-2}\langle n_2\rangle^{-2}\Big) \\
&\qquad \times \Big(\sum_{(n_j)_{j\in\mathcal{P}'}}^{*} 1\{|n_{567}| \sim N_{567}\}\Big(\prod_{j\in\mathcal{P}'} \langle n_j\rangle^{\eta/6}\Big)\Phi(n_5,n_6,n_7)^2\Big)\Big].
\end{aligned}$$

We then use a direct calculation to bound the first inner factor and to estimate the sum over n_5, n_6, and n_7. The total contribution is bounded by $\log(2+N_4)^2 N_{1234567}^{2(s-1+\eta)} N_{567}^{-2(\beta-\eta)} \lesssim \log(2+N_4)^2 N_{1234567}^{2(s-1+\eta)}$, which is acceptable.

Case 3: $(4,5) \in \mathscr{P}$. We let $\mathscr{P}'$ be the pairing on $\{1,2,3,6,7\}$ obtained by removing the pair $(4,5)$ from $\mathscr{P}$. We also understand the condition $j \notin \mathscr{P}'$ as a subset of $\{1,2,3,6,7\}$. By first using (A.5) and then Cauchy–Schwarz, we have

$$\begin{aligned}
&\sum_{(n_j)_{j\notin\mathscr{P}}} \langle n_{\mathrm{nr}}\rangle^{2(s-1)} \Big(\sum^{*}_{(n_j)_{j\in\mathscr{P}}} 1\{|n_{1234567}|\sim N_{1234567}\} 1\{|n_{567}|\sim N_{567}\} 1\{|n_4|\sim N_4\} \\
&\qquad\qquad \times \Phi(n_1,n_2,n_3)\langle n_4\rangle^{-1}\Phi(n_5,n_6,n_7)\Big)^2 \\
&\lesssim \log(2+N_4)^2 N_{1234567}^{2(s-1+\eta)} \sum_{(n_j)_{j\notin\mathscr{P}'}} \langle n_{\mathrm{nr}}\rangle^{-2\eta} \\
&\qquad \times \Big(\sum^{*}_{(n_j)_{j\in\mathscr{P}'}} \Phi(n_1,n_2,n_3)\langle n_{67}\rangle^{-1}\langle n_6\rangle^{-1}\langle n_7\rangle^{-1}\Big)^2.
\end{aligned}$$

Arguing similarly to Case 2, we obtain an upper bound by $\log(2+N_4)^2 N_{1234567}^{2(s-1+\eta)}$. While this bound does not contain the gain in N_{567}, it is still acceptable. ∎

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